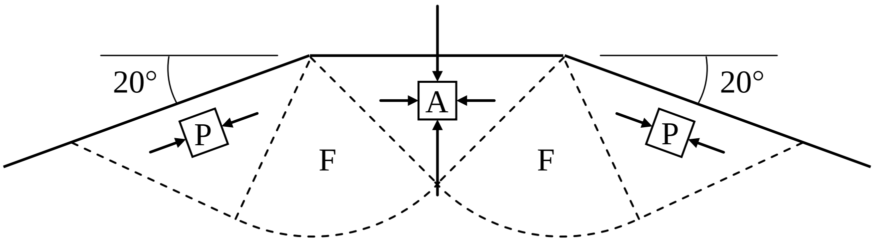
B10 Soil Mechanics Applications
Examples Sheet 1
Solutions (CMM)

1.

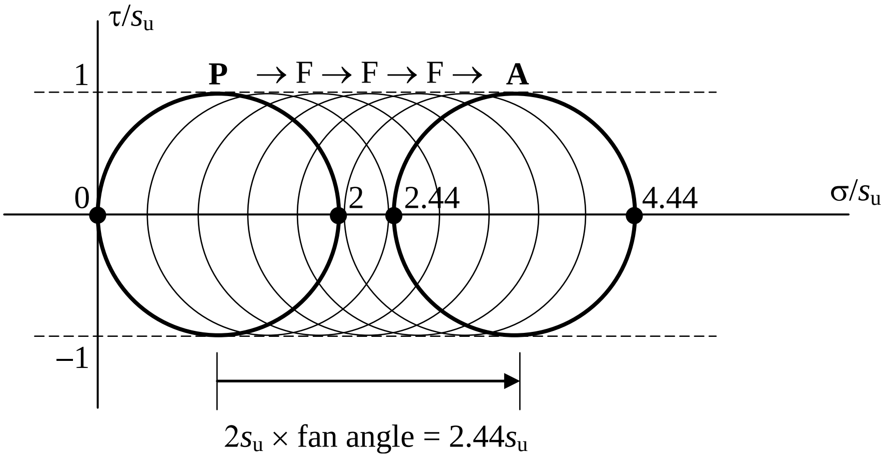


The optimal lower bound stress field consists of two passive zones (P), two fan zones (F) and an active zone (A). In all zones the full shear strength of the soil is used, i.e. the Mohr's circles of stress all touch the undrained strength envelope. Analysis of principal stresses:

P: $\sigma_{max} = 2s_u$ (parallel to surface), $\sigma_{min} = 0$ (perpendicular to surface).

F: smooth rotation of the principal stresses through 70° = 7π/18 rad, hence they increase by $2s_u \times 7\pi/18 = 7\pi s_u/9 \approx 2.44s_u$. See eqn (7) in the lecture notes.

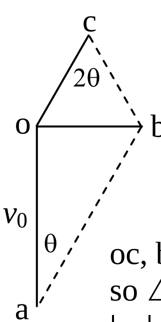
A: $\sigma_{max} = 2s_u + 2.44s_u = 4.44s_u$ (vertical), $\sigma_{min} = 0 + 2.44s_u = 2.44s_u$ (horizontal).



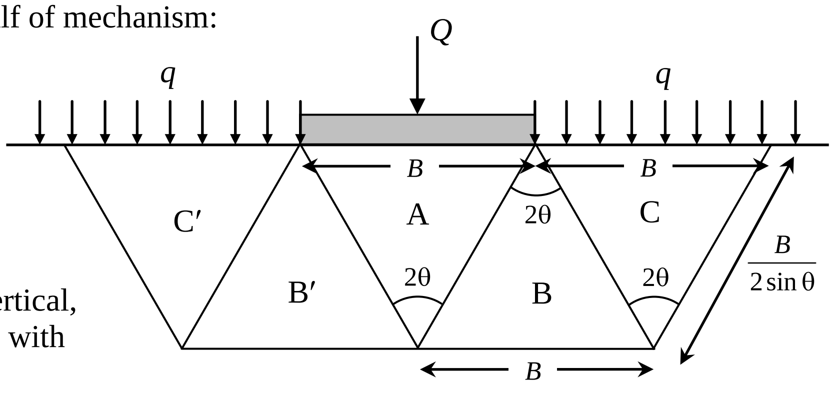
So the bearing capacity $Q = B \times (\sigma_{max}$ in active zone$) = 4.44Bs_u$.

