Deep Koopman Linearization of Nonlinear Dynamical System

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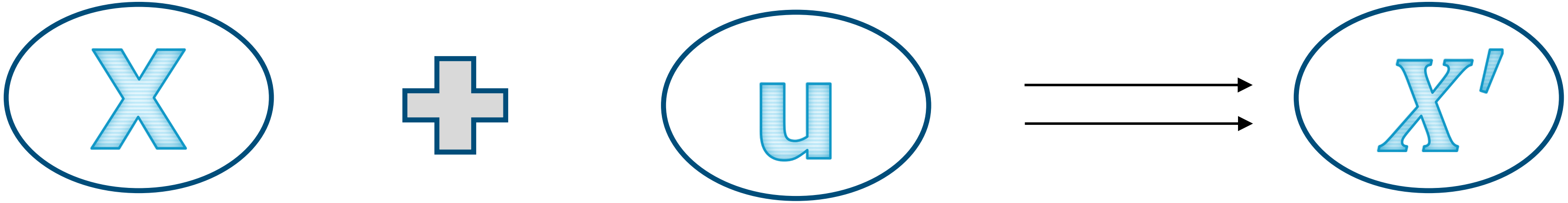
System and Control



State

Applying Disturbance/ Control/ Action/ Doing of any

New State

In mathematical expression, $x_{k+1} = F(x_k, u)$

For continuous system, $\dot{X} = A.X + B.u$ $\rightarrow$ integrate it

Koopman Theory & Benefits:

- The Koopman operator is a linear operator that describes the evolution of scalar observables (i.e., measurement functions of the states) in an infinite dimensional Hilbert space.

- Nonlinear states can be represented as linear in higher dimensional space. The variable which is not part of the dynamics will automatically eliminated by having zero eigen values.

- Deep Learning Model requires unbiased data which would be larger in size. Koopman Theory models capture the dynamics with lesser amount of data.

Koopman Operator (K):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathcal{K}_t g = g \circ \mathbf{F}_t$$

$$\mathcal{K}_{\Delta t} g(\mathbf{x}_k) = g(\mathbf{F}_{\Delta t}(\mathbf{x}_k)) = g(\mathbf{x}_{k+1})$$

$$g(\mathbf{x}_{k+1}) = \mathcal{K}_{\Delta t} g(\mathbf{x}_k)$$

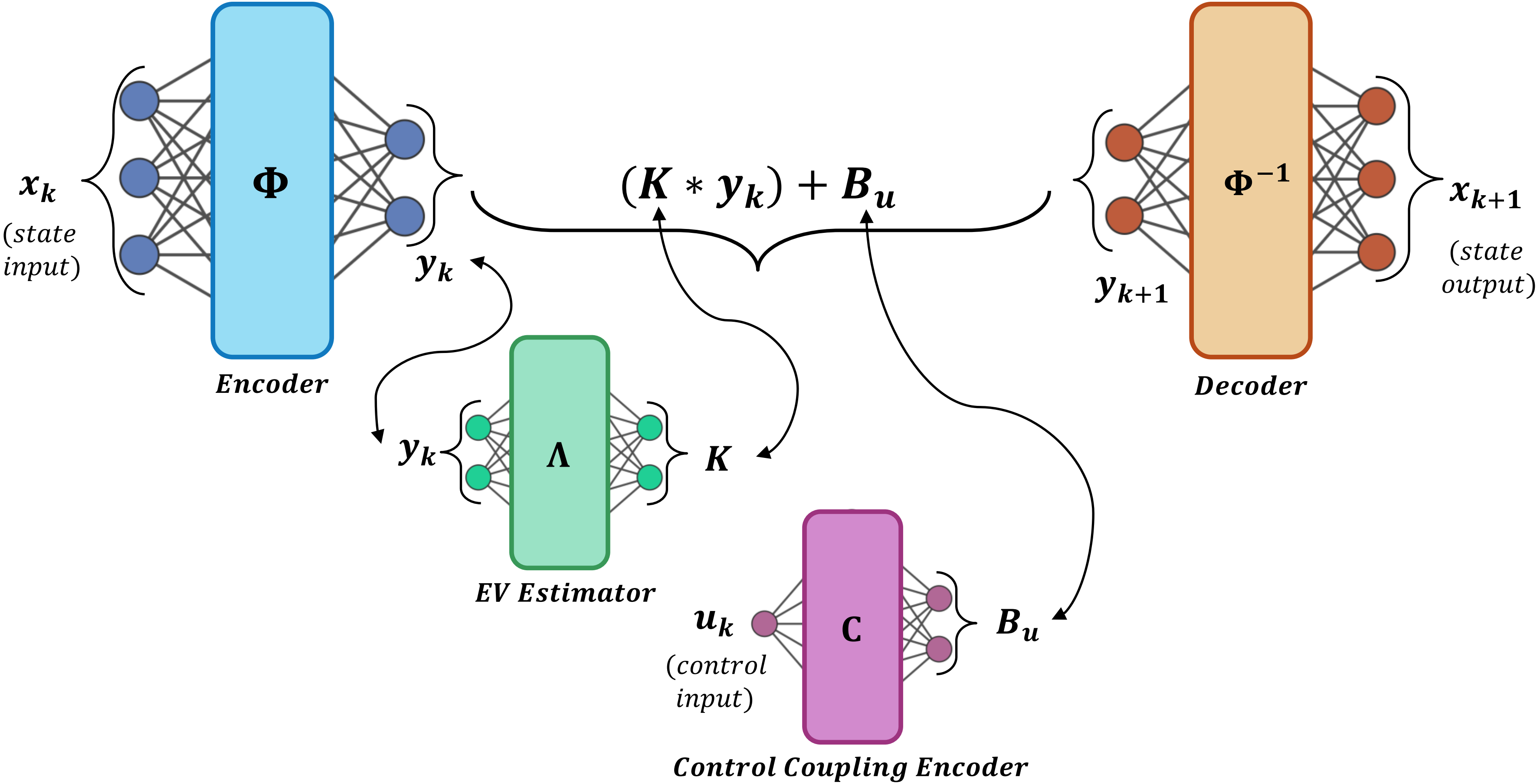
- Dynamic Mode Decomposition (DMD)
- Extended Dynamic Mode Decomposition (eDMD)

