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Module S: Systems of ODEs

How can we solve and apply systems of linear ODEs?

At the end of this module, students will be able to...

S1. Solving systems. ...solve systems of constant coefficient ODEs

S2. Modeling interacting populations. ...model the populations of two interacting populations with a system of ODEs

S3. Modeling coupled oscillators. ...model systems of coupled mechanical oscillators using a system of ODEs

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Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

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Module S Section 1

Activity S.1.1 *(∼10 min)*
Consider the countries of Transia and Wakanda: each year, 8% of people living in Transia move to Wakanda, and 3% of Wakandans move to Transia.

Let T be the population of Transia, and W the population of Wakanda (both are functions of time, t.

Write down two differential equations modelling the population changes $\frac{dT}{dt}$ and $\frac{dW}{dt}$.

Activity S.1.2 *(∼5 min)*
This problem resulted in a **system of linear differential equations**, namely

$$T' = 0.03W - 0.08T$$
$$W' = 0.08T - 0.03W$$

Rewrite this system using differential operators.

Activity S.1.3 *(∼15 min)*
Solve the system

$$(D + 0.08)T - (0.03)W = 0$$
$$-0.08T + (D + 0.03)W = 0$$

Observation S.1.4
Because D is linear, $a(D+b) = (D+b)a$ for constants a, b. This is not true in general!

Thus, for any **constant coefficient linear systems of differential equations**, we can use our typical elimination technique.

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Activity S.1.5 *(∼15 min)*
Solve the system

$$x' = 5x - 2y$$
$$y' = 6y - 3x$$

with initial conditions $x(0) = 2$, $y(0) = -1$.

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Module S Section 2

Activity S.2.1 *(∼20 min)*
Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Activity S.2.1 *(∼20 min)*
Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Activity S.2.1 *(∼20 min)*
Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators
Part 2: Use elimination to eliminate a variable

Activity S.2.1 *(∼20 min)*
Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators
Part 2: Use elimination to eliminate a variable
Part 3: Solve the resulting second order ODE in one variable.

Activity S.2.1 *(∼20 min)*
Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators
Part 2: Use elimination to eliminate a variable
Part 3: Solve the resulting second order ODE in one variable.
Part 4: Find the solution for the other variable.

Activity S.2.2 *(∼10 min)*
Solve the system

$$x' = 3x - 2y + \sin(t)$$
$$y' = 4x - y - \cos(t)$$

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Module S Section 3

Activity S.3.1 *(∼5 min)*
Consider a forest of bamboo that grows unimpeded by other organisms. Which ODE models the size of the population best (all constants are positive)?

(a) $\frac{dB}{dt} = k$
(b) $\frac{dB}{dt} = kB$
(c) $\frac{dB}{dt} = kB - aB^2$
(d) $\frac{dB}{dt} = kB^2$

Activity S.3.2 *(∼5 min)*
The model

$$\frac{dB}{dt} = kB$$

models an ideal growth, free from competition (e.g. if population is sparse).

The model

$$\frac{dB}{dt} = kB - aB^2$$

models competitive growth.

Observe that both models are autonomous. Draw a phase line for each model, and describe the possible long term behaviors.

Activity S.3.3 *(∼10 min)*
Which of the following best models the bamboo population in the presence of a panda population (P)?

(a) $\frac{dB}{dt} = kB - aB^2$
(b) $\frac{dB}{dt} = kB - aB^2 - cP$
(c) $\frac{dB}{dt} = kB - aB^2 - cP^2$
(d) $\frac{dB}{dt} = kB - aB^2 - cBP$

Activity S.3.4 *(∼5 min)*
Which of the following best models the (sparse) Panda population in the bamboo forest?

(a) $\frac{dP}{dt} = -dP$
(b) $\frac{dP}{dt} = -dP + fBP$
(c) $\frac{dP}{dt} = -dP - fBP$
(d) $\frac{dP}{dt} = -dP - fBP - gP^2$

Observation S.3.5
The interacting bamboo and Panda populations are modelled by the **autonomous system**

$$\frac{dB}{dt} = kb - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

These are referred to as **Lotka-Volterra equations**

Activity S.3.6 *(∼10 min)*
Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Activity S.3.6 *(∼10 min)*
Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Part 1: When is $\frac{dB}{dt}$ zero?

