EE22S044

SIMULATION ASSIGNMENT

Comparison of LMS, ϵ- NLMS and RLS algorithms

In [17]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [2]:
```
M = 4 #Channel size
N = 600 #Iterations
n_ensembles = 300 #Number of ensembles
```

In [18]:
```
C = np.array([1,0.5,-1,2]) # impulse response of a chanel H(Z) = 1 + 0.5Z^-1 + -1*Z^-2 + 2 * Z^-3
```

Adapting the notation from Adaptive Filters by Ali H Sayed, section 8.1, we generate the data from the following relation:

$d = uw + v$

Here d is a column vector, u is a 2 dimensional matrix, w is a column vector and v is AWGN random vector with variance σ_v^2

For our problem of channel estimation, w is simply the channel impulse response and u is contains time embedded input stacked as rows(i.e. matrix form for linear convolution). $\sigma_v^2 = 0.01$

We will start with an initial guess of zero for w $(w_{-1} = 0^T)$ and see how error(prediction) evolves with iterations for the three algorithms. Each algorithm is stochastic, since we do not have complete knowledge of the gradient.

In [4]:
```
X = np.random.randn(n_ensembles,N)
U = np.zeros((n_ensembles,N,M))
U[:,:,0] = X[:,:]
U[:,1:,1] = X[:,0:-1]
U[:,2:,2] = X[:,0:-2]
U[:,3:,3] = X[:,0:-3]
D = U@C+0.1*np.random.randn(n_ensembles,N)
```

LMS Algorthiam

update equations for LMS algoritham is given below

- $e_i = d_i - u_i w_{i-1}$
- $w_i = w_{i-1} + \mu u_i^H e_i$

Here d_i is a scalar, the i^{th} of column vector d or value of d[n] at instant i. w_i is our guess at time instant i and u_i is the ith row of matrix u. μ is the step size for descent.

This is a stochastic algorithm where the values for correlation matrices of u and of u with d are taken from instantaneous values.

Code for the LMS algorithm:

In [5]:
```
#LMS - algoritham
mu = 0.01
w_lms = np.zeros((n_ensembles,M,N+1))
err_lms = np.zeros((n_ensembles,N))
for n in range(n_ensembles):
    for i in range(N):
        err_lms[n,i] = D[n,i]-U[n,i]@w_lms[n,:,i]
        w_lms[n,:,i+1] = w_lms[n,:,i]+mu*((U[n,i].T)*err_lms[n,i])

temp = err_lms**2
mse_lms = np.mean(temp,axis=0)
```

ϵ-NLMS Algoritham

The update rule is given by

- $e_i = d_i - u_i w_{i-1}$
- $w_i = w_{i-1} + \frac{\mu}{\epsilon + ||u_i||^2} u_i^H e_i$

Code:

In [6]:
```
#NLMS-Algoritham
mu_nlms = 0.2
eps_nlms = 0.001
w_nlms = np.zeros((n_ensembles,M,N+1))
err_nlms = np.zeros((n_ensembles,N))
for n in range(n_ensembles):
    for i in range(N):
        err_nlms[n,i] = D[n,i]-U[n,i]@w_nlms[n,:,i]
        w_nlms[n,:,i+1] = w_nlms[n,:,i]+(mu_nlms/(eps_nlms+U[n,i]@U[n,i].T))*((U[n,i].T)*err_nlms[n,i])

temp = err_nlms**2
mse_nlms = np.mean(temp,axis=0)
```

RLS Algoritham

update equations of RLS is given by

- $w_i = w_{i-1} + \mu(\epsilon I + R_u)^{-1}(R_{du} - R_u w_{i-1})$

Here the instantaneous approximation is used for $(R_{du} - R_u w_{i-1})$.

