READING

7

Statistical Concepts and Market Returns

by Richard A. DeFusco, PhD, CFA, Dennis W. McLeavey, DBA, CFA, Jerald E. Pinto, PhD, CFA, and David E. Runkle, PhD, CFA

Richard A. DeFusco, PhD, CFA, is at the University of Lincoln-Nebraska (USA). Dennis W. McLeavey, DBA, CFA, is at the University of Rhode Island (USA). Jerald E. Pinto, PhD, CFA, is at CFA Institute (USA). David E. Runkle, PhD, CFA, is at Trilogy Global Advisors (USA).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**a.** distinguish between descriptive statistics and inferential statistics, between a population and a sample, and among the types of measurement scales;
☐	**b.** define a parameter, a sample statistic, and a frequency distribution;
☐	**c.** calculate and interpret relative frequencies and cumulative relative frequencies, given a frequency distribution;
☐	**d.** describe the properties of a data set presented as a histogram or a frequency polygon;
☐	**e.** calculate and interpret measures of central tendency, including the population mean, sample mean, arithmetic mean, weighted average or mean, geometric mean, harmonic mean, median, and mode;
☐	**f.** calculate and interpret quartiles, quintiles, deciles, and percentiles;
☐	**g.** calculate and interpret 1) a range and a mean absolute deviation and 2) the variance and standard deviation of a population and of a sample;
☐	**h.** calculate and interpret the proportion of observations falling within a specified number of standard deviations of the mean using Chebyshev's inequality;
☐	**i.** calculate and interpret the coefficient of variation;
☐	**j.** explain skewness and the meaning of a positively or negatively skewed return distribution;

(continued)

LEARNING OUTCOMES

Mastery	The candidate should be able to:
☐	**k.** describe the relative locations of the mean, median, and mode for a unimodal, nonsymmetrical distribution;
☐	**l.** explain measures of sample skewness and kurtosis;
☐	**m.** compare the use of arithmetic and geometric means when analyzing investment returns.

1 INTRODUCTION

Statistical methods provide a powerful set of tools for analyzing data and drawing conclusions from them. Whether we are analyzing asset returns, earnings growth rates, commodity prices, or any other financial data, statistical tools help us quantify and communicate the data's important features. This reading presents the basics of describing and analyzing data, the branch of statistics known as descriptive statistics. The reading supplies a set of useful concepts and tools, illustrated in a variety of investment contexts. One theme of our presentation, reflected in the reading's title, is the demonstration of the statistical methods that allow us to summarize return distributions.[1] We explore four properties of return distributions:

- where the returns are centered (central tendency);
- how far returns are dispersed from their center (dispersion);
- whether the distribution of returns is symmetrically shaped or lopsided (skewness); and
- whether extreme outcomes are likely (kurtosis).

These same concepts are generally applicable to the distributions of other types of data, too.

The reading is organized as follows. After defining some basic concepts in Section 2, in Sections 3 and 4 we discuss the presentation of data: Section 3 describes the organization of data in a table format, and Section 4 describes the graphic presentation of data. We then turn to the quantitative description of how data are distributed: Section 5 focuses on measures that quantify where data are centered, or measures of central tendency. Section 6 presents other measures that describe the location of data. Section 7 presents measures that quantify the degree to which data are dispersed. Sections 8 and 9 describe additional measures that provide a more accurate picture of data. Section 10 provides investment applications of concepts introduced in Section 5.

2 SOME FUNDAMENTAL CONCEPTS

Before starting the study of statistics with this reading, it may be helpful to examine a picture of the overall field. In the following, we briefly describe the scope of statistics and its branches of study. We explain the concepts of population and sample. Data

1 Ibbotson Associates (www.ibbotson.com) generously provided some of the data used in this reading. We also draw on Dimson, Marsh, and Staunton's (2011) history and study of world markets as well as other sources.

come in a variety of types, affecting the ways they can be measured and the appropriate statistical methods for analyzing them. We conclude by discussing the basic types of data measurement.

2.1 The Nature of Statistics

The term **statistics** can have two broad meanings, one referring to data and the other to method. A company's average earnings per share (EPS) for the last 20 quarters, or its average returns for the past 10 years, are statistics. We may also analyze historical EPS to forecast future EPS, or use the company's past returns to infer its risk. The totality of methods we employ to collect and analyze data is also called statistics.

Statistical methods include descriptive statistics and statistical inference (inferential statistics). **Descriptive statistics** is the study of how data can be summarized effectively to describe the important aspects of large data sets. By consolidating a mass of numerical details, descriptive statistics turns data into information. **Statistical inference** involves making forecasts, estimates, or judgments about a larger group from the smaller group actually observed. The foundation for statistical inference is probability theory, and both statistical inference and probability theory will be discussed in later readings. Our focus in this reading is solely on descriptive statistics.

2.2 Populations and Samples

Throughout the study of statistics we make a critical distinction between a population and a sample. In this section, we explain these two terms as well as the related terms "parameter" and "sample statistic."[2]

- **Definition of Population.** A **population** is defined as all members of a specified group.

Any descriptive measure of a population characteristic is called a **parameter**. Although a population can have many parameters, investment analysts are usually concerned with only a few, such as the mean value, the range of investment returns, and the variance.

Even if it is possible to observe all the members of a population, it is often too expensive in terms of time or money to attempt to do so. For example, if the population is all telecommunications customers worldwide and an analyst is interested in their purchasing plans, she will find it too costly to observe the entire population. The analyst can address this situation by taking a sample of the population.

- **Definition of Sample.** A **sample** is a subset of a population.

In taking a sample, the analyst hopes it is characteristic of the population. The field of statistics known as sampling deals with taking samples in appropriate ways to achieve the objective of representing the population well. A later reading addresses the details of sampling.

Earlier, we mentioned statistics in the sense of referring to data. Just as a parameter is a descriptive measure of a population characteristic, a sample statistic (statistic, for short) is a descriptive measure of a sample characteristic.

- **Definition of Sample Statistic.** A **sample statistic** (or **statistic**) is a quantity computed from or used to describe a sample.

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2 This reading introduces many statistical concepts and formulas. To make it easy to locate them, we have set off some of the more important ones with bullet points.

We devote much of this reading to explaining and illustrating the use of statistics in this sense. The concept is critical also in statistical inference, which addresses such problems as estimating an unknown population parameter using a sample statistic.

2.3 Measurement Scales

To choose the appropriate statistical methods for summarizing and analyzing data, we need to distinguish among different **measurement scales** or levels of measurement. All data measurements are taken on one of four major scales: nominal, ordinal, interval, or ratio.

Nominal scales represent the weakest level of measurement: They categorize data but do not rank them. If we assigned integers to mutual funds that follow different investment strategies, the number 1 might refer to a small-cap value fund, the number 2 to a large-cap value fund, and so on for each possible style. This nominal scale categorizes the funds according to their style but does not rank them.

Ordinal scales reflect a stronger level of measurement. Ordinal scales sort data into categories that are ordered with respect to some characteristic. For example, the Morningstar and Standard & Poor's star ratings for mutual funds represent an ordinal scale in which one star represents a group of funds judged to have had relatively the worst performance, with two, three, four, and five stars representing groups with increasingly better performance, as evaluated by those services.

An ordinal scale may also involve numbers to identify categories. For example, in ranking balanced mutual funds based on their five-year cumulative return, we might assign the number 1 to the top 10 percent of funds, and so on, so that the number 10 represents the bottom 10 percent of funds. The ordinal scale is stronger than the nominal scale because it reveals that a fund ranked 1 performed better than a fund ranked 2. The scale tells us nothing, however, about the difference in performance between funds ranked 1 and 2 compared with the difference in performance between funds ranked 3 and 4, or 9 and 10.

Interval scales provide not only ranking but also assurance that the differences between scale values are equal. As a result, scale values can be added and subtracted meaningfully. The Celsius and Fahrenheit scales are interval measurement scales. The difference in temperature between 10°C and 11°C is the same amount as the difference between 40°C and 41°C. We can state accurately that 12°C = 9°C + 3°C, for example. Nevertheless, the zero point of an interval scale does not reflect complete absence of what is being measured; it is not a true zero point or natural zero. Zero degrees Celsius corresponds to the freezing point of water, not the absence of temperature. As a consequence of the absence of a true zero point, we cannot meaningfully form ratios on interval scales.

As an example, 50°C, although five times as large a number as 10°C, does not represent five times as much temperature. Also, questionnaire scales are often treated as interval scales. If an investor is asked to rank his risk aversion on a scale from 1 (extremely risk-averse) to 7 (extremely risk-loving), the difference between a response of 1 and a response of 2 is sometimes assumed to represent the same difference in risk aversion as the difference between a response of 6 and a response of 7. When that assumption can be justified, the data are measured on an interval scale.

Ratio scales represent the strongest level of measurement. They have all the characteristics of interval measurement scales as well as a true zero point as the origin. With ratio scales, we can meaningfully compute ratios as well as meaningfully add and subtract amounts within the scale. As a result, we can apply the widest range of statistical tools to data measured on a ratio scale. Rates of return are measured on a ratio scale, as is money. If we have twice as much money, then we have twice the purchasing power. Note that the scale has a natural zero—zero means no money.

Now that we have addressed the important preliminaries, we can discuss summarizing and describing data.

EXAMPLE 1

Identifying Scales of Measurement

State the scale of measurement for each of the following:

1 Credit ratings for bond issues.[3]
2 Cash dividends per share.
3 Hedge fund classification types.[4]
4 Bond maturity in years.

Solution to 1:

Credit ratings are measured on an ordinal scale. A rating places a bond issue in a category, and the categories are ordered with respect to the expected probability of default. But the difference in the expected probability of default between AA− and A+, for example, is not necessarily equal to that between BB− and B+. In other words, letter credit ratings are not measured on an interval scale.

Solution to 2:

Cash dividends per share are measured on a ratio scale. For this variable, 0 represents the complete absence of dividends; it is a true zero point.

Solution to 3:

Hedge fund classification types are measured on a nominal scale. Each type groups together hedge funds with similar investment strategies. In contrast to credit ratings for bonds, however, hedge fund classification schemes do not involve a ranking. Thus such classification schemes are not measured on an ordinal scale.

Solution to 4:

Bond maturity is measured on a ratio scale.

SUMMARIZING DATA USING FREQUENCY DISTRIBUTIONS 3

In this section, we discuss one of the simplest ways to summarize data—the frequency distribution.

- **Definition of Frequency Distribution.** A **frequency distribution** is a tabular display of data summarized into a relatively small number of intervals.

3 Credit ratings for a bond issue gauge the bond issuer's ability to meet the promised principal and interest payments on the bond. For example, one rating agency, Standard & Poor's, assigns bond issues to one of the following ratings, given in descending order of credit quality (increasing probability of default): AAA, AA+, AA, AA−, A+, A, A−, BBB+, BBB, BBB−, BB+, BB, BB−, B+, B, B−, CCC+, CCC−, CC, C, D. For more information on credit risk and credit ratings, see the Level I CFA Program curriculum reading "Fixed-Income Securities: Defining Elements."

4 "Hedge fund" refers to investment vehicles with legal structures that result in less regulatory oversight than other pooled investment vehicles such as mutual funds. Hedge fund classification types group hedge funds by the kind of investment strategy they pursue.

Frequency distributions help in the analysis of large amounts of statistical data, and they work with all types of measurement scales.

Rates of return are the fundamental units that analysts and portfolio managers use for making investment decisions and we can use frequency distributions to summarize rates of return. When we analyze rates of return, our starting point is the holding period return (also called the total return).

- **Holding Period Return Formula.** The holding period return for time period t, R_t, is

$$R_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}} \tag{1}$$

where

P_t = price per share at the end of time period t

P_{t-1} = price per share at the end of time period $t - 1$, the time period immediately preceding time period t

D_t = cash distributions received during time period t

Thus the holding period return for time period t is the capital gain (or loss) plus distributions divided by the beginning-period price. (For common stocks, the distribution is a dividend; for bonds, the distribution is a coupon payment.) Equation 1 can be used to define the holding period return on any asset for a day, week, month, or year simply by changing the interpretation of the time interval between successive values of the time index, t.

The holding period return, as defined in Equation 1, has two important characteristics. First, it has an element of time attached to it. For example, if a monthly time interval is used between successive observations for price, then the rate of return is a monthly figure. Second, rate of return has no currency unit attached to it. For instance, suppose that prices are denominated in euros. The numerator and denominator of Equation 1 would be expressed in euros, and the resulting ratio would not have any units because the units in the numerator and denominator would cancel one another. This result holds regardless of the currency in which prices are denominated.[5]

With these concerns noted, we now turn to the frequency distribution of the holding period returns on the S&P 500 Index.[6] First, we examine annual rates of return; then we look at monthly rates of return. The annual rates of return on the S&P 500 calculated with Equation 1 span the period January 1926 to December 2017, for a total of 92 annual observations. Monthly return data cover the period January 1926 to December 2017, for a total of 1,104 monthly observations.

We can state a basic procedure for constructing a frequency distribution as follows.

Construction of a Frequency Distribution.

1 Sort the data in ascending order.

2 Calculate the range of the data, defined as Range = Maximum value − Minimum value.

3 Decide on the number of intervals in the frequency distribution, k.

5 Note, however, that if price and cash distributions in the expression for holding period return were not in one's home currency, one would generally convert those variables to one's home currency before calculating the holding period return. Because of exchange rate fluctuations during the holding period, holding period returns on an asset computed in different currencies would generally differ.

6 For January 1926 to December 2012, we use the total return series on the S&P 500 provided by Ibbotson Associates. For January 2013 to December 2017, we use the total return series on the S&P 500 from S&P Dow Jones Indices LLC (https://us.spindices.com/indices/equity/sp-500, accessed 31 October 2018).

4 Determine interval width as Range/k.
5 Determine the intervals by successively adding the interval width to the minimum value, to determine the ending points of intervals, stopping after reaching an interval that includes the maximum value.
6 Count the number of observations falling in each interval.
7 Construct a table of the intervals listed from smallest to largest that shows the number of observations falling in each interval.

In Step 4, when rounding the interval width, round up rather than down, to ensure that the final interval includes the maximum value of the data.

As the above procedure makes clear, a frequency distribution groups data into a set of intervals.[7] An **interval** is a set of values within which an observation falls. Each observation falls into only one interval, and the total number of intervals covers all the values represented in the data. The actual number of observations in a given interval is called the **absolute frequency**, or simply the frequency. The frequency distribution is the list of intervals together with the corresponding measures of frequency.

To illustrate the basic procedure, suppose we have 12 observations sorted in ascending order: −4.57, −4.04, −1.64, 0.28, 1.34, 2.35, 2.38, 4.28, 4.42, 4.68, 7.16, and 11.43. The minimum observation is −4.57 and the maximum observation is +11.43, so the range is +11.43 − (−4.57) = 16. If we set k = 4, the interval width is 16/4 = 4. Exhibit 1 shows the repeated addition of the interval width of 4 to determine the endpoints for the intervals (Step 5).

Exhibit 1 Endpoints of Intervals

−4.57	+	4.00	=	−0.57
−0.57	+	4.00	=	3.43
3.43	+	4.00	=	7.43
7.4	+	4.00	=	11.43

Thus the intervals are [−4.57 to −0.57), [−0.57 to 3.43), [3.43 to 7.43), and [7.43 to 11.43].[8] Exhibit 2 summarizes Steps 5 through 7.

Exhibit 2 Frequency Distribution

Interval				Absolute Frequency
A	−4.57	≤ observation <	−0.57	3
B	−0.57	≤ observation <	3.43	4
C	3.43	≤ observation <	7.43	4
D	7.43	≤ observation ≤	11.43	1

Note that the intervals do not overlap, so each observation can be placed uniquely into one interval.

7 Intervals are also sometimes called classes, ranges, or bins.
8 The notation [−4.57 to −0.57) means −4.57 ≤ observation < −0.57. In this context, a square bracket indicates that the endpoint is included in the interval.

In practice, we may want to refine the above basic procedure. For example, we may want the intervals to begin and end with whole numbers for ease of interpretation. We also need to explain the choice of the number of intervals, k. We turn to these issues in discussing the construction of frequency distributions for the S&P 500.

We first consider the case of constructing a frequency distribution for the annual returns on the S&P 500 over the period 1926 to 2017. During that period, the return on the S&P 500 had a minimum value of −43.35 percent (in 1931) and a maximum value of +53.97 percent (in 1933). Thus the range of the data was +54% − (−43%) = 97%, approximately. The question now is the number k of intervals into which we should group observations. Although some guidelines for setting k have been suggested in statistical literature, the setting of a useful value for k often involves inspecting the data and exercising judgment. How much detail should we include? If we use too few intervals, we will summarize too much and lose pertinent characteristics. If we use too many intervals, we may not summarize enough.

We can establish an appropriate value for k by evaluating the usefulness of the resulting interval width. A large number of empty intervals may indicate that we are trying to organize the data to present too much detail. Starting with a relatively small interval width, we can see whether or not the intervals are mostly empty and whether or not the value of k associated with that interval width is too large. If intervals are mostly empty or k is very large, we can consider increasingly larger intervals (smaller values of k) until we have a frequency distribution that effectively summarizes the distribution. For the annual S&P 500 series, return intervals of 1 percent width would result in 97 intervals and many of them would be empty because we have only 92 annual observations. We need to keep in mind that the purpose of a frequency distribution is to *summarize* the data. Suppose that for ease of interpretation we want to use an interval width stated in whole rather than fractional percents. A 2 percent interval width would have many fewer empty intervals than a 1 percent interval width and effectively summarize the data. A 2 percent interval width would be associated with 97/2 = 48.5 intervals, which we can round up to 49 intervals. That number of intervals will cover 2% × 49 = 98%. We can confirm that if we start the smallest 2 percent interval at the whole number −44.0 percent, the final interval ends at −44.0% + 98% = 54% and includes the maximum return in the sample, 53.99 percent. In so constructing the frequency distribution, we will also have intervals that end and begin at a value of 0 percent, allowing us to count the negative and positive returns in the data. Without too much work, we have found an effective way to summarize the data. We will use return intervals of 2 percent, beginning with $-44\% \le R_t < -42\%$ (given as "−44% to −42%" in the exhibit) and ending with $52\% \le R_t \le 54\%$. Exhibit 3 shows the frequency distribution for the annual total returns on the S&P 500.

Exhibit 3 includes three other useful ways to present data, which we can compute once we have established the frequency distribution: the relative frequency, the cumulative frequency (also called the cumulative absolute frequency), and the cumulative relative frequency.

- **Definition of Relative Frequency.** The **relative frequency** is the absolute frequency of each interval divided by the total number of observations.

The **cumulative relative frequency** cumulates (adds up) the relative frequencies as we move from the first to the last interval. It tells us the fraction of observations that are less than the upper limit of each interval. Examining the frequency distribution given in Exhibit 3, we see that the first return interval, −44 percent to −42 percent, has one observation; its relative frequency is 1/92 or 1.09 percent. The cumulative frequency for this interval is 1 because only one observation is less than −42 percent. The cumulative relative frequency is thus 1/92 or 1.09 percent. The next return interval has zero observations; therefore, its cumulative frequency is 0 plus 1 and its cumulative relative frequency is 1.09 percent (the cumulative relative frequency from the previous interval). We can find the other cumulative frequencies by adding the (absolute) frequency to the previous cumulative frequency. The cumulative frequency, then, tells us the number of observations that are less than the upper limit of each return interval.

As Exhibit 3 shows, return intervals have frequencies from 0 to 7 in this sample. The interval encompassing returns between −10 percent and −8 percent ($-10\% \leq R_t < -8\%$) has the most observations, seven. Next most frequent are returns between 4 percent and 6 percent ($4\% \leq R_t < 6\%$) and between18 percent and 20 percent ($18\% \leq R_t < 20\%$), with six observations in each interval. From the cumulative frequency column, we see that the number of negative returns is 24. The number of positive returns must then be equal to 92 − 24, or 68. We can express the number of positive and negative outcomes as a percentage of the total to get a sense of the risk inherent in investing in the stock market. During the 92-year period, the S&P 500 had negative annual returns 26.09 percent of the time (that is, 24/92). This result appears in the fifth column of Exhibit 3, which reports the cumulative relative frequency.

The frequency distribution gives us a sense of not only where most of the observations lie but also whether the distribution is evenly distributed, lopsided, or otherwise distinctive. In the case of the S&P 500, we can see that more than half of the outcomes are positive and most of those annual returns are larger than 10 percent. (Only 15 of the 68 positive annual returns—about 16 percent—were between 0 and 10 percent.)

Exhibit 3 permits us to make an important further point about the choice of the number of intervals related to equity returns in particular. From the frequency distribution in Exhibit 3, we can see that only six outcomes fall between −44 percent to −16 percent and only five outcomes fall between 38 percent to 54 percent. Stock return data are frequently characterized by a few very large or small outcomes. We could have collapsed the return intervals in the tails of the frequency distribution by choosing a smaller value of *k*, but then we would have lost the information about how extremely poorly or well the stock market had performed. A risk manager may need to know the worst possible outcomes and thus may want to have detailed information on the tails (the extreme values). A frequency distribution with a relatively large value of *k* is useful for that. A portfolio manager or analyst may be equally interested in detailed information on the tails; however, if the manager or analyst wants a picture only of where most of the observations lie, he might prefer to use an interval width of 4 percent (25 intervals beginning at −44 percent), for example.

The frequency distribution for monthly returns on the S&P 500 looks quite different from that for annual returns. The monthly return series from January 1926 to December 2017 has 1,104 observations. Returns range from a minimum of approximately −30 percent to a maximum of approximately +43 percent. With such a large quantity of monthly data we must summarize to get a sense of the distribution, and so we group the data into 37 equally spaced return intervals of 2 percent. The gains from summarizing in this way are substantial. Exhibit 4 presents the resulting frequency distribution. The absolute frequencies appear in the second column, followed by the relative frequencies. The relative frequencies are rounded to two decimal places. The cumulative absolute and cumulative relative frequencies appear in the fourth and fifth columns, respectively.

Exhibit 3 Frequency Distribution for the Annual Total Return on the S&P 500, 1926–2017

Return Interval (%)	Frequency	Relative Frequency (%)	Cumulative Frequency	Cumulative Relative Frequency (%)	Return Interval (%)	Frequency	Relative Frequency (%)	Cumulative Frequency	Cumulative Relative Frequency (%)
−44.0 to −42.0	1	1.09	1	1.09	4.0 to 6.0	6	6.52	34	36.96
−42.0 to −40.0	0	0.00	1	1.09	6.0 to 8.0	4	4.35	38	41.30
−40.0 to −38.0	0	0.00	1	1.09	8.0 to 10.0	1	1.09	39	42.39
−38.0 to −36.0	1	1.09	2	2.17	10.0 to 12.0	5	5.43	44	47.83
−36.0 to −34.0	1	1.09	3	3.26	12.0 to 14.0	2	2.17	46	50.00
−34.0 to −32.0	0	0.00	3	3.26	14.0 to 16.0	4	4.35	50	54.35
−32.0 to −30.0	0	0.00	3	3.26	16.0 to 18.0	2	2.17	52	56.52
−30.0 to −28.0	0	0.00	3	3.26	18.0 to 20.0	6	6.52	58	63.04
−28.0 to −26.0	1	1.09	4	4.35	20.0 to 22.0	4	4.35	62	67.39
−26.0 to −24.0	1	1.09	5	5.43	22.0 to 24.0	5	5.43	67	72.83
−24.0 to −22.0	1	1.09	6	6.52	24.0 to 26.0	2	2.17	69	75.00
−22.0 to −20.0	0	0.00	6	6.52	26.0 to 28.0	2	2.17	71	77.17
−20.0 to −18.0	0	0.00	6	6.52	28.0 to 30.0	2	2.17	73	79.35
−18.0 to −16.0	0	0.00	6	6.52	30.0 to 32.0	5	5.43	78	84.78
−16.0 to −14.0	1	1.09	7	7.61	32.0 to 34.0	5	5.43	83	90.22
−14.0 to −12.0	0	0.00	7	7.61	34.0 to 36.0	0	0.00	83	90.22
−12.0 to −10.0	4	4.35	11	11.96	36.0 to 38.0	4	4.35	87	94.57
−10.0 to −8.0	7	7.61	18	19.57	38.0 to 40.0	0	0.00	87	94.57
−8.0 to −6.0	1	1.09	19	20.65	40.0 to 42.0	0	0.00	87	94.57
−6.0 to −4.0	1	1.09	20	21.74	42.0 to 44.0	2	2.17	89	96.74
−4.0 to −2.0	1	1.09	21	22.83	44.0 to 46.0	0	0.00	89	96.74
−2.0 to 0.0	3	3.26	24	26.09	46.0 to 48.0	1	1.09	90	97.83
0.0 to 2.0	3	3.26	27	29.35	48.0 to 50.0	0	0.00	90	97.83
2.0 to 4.0	1	1.09	28	30.43	50.0 to 52.0	0	0.00	90	97.83
					52.0 to 54.0	2	2.17	92	100.00

Note: The lower class limit is the weak inequality (≤) and the upper class limit is the strong inequality (<). Cumulative relative frequency totals reflect calculations using full precision, with results rounded to two decimal places.

Sources: Ibbotson Associates and S&P Dow Jones Indices LLC.

