# Module S: Systems of ODEs

# How can we solve and apply systems of linear ODEs?

At the end of this module, students will be able to...

**S1.** **Solving systems.** ...solve systems of constant coefficient ODEs

**S2.** **Modeling interacting populations.** ...model the populations of two interacting populations with a system of ODEs

**S3.** **Modeling coupled oscillators.** ...model systems of coupled mechanical oscillators using a system of ODEs

**Readiness Assurance Outcomes**

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

# Module S Section 1

**Activity S.1.1** *(∼10 min)*
Consider the countries of Transia and Wakanda: each year, 8% of people living in Transia move to Wakanda, and 3% of Wakandans move to Transia.

Let $T$ be the population of Transia, and $W$ the population of Wakanda (both are functions of time, $t$.

Write down two differential equations modelling the population changes $\frac{dT}{dt}$ and $\frac{dW}{dt}$.

**Activity S.1.2** *(∼5 min)*

This problem resulted in a **system of linear differential equations**, namely

$$
\begin{aligned}
T' &= 0.03W - 0.08T \\
W' &= 0.08T - 0.03W
\end{aligned}
$$

Rewrite this system using differential operators.

**Activity S.1.3** *(∼15 min)*

Solve the system

$$\begin{aligned}(D+0.08)T-(0.03)W&=0\\ -0.08T+(D+0.03)W&=0\end{aligned}$$

**Observation S.1.4**

Because $D$ is linear, $a(D+b) = (D+b)a$ for constants $a, b$. This is not true in general!

Thus, for any **constant coefficient linear systems of differential equations**, we can use our typical elimination technique.

**Activity S.1.5** *(∼15 min)*
Solve the system

$$\begin{aligned} x' &= 5x - 2y \\ y' &= 6y - 3x \end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = -1$.

# Module S Section 2

**Activity S.2.1** *(∼20 min)*

Solve the system

$$\begin{aligned} x' &= 3x - 4y + 1 \\ y' &= 4x - 7y + 10t \end{aligned}$$

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*Part 1:* Rewrite the system using differential operators

**Activity S.2.1** *(∼20 min)*
Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

*Part 1:* Rewrite the system using differential operators
*Part 2:* Use elimination to eliminate a variable

**Activity S.2.1** *(∼20 min)*
Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

*Part 1:* Rewrite the system using differential operators
*Part 2:* Use elimination to eliminate a variable
*Part 3:* Solve the resulting second order ODE in one variable.

**Activity S.2.1** *(∼20 min)*

Solve the system

$$\begin{aligned} x' &= 3x - 4y + 1 \\ y' &= 4x - 7y + 10t \end{aligned}$$

*Part 1:* Rewrite the system using differential operators

*Part 2:* Use elimination to eliminate a variable

*Part 3:* Solve the resulting second order ODE in one variable.

*Part 4:* Find the solution for the other variable.

**Activity S.2.2** *(∼10 min)*

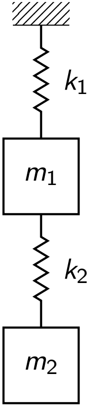
Solve the system

$$\begin{aligned} x' &= 3x - 2y + \sin(t) \\ y' &= 4x - y - \cos(t) \end{aligned}$$

# Module S Section 3

**Activity S.3.1** *(∼10 min)*

Consider two coupled masses with two springs.




Let $x_1$ be the position of the upper mass, and $x_2$ the position of the lower mass. Which ODE models the forces acting on the **lower** mass?

(A) $m_2x_2'' + k_2x_2 = 0$

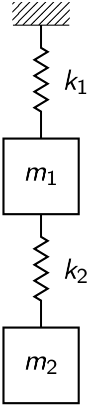
(B) $m_2x_2'' + k_2x_1 = 0$

(C) $m_2x_2'' + k_2(x_2 - x_1) = 0$

(D) $m_2x_2'' + k_2(x_1 - x_2) = 0$

**Activity S.3.2** *(∼5 min)*

Consider two coupled masses with two springs.




Let $x_1$ be the position of the upper mass, and $x_2$ the position of the lower mass. Which ODE models the forces acting on the **upper** mass?

