Module C: Constant coefficient linear ODEs

Standards for this Module

How can we solve and apply linear constant coefficient ODEs? At the end of this module, students will be able to...

C1. Constant coefficient first order. ...find the general solution to a first order constant coefficient ODE.

C2. Modeling motion in viscous fluids. ...model the motion of a falling object with linear drag

C3. Homogeneous constant coefficient second order. ...find the general solution to a homogeneous second order constant coefficient ODE.

C4. IVPs. ...solve initial value problems for constant coefficient ODEs

C5. Non-homogenous constant coefficient second order. ...find the general solution to a non-homogeneous second order constant coefficient ODE

C6. Modeling oscillators. ...model (free or forced, damped or undamped) mechanical oscillators with a second order ODE

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Describe Newton's laws in terms of differential equations.
- Find all roots of a quadratic polynomial.
- Use Euler's theorem to relate $\sin(t)$, $\cos(t)$, and e^t.
- Use Euler's theorem to simplify complex exponentials.
- Use substitution to compute indefinite integrals.
- Use integration by parts to compute indefinite integrals.
- Solve systems of two linear equations in two variables.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Describe Newtons laws in terms of differential equations. `https://youtu.be/cioi4lRrAzw`
- Find all roots of a quadratic polynomial. `https://youtu.be/2ZzuZvz33X0` `https://youtu.be/TV5kDqiJ1Os`
- Use Eulers theorem to relate $\sin(t)$, $\cos(t)$, and e^t and to simplify complex exponentials. `https://youtu.be/F_0yfvm0UoU` `https://youtu.be/sn3orkHWqUQ`
- Use substitution to compute indefinite integrals. `https://youtu.be/b76wePnIBdU`
- Use integration by parts to compute indefinite integrals. `https://youtu.be/bZ8YAHDTFJ8`
- Solve systems of two linear equations in two variables. `https://youtu.be/Y6JsEja15Vk`

Section C.1

Activity C.1.1 *(∼5 min)* Why don't clouds fall out of the sky?

(a) They are lighter than air

(b) Wind keeps them from falling

(c) Electrostatic charge

(d) They do fall, just very slowly

Activity C.1.2 *(∼5 min)* List all of the forces acting on a tiny droplet of water falling from the sky.

Activity C.1.3 *(∼5 min)* Tiny droplets of water obey **Stoke's law**, which says that air resistance is proportional to (the magnitude of) velocity.

- Let v be the velocity of a droplet of water (positive for upward, negative for downward).
- Let $g > 0$ be the magnitude of acceleration due to gravity and $b > 0$ be another positive constant.

Apply Newton's second law (force = mass × acceleration) to determine which of the following **ordinary differential equations (ODEs)** models the velocity of a falling droplet of water.

(a) $v' = g - v$

(b) $v' = g + v$

(c) $mv' = -mg - bv$

(d) $mv' = -mg + bv$

Observation C.1.4 The modeling equation

$$mv' = -mg - bv$$

may be obtained by splitting the total force into gravity and air resistance:

$$F = F_g + F_r$$

Then $F = ma = mv'$ and $F_g = m(-g) = -mg$ are the result of Newton's second law, and $F_r = -bv$ holds because it should be (a) in the opposite direction of velocity and (b) a constant multiple of velocity.
Note that this equation may be rearranged as follows to group v and its derivative v' together on the left-hand side:

$$v' + \left(\frac{b}{m}\right) v = -g$$

Definition C.1.5 A **first order constant coefficient** differential equation can be written in the form

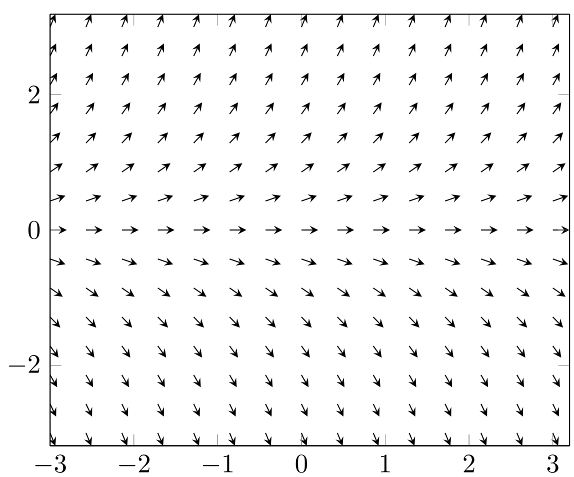
$$y' + by = f(x),$$

or equivalently,

$$\frac{dy}{dx} + by = f(x).$$

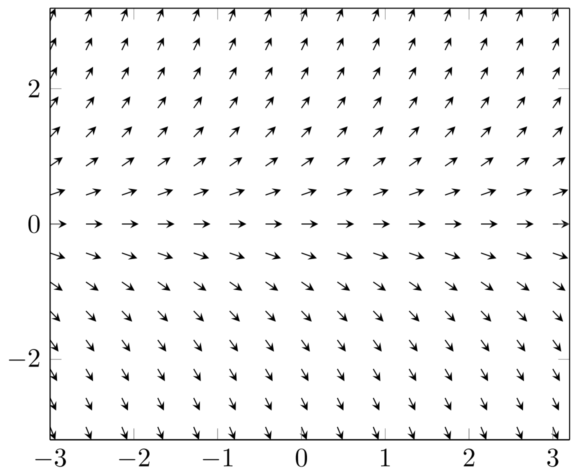
We will use both notations interchangeably.
Here, **first order** refers to the fact that the highest derivative we see is the first derivative of y.

Observation C.1.6 Consider the differential equation $y' = y$.
A useful way to visualize a first order differential equation is by a **slope field**



Each arrow represents the slope of a solution **trajectory** through that point.

Activity C.1.7 *(∼5 min)* Consider the differential equation $y' = y$ with slope field below.



