Differential Equations

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Module C: Constant coefficient linear ODEs

How can we solve and apply linear constant coefficient ODEs?

At the end of this module, students will be able to...

C1. Modeling motion in viscous fluids. ...

C2. Constant coefficient first order. ...

C3. Modeling oscillators. ...

C4. Homogeneous constant coefficient second order. ...

C5. Non-homogenous constant coefficient second order. ...

C6. IVPs. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Find all roots of a quadratic polynomial.
- Use Euler's theorem to relate $\sin(t)$, $\cos(t)$, and e^t.
- Use Euler's theorem to simplify complex exponentials.
- Describe Newton's laws in terms of differential equations.
- Use substitution to compute indefinite integrals.
- Use integration by parts to compute indefinite integrals.
- Solve systems of two linear equations in two variables.

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): `http://bit.ly/2l21etm`
- Solving linear systems with substitution (Khan Academy): `http://bit.ly/1SlMpix`
- Set builder notation: `https://youtu.be/xnfUZ-NTsCE`

Module C Section 1

Activity C.1.1 *(∼5 min)*

Why don't clouds fall out of the sky?

(a) They are lighter than air
(b) Wind keeps them from falling
(c) Electorstatic charge
(d) They do fall, just very slowly

Activity C.1.2 *(∼5 min)*

List all of the forces acting on a tiny droplet of water falling from the sky.

Activity C.1.3 *($\sim$5 min)*

Tiny droplets of water obey **Stoke's law**, which says that air resistance is proportional to velocity.

Use Newton's laws to write a differential equation that models the velocity of a falling droplet of water.

Definition C.1.4

A **first order constant coefficient** differential equation can be written in the form

$$y' + by = c,$$

or equivalently,

$$\frac{dy}{dx} + by = c.$$

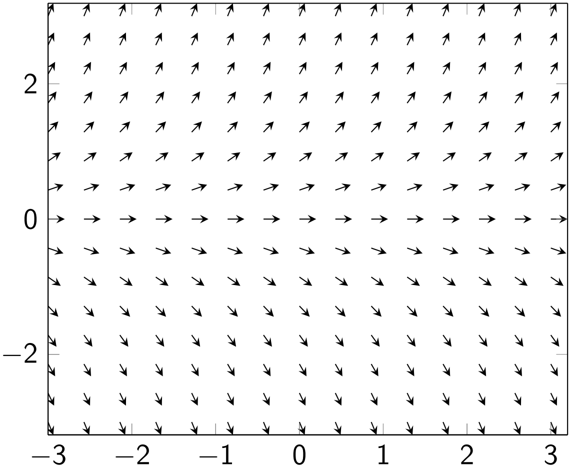
We will use both notations interchangeably.

Here, **first order** refers to the fact that the highest derivative we see is the first derivative of y.

Observation C.1.5

Consider the differential equation $y' = y$.

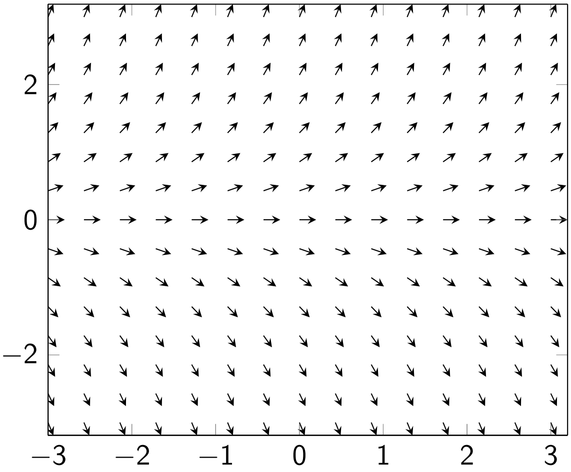
A useful way to visualize a first order differential equation is by a **slope field**



Each arrow represents the slope of a solution **trajectory** through that point.

Activity C.1.6 *(∼5 min)*

Consider the differential equation $y' = y$ with slope field below.



