Exercise 1. An element of a ring is called **nilpotent** if $a^n = 0$ for some natural number n. Show that the set of nilpotent elements of a commutative ring form an ideal.

Exercise 2. Let R be a commutative ring, and $I \subset R$ an ideal. Show that the **radical** of I, defined by

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$$

is an ideal.

Exercise 3. Let R be a commutative ring, and let $X \subset R$ be a nonempty subset. Show that the **annihilator** of X, defined by

$$\operatorname{Ann}(X) = \{r \in R \mid rx = 0 \text{ for all } x \in X\}$$

is an ideal.

Exercise 4. Let $f : R \to S$ be a ring homomorphism, and let $I \subset R$ and $J \subset S$ be ideals.

(a) Show that $f^{-1}(J)$ is always an ideal that contains $\ker f$.

(b) Show that if f is surjective, then $f(I)$ is an ideal in S.

Exercise 5. Show that every finite integral domain is a field.

Exercise 6. Prove the third isomorphism theorem for rings.