Part 1: Draw a trajectory through the point $(0, 1)$.
Part 2: Draw a trajectory through the point $(-1, -1)$.

Activity C.1.8 *(∼15 min)* Consider the differential equation $y' = y$.
Part 1: Find a solution to $y' = y$.
Part 2: Modify this solution to write an expression describing **all** solutions to $y' = y$.

Definition C.1.9 A differential equation will have many solutions. Each individual solution is said to be a **particular solution**, while the **general solution** encompasses **all** of these by using parameters such as C, k, c_0, c_1 and so on. For example:

- The general solution to the differential equation $y' = 2x - 3$ is $y = x^2 - 3x + C$ (as done in Calculus courses).
- The general solution for $y' = y$ is $y = ke^x$ (as done in the previous activity).

Activity C.1.10 *(∼15 min)* Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.
Part 1: Solve $y' = 2y$.
Part 2: Solve $y' = y + 2$.

Activity C.1.11 *(∼15 min)* Find the solution for $y' = y + 2$ directly.
Simple idea: Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.
Part 1: Use the Product Rule to find $y_p' = \frac{d}{dx}[ve^x]$.
Part 2: Substitute y_p and y_p' into the equation $y' = y + 2$.
Part 3: Solve for v', and integrate to find v.
Part 4: Find y_p.

Observation C.1.12 The technique outlined in the previous activity is called **variation of parameters**. If y_0 is a particular solution of the **homogeneous** equation, assume that a particular solution of the **non-homogeneous** equation has the form $y_p = vy_0$, and then determine what v must be.
Example:

$$\begin{aligned} y' + 3y &= 0 && \text{homogeneous} \\ y' + 3y &= x && \text{non-homogeneous} \end{aligned}$$

Note that each term of the homogeneous equation includes y or it derivatives.

Activity C.1.13 *(~20 min)* Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$\begin{aligned} y' + 3y &= 0 && \text{homogeneous} \\ y' + 3y &= x && \text{non-homogeneous} \end{aligned}$$

Part 1: Modify e^x to find the general solution y_h for the homogeneous equation.
Part 2: Choose a particular solution y_0 for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** v. Substitute y_p into non-homogeneous equation and simplify.
Part 3: Determine v, and then determine y_p.

Observation C.1.14 Since $y_h = ke^{-3x}$ was the general solution of $y' + 3y = 0$, and $y_p = \frac{x}{3} - \frac{1}{9}$ is a particular solution of $y' + 3y = x$,

$$y = y_h + y_p = \left(ke^{-3x}\right) + \left(\frac{x}{3} - \frac{1}{9}\right)$$

is a solution to $y' + 3y = x$:

$$\frac{d}{dx}[y_h + y_p] + 3(y_h + y_p) = (y_h' + 3y_h) + (y_p' + 3y_p) = 0 + x = x$$

Fact C.1.15 Let a be a constant real number. Every constant coefficient first order ODE

$$y' + ay = f(x)$$

has the general solution

$$y = y_h + y_p$$

where y_h is the general solution to the homogeneous equation $y' + ay = 0$ and y_p is a particular solution to $y' + ay = f(t)$.

Activity C.1.16 *(∼15 min)* Find the general solution to $y' = 2y + x + 1$ using variation of parameters:

- Write the homogeneous equation and find its general solution y_h.
- Use a particular solution y_0 for the homogeneous equation to find a particular solution $y_p = vy_0$ for the original equation.
- Then $y = y_h + y_p$ gives the general solution to the equation.

Section C.2

Observation C.2.1 Recall that we can model the velocity of a water droplet in a cloud by

$$mv' = -mg - bv$$

where negative numbers represent downward motion, $m > 0$ is the mass of the droplet, $g > 0$ is the magnitude of acceleration due to gravity, and $b > 0$ is the proportion of wind resistance to speed.

Activity C.2.2 *(∼20 min)* A water droplet with a radius of $10\,\mu$m has a mass of about 4×10^{-15} kg. It is determined in a laboratory that for a droplet this size, the constant b has a value of 3×10^{-3} kg/s, and it is known that g is approximately $9.8\,\mathrm{m/s}^2$.
Complete the following tasks to study the motion of this droplet.
Part 1: Rewrite $mv' = -mg - bv$ in the form of $v' + av = \ ?$ for some value of a.
Part 2: Find the general solution of this ODE in terms of a and g. (Let $v_p = wv_0$ when using variation of parameters to avoid confusion.)
Part 3: Due to wind resistence, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of a and g?
Part 4: If the droplet starts from rest ($v = 0$ when $t = 0$), what is its velocity after 0.01 s? Use a calculator to compute the answer in m/s.

Definition C.2.3 The last part of the previous activity is an example of an **Initial Value Problem (IVP)**; we were given the initial value of the velocity in addition to our differential equation.

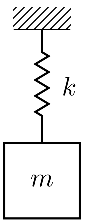
$$v' + (b/m)v = -g \qquad v(0) = 0$$

Physical scenarios often produce IVPs with a unique solution.

Section C.3

Observation C.3.1 What happens when your tire hits a pothole?
`https://prof.clontz.org/assets/img/good-bad-shocks.gif`

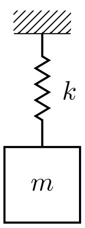
Activity C.3.2 *(∼5 min)* **Hooke's law** says that the force exerted by the spring is proportional to the distance the spring is stretched from its natural length, given by a spring coefficient $k > 0$.



Let y measure the displacement of the mass from the spring's natural length. Write a differential equation modeling the displacement of the m kg mass, assuming that the only force acting on the mass comes from the spring.