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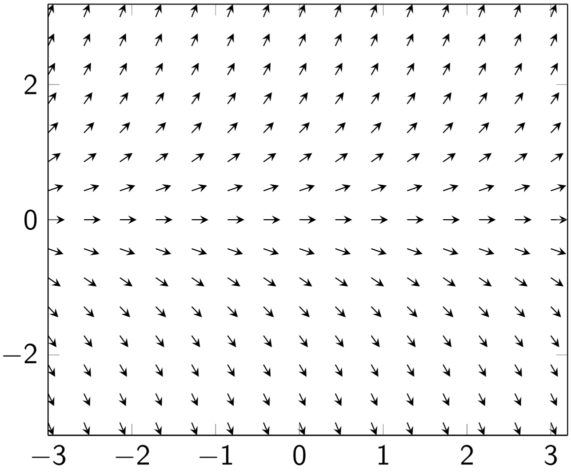

Part 1: Draw a trajectory through the point $(0, 1)$.

Activity C.1.6 *(∼5 min)*

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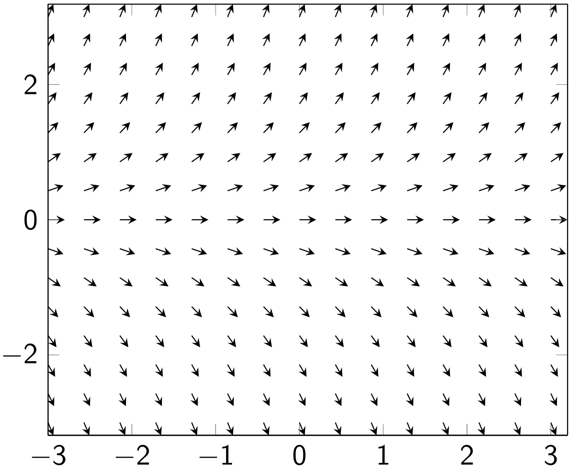

Part 1: Draw a trajectory through the point $(0, 1)$.

Part 2: Draw a trajectory through the point $(-1, -1)$.

Activity C.1.7 *(∼15 min)*
Consider the differential equation $y' = y$.

Activity C.1.7 *(∼15 min)*
Consider the differential equation $y' = y$.
Part 1: Find a solution to $y' = y$.

Activity C.1.7 *(∼15 min)*

Consider the differential equation $y' = y$.

Part 1: Find a solution to $y' = y$.

Part 2: Find **all** solutions to $y' = y$.

Definition C.1.8

A differential equation will have many solutions. The **general solution** encompasses all of these by using parameters such as C, k, c_0, c_1 and so on. For example:

- The general solution to the differential equation $y' = 2x - 3$ is $y = x^2 - 3x + C$ (as done in Calculus courses).
- The general solution for $y' = y$ is $y = c_0 e^x$ (as done in the previous activity).

Activity C.1.9 *(∼15 min)*

Adapt the solution $y = c_0 e^x$ for $y' = y$ to find a general solutions for the following differential equations.

Activity C.1.9 *(∼15 min)*

Adapt the solution $y = c_0 e^x$ for $y' = y$ to find a general solutions for the following differential equations.

Part 1: Solve $y' = 2y$.

Activity C.1.9 *(∼15 min)*
Adapt the solution $y = c_0 e^x$ for $y' = y$ to find a general solutions for the following differential equations.
Part 1: Solve $y' = 2y$.
Part 2: Solve $y' = y + 2$.

Module C Section 2

Observation C.2.1

Recall the last activity from yesterday:

Solve $y' = y + 2$

This is very similar to the equation $y' = y$, which we know how to solve.

Activity C.2.2 *(∼15 min)*
Solve $y' = y + 2$

Simple idea: Since e^t is a solution of $y' = y$, we suppose a solution is of the form $y_p = \mu e^t$ for some function μ.

Activity C.2.2 *(∼15 min)*
Solve $y' = y + 2$

Simple idea: Since e^t is a solution of $y' = y$, we suppose a solution is of the form $y_p = \mu e^t$ for some function μ.
Part 1: Subsitute y_p into the equation $y' = y + 2$ and simplify.

Activity C.2.2 *(∼15 min)*
Solve $y' = y + 2$

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Part 1: Subsitute y_p into the equation $y' = y + 2$ and simplify.
Part 2: Find μ.

Activity C.2.2 *(∼15 min)*
Solve $y' = y + 2$

Simple idea: Since e^t is a solution of $y' = y$, we suppose a solution is of the form $y_p = \mu e^t$ for some function μ.
Part 1: Subsitute y_p into the equation $y' = y + 2$ and simplify.
Part 2: Find μ.
Part 3: Find y_p.