Based on the literature survey, Parsimonious Approach to the model development is limited.

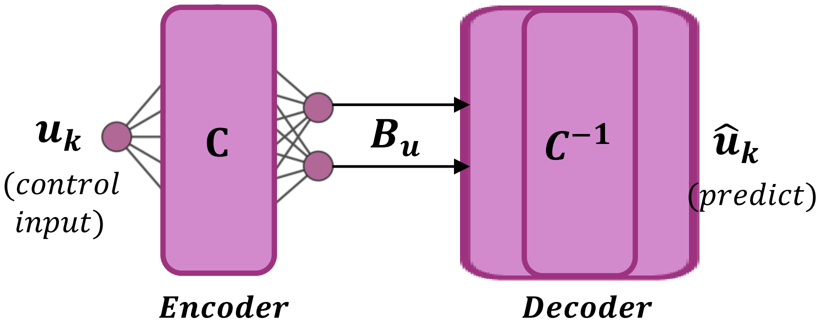
Dynamic Mode Decomposition – *(for observation)*

- $The\ Motivation\ is \quad \rightarrow \quad X' = A.X$
- $Step\ 1:$ $\quad X = U\,\Sigma\,V^T$
- $Step\ 2:$ $\quad X' = A\,U\,\Sigma\,V^T$
- $Step\ 3:$ $\quad \tilde{A} = U^T\,X'\,V\,\Sigma^{-1} = U^T\,A\,U$
- $Step\ 4:$ $\quad \tilde{A}.W = \mathrm{W}.\Lambda$
- $Step\ 5:$ $\quad \Phi = X'V\Sigma^{-1}\,W$

- Finally dynamical system of coupled spatial temporal modes
  - $\boldsymbol{\hat{X}(k\,\Delta t) = \Phi\,\Lambda^t b_0}$



x_k
(state input)
Φ
y_k
Encoder
$(K * y_k) + B_u$
y_k
Λ
K
EV Estimator
u_k
(control input)
C
B_u
Control Coupling Encoder
y_{k+1}
Φ^{-1}
x_{k+1}
(state output)
Decoder


u_k
$(control\ input)$
C
B_u
C^{-1}
$\hat{u}_k$
$(predict)$
$Encoder$
$Decoder$

Loss Functions :

- Loss Function of Decoder Network:

$$L\left(\phi^{-1}\right) = \left\|x_{k+1} - \phi^{-1}((K * y_k) + B_u)\right\|$$