Observation C.3.3 Since the spring acts on the mass in the opposite direction of displacement, we may model the mass-spring system with

$$my'' = -ky.$$

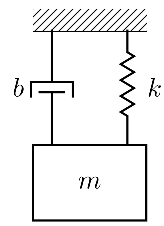


Activity C.3.4 *(∼15 min)* Consider the mass-spring equation $my'' = -ky$ where $m = k = 1$:

$$y'' = -y.$$

Part 1: Find a solution.
Part 2: Find the general solution.
Part 3: Describe the long term behavior of the mass-spring system.

Activity C.3.5 *(∼5 min)* The general solution $y = c_1 \cos(t) + c_2 \sin(t)$ models infinitely oscillating behavior, but in applications this does not occur.
Thus, a damper (a.k.a. dashpot) is often considered, which provides a force proportional to velocity, given by the coefficient $b > 0$. For example, friction may act as a damper to a mass-spring system.



Write a differential equation modeling the displacement of a mass in a **damped** mass-spring system.

Observation C.3.6 The damped mass-spring system can be modelled by

$$my'' = -by' - ky.$$

Here m is the mass, k is the spring constant, and b is the damping constant. We can rearrange this as

$$y'' + By' + Ky = 0$$

where $B = \frac{b}{m}$ and $K = \frac{k}{m}$.
This is a **homogeneous second order constant coefficient** differential equation. Here, **homogeneous** refers to the 0 on the right hand side of the equation.

Activity C.3.7 *(∼15 min)* Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.
Part 2: Find the general solution.
Part 3: Describe the long term behavior of the solutions.

Observation C.3.8 It is sometimes useful to think in terms of **differential operators**.

- We will use D to represent a derivative.So for any function y,

$$D(y) = \frac{\partial y}{\partial x} = y'.$$

- D^2 will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use I for the identity operator, so $I(y) = y$. (It can be thought of as $I = D^0$, take the derivative zero times.)

In this language, the differential equation $y' + 3y = 0$ can be rewritten as $D(y) + 3I(y) = 0$, or more simply $(D + 3I)(y) = 0$.
Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator $D + 3I$: all the functions y that the transformation $D + 3I$ turns into the zero function.

Activity C.3.9 *(∼5 min)* Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

a) $D - 4I$

b) $D + 4I$

c) $D^2 - 4I$

d) $D^2 + 4D$

Activity C.3.10 *(∼5 min)* The kernel of the differential operator $D-4I$ whose kernel is the general solution of the ODE $y' = 4y$. What is its general solution?

a) $y = ke^{-4x}$

b) $y = ke^{4x}$

c) $y = 4x + k$

d) $y = 4$

Activity C.3.11 *(∼5 min)* What are ODE and general solution given by the kernel of the differential operator $D - aI$ for a real number a?

a) $y' - ay = 0$ and $y = ke^{ax}$.

b) $y' + ay = 0$ and $y = ke^{-ax}$.

c) $y' - a = 0$ and $y = ax + k$.

d) $y'' + a = 0$ and $y = -\frac{a}{2}x^2 + kx + l$.

Observation C.3.12 The kernel of the differential operator $D-aI$ is given by the general solution $y = ke^{ax}$.

Activity C.3.13 *(∼15 min)* Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Use I, D, D^2 to write a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two simpler operators, as you would a polynomial. (This works because the order of applying the transformations D and I doesn't matter).
Part 3: Find the general solution for each factor, and then combine to find the general solution to the overall ODE.
Part 4: Check that your general solution is valid by computing y', y'' and plugging into $y'' + 5y' + 6y = 0$.

Observation C.3.14 The kernel of $(D + 3I)(D + 2I)$ is given by $y = k_1 e^{-3t} + k_2 e^{-2t}$.
In general for $\alpha \neq \beta$, the kernel of $(D - \alpha I)(D - \beta I)$ is given by $y = k_1 e^{at} + k_2 e^{bt}$.

Activity C.3.15 *(∼10 min)* Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

Activity C.3.16 *(∼15 min)* Recall that the general solution to $y''+y = 0$ is given by $y = c_1 \sin(x)+c_2 \cos(x)$. Show how to find this solution using the differential operator $D^2 + 1$.

Activity C.3.17 *(∼15 min)* Consider the ODE

$$y'' + 2y' + 5y = 0$$

.
Part 1: Find its general solution using complex numbers.
Part 2: Describe the general solution only involving real numbers.

Activity C.3.18 *(∼5 min)* Which of these are solutions to the following ODE?

$$y'' - 4y' + 4y = 0$$

a) $y = e^{2t}$, where $y' = 2e^{2t}$ and $y'' = 4e^{2t}$

b) $y = te^{2t}$, where $y' = e^{2t} + 2te^{2t}$ and $y'' = 4e^{2t} + 4e^{2t}$

c) $y = e^{2t} + te^{2t}$, where $y' = 3e^{2t} + 2te^{2t}$ and $y'' = 8e^{2t} + 4e^{2t}$

d) All of the above

Observation C.3.19 To solve $y'' - 4y' + 4y = 0$, we need to find the kernel of $(D - 2I)(D - 2I) = (D - 2I)^2$.

- The kernel of $D - 2I$ is given by ke^{2x}.
- But if $(D - 2I)(y) = e^{2t}$, then $(D - 2I)(D - 2I)(y) = (D - 2I)(e^{2t}) = 0$ also.
- That means the kernel of $(D - 2I)^2$ is given by both $(D - 2I)(y) = 0$ and $(D - 2I)(y) = e^{2t}$.

Activity C.3.20 *(∼15 min)* Solve $(D-2I)(y)=e^{2x}$.

Observation C.3.21 Since $(D-2I)(y)=0$ solves to ke^{2t} and $(D-2I)(y)=e^{2t}$ solves to kte^{2t}, we have shown that the general solution of

$$y''-4y'+4y=0$$

is

$$y=c_0e^{2t}+c_1te^{2t}.$$

Activity C.3.22 *(∼10 min)* Consider the homogeneous second order constant coefficient ODE

$$ay''+by'+cy=0.$$

If r is a number such that $ar^2+br+c=0$, what can you conclude?