(A) $m_1 x_1'' + k_1 x_1 = 0$

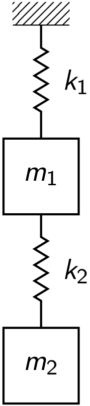
(B) $m_1 x_1'' + k_1 x_1 - k_2 x_2 = 0$

(C) $m_1 x_1'' + k_1 x_1 + k_2(x_2 - x_1) = 0$

(D) $m_1 x_1'' + k_1 x_1 + k_2(x_1 - x_2) = 0$

## **Activity S.3.3** *(∼30 min)*

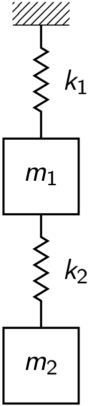
Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s^2}$, and $k_2 = 2\mathrm{kg/s^2}$. Then our model is




$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

## Activity S.3.3 *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s^2}$, and $k_2 = 2\mathrm{kg/s^2}$. Then our model is

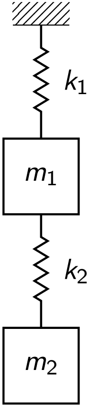



$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

*Part 1:* Rewrite the system using differential operators.

**Activity S.3.3** *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg}/\mathrm{s}^2$, and $k_2 = 2\mathrm{kg}/\mathrm{s}^2$. Then our model is



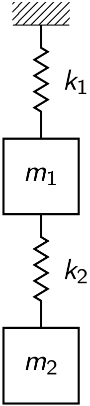

$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

*Part 1:* Rewrite the system using differential operators.

*Part 2:* Use elimination to write a single fourth order ODE for $x_1$.

## Activity S.3.3 *(~30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s}^2$, and $k_2 = 2\mathrm{kg/s}^2$. Then our model is



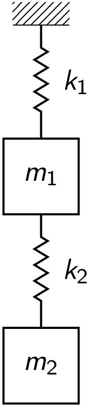

$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

*Part 1:* Rewrite the system using differential operators.
*Part 2:* Use elimination to write a single fourth order ODE for $x_1$.
*Part 3:* Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

## Activity S.3.3 *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s^2}$, and $k_2 = 2\mathrm{kg/s^2}$. Then our model is




$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

*Part 1:* Rewrite the system using differential operators.
*Part 2:* Use elimination to write a single fourth order ODE for $x_1$.
*Part 3:* Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

*Part 4:* Determine a function for $x_2$ as well.

# Module S Section 4

**Activity S.4.1** *(∼5 min)*

Consider a forest of bamboo that grows unimpeded by other organisms. Which ODE models the size of the population best (all constants are positive)?

(a) $\frac{dB}{dt} = k$

(b) $\frac{dB}{dt} = kB$

(c) $\frac{dB}{dt} = kB - aB^2$

(d) $\frac{dB}{dt} = kB^2$

**Activity S.4.2** *(~5 min)*

The model

$$\frac{dB}{dt} = kB$$

models an ideal growth, free from competition (e.g. if population is sparse).

The model

$$\frac{dB}{dt} = kB - aB^2$$

models competitive growth.

Observe that both models are autonomous. Draw a phase line for each model, and describe the possible long term behaviors.

**Activity S.4.3** *(∼10 min)*

Which of the following best models the bamboo population in the presence of a panda population $(P)$?

(a) $\frac{dB}{dt} = kB - aB^2$

(b) $\frac{dB}{dt} = kB - aB^2 - cP$

(c) $\frac{dB}{dt} = kB - aB^2 - cP^2$

(d) $\frac{dB}{dt} = kB - aB^2 - cBP$

**Activity S.4.4** *(∼5 min)*

Which of the following best models the (sparse) Panda population in the bamboo forest?

(a) $\frac{dP}{dt} = -dP$

(b) $\frac{dP}{dt} = -dP + fBP$

(c) $\frac{dP}{dt} = -dP - fBP$

(d) $\frac{dP}{dt} = -dP - fBP - gP^2$

**Observation S.4.5**

The interacting bamboo and Panda populations are modelled by the **autonomous system**

$$\frac{dB}{dt} = kb - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

These are referred to as **Lotka-Volterra equations**

**Activity S.4.6** *(∼10 min)*

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

**Activity S.4.6** *(∼10 min)*

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

*Part 1:* When is $\frac{dB}{dt}$ zero?