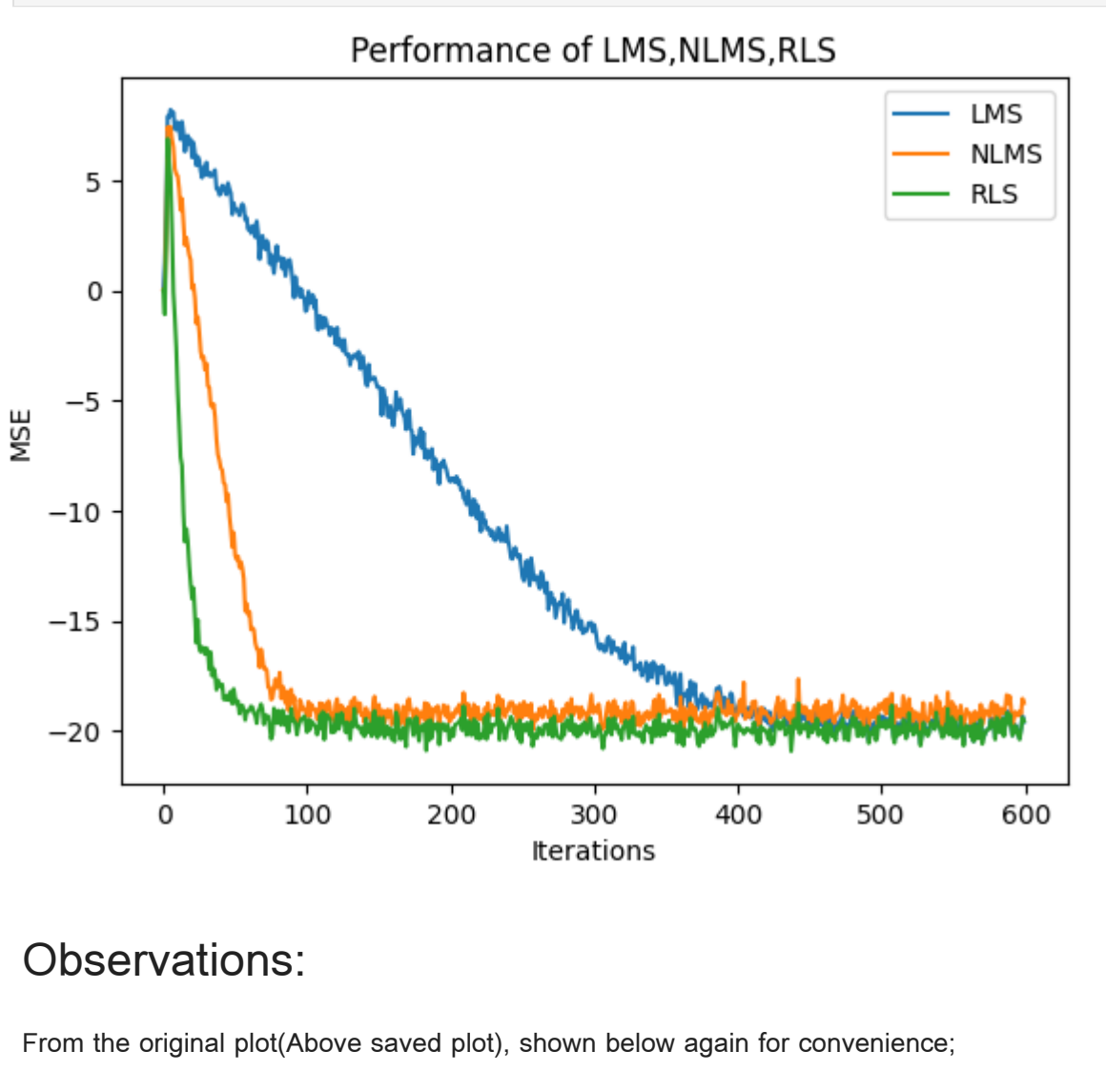
But for calculation of R_u, an exponentially weighted sample average is used.

The final update rule turns out to be:

- $P_i = \lambda^{-1}[P_{i-1} - \frac{\lambda^{-1} P_{i-1} u_i^H u_i P_{i-1}}{1+\lambda^{-1} u_i P_{i-1} u_i^H}]$
- $e_i = d_i - u_i w_{i-1}$
- $w_i = w_{i-1} + P_i u_i^H e_i$