Exhibit 4 Frequency Distribution for the Monthly Total Return on the S&P 500, January 1926 to December 2017

Return Interval (%)	Absolute Frequency	Relative Frequency (%)	Cumulative Absolute Frequency	Cumulative Relative Frequency (%)
−30.0 to −28.0	1	0.09	1	0.09
−28.0 to −26.0	0	0.00	1	0.09
−26.0 to −24.0	1	0.09	2	0.18
−24.0 to −22.0	1	0.09	3	0.27
−22.0 to −20.0	2	0.18	5	0.45
−20.0 to −18.0	2	0.18	7	0.63
−18.0 to −16.0	3	0.27	10	0.91
−16.0 to −14.0	2	0.18	12	1.09
−14.0 to −12.0	6	0.54	18	1.63
−12.0 to −10.0	7	0.63	25	2.26
−10.0 to −8.0	23	2.08	48	4.35
−8.0 to −6.0	35	3.17	83	7.52
−6.0 to −4.0	60	5.43	143	12.95
−4.0 to −2.0	102	9.24	245	22.19
−2.0 to 0.0	166	15.04	411	37.23
0.0 to 2.0	240	21.74	651	58.97
2.0 to 4.0	190	17.21	841	76.18
4.0 to 6.0	143	12.95	984	89.13
6.0 to 8.0	64	5.80	1,048	94.93
8.0 to 10.0	26	2.36	1,074	97.28
10.0 to 12.0	15	1.36	1,089	98.64
12.0 to 14.0	6	0.54	1,095	99.18
14.0 to 16.0	2	0.18	1,097	99.37
16.0 to 18.0	3	0.27	1,100	99.64
18.0 to 20.0	0	0.00	1,100	99.64
20.0 to 22.0	0	0.00	1,100	99.64
22.0 to 24.0	0	0.00	1,100	99.64
24.0 to 26.0	1	0.09	1,101	99.73
26.0 to 28.0	0	0.00	1,101	99.73
28.0 to 30.0	0	0.00	1,101	99.73
30.0 to 32.0	0	0.00	1,101	99.73
32.0 to 34.0	0	0.00	1,101	99.73
34.0 to 36.0	0	0.00	1,101	99.73
36.0 to 38.0	0	0.00	1,101	99.73
38.0 to 40.0	2	0.18	1,103	99.91
40.0 to 42.0	0	0.00	1,103	99.91
42.0 to 44.0	1	0.09	1,104	100.00

Note: The lower class limit is the weak inequality (≤) and the upper class limit is the strong inequality (<). The relative frequency is the absolute frequency or cumulative frequency divided by the total number of observations. Cumulative relative frequency totals reflect calculations using full precision, with results rounded to two decimal places.

(continued)

Exhibit 4 (Continued)

Sources: Ibbotson Associates and S&P Dow Jones Indices LLC.

The advantage of a frequency distribution is evident in Exhibit 4, which tells us that the vast majority of observations (739/1,104 = 67 percent) lie in the four intervals spanning −2 percent to +6 percent. Altogether, we have 411 negative returns and 693 positive returns. Almost 63 percent of the monthly outcomes are positive. Looking at the cumulative relative frequency in the last column, we see that the interval −2 percent to 0 percent shows a cumulative frequency of 37.23 percent, for an upper return limit of 0 percent. This means that 37.23 percent of the observations lie below the level of 0 percent. We can also see that not many observations are greater than +12 percent or less than −12 percent. Note that the frequency distributions of annual and monthly returns are not directly comparable. On average, we should expect the returns measured at shorter intervals (for example, months) to be smaller than returns measured over longer periods (for example, years).

Next, we construct a frequency distribution of average inflation-adjusted returns over 1900–2010 for 19 major equity markets.

EXAMPLE 2

Constructing a Frequency Distribution

How have equities rewarded investors in different countries in the long run? To answer this question, we could examine the average annual returns directly.[9] The worth of a nominal level of return depends on changes in the purchasing power of money, however, and internationally there have been a variety of experiences with price inflation. It is preferable, therefore, to compare the average real or inflation-adjusted returns earned by investors in different countries. Dimson, Marsh, and Staunton (2011) presented authoritative evidence on asset returns in 19 countries for the 111 years 1900–2010. Exhibit 5 excerpts their findings for average inflation-adjusted returns.

Exhibit 5 Real (Inflation-Adjusted) Equity Returns: Nineteen Major Equity Markets, 1900–2017

Country	Arithmetic Mean (%)
Australia	8.3
Belgium	5.3
Canada	7.1
Denmark	7.4
Finland	9.3
France	5.8
Germany	8.2
Ireland	7.0

9 The average or arithmetic mean of a set of values equals the sum of the values divided by the number of values summed. To find the arithmetic mean of 111 annual returns, for example, we sum the 111 annual returns and then divide the total by 111. Among the most familiar of statistical concepts, the arithmetic mean is explained in more detail later in the reading.

Exhibit 5 (Continued)

Country	Arithmetic Mean (%)
Italy	6.0
Japan	8.8
Netherlands	7.2
New Zealand	8.0
Norway	7.3
South Africa	9.4
Spain	5.8
Sweden	8.0
Switzerland	6.3
United Kingdom	7.3
United States	8.4

Source: Dimson, Marsh, and Staunton (2018), Table 1.

Exhibit 6 summarizes the data in Exhibit 5 into five intervals spanning 5 percent to 10 percent. With nineteen markets, the relative frequency for the 5.0 to 6.0 percent return interval is calculated as 3/19 = 15.79 percent, for example.

Exhibit 6 Frequency Distribution of Average Real Equity Returns

Return Interval (%)	Absolute Frequency	Relative Frequency (%)	Cumulative Absolute Frequency	Cumulative Relative Frequency (%)
5.0 to 6.0	3	15.79	3	15.79
6.0 to 7.0	2	10.53	5	26.32
7.0 to 8.0	6	31.58	11	57.90
8.0 to 9.0	6	31.58	17	89.47
9.0 to 10	2	10.53	19	100.00

As Exhibit 6 shows, there is substantial variation internationally of average real equity returns. Nearly one-third of the observations fall in the 7.0 to 8.0 percent interval and nearly one-third fall in the 8.0 to 9.0 percent interval, each having a relative frequency of 31.58 percent. The other three intervals each has two or three observations.

4 THE GRAPHIC PRESENTATION OF DATA

A graphical display of data allows us to visualize important characteristics quickly. For example, we may see that the distribution is symmetrically shaped, and this finding may influence which probability distribution we use to describe the data. In this section, we discuss the histogram, the frequency polygon, and the cumulative frequency distribution as methods for displaying data graphically. We construct all of these graphic presentations with the information contained in the frequency distribution of the S&P 500 shown in either Exhibit 3 or Exhibit 4.

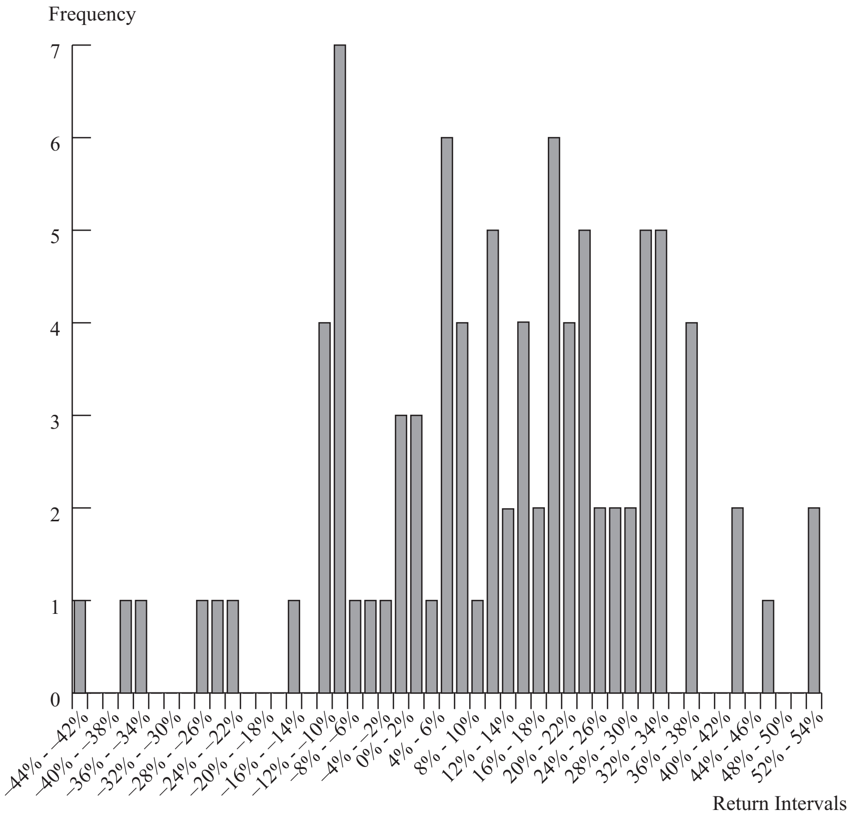
4.1 The Histogram

A histogram is the graphical equivalent of a frequency distribution.

- **Definition of Histogram.** A **histogram** is a bar chart of data that have been grouped into a frequency distribution.

The advantage of the visual display is that we can see quickly where most of the observations lie. To see how a histogram is constructed, look at the return interval 18% ≤ R_t < 20% in Exhibit 3. This interval has an absolute frequency of 6. Therefore, we erect a bar or rectangle with a height of 6 over that return interval on the horizontal axis. Continuing with this process for all other return intervals yields a histogram. Exhibit 7 presents the histogram of the annual total return series on the S&P 500 from 1926 to 2017.

Exhibit 7 Histogram of S&P 500 Annual Total Returns: 1926 to 2017

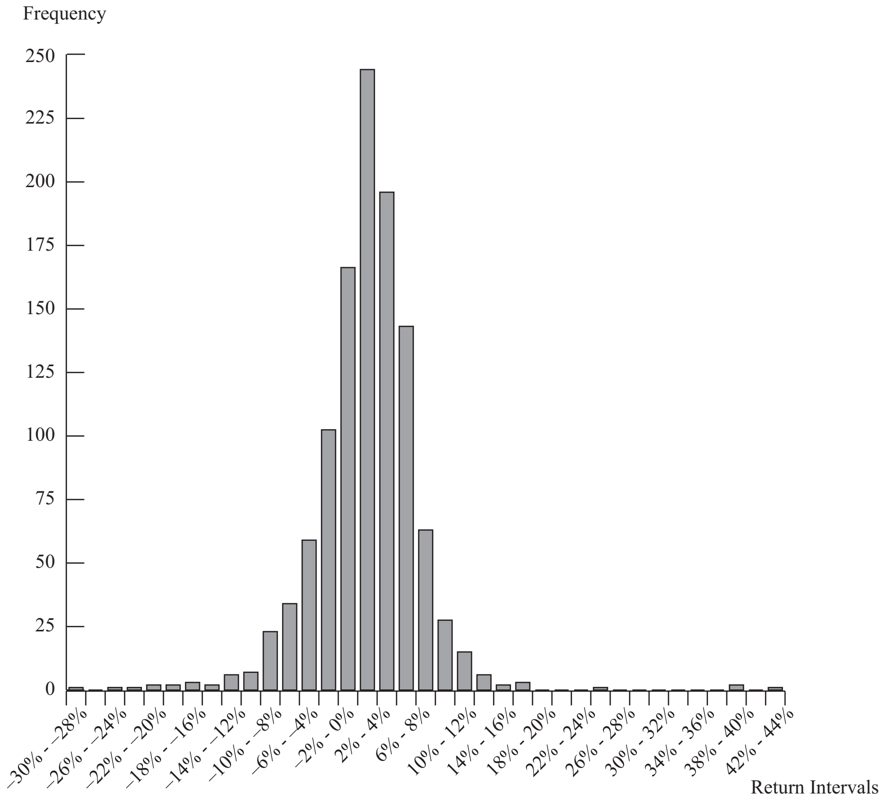


Note: Because of space limitations, only every other return interval is labeled below the horizontal axis.
Sources: Ibbotson Associates and S&P Dow Jones Indices LLC

In the histogram in Exhibit 7, the height of each bar represents the absolute frequency for each return interval. The return interval −10% ≤ R_t < −8% has a frequency of 7 and is represented by the tallest bar in the histogram. Because there are no gaps between the interval limits, there are no gaps between the bars of the histogram. Many of the return intervals have zero frequency; therefore, they have no height in the histogram.

Exhibit 8 presents the histogram for the distribution of monthly returns on the S&P 500. Somewhat more symmetrically shaped than the histogram of annual returns shown in Exhibit 7, this histogram also appears more bell-shaped than the distribution of annual returns.

Exhibit 8 Histogram of S&P 500 Monthly Total Returns: January 1926 to December 2017

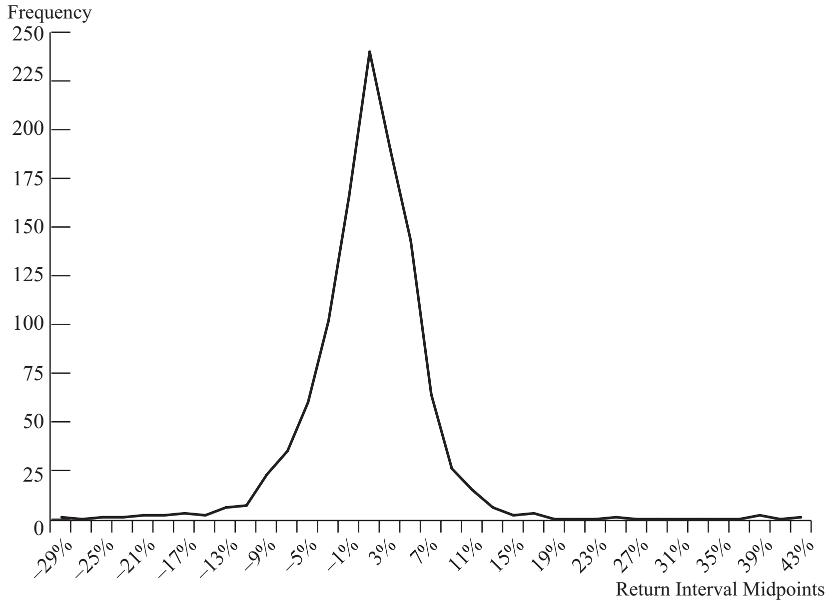


Sources: Ibbotson Associates and S&P Dow Jones Indices LLC

4.2 The Frequency Polygon and the Cumulative Frequency Distribution

Two other graphical tools for displaying data are the frequency polygon and the cumulative frequency distribution. To construct a **frequency polygon**, we plot the midpoint of each interval on the x-axis and the absolute frequency for that interval on the y-axis; we then connect neighboring points with a straight line. Exhibit 9 shows the frequency polygon for the 1,104 monthly returns for the S&P 500 from January 1926 to December 2017.

Exhibit 9 Frequency Polygon of S&P 500 Monthly Total Returns: January 1926 to December 2017



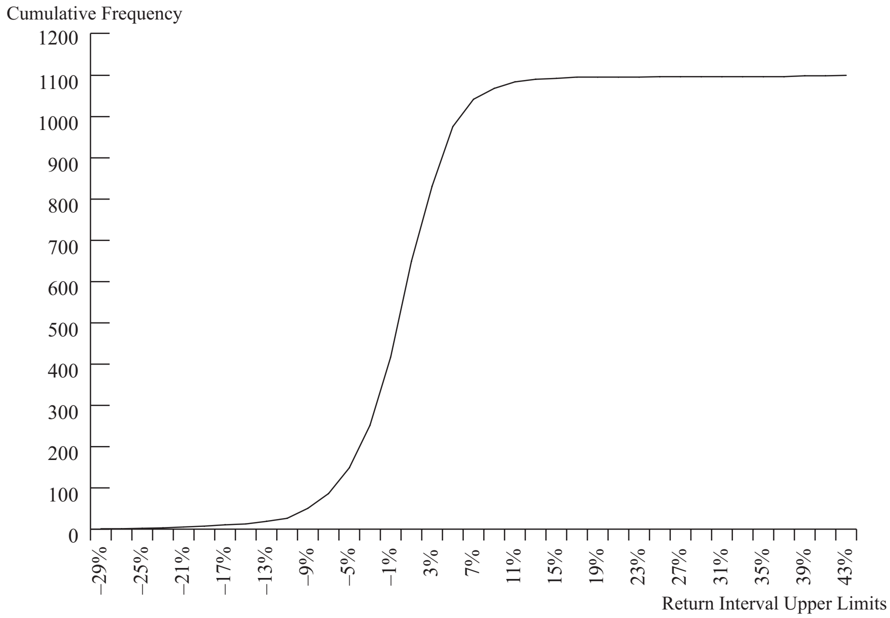
Sources: Ibbotson Associates and S&P Dow Jones Indices LLC

In Exhibit 9, we have replaced the bars in the histogram with points connected with straight lines. For example, the return interval 0 percent to 2 percent has an absolute frequency of 240. In the frequency polygon, we plot the return-interval midpoint of 1 percent and a frequency of 240. We plot all other points in a similar way.[10] This form of visual display adds a degree of continuity to the representation of the distribution.

Another form of line graph is the cumulative frequency distribution. Such a graph can plot either the cumulative absolute or cumulative relative frequency against the upper interval limit. The cumulative frequency distribution allows us to see how many or what percent of the observations lie below a certain value. To construct the cumulative frequency distribution, we graph the returns in the fourth or fifth column of Exhibit 4 against the upper limit of each return interval. Exhibit 10 presents a graph of the cumulative absolute distribution for the monthly returns on the S&P 500. Notice that the cumulative distribution tends to flatten out when returns are extremely negative or extremely positive. The steep slope in the middle of Exhibit 10 reflects the fact that most of the observations lie in the neighborhood of –2 percent to 6 percent.

10 Even though the upper limit on the interval is not a return falling in the interval, we still average it with the lower limit to determine the midpoint.

Exhibit 10 Cumulative Absolute Frequency Distribution of S&P 500 Monthly Total Returns: January 1926 to December 2017



Sources: Ibbotson Associates and S&P Dow Jones Indices LLC

We can further examine the relationship between the relative frequency and the cumulative relative frequency by looking at the two return intervals reproduced in Exhibit 11. The first return interval (0 percent to 2 percent) has a cumulative relative frequency of 58.97 percent. The next return interval (2 percent to 4 percent) has a cumulative relative frequency of 76.18 percent. The change in the cumulative relative frequency as we move from one interval to the next is the next interval's relative frequency. For instance, as we go from the first return interval (0 percent to 2 percent) to the next return interval (2 percent to 4 percent), the change in the cumulative relative frequency is 76.18% − 58.97% = 17.21%. (Values in the exhibit have been rounded to two decimal places.) The fact that the slope is steep indicates that these frequencies are large. As you can see in the graph of the cumulative distribution, the slope of the curve changes as we move from the first return interval to the last. A fairly small slope for the cumulative distribution for the first few return intervals tells us that these return intervals do not contain many observations. You can go back to the frequency distribution in Exhibit 4 and verify that the cumulative absolute frequency is only 25 observations (the cumulative relative frequency is 2.39 percent) up to the 10th return interval (−12 percent to −10 percent). In essence, the slope of the cumulative absolute distribution at any particular interval is proportional to the number of observations in that interval.

Exhibit 11 Selected Class Frequencies for the S&P 500 Monthly Returns

Return Interval (%)	Absolute Frequency	Relative Frequency (%)	Cumulative Absolute Frequency	Cumulative Relative Frequency (%)
0.0 to 2.0	240	21.74	651	58.97
2.0 to 4.0	190	17.21	841	76.18

5 MEASURES OF CENTRAL TENDENCY

So far, we have discussed methods we can use to organize and present data so that they are more understandable. The frequency distribution of an asset class's return series, for example, reveals the nature of the risks that investors may encounter in a particular asset class. As an illustration, the histogram for the annual returns on the S&P 500 clearly shows that large positive and negative annual returns are common. Although frequency distributions and histograms provide a convenient way to summarize a series of observations, these methods are just a first step toward describing the data. In this section we discuss the use of quantitative measures that explain characteristics of data. Our focus is on measures of central tendency and other measures of location or location parameters. A **measure of central tendency** specifies where the data are centered. Measures of central tendency are probably more widely used than any other statistical measure because they can be computed and applied easily. **Measures of location** include not only measures of central tendency but other measures that illustrate the location or distribution of data.

In the following subsections we explain the common measures of central tendency—the arithmetic mean, the median, the mode, the weighted mean, and the geometric mean. We also explain other useful measures of location, including quartiles, quintiles, deciles, and percentiles.

5.1 The Arithmetic Mean

Analysts and portfolio managers often want one number that describes a representative possible outcome of an investment decision. The arithmetic mean is by far the most frequently used measure of the middle or center of data.

- **Definition of Arithmetic Mean.** The **arithmetic mean** is the sum of the observations divided by the number of observations.

We can compute the arithmetic mean for both populations and samples, known as the population mean and the sample mean, respectively.

5.1.1 *The Population Mean*

The population mean is the arithmetic mean computed for a population. If we can define a population adequately, then we can calculate the population mean as the arithmetic mean of all the observations or values in the population. For example, analysts examining the year-over-year growth in same-store sales of major US wholesale clubs might define the population of interest to include only three companies: BJ's Wholesale

Club, Costco Wholesale Corporation, and Sam's Club, part of Wal-Mart Stores.[11] As another example, if a portfolio manager's investment universe (the set of securities he or she must choose from) is the Nikkei 225 Index, the relevant population is the 225 shares on the First Section of the Tokyo Stock Exchange that compose the Nikkei.

- **Population Mean Formula.** The **population mean**, μ, is the arithmetic mean value of a population. For a finite population, the population mean is

$$\mu = \frac{\sum_{i=1}^{N} X_i}{N} \tag{2}$$

where N is the number of observations in the entire population and X_i is the ith observation.

The population mean is an example of a parameter. The population mean is unique; that is, a given population has only one mean. To illustrate the calculation, we can take the case of the population mean of profit as a percentage of revenue of US companies running major wholesale clubs for 2018. During the year, profit as a percentage of revenue was about 0 percent for BJ's Wholesale club (according to https://investors.bjs.com/), and 2.1 percent and 2.0 percent for Costco Wholesale Corporation, and Wal-Mart Stores, respectively (according to the Fortune 500 list for 2018). Thus the population mean profit as a percentage of revenue was μ = (0.0 + 2.1 + 2.0)/3 = 4.1/3 = 1.37 percent.

5.1.2 *The Sample Mean*

The sample mean is the arithmetic mean computed for a sample. Many times we cannot observe every member of a set; instead, we observe a subset or sample of the population. The concept of the mean can be applied to the observations in a sample with a slight change in notation.

- **Sample Mean Formula.** The **sample mean** or average, $\bar{X}$ (read "X-bar"), is the arithmetic mean value of a sample:

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \tag{3}$$

where n is the number of observations in the sample.

Equation 3 tells us to sum the values of the observations (X_i) and divide the sum by the number of observations. For example, if a sample of price-to-earnings (P/E) multiples for six publicly traded companies contains the values 35, 30, 22, 18, 15, and 12, the sample mean P/E is 132/6 = 22. The sample mean is also called the arithmetic average.[12] As we discussed earlier, the sample mean is a statistic (that is, a descriptive measure of a sample).

Means can be computed for individual units or over time. For instance, the sample might be the return on equity (ROE) in a given year for the 100 companies in the FTSE Eurotop 100, an index of Europe's 100 largest companies. In this case, we calculate mean ROE in that year as an average across 100 individual units. When we examine the characteristics of some units at a specific point in time (such as ROE for the FTSE

11 A wholesale club implements a store format dedicated mostly to bulk sales in warehouse-sized stores to customers who pay membership dues. As of the early 2010s, those three wholesale clubs dominated the segment in the United States.

12 Statisticians prefer the term "mean" to "average." Some writers refer to all measures of central tendency (including the median and mode) as averages. The term "mean" avoids any possibility of confusion.

Eurotop 100), we are examining **cross-sectional data**. The mean of these observations is called a cross-sectional mean. On the other hand, if our sample consists of the historical monthly returns on the FTSE Eurotop 100 for the past five years, then we have **time-series data**. The mean of these observations is called a time-series mean. We will examine specialized statistical methods related to the behavior of time series in the reading on times-series analysis.

Next, we show an example of finding the sample mean return for 16 European equity markets for 2012. In this case, the mean is cross-sectional because we are averaging individual country returns.

EXAMPLE 3

Calculating a Cross-Sectional Mean

The MSCI EAFE (Europe, Australasia, and Far East) Index is a free float-adjusted market capitalization index designed to measure developed-market equity performance excluding the United States and Canada.[13] As of October 2018, the EAFE consisted of 21 developed market country indexes, including indexes for 15 European markets, 2 Australasian markets (Australia and New Zealand), 3 Far Eastern markets (Hong Kong SAR, Japan, and Singapore), and Israel.

Suppose we are interested in the local currency performance over the past five years of the 15 European markets in the EAFE as of 30 October, 2018. We want to find the sample mean total annual return across these 16 markets over the past five years. The return series reported in Exhibit 12 are in local currency (that is, returns are for investors living in the country). Because this return is not stated in any single investor's home currency, it is not a return any single investor would earn. Rather, it is an average of returns in local currencies of the 16 countries.

Exhibit 12 Total Annual Returns over Past Five Years for European Equity Markets, 30 October 2018

Market	Total Return in Local Currency (%)
Austria	5.44
Belgium	5.70
Denmark	10.05
Finland	9.94
France	7.43
Germany	5.01
Ireland	6.81
Italy	2.32
Netherlands	9.14
Norway	7.20
Portugal	−1.35
Spain	1.68
Sweden	8.16

13 The term "free float adjusted" means that the weights of companies in the index reflect the value of the shares actually available for investment.