(a) e^{rt} is a solution.

(b) e^{-rt} is a solution.

(c) te^{rt} is a solution.

(d) There are no solutions.

Activity C.3.23 *(∼5 min)* Consider the homogeneous second order constant coefficient ODE

$$ay''+by'+cy=0.$$

When does the general solution have the form $c_0e^{rt}+c_1te^{rt}$?

(a) When the polynomial ax^2+bx+c has two distinct real roots.

(b) When the polynomial ax^2+bx+c has a repeated real root.

(c) When the polynomial ax^2+bx+c has two distinct non-real roots.

(d) When the polynomial ax^2+bx+c has a repeated non-real root.

Observation C.3.24 Consider the homogeneous second order constant coefficient ODE

$$ay''+by'+cy=0$$

given by the differential operator $aD^2+bD+cI$. Let r be a (possibly non-real) solution to $ax^2+bx+c=0$:

- e^{rt} is a particular solution of the ODE.
- If r is a double root, te^{rt} is also a particular solution.
- if $r=\alpha+\beta i$ is not real, Euler's formula allows us to express the real-valued solutions in terms of $\sin(\beta t)$ and $\cos(\beta t)$.

Due to the usefulness of its solutions, $ax^2+bx+c=0$ is called the **auxiliary equation** for this ODE.

Section C.4

Remark C.4.1 While first or second-order constant-coefficient ODEs usually solve to general solutions such as $y = c_1e^t + c_2e^{-2t}$, the values of the parameters c_1, c_2 may be determined when given additional information.

Activity C.4.2 *(∼10 min)* Solve the IVP

$$y' + 3y = 0, \qquad y(0) = 2.$$

Activity C.4.3 *(∼15 min)* Solve $y'' - 6y' + 9y = 0$ where $y(0) = 2$ and $y(1) = \frac{3}{e^3}$.

Activity C.4.4 *(∼15 min)* Solve $y'' - 6y' + 8y = 0$ where $y(0) = 1$ and $y'(0) = -2$.

Section C.5

¡¡¡¡¡¡ HEAD =======
Observation C.5.1 It is sometimes useful to think in terms of **differential operators**.

- We will use D to represent a derivative; another common notation is $\frac{\partial}{\partial x}$. So for any function y,

$$D(y) = \frac{\partial y}{\partial x} = y'.$$

- D^2 will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use I for the identity operator; it does nothing to a function. That is, $I(y) = y$. It can be thought of as $I = D^0$ (i.e. differentiate zero times).

In this language, the differential equation $y'+3y = 0$ can be rewritten as $D(y)+3I(y) = 0$, or $(D+3I)(y) = 0$. Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator $D + 3I$.

Activity C.5.2 *(∼5 min)* What is the kernel of $D - I$?
Part 1: Write a differential equation that corresponds to this question.
Part 2: Find the general solution of this differential equation.

Activity C.5.3 *(∼5 min)* Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

Activity C.5.4 *(∼10 min)* Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).
Part 3: Find the general solution of the ODE.

Observation C.5.5 If we let $\mathcal{L} = D^2 + 5D + 6I$, we can write the ODE

$$y'' + 5y + 6y = 0$$

as

$$\mathcal{L}(y) = 0.$$

Note that such an $\mathcal{L}$ is always a **linear transformation**.

¿¿¿¿¿¿¿ master
Activity C.5.6 *(∼5 min)* Find the general solution to

$$y'' + y' - 12y = 0.$$

Section C.6

¡¡¡¡¡¡¡ HEAD =======
Activity C.6.1 *(∼5 min)* Consider the ODE

$$y'' + 5y - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).
Part 3: Find the general solution of the ODE.

Activity C.6.2 *(∼5 min)* Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

Activity C.6.3 *(∼5 min)* An **Initial Value Problem (IVP)** consists of an ODE along with some initial conditions that allow you to determine a single solution.
Solve the IVP

$$2y'' + 7y' + 6y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

.

Activity C.6.4 *(∼5 min)* Solve the ODE

$$y'' + y = 0.$$

Activity C.6.5 *(∼15 min)* Consider the ODE

$$y'' + 2y' + 5y = 0$$

.
Part 1: Find the general solution.
Part 2: Describe the long-term behavior of the solutions.

¿¿¿¿¿¿¿ master
Observation C.6.6 Solving $y'' + 2y' + 5y = 0$ produced a general solution

$$y = c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t}.$$

Applying Euler's formula yields

$$\begin{aligned} y &= c_1 e^{-t}\left(\cos(2t) + i\sin(2t)\right) + c_2 e^{-t}\left(\cos(2t) - i\sin(2t)\right) \\ &= (c_1 + c_2)e^{-t}\cos(2t) i(c_1 - c_2)e^{-t}\sin(2t) \end{aligned}$$

which we can rewrite (letting $k_1 = c_1 + c_2$ and $k_2 = i(c_1 - c_2)$) as

$$y = k_1 e^{-t}\cos(2t) + k_2 e^{-t}\sin(2t).$$

Activity C.6.7 *($\sim$15 min)* Solve the IVP

$$y'' + 6y' + 34y = 0, \qquad y(0) = 2, \qquad y'(0) = 4.$$

Section C.7

Section C.8

Observation C.8.1 Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ax^2 + bx + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that te^{rt} is also a solution.
- if $r = a + bi$ is not real, Euler's formula allows us to express the solution in terms of e^{at}, $\sin(bt)$, and $\cos(bt)$.

Activity C.8.2 *(∼15 min)* Consider the following scenario: a mass of 4 kg suspended from a damped spring with spring constant $k = 2$ kg/s^2 and damping constant $b = 6$ kg/s.
The mass is pulled down 0.3 m and released from rest.
Part 1: Write down an ODE modelling this scenario, and find the general solution.
Part 2: Use the initial conditions $y(0) = -0.3$ and $y'(0) = 0$ to find particular values of the constants.

