Linear Algebra

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Module I: Introduction

Remark I.0.1

This brief module gives an overview for the course.

Module I Section 0

Remark I.0.1
What is Linear Algebra?
Linear algebra is the study of **linear maps**.

- In Calculus, you learn how to approximate any function by a linear function.
- In Linear Algebra, we learn about how linear maps behave.
- Combining the two, we can approximate how any function behaves.

Remark I.0.2
What is Linear Algebra good for?

- Linear algebra is used throughout several fields in higher mathematics.
- In computer graphics, linear algebra is used to help represent 3D objects in a 2D grid of pixels.
- Linear algebra is used to approximate differential equation solutions in a vast number of engineering applications (e.g. fluid flows, vibrations, heat transfer) whose solutions are very difficult (or impossible) to find precisely.
- Google's search engine is based on its Page Rank algorithm, which ranks websites by computing an eigenvector of a matrix.

Remark I.0.3

What will I learn in this class?

By the end of this class, you will be able to:

- Solve systems of linear equations. (Module E)
- Identify vector spaces and their properties. (Module V)
- Analyze the structure of vector spaces and sets of vectors. (Module S)
- Use and apply the algebraic properties of linear transformations. (Module A)
- Perform fundamental operations in the algebra of matrices. (Module M)
- Use and apply the geometric properties of linear transformations. (Module G)

Module C: Constant coefficient linear ODEs

How can we solve and apply linear constant coefficient ODEs?

At the end of this module, students will be able to...

C1. **Sketching trajectories.** ...

C2. **Constant coefficient first order.** ...

C3. **Homogeneous constant coefficient second order.** ...

C4. **Non-homogenous constant coefficient second order.** ...

C5. **IVPs.** ...

C6. **Modeling motion in viscous fluids.** ...

C7. **Modeling oscillators.** ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.
- Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): http://bit.ly/2l21etm
- Solving linear systems with substitution (Khan Academy): http://bit.ly/1SlMpix
- Set builder notation: https://youtu.be/xnfUZ-NTsCE

Module C Section 0

Definition C.0.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

Remark C.0.2

In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition C.0.3

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned}
a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n \; &= \; b_1\\
a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n \; &= \; b_2\\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1\\ s_2\\ \vdots\\ s_n \end{bmatrix} \;\middle|\; \begin{bmatrix} s_1\\ s_2\\ \vdots\\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

Remark C.0.4

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Definition C.0.5

A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

Fact C.0.6

All linear systems are one of the following:

- **Consistent with one solution**: its solution set contains a single vector, e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix}\middle|\, a \in \mathbb{R}\right\}$
- **Inconsistent**: its solution set is the empty set $\{\} = \emptyset$

Activity C.0.7 *(∼10 min)*
All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is $\emptyset$.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity C.0.8 *(∼10 min)*

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity C.0.8 *(∼10 min)*

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.

Activity C.0.8 *(∼10 min)*

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.

Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$ for the linear system.

Activity C.0.9 *(∼10 min)*
Consider the following linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & &- x_4 &= 3 \\ & & & x_3 &+ 4x_4 &= -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation C.0.10

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{aligned} -2x_1 - 4x_2 + x_3 - 4x_4 &= -8 \\ x_1 + 2x_2 + 2x_3 + 12x_4 &= -1 \\ x_1 + 2x_2 + x_3 + 8x_4 &= 1 \end{aligned}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Module C Section 1

Remark C.1.1

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition C.1.2

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
\qquad
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]
$$

Example C.1.3

The corresopnding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Definition C.1.4

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \qquad\qquad \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity C.1.5 *(∼10 min)*

Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.
b) Swap two columns.
c) Add a constant to every term in a row.
d) Multiply a row by a nonzero constant.
e) Add a constant multiple of one row to another row.
f) Replace a column with zeros.
g) Replace a row with zeros.

Definition C.1.6

The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity C.1.7 *(∼10 min)*

Consider the following (equivalent) linear systems.