The second problem is similar except the fan angle is now 120° = 2π/3 rad, so the passive Mohr's circle is shifted by $2s_u \times 2\pi/3 = 4\pi s_u/3 \approx 4.19s_u$ to the right. So the bearing capacity $Q = B \times (\sigma_{max}$ in active zone$) = 6.19Bs_u$.

2. Velocity diagram for RH half of mechanism:



oc, bc both at θ to vertical, so △ocb is isosceles with |oc| = |bc|



Let $|\text{oa}| = v_0$, so other block speeds are $|\text{ob}| = v_0 \tan\theta$ and $|\text{oc}| = \dfrac{v_0 \tan\theta}{2} \Big/ \sin\theta = \dfrac{v_0}{2\cos\theta}$.

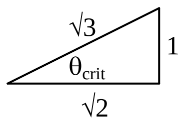
Velocity jumps between blocks have magnitude $|\text{ab}| = \dfrac{v_0}{\cos\theta}$ and $|\text{bc}| = |\text{oc}| = \dfrac{v_0}{2\cos\theta}$.

$$
\begin{aligned}
W_{\text{int}} &= \underbrace{2\times}_{\text{symm}} s_\text{u}\left[\underbrace{v_0 \tan\theta \times B}_{\text{B rel O}} + \underbrace{\frac{v_0}{2\cos\theta}\times\frac{B}{2\sin\theta}}_{\text{C rel O}} + \underbrace{\frac{v_0}{\cos\theta}\times\frac{B}{2\sin\theta}}_{\text{B rel A}} + \underbrace{\frac{v_0}{2\cos\theta}\times\frac{B}{2\sin\theta}}_{\text{C rel B}}\right] \\
&= 2Bs_\text{u}v_0\left[\tan\theta + \frac{1}{\sin\theta\cos\theta}\right] \\
&= 2Bs_\text{u}v_0\left[\tan\theta + 2\csc 2\theta\right]
\end{aligned}
$$

Now the vertical component of the velocity of blocks C and C′ is $v_0/2$ upwards, so the downward surcharge q does negative work. Gravity does positive work on block A, zero work on blocks B and B′, and negative work on blocks C and C′.

$$
\begin{aligned}
W_{\text{ext}} &= \underbrace{Q\times v_0}_{\text{load}} - \underbrace{2\times qB\times\frac{v_0}{2}}_{\text{surcharge}} + \underbrace{\gamma A\times v_0}_{\text{grav A}} - \underbrace{2\times\gamma A\times\frac{v_0}{2}}_{\text{grav C, C}'} \qquad \text{where } A = \text{block area (all same)} \\
&= Qv_0 - qBv_0
\end{aligned}
$$

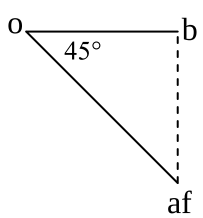
Note how there is no *net* work done by gravity. Setting $W_{\text{ext}} = W_{\text{int}}$ gives the upper bound solution $Q = 2Bs_\text{u}\left[\tan\theta + 2\csc 2\theta\right] + qB$. Differentiating w.r.t. θ, this is minimized when $\sec^2\theta - 4\cot 2\theta \csc 2\theta = 0$ which after some trigonometric manipulations gives the solution $\sin\theta_{\text{crit}} = 1\big/\sqrt{3}$. Hence (see triangle sketch) $\tan\theta_{\text{crit}} = 1\big/\sqrt{2}$ and $\cos\theta_{\text{crit}} = \sqrt{2}\big/\sqrt{3}$, thus $2\csc 2\theta_{\text{crit}} = \dfrac{1}{\sin\theta_{\text{crit}}\cos\theta_{\text{crit}}} = 3\big/\sqrt{2}$.