Definition C.8.3 In the previous problem, we needed to solve

$$4y'' + 6y' + 2y = 0, \qquad y(0) = -0.3, \qquad y'(0) = 0.$$

This is called an **Initial Value Problem (IVP)** since we are provided with initial values of y and y'.
To solve an IVP, find a general solution of the ODE, and use the initial conditions to find the values of the constants.

Activity C.8.4 *(∼15 min)* Consider a mass of 5 kg suspended from a damped spring with spring constant $k = 2$ kg/s^2 and damping constant $b = 6$kg/s.
The mass is pulled down 0.3m and released from rest. How many times does it pass back through its equillibrium state?

(a) 0

(b) 1

(c) 2

(d) Infinitely many

Observation C.8.5 It can be shown that in the **overdamped** situation, the spring might pass through the equillibrium position once (e.g. if given an initial push), but never more than once.

Section C.9

Activity C.9.1 *(~10 min)* A 1 kg mass is suspended from a spring with spring constant $k = 9$ kg/s^2. An external force is applied by an electromagnet and is modeled by the function $F(t) = \sin(t)$. Write an ODE modeling the displacement of the spring.

Observation C.9.2 In the previous activity, we encountered a **nonhomogeneous** second order constant coefficient ODE, i.e. of the form

$$ay'' + by' + cy = f(x)$$

where a, b, c are constants, and $f(x)$ is a function.
We will again use **variation of parameters** to find a particular solution.

Activity C.9.3 *(~15 min)* Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L}(y) = ay'' + by' + cy$.
Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions v_1, v_2.
Part 1: Use the product rule (twice) to compute y_p'.
Part 2: To simplify calculations, we will **assume** $v_1' y_1 + v_2' y_2 = 0$. Assuming this, compute y_p''.
Part 3: Compute $\mathcal{L}(y_p)$; simplify the ODE $\mathcal{L}(y_p) = f(x)$.

Observation C.9.4 If we can find v_1 and v_2 that satisfy

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= \frac{f}{a} \end{aligned}$$

then we have a solution. So we just need to solve this system of equations for v_1' and v_2'.

Activity C.9.5 *(~15 min)* Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.
Part 1: Find y_1 and y_2, two independent solutions of $y'' + 9y = 0$.
Part 2: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(t) \end{aligned}$$

Part 3: Write the general solution of the original nonhomogeneous ODE.

Activity C.9.6 *(~10 min)* Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.
Part 1: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(3t) \end{aligned}$$

Part 2: Write the general solution of the original nonhomogeneous ODE.

Module F: First order ODEs

Standards for this Module

How can we solve and apply first order ODEs? At the end of this module, students will be able to...

F1. Sketching trajectories. ...given a slope field, sketch a trajectory of a solution to a first order ODE

F2. Separable ODEs. ...find the general solution to a separable first order ODE

F3. Modeling motion. ...model the motion of an object with quadratic drag

F4. Autonomous ODEs. ...find and classify the equillibria of an autonomous first order ODE, and describe the long term behavior of solutions

F5. First order linear ODEs. ...find the general solution to a first order linear ODE

F6. Exact ODES. ...find the general solution to an exact first order ODE

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine the intervals on which a polynomial is positive, negative, or zero.
- Use integration techniques like substitution, integration by parts, and partial fraction decomposition to compute indefinite integrals.
- Determine when a vector field is conservative.
- Find the potential function of a conservative vector field.
- Use variation of parameters to solve non-homogeneous first order constant coefficient ODEs (Standard C1)

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Determine the intervals on which a polynomial is positive, negative, or zero. `https://youtu.be/jGa0GJjwQh8`
- Use integration techniques like substitution, integration by parts, and partial fraction decomposition to compute indefinite integrals. `https://youtu.be/b76wePnIBdU` `https://youtu.be/bZ8YAHDTFJ8` `https://youtu.be/qMX4vRhXBOE`
- Determine when a vector field is conservative. `https://youtu.be/gAb1ZTD41wo`
- Find the potential function of a conservative vector field. `https://youtu.be/nY4mW_R-T40`
- Use variation of parameters to solve non-homogeneous ODEs when given the solution to the corresponding homogeneous ODE (Standard C5)

Section F.1

Definition F.1.1 A **first order ODE** is an equation involving (for a function $y(x)$) only y', y, and x.
We will most often deal with **explicit first order ODEs**, which can be written in the form

$$y' = f(y, x)$$

for some function $f(y, x)$.

Activity F.1.2 *(~5 min)* Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.
Part 1: Compute y' at each of the points $(1, 1)$, $(2, 1)$, $(3, -2)$, and $(4, -7)$.
Part 2:
Let $y_0(x)$ be a solution that passes through the point $(1, 1)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$?

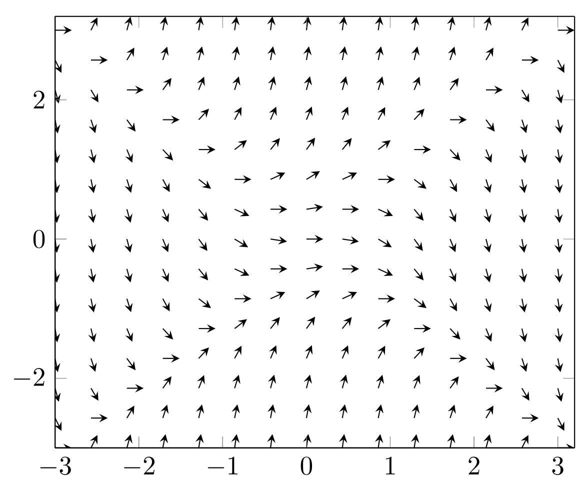
(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

