B5 - Non-Linear Material Behaviour

Solutions

1. For $\sigma \leq \sigma_y$ the response is elastic and

$$\varepsilon = \frac{\sigma}{E}$$

$$\therefore \quad \varepsilon_y = \frac{\sigma_y}{E} = \frac{250 \times 10^6}{210 \times 10^9} = 1.19 \times 10^{-3}$$

For $\sigma > \sigma_y$ response is elasto-plastic, then

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p = \frac{d\sigma}{E} + \frac{d\sigma}{h} = \frac{h+E}{Eh}\, d\sigma$$

$$\therefore \quad d\sigma = \frac{Eh}{h+E}\, d\varepsilon$$

Therefore stress to achieve a strain of 0.1 is

$$\sigma = \sigma_y + \int_{\varepsilon_y}^{0.1} \frac{Eh}{h+E}\, d\varepsilon = \sigma_y + \frac{Eh}{h+E}\left(0.1 - 1.19\times10^{-3}\right)$$

$$= 250 + \frac{210\times10^3 \times 50}{210\times10^3 + 50}\left(0.1 - 1.19\times10^{-3}\right)$$

$= 254.9$ MPa (Note this is approximately equal to $250 + h \times 0.1$
$= 250 \times 5 = 255$ MPa)

2. Note $\sigma_1 = s_1 + \sigma_m$
$\sigma_2 = s_2 + \sigma_m$
$\sigma_3 = s_3 + \sigma_m$

$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$

$$\therefore \sqrt{\tfrac{1}{2}\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\}} = \sqrt{\tfrac{1}{2}\{(s_1-s_2)^2+(s_2-s_3)^2+(s_1-s_3)^2\}}$$

$$= \sqrt{\tfrac{3}{2}(s_1^2+s_2^2+s_3^2) - \tfrac{1}{2}(s_1+s_2+s_3)^2}$$

$$= \sqrt{\tfrac{3}{2}(s_1^2+s_2^2+s_3^2)} = \sqrt{\tfrac{3}{2} s_{ij} s_{ij}}$$

since $s_1 + s_2 + s_3 = \sigma_1 + \sigma_2 + \sigma_3 - 3\sigma_m = 0$

3. Note in principal co-ordinates

$$d\varepsilon_1^p = \frac{3}{2}\mu \frac{s_1}{\sigma_e} \qquad \text{etc}$$

$\therefore$

$$dp = \left[(s_1-s_2)^2 + (s_2-s_3)^2 + (s_3-s_1)^2\right]^{1/2} \frac{\sqrt{2}}{3}\,\frac{3}{2}\,\frac{\mu}{\sigma_e}$$

$$= \left[\tfrac{1}{2}\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]^{1/2} \frac{\mu}{\sigma_e}$$

$$= \sigma_e \frac{\mu}{\sigma_e} = \mu.$$

4. $$\sigma_e = \sqrt{\tfrac{1}{2}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3^2) + (\sigma_1-\sigma_3)^2\right\}}$$

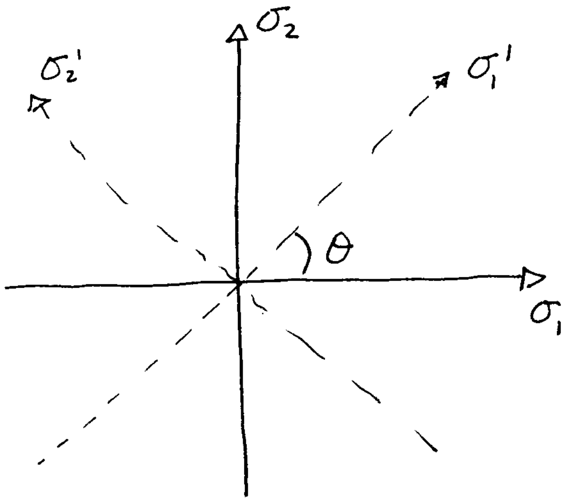
Plane stress $\sigma_3 = 0$

$$\therefore \quad \sigma_e = \sqrt{\tfrac{1}{2}\left\{\sigma_1^2+\sigma_2^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_1^2\right\}}$$

$$= \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$$

At yield $\sigma_e = \sigma_y$

or $$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2 \qquad (1)$$



Consider a set of axes (σ_1', σ_2') that are rotated through an angle $\theta = 45°$ w.r.t. the axes (σ_1, σ_2) in stress space.

