1. Consider the Vandermonde matrix V, i.e.,

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \end{bmatrix}$$

   - Show that $\det(V)$ is a polynomial in the variables $x_0, x_1, \ldots, x_n$ with degree $\dfrac{n(n+1)}{2}$.
   - Show that if $x_i = x_j$ for $i \neq j$, then $\det(V) = 0$.
   - Hence, conclude that $(x_i - x_j)$ is a factor of $\det(V)$.
   - Hence, conclude that $\det(V) = C\left(\displaystyle\prod_{1\leq j<i\leq n}(x_i - x_j)\right)$, where C is a constant.
   - Compare the coefficient of $x_1 x_2^2 x_3^3 \cdots x_n^n$ to conclude that $C = 1$.

2. Consider uniformly spaced nodes ($x_k = -1 + (2k+1)/n$ for $k \in \{0, 1, 2, \ldots, n-1\}$) and Chebyshev nodes ($y_k = \sin(\pi x_k/2)$ for $k \in \{0, 1, 2, \ldots, n-1\}$). For both these sets of nodes perform the following:
   - Plot the condition number of these Vandermonde matrices as a function of n. (Use semilogy to plot, i.e., the Y axis is the log(condition number).) Comment on how the condition number scales with n.
   - Consider the function $f(x) = \dfrac{1}{1+25x^2}$. This is called the Runge function. For $n \in \{5, 10, 20, 50\}$, obtain and plot the interpolant by
     - Solving the linear system
     - Using fundamental Lagrange polynomials, i.e., $\ell_j(x) = \dfrac{\displaystyle\prod_{k\neq j}(x - x_k)}{\displaystyle\prod_{k\neq j}(x_j - x_k)}$

     Comment on the interpolant you observe.
   - What is the cost of evaluating the interpolant at a point x as a function of n?
   - Based on the above observation, which method would you prefer for polynomial approximation?

3. Show that for any set of interpolation nodes, we have

$$\sum_{j=0}^{n} x_j^m \ell_j(x) = x^m$$

   for all $m \in \{0, 1, 2, \ldots, n\}$.

4. Recall that a function $f(x)$ on $[-1, 1]$ is α-Hölder continuous if for all $x, y \in [-1, 1]$, we have $|f(x) - f(y)| \leq C\,|x - y|^{\alpha}$ for some $\alpha, C \in \mathbb{R}^+$ and is Lipschitz continuous if $\alpha = 1$. We will denote α-Hölder continuous functions on $[-1, 1]$ as $H^{\alpha}([-1, 1])$.
   Prove that if $\alpha < \beta$, then $H^{\alpha}([-1, 1]) \supset H^{\beta}([-1, 1])$.

5. Give examples (with proofs as to why the examples are correct) of function on $[-1, 1]$ for the following:
   - Continuous but not Hölder continuous for any $\alpha > 0$
   - Lipschitz but not differentiable
   - Differentiable but its derivative is not continuous