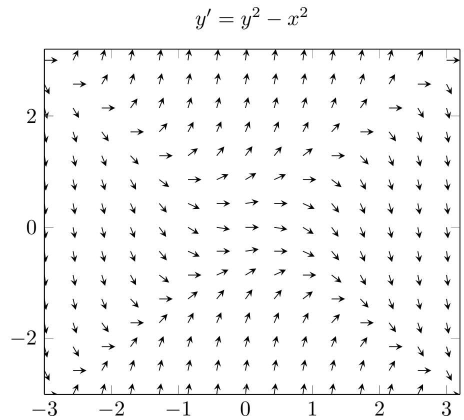
(C) $\lim_{x\to\infty} y_0(x) = \infty$

Definition F.1.3 These kinds of questions are easier to answer if we draw a **slope field** (sometimes called a **direction field**.
To draw one, draw a small line segment or arrow with the correct slope at each point.

$$y' = y^2 - x^2$$



Activity F.1.4 *(∼5 min)*



Let $y_1(x)$ be a solution that passes through the point $(2, 1)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$?

(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

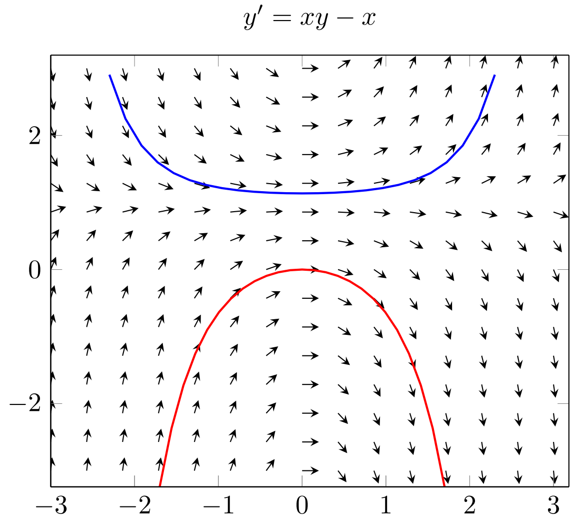
(C) $\lim_{x\to\infty} y_0(x) = \infty$

Activity F.1.5 *(∼15 min)* Consider the ODE

$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.
Part 2: Draw a solution that passes through the point (0,0).
Part 3: Draw a solution that passes through the point (-2,2).

Observation F.1.6



Observation F.1.7 How can we solve $y' = xy - x$ exactly?
Notice $xy - x = x(y-1)$, so we can write $y' = x(y-1)$.
Write

$$\frac{y'}{y-1} = x.$$

This is called a **separable** DE.

Observation F.1.8 Integrate both sides (and switch to Leibniz notation):

$$\int \frac{1}{y-1}\frac{dy}{dx}\ dx = \int x\ dx.$$

The substitution rule (i.e. chain rule) says this is equivalent to

$$\int \frac{1}{y-1} dy = \int x\ dx.$$

Thus, $\ln|y-1| = \frac{1}{2}x^2 + c$. Exponentiating, we have

$$|y-1| = e^{\frac{1}{2}x^2+c} = e^{\frac{1}{2}x^2}e^c = c_0 e^{\frac{1}{2}x^2}.$$

Allowing c_0 to take on negative values, we can drop the absolute value sign, and obtain

$$y = 1 + c_0 e^{\frac{1}{2}x^2}.$$

Activity F.1.9 *(∼10 min)* Find the general solution to

$$y' = xy + y.$$

Activity F.1.10 *(∼10 min)* Solve the IVP

$$y' = \frac{x}{y}, \qquad y(0) = -1.$$

Section F.2

Activity F.2.1 *(∼5 min)* In Module C, we discussed that tiny spherical objects like droplets of water obey Stoke's law: drag is proportional to velocity (speed). But for larger objects, a better model incorporates **quadratic drag**, i.e. drag is proportional to the square of velocity.
Which of the following ODEs models the velocity of a falling object subject to quadratic drag?

(a) $mv' = -mg + bv$

(b) $mv' = -mg - bv$

(c) $mv' = -mg + bv^2$

(d) $mv' = -mg - bv^2$

Activity F.2.2 *(∼10 min)* Consider our model of a falling object under quadratic drag

$$mv' = -mg + bv^2.$$

Part 1: For what value of v will the change in velocity be 0?
Part 2: Suppose the object is currently falling at a rate slower than this speed. What will happen?

(a) It will slow down

(b) It will keep falling at the same speed.

(c) It will speed up

Observation F.2.3 This equillibrium speed is called the **terminal velocity**.

Activity F.2.4 *(∼5 min)* Consider the following question:
A penny is dropped off the top of the Empire State Building. How fast will it be going when it hits the ground?
What information do we need to answer this question?

Observation F.2.5 The mass of a penny is 2.5g. The Empire State Building is (roughly) 400m tall.
The terminal velocity of a penny is about 25m/s.

Activity F.2.6 *(∼20 min)* We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.
Part 1: Solve for b in terms of v_t, m, g, and substitute this in to our model $v' = -g + \frac{b}{m}v^2$.
Part 2: Solve this separable ODE
Hint: $\frac{1}{v_t^2 - v^2} = \frac{2}{v_t}\left(\frac{1}{v_t - v} + \frac{1}{v_t + v}\right)$
Part 3: How fast is the penny going after 10 seconds?

Section F.3

Observation F.3.1 There are two very simple kinds of separable ODEs.
Equations of the form $y' = f(x)$ can be solved immediately by integrating and produce explicit solutions.
Equations of the form $y' = f(y)$ are often impossible or difficult to solve explicitly. They are called **autonomous** equations.

Activity F.3.2 *(~10 min)* Consider the autonomous equation

$$y' = y^2.$$

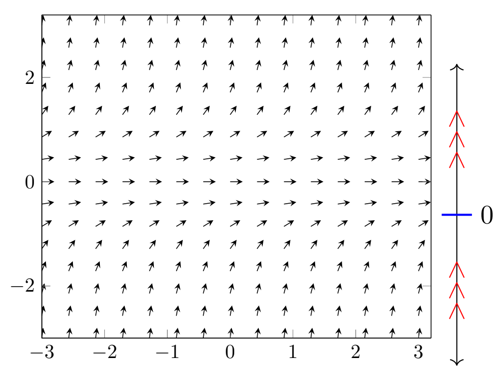
Part 1: Draw a slope field
Part 2: Suppose a solution goes through the point $y(10) = 50$. What can you say about $y(11)$?

(a) $y(10) < y(11)$

(b) $y(10) = y(11)$

(c) $y(10) > y(11)$

Observation F.3.3 Since the slopes do not change when moving horizontally (i.e. in the x direction), we often collapse the slope field onto the y-axis.



This is called a **phase line**.

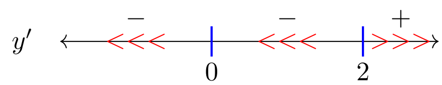
Activity F.3.4 *(~10 min)* Consider the autonomous equation

$$y' = y^2(y - 2).$$

Part 1: Draw a number line for y', indicating where it is positive or negative.
Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?
Part 3: What can you say about the long term behavior of a solution passing through $y(2) = 0.001$?
Part 4: What can you say about the long term behavior of a solution passing through $y(2) = -0.001$?

Section F.4

Definition F.4.1 Recall from last week: the **phase line** is a useful way to visualize the long term behavior of an autonomous DE.
For example, here is a phase line for the autonomous DE $y' = y^2(y-2)$.



Activity F.4.2 *(∼15 min)* Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.
Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.
Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.
Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.
Part 5: Describe the long term behavior of solutions passing near the point $y(3) = 0$.
Part 6: Describe the long term behavior of solutions passing near the point $y(11) = 2$.

Definition F.4.3 The **critical points** of an autonomous DE are the numbers that give rise to equillibrium solutions (e.g. $0, -1, 2$ in the previous problem).
A **source** is an unstable equillibrium in which all nearby trajectories move away in the limit.
A **sink** is a stable equillibrium in which all nearby trajectories approach the equillibrium in the limit.
There are also unstable equillibria in which some nearby trajectories return, while others diverge, analagous to a saddle point.

Activity F.4.4 *(∼15 min)* Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

Part 1: Find and classify all of the critical points.
Part 2: Describe the long term behavior of solutions passing near the point $y(1) = 1.5$.

Activity F.4.5 *(∼15 min)* Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Part 1: Find and classify all of the critical points.
Part 2: Describe the long term behavior of solutions passing near the point $y(0) = 0.5$.
Part 3: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

Section F.5

Observation F.5.1 In module C, we solved **constant coefficient linear ODEs**.
Today we will observe that our existing techniques allow us to solve all **first order linear ODES**, i.e. ODEs of the form

$$a(x)y' + b(x)y + c(x) = 0.$$

Such equations can always be rewritten (by rearranging and dividing by $a(x)$) in **standard form**:

$$y' + P(x)y = Q(x).$$

Activity F.5.2 *(∼20 min)* Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

Part 1: Solve the **homogeneous** first order linear ODE

$$y' + \frac{1}{x}y = 0.$$

Part 2: Use variation of parameters to the solve the original ODE

Activity F.5.3 *(∼15 min)* Solve the first order linear ODE

$$\frac{1}{x}y' - \frac{2}{x^2}y - x\cos(x) = 0.$$

Activity F.5.4 *(∼15 min)* Solve

$$(x+1)y' + y = x.$$

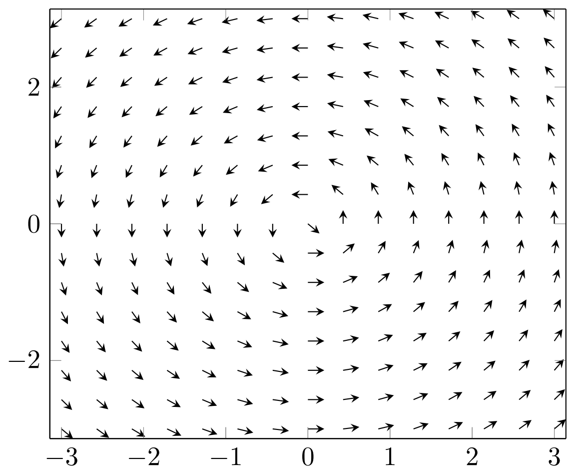
Remark F.5.5 The book provides a different technique (multiplying by an integrating factor); however, the method presented here does not require memorizing anything new.

Section F.6

Observation F.6.1 A vector field $\langle P, Q\rangle$ corresponds to the slope field of the differential equation

$$\frac{dy}{dx} = \frac{Q}{P}.$$

Thus, a solution to this ODE describes the path taken by the particle in this fluid flow.



Activity F.6.2 *(∼10 min)* Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2 y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2 y \frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2y^2 + x$.
Part 1: Compute $\nabla\phi$.
Part 2: Differentiate the equation $\phi(x, y) = c$ with respect to x.
Part 3: Solve the ODE $(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0$.

Definition F.6.3 If $\langle M, N\rangle$ is a conservative vector field, the ODE

$$M + N\frac{dy}{dx} = 0$$

is called **exact**. This ODE can also be written

$$\frac{dy}{dx} = \frac{-M}{N}.$$

If $\phi(x, y)$ is a potential function of $\langle M, N\rangle$, the general solution to the ODE is $\phi(x, y) = c$.
Careful: The slope field of the ODE $\frac{dy}{dx} = \frac{-M}{N}$ is the vector field $\langle -N, M\rangle$!