$$\begin{bmatrix}\sigma_1' \\ \sigma_2'\end{bmatrix} = \begin{bmatrix} R(\theta) \end{bmatrix} \begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix}$$

$$R(\theta) = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$$ Rotation Matrix

and $$\begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix} = \begin{bmatrix} R(-\theta) \end{bmatrix} \begin{bmatrix}\sigma_1' \\ \sigma_2'\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \begin{bmatrix}\sigma_1' \\ \sigma_2'\end{bmatrix}$$

ie $$\sigma_1 = \frac{\sigma_1' - \sigma_2'}{\sqrt{2}}$$

$$\sigma_2 = \frac{\sigma_1' + \sigma_2'}{\sqrt{2}}$$

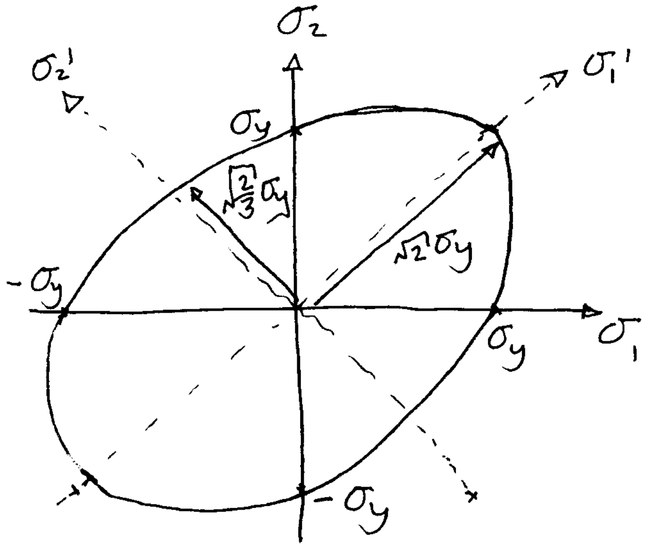
for $\theta = 45°$

Substitute into 1

$$\frac{(\sigma_1' - \sigma_2')^2}{2} + \frac{(\sigma' + \sigma_2')^2}{2} - \frac{(\sigma_1' - \sigma_2')(\sigma_1' + \sigma_2')}{2} = \sigma_y^2$$

ie
$$\frac{\sigma_1'^2}{2\sigma_y^2} + \frac{3\sigma_2'^2}{2\sigma_y^2} = 1$$

This is the equation of an ellipse with major axis $= \sqrt{2}\,\sigma_y$ and minor axis $b = \sqrt{\frac{2}{3}}\,\sigma_y$



Yield surface.

For uniaxial loading – $\sigma_2 = \sigma_3 = 0$

$$d\varepsilon_1^P = \frac{3}{2}\mu\frac{s_1}{\sigma_e} = \mu\frac{\sigma_1}{\sigma_1}$$

$$= \mu = dp$$

$$\sigma_m = \frac{1}{3}\sigma_1$$

$$s_1 = \sigma_1 - \sigma_m = \frac{2}{3}\sigma_1$$

$$d\varepsilon_2^P = d\varepsilon_3^P = -\frac{\mu}{2}\frac{\sigma_1}{\sigma_1}$$

$$= -\frac{\mu}{2} = -\frac{dp}{2}$$

$$s_2 = s_3 = 0 - \sigma_m = -\frac{1}{3}\sigma_1$$

and $\sigma_e = \sigma_1$

5.

If stress remains on yield surface

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial p} dp = 0 \qquad (1)$$

Note $\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial \sigma_e}{\partial \sigma_{ij}} = \frac{\partial \sigma_e}{\partial S_{ij}} = \frac{3}{2} \frac{S_{ij}}{\sigma_e}$

since $\sigma_e = \frac{3}{2} S_{ij} S_{ij}$

and $\frac{\partial f}{\partial p} = -\frac{\partial r}{\partial p} = -b(q-r)$

$\therefore$ (1) becomes

$$\frac{3}{2} \frac{S_{ij}}{\sigma_e} \cdot d\sigma_{ij} - b(q-r)\, dp = 0 \qquad (2)$$

For uniaxial loading

$$\begin{aligned} S_{ij}\, d\sigma_{ij} &= S_{11}\, d\sigma_{11} + S_{22}\, d\sigma_{22} + S_{33}\, d\sigma_{33} \text{ etc} \\ &= S_{11}\, d\sigma_{11} \qquad (\text{since all other } d\sigma_{ij} = 0) \\ &= \frac{2}{3} \sigma\, d\sigma \end{aligned}$$

Also $\sigma_e = \sigma$

Therefore (2) becomes

$$d\sigma - b(q-r)\, dp = 0 \qquad (3)$$

Note
$$\begin{aligned} d\varepsilon &= d\varepsilon^e + d\varepsilon^p = d\varepsilon^e + dp \\ &= \frac{d\sigma}{E} + \frac{d\sigma}{b(q-r)} \qquad (\text{from } (3)) \end{aligned}$$

$$\therefore \quad d\sigma = \frac{E\, b(q-r)}{E + b(q-r)}\, d\varepsilon = E\left(1 - \frac{E}{E + b(q-r)}\right) d\varepsilon.$$

Note $\quad dr = b(q-r)\,dp$

$$\int_0^r \frac{dr}{(q-r)} = \int_0^p b\,dp$$

$$\Big[-\ln(q-r)\Big]_0^r = \Big[bp\Big]_0^p$$

$$\ln\left(\frac{q}{q-r}\right) = bp$$

$$\frac{q-r}{q} = \exp -bp$$

$\therefore$ if $b \to \infty$ $\quad r \to q$ $\quad$ then $(q-r) \to 0$

and $\quad d\sigma = E\left(1 - \frac{E}{E}\right) d\varepsilon = 0$

ie no further increase in stress.