Observation C.2.3

This technique is called **variation of parameters**. If y_0 is a solution of the **homogeneous** equation, we suppose a solution of the **non-homogeneous** equation has the form $y_p = \mu y_0$, and then determine what μ must be.

Example:

$y' + 3y = 0$	homogeneous
$y' + 3y = x$	non-homogeneous

Activity C.2.4 *(∼20 min)*

Solve $y' = x - 3y$.

Activity C.2.4 *(∼20 min)*
Solve $y' = x - 3y$.
Part 1: Solve the homogeneous equation $y' + 3y = 0$.

Activity C.2.4 *(∼20 min)*
Solve $y' = x - 3y$.
Part 1: Solve the homogeneous equation $y' + 3y = 0$.
Part 2: If y_0 is a solution of the homogeneous equation, let $y_p = \mu y_0$ for some **function** μ. Substitute y_p in to original equation and simplify.

Activity C.2.4 *(∼20 min)*

Solve $y' = x - 3y$.

Part 1: Solve the homogeneous equation $y' + 3y = 0$.

Part 2: If y_0 is a solution of the homogeneous equation, let $y_p = \mu y_0$ for some **function** μ. Substitute y_p in to original equation and simplify.

Part 3: Determine μ, and then determine y_p.

Observation C.2.5

Since $y_0 = c_0 e^{-3x}$ was the general solution of the homogeneous equation, and $y_p = \frac{x}{3} - \frac{1}{9}$ is a particular solution of the non-homogeneous equation, a general solution to the non-homegenous equation

$$y' + 3y = x$$

is

$$y_0 + y_p = c_0 e^{-3x} + \frac{x}{3} - \frac{1}{9}.$$

Activity C.2.6 *(∼15 min)*

Find the general solution to $y' = 2y + x + 1$.

Module C Section 3

Observation C.3.1

Recall that we can model the velocity of a water droplet in a cloud by

$$mv' = -mg - bv$$

where here, negative denotes downward velocity. m is the mass, g is Newton's gravitational constant, and b is a physical constant (like a coefficient of friction).

Activity C.3.2 *(∼25 min)*
A water droplet with a radius of 10 $\mu\mathrm{m}$ has a mass of about $4 \times 10^{-15}\mathrm{kg}$. It is determined in a laboratory that for a droplet this size, the constant b has a value of $0.003\mathrm{kg/s}$.

Activity C.3.2 *(∼25 min)*
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Part 1: Write down the differential equation modelling this scenario.

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Part 2: Find the general solution of this ODE.

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Part 1: Write down the differential equation modelling this scenario.

Part 2: Find the general solution of this ODE.

Part 3: What is the terminal velocity of the droplet?

Activity C.3.2 *(∼25 min)*
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Part 1: Write down the differential equation modelling this scenario.

Part 2: Find the general solution of this ODE.

Part 3: What is the terminal velocity of the droplet?

Part 4: If the droplet starts from rest ($v = 0$), what is its velocity after 0.01 s?

Definition C.3.3

The second part of the previous activity is an example of an **Initial Value Problem (IVP)**; we were given the initial value of the velocity in addition to our differential equation.

Activity C.3.4 *(∼10 min)*

Solve the IVP

$$y' + 3y = 0, y(0) = 2.$$

Activity C.3.5 *(∼10 min)*
Solve the IVP

$$y' - 2y = 2, y(0) = 1.$$

Activity C.3.6 *(~5 min)*

Solve the IVP

$$y' - 2y = 2, y(2) = 1.$$

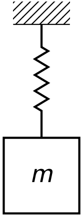
Module C Section 4

Observation C.4.1

What happens when your tire hits a pothole?

Activity C.4.2 *(∼5 min)*

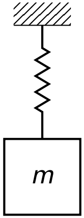
More abstractly, let's attach a mass (weighing $m\mathrm{kg}$) to a spring.



List all forces acting on the mass.

Activity C.4.3 *(∼5 min)*

Hooke's law says that the force exerted by the spring is proportional to the distance the spring is stretched.

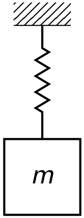


Write a differential equation modeling the displacement of the mass.