Exhibit 12 (Continued)

Market	Total Return in Local Currency (%)
Switzerland	5.37
United Kingdom	4.46

Source: www.msci.com.

Using the data in Exhibit 12, calculate the sample mean return for the 15 equity markets.

Solution:

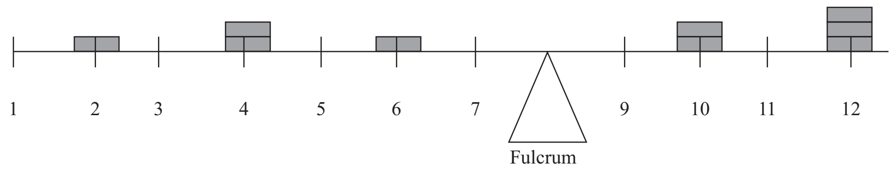
The calculation applies Equation 3 to the returns in Exhibit 12: (5.44 + 5.70 + 10.05 + 9.94 + 7.43 + 5.01 + 6.81 _ 2.32 + 9.14 + 7.20 − 1.35 + 1.68 + 8.16 +5.37 + 4.46)/15 = 187.36/15 = 5.82 percent.

In Example 3, we can verify that eight markets had returns less than the mean and seven had returns that were greater. We should not expect any of the actual observations to equal the mean, because sample means provide only a summary of the data being analyzed. Also, although in this example the number of values below the mean is quite close to the number of values above the mean, that need not be the case. As an analyst, you will often need to find a few numbers that describe the characteristics of the distribution. The mean is generally the statistic that you will use as a measure of the typical outcome for a distribution. You can then use the mean to compare the performance of two different markets. For example, you might be interested in comparing the stock market performance of investments in Pacific Rim countries with investments in European countries. You can use the mean returns in these markets to compare investment results.

5.1.3 *Properties of the Arithmetic Mean*

The arithmetic mean can be likened to the center of gravity of an object. Exhibit 13 expresses this analogy graphically by plotting nine hypothetical observations on a bar. The nine observations are 2, 4, 4, 6, 10, 10, 12, 12, and 12; the arithmetic mean is 72/9 = 8. The observations are plotted on the bar with various heights based on their frequency (that is, 2 is one unit high, 4 is two units high, and so on). When the bar is placed on a fulcrum, it balances only when the fulcrum is located at the point on the scale that corresponds to the arithmetic mean.

Exhibit 13 Center of Gravity Analogy for the Arithmetic Mean



When the fulcrum is placed at 8, the bar is perfectly balanced.

As analysts, we often use the mean return as a measure of the typical outcome for an asset. As in the example above, however, some outcomes are above the mean and some are below it. We can calculate the distance between the mean and each outcome and call it a deviation. Mathematically, it is always true that the sum of the deviations around the mean equals 0. We can see this by using the definition of the arithmetic mean shown in Equation 3, multiplying both sides of the equation by n: $n\overline{X} = \sum_{i=1}^{n} X_i$. The sum of the deviations from the mean can thus be calculated as follows:

$$\sum_{i=1}^{n}\left(X_i - \overline{X}\right) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X} = \sum_{i=1}^{n} X_i - n\overline{X} = 0$$

Deviations from the arithmetic mean are important information because they indicate risk. The concept of deviations around the mean forms the foundation for the more complex concepts of variance, skewness, and kurtosis, which we will discuss later in this reading.

An advantage of the arithmetic mean over two other measures of central tendency, the median and mode, is that the mean uses all the information about the size and magnitude of the observations. The mean is also easy to work with mathematically.

A property and potential drawback of the arithmetic mean is its sensitivity to extreme values. Because all observations are used to compute the mean, the arithmetic mean can be pulled sharply upward or downward by extremely large or small observations, respectively. For example, suppose we compute the arithmetic mean of the following seven numbers: 1, 2, 3, 4, 5, 6, and 1,000. The mean is 1,021/7 = 145.86 or approximately 146. Because the magnitude of the mean, 146, is so much larger than that of the bulk of the observations (the first six), we might question how well it represents the location of the data. In practice, although an extreme value or outlier in a financial dataset may only represent a rare value in the population, it may also reflect an error in recording the value of an observation, or an observation generated from a different population from that producing the other observations in the sample. In the latter two cases in particular, the arithmetic mean could be misleading. Perhaps the most common approach in such cases is to report the median in place of or in addition to the mean.[14] We discuss the median next.

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14 Other approaches to handling extreme values involve variations of the arithmetic mean. The **trimmed mean** is computed by excluding a stated small percentage of the lowest and highest values and then computing an arithmetic mean of the remaining values. For example, a 5 percent trimmed mean discards the lowest 2.5 percent and the largest 2.5 percent of values and computes the mean of the remaining 95 percent of values. A trimmed mean is used in sports competitions when judges' lowest and highest scores are discarded in computing a contestant's score. A **Winsorized mean** assigns a stated percent of the lowest values equal to one specified low value, and a stated percent of the highest values equal to one specified high value, then computes a mean from the restated data. For example, a 95 percent Winsorized mean sets the bottom 2.5 percent of values equal to the 2.5th percentile value and the upper 2.5 percent of values equal to the 97.5th percentile value. (Percentile values are defined later.)

5.2 The Median

A second important measure of central tendency is the median.

- **Definition of Median.** The **median** is the value of the middle item of a set of items that has been sorted into ascending or descending order. In an odd-numbered sample of n items, the median occupies the $(n + 1)/2$ position. In an even-numbered sample, we define the median as the mean of the values of items occupying the $n/2$ and $(n + 2)/2$ positions (the two middle items).[15]

Earlier we gave the profit as a percentage of revenue of three wholesale clubs as 0.0, 2.0, and 2.1. With an odd number of observations ($n = 3$), the median occupies the $(n + 1)/2 = 4/2 =$ 2nd position. The median was 2.0 percent. The value of 2.0 percent is the "middlemost" observation: One lies above it, and one lies below it. Whether we use the calculation for an even- or odd-numbered sample, an equal number of observations lie above and below the median. A distribution has only one median.

A potential advantage of the median is that, unlike the mean, extreme values do not affect it. The median, however, does not use all the information about the size and magnitude of the observations; it focuses only on the relative position of the ranked observations. Calculating the median is also more complex; to do so, we need to order the observations from smallest to largest, determine whether the sample size is even or odd and, on that basis, apply one of two calculations. Mathematicians express this disadvantage by saying that the median is less mathematically tractable than the mean.

To demonstrate finding the median, we use the data from Example 3, reproduced in Exhibit 14 in ascending order of the annual total return over the past five years for European equities. Because this sample has 15 observations, the median the value in the sorted array that occupies the $(15 + 1)/2 =$ 8thposition.[16] Belgium's return occupies the eighth position with a return of 5.70 percent, and is the median. Note that the median is not influenced by extremely large or small outcomes. Had Portugal's total return been a much lower value or Denmark's total return a much larger value, the median would not have changed. Using a context that arises often in practice, Example 4 shows how to use the mean and median in a sample with extreme values.

Exhibit 14 Total Annual Returns over Past Five Years for European Equity Markets, 30 October 2018 (in Ascending Order)

No.	Market	Total Return in Local Currency (%)
1	Portugal	–1.35
2	Spain	1.68
3	Italy	2.32
4	United Kingdom	4.46
5	Germany	5.01
6	Switzerland	5.37
7	Austria	5.44

(continued)

15 The notation M_d is occasionally used for the median. Just as for the mean, we may distinguish between a population median and a sample median. With the understanding that a population median divides a population in half while a sample median divides a sample in half, we follow general usage in using the term "median" without qualification, for the sake of brevity.

16 If a sample has an even number of observations, the median is the mean of the two values in the middle. For example, if our sample in Example 3 had 16 countries instead of 15, the median would be the mean of the values in the sorted array that occupy the 8th and the 9th positions.

Exhibit 14 (Continued)

No.	Market	Total Return in Local Currency (%)
8	Belgium	5.70
9	Ireland	6.81
10	Norway	7.20
11	France	7.43
12	Sweden	8.16
13	Netherlands	9.14
14	Finland	9.94
15	Denmark	10.05

Source: www.msci.com.

EXAMPLE 4

Median and Arithmetic Mean: The Case of the Price–Earnings Ratio

Suppose a client asks you for a valuation analysis on the seven-stock US common stock portfolio given in Exhibit 15. The stocks are equally weighted in the portfolio. One valuation measure that you use is P/E, the ratio of share price to earnings per share (EPS). Many variations exist for the denominator in the P/E, but you are examining trailing twelve month (TTM) P/E defined as current price divided by the EPS for the company for the last twelve months ("EPS (TTM)" in the exhibit).[17] The values in Exhibit 15 are as of 31 October 2018. For comparison purposes, the average current trailing twelve month P/E on the companies in the S&P 500 index was 22.14 at that time.

Exhibit 15 P/Es for a Client Portfolio

Stock	EPS (TTM)	P/E (TTM)
Caterpillar, Inc.	5.18	23.44
Dunkin' Brands Group, Inc.	4.12	17.62
Ford Motor Company	1.69	5.65
General Dynamics	9.89	17.46
McDonald's Corporation	6.82	25.95
Salesforce.com	0.96	143.11
Spirit Airlines	1.40	22.95

Source: finance.yahoo.com.

17 For more information on price multiples, see the Level I CFA Program curriculum reading "Equity Valuation: Concepts and Basic Tools."

Using the data in Exhibit 15, address the following:

1 Calculate the arithmetic mean P/E.

2 Calculate the median P/E.

3 Evaluate the mean and median P/Es as measures of central tendency for the above portfolio.

Solution to 1:

The mean P/E is (23.44 + 17.62 + 5.65 + 17.46 25.95 + 143.11 + 22.95)/7 = 256.18/7 = 36.60.

Solution to 2:

The P/Es listed in ascending order are:

5.65	17.46	17.62	22.95	23.44	25.95	143.11

The sample has an odd number of observations with $n = 7$, so the median occupies the $(n + 1)/2 = 8/2 = 4$th position in the sorted list. Therefore, the median P/E is 22.95.

Solution to 3:

Salesforce.com's P/E of approximately 143 tremendously influences the value of the portfolio's arithmetic mean P/E. The mean P/E of about 37 is much larger than the P/E of six of the seven stocks in the portfolio. The mean P/E also misleadingly suggests an orientation to stocks with high P/Es. The mean P/E of the stocks excluding Salesforce.com, or excluding the largest- and smallest-P/E stocks (Salesforce.com and Ford Motor Company), is below the average P/E of 22.14 for the companies in the S&P 500 Index. The median P/E of 22.95 appears to better represent the central tendency of the P/Es than the mean P/E of 36.60.

It frequently happens that when a company's EPS is quite low—at a low point in the business cycle, for example—its P/E is extremely high. The high P/E in those circumstances reflects an anticipated future recovery of earnings. Extreme P/E values need to be investigated and handled with care. For reasons related to this example, analysts often use the median of price multiples to characterize the valuation of industry groups.

5.3 The Mode

The third important measure of central tendency is the mode.

- **Definition of Mode.** The **mode** is the most frequently occurring value in a distribution.[18]

A distribution can have more than one mode, or even no mode. When a distribution has one most frequently occurring value, the distribution is said to be unimodal. If a distribution has two most frequently occurring values, then it has two modes and we say it is bimodal. If the distribution has three most frequently occurring values, then it is trimodal. When all the values in a data set are different, the distribution has no mode because no value occurs more frequently than any other value.

18 The notation M_o is occasionally used for the mode. Just as for the mean and the median, we may distinguish between a population mode and a sample mode. With the understanding that a population mode is the value with the greatest probability of occurrence, while a sample mode is the most frequently occurring value in the sample, we follow general usage in using the term "mode" without qualification, for the sake of brevity.

Stock return data and other data from continuous distributions may not have a modal outcome. When such data are grouped into intervals, however, we often find an interval (possibly more than one) with the highest frequency: the **modal interval** (or intervals). For example, the frequency distribution for the monthly returns on the S&P 500 has a modal interval of 0 percent to 2 percent, as shown in Exhibit 8; this return interval has 240 observations out of a total of 1,104. The modal interval always has the highest bar in the histogram.

The mode is the only measure of central tendency that can be used with nominal data. When we categorize mutual funds into different styles and assign a number to each style, the mode of these categorized data is the most frequent mutual fund style.

5.4 Other Concepts of Mean

Earlier we explained the arithmetic mean, which is a fundamental concept for describing the central tendency of data. Other concepts of mean are very important in investments, however. In the following, we discuss such concepts.

EXAMPLE 5

Calculating a Mode

Exhibit 16 gives the credit ratings on senior unsecured debt as of November 2018 of six US department stores rated by Moody's Investors Service. In descending order of credit quality (increasing expected probability of default), Moody's ratings are Aaa, Aa1, Aa2, Aa3, A1, A2, A3, Baa1, Baa2, Baa3, Ba1, Ba2, Ba3, B1, B2, B3, Caa1, Caa2, Caa3, Ca, and C.[19]

Exhibit 16 Senior Unsecured Debt Ratings: US Department Stores, November 2018

Company	Credit Rating
Dillard's, Inc.	Baa3
Kohl's Corporation	Baa2
Macy's, Inc.	Baa3
Neiman Marcus Group Ltd.	Caa3
Nordstrom, Inc.	Baa1
Penney, JC, Corporation, Inc.	Caa2

Source: www.moodys.com.

Using the data in Exhibit 16, address the following concerning the senior unsecured debt of US department stores:

1 State the modal credit rating.

2 State the median credit rating.

19 For more information on credit risk and credit ratings, see Petitt, Pinto, and Pirie (2015).

Solution to 1:

The group of companies represents six distinct credit ratings, ranging from Baa1 to Caa3. To make our task easy, we first organize the ratings into a frequency distribution.

Exhibit 17 Senior Unsecured Debt Ratings: US Department Stores, Distribution of Credit Ratings

Credit Rating	Frequency
Baa1	1
Baa2	1
Baa3	2
Caa2	1
Caa3	1

Credit rating Baa3 has a frequency of 2, and the other four ratings have a frequency of 1. Therefore, the credit rating of US department stores in November 2018 was unimodal, with Baa3 being the mode. Moody's considers bonds rated Baa to be of moderate credit risk, Ba to be of substantial credit risk, and B to be of high credit risk.

Solution to 2:

For the group n = 6, an odd number. The group's median occupies the mean of the 6/2 = 3rd and 8/2 = 4th position. We see from Exhibit 17 that Baa3 occupies both the third and the fourth positions. Therefore the median credit rating at November 2018 was Baa3.

5.4.1 *The Weighted Mean*

The concept of weighted mean arises repeatedly in portfolio analysis. In the arithmetic mean, all observations are equally weighted by the factor $1/n$ (or $1/N$). In working with portfolios, we need the more general concept of weighted mean to allow different weights on different observations.

To illustrate the weighted mean concept, an investment manager with $100 million to invest might allocate $70 million to equities and $30 million to bonds. The portfolio has a weight of 0.70 on stocks and 0.30 on bonds. How do we calculate the return on this portfolio? The portfolio's return clearly involves an averaging of the returns on the stock and bond investments. The mean that we compute, however, must reflect the fact that stocks have a 70 percent weight in the portfolio and bonds have a 30 percent weight. The way to reflect this weighting is to multiply the return on the stock investment by 0.70 and the return on the bond investment by 0.30, then sum the two results. This sum is an example of a weighted mean. It would be incorrect to take an arithmetic mean of the return on the stock and bond investments, equally weighting the returns on the two asset classes.

Consider a portfolio invested in Canadian stocks and bonds. The stock component of the portfolio includes the RBC Canadian Index Fund, which tracks the performance of the S&P/TSX Composite Total Return Index. The bond component of the portfolio includes the RBC Bond Fund, which invests in high-quality fixed-income securities issued by Canadian governments and corporations. The portfolio manager allocates 60 percent of the portfolio to the Canadian stock fund and 40 percent to the Canadian bond fund. Exhibit 18 presents total returns for these funds for a five-year period.

Exhibit 18 Returns for Canadian Equity and Bond Funds

Year	Equity Fund (%)	Bond Fund (%)
Year 1	−33.1	−0.1
Year 2	34.1	11.0
Year 3	16.8	6.4
Year 4	−9.2	8.4
Year 5	6.4	3.8

Source: funds.rbcgam.com.

- **Weighted Mean Formula.** The **weighted mean** $\overline{X}_w$ (read "X-bar sub-w"), for a set of observations $X_1, X_2, \ldots, X_n$ with corresponding weights of $w_1, w_2, \ldots, w_n$ is computed as

$$\overline{X}_w = \sum_{i=1}^{n} w_i X_i \tag{4}$$

where the sum of the weights equals 1; that is, $\sum_i w_i = 1$.

In the context of portfolios, a positive weight represents an asset held long and a negative weight represents an asset held short.[20]

The return on the portfolio under consideration is the weighted average of the return on the Canadian stock fund and the Canadian bond fund (the weight of the stock fund is 0.60; that of the bond fund is 0.40). We find, using Equation 4, that

$$\begin{aligned} \text{Portfolio return for Year 1} &= w_{\text{stock}}R_{\text{stock}} + w_{\text{bonds}}R_{\text{bonds}} \\ &= 0.60(-33.1) + 0.40(-0.1) \\ &= -19.9\% \end{aligned}$$

It should be clear that the correct mean to compute in this example is the weighted mean and not the arithmetic mean. If we had computed the arithmetic mean for Year 1, we would have calculated a return equal to ½(−33.1%) + ½(−0.1%) = (−33.1% − 0.1%)/2 = −16.6%. Given that the portfolio manager invested 60 percent in stocks and 40 percent in bonds, the arithmetic mean would underweight the investment in stocks and overweight the investment in bonds, resulting in a number for portfolio return that is too high by 3.3 percentage points (−16.6% − (−19.9%) = −16.6% + 19.9%).

Now suppose that the portfolio manager maintains constant weights of 60 percent in stocks and 40 percent in bonds for all five years. This method is called a constant-proportions strategy. Because value is price multiplied by quantity, price fluctuation causes portfolio weights to change. As a result, the constant-proportions strategy

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20 The formula for the weighted mean can be compared to the formula for the arithmetic mean. For a set of observations $X_1, X_2, \ldots, X_n$, let the weights $w_1, w_2, \ldots, w_n$ all equal $1/n$. Under this assumption, the formula for the weighted mean is $(1/n)\sum_{i=1}^{n} X_i$. This is the formula for the arithmetic mean. Therefore, the arithmetic mean is a special case of the weighted mean in which all the weights are equal.

requires rebalancing to restore the weights in stocks and bonds to their target levels. Assuming that the portfolio manager is able to accomplish the necessary rebalancing, we can compute the portfolio returns in Years 2, 3, 4, and 5 with Equation 4 as follows:

Portfolio return for Year 2 = 0.60(34.1) + 0.40(11.0) = 24.9%

Portfolio return for Year 3 = 0.60(16.8) + 0.40(6.4) = 12.6%

Portfolio return for Year 4 = 0.60(−9.2) + 0.40(8.4) = −2.2%

Portfolio return for Year 5 = 0.60(6.4) + 0.40(3.8) = 5.4%

We can now find the time-series mean of the returns for Year 1 through Year 5 using Equation 3 for the arithmetic mean. The time-series mean total return for the portfolio is (−19.9 + 24.9 + 12.6 − 2.2 + 5.4)/5 = 20.8/5 = 4.2 percent.

Instead of calculating the portfolio time-series mean return from portfolio annual returns, we can calculate the arithmetic mean stock and bond fund returns for the five years and then apply the portfolio weights of 0.60 and 0.40, respectively, to those values. The mean stock fund return is (−33.1 + 34.1 + 16.8 − 9.2 + 6.4)/5 = 15.0/5 = 3.0 percent. The mean bond fund return is (−0.1 + 11.0 + 6.4 + 8.4 + 3.8)/ 5 = 29.5/5 = 5.9 percent. Therefore, the mean total return for the portfolio is 0.60(3.0) + 0.40(5.9) = 4.2 percent, which agrees with our previous calculation.

EXAMPLE 6

Portfolio Return as a Weighted Mean

Exhibit 19 gives information on the asset allocation of the pension plan of the Canadian Broadcasting Corporation in 2017 as well as the one-year returns on these asset classes in 2017.[21]

Exhibit 19 Asset Allocation for the Pension Plan of the Canadian Broadcasting Corporation in 2017

Asset Class	Asset Allocation (Weight)	Asset Class Return (%)
Cash and short-term investments	4.7	1.2
Nominal bonds	29.0	8.0
Real return bonds	11.8	1.2
Canadian equities	10.5	8.2
Global equities	24.8	15.4
Strategic investments	19.0	15.6
Bond overlay	0.2	5.7

Source: Canadian Broadcasting Corporation Pension Plan, 2017 Annual Report

Using the information in Exhibit 19, calculate the mean return earned by the pension plan in 2017.

21 In Exhibit 19, strategic investments include investments in property, private equity, and hedge fund investments. Bond overlay consists of derivatives used to hedge interest rate and inflation changes.

Solution:

Converting the percent asset allocation to decimal form, we find the mean return as a weighted average of the asset class returns. We have

$$\begin{aligned}\text{Mean portfolio return} &= 0.047(1.2\%) + 0.290(8.0\%) + 0.118(1.2\%) \\ &\quad + 0.105(8.2\%) + 0.248(15.4\%) + 0.190(15.6\%) \\ &\quad + 0.002(5.7\%) \\ &= 0.056\% + 2.320\% + 0.142\% + 0.861\% + 3.819\% + \\ &\quad 2.964\% + 0.011\% \\ &= 10.2 \text{ percent}\end{aligned}$$

The previous examples illustrate the general principle that a portfolio return is a weighted sum. Specifically, a portfolio's return is the weighted average of the returns on the assets in the portfolio; the weight applied to each asset's return is the fraction of the portfolio invested in that asset.

Market indexes are computed as weighted averages. For market-capitalization indexes such as the CAC-40 in France or the TOPIX in Japan or the S&P 500 in the United States, each included stock receives a weight corresponding to its outstanding market value divided by the total market value of all stocks in the index.

Our illustrations of weighted mean use past data, but they might just as well use forward-looking data. When we take a weighted average of forward-looking data, the weighted mean is called **expected value**. Suppose we make one forecast for the year-end level of the S&P 500 assuming economic expansion and another forecast for the year-end level of the S&P 500 assuming economic contraction. If we multiply the first forecast by the probability of expansion and the second forecast by the probability of contraction and then add these weighted forecasts, we are calculating the expected value of the S&P 500 at year-end. If we take a weighted average of possible future returns on the S&P 500, we are computing the S&P 500's expected return. The probabilities must sum to 1, satisfying the condition on the weights in the expression for weighted mean, Equation 4.

5.4.2 *The Geometric Mean*

The geometric mean is most frequently used to average rates of change over time or to compute the growth rate of a variable. In investments, we frequently use the geometric mean to average a time series of rates of return on an asset or a portfolio, or to compute the growth rate of a financial variable such as earnings or sales. In the reading on the time value of money, for instance, we computed a sales growth rate. That growth rate was a geometric mean. Because of the subject's importance, in a later section we will return to the use of the geometric mean and offer practical perspectives on its use. The geometric mean is defined by the following formula.

- **Geometric Mean Formula.** The **geometric mean**, G, of a set of observations $X_1, X_2, \ldots, X_n$ is

$$G = \sqrt[n]{X_1 X_2 X_3 \ldots X_n} \quad \text{with } X_i \geq 0 \text{ for } i = 1,2,\ldots,n. \tag{5}$$

Equation 5 has a solution, and the geometric mean exists, only if the product under the radical sign is non-negative. We impose the restriction that all the observations X_i in Equation 5 are greater than or equal to zero. We can solve for the geometric mean using Equation 5 directly with any calculator that has an exponentiation key (on most calculators, y^x). We can also solve for the geometric mean using natural logarithms. Equation 5 can also be stated as

$$\ln G = \frac{1}{n}\ln\left(X_1 X_2 X_3 \ldots X_n\right)$$

or as

$$\ln G = \frac{\sum_{i=1}^{n} \ln X_i}{n}$$

When we have computed ln G, then $G = e^{\ln G}$ (on most calculators, the key for this step is e^x).

Risky assets can have negative returns up to −100 percent (if their price falls to zero), so we must take some care in defining the relevant variables to average in computing a geometric mean. We cannot just use the product of the returns for the sample and then take the nth root because the returns for any period could be negative. We must redefine the returns to make them positive. We do this by adding 1.0 to the returns expressed as decimals. The term $(1 + R_t)$ represents the year-ending value relative to an initial unit of investment at the beginning of the year. As long as we use $(1 + R_t)$, the observations will never be negative because the biggest negative return is −100 percent. The result is the geometric mean of $1 + R_t$; by then subtracting 1.0 from this result, we obtain the geometric mean of the individual returns R_t. For example, the returns on RBC Canadian Index Fund during a five-year period were given in Exhibit 18 as −0.331, 0.341, 0.168, −0.092, and 0.064, putting the returns into decimal form. Adding 1.0 to those returns produces 0.669, 1.341, 1.168, 0.908, and 1.064. Using Equation 5 we have $\sqrt[5]{(0.669)(1.341)(1.168)(0.908)(1.064)} = \sqrt[5]{1.012337} = 1.002455$.

