1. Consider the Runge function as before, $f(x) = \dfrac{1}{1+25x^2}$, where $x \in [-1,1]$. Approximate it using Chebyshev function and cubic splines using $\{5, 9, 17, 33, 65, 129, 257\}$ equally spaced nodes. Define the error due to interpolation as

$$e_n = \max_{x \in [-1,1]} \left\| f(x) - \tilde{f}_n(x) \right\|$$

where $\tilde{f}_n(x)$ is the approximation using the n nodes. To obtain e_n evaluate $\left\| f(x) - \tilde{f}_n(x) \right\|$ at $100,000$ equi-spaced points on $[-1,1]$ and take the maximum value. Tabulate and plot these errors. In the plot, let the X axis be number of nodes and Y axis be $\log(e_n)$. Comment on the behavior of the error as a function of n and which interpolant converges faster. What happens when you change the function to $1-|x|$? Again, comment on the behavior of the error and convergence of both interpolants.

2. The electric potential due to a unit charge is given by $1/R$, where R is the distance from the charge.
   - If the point charge is located at $(0,0,a)$, compute the potential at a point (r, θ, ϕ) given in spherical coorindates.
   - Show that the potential can be written as $\dfrac{1}{r}\displaystyle\sum_{k=0}^{\infty}\left(\frac{a}{r}\right)^k Q_k(\cos(\theta))$, where Q_k is the k^{th} Legendre polynomial and $a/r < 1$.
   - For $r > 2a$, how many terms in the expansion do we need to attain machine epsilon in double precision arithmetic, i.e., 2^{-53}?

3. Let P_n be the set of all polynomials of degree not exceeding n. Show that the polynomial $p \in P_n$ that minimizes

$$\int_{-1}^{1} \left(x^{n+1} - p(x)\right)^2 dx$$

is $p(x) = x^{n+1} - q_{n+1}(x)$, where $q_{n+1}(x)$ is the Legendre poylnomial Q_{n+1} scaled so that it has leading coefficient unity.