- Loss Function of Control Coupling Encoder Network:

$$L(C) = \|\phi(x_{k+1}) - (K * y_k) - B_u\|$$

- Loss Function of Eigen Value Estimator Network:

$$L(\Lambda) = \|\phi(x_{k+1}) - (K * y_k) - B_u\|$$

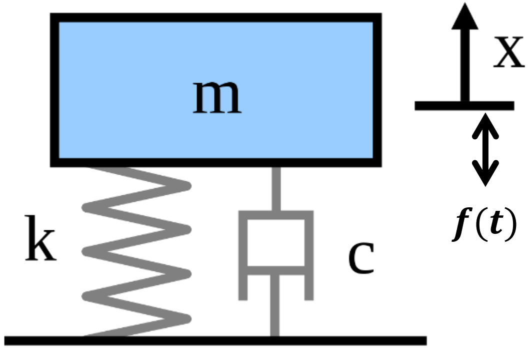
- Loss Function of Encoder Network:

$$L(\phi) = \left\|x_k - \phi^{-1}(\phi(x_k))\right\|$$

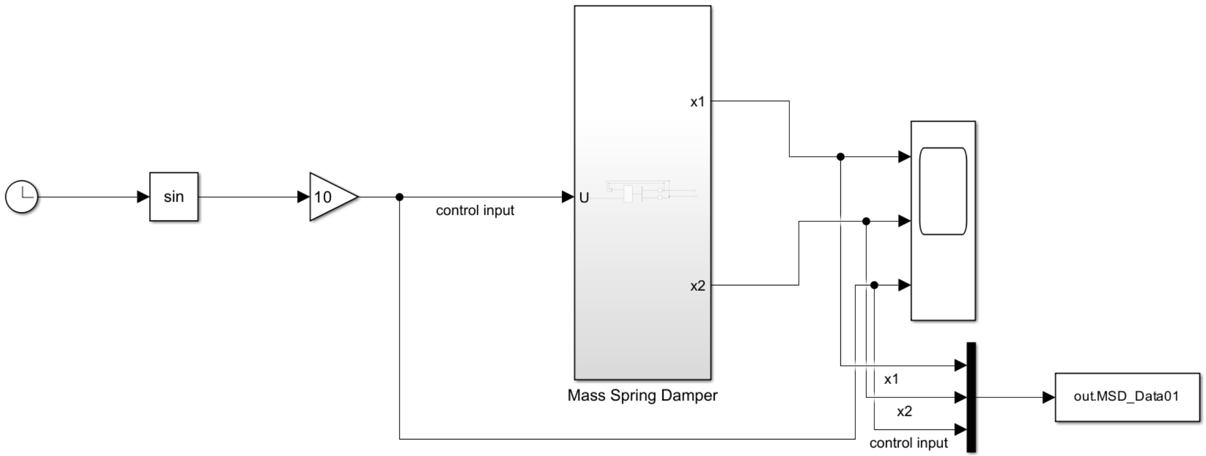