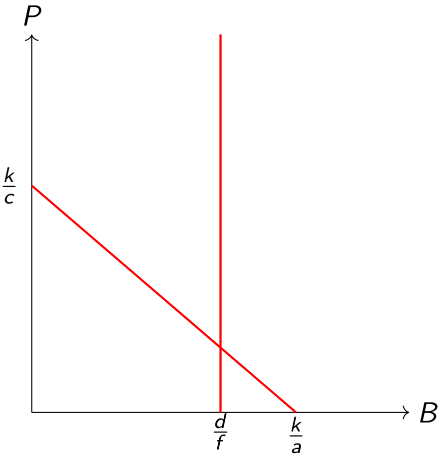
**Activity S.4.6** *($\sim$10 min)*
Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

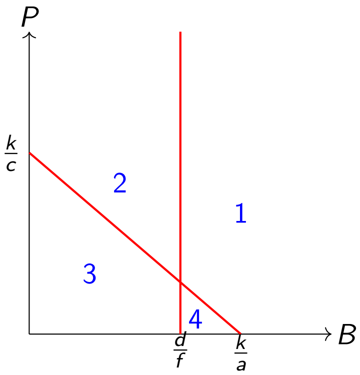
*Part 1:* When is $\frac{dB}{dt}$ zero?
*Part 2:* When is $\frac{dP}{dt}$ zero?

## Observation S.4.7

These lines where the population of one species is unchanging are called **isoclines**

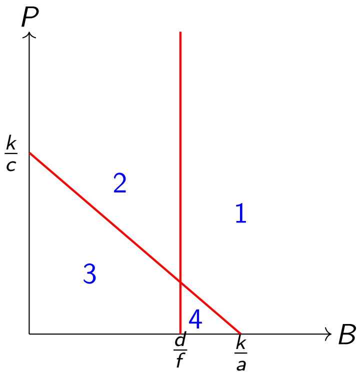


# Activity S.4.8 *(∼15 min)*



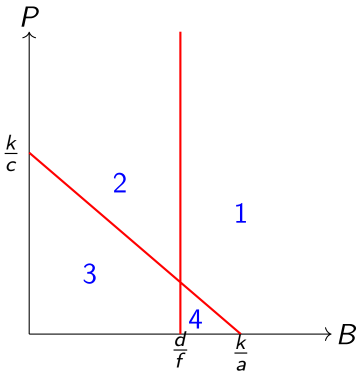

For each of the four regions

# Activity S.4.8 *(∼15 min)*



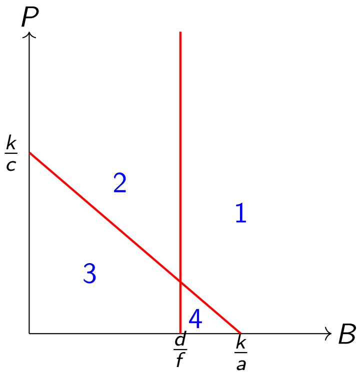

For each of the four regions
*Part 1:* Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

# Activity S.4.8 *(∼15 min)*




For each of the four regions
*Part 1:* Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.
*Part 2:* Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

# Activity S.4.8 *(∼15 min)*




For each of the four regions

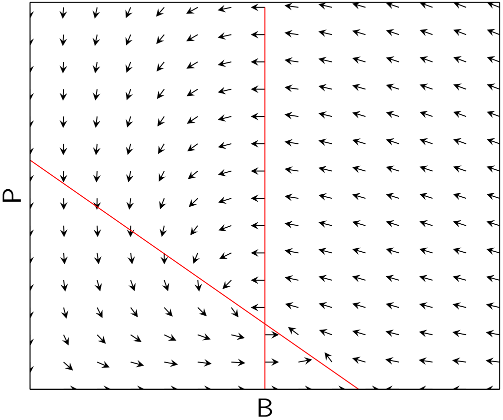
*Part 1:* Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

*Part 2:* Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

*Part 3:* Describe the general shape of the trajectories.

# Observation S.4.9

Plotting the slope field with software makes it more clear that the trajectories are closed curves.