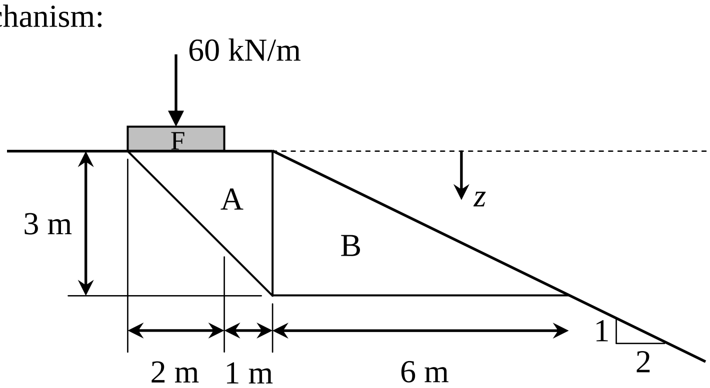
Back-substituting with $\theta = \theta_{\text{crit}}$ we get $Q = 2Bs_\text{u}\left[1\big/\sqrt{2} + 3\big/\sqrt{2}\right] + qB = 4\sqrt{2}Bs_\text{u} + qB$.

For part (b), put the bearing capacity in the form $Q/B = 4\sqrt{2}s_\text{u} + q$ and compare with Terzaghi's equation, eqn (17) in notes. Evidently $N_c = 4\sqrt{2}$ (10% higher than exact solution 2 + π), $N_q = 1$ (same as exact solution) and $N_\gamma = 0$ (same as exact solution).

3. Velocity diagram for two block mechanism:



Note: the footing F is attached to block A and moves with it



Let v_0 be the *downward component* of the velocity of the footing, so the block speeds are $|\mathrm{oa}| = \sqrt{2}v_0$ and $|\mathrm{ob}| = v_0$, and the velocity jump between blocks A and B has magnitude $|\mathrm{ab}| = v_0$. The shear strength at a depth of 3 m is 16 kPa. Note also that the *average* shear strength between the surface and a depth of 3 m is 13 kPa (this will be needed to calculate the dissipation along the edges of block A). Using units of kN and m we have:

$$\begin{aligned} W_{\mathrm{int}} &= \overbrace{13 \times \sqrt{2}v_0 \times 3\sqrt{2}}^{\text{A rel O}} + \overbrace{16 \times v_0 \times 6}^{\text{B rel O}} + \overbrace{13 \times v_0 \times 3}^{\text{B rel A}} \\ &= 213 v_0 \end{aligned}$$

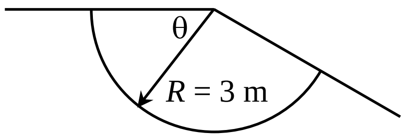
Gravity does positive work on block A (which has downward velocity component v_0) but no work on block B (which is moving horizontally).

$$\begin{aligned} W_{\mathrm{ext}} &= \overbrace{Q \times v_0}^{\text{load}} + \overbrace{20 \times 4.5 \times v_0}^{\text{grav A}} \qquad \text{since } \gamma = 20 \text{ kN/m}^3 \text{ and area of block A} = 4.5 \text{ m}^2 \\ &= Q v_0 + 90 v_0 \end{aligned}$$

Setting $W_{\mathrm{ext}} = W_{\mathrm{int}}$ gives Q = 123 kN/m. Thus the two block mechanism gives an upper bound factor of safety = 123/60 = 2.05.

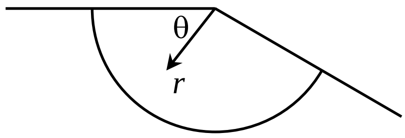
For the other mechanism, let the angular velocity of the rotating block be ω (anticlockwise +ve). The velocity jump along the slip surface is $R\omega$ where R = 3 m, so the dissipation is

$$\begin{aligned} W_{\mathrm{int}} &= \int_{\theta=0}^{\theta_{\max}} s_{\mathrm{u}} \times R\omega \times R\mathrm{d}\theta \\ &= \int_{\theta=0}^{\pi - \tan^{-1}(1/2)} \left(10 + 2 \times 3 \sin\theta\right) \times 3\omega \times 3\mathrm{d}\theta \\ &= 343.3\omega \end{aligned}$$



