MA2040: Probability, Statistics and Stochastic Processes
Problem Set-III

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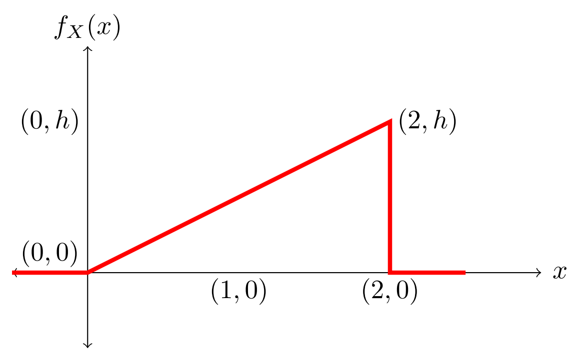
1. If $X_1, X_2, \ldots, X_n$ are independent random variables having the same probability density function $f_X(x)$, what is the probability density function for the random variable $Y = \min\{X_1, X_2, \ldots, X_n\}$?
Solution:

$$P(Y \geq y) = P(X_1 \geq y, X_2 \geq y, \ldots, X_n \geq y) = P(X_1 \geq y)P(X_2 \geq y)\ldots P(X_n \geq y)$$

Hence, we obtain that

$$1 - F_Y(y) = (1 - F_X(y))^n$$
$$f_Y(y) = n f_X(y) (1 - F_X(y))^{n-1}$$

2. A random variable X has a probability density function as shown below.



(a) Determine h
Solution: Area equals one implies $h = 1$

(b) Determine the cumulative distribution function

Solution: $F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2/4 & \text{if } x \in [0,2] \\ 1 & \text{if } x \geq 2 \end{cases}$

(c) Compute the mean

Solution: We have

$$\mathbb{E}(X) = \int_0^2 x f_X(x) dx = \int_0^2 x \cdot \frac{x}{2} dx = 4/3$$

(d) Compute the variance

Solution:

$$\mathbb{E}\left(X^2\right) = \int_0^2 x^2 f_X(x)dx = \int_0^2 x^2 \cdot \frac{x}{2}dx = 2 \implies \text{Var}\left(X\right) = 2/9$$

(e) Determine the probability that $X \in (1, 2)$.

Solution:

$$\mathbb{P}\left(X \in (1, 2)\right) = \int_1^2 f_X(x)dx = \int_1^2 x/2dx = 3/4$$

3. The median m of a probability density function is defined as the value of m such that

$$\int_{-\infty}^m f(x)dx = \int_m^\infty f(x)dx = 1/2$$

Essentially, the median splits the distribution into two equal halves. Prove that the median is the best predictor if one wants to minimize the expected value of the absolute error, i.e., $\mathbb{E}\left(|X - c|\right)$ is minimized when c is the median of the underlying distribution.

Solution: We have

$$\mathbb{E}\left(|X - c|\right) = \int_{-\infty}^c \left(c - x\right) f_X(x)dx + \int_c^\infty \left(x - c\right) f_X(x)dx$$

Differentiating with respect to c, we obtain

$$\int_{-\infty}^c f_X(x)dx = \int_c^\infty f_X(x)dx$$

We also have that

$$\int_{-\infty}^c f_X(x)dx + \int_c^\infty f_X(x)dx = 1$$

Hence, this gives us that

$$\int_{-\infty}^c f_X(x)dx = \int_c^\infty f_X(x)dx = 1/2$$

4. Let X be a random variable, whose pdf is given by

$$f_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ xe^{-x^2/2} & \text{if } x > 0 \end{cases}$$

Find the pdf for the random variable $Y = X^2$.

Solution: Since $Y = X^2$ is an increasing function on $[0, \infty)$, we have

$$f_Y(y) = f_X(\sqrt{y})\frac{dx}{dy} = \frac{f_X(\sqrt{y})}{2\sqrt{y}} = 1/2e^{-y/2}$$

5. Let X be a uniform random variable on the interval $[0, 1]$. Consider the random variable $Y = g\left(X\right)$, where

$$g(x) = \begin{cases} 1 & \text{if } x \leq 1/3 \\ 2 & \text{else} \end{cases}$$

Find the probability mass function of Y and compute its expected value.

Solution:

$$\mathbb{P}\left(Y = 1\right) = \mathbb{P}\left(X \leq 1/3\right) = 1/3$$
$$\mathbb{P}\left(Y = 2\right) = \mathbb{P}\left(X \geq 1/3\right) = 2/3$$

6. Show the expected value of a random variable X can also be obtained as

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\,dx - \int_0^\infty \mathbb{P}(X < -x)\,dx$$

Solution: We have $X = X^+ - X^-$, where $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$. Note that

$$X^+ = \int_0^\infty I(X > t)\,dt$$

and

$$X^- = \int_{-\infty}^0 I(X \leq t)\,dt$$

where I is the indicator function that takes the value 1, when the argument is true and takes the value 0, when the argument is false. We have

$$\mathbb{E}(X) = \mathbb{E}\left(X^+ - X^-\right)$$

which gives us what we want. Note that expectation of the indicator function is nothing but the probability of the argument.

7. A defective coin minting machine produces coins whose probability of heads is a random variable Y with PDF

$$f_Y(y) = \begin{cases} y\exp(y) & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

8. Let the random variables X and Y have a joint PDF, which is uniform over the triangles with vertices $(0,0)$, $(0,1)$ and $(1,0)$.

   (a) Find the joint PDF of X and Y.

   Solution: $f_{XY}(x,y) = 2$ over the triangle.

   (b) Find the marginal PDFs.

   Solution:

$$f_X(x) = \int_{y=0}^{1-x} 2dy = 2(1-x)$$
$$f_Y(y) = \int_{x=0}^{1-y} 2dx = 2(1-y)$$

   (c) Find the conditional PDFs.

   Solution:

$$f_{X|Y=y} = \frac{1}{1-y}$$
$$f_{Y|X=x} = \frac{1}{1-x}$$

9. Chennai's temperature is modeled as a normal random variable with a mean temperature of 34°C and a standard deviation of 5°C. What is the probability that the temperature at a randomly chosen time will exceed 45°C?

Solution: $z = \dfrac{45-34}{5} = 2.2$. $\Phi(2.2) = 0.9861$. Hence, desired probability is $1 - 0.9861 = 0.0139$.

10. A surface is ruled with parallel lines, which are at a distance d from each other. Suppose that we throw a needle of length l on the surface at random. What is the probability that the needle with intersect one of the lines? (NOTE: You will need to treat the case $d < l$ and $d > l$ separately.)

    Solution: Page 161. Bertsekas.

$$d > l : \frac{2l}{\pi d}$$

$$d < l : \frac{2}{\pi} \arccos\left(\frac{d}{l}\right) + \frac{2}{\pi}\frac{l}{d}\left(1 - \sqrt{1 - \left(\frac{d}{l}\right)^2}\right)$$

11. Consider two continuous random variables Y and Z and a random variable X that is equal to Y with a probability p and equals Z with a probability $1-p$. Obtain the pdf of X interms of the pdf's of Y and Z.

    Solution:

$$F_X(x) = \mathbb{P}(X \le x) = p\mathbb{P}(Y \le x) + (1-p)\,\mathbb{P}(Z \le x)$$

Hence,

$$f_X(x) = pf_Y(x) + (1-p)\,f_Z(x)$$