Contents

Problem Set 1

Exercise 1.1. Show the following directly from the definitions:

(a) If G is a monoid, the identity element is unique.

(b) If G is a group, every element has a unique inverse.

Solution

□

Exercise 1.2. Let G be a semigroup. Show that G is a group if and only if for all $a, b \in G$, the equations $ax = b$ and $ya = b$ have solutions in G.

Solution

□

Exercise 1.3. Let G be a group. Show that the following are equivalent:

(i) G is abelian.

(ii) $(ab)^2 = a^2b^2$ for all $a, b \in G$.

(iii) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

(iv) $(ab)^n = a^n b^n$ for three consecutive integers n and for all $a, b \in G$.

Solution

□

Exercise 1.4. Let G be a semigroup. G is called *left cancellative* if $ab = ac$ implies $b = c$ (for all $a, b, c \in G$), and is called *right cancellative* if $ba = ca$ implies $b = c$ (again for all $a, b, c \in G$. A semigroup that is both left and right cancellative is just called *cancellative*.

(a) Show that every finite cancellative semigroup is a group.

(b) Give an example of an infinite cancellative semigroup that is not a group.

Solution

□

Exercise 1.5. Let G be a cyclic group. Show that every homomorphic image of G and every subgroup of G is also cyclic.

Solution

□

Exercise 1.6. Let G be an abelian group of order pq for coprime $p, q \in \mathbb{N}$. Show that if G contains elements of both order p and order q, then G must be cyclic.

Solution

□

Exercise 1.7. Let $f : G \to H$ be a group homomorphism. Suppose $a \in G$ such that $f(a) \in H$ has finite order. Show that if $|a|$ is finite, then $|f(a)|$ divides $|a|$.

Solution

□

Exercise 1.8. Show that every group with a finite number of subgroups is itself finite.

Solution

□

Exercise 1.9. Let H, K be subgroups of a group G. Show that HK is a subgroup if and only if $HK = KH$.

Solution

□

Problem Set 2

Exercise 2.1. Let G be a group. Show that the following are equivalent:

(i) $|G|$ is prime

(ii) G has exactly two subgroups (G and the trivial subgroup).

(iii) $G \cong \mathbb{Z}_p$ for some prime p.

Solution

□

Exercise 2.2. Let G be a group, and let H and K subgroups of finite index. Show that if $[G : H]$ and $[G : K]$ are coprime, then $G = HK$.

Solution

□

Exercise 2.3. Let H, K, and N be subgroups of a group G. Show that if $H < N$, then $HK \cap N = H(K \cap N)$.

Solution

□

Exercise 2.4. Show that every subgroup of index 2 is normal.

Solution

□

Exercise 2.5. Let $N = \{\sigma \in S_4 \mid \sigma(4) = 4\}$. Determine if N is a normal subgroup of S_4 or not.

Solution

□

Exercise 2.6. Let $Q = \langle i, j \mid i^4 = e, i^2 = j^2, iji = j \rangle$ (this is called the *quaternion group*). Show that every subgroup of Q is normal.

Solution

□

Problem Set 3

Exercise 3.1. If G is a group, the *center* of G is the group

$$Z(G) = \{a \in G \mid ax = xa \text{ for all } x \in G\}.$$

Show that the center is a normal subgroup of G.

Solution

$\square$

Exercise 3.2. Consider the subgroup $N = \langle (123) \rangle$ of S_3.

(a) Show that N is a normal subgroup.

(b) Describe S_3/N.

Solution

$\square$

Exercise 3.3. Let $f : G \to H$ be a group homomorphism, and set $N = \ker f$. Let $K < G$ be any subgroup.

(a) Show that $f^{-1}(f(K)) = KN$.

(b) Show that $f^{-1}(f(K)) = K$ if and only if $N < K$.

Solution

$\square$

Exercise 3.4. Consider the subgroups $H = \langle 7 \rangle$ and $K = \langle 42 \rangle$ of $\mathbb{Z}$. Note that $K \lhd H$; describe H/K.