# Module S Section 5

**Activity S.5.1** *(∼5 min)*

Consider populations of Green Sunfish ($G$) and Bluegills ($B$) in the same lake. They compete for the same food.

Which system of ODEs would model this interaction best?

(A)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

(B)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = -0.1B - 0.003B^2 + 0.005BG$$

(C)

$$\frac{dG}{dt} = -0.1G - 0.002G^2 + 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

(D)

**Activity S.5.2** *(∼15 min)*
Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

**Activity S.5.2** *(∼15 min)*
Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

*Part 1:* Plot the isoclines for each species.

**Activity S.5.2** *(~15 min)*

Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

*Part 1:* Plot the isoclines for each species.

*Part 2:* If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?

**Activity S.5.2** *(∼15 min)*

Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

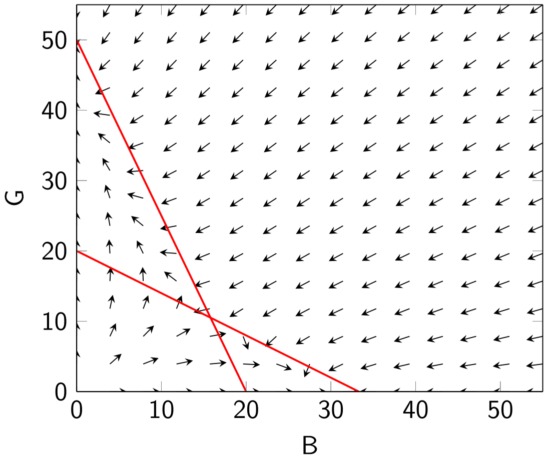
*Part 1:* Plot the isoclines for each species.

*Part 2:* If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?

*Part 3:* If the lake is stocked with 25 Bluegills and 5 Greenfish, what will happen?

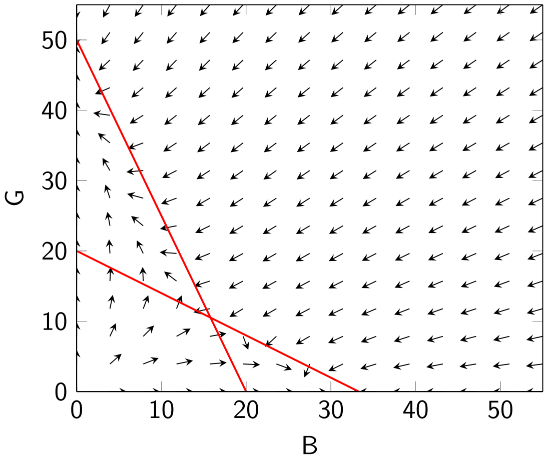
**Activity S.5.3** *(∼10 min)*

Plotting the slope field along with the isoclines makes the unstable behavior more clear.



**Activity S.5.3** *(∼10 min)*

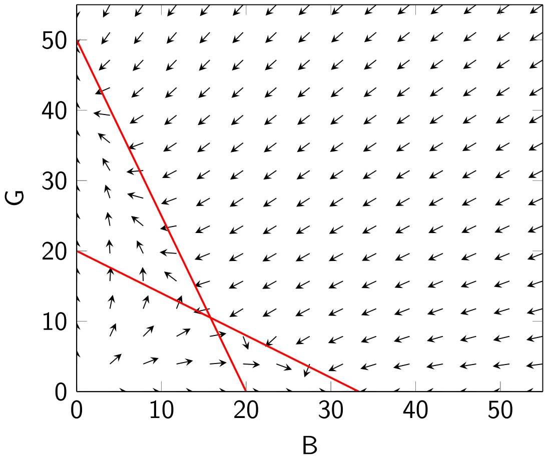
Plotting the slope field along with the isoclines makes the unstable behavior more clear.




*Part 1:* If the lake is stocked with 20 of each species, what will happen?

**Activity S.5.3** *(∼10 min)*

Plotting the slope field along with the isoclines makes the unstable behavior more clear.




*Part 1:* If the lake is stocked with 20 of each species, what will happen?

*Part 2:* If the lake is stocked with 30 Bluegills and 10 Greenfish, what will happen?