MA2040: Probability, Statistics and Stochastic Processes
Problem Set-I

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1. A six-sided die is made in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4.
Solution: The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let the probability of each odd face be p. We then have the probability of each even face to be $2p$. We have
$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\{1\}) + \mathbb{P}(\{2\}) + \mathbb{P}(\{3\}) + \mathbb{P}(\{4\}) + \mathbb{P}(\{5\}) + \mathbb{P}(\{6\})$$
This gives us that
$$9p = 1 \implies p = 1/9$$
$$\mathbb{P}(\text{Outcome is less than } 4) = \mathbb{P}(\{1\}) + \mathbb{P}(\{2\}) + \mathbb{P}(\{3\}) = p + 2p + p = 4p = 4/9$$

2. Let $S_1, S_2, \ldots, S_n$ be a partition of the sample space Ω.
   (a) Show that for any event A,
   $$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap S_i)$$
   Solution: Since $S_1, S_2, \ldots, S_n$ is a partition of Ω, we have $\Omega = \displaystyle\bigcup_{i=1}^{n} S_i$ and $S_i \bigcap S_j = \emptyset$, whenever $i \neq j$. We have
   $$A = A \bigcap \Omega = A \bigcap \left(\bigcup_{i=1}^{n} S_i\right) = \bigcup_{i=1}^{n} \left(A \bigcap S_i\right)$$
   Since S_i's are mutually disjoint, so are $A \bigcap S_i$'s. Hence, we have that
   $$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{n} \left(A \bigcap S_i\right)\right) = \sum_{i=1}^{n} \mathbb{P}\left(A \bigcap S_i\right)$$
   (b) Use the previous part to show that, for events A, B and C,
   $$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B^c \cap C^c) - \mathbb{P}(A \cap B \cap C)$$
   Solution: Note that our sample space can be paritioned into two disjoint sets $S_1 = B \bigcup C$ and $S_2 = \left(B \bigcup C\right)^c = B^c \bigcap C^c$. From previous part, we hence have
   $$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(A \cap S_1) + \mathbb{P}(A \cap S_2) = \mathbb{P}(A \cap (B \cup C)) + \mathbb{P}(A \cap (B^c \cap C^c)) && (1) \\ &= \mathbb{P}((A \cap B) \cup (A \cap C)) + \mathbb{P}(A \cap B^c \cap C^c) && (2) \\ &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}((A \cap B) \cap (A \cap C)) + \mathbb{P}(A \cap B^c \cap C^c) && (3) \\ &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B^c \cap C^c) - \mathbb{P}(A \cap B \cap C) && (4) \end{aligned}$$

3. (a) Prove that for any two events A and B, we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$$

Solution: Recall that $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$. Since $\mathbb{P}(A \cup B) \leq 1$, we obtain that $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$.

(b) Using the above, establish the following generalization:

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n) - (n-1)$$

Solution: This is immediately obtained by induction. Let $S_{n+1} = A_1 \cap A_2 \cap \cdots A_n$. From the previous part, we have

$$\mathbb{P}(S_{n+1}) = \mathbb{P}(S_n \cap A_n) \geq \mathbb{P}(S_n) + \mathbb{P}(A_n) - 1$$

And now by the induction, we have that $\mathbb{P}(S_n) \geq \displaystyle\sum_{i=1}^{n-1} \mathbb{P}(A_i) - (n-2)$ and hence we obtain what we want.

4. Let $\Omega = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 2019\}$ and x_1, x_2, x_3 are positive integers. Assuming all the elements in Ω are equally likely to be drawn, what is the probability of having x_1, x_2 and x_3 (all three) to be odd?