This number is 1 plus the geometric mean rate of return. Subtracting 1.0 from this result, we have 1.002455 − 1.0 = 0.002455 or approximately 0.25 percent. The geometric mean return of RBC Canadian Index Fund during the five-year period was 0.25 percent.

An equation that summarizes the calculation of the geometric mean return, R_G, is a slightly modified version of Equation 5 in which the X_i represent "1 + return in decimal form." Because geometric mean returns use time series, we use a subscript t indexing time as well.

$$1 + R_G = \sqrt[T]{\left(1 + R_1\right)\left(1 + R_2\right)\ldots\left(1 + R_T\right)}$$

$$1 + R_G = \left[\prod_{t=1}^{T}\left(1 + R_t\right)\right]^{\frac{1}{T}}$$

which leads to the following formula.

- **Geometric Mean Return Formula.** Given a time series of holding period returns R_t, $t = 1, 2, \ldots, T$, the geometric mean return over the time period spanned by the returns R_1 through R_T is

$$R_G = \left[\prod_{t=1}^{T}\left(1 + R_t\right)\right]^{\frac{1}{T}} - 1 \tag{6}$$

We can use Equation 6 to solve for the geometric mean return for any return data series. Geometric mean returns are also referred to as compound returns. If the returns being averaged in Equation 6 have a monthly frequency, for example, we may call the geometric mean monthly return the compound monthly return. The next example illustrates the computation of the geometric mean while contrasting the geometric and arithmetic means.

EXAMPLE 7

Geometric and Arithmetic Mean Returns (1)

As a mutual fund analyst, you are examining, in 2018, the most recent five years of total returns for two US large-cap value equity mutual funds.

Exhibit 20 Total Returns for Two Mutual Funds, 2013–2017

Year	Selected American Shares (SLASX)	T. Rowe Price Equity Income (PRFDX)
2013	34.90%	31.69%
2014	6.13	7.75
2015	2.69	−7.56
2016	11.66	18.25
2017	21.77	16.18

Source: performance.morningstar.com.

Based on the data in Exhibit 20, address the following:

1 Calculate the geometric mean return of SLASX.

2 Calculate the arithmetic mean return of SLASX and contrast it to the fund's geometric mean return.

3 Calculate the geometric mean return of PRFDX.

4 Calculate the arithmetic mean return of PRFDX and contrast it to the fund's geometric mean return.

Solution to 1:

Converting the returns on SLASX to decimal form and adding 1.0 to each return produces 1.3490, 1.0613, 1.0269, 1.1166, and 1.2177. We use Equation 6 to find SLASX's geometric mean return:

$$\begin{aligned} R_G &= \sqrt[5]{(1.3490)(1.0613)(1.0269)(1.1166)(1.2177)} - 1 \\ &= \sqrt[5]{1.999016} - 1 \\ &= 1.148585 - 1 \\ &= 14.86\% \end{aligned}$$

Solution to 2:

For SLASX, $\overline{R}$ = (34.90 + 6.13 + 2.69 − 11.66 +21.77)/ 5 = 77.15/5 = 15.43%. The arithmetic mean return for SLASX exceeds the geometric mean return by 15.43 − 14.86 = 0.57% or 57 basis points.

Solution to 3:

Converting the returns on PRFDX to decimal form and adding 1.0 to each return produces 1.3169, 1.0775, 0.9244. 1.1825, and 1.1618. We use Equation 6 to find PRFDX's geometric mean return:

$$\begin{aligned} R_G &= \sqrt[5]{(1.3169)(1.0775)(0.9244)(1.1825)(1.1618)} - 1 \\ &= \sqrt[5]{1.802032} - 1 \\ &= 1.125000 - 1 \\ &= 12.50\% \end{aligned}$$

Solution to 4:

PRFDX, $\bar{R}$ = (31.69 + 7.75 − 7.56 + 18.25 + 6.18)/5 = 66.31/5 = 13.26%. The arithmetic mean for PRFDX exceeds the geometric mean return by 13.26 − 12.50 = 0.76% or 76 basis points. The exhibit below summarizes the findings.

Exhibit 21 Mutual Fund Arithmetic and Geometric Mean Returns: Summary of Findings

Fund	Arithmetic Mean (%)	Geometric Mean (%)
SLASX	15.43	14.86
PRFDX	13.26	12.50

In Example 7, for both mutual funds, the geometric mean return was less than the arithmetic mean return. In fact, the geometric mean is always less than or equal to the arithmetic mean.[22] The only time that the two means will be equal is when there is no variability in the observations—that is, when all the observations in the series are the same.[23] In Example 7, there was variability in the funds' returns; thus for both funds, the geometric mean was strictly less than the arithmetic mean. In general, the difference between the arithmetic and geometric means increases with the variability in the period-by-period observations.[24] This relationship is also illustrated by Example 7. Casual inspection suggests that the returns of PRFDX are somewhat more variable than those of SLASX, and consequently, the spread between the arithmetic and geometric mean returns is larger for PRFDX (76 basis points) than for SLASX (57 basis points).[25] Arithmetic and geometric returns need not always rank funds similarly, however, in this example, SLASX has both higher arithmetic and geometric mean returns than PRFDX. However, the difference between the geometric mean returns of the two funds (2.36%) is greater than the difference between the arithmetic mean returns of the two funds (2.17%). How should the analyst interpret these results?

22 This statement can be proved using Jensen's inequality that the average value of a function is less than or equal to the function evaluated at the mean if the function is concave from below—the case for ln(X).

23 For instance, suppose the return for each of the three years is 10 percent. The arithmetic mean is 10 percent. To find the geometric mean, we first express the returns as $(1 + R_t)$ and then find the geometric mean: $[(1.10)(1.10)(1.10)]^{1/3} - 1.0$ = 10 percent. The two means are the same.

24 We will soon introduce standard deviation as a measure of variability. Holding the arithmetic mean return constant, the geometric mean return decreases for an increase in standard deviation.

25 We will introduce formal measures of variability later. But note, for example, the 15.31 percentage point swing in returns between 2013 and 2014 for PRFDX versus the 3.44 percentage point for SLASX.

The geometric mean return represents the growth rate or compound rate of return on an investment. One dollar invested in SLASX at the beginning of 2013 would have grown to (1.3490)(1.0613)(1.0269)(1.1166)(1.2177) = \$1.9990, which is equal to 1 plus the geometric mean return compounded over five periods: $[1 + 0.148585]^5 = (1.148585)^5$ = \$1.9990, confirming that the geometric mean is the compound rate of return. For PRFDX, one dollar would have grown to a smaller amount, (1.3169)(1.0775)(0.9244)(1.1825)(1.1618) = \$1.8020, equal to $(1.125000)^5$. With its focus on the profitability of an investment over a multiperiod horizon, the geometric mean is of key interest to investors. The arithmetic mean return, focusing on average single-period performance, is also of interest. Both arithmetic and geometric means have a role to play in investment management, and both are often reported for return series. Example 8 highlights these points in a simple context.

EXAMPLE 8

Geometric and Arithmetic Mean Returns (2)

A hypothetical investment in a single stock initially costs €100. One year later, the stock is trading at €200. At the end of the second year, the stock price falls back to the original purchase price of €100. No dividends are paid during the two-year period. Calculate the arithmetic and geometric mean annual returns.

Solution:

First, we need to find the Year 1 and Year 2 annual returns with Equation 1.

Return in Year 1 = 200/100 – 1 = 100%

Return in Year 2 = 100/200 – 1 = –50%

The arithmetic mean of the annual returns is (100% – 50%)/2 = 25%.

Before we find the geometric mean, we must convert the percentage rates of return to $(1 + R_t)$. After this adjustment, the geometric mean from Equation 6 is $\sqrt{2.0 \times 0.50} - 1 = 0$ percent.

The geometric mean return of 0 percent accurately reflects that the ending value of the investment in Year 2 equals the starting value in Year 1. The compound rate of return on the investment is 0 percent. The arithmetic mean return reflects the average of the one-year returns.

5.4.3 *The Harmonic Mean*

The arithmetic mean, the weighted mean, and the geometric mean are the most frequently used concepts of mean in investments. A fourth concept, the **harmonic mean**, $\bar{X}_H$, is appropriate in a limited number of applications.[26]

- **Harmonic Mean Formula.** The harmonic mean of a set of observations $X_1, X_2, \ldots, X_n$ is

$$\bar{X}_H = n \Big/ \sum_{i=1}^{n} (1/X_i) \quad \text{(7)}$$

with $X_i > 0$ for $i = 1,2,\ldots,n$

26 The terminology "harmonic" arises from its use relative to a type of series involving reciprocals known as a harmonic series.

The harmonic mean is the value obtained by summing the reciprocals of the observations—terms of the form $1/X_i$—then averaging that sum by dividing it by the number of observations n, and, finally, taking the reciprocal of the average.

The harmonic mean may be viewed as a special type of weighted mean in which an observation's weight is inversely proportional to its magnitude. The harmonic mean is a relatively specialized concept of the mean that is appropriate when averaging ratios ("amount per unit") when the ratios are repeatedly applied to a fixed quantity to yield a variable number of units. The concept is best explained through an illustration. A well-known application arises in the investment strategy known as **cost averaging**, which involves the periodic investment of a fixed amount of money. In this application, the ratios we are averaging are prices per share at purchases dates, and we are applying those prices to a constant amount of money to yield a variable number of shares.

Suppose an investor purchases €1,000 of a security each month for n = 2 months. The share prices are €10 and €15 at the two purchase dates. What is the average price paid for the security?

In this example, in the first month we purchase €1,000/€10 = 100 shares and in the second month we purchase €1,000/€15 = 66.67, or 166.67 shares in total. Dividing the total euro amount invested, €2,000, by the total number of shares purchased, 166.67, gives an average price paid of €2,000/166.67 = €12. The average price paid is in fact the harmonic mean of the asset's prices at the purchase dates. Using Equation 7, the harmonic mean price is 2/[(1/10) + (1/15)] = €12. The value €12 is less than the arithmetic mean purchase price (€10 + €15)/2 = €12.5. However, we could find the correct value of €12 using the weighted mean formula, where the weights on the purchase prices equal the shares purchased at a given price as a proportion of the total shares purchased. In our example, the calculation would be (100/166.67)€10.00 + (66.67/166.67)€15.00 = €12. If we had invested varying amounts of money at each date, we could not use the harmonic mean formula. We could, however, still use the weighted mean formula in a manner similar to that just described.

A mathematical fact concerning the harmonic, geometric, and arithmetic means is that unless all the observations in a data set have the same value, the harmonic mean is less than the geometric mean, which in turn is less than the arithmetic mean. In the illustration given, the harmonic mean price was indeed less than the arithmetic mean price.

OTHER MEASURES OF LOCATION: QUANTILES

6

Having discussed measures of central tendency, we now examine an approach to describing the location of data that involves identifying values at or below which specified proportions of the data lie. For example, establishing that 25, 50, and 75 percent of the annual returns on a portfolio are at or below the values −0.05, 0.16, and 0.25, respectively, provides concise information about the distribution of portfolio returns. Statisticians use the word **quantile** (or **fractile**) as the most general term for a value at or below which a stated fraction of the data lies. In the following, we describe the most commonly used quantiles—quartiles, quintiles, deciles, and percentiles—and their application in investments.

6.1 Quartiles, Quintiles, Deciles, and Percentiles

We know that the median divides a distribution in half. We can define other dividing lines that split the distribution into smaller sizes. **Quartiles** divide the distribution into quarters, **quintiles** into fifths, deciles into tenths, and **percentiles** into hundredths.

Given a set of observations, the yth percentile is the value at or below which y percent of observations lie. Percentiles are used frequently, and the other measures can be defined with respect to them. For example, the first quartile (Q_1) divides a distribution such that 25 percent of the observations lie at or below it; therefore, the first quartile is also the 25th percentile. The second quartile (Q_2) represents the 50th percentile, and the third quartile (Q_3) represents the 75th percentile because 75 percent of the observations lie at or below it.

When dealing with actual data, we often find that we need to approximate the value of a percentile. For example, if we are interested in the value of the 75th percentile, we may find that no observation divides the sample such that exactly 75 percent of the observations lie at or below that value. The following procedure, however, can help us determine or estimate a percentile. The procedure involves first locating the position of the percentile within the set of observations and then determining (or estimating) the value associated with that position.

Let P_y be the value at or below which y percent of the distribution lies, or the yth percentile. (For example, P_{18} is the point at or below which 18 percent of the observations lie; 100 − 18 = 82 percent are greater than P_{18}.) The formula for the position of a percentile in an array with n entries sorted in ascending order is

$$L_y = (n+1)\frac{y}{100} \tag{8}$$

where y is the percentage point at which we are dividing the distribution and L_y is the location (L) of the percentile (P_y) in the array sorted in ascending order. The value of L_y may or may not be a whole number. In general, as the sample size increases, the percentile location calculation becomes more accurate; in small samples it may be quite approximate.

As an example of the case in which L_y is not a whole number, suppose that we want to determine the 60th percentile of annual returns for the past five years as of 30 October 2018(Q_3 or P_{75}) for the 15 European equity markets given in Exhibit 12. According to Equation 8, the position of the 60th percentile is L_{60} = (15 + 1)(60/100) = 9.60, or between the 9th and 10th items in Exhibit 14, which ordered the returns into ascending order. The 9th item in Exhibit 14 is the return to equities in Ireland, 6.81 percent. The 10th item is the return to equities in Norway, 7.20 percent. Reflecting the "0.60" in "9.60," we would conclude that P_{60} lies 60 percent of the distance between 6.81 percent and 7.20 percent.

To summarize:

- When the location, L_y, is a whole number, the location corresponds to an actual observation. For example, if we were determining the third quartile, then L_y would have been L_{75} = (15 + 1)(75/100) = 12, and the third quartile would be $P_{75} = X_{12}$, where X_i is defined as the value of the observation in the ith ($i = L_{75}$) position of the data sorted in ascending order (i.e., P_{75} =8.16).
- When L_y is not a whole number or integer, L_y lies between the two closest integer numbers (one above and one below), and we use **linear interpolation** between those two places to determine P_y. Interpolation means estimating an unknown value on the basis of two known values that surround it (lie above and below it); the term "linear" refers to a straight-line estimate. Returning to the calculation of P_{60} for the equity returns, we found that L_y = 9.60; the next lower whole number is 9 and the next higher whole number is 10. Using linear interpolation, $P_{60} \approx X_9 + (9.60 - 9)(X_{10} - X_9)$. As above, in the 9th position is the return to equities in Ireland, so X_9 = 6.81 percent; X_{10} = 7.20 percent, the return to equities in Norway. Thus our estimate is $P_{60} \approx X_9 + (9.60 - 9.00)(X_{10} - X_9)$ = 6.81 + 0.60 [7.20 − 6.81] = 6.81 + 0.60(0.39) = 6.81 + 0.23 = 7.04 percent. In words, 6.81 and 7.20 bracket P_{60} from below and above, respectively. Because

9.60 − 9 = 0.60, using linear interpolation we move 60 percent of the distance from 6.81 to 7.20 as our estimate of P_{60}. We follow this pattern whenever L_y is a non-integer: The nearest whole numbers below and above L_y establish the positions of observations that bracket P_y and then interpolate between the values of those two observations.

Example 9 illustrates the calculation of various quantiles for the dividend yield on the components of a major European equity index.

EXAMPLE 9

Calculating Percentiles, Quartiles, and Quintiles

The EURO STOXX 50 is an index of 50 publicly traded companies, which provides a blue-chip representation of supersector leaders in the Eurozone. Exhibit 22 shows the market capitalization on the 50 component stocks in the index in November 2018. The market capitalizations are ranked in ascending order.

Exhibit 22 Market Capitalizations of the Components of the EURO STOXX 50

No.	Company	Market Cap (Euro Billion)
1	RWE	9.9
2	Carrefour	12.2
3	E.ON	15.5
4	Inditex	15.5
5	Unibail Rodamco	16.1
6	Deutsche Bank	16.7
7	Saint Gobain	17.7
8	Arcelor Mittal	19.5
9	Repsol S.A.	19.9
10	CRH	20.1
11	Banco Bilbao Vizcaya Argentaria	22.2
12	Assicurazioni Generali	22.2
13	Societe Generale	24.2
14	Engie	26.5
15	Essilor International	27.4
16	Vivendi	28.2
17	Engie	28.4
18	Intesa SanPaolo	30.3
19	Philips	31.5
20	Telefonica SA	33.0
21	Air Liquide	33.4
22	Munchener Ruckversicherungs AG	33.7
23	Banco Santander	35.4
24	Schneider Electric	35.4
25	Orange	36.6

(continued)

Exhibit 22 (Continued)

No.	Company	Market Cap (Euro Billion)
26	EIberdrola	38.4
27	Danone	40.2
28	ING Groep	40.2
29	ENEL	40.8
30	Volkswagon	43.0
31	Vinci	45.5
32	BMW	45.9
33	AXA	52.2
34	Daimler	55.8
35	Bayer	56.1
36	ENI	56.9
37	BNP Paribas	57.8
38	Unilever	61.9
39	BASF	62.5
40	Deutsche Telekom	62.6
41	ASML Holdings	63.4
42	UniCredit	65.5
43	Allianz	83.3
44	Siemens	89.6
45	Anheuser-Busch Inbev	104.7
46	Sanofi-Aventis	104.7
47	SAP AG	116.3
48	LOréal	120.8
49	Total	122.9
50	LVMH	136.5

Source: https://www.dividendmax.com/market-index-constituents/euro-stoxx-50 accessed 01 November 2018.

Using the data in Exhibit 22, address the following:

1. Calculate the 10th and 90th percentiles.
2. Calculate the first, second, and third quartiles.
3. State the value of the median.
4. How many quintiles are there, and to what percentiles do the quintiles correspond?
5. Calculate the value of the first quintile.

Solution to 1:

In this example, $n = 50$. Using Equation 8, $L_y = (n + 1)y/100$ for position of the yth percentile, so for the 10th percentile we have

$$L_{10} = (50 + 1)(10/100) = 5.1$$

L_{10} is between the fifth and sixth observations with values $X_5 = 16.1$ and $X_6 = 16.7$. The estimate of the 10th percentile (first decile) for dividend yield is

$$\begin{aligned} P_{10} \approx X_5 + (5.1 - 5)(X_6 - X_5) &= 16.1 + 0.1(16.7 - 16.1) \\ &= 16.1 + 0.1(0.6) = 16.16 \end{aligned}$$

For the 90th percentile,

$$L_{90} = (50 + 1)(90/100) = 45.9$$

L_{90} is between the 45th and 46th observations with values $X_{45} = 104.7$ and $X_{46} = 104.7$, respectively. The two values are the same. Therefore, the estimate of the 90th percentile (ninth decile) is 104.7. If the values were different, we would have computed the 90th percentile using the following formula.

$$P_{90} \approx X_{45} + (45.9 - 45)(X_{46} - X_{45})$$

Solution to 2:

The first, second, and third quartiles correspond to P_{25}, P_{50}, and P_{75}, respectively.

L_{25}	$= (51)(25/100) = 12.75$	L_{25} is between the 12th and 13th entries with values $X_{12} = 22.2$ and $X_{13} = 24.2$.
P_{25}	$= Q_1 \approx X_{12} + (12.75 - 12)(X_{13} - X_{12})$ $= 22.2 + 0.75(24.2 - 22.2)$ $= 22.2 + 0.75(2) = 23.7$	
L_{50}	$= (51)(50/100) = 25.5$	L_{50} is between the 25th and 26th entries with values, $X_{25} = 36.6$ and $X_{26} = 38.4$.
P_{50}	$= Q_2 \approx X_{25} + (25.50 - 25)(X_{26} - X_{25})$ $= 36.6 + 0.50(38.4 - 36.6)$ $= 36.6 + 0.50(1.8) = 37.5$	
L_{75}	$= (51)(75/100) = 38.25$	L_{75} is between the 38th and 39th entries with values $X_{38} = 61.9$ and $X_{39} = 62.5$.
P_{75}	$= Q_3 \approx X_{38} + (38.25 - 38)(X_{39} - X_{38})$ $= 61.9 + 0.25(62.5 - 61.9)$ $= 61.9 + 0.25(0.6) = 62.05$	

Solution to 3:

The median is the 50th percentile, 37.5. This is the same value that we would obtain by taking the mean of the $n/2 = 50/2 = 25$th item and $(n + 2)/2 = 52/2 = 26$th items, consistent with the procedure given earlier for the median of an even-numbered sample.

Solution to 4:

There are five quintiles, and they are specified by P_{20}, P_{40}, P_{60}, and P_{80}.

Solution to 5:

The first quintile is P_{20}.

$L_{20} = (50 + 1)(20/100) = 10.2$ — L_{20} is between the 10th and 11th observations with values $X_{10} = 20.1$ and $X_{11} = 22.2$.

The estimate of the first quintile is

$$\begin{aligned} P_{20} \approx X_{10} + (10.2 - 10)(X_{11} - X_{10}) &= 20.1 + 0.2(22.2 - 20.1) \\ &= 20.1 + 0.2(2.1) = 20.52. \end{aligned}$$

6.2 Quantiles in Investment Practice

In this section, we discuss the use of quantiles in investments. Quantiles are used in portfolio performance evaluation as well as in investment strategy development and research.

Investment analysts use quantiles every day to rank performance—for example, the performance of portfolios. The performance of investment managers is often characterized in terms of the quartile in which they fall relative to the performance of their peer group of managers. The Morningstar mutual fund star rankings, for example, associates the number of stars with percentiles of performance relative to similar-style mutual funds.

Another key use of quantiles is in investment research. Analysts refer to a group defined by a particular quantile as that quantile. For example, analysts often refer to the set of companies with returns falling below the 10th percentile cutoff point as the bottom return decile. Dividing data into quantiles based on some characteristic allows analysts to evaluate the impact of that characteristic on a quantity of interest. For instance, empirical finance studies commonly rank companies based on the market value of their equity and then sort them into deciles. The 1st decile contains the portfolio of those companies with the smallest market values, and the 10th decile contains those companies with the largest market value. Ranking companies by decile allows analysts to compare the performance of small companies with large ones.

We can illustrate the use of quantiles, in particular quartiles, in investment research using the example of Ibbotson et al. (2018). That study is an update of Ibbotson et al. (2013), which proposed an investment style based on liquidity—buying stocks of less liquid stocks and selling stocks of more liquid stocks. These studies compare the performance of this style with three already popular investment styles, which include (1) firm size (buying stocks of small firms and selling stocks of large firms), (2) value/growth (buying stocks of value firms, defined as firms for which the stock price is relatively low in relation to earnings per share, book value per share, or dividends per share, and selling stocks of growth firms, defined as firms for which the stock price is relatively high in relation to those same measures), and (3) momentum (buying stocks of firms with a high momentum in returns, or winners, and selling stocks of firms with a low momentum, or losers.)

Ibbotson et al. (2018) examined the top 3,500 US stocks by market capitalization for the period of 1971–2017. For each stock, they computed yearly measures of liquidity as the annual share turnover (the sum of the 12 monthly volumes divided by each month's shares outstanding), size as the year-end market capitalization, value as the trailing earnings-to-price ratio as of the year end, and momentum as the annual return. They assigned one-fourth of the total sample with the lowest liquidity in a year to Quartile 1 and the one-fourth with the highest liquidity in that year to Quartile 4. The stocks with the second-highest liquidity formed Quartile 3 and the stocks with the second-lowest liquidity, Quartile 2. Treating each quartile group as a portfolio composed of equally weighted stocks, they measured the returns on each liquidity quartile in the following year (so that the quartiles are constructed "before the fact".) The authors repeated this process for each of the other three investment styles (size, value, and momentum.) The results from Table 1 of their study are included in Exhibit 23. We have added a column with the spreads in returns from Quartile 1 to Quartile 4.

Exhibit 23 reports each investment style's geometric and arithmetic mean returns and standard deviation of returns for each quartile grouping. In each style, moving from Quartile 1 to Quartile 4, mean returns decrease. For example, the geometric mean return for the least liquid stocks is 15.16% and for the most liquid stocks is 7.70%. Only for the case of size does standard deviation decrease at each step moving from Quartile 1 to Quartile 4. Thus, the exhibit provides evidence that the investment styles generally having incremental value in explaining returns in relation to standard deviation. The authors conclude that liquidity appears to differentiate the returns approximately as well as the other styles.

Exhibit 23 Cross-Sectional Investment Style Returns (%) and Standard Deviations of Returns (%), 1972–2017

Investment Style	Q_1	Q_2	Q_3	Q_4	Spread in Return, Q1 to Q4
Size (Q1 = micro; Q4 = large)					
Geometric Mean	13.40	12.36	12.38	11.61	+1.79
Arithmetic Mean	16.49	14.90	14.40	13.11	+3.38
Standard deviation	26.13	23.63	21.01	17.67	
Value (Q1 = value; Q4 = growth)					
Geometric Mean	16.34	14.03	10.73	8.23	+8.11
Arithmetic Mean	18.62	15.70	12.73	11.86	+6.76
Standard deviation	22.47	19.34	20.50	28.24	
Momentum (Q1 = winners; Q4 = losers)					
Geometric Mean	13.17	14.60	13.74	7.89	+5.28
Arithmetic Mean	15.49	16.23	15.60	11.58	+3.91
Standard deviation	22.54	18.99	20.31	28.31	
Liquidity (Q1 = low; Q4 = high)					
Geometric Mean	15.16	14.19	12.45	7.70	+5.46
Arithmetic Mean	16.89	16.12	14.72	11.17	+5.72
Standard deviation	19.60	20.820	22.22	27.19	

Note: Each investment style portfolio contains as average of 742 stocks a year.
Source: Ibbotson et al. (2018)

To address the concern that liquidity may simply be a proxy for firm size, with investing in less liquid firms being equivalent to investing in small firms, the authors examined how less liquid stocks performed relative to more liquid stocks while controlling for firm size. This step involved constructing equally-weighted double-sorted portfolios in firm size and liquidity quartiles. That is, they constructed 16 different liquidity and size portfolios (4 × 4 = 16) and investigated the interaction between these two styles. The results from Table 2 of their article are included in Exhibit 24. We have added a column with the spreads in returns from Quartile 1 to Quartile 4 for each size category.

