Preliminaries

Axiom of Choice Let $(S_i)_{i \in I}$ be an indexed family of non-empty sets. Then there exists a "choice function", i.e. an indexed family $(x_i)_{i \in I}$ such that $x_i \in S_i$

Well Ordering Principle Every set has a well-ordering, i.e. an order s.t. every nonempty subset has a least element.

Zorn's Lemma Let A be a non-empty partially ordered set s.t. every chain in A has an upper bound in A. Then A has a maximal element

Thm AC, well-ordering, and Zorn are all equivalent + independent of ZF.

Ex Thm Every vector space has a basis.

Pf Let V be a vector space. Let C be the collection of all linearly independent subsets of V.

Observe: If $S_1 \subset S_2 \subset S_3 \subset \ldots$ is a chain in C,

then $\bigcup_{i \in \mathbb{N}} S_i$ is linearly independent, hence an upper bound.

Zorn $\Rightarrow$ C has a maximal element $\mathcal{B}$.

Claim $V = \operatorname{span} \mathcal{B}$.

Pf S'pose not: let $v \in V \setminus \operatorname{span} \mathcal{B}$.

Then $\mathcal{B} \cup \{v\}$ is linearly independent $\Rightarrow$ $\mathcal{B}$ is not maximal ↯

∎

Chapter 1

Def (i) A semigroup is a set G with an associative operation

(ii) A monoid is a semigroup G with an identity element, i.e. an element $e \in G$ s.t. $ex = xe = x$ for all $x \in G$.

(iii) A group is a monoid G in which every element has an inverse, i.e. for each $x \in G$, there exists $x^{-1} \in G$ s.t. $xx^{-1} = x^{-1}x = e$.

Remark Identity and inverses must be unique

Def A group G is called abelian if the operation is commutative, i.e. $xy = yx$ for all $x, y \in G$.

Ex Classify as semigroup / monoid / group : $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ (under $+$)
$\mathbb{Z}$, $2\mathbb{Z}$, $\mathbb{Z}\setminus\{0\}$, $\mathbb{Q}$, $\mathbb{Q}\setminus\{0\}$ (under $\cdot$)

Prop 1.3 Let G be a semigroup. Then G is a group if & only if left* inverses exist and a left* identity exists, i.e.
(i) there exists $e \in G$ s.t. $ex = x$ for all $x \in G$.
(ii) for each $x \in G$, there exists x^{-1} s.t. $x^{-1}x = e$.

Remark Also true for "right".

Ex Dihedral group $D_n = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$
Symmetries of regular n-gon

Ex Symmetric group

$S_n = \{\text{bijections of } \{1, \dots, n\}\}$ with composition as operation

Notation 1 $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix} \in S_5$

Notation 2 (cycle notation) $(1\,3\,4\,2) \in S_5$

Ex $(1\,2)(1\,3\,4\,2\,5) = (1\,3\,4)(2\,5)$

Fact Every element of S_n can be written as a product of disjoint cycles.

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Def Let G, H be semigroups (resp. monoids, resp. groups). A homomorphism is a map $f: G \to H$ satisfying $f(ab) = f(a)f(b)$ for all $a, b \in G$.

- If f is injective, it is called a monomorphism*
- If f is surjective, it is called an epimorphism*
- If f is bijective, it is called an isomorphism
- If $f: G \to G$, f is called an endomorphism
- An isomorphism $f: G \to G$ is called an automorphism.

Ex $\det : GL_n(k) \to k^*$ is a homomorphism

Ex If A is an abelian group, the map $a \mapsto a^{-1}$ is an automorphism.
The map $a \mapsto a^2$ is an endomorphism.

Def Let $f: G \to H$ be a homomorphism.

- The Kernel of f is $\operatorname{Ker} f = \{g \in G \mid f(g) = e\}$
- The image of f is $\operatorname{Im} f = \{h \in H \mid h = f(g) \text{ for some } g \in G\}$

Ex $\operatorname{Ker} \det = SL_n(k)$

Thm 2.3 Let $f: G \to H$ be a group homomorphism.

(i) f is injective $\iff$ $\operatorname{Ker} f = \{e\}$

(ii) f is ~~an~~ bijective $\iff$ there exists a homomorphism $f^{-1}: H \to G$
s.t. $ff^{-1} = 1_H$ and $f^{-1}f = 1_G$

Def Let G be a group, and $H \subset G$ a subset. If H is a group, then H is called a __subgroup__ and we write $H < G$

Fact If G a group, $H \subset G$ a subset, then H is a subgroup $\iff$ H closed under ~~operation~~. mult + inversion

Ex $\{e\}$, G are always subgroups of G.

Ex $\{1, r, r^2, r^3, \dots r^{n-1}\}$ is a subgroup of D_n

Cor 2.6 Any intersection of subgroups is a subgroup.

Def Let G be a group, and $X \subset G$ a subset.
then
$$\langle X \rangle = \bigcap_{\substack{H_i < G \\ X \subset H_i}} H_i$$
is the __subgroup generated by X__

Thm 2.8 $\langle X \rangle = \{ a_1^{n_1} a_2^{n_2} \dots a_k^{n_k} \mid a_i \in X, n_i \in \mathbb{Z} \}$

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Thm Every subgroup of $\mathbb{Z}$ is cyclic.

Thm Every infinite cyclic group is isomorphic to $\mathbb{Z}$. Every finite cyclic group is isomorphic to $\mathbb{Z}_m$.

Thm Let $G = \langle a \rangle$ be a cyclic group. If G is infinite, a and a^{-1} are the only generators of G. If $|G| = m$, then $\langle a^k \rangle = G \iff (k, m) = 1$

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Recall: Congruence in $\mathbb{Z}$ modulo m (or $\langle m \rangle$)

$a \equiv b \pmod{m} \iff a-b \equiv 0 \pmod{m} \iff m \mid a-b \iff a-b \in \langle m \rangle$

Def Let G be a group, $H \leq G$. Let $a, b \in G$.

a is right congruent to b modulo H if $ab^{-1} \in H$

a is left congruent to b modulo H if $a^{-1}b \in H$

Thm 4.2 (i) These are equivalence relations

(ii) The equivalence classes are the right (resp. left) cosets $Ha = \{ha \mid h \in H\}$

(iii) $|Ha| = |H| = |aH|$ for all $a \in G$.

Cor 4.3 (i+ii) The right (resp left) cosets partition G.

(iii) For all $a, b \in G$ $Ha = Hb \iff ab^{-1} \in H$

$aH = bH \iff a^{-1}b \in H$

(iv) The left and right cosets are in bijection ($Ha \mapsto a^{-1}H$)

Def The index of H in G is the cardinality of the set of distinct cosets denoted $[G:H]$

Ex $[\mathbb{Z} : \langle m \rangle] = m$

Ex $[G:G] = 1$ $[G : \langle e \rangle] = |G|$

Thm 4.5 Let $K < H < G$ be groups. Then $[G:K] = [G:H][H:K]$

Pf Write $G = \coprod_{i \in I} Ha_i$ as a partition of right cosets, so $|I| = [G:H]$

$$H = \coprod_{j \in J} K b_j \qquad \text{so } |J| = [H:K]$$

Then $$G = \coprod_{\substack{i \in I \\ j \in J}} K b_j a_i$$ ↖ Have not shown disjoint yet!

Suppose $K b_j a_i = K b_r a_t$, i.e. $\underbrace{b_j a_i}_{\substack{\cap \\ Ha_i \\ \text{since } b_j \in H}} = \underbrace{k b_r a_t}_{\substack{\cap \\ Ha_t \\ \text{since } kb_r \in H}}$ for some $k \in K$.

$\Rightarrow Ha_i = Ha_t \Rightarrow a_i = a_t$

Then $b_j = k b_r$, so $K b_j = K b_r \Rightarrow b_j = b_r$. ☒

Cor 4.6 (Lagrange's Theorem) If $H < G$, then $|G| = [G:H]\,|H|$.

In particular, if G is finite, then $|a| \mid |G|$ for all $a \in G$.

Notation Let G be a group, H, K subsets of G.

$$HK = \{ab \mid a \in H, b \in K\}$$

Remark HK is usually not a subgroup! Even if H, K are subgroups.

Thm 4.7 Let G be a group, and $H, K < G$ be finite. Then $|HK| = \dfrac{|H|\,|K|}{|H \cap K|}$

Pf Let $C = H \cap K$. $C < K$, let $n = [K:C] = \dfrac{|K|}{|C|} = \dfrac{|K|}{|H \cap K|}$ (by Lagrange)

So $K = Ck_1 \amalg Ck_2 \amalg Ck_3 \amalg \dots \amalg Ck_n$ for some $k_i \in K$

Claim $HK = Hk_1 \amalg Hk_2 \amalg \dots \amalg Hk_n$

$\left(\text{claim} \Rightarrow |HK| = |H|\,n = \dfrac{|H|\,|K|}{|H \cap K|}\right)$

<u>Pf of claim</u> Need to show

(1) HK_i and HK_j are disjoint

(2) $HK \subset HK_1 \amalg \dots \amalg HK_n$

(3) $HK \supset HK_1 \amalg \dots \amalg HK_n$ (immediate)

(1) Suppose ~~...~~ $h_i K_i = h_j K_j$

~~...~~ Then $h_j^{-1} h_i = K_j K_i^{-1} \in C$

$\Rightarrow K_j \in CK_i \Rightarrow K_j = K_i$

(2) Let $hK \in HK$ $(h \in H, K \in K)$

Then $K = cK_i$ for some i, $c \in C$

Then $hK = (hc)K_i \in HK_i$ $\square$

<u>Prop 4.8</u> Let G be a group, $H, K < G$, and suppose HK is a subgroup.

Then $[HK : K] = [H : H \cap K]$ and $[HK : H] = [$~~...~~ $K : H \cap K]$

$$\begin{tikzcd} & HK \arrow[dl, dash] \arrow[dr, dash] & \\ H \arrow[dr, dash] & & K \arrow[dl, dash] \\ & H\cap K & \end{tikzcd}$$

HW: $HK = KH$

<u>Pf</u> We will construct bijection $\varphi : \{\text{right cosets of } H \cap K \text{ in } H\} \longrightarrow \{\text{right cosets of } K \text{ in } KH\}$

$$\varphi((H \cap K)h) = Kh$$

<u>well defined</u> S'pose $(H\cap K)h_1 = (H \cap K)h_2$, i.e. $h_1 h_2^{-1} \in H \cap K < K$, so $Kh_1 = Kh_2$

<u>Surjective</u> clear

<u>Injective</u> S'pose $\varphi((H\cap K)h_1) = \varphi((H \cap K)h_2)$

$Kh_1 = Kh_2$

$h_1 h_2^{-1} \in K$, so $h_1 h_2^{-1} \in H \cap K$, so $(H\cap K)h_1 = (H \cap K)h_2$ $\square$

Prop 4.9 Let G be a group, $H, K < G$ s.t. HK is a subgroup. If H, K are finite index in HK, then $[HK : H \cap K] = [HK : H][HK : K]$

Pf Thm 4.5 + Prop 4.8 $\square$

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Thm 5.1 Let N be a subgroup of a group G. TFAE

(i) Left cosets are right cosets

(ii) $aN = Na$ for all $a \in G$

(iii) $aNa^{-1} = N$ for all $a \in G$.