6. From above

$$1 - \frac{r}{q} = \exp -bp$$

$$\therefore \quad r = q(1 - \exp -bp)$$

Yield condition $\quad \sigma - r - \sigma_y = 0$

$$\therefore \quad \sigma = \sigma_y + q(1 - \exp -bp)$$

7. (i) Isotropic Hardening

Note $\beta E = \dfrac{hE}{E+h}$

$\therefore \quad \beta(E+h) = h \qquad$ ie $\quad h = \dfrac{\beta E}{1-\beta} = 0.0204\,E$

Now $\quad \sigma_1 = \sigma_2 = \sigma \qquad \sigma_3 = 0$

$$\sigma_e = \sqrt{\tfrac{1}{2}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}} = \sigma$$

$\sigma_m = \dfrac{2}{3}\sigma \qquad s_1 = \sigma - \dfrac{2}{3}\sigma = \dfrac{1}{3}\sigma = s_2$

$s_3 = 0 - \dfrac{2}{3}\sigma = -\dfrac{2}{3}\sigma.$

Yield condition

$$f = f(\sigma_{ij}, p) = \sigma_e - hp - \sigma_y = 0$$

$$df = \frac{\partial \sigma_e}{\partial \sigma_{ij}} d\sigma_{ij} - h\,dp = 0$$

$$= \frac{\partial \sigma_e}{\partial s_{ij}} d\sigma_{ij} - h\,dp = 0$$

$$\frac{\partial \sigma_e}{\partial s_{ij}} = \frac{3}{2}\frac{s_{ij}}{\sigma_e}$$

$$\therefore \quad \frac{\partial \sigma_e}{\partial s_{ij}} d\sigma_{ij} = \frac{3}{2}\frac{s_1\, d\sigma_1}{\sigma_e} + \frac{3}{2}\frac{s_2\, d\sigma_2}{\sigma_e}$$

$$= \frac{3\sigma}{2\sigma}\left(\frac{1}{3}d\sigma + \frac{1}{3}d\sigma\right) = d\sigma$$

$$\therefore \quad dp = \frac{d\sigma}{h}$$

ie $$p = \frac{\sigma - \sigma_y}{h} = \frac{100}{0.0204 \times 200 \times 10^3} = 0.0245$$

Now $d\varepsilon_1^p = \frac{3}{2}\frac{s_1}{\sigma_e} dp = \frac{1}{2} dp$

and $d\varepsilon_2^p = \frac{1}{2} dp$

Note $\varepsilon_1^e = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = (1-\nu)\frac{\sigma}{E} = \frac{(1-\nu)\,300}{200\times10^3}$

$= 1.05\times10^{-3}$

$\therefore \quad \varepsilon_1 = \varepsilon_1^e + \varepsilon_1^p = \left(\frac{24.5}{2} + 1.05\right)\times10^{-3}$

$= 13.3\times10^{-3} = \varepsilon_2$

After loading, yield condition is

$$f = \sigma_e - 100 - \sigma_y = 0$$

$\therefore$ for uniaxial loading $\sigma_e = \sigma_1$

At yield $\sigma_1 = 100 + 200 = 300$ MPa

(ii) From lecture notes

$$\beta E = \frac{\frac{3}{2}c * E}{\frac{3}{2}c + E}$$

where $dx'_{ij} = c\, d\varepsilon_{ij}^p$ is the law for the translation of the yield surface

$\therefore \quad c = \frac{2}{3}\times 0.0204\,E$

$= 0.0136\,E$

Yield condition

$$f = \bar{\sigma} - \sigma_y = 0$$

where $\bar{\sigma}^2 = \frac{3}{2}(s_{ij} - x'_{ij})(s_{ij} - x'_{ij})$

For equi-biaxial loading

$$S_1 = S_2 = \frac{1}{3}\sigma \qquad S_3 = -\frac{2}{3}\sigma$$

Also $d\varepsilon_1^p = d\varepsilon_2^p$, therefore $dx_1' = dx_2' = dx$

and $x_1' = x_2' = x$

Also $d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p = 0$

$\therefore \quad dx_1' + dx_2' + dx_3' = 0$

and $\quad x_1' + x_2' + x_3' = 0$

$\therefore \quad x_3 = -(x_1' + x_2') = -2x$

Yield condition becomes

$$\bar{\sigma}^2 = \frac{3}{2}\left[(S_1 - x_1')^2 + (S_2 - x_2')^2 + (S_3 - x_3')^2\right]$$

$$= \frac{3}{2}\left[\left(\frac{1}{3}\sigma - x\right)^2 \times 2 + \left(\frac{2}{3}\sigma - 2x\right)^2\right]$$

$$= \left[(\sigma - 3x)^2\right]$$

ie $\quad f = (\sigma - 3x) - \sigma_y = 0$

Note $\quad x = x_1' = c\,\varepsilon_1^p$

$$\therefore \quad \varepsilon_1^p = \frac{\sigma - \sigma_y}{3c} = \frac{100}{3 \times 0.0136 \times 200 \times 10^3}$$

$$= 0.01225$$

$$\varepsilon_1 = \varepsilon_1^p + \varepsilon_1^e = 0.01225 + 1.05 \times 10^{-3}$$

$$= 13.3 \times 10^{-3} = \varepsilon_2$$

Note: this is exactly the same as for isotropic hardening - ie the two results are the same for proportional loading.