External work is done by the applied load (located at a radius of 2 m) and by gravity. The simplest way to deal with the latter is to consider a typical element in polar coordinates, and integrate the product of its weight ($\gamma \mathrm{d}A = \gamma r \mathrm{d}r \mathrm{d}\theta$) and the downward component of its velocity ($r\omega \times \cos\theta$) over the whole area of the rotating block. This gives

$$\begin{aligned} W_{\mathrm{ext}} &= Q \times 2\omega + \iint \gamma \mathrm{d}A \times v_{\downarrow} \\ &= 2Q\omega + \int_{\theta=0}^{\theta_{\max}} \int_{r=0}^{R} \gamma r \mathrm{d}r \mathrm{d}\theta \times r\omega\cos\theta \\ &= 2Q\omega + 20\omega \int_{\theta=0}^{\pi - \tan^{-1}(1/2)} \int_{r=0}^{3} r^2 \cos\theta \mathrm{d}r \mathrm{d}\theta \\ &= 2Q\omega + 80.5\omega \end{aligned}$$



Setting $W_{\mathrm{ext}} = W_{\mathrm{int}}$ gives Q = 131.4 kN/m, i.e. a factor of safety of 131.4/60 = 2.19. This is higher, and therefore a less critical upper bound solution, than the result from the two block mechanism.

Are further calculations needed? Definitely, since these are extremely simple mechanisms, and the factor of safety obtained from a more complicated upper bound mechanism (e.g. with 4 or 5 sliding blocks) could well be considerably lower. In practice a finite element analysis would probably be performed at this point to identify the most critical failure mechanism, and to obtain a close approximation to the exact collapse load (note that elastic-plastic FE analysis does not give a strict lower or upper bound plasticity solution, but as the mesh is refined it should converge to the exact solution). In soil mechanics there is so much uncertainty about material properties that we tend to target a safety factor of at least 2 for the ultimate limit state – particularly in situations like this where failure would be serious.

4. Terzaghi's equation is $\frac{Q}{B} = cN_c + qN_q + \frac{1}{2}\gamma B N_\gamma$. Here B = 2 m. The bearing capacity factors are given in eqns (18)-(20) in the lecture notes. See also the table about the correct use of Terzaghi's equation for drained and undrained analyses – very important!

(a) Undrained analysis, so take $c = s_u$ = 105 kPa and $\phi = \phi_u = 0$ for the strength parameters. We also require the *total* surcharge at footing level ($q = \gamma D$ = 19 kN/m^3 × 1.2 m = 22.8 kPa) and the *total* unit weight of the soil below footing level (γ = 21 kN/m^3; in fact this is not relevant since $N_\gamma = 0$ when $\phi = 0$). When $\phi = 0$, the b.c. factors are $N_c = 2 + \pi$, $N_q = 1$ and $N_\gamma = 0$, so $Q = 2 \times \left[105 \times (2+\pi) + 22.8 \times 1 + 0\right] = 1125\,\text{kN/m}$.

(b) Drained analysis, so take $c = c' = 0$ and $\phi = \phi' = 28°$ for the strength parameters. To solve the various cases we require the *effective* surcharge at footing level ($q' = \gamma' D$) and the *effective* unit weight of the soil below footing level (γ'). These are as follows:

Case	Effective unit weight γ' above footing level (kN/m^3)	Effective surcharge q' (kN/m^2)	Effective unit weight γ' below footing level (kN/m^3)
(i)	dry → 19	19 × 1.2 = 22.8	dry → 19
(ii)	dry → 19	19 × 1.2 = 22.8	sat. → 21 – 9.8 = 11.2
(iii)	sat. → 21 – 9.8 = 11.2	11.2 × 1.2 = 13.44	sat. → 21 – 9.8 = 11.2

When $\phi = 28°$, the b.c. factors are $N_c = 25.80$ (in fact not relevant here since c = 0), $N_q = 14.72$ and $N_\gamma = 10.94$, so the respective bearing capacities are