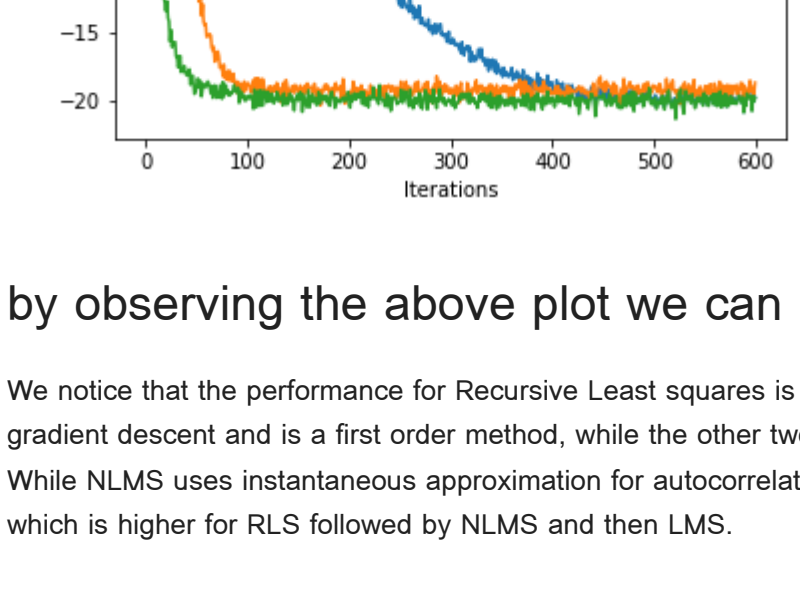
In [7]:
```
#RLS-algoritham
lambda_rls = 0.995
eps_rls = 0.995
w_rls = np.zeros((n_ensembles,M,N+1))
err_rls = np.zeros((n_ensembles,N))

l_inv = 1/lambda_rls

P = np.zeros((n_ensembles,N+1,M,M))
for n in range(n_ensembles):
    P[n,0] = (1/eps_rls)*np.eye(M)

    for i in range(N):
        err_rls[n,i] = D[n,i]-U[n,i]@w_rls[n,:,i]
        P[n,i+1] = l_inv*(P[n,i]-(l_inv/(1+l_inv*U[n,i]@P[n,i]@U[n,i].T))*P[n,i]@np.outer(U[n,i],U[n,i])@P[n,i])
        w_rls[n,:,i+1] = w_rls[n,:,i]+P[n,i+1]@((U[n,i].T)*err_rls[n,i])

temp = err_rls**2
mse_rls = np.mean(temp,axis=0)
```

In [8]:
```
plt.plot(10*np.log10(mse_lms),label='LMS')
plt.plot(10*np.log10(mse_nlms),label='NLMS')
plt.plot(10*np.log10(mse_rls),label='RLS')
plt.legend()
plt.xlabel('Iterations')
plt.ylabel('MSE')
plt.title('Performance of LMS,NLMS,RLS')
plt.savefig('q10.png')
plt.show()
```



Observations:

From the original plot(Above saved plot), shown below again for convenience;



by observing the above plot we can summarize few points those are:

We notice that the performance for Recursive Least squares is better than NLMS which is much better than that for LMS. There is a clear improvement when looking at RLS and NLMS versus LMS. LMS is based on stochastic gradient descent and is a first order method, while the other two are based on a stochastic version of Newton's method, which is second order and uses curvature information, and hence show dramatic improvements over LMS. While NLMS uses instantaneous approximation for autocorrelation matrix of u, RLS uses a weighted sample average, which gives it the advantage of using past information as well. The tradeoff is increased computational complexity, which is higher for RLS followed by NLMS and then LMS.

Error relation between three algorithams

- error(RLS) < error(NLMS) < error(LMS)

Performance Relations between three algorithams

- RLS > NLMS > LMS

Simulations for steady state performance for LMS with help of Gaussian Regressors(with and without Tapped delay line)

Theory(Taken from the Ali sayed Text book)

These are the corresponding equations that we are using in the experiment and all these are taken from the Ali Sayed book The notation used here is same as in previous problems. The exact expression for steady state excess mean squared error, when using LMS algorithm, is given by:

$\zeta^{LMS} = \lim_{i\to\infty} \frac{\mu}{2}[E||u_i||^2|e_a(i)|^2 + \sigma_v^2 Tr(R_u)]$

Here $e_a = e(i) - v(i)$

We will be using 2 approximations of this exact expression, small step size and separation principle.

For small step sizes, one neglects the contribution of the first term and the resulting EMSE becomes

$\zeta^{LMS} = \frac{\mu\sigma_v^2 Tr(R_u)}{2}$

The second approximation assumes that at steady state, $||u_i||^2$ is independent of $e_a(i)$, which leads to the following expression

$\zeta^{LMS} = \frac{\mu\sigma_v^2 Tr(R_u)}{2-\mu Tr(R_u)}$

One can see that the first approximation is a limiting case of the second mathematically, although both are derived under seemingly different assumptions(assumption 2 encompasses 1 actually).