Solution: Let A be the event we are after, i.e., $A = \{(x_1, x_2, x_3) \in \Omega : x_1, x_2, x_3 \text{ are all odd}\}$. The number of elements in Ω is given by $\dbinom{2019-1}{3-1} = \dbinom{2018}{2}$. To obtain the number of elements in A, let $x_i = 2y_i - 1$, where $y_i \in \mathbb{Z}^+$. We then need the number of positive integer solutions to

$$(2y_1 - 1) + (2y_2 - 1) + (2y_3 - 1) = 2019 \implies y_1 + y_2 + y_3 = 1011$$

The number of elements in A is therefore given by $\dbinom{1011-1}{3-1} = \dbinom{1010}{2}$. Hence, the desired probability is

$$\frac{\dbinom{1010}{2}}{\dbinom{2018}{2}} = \frac{1010 \times 1009}{2018 \times 2017} = \frac{505}{2017}$$

Remark: Note that the number of distinct positive integer solutions to $x_1 + x_2 + \cdots + x_n = N$, where $N \geq n$, $n, N \in \mathbb{Z}^+$ is given by

$$\binom{N-1}{n-1}$$

The proof is as follows. Consider a sequence of N ones, i.e., $\underbrace{111\ldots1}_{N \text{ ones}}$. A n-partition of N is obtained by inserting $n-1$ bars in the $N-1$ gaps corresponding to N ones. For example, consider $n = 3$ and $N = 6$. One possible solution is $(x_1, x_2, x_3) = (2, 3, 1)$. This can be interpreted as inserting a bar after the second 1 and another bar 3 ones later, i.e.,

$$11\Big|111\Big|1$$

Note that any insertion of $n-1$ bars in the $N-1$ gaps gives a solution (comprising of positive integers) to $x_1 + x_2 + \cdots + x_n = N$ and any solution (comprising of positive integers) to $x_1 + x_2 + \cdots + x_n = N$

can be interpreted as inserting $n-1$ bars in the $N-1$ gaps. Hence, the number of positive integer solutions to $x_1+x_2+\cdots+x_n=N$ is given by

$$\binom{N-1}{n-1}$$

5. Two fair 6-sided dice are rolled.

   (a) Given that the roll results in a sum of 4 or less, find the conditional probability that both dice show the same number.
   Solution: A be the event that both dice show the same number and B be the event that the roll results in a sum not exceeding 4. We need $\mathbb{P}(A|B)$. We have $B=\{(1,1),(1,2),(2,1),(2,2)\}$. We have $A=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$ and $A\cap B=\{(1,1),(2,2)\}$. Hence,

   $$\mathbb{P}(A|B)=\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}=\frac{2/36}{4/36}=\frac{1}{2}$$

   (b) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.
   Solution: Let C be the event that at least one die roll is 6 and D be the event that the two dice land on different numbers. We have

   $$C=\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,5),(6,4),(6,3),(6,2),(6,1)\}$$
   $$D=\Omega-\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

   Hence,

   $$\mathbb{P}(C|D)=\frac{\mathbb{P}(C\cap D)}{\mathbb{P}(D)}=\frac{10/36}{30/36}=1/3$$

6. A batch of 100 items is inspected by testing 4 randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted, if it contains exactly five defectives?
   Solution:

   $$\begin{aligned}\mathbb{P}(\text{Batch being accepted}) &= \mathbb{P}(\text{Four randomly selected items being non-defective}) \quad (5)\\ &= p_1p_2p_3p_4 \quad (6)\end{aligned}$$

   where $p_i=\mathbb{P}\left(i^{th}\text{ item being non-defective given that the previous } i-1 \text{ items are defective}\right)$. We have $p_1=\frac{95}{100}, p_2=\frac{94}{99}, p_3=\frac{93}{98}$ and $p_4=\frac{92}{97}$. Hence, the desired probability is $\frac{95}{100}\times\frac{94}{99}\times\frac{93}{98}\times\frac{92}{97}$.

7. Consider a coin that comes up heads with probability p and tails with probability $1-p$. Let q_n be the probability of obtaining even number of heads in n independent tosses. Derive a recursion that relates q_n to q_{n-1} and establish the formula

   $$q_n=\frac{1+(1-2p)^n}{2}$$

   Solution: We will condition on the n^{th} toss.