<u>Back Propagation Order:</u>

$\phi^{-1} \rightarrow C \rightarrow \Lambda \rightarrow \phi$

Test Case 1 : Mass Spring Damper System



Pictorial Representation

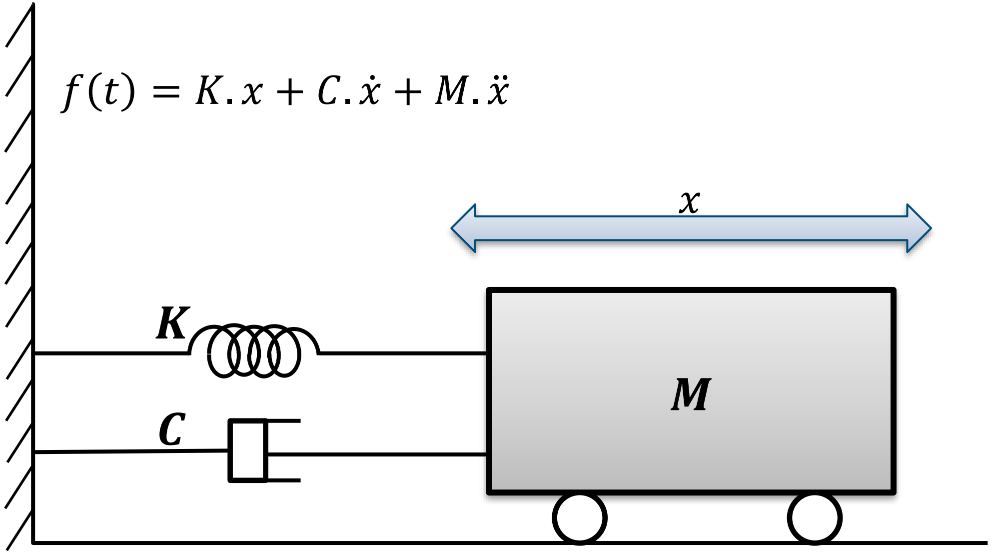


Simulink setup for Data Generation

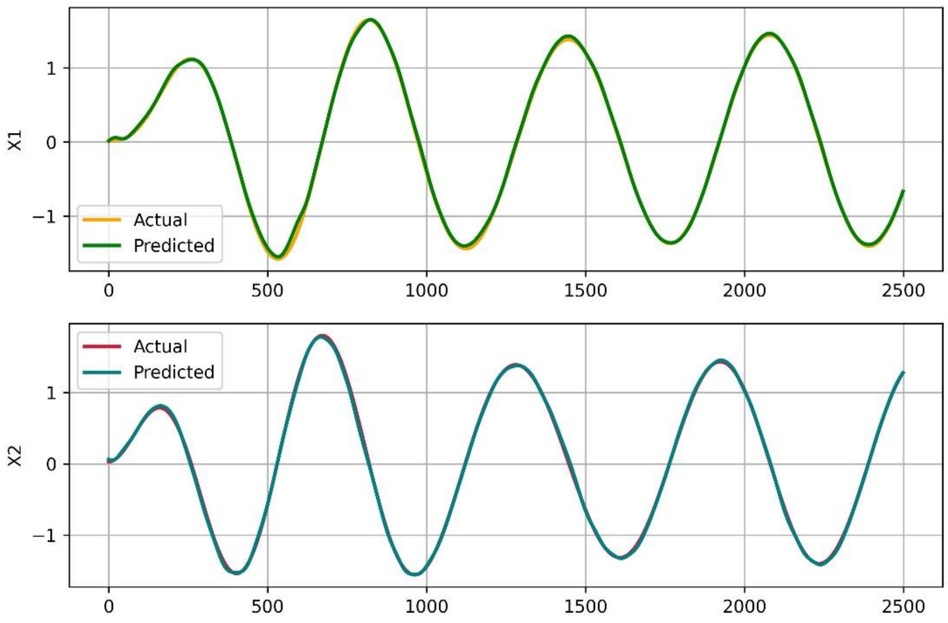
$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} . \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} f(t)$$