Activity F.6.4 *(∼10 min)* Determine which of the following ODEs are exact.

(a) $2xy + (x^2 - 2y)\frac{dy}{dx} = 0$

(b) $\frac{dy}{dx} = \frac{2xy}{x^2+2y}$

(c) $\frac{dy}{dx} = -\frac{2xy}{x^2+2y}$

Activity F.6.5 *(∼10 min)* Solve the exact ODE $2xy + (x^2 - 2y)\frac{dy}{dx} = 0$.

Section F.7

Activity F.7.1 *(∼10 min)* Determine which of the following ODEs are exact.

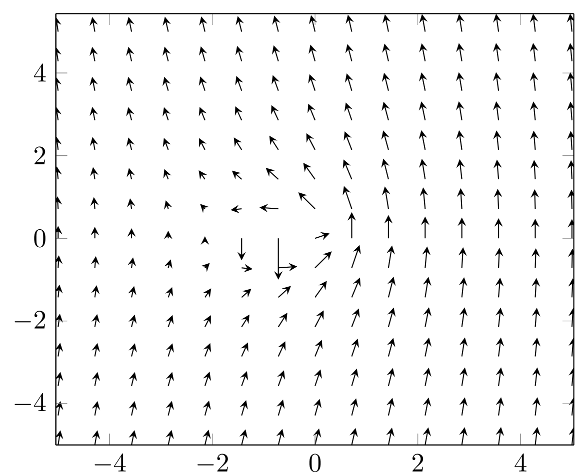
(a) $\frac{dy}{dx} = \frac{-y}{x^2+y^2+x}$

(b) $1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx} = 0$

Activity F.7.2 *(∼15 min)* Solve the exact ODE

$$1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx} = 0.$$

These solutions describe the trajectories taken by particles in the fluid flow below



Activity F.7.3 *(∼20 min)* Find solutions for the ODE

$$1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx} = 0$$

for each of the following initial conditions

(a) $y(0) = -1$.

(b) $y(-2) = -2$.

(c) $y(-4) = -4$.

Plot each of the solution curves.

Module S: Systems of ODEs

Standards for this Module

How can we solve and apply systems of linear ODEs? At the end of this module, students will be able to...

S1. Solving systems. ...solve systems of constant coefficient ODEs

S2. Modeling interacting populations. ...model the populations of two interacting populations with a system of ODEs

S3. Modeling coupled oscillators. ...model systems of coupled mechanical oscillators using a system of ODEs

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Section S.1

Activity S.1.1 *(~10 min)* Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad\qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

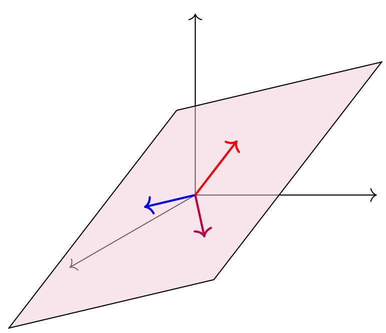
Which of the following is true?

(A) $\operatorname{span} S$ is bigger than $\operatorname{span} T$.

(B) $\operatorname{span} S$ and $\operatorname{span} T$ are the same size.

(C) $\operatorname{span} S$ is smaller than $\operatorname{span} T$.

Definition S.1.2 We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay on the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

Activity S.1.3 *(~10 min)* Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$. Suppose $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$, so the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Which of the following is true of the vector equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}$?

(A) It is consistent with one solution

(B) It is consistent with infinitely many solutions

(C) It is inconsistent.

Fact S.1.4 For any vector space, the set $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{z}$ is consistent with infinitely many solutions.

Activity S.1.5 *(∼10 min)* Find

$$\text{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Fact S.1.6 A set of Euclidean vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix}\mathbf{v}_1 & \dots & \mathbf{v}_n\end{bmatrix}$ has a column without a pivot position.

Activity S.1.7 *(∼5 min)* Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

Activity S.1.8 *(∼10 min)* Is the set of polynomials $\left\{x^3+1, x^2+2x, x^2+7x+4\right\}$ linearly dependent or linearly independent?

Activity S.1.9 *(∼5 min)* What is the largest number of vectors in $\mathbb{R}^4$ that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.10 *(∼5 min)* What is the largest number of vectors in

$$\mathcal{P}^4 = \left\{ax^4 + bx^3 + cx^2 + dx + e \,\middle|\, a, b, c, d, e \in \mathbb{R}\right\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.11 *(∼5 min)* What is the largest number of vectors in

$$\mathcal{P} = \{f(x) \,|\, f(x) \text{ is any polynomial}\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Module N: Numerical

Standards for this Module

How can we use numerical approximation methods to apply and solve unsolvable ODEs? At the end of this module, students will be able to...

N1. First Order Existence and Uniqueness. ...determine when a unique solution exists for a first order ODE

N2. Second Order Linear Existence and Uniqueness. ...determine when a unique solution exists for a second order linear ODE

N3. Systems Existence and Uniqueness. ...determine when a unique solution exists for a system of first order ODEs

N4. Euler's method for first order ODES. ...use Euler's method to find approximate solution to first order ODEs

N5. Euler's method for systems. ...use Euler's method to find approximate solutions to systems of first order ODEs

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Section N.1

Definition N.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Module D: Discontinuous functions in ODEs

Standards for this Module

How can we solve and apply ODEs involving functions that are not continuous? At the end of this module, students will be able to...

D1. Laplace Transform. ...compute the Laplace transform of a function

D2. Discontinuous ODEs. ...solve initial value problems for ODEs with discontinuous coefficients

D3. Modeling non-smooth motion. ...model the motion of an object undergoing discontinuous acceleration

D4. Modeling non-smooth oscillators. ...model mechanical oscillators undergoing discontinuous acceleration

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Section D.1

Definition D.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.