(iv) N is closed under conjugation by elements of G.

Def If N satisfies these conditions it is called a <u>normal</u> subgroup of G, denoted $N \lhd G$.

Pf (i) => (ii) Let aN be a left coset. Then $aN = Nb$ for some $b \in G$. In particular, $a \in Na \cap Nb \Rightarrow Na = Nb$. So $aN = Na$.

(ii) => (iii) Immediate.

(iii) => (iv) Immediate

(iv) => (i) Let aN be a left coset. If $b \in N$, $aba^{-1} \in N$, so $ab \in Na \Rightarrow aN \subset Na$. Similarly, $Na \subset aN$. $\square$

Ex In an abelian group, all subgroups are normal!

Ex Recall $D_4 = \langle r, s \mid r^4 = 1, s^2 = 1, srs = r^{-1} \rangle$

$N = \langle r \rangle = \{1, r, r^2, r^3\}$ is normal

$H = \langle sr \rangle$ is not normal

Remark: If $N \lhd G$ and $N < H < G$, then $N \lhd H$

Caution! $N \lhd K \lhd G$ does <u>not</u> imply $N \lhd G$!

Thm 5.3 Let G be a group, $K < G$, $N \lhd G$

(i) $N \cap K \lhd K$

(ii) $N \lhd \langle N, K \rangle$ (book uses notation $N \vee K$)

(iii) $NK = KN = \langle N, K \rangle$

(iv) If $K \lhd G$ and $K \cap N = \langle e \rangle$, then $nk = kn$ for all $k \in K$, $n \in N$.

Pf (i) Let $x \in N \cap K$, $a \in K$. Then $N \lhd G \Rightarrow axa^{-1} \in N$, $x, a \in K \Rightarrow axa^{-1} \in K$ $\Rightarrow axa^{-1} \in N \cap K$.

(ii) Remark

(iii) It suffices to show $\langle N, K \rangle = NK$ (show if NK a subgroup, $NK = KN$ (homework))

Trivial: $NK \subset \langle N, K \rangle$

Let $n_1 k_1 n_2 k_2 \dots n_r k_r \in \langle N, K \rangle$ $(n_i \in N,\ k_i \in K)$

Induct on n: If $r = 1$, $n_1 k_1 \in NK$

If $r > 1$: Assume $n_1 k_1 \dots n_{r-1} k_{r-1} = n_0 k_0 \in NK$

$$n_1 k_1 \dots n_{r-1} k_{r-1} n_r k_r = n_0 k_0 n_r k_r$$
$$= \underbrace{n_0 (k_0 n_r k_0^{-1})}_{N} \underbrace{k_0 k_r}_{K} \in NK$$

(iv) $\overbrace{\underbrace{n k n^{-1}}_{K} k^{-1}}^{\in N} \in K \cap N = \langle e \rangle$, so $nkn^{-1}k^{-1} = e \Leftrightarrow nk = kn$. $\blacksquare$

Thm 5.4 Let G be a group, $N \triangleleft G$. Then G/N (set of cosets of N) is a group of order $[G:N]$ with multiplication $(aN)(bN) = abN$.

Pf Need to show multiplication is well defined, i.e. if $aN = \tilde{a}N$, $bN = \tilde{b}N$, then $abN = \tilde{a}\tilde{b}N$.

write $\tilde{a} = an_1$ $\quad \tilde{b} = bn_2$

Then $\tilde{a}\tilde{b} = an_1bn_2 = ab(b^{-1}n_1b)n_2 \in abN$ ∎

Def G/N is called the quotient group or factor group of G by N.

Ex $\mathbb{Z}$ is abelian, so $\langle m \rangle \triangleleft \mathbb{Z}$. Then $\mathbb{Z}/\langle m \rangle$ is exactly the group of integers mod m.

Ex $D_4/\langle r \rangle = \{\langle r \rangle, s\langle r \rangle\} \cong \mathbb{Z}/\langle 2 \rangle$

Thm 5.5 (i) If $f: G \to H$ is a group hom., then $\operatorname{Ker} f \triangleleft G$.

(ii) If $N \triangleleft G$, then $\pi: G \to G/N$ is a (surjective) hom with $\operatorname{Ker} \pi = N$
$\pi(a) = aN$.

Pf (1) Let $x \in \operatorname{Ker} f$, $a \in G$. Want $axa^{-1} \in \operatorname{Ker} f$

Compute $f(axa^{-1}) = f(a)f(x)f(a^{-1}) = f(a)\,e\,f(a)^{-1} = e \implies axa^{-1} \in \operatorname{Ker} f$

(2) Let $a,b \in G$. Want $\pi(ab) = \pi(a)\pi(b)$

$\pi(ab) = abN$

$\pi(a)\pi(b) = aNbN = abN$

So π is a homomorphism

$\operatorname{Ker} \pi = \{a \in G \mid \pi(a) = eN = N\}$

$\pi(a) = N \iff aN = N \iff a \in N$ ∎

Thm 5.6 Let $f: G \to H$ be a homomorphism, $N \lhd G$. If $N \subset \operatorname{Ker} f$, then there exists a unique homomorphism $\bar{f}: G/N \to H$ such that the diagram commutes

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

Pf Define* $\bar{f}: G/N \longrightarrow H$ by $\bar{f}(aN) = f(a)$

Careful! Need to check well-defined whenever defining in terms of coset representatives

Need to check: If $aN = bN$, then $\bar{f}(aN) = \bar{f}(bN)$

$\hookrightarrow$ write $a = bn$ for some $n \in N$.

$\bar{f}(aN) = f(a) = f(bn) = f(b)f(n) = f(b) = \bar{f}(bN)$

$\uparrow$ since $N < \operatorname{Ker} f$

Is $\bar{f}$ a homomorphism? Let $aN, bN \in G/N$.

$\bar{f}(aN\, bN) = \bar{f}(abN) = f(ab)$

$\bar{f}(aN)\bar{f}(bN) = f(a)\, f(b)$ ″

$\blacksquare$

Remark $N \lhd \operatorname{Ker} f$, and $\operatorname{Ker} \bar{f} = \operatorname{Ker} f / N$

Corollary 5.7 (First Isomorphism theorem) If $f: G \to H$ is a group homomorphism, then $G/\operatorname{Ker} f \cong \operatorname{Im} f$

Pf

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & \operatorname{Im} f \\ G/\operatorname{Ker} f \arrow[ur, dashed] & \end{tikzcd}$$

Surjective by construction
Injective by remark

$\blacksquare$

Corollary 5.9 (Second Isomorphism Theorem) Let G be a group, $K \le G$, $N \trianglelefteq G$.

Then $K/N\cap K \cong NK/N$

Pf Let φ be the composition $K \hookrightarrow NK \longrightarrow NK/N$ (so $\varphi(a) = aN$)

$$K \xrightarrow{\varphi} NK/N$$

claim $\operatorname{Ker}\varphi = N \cap K$

If $a \in N\cap K$, $\varphi(a) = aN = N$ (since $a \in N$), so $a \in \operatorname{Ker}\varphi$

If $a \in \operatorname{Ker}\varphi$, $\varphi(a) = N$, so $a \in N \Rightarrow a \in N \cap K$.

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d] & NK/N \\ K/N\cap K \arrow[ur, dashed, "\bar{\varphi}"'] & \end{tikzcd}$$

Since $N\cap K = \operatorname{Ker}\varphi$, $\bar{\varphi}$ is injective.

To see φ surjective: Let $aN \in NK/N$

Since $NK = KN$, write $a = kn$ for some $k \in K$, $n \in N$.

Then $aN = knN = kN = \varphi(k)$.

$\Rightarrow \bar{\varphi}$ is an isomorphism. $\square$

Corollary 5.10 (Third Isomorphism Theorem) Let G be a group, $H \trianglelefteq G$, $K \trianglelefteq G$ with $K < H$. Then $H/K \trianglelefteq G/K$ and $G/K \big/ H/K \cong G/H$

Pf Let φ be the ~~composition~~ quotient map $G \longrightarrow G/H$

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G/H \\ G/K \arrow[ur, dashed, "\bar{\varphi}"'] & \end{tikzcd}$$

we get a surjective map $\bar{\varphi} : G/K \longrightarrow G/H$

Suppose $aK \in \operatorname{Ker}\bar{\varphi}$, so $\bar{\varphi}(aK) = H$

$\bar{\varphi}(aK) = aH$, iff $a \in H$. Thus $H/K = \operatorname{Ker}\bar{\varphi}$

$\square$

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Thm 6.3 Every element of S_n can be written uniquely* as a product of disjoint cycles

* Can permute the cycles

Corollary 6.4 The order of a permutation is the least common multiple of the orders of its disjoint cycles

Corollary 6.5 Every permutation can be written as a product of transpositions

Pf $(x_1 x_2 \cdots x_r) = (x_1 x_r)(x_1 x_{r-1}) \cdots (x_1 x_3)(x_1 x_2)$

Caution: Not unique! $(12)(13) = (31)(32)$

Def 6.6 A permutation is **even** (resp **odd**) if it can be written as ~~a~~ a product of an even (resp odd) number of transpositions.

Ex. $(132) \in S_3$ is even since $(132) = \cancel{(13)(12)}\, (12)(13)$

(In general: odd length cycles are even)

Thm 6.7 ~~Even~~ A permutation cannot be both even + odd.

Claim If τ_i are transpositions + $\tau_1 \cdots \tau_r = id$, then r is even

spose $\sigma_1 \cdots \sigma_s = \tau_1 \cdots \tau_r$ ~~[illegible]~~

then $\sigma_1 \cdots \sigma_s \tau_r^{-1} \cdots \tau_1^{-1} = id$, so $r+s$ is even (i.e. both odd or both even)

Pf of claim Suppose $\tau_1 \cdots \tau_r = id$. Induction r

Products of transpositions:

$$(ab)(ab) = id$$
$$(ab)(cd) = (cd)(ab)$$
$$(ab)(ac) = (bc)(ab)$$
$$(ab)(bc) = (bc)(ac)$$

Push 1's to far right, then 2's, etc. Induct. ∎

Thm 6.8 For $n \geq 2$, let A_n be the set of all even permutations of S_n. Then A_n is a normal subgroup of index 2 (and is the only subgroup of index 2).