$$\dot{X} = A.X + B.u$$

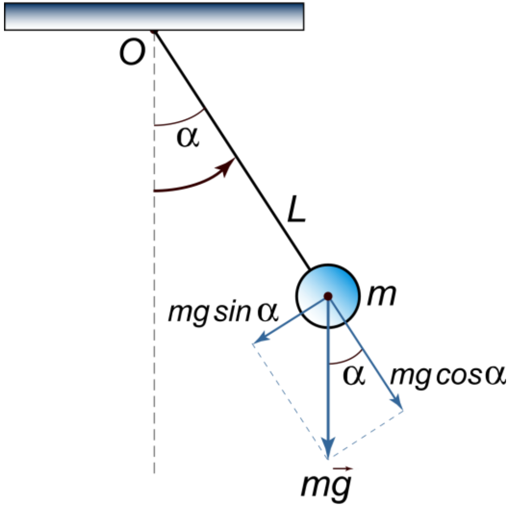
$$f(t) = K.x + C.\dot{x} + M.\ddot{x}$$



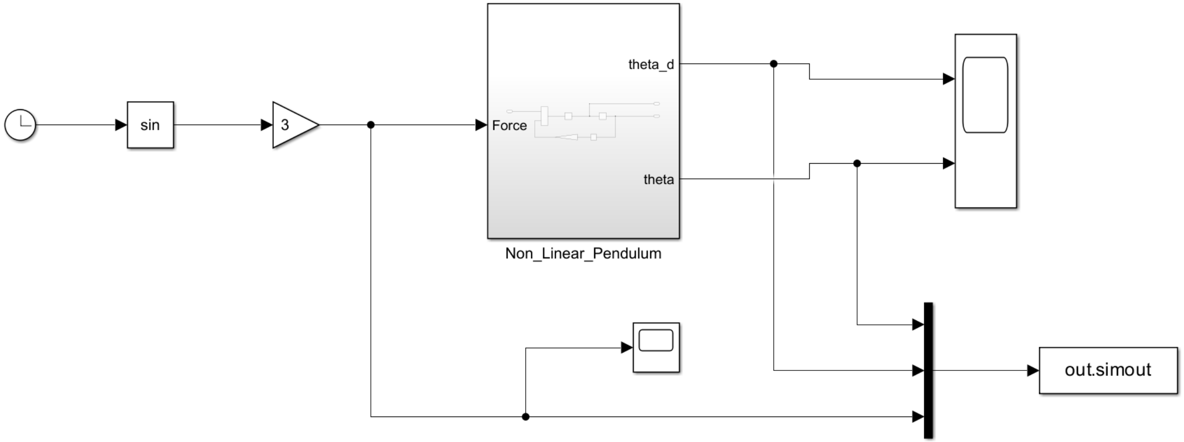
Test Case 1 : Results



Test Case 2 : Nonlinear Pendulum



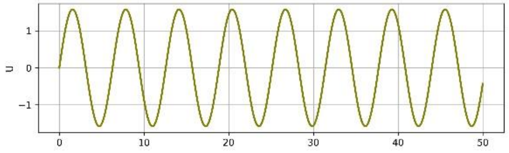
Pictorial Representation



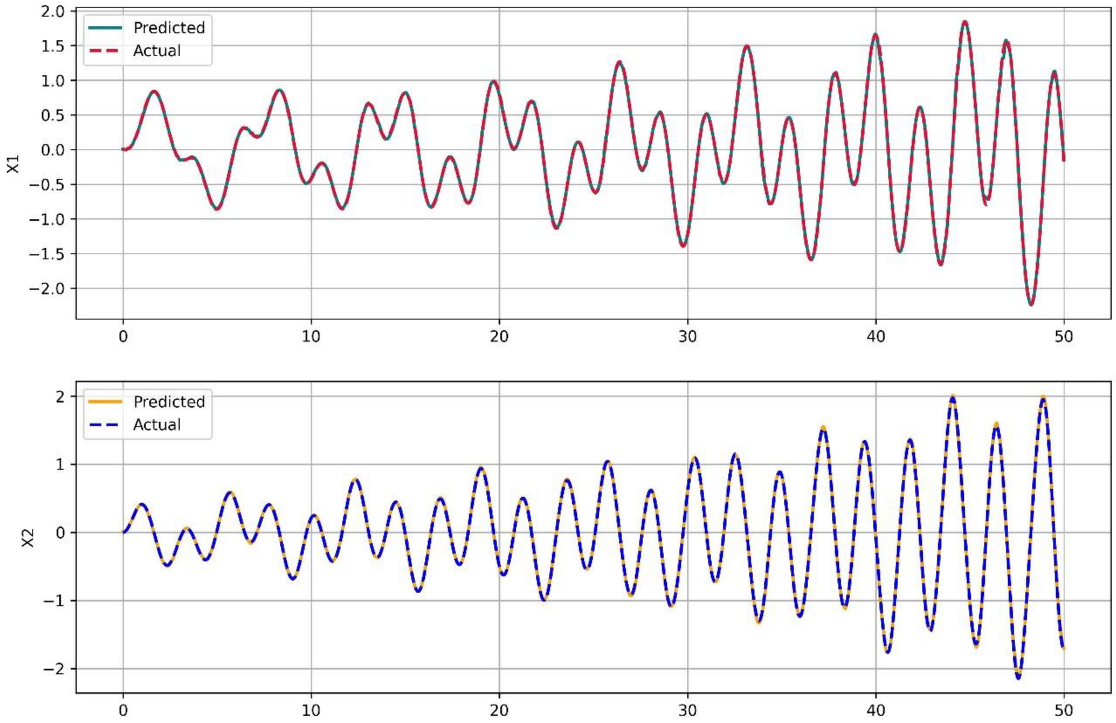
Simulink setup for Data Generation

$$\frac{d^2\alpha}{dt^2} + \frac{g}{L}\sin(\alpha) = f(t) \longrightarrow \dot{X} = f + g.u$$