Exhibit 24 Mean Annual Returns (%) and Standard Deviations of Returns (%) of Size and Liquidity Quartile Portfolios, 1972–2017

Quartile	Q_1 (Low liquidity)	Q_2	Q_3	Q_4 (High liquidity)	Spread in Return, Q_1 to Q_4
Microcap					
Geometric Mean	16.05	15.68	9.57	0.11	+15.94
Arithmetic Mean	18.38	19.23	14.77	5.23	+13.15
Standard deviation	22.67	28.52	34.54	33.07	
Small cap					
Geometric Mean	15.65	14.32	12.26	6.00	+9.65
Arithmetic Mean	17.29	16.83	15.52	10.11	+7.18
Standard deviation	19.35	23.73	26.86	30.09	
Midcap					
Geometric Mean	14.03	13.88	12.89	8.40	+5.63
Arithmetic Mean	15.38	15.54	14.99	11.87	+3.51
Standard deviation	17.59	19.47	21.54	27.35	
Large cap					
Geometric Mean	11.44	12.28	11.97	9.07	+2.37
Arithmetic Mean	12.66	13.39	13.46	11.99	+0.67
Standard deviation	16.22	15.40	17.61	24.53	

Source: Ibbotson et al. (2018)

The exhibit shows that within the quartile with the smallest firms, the low-liquidity portfolio earned an annual geometric mean return of 16.05%, in contrast to the high-liquidity portfolio return of 0.11%, producing a liquidity effect of 15.94 percentage points (1,594 basis points). While the liquidity effect is strongest for the smallest firms, it does persist in other three size quartiles also. These results indicate that size does not capture liquidity (i.e., the liquidity effect holds regardless of size group).

7 MEASURES OF DISPERSION

As the well-known researcher Fischer Black has written, "[t]he key issue in investments is estimating expected return."[27] Few would disagree with the importance of expected return or mean return in investments: The mean return tells us where returns, and investment results, are centered. To completely understand an investment, however, we also need to know how returns are dispersed around the mean. **Dispersion** is the variability around the central tendency. If mean return addresses reward, dispersion addresses risk.

In this section, we examine the most common measures of dispersion: range, mean absolute deviation, variance, and standard deviation. These are all measures of **absolute dispersion**. Absolute dispersion is the amount of variability present without comparison to any reference point or benchmark.

27 Black (1993).

These measures are used throughout investment practice. The variance or standard deviation of return is often used as a measure of risk pioneered by Nobel laureate Harry Markowitz. Other measures of dispersion, mean absolute deviation and range, are also useful in analyzing data.

7.1 The Range

We encountered range earlier when we discussed the construction of frequency distribution. The simplest of all the measures of dispersion, range can be computed with interval or ratio data.

- **Definition of Range.** The **range** is the difference between the maximum and minimum values in a data set:

$$\text{Range} = \text{Maximum value} - \text{Minimum value} \tag{9}$$

As an illustration of range, the largest monthly return for the S&P 500 in the period from January 1926 to December 2017 is 42.56 percent (in April 1933) and the smallest is −29.73 percent (in September 1931). The range of returns is thus 72.29 percent [42.56 percent − (−29.73 percent)]. An alternative definition of range reports the maximum and minimum values. This alternative definition provides more information than does the range as defined in Equation 9.

One advantage of the range is ease of computation. A disadvantage is that the range uses only two pieces of information from the distribution. It cannot tell us how the data are distributed (that is, the shape of the distribution). Because the range is the difference between the maximum and minimum returns, it can reflect extremely large or small outcomes that may not be representative of the distribution.[28]

7.2 The Mean Absolute Deviation

Measures of dispersion can be computed using all the observations in the distribution rather than just the highest and lowest. The question is, how should we measure dispersion? Our previous discussion on properties of the arithmetic mean introduced the notion of distance or deviation from the mean $\left(X_i - \bar{X}\right)$ as a fundamental piece of information used in statistics. We could compute measures of dispersion as the arithmetic average of the deviations around the mean, but we would encounter a problem: The deviations around the mean always sum to 0. If we computed the mean of the deviations, the result would also equal 0. Therefore, we need to find a way to address the problem of negative deviations canceling out positive deviations.

One solution is to examine the absolute deviations around the mean as in the mean absolute deviation.

- **Mean Absolute Deviation Formula.** The **mean absolute deviation** (MAD) for a sample is

$$\text{MAD} = \frac{\sum_{i=1}^{n}\left|X_i - \bar{X}\right|}{n} \tag{10}$$

where $\bar{X}$ is the sample mean and n is the number of observations in the sample.

28 Another distance measure of dispersion that we may encounter, the interquartile range, focuses on the middle rather than the extremes. The **interquartile range** (IQR) is the difference between the third and first quartiles of a data set: IQR = $Q_3 - Q_1$. The IQR represents the length of the interval containing the middle 50 percent of the data, with a larger interquartile range indicating greater dispersion, all else equal.

In calculating MAD, we ignore the signs of the deviations around the mean. For example, if $X_i = -11.0$ and $\bar{X} = 4.5$, the absolute value of the difference is $|-11.0 - 4.5| = |-15.5| = 15.5$. The mean absolute deviation uses all of the observations in the sample and is thus superior to the range as a measure of dispersion. One technical drawback of MAD is that it is difficult to manipulate mathematically compared with the next measure we will introduce, variance.[29] Example 10 illustrates the use of the range and the mean absolute deviation in evaluating risk.

EXAMPLE 10

The Range and the Mean Absolute Deviation

Having calculated mean returns for the two mutual funds in Example 7, the analyst is now concerned with evaluating risk.

Exhibit 20 Total Returns for Two Mutual Funds, 2013–2017 (Repeated)

Year	Selected American Shares (SLASX)	T. Rowe Price Equity Income (PRFDX)
2013	34.90%	31.69%
2014	6.13	7.75
2015	2.69	–7.56
2016	11.66	18.25
2017	21.77	16.18

Source: performance.morningstar.com.

Based on the data in Exhibit 20 on the previous page, answer the following:

1 Calculate the range of annual returns for (A) SLASX and (B) PRFDX, and state which mutual fund appears to be riskier based on these ranges.

2 Calculate the mean absolute deviation of returns on (A) SLASX and (B) PRFDX, and state which mutual fund appears to be riskier based on MAD.

Solution to 1:

A For SLASX, the largest return was 34.90 percent and the smallest was 2.69 percent. The range is thus 34.90 – 2.69 = 32.21%.

B For PFRDX, the range is 31.69 – (–7.56) = 39.25%. With a larger range of returns than SLASX, PRFDX appeared to be the riskier fund during the 2013–2017 period.

29 In some analytic work such as optimization, the calculus operation of differentiation is important. Variance as a function can be differentiated, but absolute value cannot.

Solution to 2:

A The arithmetic mean return for SLASX as calculated in Example 7 is 15.43 percent. The MAD of SLASX returns is

$$\text{MAD} = \frac{|34.90 - 15.43| + |6.13 - 15.43| + |2.69 - 15.43| + |11.66 - 15.43| + |21.77 - 15.43|}{5}$$

$$= \frac{19.47 + 9.30 + 12.74 + 3.77 + 6.34}{5}$$

$$= \frac{51.62}{5} = 10.32\%$$

B The arithmetic mean return for PRFDX as calculated in Example 7 is 13.26 percent. The MAD of PRFDX returns is

$$\text{MAD} = \frac{|31.69 - 13.26| + |7.75 - 13.26| + |-7.56 - 13.26| + |18.25 - 13.26| + |16.18 - 13.26|}{5}$$

$$= \frac{18.43 + 5.51 + 20.82 + 4.99 + 2.92}{5}$$

$$= \frac{52.67}{5} = 10.53\%$$

PRFDX, with a MAD of 10.53 percent, appears to be slightly riskier than SLASX, with a MAD of 10.32 percent.

7.3 Population Variance and Population Standard Deviation

The mean absolute deviation addressed the issue that the sum of deviations from the mean equals zero by taking the absolute value of the deviations. A second approach to the treatment of deviations is to square them. The variance and standard deviation, which are based on squared deviations, are the two most widely used measures of dispersion. **Variance** is defined as the average of the squared deviations around the mean. **Standard deviation** is the positive square root of the variance. The following discussion addresses the calculation and use of variance and standard deviation.

7.3.1 *Population Variance*

If we know every member of a population, we can compute the **population variance**. Denoted by the symbol σ^2, the population variance is the arithmetic average of the squared deviations around the mean.

- **Population Variance Formula.** The population variance is

$$\sigma^2 = \frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N} \quad (11)$$

where μ is the population mean and N is the size of the population.

Given knowledge of the population mean, μ, we can use Equation 11 to calculate the sum of the squared differences from the mean, taking account of all N items in the population, and then to find the mean squared difference by dividing the sum by N. Whether a difference from the mean is positive or negative, squaring that difference results in a positive number. Thus variance takes care of the problem of negative deviations from the mean canceling out positive deviations by the operation of squaring those deviations. The profit as a percentage of revenue for BJ's Wholesale Club, Costco, and Walmart was given earlier as 0.0, 2.1, and 2.0, respectively. We calculated the mean profit as a percentage of revenue as 1.37. Therefore, the population variance of the profit as a percentage of revenue is $(1/3)[(0.0 - 1.37)^2 + (2.1 - 1.37)^2 + (2.0 - 1.37)^2] = (1/3)(-1.37^2 + 0.73^2 + 0.63^2) = (1/3)(1.88 + 0.53 + 0.40) = (1/3)(2.81) = 0.94$.

7.3.2 *Population Standard Deviation*

Because the variance is measured in squared units, we need a way to return to the original units. We can solve this problem by using standard deviation, the square root of the variance. Standard deviation is more easily interpreted than the variance because standard deviation is expressed in the same unit of measurement as the observations.

- **Population Standard Deviation Formula.** The **population standard deviation**, defined as the positive square root of the population variance, is

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}} \qquad (12)$$

where μ is the population mean and N is the size of the population.

Using the example of the profit as a percentage of revenue for BJ's Wholesale Club, Costco, and Walmart, according to Equation 12 we would calculate the variance, 0.94, then take the square root: $\sqrt{0.94} = 0.97$.

Both the population variance and standard deviation are examples of parameters of a distribution. In later readings, we will introduce the notion of variance and standard deviation as risk measures.

In investments, we often do not know the mean of a population of interest, usually because we cannot practically identify or take measurements from each member of the population. We then estimate the population mean with the mean from a sample drawn from the population, and we calculate a sample variance or standard deviation using formulas different from Equations 11 and 12. We shall discuss these calculations in subsequent sections. However, in investments we sometimes have a defined group that we can consider to be a population. With well-defined populations, we use Equations 11 and 12, as in the following example.

EXAMPLE 11

Calculating the Population Standard Deviation

Exhibit 25 gives the P/E, the ratio of share price to projected earnings per share (EPS) for 2018 for the 10 US stocks that composed the list of *Barron's* 10 favorite stocks for 2018.[30] The identity of the stocks on the *Barron's* Top 10 list changes from year to year.

Exhibit 25 P/E: *Barron's* 10 Favorite Stocks for 2018

Fund	P/E
Ally Financial	10.7
Alphabet	24.9

30 Toward the end of each year, *Barron's* magazine annually selects its 10 favorite US stocks for the next year.

Exhibit 25 (Continued)

Fund	P/E
Anthem	17.3
Applied Materials	13.0
Berkshire Hathaway	24.6
Delta Airlines	9.7
Enterprise Products Partners	16.7
Pioneer Natural Resources	50.6
US Foods	18.1
Volkswagen	6.5

Source: Barron's (9 December 2017).

Based on the data in Exhibit 25, address the following:

1 Calculate the population mean P/E for the period used by *Barron's* for the 10 favorite stocks.

2 Calculate the population variance and population standard deviation of the P/E.

3 Explain the use of the population formulas in this example.

Solution to 1:

μ = (10.7 + 24.9 + 17.3 + 13.0 + 24.6 + 9.7 + 16.7 + 50.6 + 18.1 + 6.5)/10 = 192.1/10 = 19.21.

Solution to 2:

Having established that μ = 19.21, we can calculate $\sigma^2 = \frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}$ by first calculating the numerator in the expression and then dividing by N = 10. The numerator (the sum of the squared differences from the mean) is

$$(10 - 53)^2 + (360 - 53)^2 + (37 - 53)^2 + (20 - 53)^2 + (49 - 53)^2 + (1 - 53)^2 + (32 - 53)^2 + (72 - 53)^2 + (9 - 53)^2 + (19 - 53)^2 + (16 - 53)^2 + (11 - 53)^2 = 107{,}190$$

Thus σ^2 = 107,190/12 = 8,932.50.

$$(10.7 - 19.21)^2 + (24.9 - 19.21)^2 + (17.3 - 19.21)^2 + (13.0 - 19.21)^2 + (24.6 - 19.21)^2 + (9.7 - 19.21)^2 + (16.7 - 19.21)^2 + (50.6 - 19.21)^2 + (18.1 - 19.21)^2 + (6.5 - 19.21)^2 = 1{,}420.9$$

Thus σ^2 = 1,420.9/10 = 142.09.

To calculate standard deviation, $\sigma = \sqrt{142.09}$ = 11.92 percent. (The unit of variance is percent squared so the unit of standard deviation is percent.)

Solution to 3:

If the population is clearly defined to be the *Barron's* favorite stocks for one specific year (2018), and if the P/E is understood to refer to the specific one-year period reported upon by *Barron's*, the application of the population formulas to variance and standard deviation is appropriate. The results of 142.09 and 11.92 are, respectively, the cross-sectional variance and standard deviation in the P/E for the 2018 *Barron's* favorite stocks list.[31]

7.4 Sample Variance and Sample Standard Deviation

In the following discussion, note the switch to Roman letters to symbolize sample quantities.

7.4.1 *Sample Variance*

In many instances in investment management, a subset or sample of the population is all that we can observe. When we deal with samples, the summary measures are called statistics. The statistic that measures the dispersion in a sample is called the sample variance.

- **Sample Variance Formula.** The **sample variance** is

$$s^2 = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{n-1} \tag{13}$$

where $\bar{X}$ is the sample mean and n is the number of observations in the sample.

Equation 13 tells us to take the following steps to compute the sample variance:

i. Calculate the sample mean, $\bar{X}$.

ii. Calculate each observation's squared deviation from the sample mean, $\left(X_i - \bar{X}\right)^2$.

iii. Sum the squared deviations from the mean: $\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$.

iv. Divide the sum of squared deviations from the mean by $n - 1$: $\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 / (n-1)$.

We will illustrate the calculation of the sample variance and the sample standard deviation in Example 12.

We use the notation s^2 for the sample variance to distinguish it from population variance, σ^2. The formula for sample variance is nearly the same as that for population variance except for the use of the sample mean, $\bar{X}$, in place of the population mean, μ, and a different divisor. In the case of the population variance, we divide by the size of the population, N. For the sample variance, however, we divide by the sample size minus 1, or $n - 1$. By using $n - 1$ (rather than n) as the divisor, we improve the statistical properties of the sample variance. In statistical terms, the sample variance defined in Equation 13 is an unbiased estimator of the population variance.[32] The quantity $n - 1$ is also known as the number of degrees of freedom in estimating the population variance. To estimate the population variance with s^2, we must first calculate the mean. Once we have computed the sample mean, there are only $n - 1$ independent deviations from it.

31 In fact, we could not properly use the Honor Roll funds to estimate the population variance of the P/E (for example) of any other differently defined population, because the favorite stocks are not a random sample from any larger population of US stocks.

32 We discuss this concept further in the reading on sampling.

7.4.2 *Sample Standard Deviation*

Just as we computed a population standard deviation, we can compute a sample standard deviation by taking the positive square root of the sample variance.

■ **Sample Standard Deviation Formula.** The **sample standard deviation**, s, is

$$s = \sqrt{\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n-1}} \qquad (14)$$

where $\overline{X}$ is the sample mean and n is the number of observations in the sample.

To calculate the sample standard deviation, we first compute the sample variance using the steps given. We then take the square root of the sample variance. Example 12 illustrates the calculation of the sample variance and standard deviation for the two mutual funds introduced earlier.

EXAMPLE 12

Calculating Sample Variance and Sample Standard Deviation

After calculating the geometric and arithmetic mean returns of two mutual funds in Example 7, we calculated two measures of dispersions for those funds, the range and mean absolute deviation of returns, in Example 10. We now calculate the sample variance and sample standard deviation of returns for those same two funds.

Exhibit 20 Total Returns for Two Mutual Funds, 2013–2017 (Repeated)

Year	Selected American Shares (SLASX)	T. Rowe Price Equity Income (PRFDX)
2013	34.90%	31.69%
2014	6.13	7.75
2015	2.69	–7.56
2016	11.66	18.25
2017	21.77	16.18

Source: performance.morningstar.com.

Based on the data in Exhibit 20 repeated above, answer the following:

1 Calculate the sample variance of return for (A) SLASX and (B) PRFDX.

2 Calculate sample standard deviation of return for (A) SLASX and (B) PRFDX.

3 Contrast the dispersion of returns as measured by standard deviation of return and mean absolute deviation of return for each of the two funds.

Solution to 1:

To calculate the sample variance, we use Equation 13. (Deviation answers are all given in percent squared.)

A SLASX

i. The sample mean is
$\bar{R}$ = (34.90 + 6.13 + 2.69 − 11.66 +21.77)/ 5 = 77.15/5 = 15.43%.

ii. The squared deviations from the mean are

$(34.90 - 15.43)^2 = (19.47)^2 = 379.08$

$(6.13 - 15.43)^2 = (-9.30)^2 = 86.49$

$(2.69 - 15.43)^2 = (-12.74)^2 = 162.31$

$(11.66 - 15.43)^2 = (-3.77)^2 = 14.21$

$(21.77 - 15.43)^2 = (6.34)^2 = 40.20$

iii. The sum of the squared deviations from the mean is 379.08 + 86.49 + 162.31 + 14.21 + 40.20 = 682.29.

iv. Divide the sum of the squared deviations from the mean by $n - 1$: 682.29/(5 − 1) = 682.29/4 = 170.57

B PRFDX

i. The sample mean is
$\bar{R}$ = (31.69 + 7.75 − 7.56 + 18.25 + 6.18)/5 = 66.30/5 = 13.26%.

ii. The squared deviations from the mean are

$(31.69 - 13.26)^2 = (18.43)^2 = 339.59$

$(7.75 - 13.26)^2 = (-0.725.51)^2 = 30.38$

$(-7.56 - 13.26)^2 = (-20.82)^2 = 433.56$

$(18.25 - 13.26)^2 = (4.99)^2 = 24.88$

$(16.18 - 13.26)^2 = (2.92)^2 = 8.51$

iii. The sum of the squared deviations from the mean is 339.59 + 30.38 + 433.56 + 24.88 + 8.51 = 836.92.

iv. Divide the sum of the squared deviations from the mean by $n - 1$: 836.92/4 = 209.23

Solution to 2:

To find the standard deviation, we take the positive square root of variance.

A For SLASX, $s = \sqrt{170.57}$ = 13.1%.

B For PRFDX, $s = \sqrt{209.23}$ = 14.5%.

Solution to 3:

Exhibit 26 summarizes the results from Part 2 for standard deviation and incorporates the results for MAD from Example 10.

Exhibit 26 Two Mutual Funds: Comparison of Standard Deviation and Mean Absolute Deviation

Fund	Standard Deviation (%)	Mean Absolute Deviation (%)
SLASX	13.1	10.3
PRFDX	14.5	10.5

Note that the mean absolute deviation is less than the standard deviation. The mean absolute deviation will always be less than or equal to the standard deviation because the standard deviation gives more weight to large deviations than to small ones (remember, the deviations are squared).

Because the standard deviation is a measure of dispersion about the arithmetic mean, we usually present the arithmetic mean and standard deviation together when summarizing data. When we are dealing with data that represent a time series of percent changes, presenting the geometric mean—representing the compound rate of growth—is also very helpful. Exhibit 27 presents the historical geometric and arithmetic mean returns, along with the historical standard deviation of returns, for the S&P 500 annual and monthly returns. We present these statistics for nominal (rather than inflation-adjusted) returns so we can observe the original magnitudes of the returns.

Exhibit 27 Equity Market Returns: Means and Standard Deviations

Return Series	Geometric Mean (%)	Arithmetic Mean (%)	Standard Deviation
1926–2017			
S&P 500 (Annual)	10.16	12.06	19.79
S&P 500 (Monthly)	0.81	0.95	5.39

Sources: Ibbotson Associates and S&P Dow Jones Indices LLC.

7.5 Semivariance, Semideviation, and Related Concepts

An asset's variance or standard deviation of returns is often interpreted as a measure of the asset's risk. Variance and standard deviation of returns take account of returns above and below the mean, but investors are concerned only with downside risk, for example, returns below the mean. As a result, analysts have developed semivariance, semideviation, and related dispersion measures that focus on downside risk. **Semivariance** is defined as the average squared deviation below the mean. **Semideviation** (sometimes called semistandard deviation) is the positive square root of semivariance.[33] To compute the sample semivariance, for example, we take the following steps:

i. Calculate the sample mean.

ii. Identify the observations that are smaller than or equal to the mean (discarding observations greater than the mean).

iii. Compute the sum of the squared negative deviations from the mean (using the observations that are smaller than or equal to the mean).

iv. Divide the sum of the squared negative deviations from Step iii by the *total* sample size minus 1: $n - 1$. A formula for semivariance approximating the unbiased estimator is

$$\sum_{\text{for all } X_i \le \bar{X}} \left(X_i - \bar{X}\right)^2 \Big/ (n-1)$$

33 This is an informal treatment of these two measures; see the survey article by N. Fred Choobinbeh (2005) for a rigorous treatment.

To take the case of Selected American Shares with returns (in percent) of 34.90, 6.13, 2.69, 11.66, and 21.77, we earlier calculated a mean return of 15.43 percent. Three returns, 2.69, 6.13, and 11.66, are smaller than 15.43. We compute the sum of the squared negative deviations from the mean as $(2.69 - 15.43)^2 + (6.13 - 15.43)^2 + (11.66 - 15.43)^2 = -12.74^2 + -9.30^2 + -3.77^2 = 162.31 + 86.49 + 14.21 = 263.01$. With $n - 1 = 4$, we conclude that semivariance is 263.01/4 = 65.75 and that semideviation is $\sqrt{65.76} = 8.1$ percent, approximately. The semideviation of 8.1 percent is less than the standard deviation of 13.1 percent. From this downside risk perspective, therefore, standard deviation overstates risk.

In practice, we may be concerned with values of return (or another variable) below some level other than the mean. For example, if our return objective is 11.50 percent annually, we may be concerned particularly with returns below 11.50 percent a year. We can call 11.50 percent the target. The name **target semivariance** has been given to average squared deviation below a stated target, and **target semideviation** is its positive square root. To calculate a sample target semivariance, we specify the target as a first step. After identifying observations below the target, we find the sum of the squared negative deviations from the target and divide that sum by the number of observations minus 1. A formula for target semivariance is

$$\sum_{\text{for all } X_i \le B} \left(X_i - B\right)^2 \Big/ (n-1)$$

where B is the target and n is the number of observations. With a target return of 11.50 percent, we find in the case of Selected American Shares that three returns (2.69 and 6.13) were below the target. The target semivariance is $[(2.69 - 11.50)^2 + (6.13 - 11.50)^2]/(5 - 1) = 26.61$, and the target semideviation is $\sqrt{26.61} = 5.2$ percent, approximately.

When return distributions are symmetric, semivariance and variance are effectively equivalent. For asymmetric distributions, variance and semivariance rank prospects' risk differently.[34] Semivariance (or semideviation) and target semivariance (or target semideviation) have intuitive appeal, but they are harder to work with mathematically than variance.[35] Variance or standard deviation enters into the definition of many of the most commonly used finance risk concepts, such as the Sharpe ratio and beta. Perhaps because of these reasons, variance (or standard deviation) is much more frequently used in investment practice.

7.6 Chebyshev's Inequality

The Russian mathematician Pafnuty Chebyshev developed an inequality using standard deviation as a measure of dispersion. The inequality gives the proportion of values within k standard deviations of the mean.

- **Definition of Chebyshev's Inequality.** According to Chebyshev's inequality, for any distribution with finite variance, the proportion of the observations within k standard deviations of the arithmetic mean is at least $1 - 1/k^2$ for all $k > 1$.

Exhibit 28 illustrates the proportion of the observations that must lie within a certain number of standard deviations around the sample mean.

34 We discuss skewness later in this reading.

35 As discussed in the reading on probability concepts and the various readings on portfolio concepts, we can find a portfolio's variance as a straightforward function of the variances and correlations of the component securities. There is no similar procedure for semivariance and target semivariance. We also cannot take the derivative of semivariance or target semivariance.