Pf Define $sgn : S_n \longrightarrow \mathbb{Z}_2$ is a homomorphism with Kernel A_n.

Exercise It is the only subgroup of index 2 ∎

Def A_n is called the <u>alternating group</u>

Def A group G is called <u>simple</u> if it has no proper normal subgroups

Ex $\mathbb{Z}_p$ for prime p are precisely the simple abelian groups

Thm 6.10 A_n is simple if and only if $n \neq 4$

Lemma $\sigma(x_1 x_2 \ldots x_r)\sigma^{-1} = (\sigma(x_1)\ \sigma(x_2)\ \ldots\ \sigma(x_r))$

Ex Let $\sigma = (123)$

$\sigma(15234)\sigma^{-1} = (25314)$

$(123)(15234)(321) = (14253)$

Lemma If $n \geq 5$, all 3-cycles are conjugate in A_n

Pf By lemma, conjugate in $\underline{\underline{S_n}}$

i.e. If τ_1, τ_2 are 3-cycles $\tau_1 = \sigma\tau_2\sigma^{-1}$ for some $\sigma \in S_n$

If σ is odd: choose 2 elements a, b not appearing in τ_2

then $\tilde{\sigma} = \sigma(ab)$ is even, and $\tilde{\sigma}\tau_2\tilde{\sigma}^{-1} = \sigma(ab)\tau_2(ab)\sigma^{-1}$

$= \sigma\tau_2\sigma^{-1}$

$= \tau_1$ ∎

Lemma Let $n \geq 5$. If $N \lhd A_n$ and N contains a 3-cycle, then $N = A_n$

Pf It suffices to show that A_n is generated by 3-cycles

Claim A product of two transpositions is generated by 3-cycles.

Pf **Case 1** $(ab)(cd) = (acb)(acd)$

Case 2 $(ab)(ac) = (acb)$

~~Case~~ **Case 3** $(ab)(ab) = id.$ $\boxtimes$

Pf of Thm 6.10 Suppose $H \lhd A_n$ is nontrivial. We will show H contains a 3-cycle.

~~Cases~~ Cases: Disjoint cycle structure of elements of H

Case 1 Cycle of length $r \geq 4$

WLOG $\sigma = (1\,2\,3 \ldots r)\tau$

Let $\delta = (1\,2\,3)$

$H \ni \sigma^{-1}\delta\sigma\delta^{-1} = \tau^{-1}(r \ldots 3\,2\,1)(1\,2\,3)(1\,2\,3 \ldots r)\tau(3\,2\,1)$

$= (1\ 3\ r)$

Case 2 Multiple 3-cycles

WLOG $\sigma = (1\,2\,3)(4\,5\,6)\tau$

Let $\delta = (1\,2\,4)$

$H \ni \sigma^{-1}\delta\sigma\delta^{-1} = \tau^{-1}(6\,5\,4)(3\,2\,1)(1\,2\,4)(1\,2\,3)(4\,5\,6)\tau(4\,2\,1)$

$= (1\ 4\ 2\ 6\ 3)$

Apply Case 1

<u>Case 3</u> Single 3-cycle

wLOG $\sigma = (123)\tau$

$N \ni \sigma^2 = (123)\tau(123)\tau = (123)^2\tau^2 = (123)^2 = (123)^{-1} = (321)$

<u>Case 4</u> Product of transpositions

wLOG $\sigma = (12)(34)\tau$

Let $\delta = (123)$

$H \ni \sigma^{-1}\delta\sigma\delta^{-1} = \tau(34)(12)(123)(12)(34)\tau(321)$

$= (13)(24)$

Call this $\sigma_0 \in H$

Let $\delta_0 = (135)$

$\sigma_0^{-1}\delta_0\sigma_0\delta_0^{-1} = (13)(24)(135)(13)(24)(531)$

$= (135)$

$\boxtimes$

Def Let G, H be groups. The <u>direct product</u> (or <u>direct sum</u>) $G \times H$ is the group

$$G \times H = \{ (g,h) \mid g \in G, h \in H \}$$

with operation $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

Ex $\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{ (0,0), (0,1), (0,2), (1,0), (1,1), (1,2) \}$

$$(1,1) + (0,2) = (1+0, 1+2) = (1,0)$$

Fact $|G \times H| = |G|\,|H|$

Natural homomorphisms

$$\begin{tikzcd} & G \times H \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & H \end{tikzcd}$$

$\pi_1(g,h) = g$

$\pi_2(g,h) = h$

$$\begin{tikzcd} & G \times H & \\ G \arrow[ur, "i_1"] & & H \arrow[ul, "i_2"'] \end{tikzcd}$$

$i_1(g) = (g, e_H)$

$i_2(h) = (e_G, h)$

Observe $\ker \pi_1 = i_1(G) \cong G$

$\ker \pi_2 = i_2(H) \cong H$

$G \times H / G \cong H$

$G \times H / H \cong G$

Remark $G \times H$ is generated by $i_1(G)$, $i_2(H)$

Def Let $\{ G_i \}_{i \in I}$ be a collection of groups.

Then $\prod_{i \in I} G_i$ is a group called the <u>direct product</u> of $\{ G_i \}_{i \in I}$

Thm 8.2 The direct product is a categorical product.

Special Case Let G_1, G_2 be groups, and suppose H is a group with $\varphi_1 : H \to G_1$
$\varphi_2 : H \to G_2$.

There exists unique $\varphi : H \to G_1 \times G_2$ s.t. $\pi_i \varphi = \varphi_i$

$$\begin{tikzcd} & H \arrow[dl, "\varphi_1"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "\varphi_2"] & \\ G_1 & G_1 \times G_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & G_2 \end{tikzcd}$$

Pf $\varphi = (i_1 \varphi_1, i_2 \varphi_2)$

$\square$

Ex Let $G = \prod_{n \in \mathbb{N}} \mathbb{Z}_2$

Let $H = \langle i_n(\mathbb{Z}) \mid n \in \mathbb{N} \rangle$

$= \langle (1, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, 0, \ldots), \ldots \rangle$

Does $H = G$?

Def The <u>direct sum</u> (or <u>weak direct product</u>) is the subgroup of $\prod_{i \in I} G_i$ generated by the G_i.

It consists of elements with <u>finitely many</u> terms not equal to the identity.

Ex $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$

$\prod_{i \in \mathbb{N}} \mathbb{Z}$

Ex Is $D_4 = \langle r, s \mid r^4 = 1, s=1, srs = r^{-1} \rangle$ a direct product?

$D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$N = \langle r \rangle$ $\qquad H = \langle s \rangle$

Not normal!

Note that $D_4 = \underset{HN}{\underset{\parallel}{NH}}$ and $N \cap H = \langle e \rangle$

Every element of D_4 can be written uniquely as hn for some $h \in H$, $n \in N$.

Thm Let $N \trianglelefteq G$, $H \leq G$. Then TFAE

(1) $G = NH = HN$ and $N \cap H = \langle e \rangle$

(2) Every element of G can be written uniquely as nh for some $n \in N$, $h \in H$.

(3) Every element of G can be written uniquely as hn for some $h \in H$, $n \in N$

(4) There exists a split exact sequence

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r] & H \arrow[r] \arrow[l, bend left, dashed] & 1 \end{tikzcd}$$

Def Such a G is called the semidirect product of N and H, written $G = N \rtimes H$.

Pf (1) $\Rightarrow$ (2) uniqueness: Suppose $n_1 h_1 = n_2 h_2$ for some $n_1, n_2 \in N$, $h_1, h_2 \in H$

Then $n_2^{-1} n_1 = h_2 h_1^{-1} \in N \cap H = \langle e \rangle$

Thus $n_2^{-1} n_1 = e$ $\quad h_2 h_1^{-1} = e$

$n_1 = n_2$ $\quad h_1 = h_2$

(2) $\Rightarrow$ (3) $\quad (nh)^{-1} = h^{-1} n^{-1}$

(3) $\Rightarrow$ (4)

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\alpha"] & G \arrow[r, "\beta"] & H \arrow[r] \arrow[l, bend left, dashed, "\sigma"] & 1 \end{tikzcd}$$

$\alpha =$ inclusion (injective)

$\sigma =$ inclusion

Define $\beta : G \to H$ by $\beta(hn) = h$.

Is β a homomorphism?

Let $h_1 n_1, h_2 n_2 \in G$ $\quad (h_1, h_2 \in H, n_1, n_2 \in N)$

$\beta(h_1 n_1) = h_1 \quad \beta(h_2 n_2) = h_2$

$$\beta(h_1 n_1 h_2 n_2) = \beta(h_1 h_2 \underbrace{\underbrace{h_2^{-1} n_1 h_2}\underbrace{n_2}}_{N}) = h_1 h_2 = \beta(h_1 n_1)\beta(h_2 n_2)$$

Note β surjective, and $\beta \circ \sigma = id$. Also $\operatorname{Ker}\beta = \operatorname{Im}\alpha$

(4) $\Rightarrow$ (1) Suppose

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\alpha"] & G \arrow[r, "\beta"] & H \arrow[r] \arrow[l, bend left, dashed, "\sigma"] & 1 \end{tikzcd}$$

is a split exact sequence.

Let $x \in G$. We want to break it down into a H part and a N part.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma(H) \qquad\qquad \alpha(N)$

Set $h = \sigma\beta(x) \in \sigma(H)$

<u>Claim</u> $xh^{-1} \in \operatorname{Ker}\beta = \alpha(N)$

(Then $x \in \alpha(N)\sigma(H) \cong NH$)

<u>pf</u> $\beta(xh^{-1}) = \beta(x\,\sigma\beta(x^{-1}))$

$= \beta(x)\,\beta\sigma\beta(x^{-1})$

$= \beta(x)\beta(x^{-1})$

$= e$

Need to check $\alpha(N) \cap \sigma(H) = \langle e \rangle$.

Let $x \in \alpha(N) \cap \sigma(H)$. Then $x = \sigma(y)$ for some $y \in H$.

Since $x \in \alpha(N)$, $e_H = \beta(x) = \beta\sigma(y) = y$, so $x = \sigma(e) = e$ ☐

<u>Cor</u> If $G = N \rtimes H$, then $H \cong G/N$

Def Let X be a set.