Test Case 2 : result

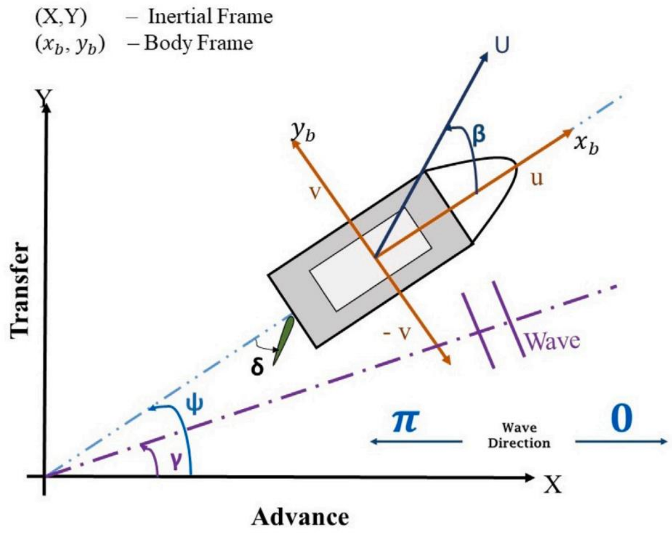


Control Input to the system



State Prediction

Test Case 4 : Ship Model with Second Order waves



Pictorial Representation

Dynamical Governing Equation:

$$(m+m_x)\dot{u} - (m+m_y)vr - x_G m r^2 = X_H + X_R + X_P + X_S$$

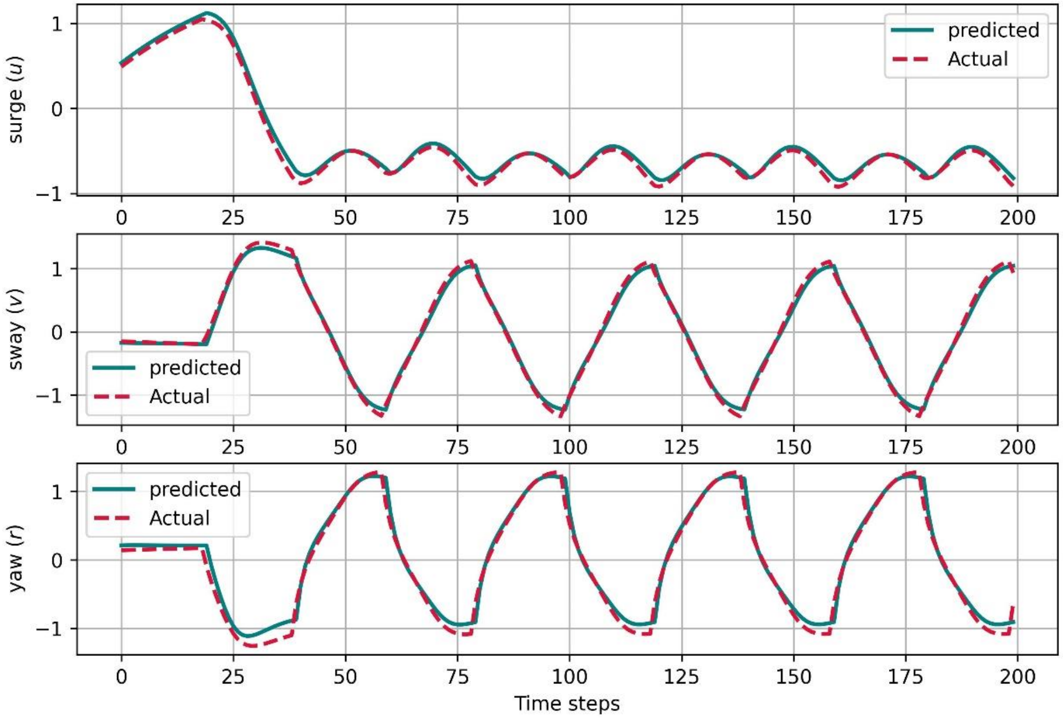
$$(m+m_y)\dot{v} - (m+m_x)ur + x_G m\dot{r} = Y_H + Y_R + Y_S$$

$$(I_z G + x_G^2 m + J_z)\dot{r} + x_G m(\dot{v} + ur) = N_H + N_R + N_S$$

Under Actuated System/ Non-affine System

We can not write such a system in $\dot{x} = A.x + B.u$ format….!

Test Case 4 : Result



Areas Need to Improve & Further Development

- Making the model to work for different sampling time
- Fine tuning of the model
- Testing with various scenarios

Test Case 3 : Nonlinear Pendulum