Solution

$\square$

Exercise 3.5. Let G be a group. Show that if $G/Z(G)$ is cyclic, then G is abelian.

Solution

$\square$

Problem Set 4

Exercise 4.1. Let $f : G \to H$ be a group homomorphism and suppose $\ker f < N < G$ for some subgroup N. Show that if H is abelian, then N must be normal.

Solution

$\square$

Exercise 4.2. Consider the set $N \leq S_4$ given by

$$N = \{(1), (12)(34), (13)(24), (14)(23)\}.$$

(a) Verify that N is a subgroup.

(b) Show that $N \lhd S_4$.

(c) Conclude that A_4 is not simple.

Solution

$\square$

Exercise 4.3. Show that A_4 has no subgroup of order 6.

Solution

$\square$

Exercise 4.4. Show that A_n is the only subgroup of S_n that has index 2 (*Hint: show that an index 2 subgroup must contain a 3-cycle*).

Solution

$\square$

Exercise 4.5. The dihedral group D_6 and the alternating group A_4 both have 12 elements. Determine if they are isomorphic or not.

Solution

$\square$

Problem Set 5

Exercise 5.1. Let G be a group with two normal subgroups, N and H. Suppose that $G = N \rtimes H$.

(a) Show that $G = H \rtimes N$.

(b) Show that $G = N \times H$.

(c) Show that G is abelian if and only if N and H are abelian.

Solution

$\square$

Exercise 5.2. Let G, H be finite cyclic groups. Show that $G \times H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

Solution

$\square$

Exercise 5.3. Show that S_3 is **not** a direct product of any of its proper subgroups.

Solution

$\square$

Exercise 5.4. Show that S_n is a semidirect product.

Solution

$\square$

Exercise 5.5. Show that a free abelian group is a free group if and only if it is cyclic.

Solution

$\square$

Exercise 5.6. Show that the direct sum of free abelian groups is a free abelian group (Note that this is not true for direct products).

Solution

$\square$

Problem Set 6

Exercise 6.1. Determine all of the Sylow 3-subgroups of S_5.

Solution

$\square$

Exercise 6.2. Show that there is no simple group of order 200 (Hint: Look for a normal Sylow p-subgroup).

Solution

$\square$

Exercise 6.3. Let G be a group of order $p^n q$ for some primes $p > q$. Show that G contains a unique normal subgroup of index q.

Solution

$\square$

Exercise 6.4. Show that every group of order $p^2 q$ for distinct primes p, q is solvable.

Solution

$\square$

Exercise 6.5. Let G be a group, $K \leq G$, and $H \trianglelefteq G$. Show that if H and K are solvable, then HK is solvable as well.

Solution

$\square$

Exercise 6.6. Use the Sylow theorems to show that every group of order 72 is solvable.

Solution

$\square$

Problem Set 7

Exercise 7.1. An element of a ring is called **nilpotent** if $a^n = 0$ for some natural number n. Show that the set of nilpotent elements of a commutative ring form an ideal.

Solution

$\square$

Exercise 7.2. Let R be a commutative ring, and $I \subset R$ an ideal. Show that the **radical** of I, defined by

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$$

is an ideal.

Solution

$\square$

Exercise 7.3. Let R be a commutative ring, and let $X \subset R$ be a nonempty subset. Show that the **annihilator** of X, defined by

$$\mathrm{Ann}(X) = \{r \in R \mid rx = 0 \text{ for all } x \in X\}$$

is an ideal.

Solution

$\square$

Exercise 7.4. Let $f : R \to S$ be a ring homomorphism, and let $I \subset R$ and $J \subset S$ be ideals.

(a) Show that $f^{-1}(J)$ is always an ideal that contains $\ker f$.

(b) Show that if f is surjective, then $f(I)$ is an ideal in S.

Solution

$\square$

Exercise 7.5. Show that every finite integral domain is a field.

Solution

$\square$

Exercise 7.6. Prove the third isomorphism theorem for rings.

Solution

$\square$

Acknowledgements