(b) (i) isotropic hardening - after loading in (a)

$$r + \sigma_y = 300 \text{ MPa}$$

For uniaxial loading σ_1 (ie $\sigma_2 = \sigma_3 = 0$)

$$\sigma_e = \sigma_1$$

material yields when

$$f = \sigma_e - (r + \sigma_y) = \sigma_1 - 300 = 0$$

ie $\sigma_1 = 300$ MPa at yield

(ii) kinematic hardening - after loading in (a)

$$x_1' = x = \frac{100}{3} = 33.3 \text{ MPa} = x_2'$$

$$x_3' = -2x_1' = -66.7 \text{ MPa}$$

For uniaxial loading σ_1

$$S_1 = \frac{2}{3}\sigma_1 \qquad S_2 = S_3 = -\frac{1}{3}\sigma_1$$

$$\therefore \quad \bar{\sigma}^2 = \frac{3}{2}\Big[\left(\tfrac{2}{3}\sigma_1 - x\right)^2 + \left(-\tfrac{1}{3}\sigma_1 - x\right)^2$$

$$\left(-\tfrac{1}{3}\sigma_1 + 2x\right)^2\Big]$$

$$= \frac{3}{2}\Big[\frac{4}{9}\sigma_1^2 - \frac{4}{3}\sigma_1 x + x^2 + \frac{1}{9}\sigma_1^2 + \frac{2}{3}\sigma_1 x + x^2$$

$$+ \frac{1}{9}\sigma_1^2 - \frac{4}{3}\sigma_1 x + 4x^2\Big]$$

$$= \left[\sigma_1^2 - 3\sigma_1 x + 9x^2\right]$$

at yield $\quad f = \bar{\sigma} - \sigma_y = 0$

or $\quad \bar{\sigma}^2 - \sigma_y^2 = 0$

ie $\quad \sigma_1^2 - 100\sigma_1 + 100^2 - 200^2 = 0$

or $\quad \sigma_1^2 - 100\sigma_1 - 3 \times 100^2 = 0$

$$\therefore \quad \sigma_1 = 100\left(\frac{1 \pm \sqrt{1+12}}{2}\right)$$

$$= +230 \text{ MPa} \quad \text{or} \quad -130 \text{ MPa}$$

(if loaded in tension) (if loaded in compression)

8. Equilibrium eqn

$$\frac{d\sigma_r}{dr} = \frac{\sigma_\theta - \sigma_r}{r}$$

Assume $\quad \sigma_\theta - \sigma_r = \frac{2}{\sqrt{3}}\sigma_y$

Then $\quad \frac{d\sigma_r}{dr} = \frac{2}{\sqrt{3}}\frac{\sigma_y}{r}$

and $\quad \sigma_r = \frac{2}{\sqrt{3}}\sigma_y \ln r + A$

Now $\quad \sigma_r = 0 \quad$ at $r = b$

$$\therefore \quad A = -\frac{2}{\sqrt{3}}\sigma_y \ln b$$

and $\quad \sigma_r = -p_i \quad$ at $\quad r = a$

$$\therefore \quad -p_i = \frac{2}{\sqrt{3}}\sigma_y \ln\left(\frac{a}{b}\right)$$

ie $\quad p_i = \frac{2}{\sqrt{3}}\sigma_y \ln\left(\frac{b}{a}\right)$

Now $\sigma_r = \frac{2}{\sqrt{3}} \sigma_y \ln \frac{r}{b}$

and $\sigma_\theta = \frac{2}{\sqrt{3}}\sigma_y + \sigma_r$

$= \frac{2}{\sqrt{3}} \sigma_y \left(1 + \ln \frac{r}{b}\right)$

Unloading – stresses reduce elastically

For elastic body $\sigma_r = A + \frac{B}{r^2}$

$\sigma_r = 0$ at $r = b$ $\therefore$ $A = -\frac{B}{b^2}$

$\sigma_r = -p$ at $r = a$

$\therefore$ $-p = -\frac{B}{b^2} + \frac{B}{a^2} = \frac{B(b^2 - a^2)}{a^2 b^2}$