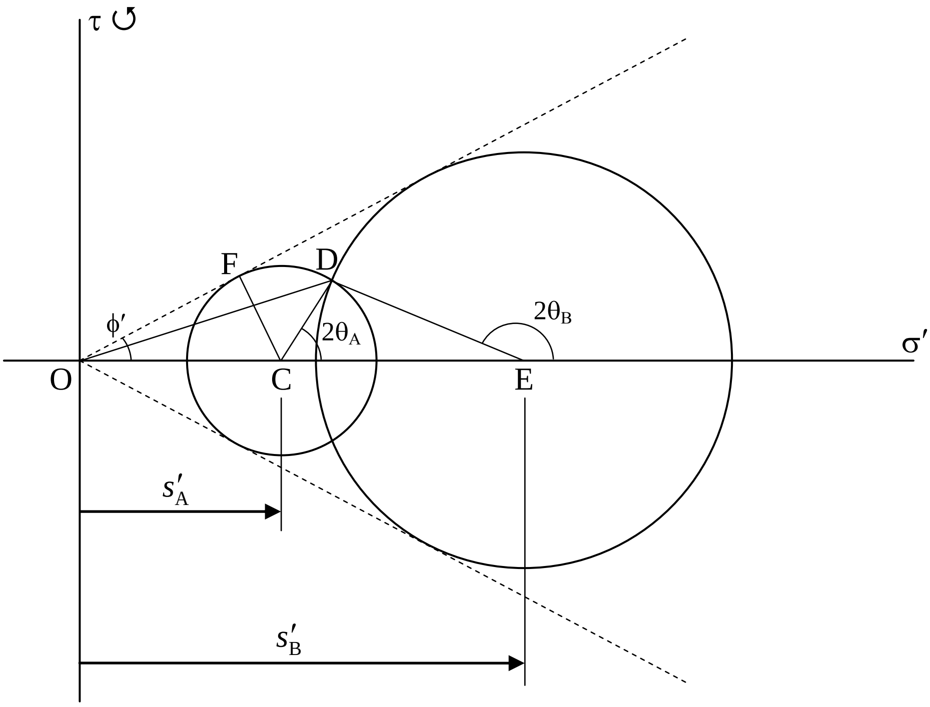
(i) $Q = 2 \times \left[0 + 22.8 \times 14.72 + \frac{1}{2} \times 19 \times 2 \times 10.94\right] = 1087\,\text{kN/m}$

(ii) $Q = 2 \times \left[0 + 22.8 \times 14.72 + \frac{1}{2} \times 11.2 \times 2 \times 10.94\right] = 916.3\,\text{kN/m}$

(iii) $Q = 2 \times \left[0 + 13.44 \times 14.72 + \frac{1}{2} \times 11.2 \times 2 \times 10.94\right] = 640.8\,\text{kN/m}$

However, in case (iii) we need to account for the fact that there is also a non-zero pore water pressure acting on the underside of the footing, providing an additional bearing capacity of $Q = \gamma_w \times D \times B = 9.8 \times 1.2 \times 2 = 23.5\,\text{kN/m}$, thus the net $Q = 640.8 + 23.5 = 664.3\,\text{kN/m}$. Although the strength of a drained soil (and hence its bearing capacity) must be worked out in terms of *effective* stresses ($\sigma' = \sigma - u$), once this calculation has been performed, overall equilibrium between the applied load Q and soil underneath the footing must of course be formulated in terms of *total* stresses ($\sigma = \sigma' + u$). Hence the need to "add u back on".

5. The analysis is similar to that on p. 7 of the lecture notes, but with the drained strength envelope instead of the undrained one.



As before we have $\angle CDE = 2\theta_B - 2\theta_A$.

Note CD = CF = radius of Mohr's circle for side A = $s'_A \sin\phi'$.

Applying the sine rule to △CDE,

$$\frac{s'_B - s'_A}{\sin(2\theta_B - 2\theta_A)} = \frac{s'_A \sin\phi'}{\sin(\pi - 2\theta_B)}$$

$$\frac{\delta s'}{\sin(2\delta\theta)} = \frac{s'_A \sin\phi}{\sin(2\theta_B)}$$

In the case of an infinitesimal discontinuity, i.e. as δθ becomes very small, it is quite easy to see that point D on the Mohr diagram approaches the strength envelope, such that △ODE becomes (in the limit) a right-angled triangle with ∠ODE = $\pi/2$. Hence as $\delta\theta \to 0$, the exterior angle $2\theta_B \to \pi/2 + \phi'$.

Inserting these limiting values,

$$\frac{\delta s'}{\sin(2\delta\theta)} = \frac{s'_A \sin\phi'}{\sin(\pi/2 + \phi')}$$

$$\frac{\delta s'}{\sin(2\delta\theta)} = \frac{s'_A \sin\phi'}{\cos\phi'} = s'_A \tan\phi'$$

Finally, as $\delta\theta \to 0$ we can replace $\sin(2\delta\theta)$ by $2\delta\theta$. This gives

$$\frac{\delta s'}{2\delta\theta} = s'_A \tan\phi'$$

$$\frac{\delta s'}{s'_A} = 2\tan\phi'\delta\theta$$

which in the infinitesimal limit becomes $\dfrac{ds'}{s'} = 2\tan\phi'\, d\theta$, as required.