Therefore, one expects the plots for second expression to be closer to simulations over a wider range of step size, something we observe later.

Simulation for regressors without shift structure (No tapped delay line for U(i) genration)

For the first problem, we have to generate data corresponding to regressors with no shift structure.

The assumptions are taken from model 15.16 from the textbook.

We generate the data as:

$d = uw_o + v$

Since the model 15.16 assumes existence of w_o, we generate w_o randomly. Upon experimentation with different values for w_o, we found no change in the plots for mean squared error, so we proceed with a randomly chosen vector.

Also here the regressors do not have shift structure, each row of u is a random vector independent of the other. Furthermore, we given that the eigenvalue spread of autocorrelation matrix of u is 5. Similarity transformations do not affect the trace and the steady state error, so we take R_u to be diagonal whose entries are linearly spaced between 1 and 5.

Important point to note here again is that we need to add an extra $sigma_v^2$ to the EMSE, which is the excess mean squred error, to get mean squared error.

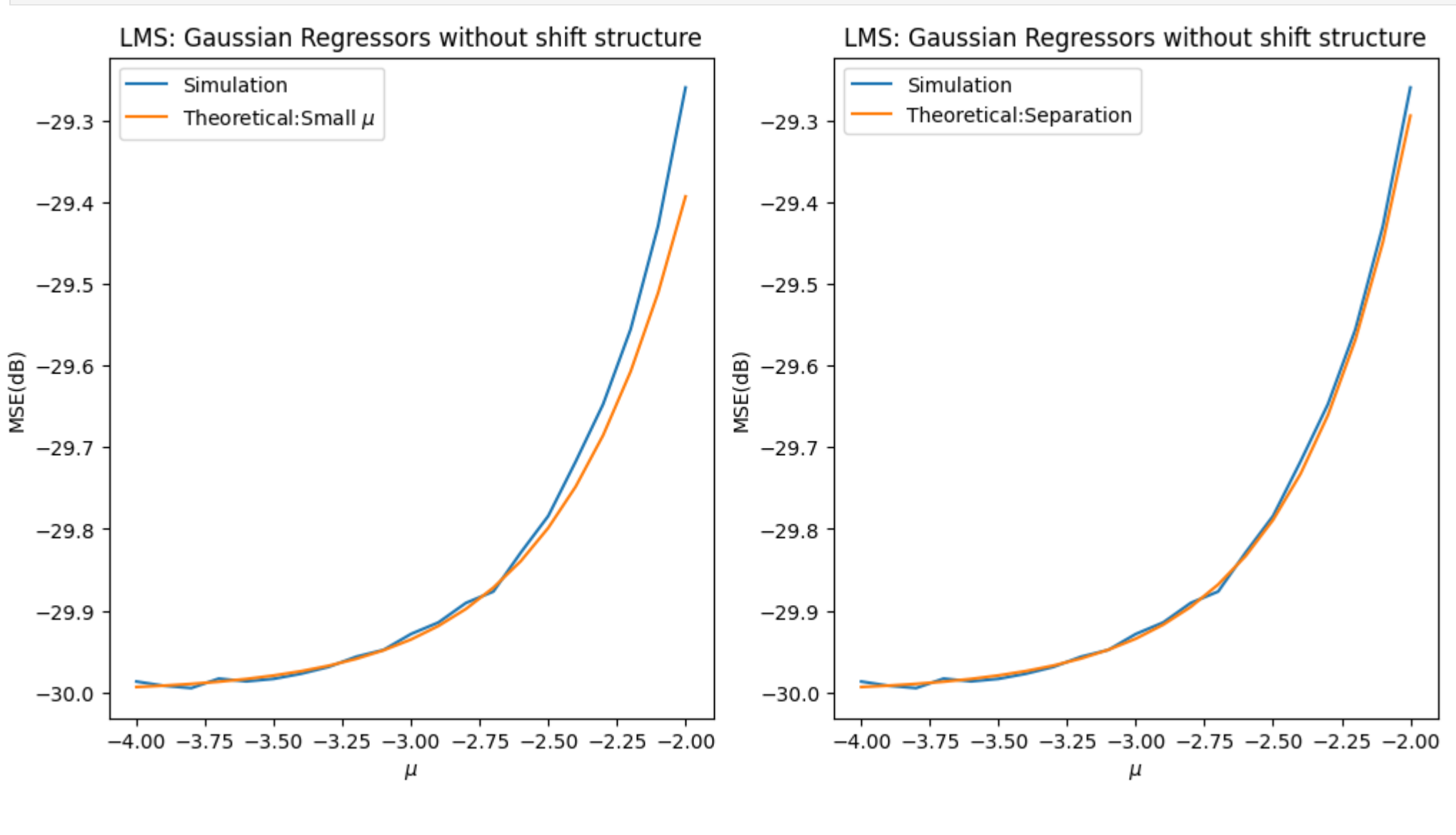
With this model, we proceed with the code below