$\therefore$ $B = -\frac{pa^2b^2}{(b^2-a^2)}$

and $A = \frac{pa^2}{(b^2-a^2)}$

$\therefore$ $\sigma_r = \frac{pa^2}{(b^2-a^2)}\left(1 - \frac{b^2}{r^2}\right)$

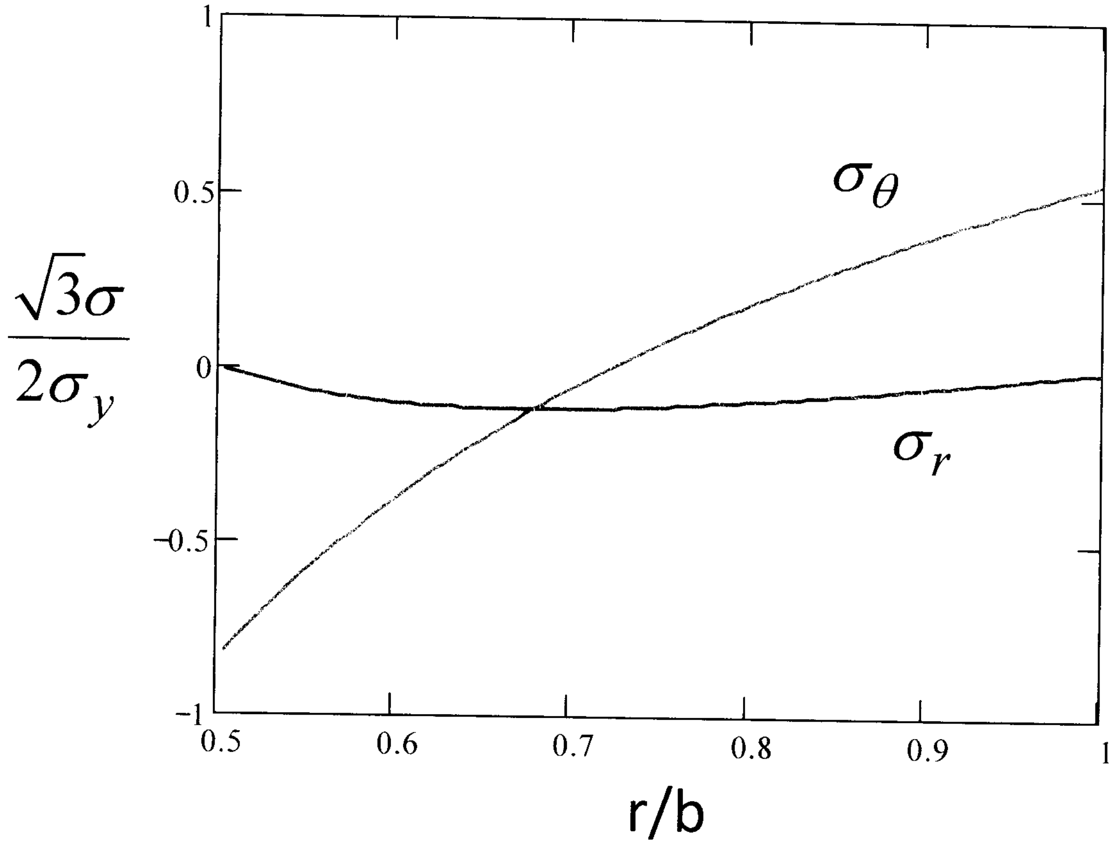
and $\sigma_\theta = A - \frac{B}{r^2} = \frac{pa^2}{(b^2-a^2)}\left(1 + \frac{b^2}{r^2}\right)$

$\therefore$ unloading from $p = p_L = \frac{2\sigma_y \ln\left(\frac{b}{a}\right)}{\sqrt{3}}$ gives

a stress state

$$\sigma_r = \frac{2}{\sqrt{3}}\sigma_y\left[\ln\left(\frac{r}{b}\right) - \ln\left(\frac{b}{a}\right)\cdot\frac{1}{\left(\frac{b^2}{a^2}-1\right)}\left(1 - \frac{b^2}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{2}{\sqrt{3}}\sigma_y\left[1 + \ln\left(\frac{r}{b}\right) - \ln\left(\frac{b}{a}\right)\cdot\frac{1}{\left(\frac{b^2}{a^2}-1\right)}\left(1 + \frac{b^2}{r^2}\right)\right]$$



These fields are plotted above for $\frac{b}{a} = 2$.

If pressure is increased, stresses increase elastically until $p = p_L$, when $\sigma_e = \sigma_y$ throughout the wall thickness.

9. (a) Let du be the horizontal displacement of block A.