   $$\begin{aligned}q_n &= \mathbb{P}(\text{Obtaining even number of heads in } n \text{ independent tosses} \mid \text{Last toss is a head})\,\mathbb{P}(\text{Last toss is a head}) \quad (7)\\ &+ \mathbb{P}(\text{Obtaining even number of heads in } n \text{ independent tosses} \mid \text{Last toss is a tail})\,\mathbb{P}(\text{Last toss is a tail}) \quad (8)\\ &= \mathbb{P}(\text{Obtaining odd number of heads in } n-1 \text{ independent tosses})\cdot\mathbb{P}(\text{Last toss is a head}) \quad (9)\\ &+ \mathbb{P}(\text{Obtaining even number of heads in } n-1 \text{ independent tosses})\cdot\mathbb{P}(\text{Last toss is a tail}) \quad (10)\\ &= (1-q_{n-1})\,p+q_{n-1}(1-p)=p+(1-2p)\,q_{n-1} \quad (11)\end{aligned}$$

Note that $q_1 = 1 - p$. Hence, we have

$$\begin{aligned}
q_n &= p + (1-2p)\left(p + (1-2p)\, q_{n-2}\right) = p + p\,(1-2p) + (1-2p)^2\, q_{n-2} && (12)\\
&= p + p\,(1-2p) + p\,(1-2p)^2 + p\,(1-2p)^3 + \cdots + (1-2p)^{n-1}\, q_1 && (13)\\
&= p \sum_{k=0}^{n-2} (1-2p)^k + (1-p)\,(1-2p)^{n-1} && (14)\\
&= p\left(\frac{1-(1-2p)^{n-1}}{2p}\right) + (1-p)\,(1-2p)^{n-1} && (15)\\
&= \frac{1-(1-2p)^{n-1}}{2} + \frac{2\,(1-p)\,(1-2p)^{n-1}}{2} && (16)\\
&= \frac{1+(1-2p)^{n-1}\,(2-2p-1)}{2} && (17)\\
&= \frac{1+(1-2p)^{n}}{2} && (18)
\end{aligned}$$

8. Two playes X and Y alternately roll a pair of unbiased dice. X wins if on a throw he gets a sum of 6 before Y gets a sum of 7; Y wins if he obtains a sum of 7 before X obtains a sum of 6; If X begins the game, prove that his probability of winning is $30/61$.
Solution:

$$\mathbb{P}\left(X \text{ winning}\right) = \sum_{n=1}^{\infty} \mathbb{P}\left(X \text{ winning on the } n^{th} \text{ attempt}\right) = \sum_{n=1}^{\infty} p_n$$

$$p_n = \mathbb{P}\left(X \text{ doesn't roll a sum of 6 and Y doesn't roll a sum of 7 in the previous } n-1 \text{ attempts}\right)$$

$$p_n = \sum_{n=1}^{\infty} p^{n-1} q^{n-1}\,(1-p)$$

where p is the probability of not getting a sum of 6 and q is the probability of not getting a sum of 7. We have $p = 1 - 5/36 = 31/36$ and $q = 1 - 6/36 = 5/6$. Hence, the desired probability is

$$\sum_{n=1}^{\infty} \left(\frac{31}{36}\right)^{n-1} \left(\frac{5}{6}\right)^{n-1} \times \frac{5}{36} = \frac{5}{36} \times \frac{1}{1 - \dfrac{155}{216}} = \frac{30}{61}$$

9. In a deck of cards, let A be the event of drawing a spade and B be the event of drawing a king. Assuming that all cards are equally likely to be drawn, obtain $\mathbb{P}(A|B)$, $\mathbb{P}(B|A)$. Are A and B independent?
Solution: $\mathbb{P}(A) = \dfrac{13}{52} = \dfrac{1}{4}$, $\mathbb{P}(B) = \dfrac{4}{52} = \dfrac{1}{13}$, $\mathbb{P}\left(A \cap B\right) = \dfrac{1}{52}$, $\mathbb{P}\left(A|B\right) = \dfrac{\mathbb{P}\left(A \cap B\right)}{\mathbb{P}\left(B\right)} = \dfrac{1/52}{4/52} = \dfrac{1}{4}$, $\mathbb{P}\left(B|A\right) = \dfrac{\mathbb{P}\left(A \cap B\right)}{\mathbb{P}(A)} = \dfrac{1/52}{13/52} = \dfrac{1}{13}$. Hence, we see that

$$\mathbb{P}\left(A|B\right) = \mathbb{P}\left(A\right)$$

and hence A, B are independent.