In [19]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [10]:
```
num_iter = 10**6
num_ensembles = 300
sigma_v = np.sqrt(0.001)
M = 10
w_o = 3*np.random.randn(M)
```

In [11]:
```
## genrating the data with corresponding given values of problem specifications( with out shift structure)
mu = np.logspace(-4,-2,21)
R_u_diag = np.linspace(1,5,M)
err_lms = np.zeros((len(mu),num_iter))

w_lms = np.zeros((len(mu),num_ensembles,M))

for i in range(num_iter):
    u_iter = np.random.randn(len(mu),num_ensembles,M)
    u_iter = u_iter*np.sqrt(R_u_diag)

    d_iter = np.sum(u_iter*w_o,axis=2)+sigma_v*np.random.randn(len(mu),num_ensembles)

    pred_temp = np.sum(u_iter*w_lms,axis=2)
    err_temp = d_iter-pred_temp

    err_lms[:,i] = np.mean(err_temp**2,axis=1)

    vector_corr_temp = mu*(np.swapaxes(u_iter,0,2))

    w_lms+=np.swapaxes(vector_corr_temp*(err_temp.T),0,2)
err_lms_exp = np.mean(err_lms[0:,-5000:],axis=1)
```

In [12]:
```
## plotting graphs for both Theortical and experimental simulations
err_lms_small_mu = mu*(sigma_v**2)*np.sum(R_u_diag)/2+(sigma_v)**2
err_lms_separation = mu*(sigma_v**2)*np.sum(R_u_diag)/(2-mu*np.sum(R_u_diag))+(sigma_v)**2

fig, ax = plt.subplots(1,2,figsize=(12,6))

ax[0].plot(np.log10(mu),10*np.log10(err_lms_exp),label='Simulation')
ax[0].plot(np.log10(mu),10*np.log10(err_lms_small_mu),label='Theoretical:Small '+r'$\mu$')

ax[0].set_xlabel(r'$\mu$')
ax[0].set_ylabel('MSE(dB)')
ax[0].set_title('LMS: Gaussian Regressors without shift structure')
ax[0].legend()

ax[1].plot(np.log10(mu),10*np.log10(err_lms_exp),label='Simulation')
ax[1].plot(np.log10(mu),10*np.log10(err_lms_separation),label='Theoretical:Separation')
ax[1].set_xlabel(r'$\mu$')
ax[1].set_ylabel('MSE(dB)')
ax[1].set_title('LMS: Gaussian Regressors without shift structure')
ax[1].legend()

plt.savefig('noshiftstr.png')
plt.show()
```



Simulation for regressors with shift structure (Presence of tapped Delay line for genration of U(i)

Again, here too, we take w_o to be a randomly generate row vector as before.

But now, u_i are regressors with shift structure. Each u_i is generated from a tap delay line from a sequence $u(i)$. u(i) is generated from passing an iid white sequence of unit variance through an IIR filter.

$u(i) = au(i-1) + \sqrt{1-a^2}s(i)$ (which is derived from the transfer function given in problem)

- Important point to note here again is that we need to add an extra $sigma_v^2$ to the EMSE, which is the excess mean squred error, to get mean squared error.