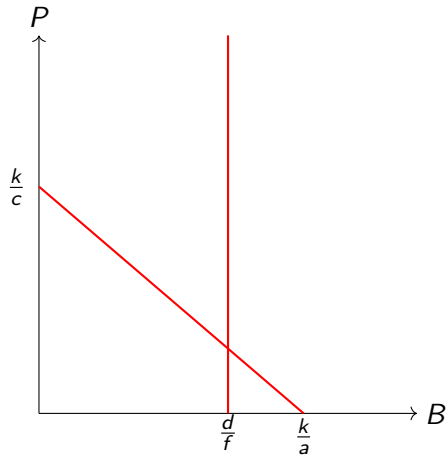
Activity S.3.6 *(∼10 min)*
Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

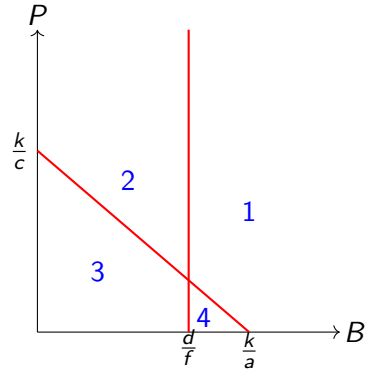
Part 1: When is $\frac{dB}{dt}$ zero?
Part 2: When is $\frac{dP}{dt}$ zero?

Observation S.3.7

These lines where the population of one species is unchanging are called **isoclines**

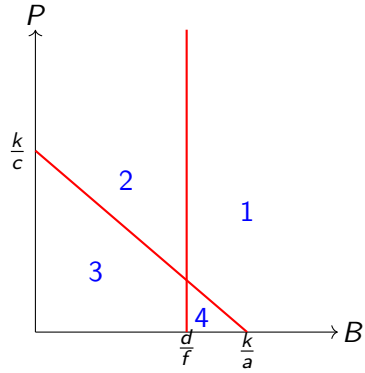


Activity S.3.8 *(∼15 min)*



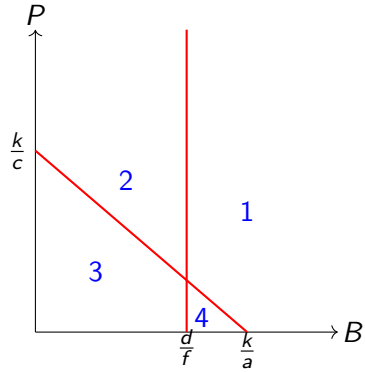
For each of the four regions

Activity S.3.8 *(∼15 min)*



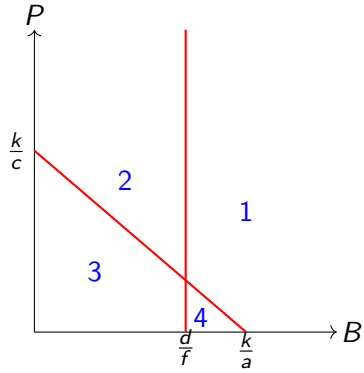
For each of the four regions
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Activity S.3.8 *(∼15 min)*



For each of the four regions
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.
Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

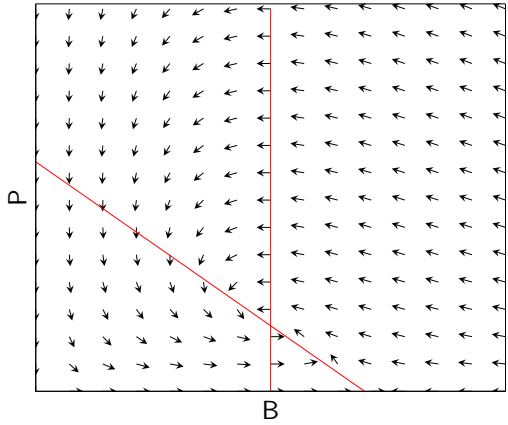
Activity S.3.8 *(∼15 min)*



For each of the four regions
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.
Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).
Part 3: Describe the general shape of the trajectories.

Observation S.3.9

Plotting the slope field with software makes it more clear that the trajectories are closed curves.



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Activity S.4.1 *(∼20 min)*

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Activity S.5.1 *(∼20 min)*