Observation C.4.4

There is an equillibrium point where the force of gravity balances the spring force. If we measure displacement from this point, we can model the mass-spring system by

$$my'' = ky.$$



Activity C.4.5 *(∼15 min)*

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

Activity C.4.5 *(∼15 min)*
Consider the (numerically simplified) mass-spring equation

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Part 1: Find a solution.

Activity C.4.5 *(∼15 min)*

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$$y'' = -y.$$

Part 1: Find a solution.

Part 2: Find the general solution.

Activity C.4.5 *(∼15 min)*
Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

Part 1: Find a solution.
Part 2: Find the general solution.
Part 3: Describe the long term behavior of the mass-spring system.

Activity C.4.6 *(∼5 min)*
In applications, this infinitely oscillating behavior is often inappropriate.

Thus, a damper (dashpot) is often incorporated. This provides a force proportional to the velocity.

Write a differential equation modeling the displacement of a mass in a **damped** mass-spring system.

Definition C.4.7

A **homogeneous second order constant coefficient** differential equation can be written in the form

$$ay'' + by' + cy = 0.$$

Here, **homogeneous** refers to the 0 on the right hand side of the equation.

Activity C.4.8 *(∼15 min)*
Consider the second order constant coefficient equation

$$y'' = y.$$

Activity C.4.8 *(∼15 min)*

Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.

Activity C.4.8 *(∼15 min)*

Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.

Part 2: Find the general solution.

Activity C.4.8 *(∼15 min)*

Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.
Part 2: Find the general solution.
Part 3: Describe the long term behavior of the solutions.

Module C Section 5

Observation C.5.1

It is sometimes useful to think in terms of **differential operators**.

- We will use D to represent a derivative; another common notation is $\frac{\partial}{\partial x}$. So for any function y,

$$D(y) = \frac{\partial y}{\partial x} = y'.$$

- D^2 will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use I for the identity operator; it does nothing to a function. That is, $I(y) = y$. It can be thought of as $I = D^0$ (i.e. differentiate zero times).

In this language, the differential equation $y' + 3y = 0$ can be rewritten as $D(y) + 3I(y) = 0$, or $(D + 3I)(y) = 0$.

Thus, the question of solving the homogeneous differential equation is the question of finding the kernel of the differential operator $D + 3I$.

Activity C.5.2 *(∼5 min)*

What is the kernel of $D - I$?

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Part 1: Write a differential equation that corresponds to this question.

Activity C.5.2 *(∼5 min)*
What is the kernel of $D - I$?

Part 1: Write a differential equation that corresponds to this question.
Part 2: Find the general solution of this differential equation.

Activity C.5.3 *($\sim$5 min)*

Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

Activity C.5.4 *(∼10 min)*
Consider the ODE

$$y'' + 5y + 6y = 0.$$

Activity C.5.4 *(∼10 min)*
Consider the ODE

$$y'' + 5y + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Activity C.5.4 *(∼10 min)*
Consider the ODE

$$y'' + 5y + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

Activity C.5.4 *(∼10 min)*
Consider the ODE

$$y'' + 5y + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).
Part 3: Find the general solution of the ODE.

Observation C.5.5

If we let $\mathcal{L} = D^2 + 5D + 6I$, we can write the ODE

$$y'' + 5y + 6y = 0$$

as

$$\mathcal{L}(y) = 0.$$

Note that such an $\mathcal{L}$ is always a **linear transformation**.

Module C Section 6

Activity C.6.1 *(∼10 min)*
Consider the ODE

$$y'' + 5y - 6y = 0.$$

Activity C.6.1 *(∼10 min)*
Consider the ODE

$$y'' + 5y - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

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Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
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Activity C.6.1 *(∼10 min)*
Consider the ODE

$$y'' + 5y - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).
Part 3: Find the general solution of the ODE.

Activity C.6.2 *(∼10 min)*
Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

Activity C.6.3 *(∼15 min)*
Solve the ODE

$$y'' + y = 0.$$

Activity C.6.4 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Activity C.6.4 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Part 1: Find the general solution.

Activity C.6.4 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Part 1: Find the general solution.
Part 2: Describe the long-term behavior of the solutions.

Module C Section 7

Activity C.7.1 *(~10 min)*
Solve the ODE

$$y'' - 4y' + 4y = 0.$$

Observation C.7.2

To solve this, we need to find the kernel of $(D-2I)(D-2I)$.