Exhibit 28 Proportions from Chebyshev's Inequality

k	Interval around the Sample Mean	Proportion (%)
1.25	$\bar{X} \pm 1.25s$	36
1.50	$\bar{X} \pm 1.50s$	56
2.00	$\bar{X} \pm 2s$	75
2.50	$\bar{X} \pm 2.50s$	84
3.00	$\bar{X} \pm 3s$	89
4.00	$\bar{X} \pm 4s$	94

Note: Standard deviation is denoted as *s*.

When $k = 1.25$, for example, the inequality states that the minimum proportion of the observations that lie within $\pm$ 1.25*s* is $1 - 1/(1.25)^2 = 1 - 0.64 = 0.36$ or 36 percent.

The most frequently cited facts that result from Chebyshev's inequality are that a two-standard-deviation interval around the mean must contain at least 75 percent of the observations, and a three-standard-deviation interval around the mean must contain at least 89 percent of the observations, no matter how the data are distributed.

The importance of Chebyshev's inequality stems from its generality. The inequality holds for samples and populations and for discrete and continuous data regardless of the shape of the distribution. As we shall see in the reading on sampling, we can make much more precise interval statements if we can assume that the sample is drawn from a population that follows a specific distribution called the normal distribution. Frequently, however, we cannot confidently assume that distribution.

The next example illustrates the use of Chebyshev's inequality.

EXAMPLE 13

Applying Chebyshev's Inequality

According to Exhibit 27, the arithmetic mean monthly return and standard deviation of monthly returns on the S&P 500 were 0.95 percent and 5.39 percent, respectively, during the 1926–2017 period, totaling 1,104 monthly observations. Using this information, address the following:

1 Calculate the endpoints of the interval that must contain at least 75 percent of monthly returns according to Chebyshev's inequality.

2 What are the minimum and maximum number of observations that must lie in the interval computed in Part 1, according to Chebyshev's inequality?

Solution to 1:

According to Chebyshev's inequality, at least 75 percent of the observations must lie within two standard deviations of the mean, $\bar{X} \pm 2s$. For the monthly S&P 500 return series, we have 0.95% ± 2(5.39%) = 0.95% ± 10.78%. Thus the lower endpoint of the interval that must contain at least 75 percent of the observations is 0.95% − 10.78% = −9.83%, and the upper endpoint is 0.95% + 10.78% = 11.73%.

Solution to 2:

For a sample size of 1,104, at least 0.75(1,104) = 828 observations must lie in the interval from −9.83% to 11.73% that we computed in Part 1. Chebyshev's inequality gives the minimum percentage of observations that must fall within a given interval around the mean, but it does not give the maximum percentage. Exhibit 4, which gave the frequency distribution of monthly returns on the S&P 500, is excerpted below. The data in the excerpted table are consistent with the prediction of Chebyshev's inequality. The set of intervals running from −10.0% to 12.0% is about equal in width to the two-standard-deviation interval −9.83% to 11.73%. A total of 1,064 observations (approximately 96 percent of observations) fall in the range from −10.0% to 12.0%.

Exhibit 4 Frequency Distribution for the Monthly Total Return on the S&P 500, January 1926 to December 2017 (Excerpt)

Return Interval (%)	Absolute Frequency
−10.0 to −8.0	23
−8.0 to −6.0	35
−6.0 to −4.0	60
−4.0 to −2.0	102
−2.0 to 0.0	166
0.0 to 2.0	240
2.0 to 4.0	190
4.0 to 6.0	143
6.0 to 8.0	64
8.0 to 10.0	26
10.0 to 12.0	15
	1,064

7.7 Coefficient of Variation

We noted earlier that standard deviation is more easily interpreted than variance because standard deviation uses the same units of measurement as the observations. We may sometimes find it difficult to interpret what standard deviation means in terms of the relative degree of variability of different sets of data, however, either because the data sets have markedly different means or because the data sets have different units of measurement. In this section we explain a measure of relative dispersion, the coefficient of variation that can be useful in such situations. **Relative dispersion** is the amount of dispersion relative to a reference value or benchmark.

We can illustrate the problem of interpreting the standard deviation of data sets with markedly different means using two hypothetical samples of companies. The first sample, composed of small companies, includes companies with last year's sales of €50 million, €75 million, €65 million, and €90 million. The second sample, composed of large companies, includes companies with last year's sales of €800 million, €825 million, €815 million, and €840 million. We can verify using Equation 14 that the standard deviation of sales in both samples is €16.8 million.[36] In the first sample, the largest observation, €90 million, is 80 percent larger than the smallest observation, €50 million. In the second sample, the largest observation is only 5 percent larger than the smallest observation. Informally, a standard deviation of €16.8 million represents a high degree of variability relative to the first sample, which reflects mean last year sales of €70 million, but a small degree of variability relative to the second sample, which reflects mean last year sales of €820 million.

The coefficient of variation is helpful in situations such as that just described.

- **Coefficient of Variation Formula.** The **coefficient of variation**, CV, is the ratio of the standard deviation of a set of observations to their mean value:[37]

$$\mathrm{CV} = s/\overline{X} \tag{15}$$

- where s is the sample standard deviation and $\overline{X}$ is the sample mean.

When the observations are returns, for example, the coefficient of variation measures the amount of risk (standard deviation) per unit of mean return. Expressing the magnitude of variation among observations relative to their average size, the coefficient of variation permits direct comparisons of dispersion across different data sets. Reflecting the correction for scale, the coefficient of variation is a scale-free measure (that is, it has no units of measurement).

We can illustrate the application of the coefficient of variation using our earlier example of two samples of companies. The coefficient of variation for the first sample is (€16.8 million)/(€70 million) = 0.24; the coefficient of variation for the second sample is (€16.8 million)/(€820 million) = 0.02. This confirms our intuition that the first sample had much greater variability in sales than the second sample. Note that 0.24 and 0.02 are pure numbers in the sense that they are free of units of measurement (because we divided the standard deviation by the mean, which is measured in the same units as the standard deviation). If we need to compare the dispersion among data sets stated in different units of measurement, the coefficient of variation can be useful because it is free from units of measurement. Example 14 illustrates the calculation of the coefficient of variation.

EXAMPLE 14

Calculating the Coefficient of Variation

Exhibit 29 summarizes annual mean returns and standard deviations computed using monthly return data for major stock market indexes of four Asia-Pacific markets. The indexes are: S&P/ASX 200 Index (Australia), Hang Seng Index (Hong Kong SAR), Nikkei 225 Index (Japan), and KOSPI Composite Index (South Korea).

36 The second sample was created by adding €750 million to each observation in the first sample. Standard deviation (and variance) has the property of remaining unchanged if we add a constant amount to each observation.

37 The reader will also encounter CV defined as $100(s/\overline{X})$, which states CV as a percentage.

Exhibit 29 Arithmetic Mean Annual Return and Standard Deviation of Returns, Asia-Pacific Stock Markets, 2013–2017

Market	Arithmetic Mean Return (%)	Standard Deviation of Return (%)
Australia	5.3	11.9
Hong Kong SAR	5.6	15.8
Japan	15.7	16.3
South Korea	4.2	8.6

Source: finance.yahoo.com, accessed 26 November 2018.

Using the information in Exhibit 29, address the following:

1 Calculate the coefficient of variation for each market given.

2 Rank the markets from most risky to least risky using CV as a measure of relative dispersion.

3 Determine whether there is more difference between the absolute or the relative riskiness of the Australia and South Korea markets. Use the standard deviation as a measure of absolute risk and CV as a measure of relative risk.

Solution to 1:

Australia: CV = 11.9%/5.3% = 2.245

Hong Kong SAR: CV = 15.8%/5.6% = 2.821

Japan: CV = 16.3%/15.7% = 1.038

South Korea: CV = 8.6%/4.2% = 2.048

Solution to 2:

Based on CV, the ranking for the 2013–2017 period examined is Hong Kong SAR (most risky), Australia, South Korea, and Japan (least risky).

Solution to 3:

As measured both by standard deviation and CV, Australia market was riskier than the South Korea market. The standard deviation of Australian returns was (11.9 – 8.6)/8.6 = 0.384 or about 38 percent larger than South Korean returns, compared with a difference in the CV of (2.245 – 2.048)/2.048 = 0.097 or about 10 percent. Thus, the CVs reveal less difference between Australian and South Korean return variability than that suggested by the standard deviations alone.

8 SYMMETRY AND SKEWNESS IN RETURN DISTRIBUTIONS

Mean and variance may not adequately describe an investment's distribution of returns. In calculations of variance, for example, the deviations around the mean are squared, so we do not know whether large deviations are likely to be positive or negative.

We need to go beyond measures of central tendency and dispersion to reveal other important characteristics of the distribution. One important characteristic of interest to analysts is the degree of symmetry in return distributions.

If a return distribution is symmetrical about its mean, then each side of the distribution is a mirror image of the other. Thus equal loss and gain intervals exhibit the same frequencies. Losses from −5 percent to −3 percent, for example, occur with about the same frequency as gains from 3 percent to 5 percent.

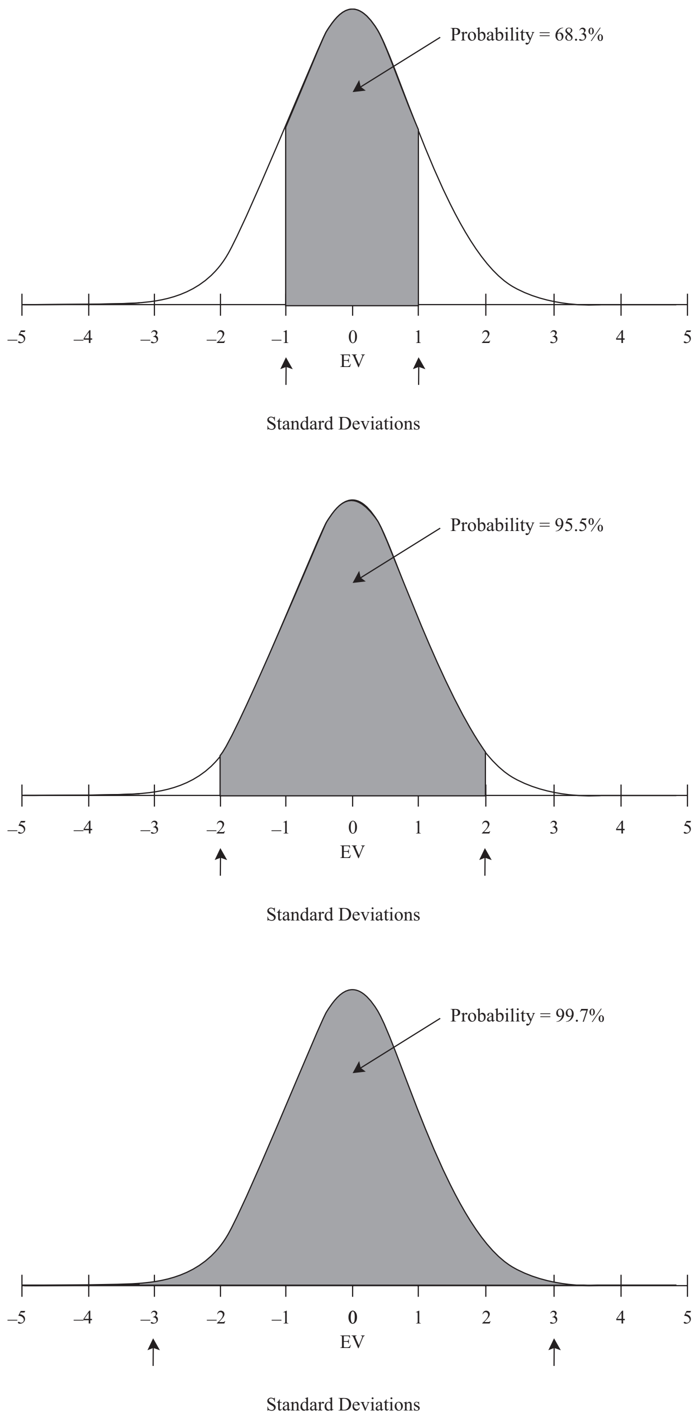
One of the most important distributions is the normal distribution, depicted in Exhibit 30. This symmetrical, bell-shaped distribution plays a central role in the mean–variance model of portfolio selection; it is also used extensively in financial risk management. The normal distribution has the following characteristics:

- Its mean and median are equal.
- It is completely described by two parameters—its mean and variance.
- Roughly 68 percent of its observations lie between plus and minus one standard deviation from the mean; 95 percent lie between plus and minus two standard deviations; and 99 percent lie between plus and minus three standard deviations.

A distribution that is not symmetrical is called **skewed**. A return distribution with positive skew has frequent small losses and a few extreme gains. A return distribution with negative skew has frequent small gains and a few extreme losses. Exhibit 31 shows continuous positively and negatively skewed distributions. The continuous positively skewed distribution shown has a long tail on its right side; the continuous negatively skewed distribution shown has a long tail on its left side. For the continuous positively skewed unimodal distribution, the mode is less than the median, which is less than the mean. For the continuous negatively skewed unimodal distribution, the mean is less than the median, which is less than the mode.[38] Investors should be attracted by a positive skew because the mean return falls above the median. Relative to the mean return, positive skew amounts to a limited, though frequent, downside compared with a somewhat unlimited, but less frequent, upside.

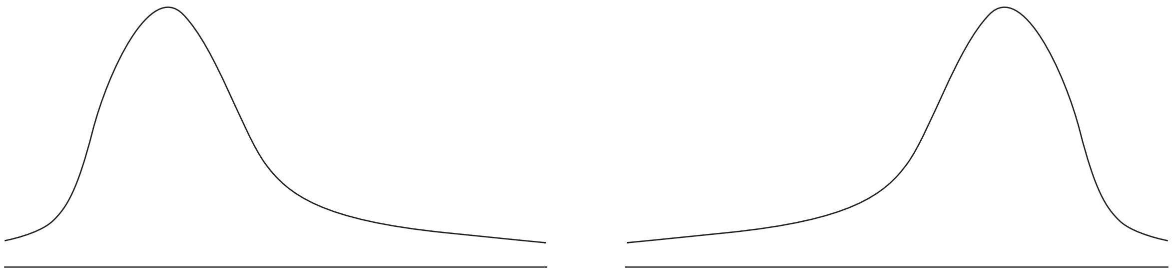
38 As a mnemonic, in this case the mean, median, and mode occur in the same order as they would be listed in a dictionary. Von Hippel (2005) explores exceptions to the relationships among these measures for distributions that vary from those shown in Exhibit 31.

Exhibit 30 Properties of a Normal Distribution (EV 5 Expected Value)



Source: Reprinted from *Fixed Income Analysis.* Copyright CFA Institute.

Exhibit 31 Properties of a Skewed Distribution



Distribution Skewed to the Right (Positively Skewed)

Distribution Skewed to the Left (Negatively Skewed)

Source: Reprinted from *Fixed Income Analysis.* Copyright CFA Institute.

Skewness is the name given to a statistical measure of skew. (The word "skewness" is also sometimes used interchangeably for "skew.") Like variance, skewness is computed using each observation's deviation from its mean. **Skewness** (sometimes referred to as relative skewness) is computed as the average cubed deviation from the mean standardized by dividing by the standard deviation cubed to make the measure free of scale.[39] A symmetric distribution has skewness of 0, a positively skewed distribution has positive skewness, and a negatively skewed distribution has negative skewness, as given by this measure.

We can illustrate the principle behind the measure by focusing on the numerator. Cubing, unlike squaring, preserves the sign of the deviations from the mean. If a distribution is positively skewed with a mean greater than its median, then more than half of the deviations from the mean are negative and less than half are positive. In order for the sum to be positive, the losses must be small and likely, and the gains less likely but more extreme. Therefore, if skewness is positive, the average magnitude of positive deviations is larger than the average magnitude of negative deviations.

A simple example illustrates that a symmetrical distribution has a skewness measure equal to 0. Suppose we have the following data: 1, 2, 3, 4, 5, 6, 7, 8, and 9. The mean outcome is 5, and the deviations are −4, −3, −2, −1, 0, 1, 2, 3, and 4. Cubing the deviations yields −64, −27, −8, −1, 0, 1, 8, 27, and 64, with a sum of 0. The numerator of skewness (and so skewness itself) is thus equal to 0, supporting our claim. Below we give the formula for computing skewness from a sample.

- **Sample Skewness Formula. Sample skewness** (also called sample relative skewness), S_K, is

$$S_K = \left[\frac{n}{(n-1)(n-2)}\right]\frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^3}{s^3} \tag{16}$$

where n is the number of observations in the sample and s is the sample standard deviation.[40]

39 We are discussing a moment coefficient of skewness. Some textbooks present the Pearson coefficient of skewness, equal to 3(Mean − Median)/Standard deviation, which has the drawback of involving the calculation of the median.

40 The term $n/[(n-1)(n-2)]$ in Equation 16 corrects for a downward bias in small samples.

The algebraic sign of Equation 16 indicates the direction of skew, with a negative S_K indicating a negatively skewed distribution and a positive S_K indicating a positively skewed distribution. Note that as n becomes large, the expression reduces to the mean cubed deviation, $S_K \approx \left(\frac{1}{n}\right)\frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^3}{s^3}$. As a frame of reference, for a sample size of 100 or larger taken from a normal distribution, a skewness coefficient of ±0.5 would be considered unusually large.

Exhibit 32 shows several summary statistics for the annual and monthly returns on the S&P 500. Earlier we discussed the arithmetic mean return and standard deviation of return, and we shall shortly discuss kurtosis.

Exhibit 32 S&P 500 Annual and Monthly Total Returns, 1926–2017: Summary Statistics

Return Series	Number of Periods	Arithmetic Mean (%)	Standard Deviation (%)	Skewness	Excess Kurtosis
S&P 500 (Annual)	92	12.06	19.79	−0.4019	0.1044
S&P 500 (Monthly)	1,104	0.95	5.5039	0.3387	9.7709

Sources: Ibbotson Associates and S&P Dow Jones Indices LLC.

Exhibit 32 reveals that S&P 500 annual returns during this period were negatively skewed while monthly returns were positively skewed, and the magnitude of skewness was greater for the annual series. We would find for other market series that the shape of the distribution of returns often depends on the holding period examined.

Some researchers believe that investors should prefer positive skewness, all else equal—that is, they should prefer portfolios with distributions offering a relatively large frequency of unusually large payoffs.[41] Different investment strategies may tend to introduce different types and amounts of skewness into returns. Example 15 illustrates the calculation of skewness for a managed portfolio.

EXAMPLE 15

Calculating Skewness for a Mutual Fund

Exhibit 33 presents 10 years of annual returns on the T. Rowe Price Equity Income Fund (PRFDX).

Exhibit 33 Annual Rates of Return: T. Rowe Price Equity Income, 2008–2017

Year	Return (%)
2008	−35.75
2009	25.62
2010	15.15
2011	−0.72

41 For more on the role of skewness in portfolio selection, see Reilly and Brown (2018) and Elton, Gruber, Brown, and Goetzmann (2013) and the references therein.

Exhibit 33 (Continued)

Year	Return (%)
2012	17.25
2013	31.69
2014	7.75
2015	−7.76
2016	18.25
2017	16.18

Source: performance.morningstar.com.

Using the information in Exhibit 33, address the following:

1 Calculate the skewness of PRFDX showing two decimal places.

2 Characterize the shape of the distribution of PRFDX returns based on your answer to Part 1.

Solution to 1:

To calculate skewness, we find the sum of the cubed deviations from the mean, divide by the standard deviation cubed, and then multiply that result by $n/[(n - 1)(n - 2)]$. Exhibit 34 gives the calculations.

Exhibit 34 Calculating Skewness for PRFDX

Year	R_t	$R_t - \overline{R}$	$\left(R_t - \overline{R}\right)^3$
2008	−35.75	−44.52	−88,239.993
2009	25.62	16.85	4,784.094
2010	15.15	6.38	259.694
2011	−0.72	−9.49	−854.670
2012	17.25	8.48	609.800
2013	31.69	22.92	12,040.481
2014	7.75	−1.02	−1.061
2015	−7.76	−16.53	−4,516.672
2016	18.25	9.48	851.971
2017	16.18	7.41	406.869
n =	10		
$\overline{R}$ =	8.77%		
		Sum =	−74,659.487
s =	19.47%	s^3 =	7,380.705
		Sum/s^3 =	−10.1155
		$n/[(n - 1)(n - 2)]$ =	0.1389
		Skewness =	**−1.40**

Source: performance.morningstar.com.

Using Equation 16, the calculation is:

$$S_k = \left[\frac{10}{(9)(8)}\right]\frac{-74{,}659.487}{19.47^3} = -1.40$$

Solution to 2:

Based on this small sample, the distribution of annual returns for the fund appears to be negatively skewed. In this example, four deviations are negative and six are positive. While, there are more positive deviations, they are much more than offset by a huge negative deviation in 2008, when the stock markets sharply went down as a consequence of the global financial crisis. The result is that skewness is a negative number, implying that the distribution is skewed to the left.

9 KURTOSIS IN RETURN DISTRIBUTIONS

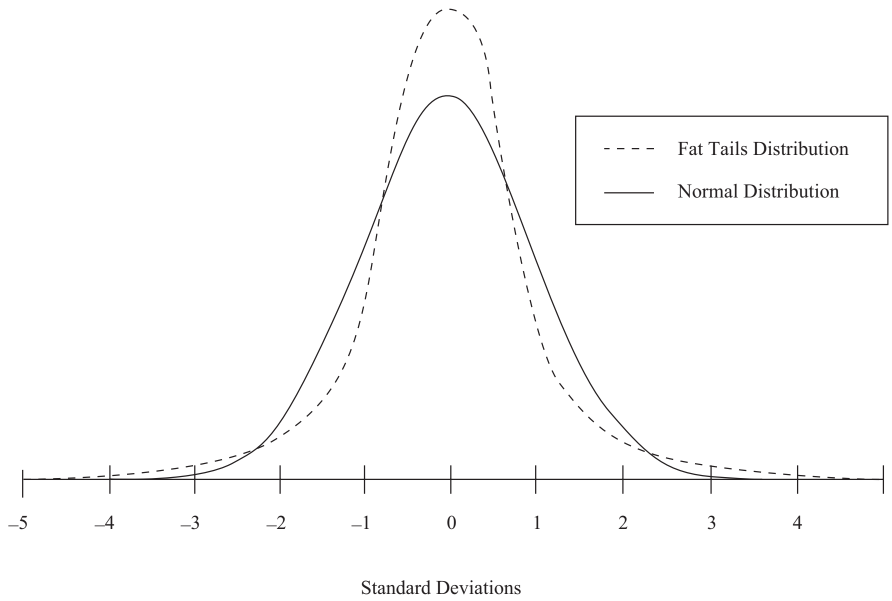
In the previous section, we discussed how to determine whether a return distribution deviates from a normal distribution because of skewness. One other way in which a return distribution might differ from a normal distribution is its relative tendency to generate large deviations from the mean. Most investors would perceive a greater chance of extremely large deviations from the mean as increasing risk.

Kurtosis is a measure of the combined weight of the tails of a distribution relative to the rest of the distribution—that is, the proportion of the total probability that is in the tails. A distribution that has fatter tails than the normal distribution is called **leptokurtic**; a distribution that has thinner tails than the normal distribution is called **platykurtic**; and a distribution identical to the normal distribution as concerns relative weight in the tails is called **mesokurtic**. A leptokurtic distribution tends to generate more-frequent extremely large deviations from the mean than the normal distribution.[42]

Exhibit 35 illustrates a leptokurtic distribution. It has fatter tails than the normal distribution. By construction, the leptokurtic and normal distributions have the same mean, standard deviation, and skewness. Note that the leptokurtic distribution is more likely than the normal distribution to generate observations in the tail regions defined by the intersection of graphs near a standard deviation of about ±2.5. The leptokurtic distribution is also more likely to generate observations that are near the mean, defined here as the region ±1 standard deviation around the mean. In compensation, to have probabilities sum to 1, the leptokurtic distribution generates fewer observations in the regions between the central region and the two tail regions.

42 See Wheeler (2011) for definitions and further reading.

Exhibit 35 Leptokurtic: Fat Tailed



Source: Reprinted from *Fixed Income Analysis.* Copyright CFA Institute.

The calculation for kurtosis involves finding the average of deviations from the mean raised to the fourth power and then standardizing that average by dividing by the standard deviation raised to the fourth power.[43] For all normal distributions, kurtosis is equal to 3. Many statistical packages report estimates of **excess kurtosis**, which is kurtosis minus 3.[44] Excess kurtosis thus characterizes kurtosis relative to the normal distribution. A normal or other mesokurtic distribution has excess kurtosis equal to 0. A leptokurtic distribution has excess kurtosis greater than 0, and a platykurtic distribution has excess kurtosis less than 0. A return distribution with positive excess kurtosis—a leptokurtic return distribution—has more frequent extremely large deviations from the mean than a normal distribution. Below is the expression for computing kurtosis from a sample.

- **Sample Excess Kurtosis Formula.** The **sample excess kurtosis** is

$$K_E = \left(\frac{n(n+1)}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^4}{s^4} \right) - \frac{3(n-1)^2}{(n-2)(n-3)} \quad \textbf{(17)}$$

where n is the sample size and s is the sample standard deviation.

43 This measure is free of scale. It is always positive because the deviations are raised to the fourth power.
44 Ibbotson and some software packages, such as Microsoft Excel, label "excess kurtosis" as simply "kurtosis." This highlights the fact that one should familiarize oneself with the description of statistical quantities in any software packages that one uses.