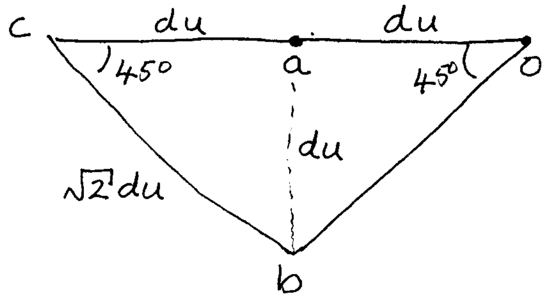
Volume in = Volume out

$\therefore \quad du\, 2h = du_c\, h$

$\uparrow$ displacement of block C

$\therefore \quad du_c = 2\,du$

Hodograph



Upper bound

$$\frac{F_L}{2}\,du \leq \sum k\, l\, du_e$$

$$= k\, l_{AB}\, du_{AB} + k\, l_{BC}\, du_{BC}$$

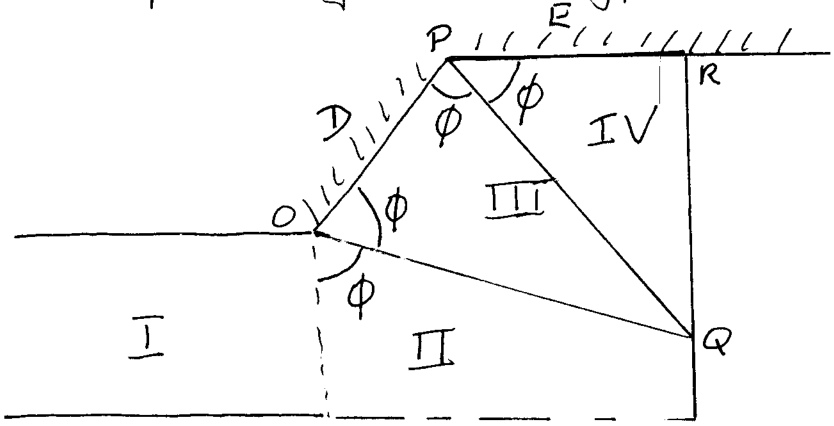
$$= k\, 2h\, du + k\sqrt{2}\,h\,\sqrt{2}\,du$$

$$= 4\,k\,h\,du$$

$$\therefore \quad F_L \leq 8\,k\,h$$

(b) Choose a stress field in which the stress is uniform in blocks of material, but the stress parallel to a boundary between any 2 blocks can be discontinuous.

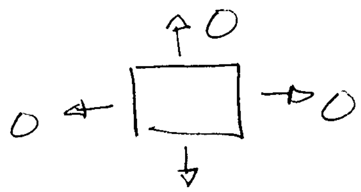
Simplest field of this type is



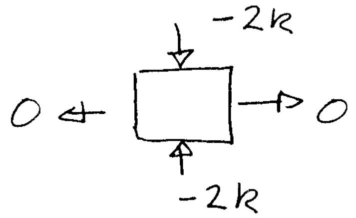
$2\phi = 135^\circ$

$\phi = 67.5^\circ$

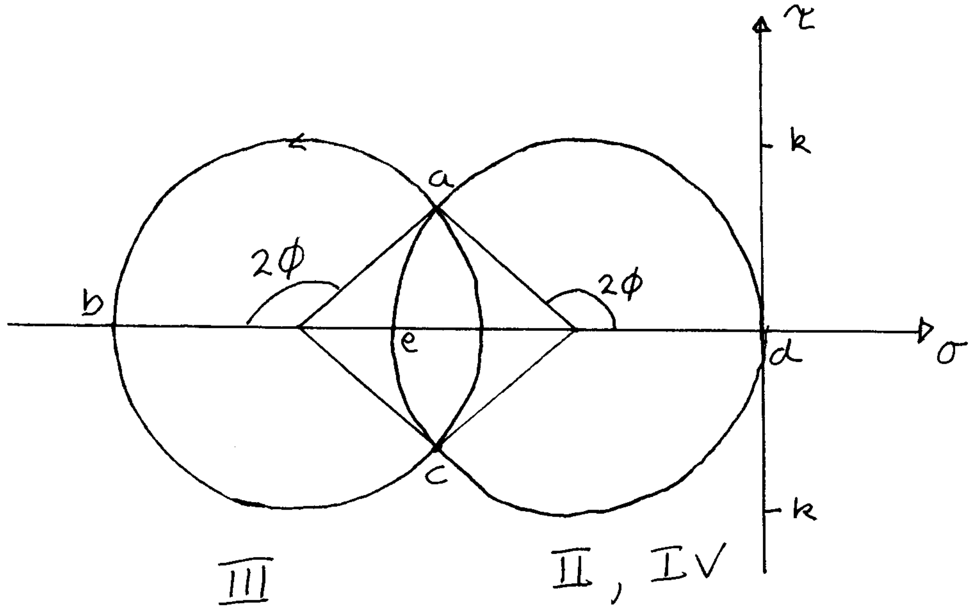
In I



In II



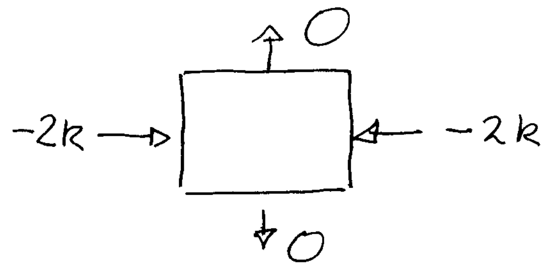
Stress state in III must be such that only the stress component parallel to boundary is discontinuous. Also, this stress state must satisfy the boundary condition that the shear stress is zero at the interface with the die wall along D.



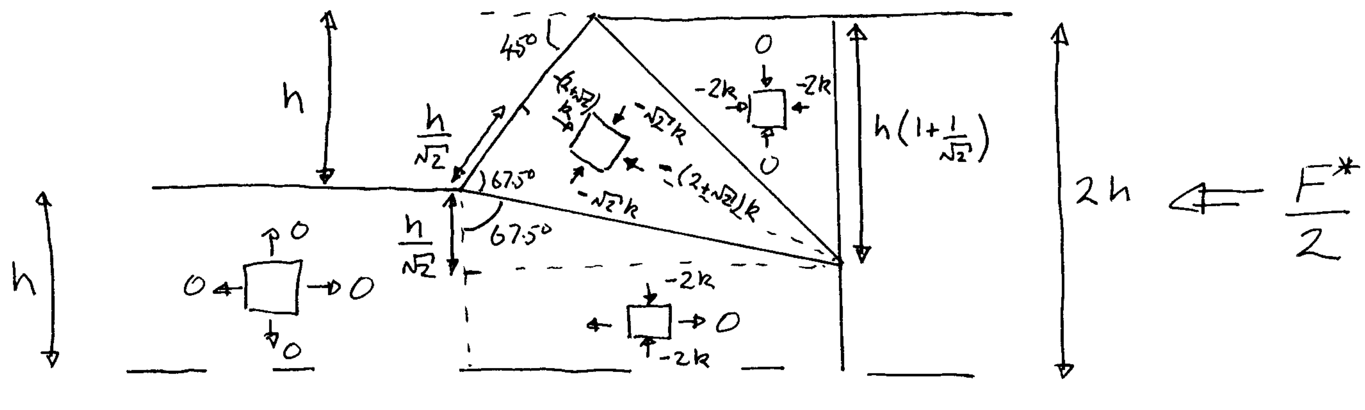
$2\phi = 135^o$

The stress state in II is shown on the Mohr's circle above. At the interface with III the shear stress along the interface and the component of stress normal to the interface are given by point a on the circle. Mohr's circle for region III must pass through this point. If the stress state is at yield in III there are two circles that can be drawn through this point that satisfy the yield condition - these are the circles labelled II and III above. We assume that the stress state in III is given by circle III. Now consider a series of plane through the corner O of the extruded slab. As we rotate the plane from the II/III interface to boundary D the stresses on the plane are given by a point that rotates around the circle from a to b.

Now continue to rotate the plane anticlockwise, but now take point P as the pivot point. The stress state on the rotating plane now translates around Mohr's circle from b to c, which corresponds with the interface III/IV. There can be a discontinuity of stress at this interface. If the stress state in IV, is given by the same Mohr's circle as for II, as the plane continues to rotate the stress state on the plane translates around Mohr's circle IV from c to d, which gives the stress on the boundary E. This satisfies the boundary condition of zero shear stress. The stress on a plane perpendicular to the boundary is given by point e. Thus the stress state in IV is given by



In summary



From geometry

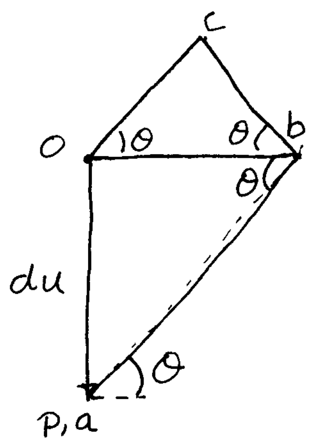
$$\text{Force} \quad \frac{F^*}{2} = h\left(1+\frac{1}{\sqrt{2}}\right)2k$$

$$F^* = 6.83\,kh \qquad \text{Lower bound}$$

$$\therefore \quad 6.83kh \le F_L \le 8kh$$

(c) Can improve upper bound by considering more complex mechanisms which consist of a larger number of blocks that slide over each other.
Similarly, can improve the lower bound by dividing the body into a larger number of blocks in which in which the stress is uniform.

10. Hodograph for half the mechanism



Internal energy dissipated

$$= \Sigma k \, l \, du_L$$

$$= 2k \left(l_{AD} \, ab + l_{OB} \, ob + l_{BC} \, bc + l_{OC} \, oc \right)$$

for 2 halves of mechanism.

From geometry $l_{AB} = l_{BC} = l_{OC} = \dfrac{D}{2\cos\theta}$

$$l_{OB} = D$$

From geometry of hodograph

$$ab = \frac{du}{\sin\theta} \qquad ob = du \cot\theta$$

$$oc = bc = \frac{du}{2} \frac{\cot\theta}{\cos\theta} = \frac{du}{2\sin\theta}$$

$$\therefore \Sigma k \, l \, du_L = 2k \left(\frac{D \, du}{2\cos\theta \sin\theta} + D \, du \frac{\cos\theta}{\sin\theta} + \frac{2D}{2\cos\theta} \frac{du}{2\sin\theta} \right)$$

$$= 2D\,du\,k\left(\frac{1}{\cos\theta\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= 2D\,du\,k\left(\frac{1+\cos^2\theta}{\cos\theta\sin\theta}\right)$$