Let X^{-1} be a set disjoint from X with $|X| = |X^{-1}|$

choose a bijection $X \longrightarrow X^{-1}$, and label the image of $x \in X$ by x^{-1}.

A <u>word</u> on X is a sequence $(a_1, a_2, a_3, \dots)$ with $a_i \in X \cup X^{-1} \cup \{1\}$ that is eventually identically 1.

The <u>empty word</u> is $(1, 1, 1, \dots)$

A word is <u>reduced</u> if a_i never equals a_{i+1}^{-1}

Ex $X = \{x, y\}$

$(x, y, x, x, x^{-1}, y, y, 1, 1, 1, \dots)$ is a word (Think: $xyxxx^{-1}yy$)

$(x, y, x, y, y, 1, 1, 1, \dots)$ is a reduced word (Think: $xyxyy$)

Usally nonempty reduced words are written of form $x_1^{\lambda_1} \cdots x_r^{\lambda_r}$ $\quad \lambda_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$

Def The set of all reduced words forms a group called the <u>free group on X</u> denoted $F(X)$

Thm 9.2 The free group is a free object in the category of groups.
In other words, if $f : X \longrightarrow G$ is a map of sets ~~to~~ to a group G, there is a unique homomorphism $\tilde{f} : F(X) \to G$

$$\begin{tikzcd} & G \\ X \arrow[ur, "f"] \arrow[r, hook] & F(X) \arrow[u, dashed, "\tilde{f}"'] \end{tikzcd}$$

Pf Define $\tilde{f}(x_1^{\lambda_1} \cdots x_r^{\lambda_r}) = f(x_1)^{\lambda_1} \cdots f(x_r)^{\lambda_r}$

Cor 9.3 Every group is the homomorphic image of a free group.

Pf Let X be a set of generators of G. (Note: $G \cong F(X)/\mathrm{Ker}\,\tilde{f}$)

$$\begin{tikzcd} & G \\ X \arrow[ur] \arrow[r, hook] & F(X) \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

∎

<u>Thm-Def 1.1</u> Let F be an abelian group. TFAE

(i) F has a nonempty <u>basis</u>, i.e. a generating set X s.t.
whenever $n_1x_1 + \dots + n_kx_k = 0$ for some $n_i \in \mathbb{Z}$, $x_i \in X$, then $n_i = 0$ for all i.
(Think: no nontrivial linear combinations make zero $\Rightarrow$ no relations among generators)

(ii) F is the direct sum of a family of infinite cyclic subgroups

(iii) F is the direct sum of copies of $\mathbb{Z}$

(iv) F is free in the category of abelian groups; i.e.
There is a nonempty set $X \hookrightarrow F$ s.t. given any abelian group G with a set map $f: X \to G$, there exists unique $\tilde{f}: F \to G$

$$\begin{tikzcd} & G \\ X \arrow[ur, "f"] \arrow[r, hook] & F \arrow[u, dashed, "\tilde{f}"'] \end{tikzcd}$$

<u>Pf</u> (i) $\Rightarrow$ (ii) If $x \in X$, then $\langle x \rangle$ is infinite (cyclic) group
Need to check: If $x_0 \in X$, then $\langle x_0 \rangle \cap \bigcup_{x \in X \setminus \{x_0\}} \langle x \rangle = 0$.
If not, $nx_0 = n_1x_1 + \dots + n_rx_r$ for some nonzero $n_i \in \mathbb{N}$, $x_i \in X$
Thus, $F = \bigoplus_{x \in X} \langle x \rangle$

(ii) $\Rightarrow$ (iii) $\mathbb{Z}$ is the only infinite cyclic group.

(iii) $\Rightarrow$ (i) Suppose $F \cong \bigoplus_{i \in I} \mathbb{Z}$. Let $X = \{(0, \dots, 0, 1, 0, \dots, 0, \dots)\}$
By construction, this is a basis.

We have shown (i), (ii), (iii) are equivalent

(i, ii, iii) $\Rightarrow$ (iv) Let X be a nonempty basis of F. Suppose G is abelian gp with $f: X \to G$.

Define $\tilde{f} : F \to G$ by $\tilde{f}(\sum n_i x_i) = \sum n_i f(x_i)$

$$\begin{tikzcd} & G \\ X \arrow[ur, "f"] \arrow[r] & F \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

(iv) $\Rightarrow$ (i, ii, iii)

We will show $F \cong \bigoplus_{x \in X} \mathbb{Z}$

we showed above $\bigoplus_{x \in X} \mathbb{Z}$ is free in categorical sense

$$\begin{tikzcd} & F \\ X \arrow[ur] \arrow[r] \arrow[dr] & \bigoplus_{x \in X} \mathbb{Z} \arrow[u, dashed, "\psi"'] \\ & F \arrow[u, dashed, "\varphi"'] \end{tikzcd}$$

Uniqueness $\Rightarrow \psi \circ \varphi = id$

So this is an isomorphism.

<u>Thm</u> A finitely generated abelian group is isomorphic to a direct sum of cyclic groups

<u>Lemma</u> If $G = \langle x_1, \ldots, x_n \rangle$ is a f.g. abelian group, then $G/\langle x_1, \ldots, x_{n-1} \rangle$ is cyclic.

<u>Pf</u> We claim $G/\langle x_1, \ldots, x_{n-1} \rangle = \langle x_n + \langle x_1, \ldots, x_{n-1} \rangle \rangle$

Let $y = a_1 x_1 + \ldots + a_n x_n \in G$.

Then $y + \langle x_1, \ldots, x_{n-1} \rangle = a_n x_n + \langle x_1, \ldots, x_{n-1} \rangle = a_n (x_n + \langle x_1, \ldots, x_{n-1} \rangle)$ ∎

<u>Pf of thm</u> Let $G = \langle x_1, \ldots, x_n \rangle$. Let $C_i = G/\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle$ be cyclic.

~~Then~~ Let $\pi_i : G \longrightarrow C_i$ be the quotient maps.

By Thm 8.2, there exists $\varphi : G \longrightarrow C_1 \oplus \ldots \oplus C_n$ that factors through each π_i.

Each π_i is surjective, so $i_i(C_i) \subset \operatorname{Im} \varphi$ for each i.

Thus φ is surjective.

Suppose $y = a_1 x_1 + \ldots + a_n x_n \in \operatorname{Ker} \varphi$ WLOG $a_i \nmid |x_i|$

Let $\sigma_i : C_1 \oplus \ldots \oplus C_n \longrightarrow C_i$ be ith projection map

Then $\sigma_i \varphi(y) = \sigma_i(0) = 0$ for every i

But $\sigma_i \varphi(y) = \pi_i(y) = \pi_i(a_1 x_1 + \ldots + a_n x_n) = a_i \pi_i(x_i)$

This is 0 only if $a_i \mid |x_i|$

$\Rightarrow$ each $a_i = 0$, so $y = 0$. Thus φ is injective ∎

Lemma 2.3 Let $m \in \mathbb{N}$, and write $m = p_1^{n_1} \cdots p_r^{n_r}$ for distinct primes p_i

Then $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{n_r}}$

Lemma If $a, b \in \mathbb{N}$ are coprime, then $\mathbb{Z}_{ab} \cong \mathbb{Z}_a \oplus \mathbb{Z}_b$

Pf Observe $\langle b \rangle = \{0, b, 2b, \ldots, (a-1)b\} \cong \mathbb{Z}_a$

$\langle a \rangle = \{0, a, 2a, \ldots, (b-1)a\} \cong \mathbb{Z}_b$

Note $\langle a \rangle \cap \langle b \rangle = 0$ (If $\lambda a = \mu b$ for some $\lambda < b, \mu < a$, then $b \mid \lambda$, $a \mid \mu$, so $\lambda = 0, \mu = 0$)

Then $\langle a \rangle \oplus \langle b \rangle$ is a subgroup of order ab, so it is all of $\mathbb{Z}_{ab}$. $\square$

Pf of Lemma 2.3 Induct on r. If $r = 1$, trivial.

If $r > 1$, $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{n_1} \cdots p_{r-1}^{n_{r-1}}} \oplus \mathbb{Z}_{p_r^{n_r}}$ by Lemma

$\cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{r-1}^{n_{r-1}}} \oplus \mathbb{Z}_{p_r^{n_r}}$ by induction hypothesis $\square$

Thm 2.2 (Fundamental Theorem of Finitely Generated Abelian Groups)

Every finitely generated abelian group is isomorphic to a direct sum of cyclic groups, each of which is infinite or of prime power order.

Pf Thm + Lemma 2.3 $\square$

Def 4.1 Let G be a group, and S a set. An <u>action</u> is a map

$$G \times S \longrightarrow S \qquad (g, x) \longmapsto g \cdot x$$

such that for all $x \in S$, $g_1, g_2 \in G$

1) $e \cdot x = x$
2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$

we say <u>G acts on S</u>, sometimes write $G \circlearrowright S$

Ex S_n acts on $\{1, \dots, n\}$

Ex $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$

$$A \cdot \vec{v} = A\vec{v}$$

Ex D_n acts on a regular n-gon

Ex $\mathbb{R}^n$ acts on itself by translation

$$v \cdot x = x + v$$

Ex Let G be a group, H a subgroup. Then H acts on G by (left) translation,

$$h \cdot g = hg$$

Ex Let G be a group, H a subgroup, $S = \{aH \mid a \in G\}$

G acts on S by translation

$$g \cdot aH = gaH$$

Ex Let $H \leq G$. H acts on G by conjugation

$$(h, g) \mapsto hgh^{-1}$$

Thm 4.2 Let G act on a set S

(i) The relation on S given by $x \sim x' \iff gx = x'$ for some $g \in G$ is an equivalence relation

(ii) If $x \in S$, $G_x := \{g \in G \mid gx = x\}$ is a subgroup

Def The equivalence classes are called <u>orbits</u> (sometimes written $G\cdot x$)

G_x is called the <u>stabilizer</u> of x.

An action is called <u>transitive</u> if there is exactly one orbit, i.e. for all $x, y \in S$ there exists $g \in G$ s.t. $g\cdot x = y$.

Ex Let G act on itself by conjugation. An orbit of $x \in G$ $\{gxg^{-1} \mid g \in G\}$ is called a <u>conjugacy class</u> of x.

Ex Let G act on its set of subgroups by conjugation. The stabilizer of a subgroup K $\quad N_G(K) = \{g \in G \mid gKg^{-1} = K\}$ is called the <u>normalizer</u> of K in G.

Note that $K \trianglelefteq G \iff N_G(K) = G$.