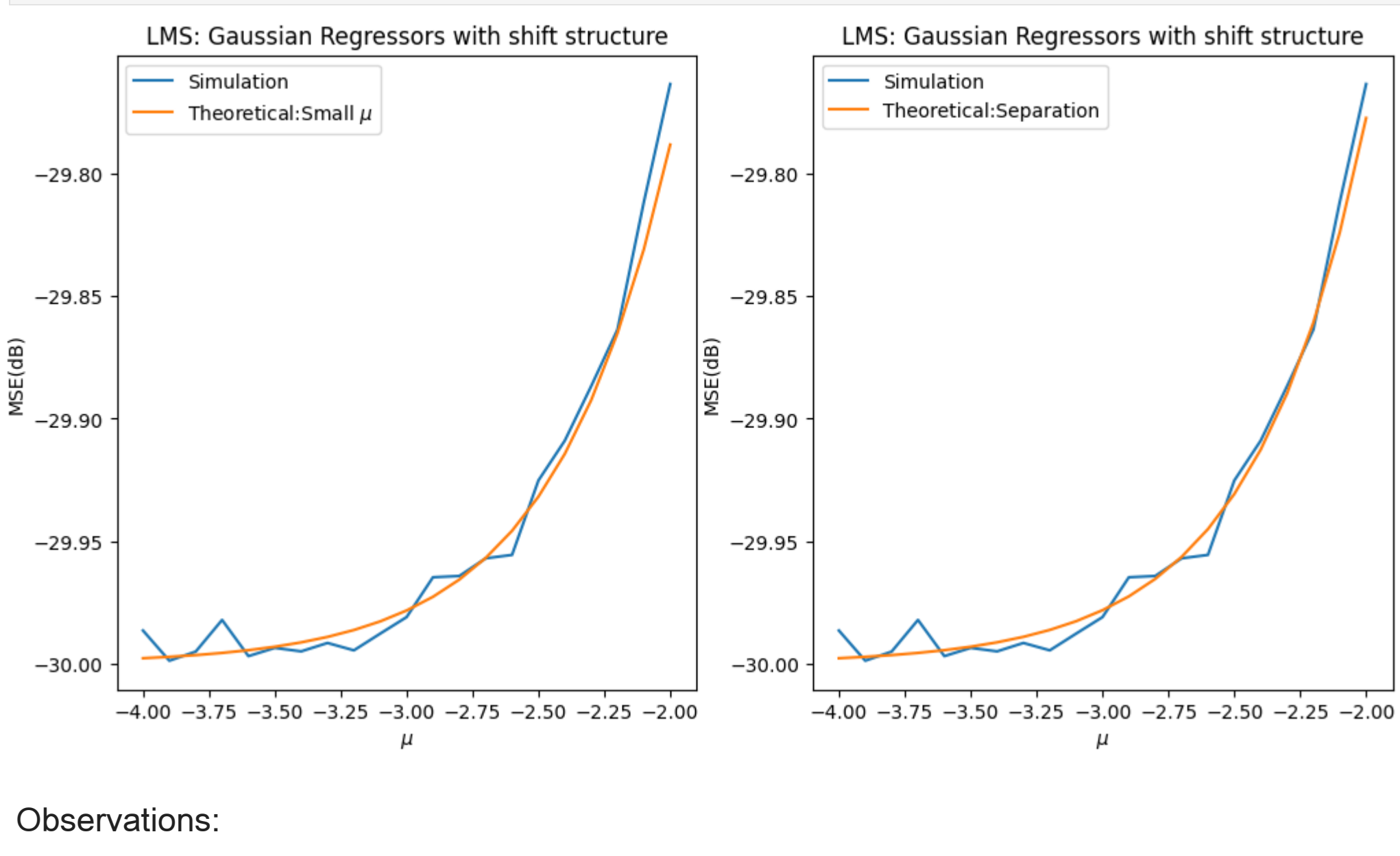
Code:

In [13]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [14]:
```
num_iter = 10**6
num_ensembles = 300
sigma_v = np.sqrt(0.001)
M = 10
w_o = 3*np.random.randn(M)
```