In Equation 17, **sample kurtosis** is the first term. Note that as n becomes large, Equation 17 approximately equals $\frac{n^2}{n^3}\frac{\sum\left(X-\bar{X}\right)^4}{s^4}-\frac{3n^2}{n^2}=\frac{1}{n}\frac{\sum\left(X-\bar{X}\right)^4}{s^4}-3$. For a sample of 100 or larger taken from a normal distribution, a sample excess kurtosis of 1.0 or larger would be considered unusually large.

Most equity return series have been found to be leptokurtic. If a return distribution has positive excess kurtosis (leptokurtosis) and we use statistical models that do not account for the fatter tails, we will underestimate the likelihood of very bad or very good outcomes. For example, the return on the S&P 500 for 19 October 1987 was 20 standard deviations away from the mean daily return. Such an outcome is possible with a normal distribution, but its likelihood is almost equal to 0. If daily returns are drawn from a normal distribution, a return four standard deviations or more away from the mean is expected once every 50 years; a return greater than five standard deviations away is expected once every 7,000 years. The return for October 1987 is more likely to have come from a distribution that had fatter tails than from a normal distribution. Looking at Exhibit 32 given earlier, the monthly return series for the S&P 500 has very large excess kurtosis, approximately 9.8. It is extremely fat-tailed relative to the normal distribution. By contrast, the annual return series has about no excess kurtosis. The results for excess kurtosis in the exhibit are consistent with research findings that the normal distribution is a better approximation for US equity returns for annual holding periods than for shorter ones (such as monthly).[45]

The following example illustrates the calculations for sample excess kurtosis for one of the two mutual funds we have been examining.

EXAMPLE 16

Calculating Sample Excess Kurtosis

Having concluded in Example 15 that the annual returns on T. Rowe Price Equity Income Fund were negatively skewed during the 2008–2017 period, what can we say about the kurtosis of the fund's return distribution? Exhibit 33 (repeated below) recaps the annual returns for the fund.

Exhibit 33 Annual Rates of Return: T. Rowe Price Equity Income, 2008–2017 (Repeated)

Year	Return (%)
2008	–35.75
2009	25.62
2010	15.15
2011	–0.72
2012	17.25
2013	31.69
2014	7.75
2015	–7.76

45 See Campbell, Lo, and MacKinlay (1997) for more details.

Exhibit 33 (Continued)

Year	Return (%)
2016	18.25
2017	16.18

Source: performance.morningstar.com.

Using the information from Exhibit 33 repeated above, address the following:

1. Calculate the sample excess kurtosis of PRFDX showing two decimal places.
2. Characterize the shape of the distribution of PRFDX returns based on your answer to Part 1 as leptokurtic, mesokurtic, or platykurtic.

Solution to 1:

To calculate excess kurtosis, we find the sum of the deviations from the mean raised to the fourth power, divide by the standard deviation raised to the fourth power, and then multiply that result by $n(n + 1)/[(n - 1)(n - 2)(n - 3)]$. This calculation determines kurtosis. Excess kurtosis is kurtosis minus $3(n - 1)^2/[(n - 2)(n - 3)]$. Exhibit 36 gives the calculations.

Exhibit 36 Calculating Kurtosis for PRFDX

Year	R_t	$R_t - \overline{R}$	$\left(R_t - \overline{R}\right)^4$
2008	−35.75	−44.52	3,928,444.507
2009	25.62	16.85	80,611.986
2010	15.15	6.38	1,656.848
2011	−0.72	−9.49	8,110.822
2012	17.25	8.48	5,171.106
2013	31.69	22.92	275,967.827
2014	7.75	−1.02	1.082
2015	−7.76	−16.53	74,660.589
2016	18.25	9.48	8,076.689
2017	16.18	7.41	3,014.899
n =	10		
$\overline{R}$ =	8.77%		
		Sum =	4,385,716.355
s =	19.47%	s^4 =	143,702.329
		Sum/s^4 =	30.519
		$n(n + 1)/[(n - 1)(n - 2)(n - 3)]$ =	0.2183
		Kurtosis =	**6.661**
		$3(n - 1)^2/[(n - 2)(n - 3)]$ =	4.34
		Excess Kurtosis =	**2.32**

Source: performance.morningstar.com.

Using Equation 17, the calculation is

$$K_E = \left[\frac{110}{(9)(8)(7)}\right]\frac{4{,}385{,}716.355}{19.47^4} - \frac{3(9)^2}{(8)(7)} = 2.32$$

Solution to 2:

The distribution of PRFDX's annual returns appears to be leptokurtic, based on a positive sample excess kurtosis. The fairly large excess kurtosis of 2.32 indicates that the distribution of PRFDX's annual returns is fat-tailed relative to the normal distribution. With a negative skewness and a positive excess kurtosis, PRFDX's annual returns do not appear to have been normally distributed during the period.[46]

10 USING GEOMETRIC AND ARITHMETIC MEANS

With the concepts of descriptive statistics in hand, we will see why the geometric mean is appropriate for making investment statements about past performance. We will also explore why the arithmetic mean is appropriate for making investment statements in a forward-looking context.

For reporting historical returns, the geometric mean has considerable appeal because it is the rate of growth or return we would have had to earn each year to match the actual, cumulative investment performance. In our simplified Example 8, for instance, we purchased a stock for €100 and two years later it was worth €100, with an intervening year at €200. The geometric mean of 0 percent is clearly the compound rate of growth during the two years. Specifically, the ending amount is the beginning amount times $(1 + R_G)^2$. The geometric mean is an excellent measure of past performance.

Example 8 illustrated how the arithmetic mean can distort our assessment of historical performance. In that example, the total performance for the two-year period was unambiguously 0 percent. With a 100 percent return for the first year and −50 percent for the second, however, the arithmetic mean was 25 percent. As we noted previously, the arithmetic mean is always greater than or equal to the geometric mean. If we want to estimate the average return over a one-period horizon, we should use the arithmetic mean because the arithmetic mean is the average of one-period returns. If we want to estimate the average returns over more than one period, however, we should use the geometric mean of returns because the geometric mean captures how the total returns are linked over time.

As a corollary to using the geometric mean for performance reporting, the use of **semilogarithmic** rather than arithmetic scales is more appropriate when graphing past performance.[47] In the context of reporting performance, a semilogarithmic graph has an arithmetic scale on the horizontal axis for time and a logarithmic scale on the vertical axis for the value of the investment. The vertical axis values are spaced according to the differences between their logarithms. Suppose we want to represent

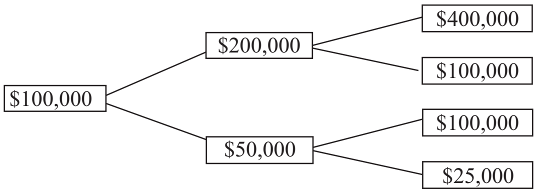
46 It is useful to know that we can conduct a Jarque–Bera (JB) statistical test of normality based on sample size n, sample skewness, and sample excess kurtosis. We can conclude that a distribution is not normal with no more than a 5 percent chance of being wrong if the quantity $\text{JB} = n\left[\left(S_K^2/6\right) + \left(K_E^2/24\right)\right]$ is 6 or greater for a sample with at least 30 observations. In this mutual fund example, we have only 10 observations and the test described is only correct based on large samples (as a guideline, for $n \geq 30$). Gujarati, Porter, and Gunasekar (2013) provides more details on this test.

47 See Campbell (1974) for more information.

£1, £10, £100, and £1,000 as values of an investment on the vertical axis. Note that each successive value represents a 10-fold increase over the previous value, and each will be equally spaced on the vertical axis because the difference in their logarithms is roughly 2.30; that is, ln 10 – ln 1 = ln 100 – ln 10 = ln 1,000 – ln 100 = 2.30. On a semilogarithmic scale, equal movements on the vertical axis reflect equal percentage changes, and growth at a constant compound rate plots as a straight line. A plot curving upward reflects increasing growth rates over time. The slopes of a plot at different points may be compared in order to judge relative growth rates.

In addition to reporting historical performance, financial analysts need to calculate expected equity risk premiums in a forward-looking context. For this purpose, the arithmetic mean is appropriate.

We can illustrate the use of the arithmetic mean in a forward-looking context with an example based on an investment's future cash flows. In contrasting the geometric and arithmetic means for discounting future cash flows, the essential issue concerns uncertainty. Suppose an investor with $100,000 faces an equal chance of a 100 percent return or a –50 percent return, represented on the tree diagram as a 50/50 chance of a 100 percent return or a –50 percent return per period. With 100 percent return in one period and –50 percent return in the other, the geometric mean return is $\sqrt{2(0.5)} - 1 = 0$.



The geometric mean return of 0 percent gives the mode or median of ending wealth after two periods and thus accurately predicts the modal or median ending wealth of $100,000 in this example. Nevertheless, the arithmetic mean return better predicts the arithmetic mean ending wealth. With equal chances of 100 percent or –50 percent returns, consider the four equally likely outcomes of $400,000, $100,000, $100,000, and $25,000 as if they actually occurred. The arithmetic mean ending wealth would be $156,250 = ($400,000 + $100,000 + $100,000 + $25,000)/4. The actual returns would be 300 percent, 0 percent, 0 percent, and –75 percent for a two-period arithmetic mean return of (300 + 0 + 0 –75)/4 = 56.25 percent. This arithmetic mean return predicts the arithmetic mean ending wealth of $100,000 × 1.5625 = $156,250. Noting that 56.25 percent for two periods is 25 percent per period, we then must discount the expected terminal wealth of $156,250 at the 25 percent arithmetic mean rate to reflect the uncertainty in the cash flows.

Uncertainty in cash flows or returns causes the arithmetic mean to be larger than the geometric mean. The more uncertain the returns, the more divergence exists between the arithmetic and geometric means. The geometric mean return approximately equals the arithmetic return minus half the variance of return.[48] Zero variance or zero uncertainty in returns would leave the geometric and arithmetic return approximately equal, but real-world uncertainty presents an arithmetic mean return larger than the geometric. For example, for the nominal annual returns on S&P 500 from 1926 to 2017, Exhibit 32 reports an arithmetic mean of 12.06 percent and standard deviation of 19.79 percent. The geometric mean of these returns is 10.16 percent. We can see the geometric mean is approximately the arithmetic mean minus half of the variance of returns: $R_G \approx 0.1206 - (1/2)(0.1979^2) = 0.1010$, or 10.10 percent.

48 See Bodie, Kane, and Marcus (2017).

SUMMARY

In this reading, we have presented descriptive statistics, the set of methods that permit us to convert raw data into useful information for investment analysis.

- A population is defined as all members of a specified group. A sample is a subset of a population.
- A parameter is any descriptive measure of a population. A sample statistic (statistic, for short) is a quantity computed from or used to describe a sample.
- Data measurements are taken using one of four major scales: nominal, ordinal, interval, or ratio. Nominal scales categorize data but do not rank them. Ordinal scales sort data into categories that are ordered with respect to some characteristic. Interval scales provide not only ranking but also assurance that the differences between scale values are equal. Ratio scales have all the characteristics of interval scales as well as a true zero point as the origin. The scale on which data are measured determines the type of analysis that can be performed on the data.
- A frequency distribution is a tabular display of data summarized into a relatively small number of intervals. Frequency distributions permit us to evaluate how data are distributed.
- The relative frequency of observations in an interval is the number of observations in the interval divided by the total number of observations. The cumulative relative frequency cumulates (adds up) the relative frequencies as we move from the first interval to the last, thus giving the fraction of the observations that are less than the upper limit of each interval.
- A histogram is a bar chart of data that have been grouped into a frequency distribution. A frequency polygon is a graph of frequency distributions obtained by drawing straight lines joining successive points representing the class frequencies.
- Sample statistics such as measures of central tendency, measures of dispersion, skewness, and kurtosis help with investment analysis, particularly in making probabilistic statements about returns.
- Measures of central tendency specify where data are centered and include the (arithmetic) mean, median, and mode (most frequently occurring value). The mean is the sum of the observations divided by the number of observations. The median is the value of the middle item (or the mean of the values of the two middle items) when the items in a set are sorted into ascending or descending order. The mean is the most frequently used measure of central tendency. The median is not influenced by extreme values and is most useful in the case of skewed distributions. The mode is the only measure of central tendency that can be used with nominal data.
- A portfolio's return is a weighted mean return computed from the returns on the individual assets, where the weight applied to each asset's return is the fraction of the portfolio invested in that asset.
- The geometric mean, G, of a set of observations $X_1, X_2, \ldots, X_n$ is $G = \sqrt[n]{X_1 X_2 X_3 \ldots X_n}$ with $X_i \geq 0$ for $i = 1, 2, \ldots, n$. The geometric mean is especially important in reporting compound growth rates for time series data.
- Quantiles such as the median, quartiles, quintiles, deciles, and percentiles are location parameters that divide a distribution into halves, quarters, fifths, tenths, and hundredths, respectively.

- Dispersion measures such as the variance, standard deviation, and mean absolute deviation (MAD) describe the variability of outcomes around the arithmetic mean.
- Range is defined as the maximum value minus the minimum value. Range has only a limited scope because it uses information from only two observations.
- MAD for a sample is $\frac{\sum_{i=1}^{n}\left|X_i - \bar{X}\right|}{n}$ where $\bar{X}$ is the sample mean and n is the number of observations in the sample.
- The variance is the average of the squared deviations around the mean, and the standard deviation is the positive square root of variance. In computing sample variance (s^2) and sample standard deviation, the average squared deviation is computed using a divisor equal to the sample size minus 1.
- The semivariance is the average squared deviation below the mean; semideviation is the positive square root of semivariance. Target semivariance is the average squared deviation below a target level; target semideviation is its positive square root. All these measures quantify downside risk.
- According to Chebyshev's inequality, the proportion of the observations within k standard deviations of the arithmetic mean is at least $1 - 1/k^2$ for all $k > 1$. Chebyshev's inequality permits us to make probabilistic statements about the proportion of observations within various intervals around the mean for any distribution with finite variance. As a result of Chebyshev's inequality, a two-standard-deviation interval around the mean must contain at least 75 percent of the observations, and a three-standard-deviation interval around the mean must contain at least 89 percent of the observations, no matter how the data are distributed.
- The coefficient of variation, CV, is the ratio of the standard deviation of a set of observations to their mean value. A scale-free measure of relative dispersion, by expressing the magnitude of variation among observations relative to their average size, the CV permits direct comparisons of dispersion across different data sets.
- Skew describes the degree to which a distribution is not symmetric about its mean. A return distribution with positive skewness has frequent small losses and a few extreme gains. A return distribution with negative skewness has frequent small gains and a few extreme losses. Zero skewness indicates a symmetric distribution of returns.
- Kurtosis measures combined weight of the tails of a distribution relative to the rest of the distribution. Distributions are characterized as leptokurtic, mesokurtic, or platykurtic according to whether there is relatively more, the same, or less weight in the tails. The calculation for kurtosis involves finding the average of deviations from the mean raised to the fourth power and then standardizing that average by the standard deviation raised to the fourth power. Excess kurtosis is kurtosis minus 3, the value of kurtosis for all normal distributions.

REFERENCES

Black, Fischer. 1993. "Estimating Expected Return." *Financial Analysts Journal*, vol. 49, no. 5:36–38.

Bodie, Zvi, Alex Kane, and Alan J. Marcus. 2017. *Investments*, 11th edition. New York: McGraw-Hill Education.

Campbell, Stephen K. 1974. *Flaws and Fallacies in Statistical Thinking*. Englewood Cliffs, NJ: Prentice-Hall.

Campbell, John, Andrew Lo, and A. Craig MacKinlay. 1997. *The Econometrics of Financial Markets*. Princeton, NJ: Princeton University Press.

Choobinbeh, N. Fred. 2005. "Semivariance" in *The Encyclopedia of Statistical Sciences*. New York: Wiley.

Dimson, Elroy, Paul Marsh, and Mike Staunton. 2011. "Equity Premiums around the World" in *Rethinking the Equity Risk Premium*. Charlottesville, VA: Research Foundation of CFA Institute.

Elton, Edwin J., Martin J. Gruber, Stephen J. Brown, and William N. Goetzmann. 2013. *Modern Portfolio Theory and Investment Analysis*, 9th edition. Hoboken, NJ: Wiley.

Forbes. 16 September 2013. "The Honor Roll." New York: Forbes Management Co., Inc.

Gujarati, Damodar N., Dawn C. Porter, and Sangeetha Gunasekar. 2013. *Basic Econometrics*, 5th edition. New York: McGraw-Hill Irwin.

Ibbotson, Roger G., Zhiwu Chen, Daniel Y.-J. Kim, and Wendy Y. Hu. 2013. "Liquidity as an Investment Style." *Financial Analysts Journal*, vol. 69, no. 3:30–44.

Reilly, Frank K., and Keith C. Brown. 2018. *Investment Analysis and Portfolio Management*, 11th edition. Mason, OH: Cengage South-Western.

Ibbotson, Roger G., and Daniel Y.-J. Kim. 2018. "Liquidity as an Investment Style. 2018 Update" available at research.zebra-capital.com ,

Von Hippel, Paul T. 2005. "Mean, Median, and Skew: Correcting a Textbook Rule." *Journal of Statistics Education*, vol. 13, no. 2. ww2.amstat.org/publications/jse/v13n2/vonhippel.html. Accessed 2 January 2018.

Wheeler, Donald J. 2011. "Problems with Skewness and Kurtosis, Part One: What Do the Shape Parameters Do?" *Quality Digest Daily*, August 1-2, 2011. https://www.spcpress.com/pdf/DJW231.pdf. Accessed 2 January 2018.

PRACTICE PROBLEMS

1 Which of the following groups *best* illustrates a sample?

A The set of all estimates for Exxon Mobil's EPS for next financial year

B The FTSE Eurotop 100 as a representation of the European stock market

C UK shares traded on Wednesday of last week that also closed above £120/share on the London Stock Exchange

2 Published ratings on stocks ranging from 1 (strong sell) to 5 (strong buy) are examples of which measurement scale?

A Ordinal

B Interval

C Nominal

3 Which of the following groups *best* illustrates a population?

A The 500 companies in the S&P 500 Index

B The NYSE-listed stocks in the Dow Jones Industrial Average

C The Lehman Aggregate Bond Index as a representation of the US bond market

4 In descriptive statistics, an example of a parameter is the:

A median of a population.

B mean of a sample of observations.

C standard deviation of a sample of observations.

5 A mutual fund has the return frequency distribution shown in the following table.

Return Interval (%)	Absolute Frequency
–10.0 to –7.0	3
–7.0 to –4.0	7
–4.0 to –1.0	10
–1.0 to +2.0	12
+2.0 to +5.0	23
+5.0 to +8.0	5

Which of the following statements is correct?

A The relative frequency of the interval "–1.0 to +2.0" is 20%.

B The relative frequency of the interval "+2.0 to +5.0" is 23%.

C The cumulative relative frequency of the interval "+5.0 to +8.0" is 91.7%.

6 An analyst is using the data in the following table to prepare a statistical report.

Portfolio's Deviations from Benchmark Return, 2003–2014 (%)			
Year 1	2.48	**Year 7**	–9.19
Year 2	–2.59	**Year 8**	–5.11
Year 3	9.47	**Year 9**	1.33

(continued)

(Continued)			
Year 4	–0.55	**Year 10**	6.84
Year 5	–1.69	**Year 11**	3.04
Year 6	–0.89	**Year 12**	4.72

The cumulative relative frequency for the interval $-1.71\% \leq x < 2.03\%$ is *closest* to:

A 0.250.

B 0.333.

C 0.583.

7 Frequency distributions summarize data in:

A a tabular display.

B overlapping intervals.

C a relatively large number of intervals.

8 Based on the table below, which of the following statements is correct?

Frequency Distributions of Sample Returns

Interval	Range	Absolute Frequency
A	–10% ≤ Observation < –5%	2
B	–5% ≤ Observation < 0%	7
C	0% ≤ Observation < 5%	15
D	5% ≤ Observation < 10%	2

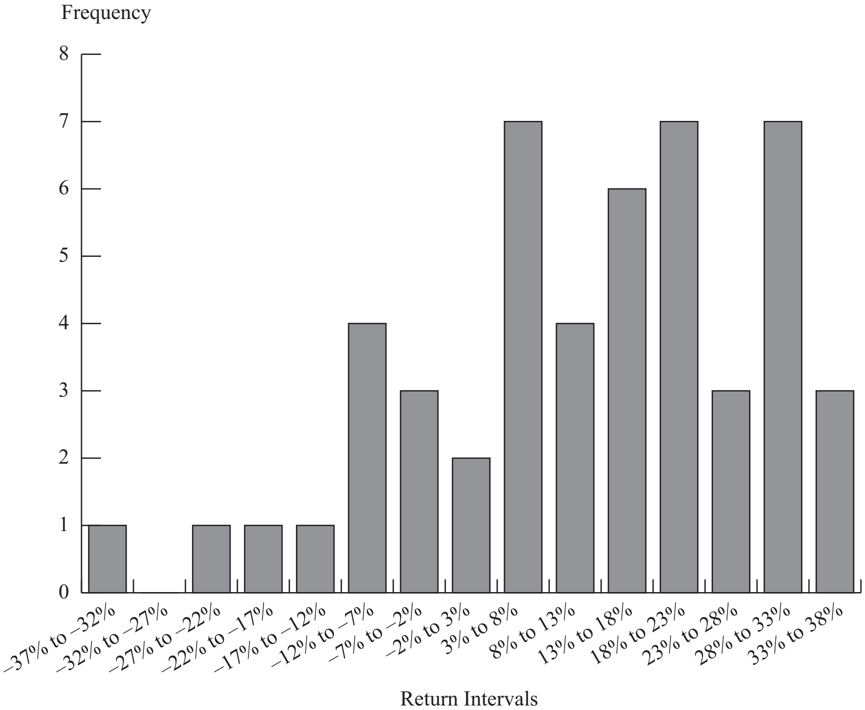
A The relative frequency of Interval C is 15.

B The cumulative frequency of Interval D is 100%.

C The cumulative relative frequency of Interval C is 92.3%.

The following information relates to Questions 9–10

The following histogram shows a distribution of the S&P 500 Index annual returns for a 50-year period:



9 The interval containing the median return is:

A 3% to 8%.

B 8% to 13%.

C 13% to 18%.

10 Based on the previous histogram, the distribution is *best* described as having:

A one mode.

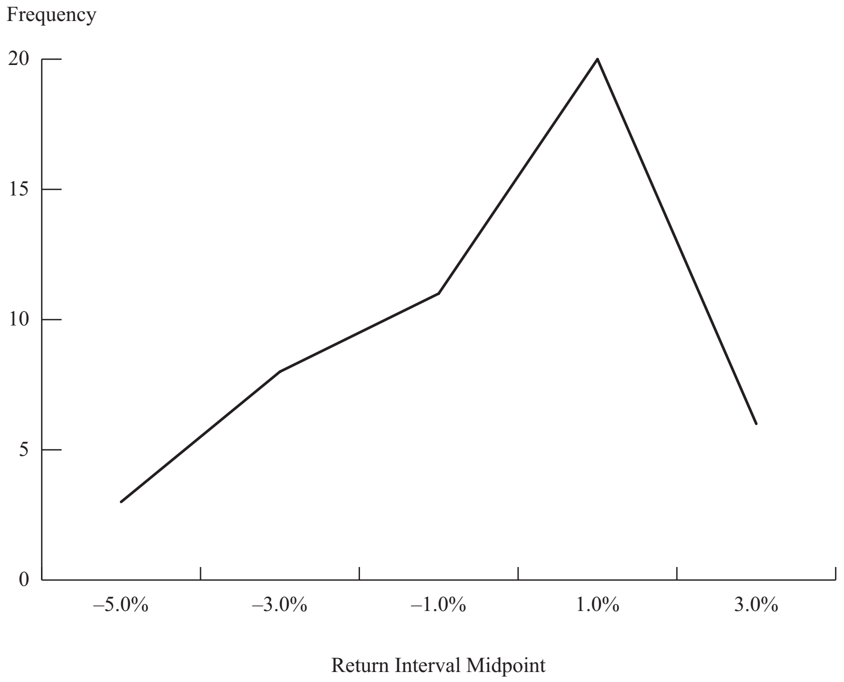
B two modes.

C three modes.

---

11 The following is a frequency polygon of monthly exchange rate changes in the US dollar/Japanese yen spot exchange rate for a four-year period. A positive change represents yen appreciation (the yen buys more dollars), and a negative change represents yen depreciation (the yen buys fewer dollars).

Monthly Changes in the US Dollar/Japanese Yen Spot Exchange Rate



Based on the chart, yen appreciation:

A occurred more than 50% of the time.

B was less frequent than yen depreciation.

C in the 0.0 to 2.0 interval occurred 20% of the time.

12 The height of a bar in a histogram represents the matching data interval's:

A relative frequency.

B absolute frequency.

C cumulative frequency.

The following table relates to Questions 13 and 14

Equity Returns for Six Companies

Company	Total Equity Return (%)
A	–4.53
B	–1.40
C	–1.20
D	–1.20

(Continued)

Company	Total Equity Return (%)
E	0.70
F	8.90

13 Based on the table, the arithmetic mean of the equity returns is *closest* to the return of:

A Company B.

B Company C.

C Company E.

14 Using the data from the table, the difference between the median and the mode is *closest* to:

A –1.41.

B 0.00.

C 1.41.

---

15 The annual returns for three portfolios are shown in the following table. Portfolios P and R were created in Year 1, Portfolio Q in Year 2.