Thm 4.3 (orbit stabilizer theorem) Suppose G acts on S. The size (cardinality) of the orbit of $x \in S$ equals the index of the stabilizer $[G:G_x]$

Pf ~~Claim~~ Define a map

$$\{gG_x\} \longrightarrow G\cdot x$$

$$gG_x \longmapsto g\cdot x$$

well defined: Suppose $gG_x = hG_x \iff g^{-1}h \in G_x$

$$\iff g^{-1}h\cdot x = x$$

$$\iff h\cdot x = g\cdot x$$

Reverse argument shows map is injective, also surjective. $\blacksquare$

Cor 4.4 Let G be a finite group, $K \leq G$.

(i) The number of elements in the conjugacy class of $x \in G$ is $[G:C_G(x)]$, where $C_G(x) = \{g \in G \mid gxg^{-1} = x\}$ is the <u>centralizer</u> of x.

(ii) If $x_1, \dots, x_n$ are representatives of the distinct conjugacy classes of G, then $|G| = \sum_{i=1}^{n} [G : C_G(x_i)]$

(iii) The number of subgroups of G conjugate to K is $[G : N_G(K)]$

Def The <u>class equation</u> is the equation $|G| = \sum_{i=1}^{n} [G : C_G(x_i)]$

Thm 4.5 Let G act on a set X. Then this induces a homomorphism $G \longrightarrow S(X)$.

Pf Let $g \in G$, Define $\tau_g \in S(X)$ by $x \longmapsto g \cdot x$

Check that τ_g is a bijection: $\tau_{g^{-1}}$ is an inverse mapping for τ_g

the map $\varphi : G \longrightarrow S(X)$ $\quad \varphi(g) = \tau_g$ is a homomorphism

$\varphi(gh) = \tau_{gh}$ $\qquad \tau_{gh}(x) = gh \cdot x$

$\varphi(g)\varphi(h) = \tau_g \tau_h$ $\qquad \tau_g(\tau_h(x)) = \tau_g(h \cdot x) = g \cdot (h \cdot x)$

Cor 4.6 (Cayley's Thm) Let G be a group. Then G embeds in a symmetric group. (is isomorphic to a subgroup of)

Pf G acts on itself by left translation, so we get a homomorphism

$$\varphi : G \longrightarrow S(G)$$

Compute $\operatorname{Ker} \varphi$: Suppose $\underset{\tau_g}{\underset{\shortparallel}{\varphi(g)}} = id$

Then $\underset{gx}{\underset{\shortparallel}{g \cdot x}} = x$ for all $x \in G$

i.e. $g = e$.

Thus $\operatorname{Ker} \varphi = \langle e \rangle$, so φ is injective.

Cor 4.7 Let G be a group.

(i) For each $g \in G$, conjugation by g induces an automorphism of G. (these are called <u>inner automorphisms</u>)

(ii) There is a homomorphism $G \longrightarrow \operatorname{Aut} G$ whose kernel is the center of G $\quad C(G) = \{g \in G \mid gx = xg$ for all $x \in G$.

Pf (i) $\gamma_g : G \longrightarrow G$, $x \longmapsto gxg^{-1}$ is an automorphism

(ii) $\gamma_g \gamma_h = \gamma_{gh}$, so the map $\begin{array}{c} G \longrightarrow \mathrm{Aut}\, G \\ g \longmapsto \gamma_g \end{array}$ is a homomorphism.

Cor 4.10 Let $H < G$, and let p be the smallest prime with $p \mid |G|$. If $[G:H] = p$, then $H \lhd G$.

Prop 4.8 Let $H < G$, and let G act on the left cosets of H by translation. Then the Kernel of the induced homomorphism $\varphi : G \longrightarrow S(\{gH\})$ is contained in H.

Pf Suppose $g \in \mathrm{Ker}\, \varphi$, so $\varphi(g) = id$

$$\varphi(g) = id = \gamma_g$$

Then

$$\gamma_g(H) = H$$

$$\gamma_g(H) = g \cdot H = gH \implies g \in H. \qquad \square$$

Cor 4.9 Let $H < G$ with $[G:H] = n$, and suppose H contains no nontrivial normal subgroup of G. Then G is isomorphic to a subgroup of S_n.

Pf Apply 4.8 to H: the map $G \longrightarrow S(\{gH\})$ must be injective.

pf of 4.10 Let X be the set of all left cosets of H in G.

Let K be the Kernel of map $G \longrightarrow S(X) \cong S_p$

$K \lhd G$, and by 4.8 $K \leq H$

Also, G/K is isomorphic to a subgroup S_p

Thus, $|G/K| \mid p!$

But no prime smaller than p divides $|G|$,

So we must have $|G/K| = p$ or $|G/K| = 1$

But $|G/K| = [G:K] = [G:H][H:K] = p[H:K]$

Thus $|G/K| = p$ and $[H:K] = 1$, i.e. $K = H$.

But K was normal in G. $\blacksquare$

Motivation: Lagrange's theorem says if $H < G$, then $|H| \mid |G|$

when is converse true? If $m \mid |G|$, when must G have a subgroup of order m?

Thm 5.2 (Cauchy's Theorem) If p is prime and $p \mid |G|$, then G has a subgroup of order p.

Lemma 5.1 Suppose H is a group of order p^n that acts on a set S.

Let $S_0 = \{x \in S \mid h \cdot x = x \text{ for all } h \in H\}$ = fixed points of action.

Then $|S| \equiv |S_0| \mod p$

Pf $S = S_0 \amalg H \cdot x_1 \amalg Hx_2 \amalg \cdots \amalg H \cdot x_r$

orbit stabilizer: $|Hx_i| = [H : H_{x_i}]$

$\uparrow$ must divide $|H| = p^n$

thus $p \mid |Hx_i|$ for all i. ∎

Pf of 5.2 Let $S = \{(a_1, \dots, a_p) \mid a_i \in G,\ a_1 a_2 \cdots a_p = e\}$

Claim $\langle (1\,2\,3 \cdots p) \rangle < S_p$ acts on S

$\cong$
$\mathbb{Z}_p$

If $(a_1, \dots, a_p) \in S$, is $(a_2, \dots, a_p, a_1) \in S$?

If $a_1 a_2 \cdots a_p = e$, then $a_2 \cdots a_p a_1 = a_1^{-1}(a_1 a_2 \cdots a_p) a_1 = a_1^{-1} e a_1 = e$

Now $S_0 = \{(a, \dots, a) \mid a \in G,\ a^p = e\}$ (fixed points)

S_0 nonempty, $(e, \dots, e) \in S_0$,

$|S_0| \equiv |S| \mod p = |G|^{p-1} \mod p \equiv 0 \mod p$ since $p \mid |G|$.

S_0 nonempty $\implies |S_0| > 1$, so there exists $a \in G \setminus \{e\}$ with $a^p = e$ ∎

Def A group is called a <u>p-group</u> if every element has order p^n for a fixed prime p and some $n \in \mathbb{N}$.

If G is a group, $H \le G$ and H is a p-group, H is called a <u>p-subgroup</u> of G.

Ex $\mathbb{Z}_{16}$ is a p-group.

Ex $\langle 3 \rangle$ is a p-subgroup of $\mathbb{Z}_{24}$

Cor 5.3 A finite group G is a p-group $\iff |G| = p^n$ for some n.

Pf $\Leftarrow$ LaGrange Theorem

$\Rightarrow$ S'pose $q \mid |G|$ for some prime q. Then Cauchy's Thm implies G has an element of order $q \Rightarrow q = p$. $\square$

Cor 5.4 Every nontrivial finite p-group has a non-trivial center.

Pf S'pose $|G| = p^n$ for some $n > 0$

Class equation: $$|G| = |Z(G)| + \sum \underbrace{[G : C_G(x_i)]}_{\text{must divide } |G| = p^n}$$

Thus $p \mid |Z(G)|$ $\square$

Lemma 5.5 Let G be finite, $H \le G$ a p-subgroup. Then $[N_G(H) : H] \equiv [G : H] \bmod p$.

Pf Let S be set of left cosets of H

H acts on S by left translation

what are fixed points?

$$xH \in S_0 \iff hxH = xH \quad \text{for all } h \in H$$
$$\iff x^{-1}hx \in H \quad \text{for all } h \in H$$
$$\iff x^{-1}Hx = H$$
$$\iff x \in N_G(H)$$

Thus $S_0 = \{xH \mid x \in N_G(H)\}$, so $|S_0| = [N_G(H) : H]$

By Lemma 5.1, $|S_0| \equiv |S| \bmod p$, and $|S| = [G:H]$ ∎

<u>Cor 5.6</u> Let G be finite, $H \leq G$ a p-subgroup, and suppose $p \mid [G:H]$. Then $N_G(H) \neq H$

<u>Pf</u> By lemma, $[N_G(H):H] \equiv [G:H] \bmod p \equiv 0 \bmod p$.
Index always positive, so $[N_G(H):H] \geq p$ ∎

<u>Thm 5.7 (First Sylow theorem)</u> Let G be a group of order $p^n m$ for a prime p, $p \nmid m$. Then G contains a subgroup of order p^i for each $1 \leq i \leq n$. Moreover, every subgroup of order p^i is normal in some subgroup of order p^{i+1}. $(i < n)$

<u>Pf</u>

<u>Claim</u> If $H < G$ is a subgroup of order p^i $(1 \leq i < n)$, there is a subgroup H_1 of order p^{i+1} with $H \triangleleft H_1$

Cauchy's Thm $\Rightarrow$ subgroup of order p, Claim + induction $\Rightarrow$ theorem.

Pf of Claim Suppose $H \leq G$, and $|H| = p^i$ for $1 \leq i < n$

Since $i < n$, $p \mid [G:H]$, so by Cor 5.6 $N_G(H) \neq H$

~~Note $N_G(H)$~~ $H \trianglelefteq N_G(H)$, so consider $N_G(H)/H$.

$$|N_G(H)/H| = [N_G(H):H] \equiv [G:H] \equiv 0 \mod p$$

Lemma 5.5

So $p \mid |N_G(H)/H|$, so it must contain a subgroup of order p,

Call it H_1/H. (for some $H_1 \leq N_G(H)$).

$H \trianglelefteq H_1$, and $|H_1| = |H|\,[H_1:H] = p^i p = p^{i+1}$.

Def Let G be a group. A <u>Sylow p-subgroup</u> or <u>p-Sylow subgroup</u> is a maximal p-subgroup of G. First Sylow theorem $\Rightarrow$ If $|G| = p^n m$, $p \nmid m$, then G has a Sylow p-subgroup of order p^n.