In [15]:
```
### with shift structers data genration and code
mu = np.logspace(-4,-2,21)
a = 0.8
c_s = np.sqrt(1-a**2)

err_lms = np.zeros((len(mu),num_iter))
w_lms = np.zeros((len(mu),num_ensembles,M))
u_iter = np.zeros((len(mu),num_ensembles,M))

for i in range(num_iter):
    u_new = a*u_iter[:,:,0]+c_s*np.random.randn(len(mu),num_ensembles)
    u_temp = np.zeros((len(mu),num_ensembles,M))
    u_temp[:,:,0] = u_new
    u_temp[:,:,1:] = u_iter[:,:,0:-1]
    u_iter = u_temp

    d_iter = np.sum(u_iter*w_o,axis=2)+sigma_v*np.random.randn(len(mu),num_ensembles)

    pred_temp = np.sum(u_iter*w_lms,axis=2)
    err_temp = d_iter-pred_temp

    err_lms[:,i] = np.mean(err_temp**2,axis=1)

    vector_corr_temp = mu*(np.swapaxes(u_iter,0,2))

    w_lms+=np.swapaxes(vector_corr_temp*(err_temp.T),0,2)
err_lms_exp = np.mean(err_lms[0:,-5000:],axis=1)
```

In [16]:
```
## graphs plotting of both theortical and experimental simulations
i = np.arange(0,M,1)
j = np.arange(0,M,1)
I, J = np.meshgrid(i,j)
R_u = a**(np.abs(I-J))

err_lms_small_mu = mu*(sigma_v**2)*np.trace(R_u)/2+(sigma_v)**2
err_lms_separation = mu*(sigma_v**2)*np.trace(R_u)/(2-mu*np.trace(R_u))+(sigma_v)**2

fig, ax = plt.subplots(1,2,figsize=(12,6))

ax[0].plot(np.log10(mu),10*np.log10(err_lms_exp),label='Simulation')
ax[0].plot(np.log10(mu),10*np.log10(err_lms_small_mu),label='Theoretical:Small '+r'$\mu$')
ax[0].set_xlabel(r'$\mu$')
ax[0].set_ylabel('MSE(dB)')
ax[0].set_title('LMS: Gaussian Regressors with shift structure')
ax[0].legend()

ax[1].plot(np.log10(mu),10*np.log10(err_lms_exp),label='Simulation')
ax[1].plot(np.log10(mu),10*np.log10(err_lms_separation),label='Theoretical:Separation')
ax[1].set_xlabel(r'$\mu$')
ax[1].set_ylabel('MSE(dB)')
ax[1].set_title('LMS: Gaussian Regressors with shift structure')
ax[1].legend()

plt.savefig('shiftstr.png')
plt.show()
```



Observations:

few observations that we have listed from the observing the above plots of the both shift and without shift structures

- The first observation we can make regarding both models is that the separation principle approximation gives better results compared to small step size approximation. As explained earlier, the small step size approximation mathematically corresponds to limit of separation principle approximation, so while it is very accurate at small step size, it differs at larger step sizes. The second approximation provides a better fit over larger ranges.
- The second observation is that for the data corresponding to regressors with shift structure, the two approximations give better fits than they did for the case with no shift structure. We try to give a reason for this here:

  The good fit for the expression for small step sizes indicates that $|e_a(i)|$ becomes very small so that the first term in the espression for exact EMSE can be neglected, and it also becomes independent of $||u_i||^2$ at steady state. But why does this occur for inputs with shift structure?

  First of all we have to realise that LMS is a stochastic algorithm. We do not have access to the correlation matrices and have to calculate these. Not only that our estimate is based on the instantaneous sample correlation, which itself is a random matrix. When we have a guess for w at some stage, the degree to which we get an improved estimate depends on how much information is shared between adjacent values of u_i.

  For the case of input with no shift structure each u_i, u_j were independent for $i \neq j$. For the case of input with shift structure, since the input is "slid" across(tap delay), there is correlation between not only adjacent rows in u matrix($u_i, u_i + 1$), but all rows within M of each other share information(M is size of tap delay). This extra information in form of correlation helps us achieve a better performance in terms of error, which leads to smaller values of $|e_a(i)|$ at steady state.

  If one thinks of w itself as a random vector at steady state, one could see the process as a Markov process. The probability that in the next iteration the entries of w will deviate more is higher if the u_i does not share any information with u_{i-1} than if u_i and u_{i-1} had common data(shift structure), in which case the new estimate is less likely to deviate more from original.

  What this results in at steady state is lesser variance in estimate of w in shift structure case vs one with no shift structure, which translates to lesser error and hence lesser value of $|e_a(i)|$.

  For this reason, in the second model, both theoretical expressions are a better fit compared to the case of no shift structure.

In [ ]: