Exercise 1. Show the following directly from the definitions:

(a) If G is a monoid, the identity element is unique.

(b) If G is a group, every element has a unique inverse.

Exercise 2. Let G be a semigroup. Show that G is a group if and only if for all $a, b \in G$, the equations $ax = b$ and $ya = b$ have solutions in G.

Exercise 3. Let G be a group. Show that the following are equivalent:

(i) G is abelian.

(ii) $(ab)^2 = a^2b^2$ for all $a, b \in G$.

(iii) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

(iv) $(ab)^n = a^nb^n$ for three consecutive integers n and for all $a, b \in G$.

Exercise 4. Let G be a semigroup. G is called *left cancellative* if $ab = ac$ implies $b = c$ (for all $a, b, c \in G$), and is called *right cancellative* if $ba = ca$ implies $b = c$ (again for all $a, b, c \in G$. A semigroup that is both left and right cancellative is just called *cancellative*.

(a) Show that every finite cancellative semigroup is a group.

(b) Give an example of an infinite cancellative semigroup that is not a group.

Exercise 5. Let G be a cyclic group. Show that every homomorphic image of G and every subgroup of G is also cyclic.

Exercise 6. Let G be an abelian group of order pq for coprime $p, q \in \mathbb{N}$. Show that if G contains elements of both order p and order q, then G must be cyclic.

Exercise 7. Let $f : G \to H$ be a group homomorphism. Suppose $a \in G$ such that $f(a) \in H$ has finite order. Show that if $|a|$ is finite, then $|f(a)|$ divides $|a|$.

Exercise 8. Show that every group with a finite number of subgroups is itself finite.

Exercise 9. Let H, K be subgroups of a group G. Show that HK is a subgroup if and only if $HK = KH$.