6. The load is spread three-dimensionally, in a pyramid-like manner. The load-bearing area at a depth z below the surface is a square of side x, where $x = 1.5 + 2 \times (z - 0.5) \times \frac{1}{3}$ m. The total stress σ_v at this depth increases by $30/x^2$ kPa. At the end of consolidation – which may take some years – the effective stress σ'_v will have increased by the same amount.

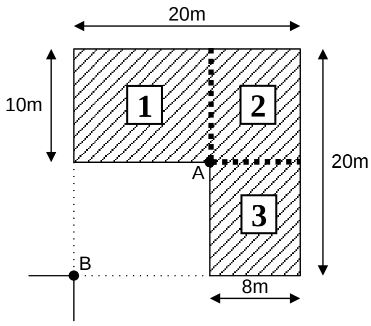
Divide the clay into three layers and tabulate:

Layer	Depth z at middle (m)	Side x (m)	Area x^2 (m^2)	$\delta\sigma_v$ (kPa)
1	2.25	2.67	7.11	4.22
2	2.75	3.00	9.00	3.33
3	3.25	3.33	11.1	2.70

The total consolidation settlement ρ is given by $\sum_{i=1}^{3} \rho_i = \sum_{i=1}^{3} h_i m_{v,i} \delta\sigma'_{v,i}$. Since h and m_v are the same for all three layers, they can be pulled out of the sum, giving

$$\rho = 0.5 \text{ m} \times 1.6 \times 10^{-3} \text{ m}^2/\text{kN} \times (4.22 + 3.33 + 2.7) \text{ kN/m}^2 = 0.0082 \text{ m} \quad [= 8.2 \text{ mm}]$$

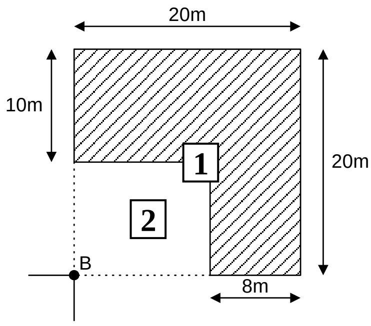
7. Vertical stress increase at depth Z = 10 m below point A:



Region	L (m)	B (m)	$m = L/Z$	$n = B/Z$	I_σ
1	12	10	1.2	1.0	0.185
2	8	10	0.8	1.0	0.160
3	8	10	0.8	1.0	0.160

$$\delta\sigma_v = 25 \times (0.185 + 0.160 + 0.160) = 12.6 \text{ kPa}$$

Vertical stress increase at depth Z = 10 m below point B:



Region	L (m)	B (m)	$m = L/Z$	$n = B/Z$	I_σ
1 (+)	20	20	2.0	2.0	0.232
2 (–)	12	10	1.2	1.0	0.185

$$\delta\sigma_v = 25 \times (0.232 - 0.185) = 1.2 \text{ kPa}$$

In this problem the soil is free-draining, so the increment in total stress is immediately transferred to the soil skeleton as effective stress (consolidation is virtually instantaneous).

8. The *in situ* vertical effective stress profile is given by:

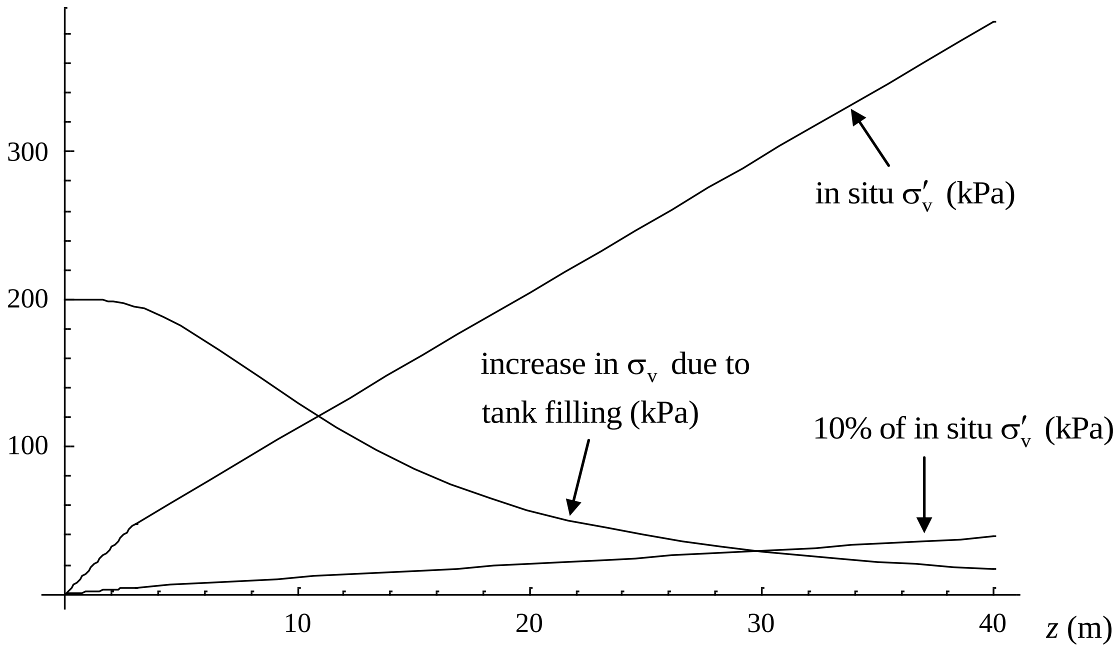
$$z \le 3\text{ m} \qquad \sigma'_v = 16z\text{ kPa}$$
$$z \ge 3\text{ m} \qquad \sigma'_v = 16\times 3 + (19-9.8)\times(z-3) = 20.4 + 9.2z\text{ kPa}$$

On the centreline, the increment of vertical total stress (and, during consolidation, of vertical effective stress as well) is given by eqn (26a) in the lecture notes:

$$\delta\sigma_v = q\times\left[1-\left(\frac{1}{1+(a/z)^2}\right)^{3/2}\right] = 200\times\left[1-\left(\frac{1}{1+(10/z)^2}\right)^{3/2}\right]\text{ kPa}$$

The depth of interest is where

$$0.1\times(20.4+9.2z) = 200\times\left[1-\left(\frac{1}{1+(10/z)^2}\right)^{3/2}\right] \quad\rightarrow\quad z = 29.8\text{ m}$$



9. (a) Settlements are required immediately after loading. In the short term, a fine-grained soil such as clay will exhibit behaviour that is essentially undrained, i.e. no volume change will occur and the soil will be (almost) incompressible. A bulk modulus of infinity corresponds to a Poisson's ratio of 0.5, see HLT, so when calculating settlements under undrained conditions it is appropriate to take $\nu_u = 0.5$.

(b) From the lecture notes, $w_{\text{corner}} = I_w \times \dfrac{qB(1-\nu)}{2G}$ where the influence factor I_w is a function of the aspect ratio L/B. The relevant aspect ratios here are 1, 3 and 5 for which I_w = 0.561, 0.89 and 1.05. The factor $\dfrac{q(1-\nu)}{2G}$ is constant and equal to $\dfrac{40\times(1-0.5)}{2\times 2000} = 0.005$ (note that 2 MPa = 2000 kPa).

$w_{\text{P}} = 0.005\times[0.561\times 2\times 4] = 0.0224$ m [= 22.4 mm]

$w_{\text{Q}} = 0.005\times[0.561\times 3 + 0.561\times 1 + 0.89\times 1\times 2] = 0.0201$ m [= 20.1 mm]

$w_{\text{R}} = 0.005\times[0.561\times 4] = 0.0112$ m [= 11.2 mm]

$w_{\text{S}} = 0.005\times[0.561\times 5 - 1.05\times 1\times 2 + 0.561\times 1] = 0.0063$ m [= 6.3 mm]

(c) Total load = 40 kPa × 16 m^2 = 640 kN. So $w_{\text{S}} = \dfrac{P(1-\nu)}{2\pi G r} = \dfrac{640\times(1-0.5)}{2\pi\times 2000\times 3\sqrt{2}} = 0.0060$ m [= 6.0 mm]. Given the crudeness of the point load approximation, this is surprisingly close to the exact solution for the square loaded area found in part (b), namely 6.3 mm.

(d) Long-term settlements due to consolidation and creep. These will often be much larger (and therefore of more concern) than the immediate settlements due to undrained shearing.