MA2040: Probability, Statistics and Stochastic Processes
Problem Set-II

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1. Assume that Carlsen meets Ding Liren in the 2020 chess championship. The championship match consists of a sequence of games and each game has three outcomes (i) Ding Liren winning, (ii) Carlsen winning, (iii) A draw. The first player to win a game wins the match. For instance, we could have a sequence of 3 draws followed by a Carlsen victory in the 4^{th} game, which would mean that Carlsen wins the Championship. The probability of a single game ending in

   (a) Carlsen's favour is 0.4

   (b) Ding Liren's favour is 0.2

   (c) draw is 0.4

   i What is the probability of Ding Liren winning the championship?

   ii What is the probability of Carlsen winning the championship?

   iii What is the Probability Mass Function for the number of games played in the championship?

2. Consider rolling a pair of fair dice. Let X denote the difference between the numbers that show up on the dice, i.e., $X = |D_1 - D_2|$, where D_i is the number that shows up on the i^{th} dice.

   - What are the possible values for X?
   - What is the probability mass function for X?
   - Find the expected value and standard deviation of X.

3. A fair die is rolled repeatedly till an odd prime appears. What is the probability that the number of rolls exceed 5?

4. Let X be a discrete random variable with mean μ and variance σ^2. Prove that

$$\mathbb{E}\left[(X-a)^2\right] = \sigma^2 + (a-\mu)^2$$

   Hence, prove that the mean (or expected value) minimizes $\mathbb{E}\left[(X-a)^2\right]$.

5. A production process is partitioned into two independent sub-processes. The probabilities of a defective component in the first and second sub-processes are 0.01 and 0.02, respectively. If 50 units are produced, what is the probability there will be fewer than 3 defective units?

6. Communication channels do not always trasmit the correct signal. Suppose that for a particular channel the error rate is 1 in 100, i.e., the probability of incorrect transmission is 1/100. If 2000 messages are sent in a given week, and it is assumed that their transmissions are independent, what is the probability that there will be at least 5 errors?

7. A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake starts at \$1 and is increased by \$1 every time heads appears. The first time tails appears, the game ends and the player wins whatever is in the pot. Thus the player wins \$1 dollar if tails appears on the first toss, \$2 dollars if heads appears on the first toss and tails on the second, \$3 dollars if heads appears on the first two tosses and tails on the third, and so on. Mathematically, the player wins \$$k$ dollars, when we have the first $k-1$ tosses to be heads and the k^{th} toss to be a tail. The casino demands a pay of \$3 to enter the game. Will you play the game?

8. Repeat the above if the price money was 2^k instead of k and the casino demands a pay of \$100 to enter the game. Will you still be willing to play the game? (For more details, look up St. Petersburg paradox)

9. If X is a discrete random variable, prove that (i) $\mathbb{E}[aX+b] = a\mathbb{E}[X]+b$ and (ii) $\mathrm{Var}[aX+b] = a^2\mathrm{Var}(X)$

10. Data shows that 5% of the individuals reserving tables at a restaurant will not appear. If the restaurant has 50 tables and takes 52 reservations, what is the probability that it will be able to accommodate everyone appearing?

11. Electrical power failures in a workplace are modeled as a Poisson experiment with a rate of one every two months.

    (a) What is the probability of having more than 10 failures in a year?

    (b) What is the probability that the number of failures in a year will differ by more than a standard deviation from the expected number?

12. Table 1 below indicates the joint probabilities

Table 1: Joint probability of weather and power cuts

	Sunny	Rainy
Power cut	0.2	0.15
No power cut	0.6	p

    (a) Find p.

    (b) What is the probability that there won't be rain for one week?

    (c) What is the probability that there will be at least one power in the next three days?

    (d) Is there a dependence between weather and power cuts?

    (e) Find the joint probability, all marginal probabilities, and all conditional probabilities.