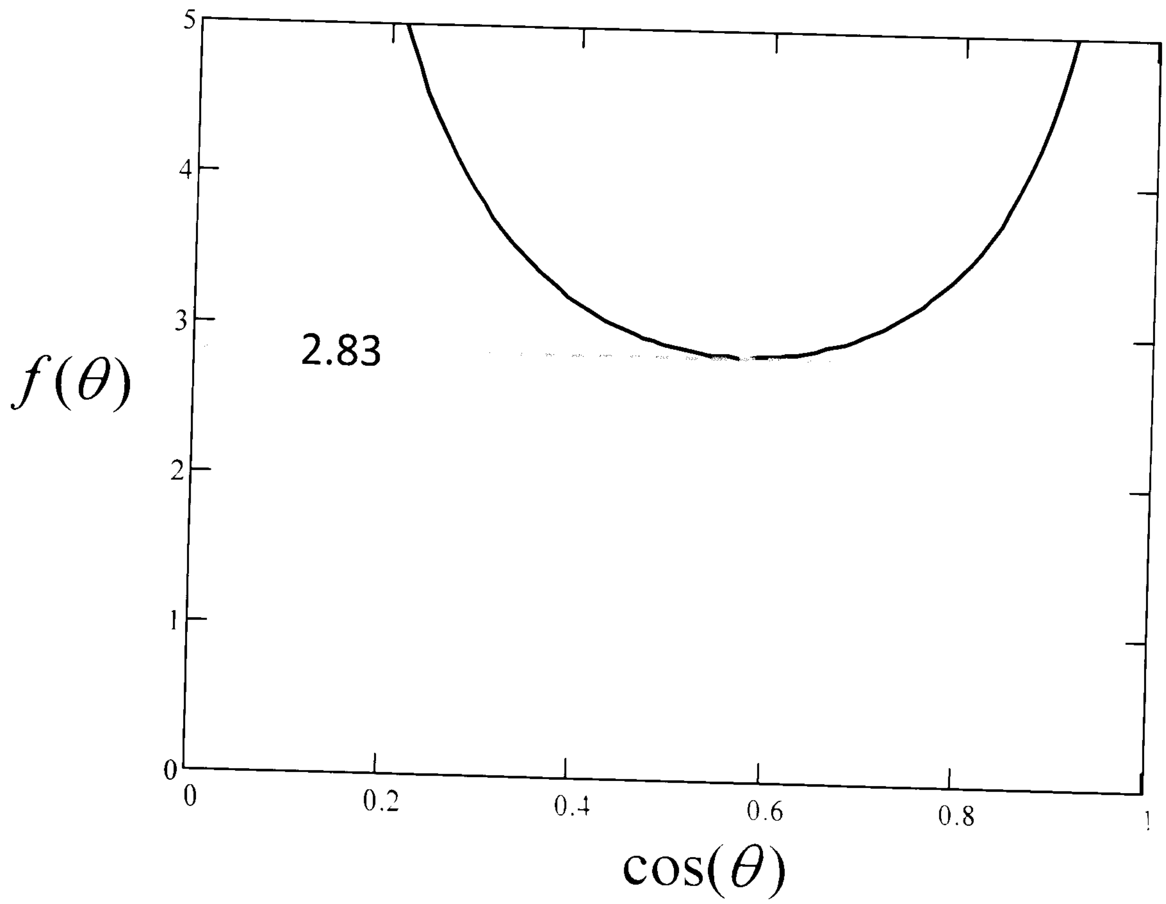
Work done by limit load

$$P_L\,du$$

$$\therefore \quad P_L\,du \leq 2du\,D\,k\left(\frac{1+\cos^2\theta}{\cos\theta\sin\theta}\right)$$

ie $\quad P_L \leq 2Dk\left(\frac{1+\cos^2\theta}{\cos\theta\sin\theta}\right) = 2Dk\,f(\theta)$

This is plotted below as a function of $\cos\theta$



The optimum bound is when

$$f(\theta) = \frac{(1+\cos^2\theta)}{\cos\theta\sin\theta}$$

is a minimum, or

$$f^{-1}(\theta) = \frac{\cos\theta\sin\theta}{(1+\cos^2\theta)}$$ is a maximum.

Write $x = \cos\theta$

Then $$f^{-1}(x) = \frac{x\,(1-x^2)^{1/2}}{(1+x^2)}$$

Max when

$$\frac{df^{-1}}{\partial x} = \frac{(1+x^2)\left[(1-x^2)^{1/2} - x^2(1-x^2)^{-1/2}\right] - 2x^2(1-x^2)^{1/2}}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\left[(1-x^2) - x^2\right] - 2x^2(1-x^2)}{(1+x^2)^2(1-x^2)^{1/2}} = 0$$

ie when $(1+x^2)(1-2x^2) - 2x^2(1-x^2) = 0$

ie when $1 - 3x^2 = 0$

ie $x = \frac{1}{\sqrt{3}}$

Therefore optimum bound when $\cos\theta = \frac{1}{\sqrt{3}}$ ie $\theta = 54.73^\circ$

Then

$$f(\theta) = 2.83$$

and

$$P_L \leq 5.66\,D\,k$$

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