(A)

$$\begin{alignedat}{4} -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ x_1 &- 2x_2 &+ 2x_3 &= 7 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(B)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(C)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ & &- 2x_3 &= -6 \end{alignedat}$$

(D)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & x_3 &= 3 \\ & &- 2x_3 &= -6 \end{alignedat}$$

(E)

$$\begin{alignedat}{4} x_1 &- 2x_2 & &= 1 \\ & & x_3 &= 3 \\ & & 0 &= 0 \end{alignedat}$$

(F)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

Activity C.1.7 *(∼10 min)*

Consider the following (equivalent) linear systems.

(A)

$$\begin{alignedat}{4} -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ x_1 &- 2x_2 &+ 2x_3 &= 7 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(C)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ & & -2x_3 &= -6 \end{alignedat}$$

(E)

$$\begin{alignedat}{4} x_1 &- 2x_2 & &= 1 \\ & & x_3 &= 3 \\ & & 0 &= 0 \end{alignedat}$$

(B)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(D)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & x_3 &= 3 \\ & & -2x_3 &= -6 \end{alignedat}$$

(F)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

Part 1: Find a solution to one of these systems.

Activity C.1.7 *(∼10 min)*
Consider the following (equivalent) linear systems.

(A)

$$\begin{alignedat}{4} -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ x_1 &- 2x_2 &+ 2x_3 &= 7 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(C)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ & & -2x_3 &= -6 \end{alignedat}$$

(E)

$$\begin{alignedat}{3} x_1 &- 2x_2 & &= 1 \\ & & x_3 &= 3 \\ & & 0 &= 0 \end{alignedat}$$

(B)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(D)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & x_3 &= 3 \\ & & -2x_3 &= -6 \end{alignedat}$$

(F)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

Part 1: Find a solution to one of these systems.
Part 2: Rank the six linear systems from most complicated to simplest.

Activity C.1.8 *(∼5 min)*

We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} -2 & 4 & -2 & -8 \\ 1 & -2 & 2 & 7 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ -2 & 4 & -2 & -8 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 3 & -6 & 4 & 15 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Activity C.1.9 *(∼10 min)*

A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Circle the leading terms in each example, and label it as RREF or not RREF.

(A)

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{array}\right]$$

(B)

$$\left[\begin{array}{ccc|c}1 & 2 & 4 & 3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{array}\right]$$

(C)

$$\left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 1 & 2 & 0 & 3\\ 0 & 0 & 1 & -1\end{array}\right]$$

(D)

$$\left[\begin{array}{ccc|c}1 & 0 & 2 & -3\\ 0 & 3 & 3 & -3\\ 0 & 0 & 0 & 0\end{array}\right]$$

(E)

$$\left[\begin{array}{ccc|c}0 & 1 & 0 & 7\\ 1 & 0 & 0 & 4\\ 0 & 0 & 0 & 0\end{array}\right]$$

(F)

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 4\\ 0 & 1 & 0 & 7\\ 0 & 0 & 1 & 0\end{array}\right]$$

Remark C.1.10

It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix into reduced row echelon form.

A video outlining how to perform the Gauss-Jordan Elimination algorithm by hand is available at `https://youtu.be/Cq0Nxk2dhhU`. Practicing several exercises outside of class using this method is recommended.

In the next section, we will learn to use technology to perform this operation for us, as will be expected when applying row-reduced matrices to solve other problems.

Module C Section 2

Activity C.2.1 *(∼10 min)*

Free browser-based technologies for mathematical computation are available online.

- Go to `http://cocalc.com` and create an account.
- Create a project titled “Linear Algebra Team X” with your appropriate team number. Add all team members as collaborators.
- Open the project and click on “New”
- Give it an appropriate name such as “Class E.2 workbook”. Make a new Jupyter notebook.
- Click on “Kernel” and make sure “Octave” is selected.
- Type `A=[1 3 4 ; 2 5 7]` and press `Shift+Enter` to store the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{bmatrix}$ in the variable A.
- Type `rref(A)` and press `Shift+Enter` to compute the reduced row echelon form of A.