- The kernel of $D-2I$ is $\left\{ce^{2t} \mid c \in \mathbb{R}\right\}$.
- However, if $(D-2I)(y) = Ae^{2t}$, then applying $D-2I$ twice will yield zero.
- So we must solve the ODE
$$y' - 2y = e^{2t}.$$

Activity C.7.3 *($\sim$15 min)*

Solve $y' - 2y = e^{2t}$.

Observation C.7.4

Thus, we have shown that the general solution of $y'' - 4y' + 4y = 0$ is $c_0 e^{2t} + c_1 te^{2t}$.

Activity C.7.5 *(∼15 min)*

Solve $y'' - 6y' + 9y = 0$.

Activity C.7.6 *(∼10 min)*

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

If r is a number such that $ar^2 + br + cr = 0$, what can you conclude?

(a) e^{rt} is a solution.

(b) e^{-rt} is a solution.

(c) te^{rt} is a solution.

(d) There are no solutions.

Activity C.7.7 *(∼5 min)*

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

When does the general solution have the form $c_0e^{rt} + te^{rt}$?

(a) When the polynomial $ax^2 + bx + c$ has two distinct real roots.

(b) When the polynomial $ax^2 + bx + c$ has a repeated real root.

(c) When the polynomial $ax^2 + bx + c$ has two distinct non-real roots.

(d) When the polynomial $ax^2 + bx + c$ has a repeated non-real root.

Observation C.7.8

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ax^2 + bx + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that te^{rt} is also a solution.
- if r is not real, Euler's formula allows us to express the solution in terms of $\sin(rt)$ and $\cos(rt)$.

Module C Section 8

Observation C.8.1

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ax^2 + bx + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that te^{rt} is also a solution.
- if r is not real, Euler's formula allows us to express the solution in terms of $\sin(rt)$ and $\cos(rt)$.

Activity C.8.2 *(∼15 min)*

Consider a mass of 4 kg suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\ \mathrm{kg/s}$.

The mass is pulled down 0.3 m and released from rest. How many times does it pass back through its equillibrium state?

(a) 0

(b) 1

(c) 2

(d) Infinitely many

Activity C.8.3 *(∼15 min)*

Consider a mass of 5 kg suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\mathrm{kg/s}$.

The mass is pulled down $0.3\mathrm{m}$ and released from rest. How many times does it pass back through its equillibrium state?

(a) 0

(b) 1

(c) 2

(d) Infinitely many

Observation C.8.4

It can be shown that in the **overdamped** situation, the spring might pass through the equillibrium position once (e.g. if given an initial push), but never more than once.

Module C Section 9

Activity C.9.1 *(∼10 min)*
A 1 kg mass is suspended from a spring with spring constant $k = 9\ \mathrm{kg/s^2}$. An external force is applied by an electromagnet and is modeled by the function $F(t) = \sin(t)$. Write an ODE modeling the displacement of the spring.

Observation C.9.2

In the previous activity, we encountered a **nonhomogeneous** second order constant coefficient ODE, i.e. of the form

$$ay'' + by' + cy = f$$

where a, b, c are constants, and $f(t)$ is a function.

We will again use **variation of parameters** to find a particular solution.

Activity C.9.3 *(∼15 min)*
Suppose y_1 and y_2 are two independent solutions of $\mathcal{L}(y) = 0$.

Our goal is to find a particular solution of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions v_1, v_2.

Activity C.9.3 *(∼15 min)*
Suppose y_1 and y_2 are two independent solutions of $\mathcal{L}(y) = 0$.

Our goal is to find a particular solution of the form $y_p = v_1y_1 + v_2y_2$ for some TBD functions v_1, v_2.

Part 1: Compute y_p'.

Activity C.9.3 *(∼15 min)*
Suppose y_1 and y_2 are two independent solutions of $\mathcal{L}(y) = 0$.

Our goal is to find a particular solution of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions v_1, v_2.

Part 1: Compute y_p'.
Part 2: To simplify calculations, we will **assume** $v_1' y_1 + v_2' y_2 = 0$. Assuming this, compute y_p''.

Activity C.9.3 *(∼15 min)*
Suppose y_1 and y_2 are two independent solutions of $\mathcal{L}(y) = 0$.

Our goal is to find a particular solution of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions v_1, v_2.

Part 1: Compute y_p'.
Part 2: To simplify calculations, we will **assume** $v_1' y_1 + v_2' y_2 = 0$. Assuming this, compute y_p''.
Part 3: Compute $\mathcal{L}(y_p)$; simplify the ODE $\mathcal{L}(y_p) = f$.

Observation C.9.4
If we can find v_1 and v_2 that satisfy

$$
\begin{aligned}
y_1 v_1' + y_2 v_2' &= 0 \\
y_1' v_1' + y_2' v_2' &= \frac{f}{a}
\end{aligned}
$$

then we have a solution. So we just need to solve this system of equations for v_1' and v_2'.

Activity C.9.5 *(∼15 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Activity C.9.5 *(∼15 min)*
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Part 1: Find y_1 and y_2, two independent solutions of $y'' + 9y = 0$.

Activity C.9.5 *(∼15 min)*
Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2, two independent solutions of $y'' + 9y = 0$.
Part 2: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(t) \end{aligned}$$

Activity C.9.5 *(∼15 min)*
Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2, two independent solutions of $y'' + 9y = 0$.
Part 2: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(t) \end{aligned}$$

Part 3: Write the general solution of the original nonhomogeneous ODE.

Activity C.9.6 *(∼10 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.

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Part 2: Write the general solution of the original nonhomogeneous ODE.

Module F: First order ODEs

How can we solve and apply first order ODEs?

At the end of this module, students will be able to...

F1. Sketching trajectories. ...

F2. Separable ODEs. ...

F3. Autonomous ODEs. ...

F4. First order linear ODEs. ...

F5. Exact ODES. ...

F6. Modeling motion. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Module F Section 1

Definition F.1.1

A **first order ODE** is an equation involving (for a function $y(x)$) only y', y, and x.

We will most often deal with **explicit first order ODEs**, which can be written in the form

$$y' = f(y, x)$$

for some function $f(y, x)$.

Activity F.1.2 *(∼5 min)*

Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.

Activity F.1.2 *(∼5 min)*

Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.

Part 1: Compute y' at each of the points $(1, 1)$, $(2, 1)$, $(3, -2)$, and $(4, -7)$.

Activity F.1.2 *(∼5 min)*

Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.

Part 1: Compute y' at each of the points $(1, 1)$, $(2, 1)$, $(3, -2)$, and $(4, -7)$.

Part 2:

Let $y_0(x)$ be a solution that passes through the point $(1, 1)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$?

(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

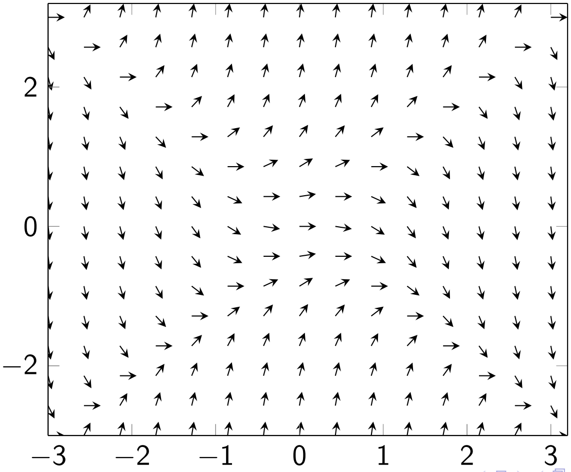
(C) $\lim_{x\to\infty} y_0(x) = \infty$

Definition F.1.3

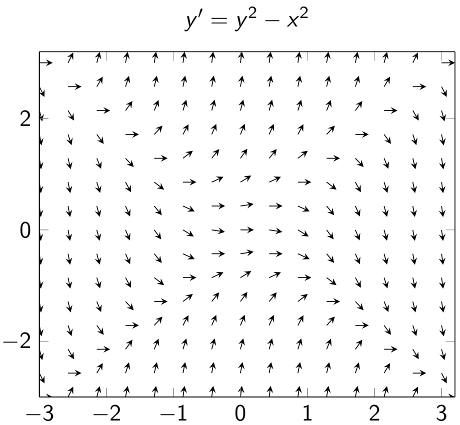
These kinds of questions are easier to answer if we draw a **slope field** (sometimes called a **direction field**.

To draw one, draw a small line segment or arrow with the correct slope at each point.

$$y' = y^2 - x^2$$



Activity F.1.4 *(∼5 min)*



Let $y_1(x)$ be a solution that passes through the point $(1, 3)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$?

(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

(C) $\lim_{x\to\infty} y_0(x) = \infty$

Activity F.1.5 *(∼15 min)*

Consider the ODE

$$y' = xy - x.$$

Activity F.1.5 *(∼15 min)*
Consider the ODE

$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.

Activity F.1.5 *(∼15 min)*
Consider the ODE

$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.
Part 2: Draw a solution that passes through the point (0,0).

Activity F.1.5 *(∼15 min)*

Consider the ODE

$$y' = xy - x.$$

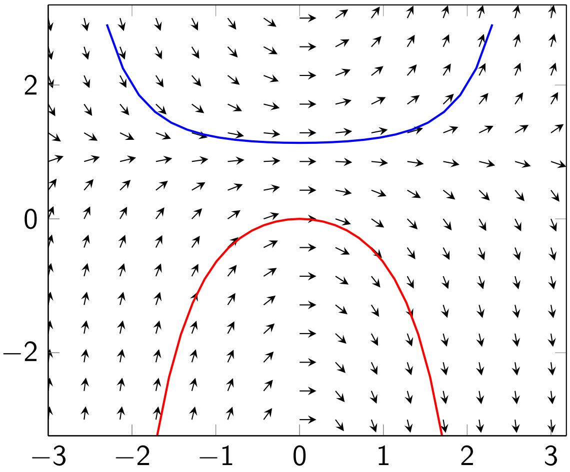
Part 1: Draw a slope field for this ODE.

Part 2: Draw a solution that passes through the point (0,0).

Part 3: Draw a solution that passes through the point (-2,2).

Observation F.1.6

$$y' = xy - x$$



Observation F.1.7

How can we solve $y' = xy - x$ exactly?

Notice $xy - x = x(y-1)$, so we can write $y' = x(y-1)$.

Write

$$\frac{y'}{y-1} = x.$$

This is called a **separable** DE.

Observation F.1.8

Integrate both sides (and switch to Leibniz notation):

$$\int \frac{1}{y-1}\frac{dy}{dx}\,dx = \int x\,dx.$$

The substitution rule (i.e. chain rule) says this is equivalent to

$$\int \frac{1}{y-1}dy = \int x\,dx.$$

Thus, $\ln|y-1| = \frac{1}{2}x^2 + c$. Exponentiating, we have

$$|y-1| = e^{\frac{1}{2}x^2+c} = e^{\frac{1}{2}x^2}e^c = c_0e^{\frac{1}{2}x^2}.$$

Allowing c_0 to take on negative values, we can drop the absolute value sign, and obtain

$$y = 1 + c_0e^{\frac{1}{2}x^2}.$$

Activity F.1.9 *(∼10 min)*
Find the general solution to
$$y' = xy + y.$$

Activity F.1.10 *(∼10 min)*

Solve the IVP

$$y' = \frac{x}{y}, y(0) = -1.$$

Module F Section 2

Observation F.2.1

There are two very simple kinds of separable ODEs.

Equations of the form $y' = f(x)$ can be solved immediately by integrating and produce explicit solutions.

Equations of the form $y' = f(y)$ are often impossible or difficult to solve explicitly. They are called **autonomous** equations.

Activity F.2.2 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2$$

.

Suppose a solution goes through the point $y(10) = 50$. What can you say about $y(11)$?

(a) $y(10) < y(11)$

(b) $y(10) = y(11)$

(c) $y(10) > y(11)$

Activity F.2.3 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2(y-2)$$

.

Activity F.2.3 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2(y-2)$$

.

Part 1: Draw a number line for y', indicating where it is positive or negative.

Activity F.2.3 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2(y-2)$$

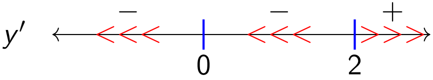
.

Part 1: Draw a number line for y', indicating where it is positive or negative.
Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?

Definition F.2.4

The **phase line** is a useful way to visualize the long term behavior of an autonomous DE.

For example, here is a phase line for the autonomous DE $y' = y^2(y-2)$.



Activity F.2.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Activity F.2.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Activity F.2.5 *(∼15 min)*
Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.
Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Activity F.2.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Activity F.2.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.

Activity F.2.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.

Part 5: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

Activity F.2.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.