Cor 5.8 Let G have order $p^n m$, p prime, $p \nmid m$. Let H be a p-subgroup of G.

(1) H is a Sylow p-subgroup $\iff |H| = p^n$

(2) Every conjugate of a Sylow p-subgroup is a Sylow p-subgroup

(3) If there is only one Sylow p-subgroup, it is a normal subgroup.

Thm 5.9 (Second Sylow Theorem)

Any two p-Sylow subgroups are conjugate.

<u>Pf</u> Let $P, Q < G$ be p-Sylow subgroups

Let $S = \{xP \mid x \in G\}$, and let Q act on S by translation.

Lemma 5.1 $\Rightarrow$ $|S_0| \equiv [G:P] \mod p$.

Since $p \nmid [G:P]$, $|S_0| > 0$

Let $xP \in S_0$, i.e. $qxP = xP$ for all $q \in Q$

$x^{-1}qxP = P$ for all $q \in Q$

$x^{-1}qx \in P$ for all $q \in Q$

$x^{-1}Qx \leq P$.

But $|x^{-1}Qx| = |Q| = |P|$, so $x^{-1}Qx = P$. $\blacksquare$

Thm 5.10 (Third Sylow Theorem)

Let G be a finite group, $P_1, \ldots, P_r$ the p-Sylow subgroups for a fixed prime p.

Then $r \equiv 1 \mod p$, and $r \mid |G|$.

<u>Pf</u> Since $P_1, \ldots, P_r$ are all the conjugates of P_1,

Orbit stabilizer $\Rightarrow$ $r = [G : N_G(P_1)]$ which must divide $|G|$.

Now let $S = \{P_1, \ldots, P_r\}$, let P_1 act on S by conjugation.

Note $P_1 \in S_0$.

Suppose $P_i \in S_0$; then $xP_ix^{-1} = P_i$ for all $x \in P_1$.

In other words $P_1 \leq N_G(P_i)$

Note that P_1, P_i are p-Sylow subgroups of $N_G(P_i)$

and $P_i \lhd N_G(P_i)$

$\Rightarrow P_i = P_1$.

So $S_0 = \{P_1\}$, Lemma 5.1 $\Rightarrow$ $r = |S| \equiv |S_0| \equiv 1 \mod p$ $\blacksquare$

Prop 6.1 Let $|G| = pq$ for primes $p > q$. Then either
$G \cong \mathbb{Z}_{pq}$ or $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ (in which case $p \equiv 1 \bmod q$)

Pf By Cauchy, let a have order p, b have order q.
Set $N = \langle a \rangle$, $H = \langle b \rangle$
Note $N \lhd G$, and $NH = G$ (since $|NH| = |N||H| = pq = |G|$), and $N \cap H = \langle e \rangle$.
Thus $G \cong N \rtimes H$. (But sometimes this is a direct product).

Suppose G has r q-sylow subgroups. Then $r \equiv 1 \bmod q$, and $r \mid pq$,
thus $r = 1$ or $r = p$.
If $r = 1$, H is normal, direct product $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_q \cong \mathbb{Z}_{pq}$.
If $r = p$, non-abelian semidirect product, and ~~[illegible]~~
$p = r \equiv 1 \bmod q$ ∎

Cor 6.2 If p is an odd prime, a group of order $2p$ is either cyclic or the dihedral group D_p.

Prop 6.3 The groups of order 8 are either abelian, D_4, or Q_8.

Pf S'pose $|G| = 8$ is nonabelian. If $|a| = 2$ for all $a \in G$, G is abelian.
So let $a \in G$ have order 4. Set $N = \langle a \rangle \lhd G$.

Case 1 Every element of $G \setminus N$ has order 2.
Let $b \in G \setminus N$, so $H = \langle b \rangle$ has order 2.
Note $H \cap N = \langle e \rangle$, and $|HN| = |H||N| = 4 \cdot 2 = 8 = |G|$, so $HN = G$.
Thus $G \cong N \rtimes H = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 = D_8$

Case 2 There exists $b \in G \setminus N$ of order 4
Let $K = \langle b \rangle$. Note $K \lhd G$
Note $|N \cap K| = \dfrac{|N||K|}{|NK|} = \dfrac{4 \cdot 4}{8} = 2$.
$\Rightarrow a^2 = b^2$

Since $N \lhd G$, $bab^{-1} \in N = \{e, a, a^2, a^3\}$

(i) $bab^{-1} = e \Rightarrow a = e$ ↯

(ii) $bab^{-1} = a \Rightarrow ba = ab \Rightarrow G$ abelian

(iii) $bab^{-1} = a^2 \Rightarrow ba^2b^{-1} = e \Rightarrow a^2 = e$ ↯

(iv) Thus, $bab^{-1} = a^3$ ☒

<u>Prop 6.4</u> The nonabelian groups of order 12 are A_4, D_6, and $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$

<u>Pf</u> Let P be a 3-Sylow subgroup. $|P| = 3$, so $[G:P] = 4$

Prop 4.8 $\Rightarrow$ $\varphi : G \longrightarrow S_4$ with $\ker\varphi \leq P$

<u>Case 1</u> $\ker\varphi = \langle e \rangle$, then $G \leq S_4$, index 2 $\Rightarrow G \cong A_4$

<u>Case 2</u> $\ker\varphi = P$, so $P \lhd G$, i.e. P is unique 3-Sylow subgroup.

Let K be a 2-Sylow subgroup, so $|K| = 4$

Note $K \cap P = \langle e \rangle$, $G = PK$ (since $|PK| = \frac{|P||K|}{|P \cap K|} = \frac{3 \cdot 4}{1} = 12 = |G|$)

$\Rightarrow G \cong P \rtimes K$

Case (a) $K \cong \mathbb{Z}_4$, $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$

Case (b) $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $G \cong \mathbb{Z}_3 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2) \cong D_6$ ☒

Motivating solvability

Quadratic: $x^2+bx+c=0$

$$\left(x+\frac{b}{2}\right)^2+c-\frac{b^2}{4}=0$$

$$x=-\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-c}$$

Cubic: $x^3+ax^2+bx+c=0$

(Early 1500s)
del Ferro
Tartaglia
Cardano

Substitute $x=y-\frac{a}{3}$

Suffices to solve depressed cubic $y^3+py+r=0$

<u>Vieté's substitution</u> Let $y=w-\frac{p}{3w}$

$$\left(w-\frac{p}{3w}\right)^3+p\left(w-\frac{p}{3w}\right)+r=0$$

$$w^3+3w^2\left(\frac{-p}{3w}\right)+3w\left(\frac{-p}{3w}\right)^2+\left(\frac{-p}{3w}\right)^3+pw-\frac{p^2}{3w}+r=0$$

$$w^3\underbrace{-pw+\frac{p^2}{3w}}-\frac{p^3}{27w^3}\underbrace{+pw-\frac{p^2}{3w}}+r=0$$

$$w^3-\frac{p^3}{27w^3}+r=0$$

$$w^6+rw^3-\frac{p^3}{27}=0 \qquad \text{Quadratic in } w^3!$$

$$w^3=\frac{-r\pm\sqrt{r^2+\frac{4p^3}{27}}}{2}$$

<u>Ex</u> $(x-1)(x-2)(x+3)=x^3-7x+6$

$$w^3=\frac{-6\pm\sqrt{36+\frac{4(-7)^3}{27}}}{2}=\frac{-6\pm\sqrt{\frac{-400}{27}}}{2}=\frac{-6\pm\frac{20}{3}\frac{i}{\sqrt{3}}}{2}=-3\pm\frac{10i}{3\sqrt{3}}$$

Quartic Formula: $x^4 + ax^3 + bx^2 + cx + d = 0$

(Ferrari, 1540)

Substitute $x = y - \frac{a}{4}$

suffices to solve $y^4 + qy^2 + ry + s = 0$

Suppose we can factor: $(y^2 + Ky + \ell)(y^2 - Ky + m) = 0$

$$\begin{aligned} &(1) \quad q = \ell + m - K^2 \\ &(2) \quad r = Km - K\ell = K(m-\ell) \\ &(3) \quad s = \ell m \end{aligned}$$

$$\begin{aligned} &(1') \quad m + \ell = q + K^2 \\ &(2') \quad m - \ell = \frac{r}{K} \end{aligned}$$

$$\begin{aligned} &(1'+2') \quad 2m = K^2 + q + \frac{r}{K} \\ &(1'-2') \quad 2\ell = K^2 + q - \frac{r}{K} \end{aligned}$$

So it suffices to find K in terms of q, r, s

(3),
$$4s = 4\ell m = \left(K^2 + q + \frac{r}{K}\right)\left(K^2 + q - \frac{r}{K}\right)$$

$$4s = K^4 + 2K^2 q + q^2 - \frac{r^2}{K^2}$$

$$0 = K^6 + 2qK^4 + (q^2 - 4s)K^2 - r^2$$

Cubic in K^2!

__Remark__ Like this proof, Ferrari's proof relies on cubic case – but proof of cubic case was not published until 1545

<u>Def V.1.1</u> If k, K are fields with $k \subset K$, K is called a <u>field extension</u> of k.

<u>Ex</u> $\mathbb{R}$ is an extension of $\mathbb{Q}$, $\mathbb{C}$ is an extension of $\mathbb{R}$ (and $\mathbb{Q}$).

<u>Def V.3.1</u> Let k be a field, $f \in k[x]$. The <u>splitting field of f over k</u> is the smallest extension in which f splits (i.e. factors into linear terms) completely. It is the smallest field containing all roots of f.

<u>Ex</u> The splitting field of x^2-2 over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, -\sqrt{2})$

<u>Ex</u> The splitting field of x^2+1 over $\mathbb{R}$ is $\mathbb{C}$

<u>Ex</u> The splitting field of x^3-2 over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, e^{\frac{2\pi i}{3}})$

<u>Def V.9.1, V.9.2</u> Let k be a field, $f \in k[x]$, and E its splitting field. f is called <u>solvable by radicals</u> if there is a chain of extensions

$k = K_0 \subset K_1 \subset K_2 \subset \dots \subset K_t$

with $E \subset K_t$ and K_{i+1}/K_i is a <u>simple radical extension</u>, i.e. $K_{i+1} = K_i(\alpha_{i+1})$ for some α_{i+1} satisfying $\alpha_{i+1}^{n_{i+1}} \in K_i$.

Idea: "Formula for roots" $\Longleftrightarrow$ "Solvable by radicals"

<u>Thm V.2.2</u> Let k be a field, $f \in k[x]$, and E its splitting field. Let $\sigma \in \mathrm{Aut}_k E$ and $\alpha \in E$ a root of f. Then $\sigma(\alpha)$ is also a root of f.