10. Let p_X denote the probability that India will play against team X in the World Cup Final 2019, given that India will qualify for the World Cup Final 2019. Let q_X be the probability of Kohli scoring a century against team X.

    (a) From the information given above, find the probability that Kohli will score a century in the World Cup Final 2019.

Teams	p_X	q_X
Afghanistan	0.02	0.50
Australia	0.10	0.30
Bangladesh	0.03	0.40
England	0.30	0.10
New Zealand	0.15	0.25
Pakistan	0.10	0.15
South Africa	0.20	0.20
SriLanka	0.05	0.25
West Indies	0.05	0.20

Solution: Let A be the event of Kohli scoring a century in the World Cup Final 2019 and E_X be the event of India playing against X in the World Cup Final 2019. Note that we have $p_X = \mathbb{P}(E_X)$ and $q_X = \mathbb{P}(A|E_X)$.

$$\begin{aligned}\mathbb{P}(A) &= \sum_{X\in\{\text{all teams}\}} \mathbb{P}(A|E_X)\,\mathbb{P}(E_X) = \sum_{X\in\{\text{all teams}\}} p_X q_X \\ &= 0.02 \times 0.50 + 0.10 \times 0.30 + 0.03 \times 0.40 + 0.30 \times 0.10 + 0.15 \times 0.25 \\ &+ 0.10 \times 0.15 + 0.20 \times 0.20 + 0.05 \times 0.25 + 0.05 \times 0.20 \\ &= 0.197\end{aligned}$$

(b) Given that Kohli scores a century in the World Cup Final 2019, which team is

i. Most likely to have played the Final along with India

ii. Least likely to have played the Final along with India

Solution: We need $\mathbb{P}(E_X|A)$. We have

$$\mathbb{P}(E_X|A) = \frac{\mathbb{P}(A|E_X)\,\mathbb{P}(E_X)}{\mathbb{P}(A)}$$

We see that South Africa is the most likely team to have played the Final against India, while West-Indies and Afghanistan are the least likely teams to have played the Final against India.

Afghanistan	$\dfrac{0.02 \times 0.50}{0.197} = \dfrac{10}{197}$
Australia	$\dfrac{0.10 \times 0.30}{0.197} = \dfrac{30}{197}$
Bangladesh	$\dfrac{0.03 \times 0.40}{0.197} = \dfrac{12}{197}$
England	$\dfrac{0.30 \times 0.10}{0.197} = \dfrac{30}{197}$
New Zealand	$\dfrac{0.15 \times 0.25}{0.197} = \dfrac{37.5}{197}$
Pakistan	$\dfrac{0.10 \times 0.15}{0.197} = \dfrac{15}{197}$
South Africa	$\dfrac{0.20 \times 0.20}{0.197} = \dfrac{40}{197}$
SriLanka	$\dfrac{0.05 \times 0.25}{0.197} = \dfrac{12.5}{197}$
West Indies	$\dfrac{0.05 \times 0.20}{0.197} = \dfrac{10}{197}$

11. A test for certain rare virus correctly predicts that the person has a virus 99% of the time and correctly identifies that the person doesn't carry a virus 98% of the time. It is known 1% of the population carries the virus. What is the probability of a person actually having the disease, if he has tested positive to the test?
Solution: Let A be the event that test predicts virus present, and B be the event that the person carries the virus. We are given that

$$\mathbb{P}(A|B) = 0.99, \mathbb{P}(A^c|B^c) = 0.98, \mathbb{P}(B) = 0.01$$

We need $\mathbb{P}(B|A)$. We have

$$\mathbb{P}(A|B^c) = 1 - \mathbb{P}(A^c|B^c) = 0.02$$

We also have that

$$\mathbb{P}(B^c) = 1 - \mathbb{P}(B) = 0.99$$

$$\mathbb{P}(A) = \mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}(A|B^c)\,\mathbb{P}(B^c) = 0.99 \times 0.01 + 0.02 \times 0.99 = 0.03 \times 0.99$$

Hence,

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\,\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{0.99 \times 0.01}{0.03 \times 0.99} = \frac{1}{3}$$