MA2040: Probability, Statistics and Stochastic Processes
Problem Set-I

Sivaram Ambikasaran

1. A six-sided die is made in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4.

2. Let $S_1, S_2, \ldots, S_n$ be a partition of the sample space Ω.

   (a) Show that for any event A,
   $$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}\left(A \cap S_i\right)$$

   (b) Use the previous part to show that, for events A, B and C,
   $$\mathbb{P}\left(A\right) = \mathbb{P}\left(A \cap B\right) + \mathbb{P}\left(A \cap C\right) + \mathbb{P}\left(A \cap B^c \cap C^c\right) - \mathbb{P}\left(A \cap B \cap C\right)$$

3. (a) Prove that for any two events A and B, we have
   $$\mathbb{P}\left(A \cap B\right) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$$

   (b) Using the above, establish the following generalization:
   $$\mathbb{P}\left(A_1 \cap A_2 \cap \cdots \cap A_n\right) \geq \mathbb{P}\left(A_1\right) + \mathbb{P}\left(A_2\right) + \cdots + \mathbb{P}\left(A_n\right) - (n-1)$$

4. Let $\Omega = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 2019\}$ and x_1, x_2, x_3 are positive integers. Assuming all the elements in Ω are equally likely to be drawn, what is the probability of having x_1, x_2 and x_3 (all three) to be odd?

5. Two fair 6-sided dice are rolled.

   (a) Given that the roll results in a sum of 4 or less, find the conditional probability that both dice show the same number.

   (b) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

6. A batch of 100 items is inspected by testing 4 randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted, if it contains exactly five defectives?

7. Consider a coin that comes up heads with probability p and tails with probability $1-p$. Let q_n be the probability of obtaining even number of heads in n independent tosses. Derive a recursion that relates q_n to q_{n-1} and establish the formula
   $$q_n = \frac{1 + (1-2p)^n}{2}$$

8. Two playes X and Y alternately roll a pair of unbiased dice. X wins if on a throw he gets a sum of 6 before Y gets a sum of 7; Y wins if he obtains a sum of 7 before X obtains a sum of 6; If X begins the game, prove that his probability of winning is 30/61.

9. In a deck of cards, let A be the event of drawing a spade and B be the event of drawing a king. Assuming that all cards are equally likely to be drawn, obtain $\mathbb{P}(A|B)$, $\mathbb{P}(B|A)$. Are A and B independent?

10. Let p_X denote the probability that India will play a team X in the World Cup Final 2019, given that India will qualify for the World Cup Final 2019. Let q_X be the probability of Kohli scoring a century against team X.

Teams	p_X	q_X
Afghanistan	0.02	0.50
Australia	0.10	0.30
Bangladesh	0.03	0.40
England	0.30	0.10
New Zealand	0.15	0.25
Pakistan	0.10	0.15
South Africa	0.20	0.20
SriLanka	0.05	0.25
West Indies	0.05	0.20

    (a) From the information given above, find the probability that Kohli will score a century in the World Cup Final 2019.

    (b) Given that Kohli scores a century in the World Cup Final 2019, which team is

        i. Most likely to have played the Final along with India

        ii. Least likely to have played the Final along with India

11. A test for certain rare virus correctly predicts that the person has a virus 99% of the time and correctly identifies that the person doesn't carry a virus 98% of the time. It is known 1% of the population carries the virus. What is the probability of a person actually having the disease, if he has tested positive to the test?