<u>Pf</u> Write $f = \sum_{i=0}^{n} a_i x^i$ for some $a_i \in k$. Then

$$0 = \sum_{i=0}^{n} a_i \alpha^i$$
$$0 = \sigma(0) = \sigma\left(\sum_{i=0}^{n} a_i \alpha^i\right) = \sum_{i=0}^{n} a_i \sigma(\alpha)^i$$

∎

Def V.2.1 Let $k \subset K$ be an extension of fields. The Galois group of K over k is $\quad Gal(K/k) := Aut_k K$

If $f \in k[x]$, the Galois group of f is the Galois group of its splitting field over k.

Thm (V.4.2) Let $f \in k[x]$ have n distinct roots. Then the Galois group of f is a subgroup of S_n.

Pf By Thm V.2.3, the Galois group acts on the set of distinct roots.

Ex Let $f(x) = (x^2-2)(x^2-3) \in \mathbb{Q}[x]$
splitting field is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash] & & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Suppose $\sigma \in Gal(\mathbb{Q}(\sqrt{3},\sqrt{2})/\mathbb{Q})$

$\sigma(\sqrt{2}) = \pm\sqrt{2}$

$\sigma(\sqrt{3}) = \pm\sqrt{3}$

$\Rightarrow Gal(\mathbb{Q}(\sqrt{3},\sqrt{2})/\mathbb{Q}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Thm (V.4.2) If $f \in k[x]$, is irreducible, its Galois group acts transitively on its roots.

Pf If α, β are two roots, $k(\alpha) \cong k(\beta)$ and this extends to an automorphism of the splitting field.

Thm A (V.2.5) Let p be prime, k a field containing a primitive p^{th} root of unity, and let $f = x^p - a \in k[x]$.

(i) f is irreducible iff none of its roots are in k

(ii) The splitting field of f is a simple radical extension

(iii) If f is irreducible, its Galois group is $\mathbb{Z}_p$

Pf (i) $\Rightarrow$ Contrapositive: If a root lies in k, f has a linear factor so is reducible.

$\Leftarrow$ Let α be a root. Then all of the roots are $\{\alpha, \omega\alpha, \omega^2\alpha, \dots, \omega^{p-1}\alpha\}$ for a primitive root of unity ω.

Suppose $f = gh$ with $\deg g = m < p$.

Write $g = a_m x^m + \dots + a_0$. We may assume $a_m = 1$, so $a_0 = \pm$ product of roots
$= \pm\, \omega^r \alpha^m$
for some $r \in \mathbb{N}$.

Since $\alpha_0 \in k$, $\omega \in k$, we see $\alpha^m \in k$.

Note that $\gcd(m, p) = 1$, so $1 = ms + pt$ for some $s, t \in \mathbb{Z}$.

Then $\alpha = \alpha^{ms+pt} = (\alpha^m)^s (\alpha^p)^t \in k$.

(ii) The roots are $\{\alpha, \omega\alpha, \dots, \omega^{p}\alpha\}$. Since $\omega \in k$, the splitting field is $k(\alpha)$.

(iii) Let E be the splitting field of f, and let $\sigma \in \mathrm{Gal}(E/k)$

Then $\sigma(\alpha) = \omega^i\alpha$ for some i, and this completely determines σ

Define
$$\mathrm{Gal}(E/k) \longrightarrow \mathbb{Z}_p$$
$$(\alpha \mapsto \omega^i\alpha) \longmapsto i$$

Immediate: homomorphism, injective

If f irreducible, $\mathrm{Gal}(E/k)$ acts transitively $\Rightarrow$ surjective. ◻

Thm B (V.2.5) Let $k \subset K \subset E$ be fields where K, E are splitting fields of $f, g \in k[x]$

Then $\mathrm{Gal}(E/K) \lhd \mathrm{Gal}(E/k)$ and

$$\mathrm{Gal}(E/k)/\mathrm{Gal}(E/K) \cong \mathrm{Gal}(K/k)$$

Pf Define
$$\Phi : \mathrm{Gal}(E/k) \longrightarrow \mathrm{Gal}(K/k)$$
$$\sigma \longmapsto \sigma|_K$$

$\ker \Phi = \{\sigma \in \mathrm{Gal}(E/k) \mid \sigma \text{ fixes } K\} = \mathrm{Gal}(E/K)$

Since any element of $\mathrm{Aut}_k K$ extends to $\mathrm{Aut}_k E$, surjective. ◻

Thm V.9.41 Let $f \in k[x]$ have degree n. Assume k contains p^{th} roots of unity for $p \leq n$. Let E be the splitting field of f over k. If f is solvable by radicals, then there exist subgroups $G_i \leq G := Gal(E/k)$ such that

(i) $G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_t = \langle e \rangle$

(ii) $G_{i+1} \trianglelefteq G_i$

(iii) G_i/G_{i+1} is cyclic of prime order for all i.

Pf Since f is solvable by radicals, there exist fields

$$k = K_0 \subset K_1 \subset \cdots \subset K_t$$

with $E \subset K_t$ and K_{i+1}/K_i a simple radical extension.

i.e. $K_{i+1} = K_i(\beta_{i+1})$ for some β_{i+1} with $\beta_{i+1}^{r_{i+1}} \in K_i$

WLOG each r_i is prime.

Let $G_i = Gal(K_t/K_i)$

Thm A $\Rightarrow$ Each K_{i+1} is a splitting field

Thm B $\Rightarrow$ $G_{i+1} \trianglelefteq G_i$

Thm B $\Rightarrow$ $G_{i+1}/G_i \cong Gal(K_{i+1}/K)$ and Thm A $\Rightarrow$ $Gal(K_{i+1}/K_i) \cong \mathbb{Z}/p$. ∎

Def (cf. 7.9) Let G be a group. A <u>subnormal series</u> is a sequence

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \dots \rhd G_n = \langle e \rangle .$$

The quotients G_i/G_{i+1} are called <u>factor groups</u>

A finite group is called <u>soluble</u> if it has a normal series with each factor group cyclic of prime order.

In this language: A polynomial is soluble by radicals iff its Galois group is soluble.
we will see S_5 is <u>not</u> soluble

Ex Let $G = \mathbb{Z}_{30}$

* (1) $G \rhd \langle 10 \rangle \rhd 1$ is a (sub)normal series

$G/\langle 10 \rangle \cong \mathbb{Z}_{10}$ $\quad \langle 10 \rangle / 1 \cong \mathbb{Z}_3$

☆ (2) $G \rhd \langle 2 \rangle \rhd \langle 6 \rangle \rhd 1$

$G/\langle 2 \rangle \cong \mathbb{Z}_2$ $\quad \langle 2 \rangle / \langle 6 \rangle \cong \mathbb{Z}_3$ $\quad \langle 6 \rangle / \langle 1 \rangle \cong \mathbb{Z}_5$ is a soluble series

Def 8.2 Let $G = G_0 \rhd G_1 \rhd \dots \rhd G_n = 1$ be a subnormal series.
A <u>refinement</u> is a subnormal series $G = H_0 \rhd H_1 \rhd H_2 \rhd \dots \rhd H_m = 1$
where $G_0, \dots, G_n$ is a subsequence of $H_0, \dots, H_m$

Ex $G = \mathbb{Z}_{30}$

$G \rhd \langle 5 \rangle \rhd \langle 10 \rangle \rhd 1$ is a refinement of *

$G/\langle 5 \rangle \cong \mathbb{Z}_5$ $\quad \langle 5 \rangle / \langle 10 \rangle \cong \mathbb{Z}_2$ $\quad \langle 10 \rangle / 1 \cong \mathbb{Z}_3$

$G \rhd \langle 2 \rangle \rhd \langle 10 \rangle \rhd 1$ is another refinement

$G/\langle 2 \rangle \cong \mathbb{Z}_2$ $\quad \langle 2 \rangle / \langle 10 \rangle \cong \mathbb{Z}_5$ $\quad \langle 10 \rangle / 1 \cong \mathbb{Z}_3$

Def 8.3 A composition series is a subnormal series in which each factor group is simple

Ex ☆ $G \rhd \langle 2 \rangle \rhd \langle 6 \rangle \rhd 1$ is a composition series

$G/\langle 2 \rangle \cong \mathbb{Z}_2 \quad \langle 2 \rangle / \langle 6 \rangle \cong \mathbb{Z}_3 \quad \langle 6 \rangle / 1 \cong \mathbb{Z}_5$

Remark A subnormal series is a composition series iff it does not admit a refinement.

Def 8.7 Two subnormal series are equivalent if there is a bijection between their sets of factor groups (up to isomorphism)

Thm 8.10 (Schreier Refinement Theorem) Let G be a group. Any two subnormal series of G admit equivalent refinements

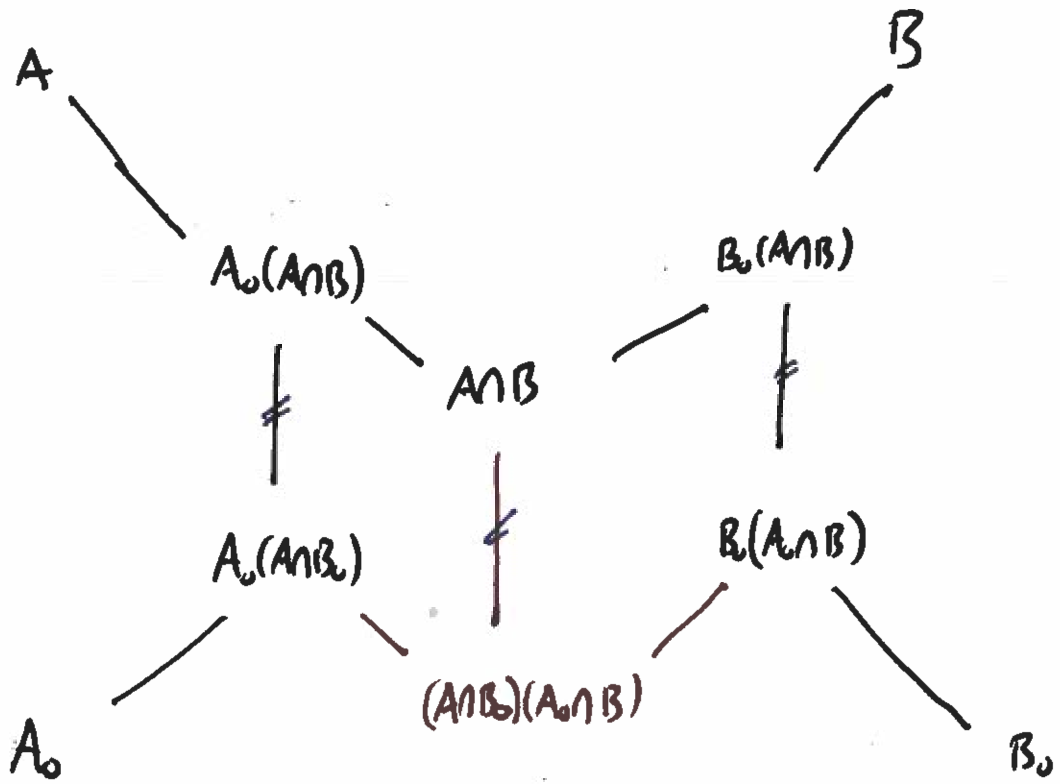
Ex * and ☆ are not equivalent, but they admit equivalent refinements

Lemma 8.9 (Zassenhaus Lemma) Let G be a group, $A_0 \lhd A \leq G$ and $B_0 \lhd B \leq G$.