Part 5: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

Part 6: Describe the long term behavior of solutions passing near the point $y(11) = 2$.

Definition F.2.6

The **critical points** of an autonomous DE are the numbers that give rise to equillibrium solutions (e.g. $0, -1, 2$ in the previous problem).

A **source** is an unstable equillibrium in which all nearby trajectories move away in the limit.

A **sink** is a stable equillibrium in which all nearby trajectories approach the equillibrium in the limit.

There are also unstable equillibria in which some nearby trajectories return, while others diverge, analagous to a saddle point.

Activity F.2.7 *(∼10 min)*

Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

Find and classify all of the critical points.

Module F Section 3

Module F Section 4

Module F Section 5

Module S: Systems of ODEs

How can we solve and apply systems of linear ODEs?

At the end of this module, students will be able to...

S1. **Solving systems.** ...

S2. **Modeling interacting populations.** ...

S3. **Modeling coupled oscillators.** ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): http://bit.ly/2y8AOwa
- Linear combinations of Euclidean vectors (Khan Academy): http://bit.ly/2nK3wne
- Adding and subtracting complex numbers (Khan Academy): http://bit.ly/1PE3ZMQ
- Adding and subtracting polynomials (Khan Academy): http://bit.ly/2d5SLGZ

Module S Section 1

Activity S.1.1 *(∼10 min)*

Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

Which of the following is true?

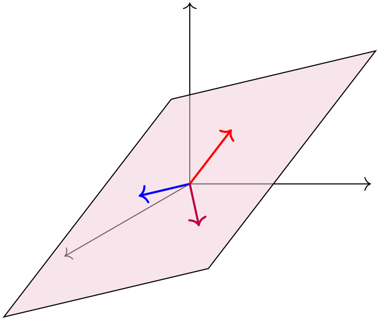
(A) span S is bigger than span T.

(B) span S and span T are the same size.

(C) span S is smaller than span T.

Definition S.1.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay on the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

Activity S.1.3 *(∼10 min)*

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$. Suppose $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$, so the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Which of the following is true of the vector equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}$?

(A) It is consistent with one solution

(B) It is consistent with infinitely many solutions

(C) It is inconsistent.

Fact S.1.4

For any vector space, the set $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{z}$ is consistent with infinitely many solutions.

Activity S.1.5 *(∼10 min)*

Find

$$\operatorname{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Fact S.1.6

A set of Euclidean vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

Activity S.1.7 *(∼5 min)*

Is the set of Euclidean vectors $\left\{\begin{bmatrix}-4\\2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}1\\10\\10\\2\\6\end{bmatrix}, \begin{bmatrix}3\\4\\7\\2\\1\end{bmatrix}\right\}$ linearly dependent or linearly independent?

Activity S.1.8 *(∼10 min)*

Is the set of polynomials $\left\{x^3+1, x^2+2x, x^2+7x+4\right\}$ linearly dependent or linearly independent?

Activity S.1.9 *(∼5 min)*

What is the largest number of vectors in $\mathbb{R}^4$ that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.10 *(∼5 min)*

What is the largest number of vectors in

$$\mathcal{P}^4 = \left\{ax^4 + bx^3 + cx^2 + dx + e \,\middle|\, a, b, c, d, e \in \mathbb{R}\right\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.11 *(∼5 min)*
What is the largest number of vectors in

$$\mathcal{P} = \{f(x) \mid f(x) \text{ is any polynomial}\}$$

that can form a linearly independent set?

(a) 3
(b) 4
(c) 5
(d) You can have infinitely many vectors and still be linearly independent.

Module N: Numerical

How can we use numerical approximation methods to apply and solve unsolvable ODEs?

At the end of this module, students will be able to...

N1. First Order Existence and Uniqueness. ...

N2. Second Order Linear Existence and Uniqueness. ...

N3. Systems Existence and Uniqueness. ...

N4. Euler's method for first order ODES. ...

N5. Euler's method for systems. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Module N Section 1

Definition N.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Module D: Discontinuous functions in ODEs

How can we solve and apply ODEs involving functions that are not continuous?

At the end of this module, students will be able to...

D1. **Laplace Transform.** ...

D2. **Discontinuous ODEs.** ...

D3. **Modeling non-smooth motion.** ...

D4. **Modeling non-smooth oscillators.** ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Module D Section 1

Definition D.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.