	Annual Portfolio Returns (%)				
	Year 1	Year 2	Year 3	Year 4	Year 5
Portfolio P	–3.0	4.0	5.0	3.0	7.0
Portfolio Q		–3.0	6.0	4.0	8.0
Portfolio R	1.0	–1.0	4.0	4.0	3.0

The median annual return from portfolio creation to 2013 for:

A Portfolio P is 4.5%.

B Portfolio Q is 4.0%.

C Portfolio R is higher than its arithmetic mean annual return.

16 Last year, an investor allocated his retirement savings in the asset classes shown in the following table.

Asset Class	Asset Allocation (%)	Asset Class Return (%)
Large-cap US equities	20.0	8.0
Small-cap US equities	40.0	12.0
Emerging market equities	25.0	–3.0
High-yield bonds	15.0	4.0

The portfolio return in 2015 is *closest to*:

A 5.1%.

B 5.3%.

C 6.3%.

17 The following table shows the annual returns for Fund Y.

	Fund Y (%)
Year 1	19.5
Year 2	–1.9
Year 3	19.7
Year 4	35.0
Year 5	5.7

The geometric mean for Fund Y is *closest* to:

A 14.9%.

B 15.6%.

C 19.5%.

18 A manager invests €5,000 annually in a security for four years at the prices shown in the following table.

	Purchase Price of Security (€)
Year 1	62.00
Year 2	76.00
Year 3	84.00
Year 4	90.00

The average price paid for the security is *closest* to:

A €76.48.

B €77.26.

C €78.00.

The following information relates to Questions 19–20

The following exhibit shows the annual MSCI World Index total returns for a 10-year period.

Year 1	15.25%	**Year 6**	30.79%
Year 2	10.02%	**Year 7**	12.34%
Year 3	20.65%	**Year 8**	–5.02%
Year 4	9.57%	**Year 9**	16.54%
Year 5	–40.33%	**Year 10**	27.37%

19 The fourth quintile return for the MSCI World Index is *closest* to:

A 20.65%.

B 26.03%.

C 27.37%.

20 For Year 6 to Year 10, the mean absolute deviation of the MSCI World Index total returns is *closest* to:

A 10.20%.

B 12.74%.

C 16.40%.

The following table relates to questions 21 and 22

10 Years of S&P 500 Total Returns (in Ascending Order)
Returns
–38.49%
–0.73%
0.00%
9.54%
11.39%
12.78%
13.41%
19.42%
23.45%
29.60%

21 The third quartile percentage of total returns is *closest* to:

A 19.42%.

B 20.43%.

C 23.45%.

22 Complete the missing entries in the table below to answer this question.

Overall Risk Measures, S&P 500 vs. Sample Portfolio		
	S&P 500	**Sample Portfolio**
Mean	8.04%	8.54%
Range	–	67.09%
MAD	–	11.78%

An analyst does a performance measurement to compare the risk of a contemporaneous sample portfolio with that of the S&P 500 by determining the ranges and mean absolute deviations (MAD) of the two investments. The comparison shows that the S&P 500 appears riskier in terms of the:

A range only.

B MAD only.

C MAD and range.

23 Annual returns and summary statistics for three funds are listed in the following table:

	Annual Returns (%)		
Year	**Fund ABC**	**Fund XYZ**	**Fund PQR**
Year 1	–20.0	–33.0	–14.0
Year 2	23.0	–12.0	–18.0
Year 3	–14.0	–12.0	6.0
Year 4	5.0	–8.0	–2.0
Year 5	–14.0	11.0	3.0
Mean	–4.0	–10.8	–5.0
Standard deviation	17.8	15.6	10.5

The fund that shows the highest dispersion is:

A Fund PQR if the measure of dispersion is the range.

B Fund XYZ if the measure of dispersion is the variance.

C Fund ABC if the measure of dispersion is the mean absolute deviation.

24 Using the information in the following table, the sample standard deviation for VWIGX is *closest* to:

2015–2017 Total Return for VWIGX

Year	Vanguard International Growth Fund (VWIGX)
2015	–0.67%
2016	1.71%
2017	42.96%

A 6.02%.

B 12.04%.

C 24.54%.

25 Over the past 240 months, an investor's portfolio had a mean monthly return of 0.79%, with a standard deviation of monthly returns of 1.16%. According to Chebyshev's inequality, the minimum number of the 240 monthly returns that fall into the range of –0.95% to 2.53% is *closest* to:

A 80.

B 107.

C 133.

26 For a distribution of 2,000 observations with finite variance, sample mean of 10.0%, and standard deviation of 4.0%, what is the minimum number of observations that will lie within 8.0% around the mean according to Chebyshev's Inequality?

A 720

B 1,500

C 1,680

27 The mean monthly return and the standard deviation for three industry sectors are shown in the following exhibit.

Sector	Mean Monthly Return (%)	Standard Deviation of Return (%)
Utilities (UTIL)	2.10	1.23
Materials (MATR)	1.25	1.35
Industrials (INDU)	3.01	1.52

Based on the coefficient of variation, the riskiest sector is:

A utilities.

B materials.

C industrials.

The following information relates to Questions 28–29

The following table shows various statistics for Portfolios 1, 2, and 3.

	Mean Return (%)	Standard Deviation of Returns (%)	Skewness	Excess Kurtosis
Portfolio 1	7.8	15.1	0.0	0.7
Portfolio 2	10.2	20.5	0.9	–1.8
Portfolio 3	12.9	29.3	–1.5	6.2

28 The skewness of Portfolio 1 indicates its mean return is *most likely*:

A less than its median.

B equal to its median.

C greater than its median.

29 Compared with a normal distribution, the distribution of returns for Portfolio 3 *most likely*:

A has less weight in the tails.

B has a greater number of extreme returns.

C has fewer small deviations from its mean.

---

30 Two portfolios have unimodal return distributions. Portfolio 1 has a skewness of 0.77, and Portfolio 2 has a skewness of –1.11.

Which of the following is correct?

A For Portfolio 1, the median is less than the mean.

B For Portfolio 1, the mode is greater than the mean.

C For Portfolio 2, the mean is greater than the median.

31 A return distribution with frequent small gains and a few extreme losses is *most likely* to be called:

A leptokurtic.

B positively skewed.

C negatively skewed.

32 Which of the following sequences *best* represents the relative sizes of the mean, median, and mode for a positively skewed unimodal distribution?

A mode ≤ median ≤ mean

B mode < median < mean

C mean < median < mode

33 A distribution with excess kurtosis less than zero is termed:

A mesokurtic.

B platykurtic.

C leptokurtic.

34 When analyzing investment returns, which of the following statements is correct?

A The geometric mean will exceed the arithmetic mean for a series with non-zero variance.

B The geometric mean measures an investment's compound rate of growth over multiple periods.

C The arithmetic mean accurately estimates an investment's terminal value over multiple periods.

35 Which of the following statistical means *best* measures a mutual fund's past performance?

A Harmonic

B Geometric

C Arithmetic

SOLUTIONS

1 B is correct. The FTSE Eurotop 100 represents a sample of all European stocks. It is a subset of the population of all European stocks.

2 A is correct. Ordinal scales sort data into categories that are ordered with respect to some characteristic and may involve numbers to identify categories but do not assure that the differences between scale values are equal. The buy rating scale indicates that a stock ranked 5 is expected to perform better than a stock ranked 4, but it tells us nothing about the performance difference between stocks ranked 4 and 5 compared with the performance difference between stocks ranked 1 and 2, and so on.

3 A is correct. A population is defined as all members of a specified group. The S&P 500 Index consists of 500 companies, so this group is the population of companies in the index.

B is incorrect because there are several Dow Jones component stocks that are not traded on the NYSE, making the NYSE group a subset of the total population of stocks included in the Dow Jones average.

C is incorrect because although the Lehman Aggregate Bond Index is representative of the US bond market, it is a sampling of bonds in that market and not the entire population of bonds in that market.

4 A is correct. Any descriptive measure of a population characteristic is referred to as a parameter.

5 A is correct. The relative frequency is the absolute frequency of each interval divided by the total number of observations. Here, the relative frequency is calculated as: (12/60) × 100 = 20%. B is incorrect because the relative frequency of this interval is (23/60) × 100 = 38.33%. C is incorrect because the cumulative relative frequency of the last interval must equal 100%.

6 C is correct. The cumulative relative frequency of an interval identifies the fraction of observations that are less than the upper limit of the given interval. It is determined by summing the relative frequencies from the lowest interval up to and including the given interval. The following exhibit shows the relative frequencies for all the intervals of the data from the previous exhibit:

Lower Limit (%)	Upper Limit (%)	Absolute Frequency	Relative Frequency	Cumulative Relative Frequency
−9.19 ≤	< −5.45	1	0.083	0.083
−5.45 ≤	< −1.71	2	0.167	0.250
−1.71 ≤	< 2.03	4	0.333	0.583
2.03 ≤	< 5.77	3	0.250	0.833
5.77 ≤	≥ 9.51	2	0.167	1.000

The interval −1.71% ≤ x < 2.03% has a cumulative relative frequency of 0.583.

7 A is correct. A frequency distribution is a tabular display of data summarized into a relatively small number of intervals.

B is incorrect because intervals cannot overlap. Each observation is placed uniquely into one interval.

C is incorrect because a frequency distribution is summarized into a relatively small number of intervals.

8 C is correct because the cumulative relative frequency of an interval tells us the fraction of all observations that are less than the upper limit of an interval. For Interval C, that would be (2 + 7 + 15)/26 = 92.3%.

A is incorrect because the relative frequency of an interval is the absolute frequency of that interval divided by the total number of observations, here 15/26 = 57.7%. The number 15 represents Interval C's absolute frequency (also known as frequency), which is simply the actual number of observations in a given interval.

B is incorrect because the cumulative frequency tells us the number of observations that are less than the upper limit of a return interval, not the percentage of observations meeting that criteria. Because Interval D is the uppermost return interval, its cumulative frequency is the total number of observations for all intervals, yielding 2 + 7 + 15 + 2 = 26 and not 100%, which is the cumulative relative frequency for Interval D.

9 C is correct. Because there are 50 data points in the histogram, the median return would be the mean of the 50/2 = 25th and (50 + 2)/2 = 26th positions. The sum of the return interval frequencies to the left of the 13% to 18% interval is 24. As a result, the 25th and 26th returns will fall in the 13% to 18% interval.

10 C is correct. The mode of a distribution with data grouped in intervals is the interval with the highest frequency. The three intervals of 3% to 8%, 18% to 23%, and 28% to 33% all have a high frequency of 7.

11 A is correct. Twenty observations lie in the interval "0.0 to 2.0," and six observations lie in the 2.0 to 4.0 interval. Together, they represent 26/48, or 54.17% of all observations, which is more than 50%.

12 B is correct. In a histogram, the height of each bar represents the absolute frequency of its associated data interval.

A is incorrect because the height of each bar in a histogram represents the absolute (not relative) frequency.

C is incorrect because the height of each bar in a histogram represents the absolute (not cumulative) frequency.

13 C is correct. The arithmetic mean equals the sum of the observations divided by the number of observations. In this case, (–4.53 –1.40 –1.20 –1.20 + 0.70 + 8.90)/6 = 1.27/6 = 0.21.

The arithmetic mean is closest to the total equity return of Company E at 0.70 for a difference of (0.70 – 0.21) = 0.49.

A is incorrect because compared with the arithmetic mean, Company B's total equity return has a difference of (–1.40 –0.21) = –1.61, which is a wider distance from the mean than Company E's total equity return.

B is incorrect because compared with the arithmetic mean, Company C's total equity return has a difference of (–1.20 – 0.21) = –1.41, which is a wider distance from the mean than Company E's total equity return.

14 B is correct. The median is the value of the middle item of a set of items sorted into ascending or descending order. In an even-numbered sample, we define the median as the mean of the values of items occupying the $n/2$ and $(n + 2)/2$ positions (the two middle items). Given Table 2 has six observations, the median is the mean of the third and fourth observations. Because both are–1.20, the median is –1.20.

The mode is the most frequently occurring value in a distribution. The only value occurring more than once is –1.20.

Because the median and the mode both equal –1.20, their difference is zero.

A is incorrect because –1.41 is the difference between both the identical mode and median with the arithmetic mean. Both differences are: [–1.20 – (0.21)] = –1.41.

C is incorrect because 1.41 is the difference between the arithmetic mean with both the identical mode and median. Both differences are: [0.21 – (–1.20)] = 1.41.

15 C is correct. The median of Portfolio R is 0.8% higher than the mean for Portfolio R.

16 C is correct. The portfolio return must be calculated as the weighted mean return, where the weights are the allocations in each asset class:

(0.20 × 8%) + (0.40 × 12%) + (0.25 × –3%) + (0.15 × 4%) = 6.25%, or ≈ 6.3%.

17 A is correct. The geometric mean return for Fund Y is found as follows:

$$\text{Fund Y} = [(1 + 0.195) \times (1 - 0.019) \times (1 + 0.197) \times (1 + 0.350) \times (1 + 0.057)]^{(1/5)} - 1$$
$$= 14.9\%.$$

18 A is correct. The harmonic mean is appropriate for determining the average price per unit. It is calculated by summing the reciprocals of the prices; then averaging that sum by dividing by the number of prices; and finally, taking the reciprocal of the average:

4/[(1/62.00) + (1/76.00) + (1/84.00) + (1/90.00)] = €76.48.

19 B is correct. Quintiles divide a distribution into fifths, with the fourth quintile occurring at the point at which 80% of the observations lie below it. The fourth quintile is equivalent to the 80th percentile. To find the yth percentile (P_y), we first must determine its location. The formula for the location (L_y) of a yth percentile in an array with n entries sorted in ascending order is $L_y = (n + 1) \times (y/100)$. In this case, $n = 10$ and $y = 80\%$, so

$$L_{80} = (10 + 1) \times (80/100) = 11 \times 0.8 = 8.8.$$

With the data arranged in ascending order (–40.33%, –5.02%, 9.57%, 10.02%, 12.34%, 15.25%, 16.54%, 20.65%, 27.37%, and 30.79%), the 8.8th position would be between the 8th and 9th entries, 20.65% and 27.37%, respectively. Using linear interpolation, $P_{80} = X_8 + (L_y - 8) \times (X_9 - X_8)$,

$$\begin{aligned} P_{80} &= 20.65 + (8.8 - 8) \times (27.37 - 20.65) \\ &= 20.65 + (0.8 \times 6.72) = 20.65 + 5.38 \\ &= 26.03\%. \end{aligned}$$

20 A is correct. The formula for mean absolute deviation (MAD) is

$$\text{MAD} = \frac{\sum_{i=1}^{n} \left| X_i - \bar{X} \right|}{n}$$

Column 1: Sum annual returns and divide by n to find the arithmetic mean $\left(\bar{X}\right)$ of 16.40%.

Column 2: Calculate the absolute value of the difference between each year's return and the mean from Column 1. Sum the results and divide by n to find the MAD.

These calculations are shown in the following exhibit:

	Column 1		Column 2
Year	Return		$\lvert X_i - \bar{X} \rvert$
Year 6	30.79%		14.39%
Year 7	12.34%		4.06%
Year 8	–5.02%		21.42%
Year 9	16.54%		0.14%
Year 10	27.37%		10.97%
Sum:	82.02%	Sum:	50.98%
n:	5	n:	5
$\bar{X}$:	16.40%	MAD:	10.20%

21 B is correct. Quartiles divide a distribution into quarters, with the third quartile occurring at the point at which 75% of the observations lie below it. The third quartile is equivalent to the 75th percentile. The formula for the location (L_y) of the yth percentile in an array with n entries sorted in ascending order is $L_y = (n + 1) \times (y/100)$. In this case, $n = 10$ and $y = 75$, so $L_{75} = (11) \times (75/100) = 11 \times 0.75 = 8.25$.

Rearranging the data in ascending order (i.e., with the lowest value at the top), the 8.25th position would be between the eighth and ninth rank order entries, 19.42% and 23.45%, respectively. Using linear interpolation, $P_{75} = X_8 + (L_{75} - 8) \times (X_9 - X_8)$, so P_{75} = 19.42% + (8.25 – 8) × (23.45% – 19.42%) = 20.428%, or 20.43%.

A is incorrect because it is the non-interpolated value of the eighth observation without the adjustment for placement at the location of the third quartile.

C is incorrect because it is the non-interpolated value of the ninth observation without the adjustment for placement at the location of the third quartile.

22 C is correct. Both the range and MAD of the S&P 500 are greater than the range and MAD of the sample portfolio. Thus both measures indicate the S&P 500 is riskier.

The range for the S&P 500 equals the distance between the lowest and highest values in the dataset. That distance for the S&P 500 is [29.60% – (–38.49%)] = 68.09%. Given that this range is larger than the range of the sample portfolio at 67.09%, the S&P 500 appears riskier than the sample portfolio.

The MAD for the S&P 500 returns equals the sum of the absolute deviations from the mean return divided by the number of observations.

$$\text{MAD} = \frac{\sum_{i=1}^{n} \lvert X_i - \bar{X} \rvert}{n}$$, where $\bar{X}$ is the sample mean and n is the number of observations in the sample.

Use the 10 observed S&P 500 returns from the table (sample mean = 8.04%) to calculate the MAD for the S&P 500 as follows:

$$\text{MAD}_{\text{S\&P500}} = \frac{\lvert 19.42 - \bar{x} \rvert + \lvert 9.54 - \bar{x} \rvert + \lvert -0.73 - \bar{x} \rvert + \lvert 11.39 - \bar{x} \rvert + \lvert 29.60 - \bar{x} \rvert + \lvert 13.41 - \bar{x} \rvert + \lvert 0.00 - \bar{x} \rvert + \lvert 12.78 - \bar{x} \rvert + \lvert 23.45 - \bar{x} \rvert + \lvert -38.49 - \bar{x} \rvert}{10}$$

$$\text{MAD}_{\text{S\&P500}} = 12.67\%$$

Given that the MAD for the S&P 500 is greater than the MAD for the sample portfolio (12.67% versus 11.78%), the S&P 500 appears riskier than the sample portfolio.

A is incorrect because although the S&P 500 is correctly identified as having the larger range, the sample portfolio has a smaller MAD.

B is incorrect because although the S&P 500 is correctly identified as having the larger MAD, the Sample Portfolio has a smaller range.

23 C is correct. The mean absolute deviation (MAD) of Fund ABC's returns is greater than the MAD of both of the other funds.

$$\text{MAD} = \frac{\sum_{i=1}^{n}\left|X_i - \bar{X}\right|}{n}, \text{ where } \bar{X} \text{ is the arithmetic mean of the series.}$$

MAD for Fund ABC =

$$\frac{|-20-(-4)| + |23-(-4)| + |-14-(-4)| + |5-(-4)| + |-14-(-4)|}{5} = 14.4\%$$

MAD for Fund XYZ =

$$\frac{|-33-(-10.8)| + |-12-(-10.8)| + |-12-(-10.8)| + |-8-(-10.8)| + |11-(-10.8)|}{5} = 9.8\%$$

MAD for Fund PQR =

$$\frac{|-14-(-5)| + |-18-(-5)| + |6-(-5)| + |-2-(-5)| + |3-(-5)|}{5} = 8.8\%$$

A and B are incorrect because the range and variance of the three funds are as follows:

	Fund ABC	Fund XYZ	Fund PQR
Range	43%	44%	24%
Variance	317	243	110

The numbers shown for variance are understood to be in "percent squared" terms so that when taking the square root, the result is standard deviation in percentage terms. Alternatively, by expressing standard deviation and variance in decimal form, one can avoid the issue of units; in decimal form, the variances for Fund ABC, Fund XYZ, and Fund PQR are 0.0317, 0.0243, and 0.0110, respectively.

24 C is correct. The sample variance is defined as sum of the squared deviations from the sample mean divided by the sample size minus one, and the sample standard deviation equals the square root of the sample variance.

The following figure summarizes the inputs for the calculation of VWGIX sample variance.

Year	VWIGX	$\left(X_i - \bar{X}\right)^2$
2015	–0.67%	2.35
2016	1.71%	1.68
2017	42.96%	8.00

(continued)

Year	VWIGX	$\left(X_i - \bar{X}\right)^2$
Sample mean $\left(\bar{X}\right)$	14.67%	
$\sum\left(X - \bar{X}\right)^2$		12.04
$\left[\sum\left(X - \bar{X}\right)^2\right]/(n-1) = \sigma^2$	[12.04/2]	6.02
$\sqrt{\left[\sum\left(X - \bar{X}\right)^2\right]/(n-1)} = \sigma$	$\sqrt{(12.04/2)}$	24.54%

The sample variance is thus calculated as $\dfrac{\left[\sum\left(X - \bar{X}\right)^2\right]}{(n-1)} = \dfrac{12.04}{2} = 6.02.$

The square root of the sample variance is the sample standard deviation.

That number is $\left(\sqrt{6.02}\right) = 24.54\%$.

A is incorrect because it is the sample variance for VWGIX, not its sample standard deviation.

B is incorrect because it represents the sum of the squared deviations from the mean, not the sample standard deviation.

25 C is correct. According to Chebyshev's inequality, the proportion of the observations within k standard deviations of the arithmetic mean is at least $1 - 1/k^2$ for all $k > 1$.

The upper limit of the range is 2.53%, which is 2.53 − 0.79 = 1.74% above the mean. The lower limit is −0.95, which is 0.79 − (−0.95) = 1.74% below the mean. As a result, k = 1.74/1.16 = 1.50 standard deviations.

Because k = 1.50, the proportion of observations within the interval is at least 1 − $1/1.5^2$ = 1 − 0.444 = 0.556, or 55.6%. Thus, the number of observations in the given range is at least 240 × 55.6%, which is ≈ 133.

26 B is correct. Observations within 8% of the sample mean will cover an interval of 8/4 or two standard deviations. Chebyshev's Inequality says the proportion of the observations P within k standard deviations of the arithmetic mean is at least $1 - 1/k^2$ for all $k > 1$. So, solving for $k = 2$: P = 1 − ¼ = 75%. Given 2,000 observations, this implies at least 1,500 will lie within 8.0% of the mean.

A is incorrect because 720 shows P = 720/2,000 = 36.0% of the observations. Using P to solve for k implies 36.0% = $1 - 1/k^2$, where k = 1.25. This result would cover an interval only 4% × 1.25 or 5% around the mean (i.e. less than two standard deviations).

C is incorrect because 1,680 shows P = 1,680/2,000 = 84.0% of the observations. Using P to solve for k implies 84.0% = $1 - 1/k^2$, where k = 2.50. This result would cover an interval of 4% × 2.5, or 10% around the mean (i.e., more than two standard deviations).

27 B is correct. The coefficient of variation (CV) is the ratio of the standard deviation to the mean, where a higher CV implies greater risk per unit of return.

$$CV_{UTIL} = \frac{s}{\bar{X}} = \frac{1.23\%}{2.10\%} = 0.59$$

$$\mathrm{CV}_{MATR} = \frac{s}{\overline{X}} = \frac{1.35\%}{1.25\%} = 1.08$$

$$\mathrm{CV}_{INDU} = \frac{s}{\overline{X}} = \frac{1.52\%}{3.01\%} = 0.51$$

28 B is correct. Portfolio 1 has a skewness of 0.0, which indicates that the portfolio's return distribution is symmetrical and thus its mean and median are equal.

29 B is correct. Portfolio 3 has positive excess kurtosis (i.e., kurtosis greater than 3), which indicates that its return distribution is leptokurtic and has fatter tails than the normal. The fatter tails mean Portfolio 3 has a greater number of extreme returns.

30 A is correct. Portfolio 1 is positively skewed, so the mean is greater than the median, which is greater than the mode.

31 C is correct. A return distribution with negative skew has frequent small gains and a few extreme losses.

A is incorrect because a leptokurtic distribution is more peaked with fatter tails, which exhibit both extreme gains and losses.

B is incorrect because a return distribution with positive skew has frequent small losses and a few extreme gains.

32 B is correct. For the positively skewed unimodal distribution, the mode is less than the median, which is less than the mean.

A is incorrect because, for the positively skewed unimodal distribution, the mode is less than the median (not less than or equal to), which is less than (not less than or equal to) the mean.

C is incorrect because, for the negatively (not positively) skewed unimodal distribution, the mean is less than the median, which is less than the mode.

33 B is correct. A platykurtic distribution has excess kurtosis less than zero.

A is incorrect because a normal or other mesokurtic distribution has excess kurtosis equal to zero.

C is incorrect because a leptokurtic distribution has excess kurtosis greater than zero.

34 B is correct. The geometric mean compounds the periodic returns of every period, giving the investor a more accurate measure of the terminal value of an investment.

35 B is correct. The geometric mean is an excellent measure of past performance. For reporting historical returns, the geometric mean has considerable appeal because it is the rate of growth or return a fund would have had to earn each year to match the actual, cumulative investment performance. To estimate the average returns over more than one period, the geometric mean captures how the total returns are linked over time.

A is incorrect because the harmonic mean is more appropriate for determining the average price per unit, not evaluating a mutual fund's return history. The average price paid is in fact the harmonic mean of the asset's prices at the purchase dates. The harmonic mean is applicable when ratios are repeatedly applied to a fixed quantity to yield a variable number of units, such as in cost averaging, which involves the periodic investment of a fixed amount of money.

C is incorrect because the arithmetic mean is more appropriate for making investment statements in a forward-looking context, not for historical returns. It can distort the assessment of historical performance, so it is better applied to estimate the average return over a one-period horizon.