(i) $A_0(A \cap B_0) \lhd A_0(A \cap B)$

(ii) $B_0(A_0 \cap B) \lhd B_0(A \cap B)$

(iii) $A_0(A \cap B) / A_0(A \cap B_0) \cong B_0(A \cap B) / B_0(A_0 \cap B)$



<u>Pf of 8.9</u> [redacted] Note $A\cap B_0 = (A\cap B)\cap B_0 \lhd A\cap B$ (since $B_0 \lhd B$)
and $A_0\cap B = A_0\cap(A\cap B) \lhd A\cap B$ (since $A_0\lhd A$)

Thus $(A\cap B_0)(A_0\cap B) \lhd A\cap B$

<u>Claim</u> $$A_0(A\cap B)/A_0(A\cap B_0) \cong A\cap B/(A\cap B_0)(A_0\cap B) \cong B_0(A\cap B)/B_0(A_0\cap B)$$

Define $$\varphi : A_0(A\cap B) \longrightarrow A\cap B/(A\cap B_0)(A_0\cap B)$$
$$ac \longmapsto \bar{c}$$

well defined: Suppose $a_1c_1 = a_2c_2$ for some $a_1,a_2\in A_0$, $c_1,c_2\in A\cap B$

$$c_1c_2^{-1} = a_1^{-1}a_2 \in \underset{\substack{\uparrow \\ \text{since } c_1c_2^{-1}\in A\cap B}}{(A\cap B)} \cap \underset{\substack{\uparrow \\ \text{since } a_1^{-1}a_2\in A_0}}{A_0} = A_0\cap B \leq (A\cap B_0)(A_0\cap B)$$

$\Rightarrow \bar{c}_1 = \bar{c}_2$

Surjective: ✓

Suppose $ac \in \operatorname{Ker}\varphi$ for some $a \in A_0$, $c \in A \cap B$.

Then $c \in (A_0 \cap B)(A \cap B_0)$, so $c = c_1 c_2$ for some $c_1 \in A_0 \cap B$, $c_2 \in A \cap B_0$)

Then $ac = (ac_1)c_2 \in A_0(A \cap B_0)$ so $\operatorname{Ker}\varphi \leq A_0(A \cap B_0)$

But $A_0(A \cap B_0) \leq \operatorname{Ker}\varphi$, so $\operatorname{Ker}\varphi = A_0(A \cap B_0)$ $\boxtimes$

<u>Pf of Thm 8.10</u> Let $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n$

$G = H_0 \rhd H_1 \rhd \cdots \rhd H_m$ be two subnormal series

Idea: Refine $G_\cdot$ by sticking $H_\cdot$ between G's

Refine $H_\cdot$ by sticking $G_\cdot$ between H's

Let $G_{n+1} = \langle e \rangle$
$H_{m+1} = \langle e \rangle$

smaller than G_0, bigger than G_1, normal seq. by Zassenhaus

$$
\begin{array}{l}
G_0 = G_1(G_0 \cap H_0) \rhd G_1(G_0 \cap H_1) \rhd G_1(G_0 \cap H_2) \rhd \cdots \rhd G_1(G_0 \cap H_m) \\
\triangledown \\
G_1 = G_2(G_1 \cap H_0) \rhd G_2(G_1 \cap H_1) \rhd G_2(G_1 \cap H_2) \rhd \cdots \rhd G_2(G_1 \cap H_m) \\
\triangledown \\
G_2 = G_3(G_2 \cap H_0) \rhd G_3(G_2 \cap H_1) \rhd G_3(G_2 \cap H_2) \rhd \cdots \rhd G_3(G_2 \cap H_m) \\
\triangledown \\
\vdots \\
\triangledown \\
G_n = G_{n+1}(G_n \cap H_0) \rhd G_{n+1}(G_n \cap H_1) \rhd G_{n+1}(G_n \cap H_2) \rhd \cdots \rhd G_{n+1}(G_n \cap H_m)
\end{array}
$$

$$
\begin{array}{ccccc}
H_0 & \rhd \; H_1 & \rhd & \cdots & \rhd \; H_m \\
\| & \| & & & \| \\
H_1(H_0 \cap G_0) & H_2(H_1 \cap G_0) & & & H_{m+1}(H_m \cap G_0) \\
\triangledown & \triangledown & & & \triangledown \\
H_1(H_0 \cap G_1) & H_2(H_1 \cap G_1) & & & H_{m+1}(H_m \cap G_1) \\
\triangledown & \triangledown & & & \triangledown \\
\vdots & \vdots & & & \vdots \\
\triangledown & \triangledown & & & \triangledown \\
H_1(H_0 \cap G_n) & H_2(H_1 \cap G_n) & & & H_{m+1}(H_m \cap G_n)
\end{array}
$$

Set $G(i,j) = G_{i+1}(G_i \cap H_j)$

$H(i,j) = H_{j+1}(H_j \cap G_i)$

Need to Show: (1) $G(i,j+1) \lhd G(i,j)$, $H(i+1,j) \lhd H(i,j)$
For $0 \le i \le n$, $0 \le j \le m$
(2) $G(i,j)/G(i,j+1) \cong H(i,j)/H(i+1,j)$

Zassenhaus!

(3) $G(i,m+1) = G(i+1,0)$ and $H(n+1,j) = H(0,j+1)$

(3): $G(i,m+1) = G_{i+1}(G_i \cap H_{m+1}) = G_{i+1} = G(i+1,0)$
$H(n+1,j) = H_{j+1}(H_j \cap G_{n+1}) = H_{j+1} = H(0,j+1)$

∎

Thm 8.11 (Jordan-Hölder Theorem) Any two composition series of a group are equivalent.

Cor 7.12 If $n \geq 5$, S_n is not soluble

Pf $S_n \rhd A_n \rhd \langle e \rangle$ is a composition series with factor groups $\mathbb{Z}/2$, A_n. But A_n is not cyclic! ∎

Ex Let $f(x) = x^5 - x - 1$. Galois group is S_5, so the roots of f cannot be expressed via radicals!

More general Def Let G be a group. G is called solvable if it has a subnormal series with abelian factor groups

Remark: Agrees with previous def for finite groups
(Refine to a composition series, quotients are the simple & abelian, i.e. cyclic of prime order)

Thm 7.11 Let $H \triangleleft G$. G is soluble iff H, G/H are both soluble.

Pf $\Rightarrow$ Let $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_n = 1$ be a soluble series for G.

(1) H is soluble: $H = G_0 \cap H \triangleright G_1 \cap H \triangleright \dots \triangleright G_n \cap H = 1$ is a subnormal series.

$$G_i \cap H / G_{i+1} \cap H \cong G_{i+1}(G_i \cap H)/G_{i+1} \quad \text{by } 2^{nd} \text{ iso thm}$$

$$\leq G_i / G_{i+1} \quad \text{which is abelian.}$$

(2) **Lemma** If $\varphi: G \to K$ a homomorphism, $\mathrm{Im}\,\varphi$ is soluble

Pf WLOG φ surjective

Then $K = \varphi(G_0) \triangleright \varphi(G_1) \triangleright \varphi(G_2) \triangleright \dots \varphi(G_n) = 1$ is subnormal series

Natural maps $G_i \longrightarrow \varphi(G_i) \longrightarrow \varphi(G_i)/\varphi(G_{i+1})$

Call this composition $\psi: G_i \longrightarrow \varphi(G_i)/\varphi(G_{i+1})$

Note $G_{i+1} \leq \mathrm{Ker}\,\psi$, so we see

$$\bar{\psi}: G_i/G_{i+1} \longrightarrow \varphi(G_i)/\varphi(G_{i+1})$$

ψ surjective $\Rightarrow$ $\bar{\psi}$ surjective

Then by 1^{st} iso thm, $\varphi(G_i)/\varphi(G_{i+1})$ isomorphic to a quotient of G_i/G_{i+1}, so abelian

$\Leftarrow$ Let $G/H = \bar{K}_0 \rhd \bar{K}_1 \rhd \cdots \rhd \bar{K}_n = \bar{1}$ be solvable series for G/H.

Correspondence theorem $\Rightarrow$ $\bar{K}_i = K_i/H$ for some $K_i \lhd K_{i-1}$

then $G = K_0 \rhd K_1 \rhd \cdots \rhd K_n = H$ and $K_i/K_{i+1} \cong {}^{K_i/H}/_{K_{i+1}/H} = {}^{\bar{K}_i}/_{\bar{K}_{i+1}}$

H solvable $\Rightarrow$ ~~HSV~~ H has a solvable series

$H = K_{n+1} \rhd K_{n+2} \rhd \cdots \rhd K_m = 1$

Then $G = K_0 \rhd K_1 \rhd \cdots \rhd K_n \rhd K_{n+1} \rhd \cdots \rhd K_m = 1$ is a solvable series. ∎

Cor (i) $H \times K$ is solvable $\Longleftrightarrow$ H, K solvable

(ii) $H \rtimes K$ is solvable $\Longleftrightarrow$ H, K solvable.

Cor If G has order pq for distinct primes p, q, then G is solvable

Pf Prop 6.1 $\Rightarrow$ $G \cong \mathbb{Z}_{pq}$ or $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ $(p > q)$ ∎

Cor Every finite p-group is solvable.

Pf Induct on $|G| = p^n$. $n=1 \Rightarrow G$ abelian, so solvable.

Class Equation $\Rightarrow$ G has a nontrivial center.

Then $Z(G)$ is abelian, hence solvable, and $G/Z(G)$ is a smaller p-group and thus solvable by induction.

Exercise Every group of order $p^2 q$ for distinct primes p, q is solvable.