Remark C.2.2

If you need to find the reduced row echelon form of a matrix during class, you are encouraged to use CoCalc's Octave interpreter.

You can change a cell from "Code" to "Markdown" or "Raw" to put comments around your calculations such as Activity numbers.

Activity C.2.3 *(∼10 min)*

Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Convert this to an augmented matrix and use CoCalc to compute its reduced row echelon form. Write these on your whiteboard, and use them to write a simpler yet equivalent linear system of equations. Then find its solution set.

Activity C.2.4 *(∼10 min)*
Consider our system of equations from above.

$$\begin{alignedat}{4}
3x_1 &\;-\;& 2x_2 &\;+\;& 13x_3 &\;=\;& 6 \\
2x_1 &\;-\;& 2x_2 &\;+\;& 10x_3 &\;=\;& 2 \\
-x_1 &&&\;-\;& 3x_3 &\;=\;& 1
\end{alignedat}$$

Convert this to an augmented matrix and use CoCalc to compute its reduced row echelon form. Write these on your whiteboard, and use them to write a simpler yet equivalent linear system of equations. Then find its solution set.

Activity C.2.5 *(∼10 min)*
Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Activity C.2.5 *(∼10 min)*

Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use CoCalc to find $\mathrm{RREF}(A)$.

Activity C.2.5 *(∼10 min)*
Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use CoCalc to find RREF(A).
Part 2: How many solutions does the corresponding linear system have?

Activity C.2.6 *(∼10 min)*

Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Activity C.2.6 *(∼10 min)*

Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Activity C.2.6 *(∼10 min)*

Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

Activity C.2.6 *(∼10 min)*

Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

Part 3: Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$ caused this?

Definition C.2.7

Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations (x_1, x_3 below). The remaining variables are called **free variables** (x_2 below).

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To efficiently solve a system in RREF form, we may assign letters to free variables and solve for the bound variables.

Activity C.2.8 *(∼10 min)*

Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

by row-reducing its augmented matrix, and then assigning letters to the free variables (given by non-pivot columns) and solving for the bound variables (given by pivot columns) in the corresponding linear system.

Observation C.2.9

The solution set to the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

may be written as

$$\left\{ \begin{bmatrix} 1+5a+2b \\ 1+2a+3b \\ a \\ 3+3b \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Remark C.2.10

Don't forget to correctly express the solution set of a linear system, using set-builder notation for consistent systems with infintely many solutions.

- **Consistent with one solution**: e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix}\middle|\, a \in \mathbb{R}\right\}$
- **Inconsistent**: $\emptyset$

Module S: Systems of ODEs

How can we solve and apply systems of linear ODEs?

At the end of this module, students will be able to...

S1. **Solving systems.** ...

S2. **Modeling interacting populations.** ...

S3. **Modeling coupled oscillators.** ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Module S Section 1

Activity S.1.1 *(~10 min)*

Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

Which of the following is true?

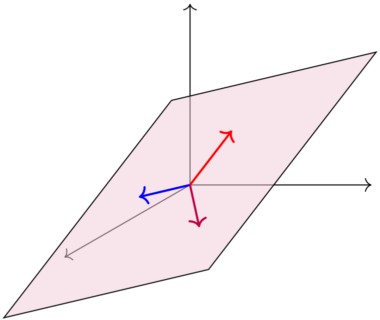
(A) span S is bigger than span T.

(B) span S and span T are the same size.

(C) span S is smaller than span T.

Definition S.1.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay on the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

Activity S.1.3 *(∼10 min)*

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$. Suppose $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$, so the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Which of the following is true of the vector equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}$?

(A) It is consistent with one solution

(B) It is consistent with infinitely many solutions

(C) It is inconsistent.

Fact S.1.4

For any vector space, the set $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{z}$ is consistent with infinitely many solutions.

Activity S.1.5 *(∼10 min)*

Find

$$\operatorname{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Fact S.1.6

A set of Euclidean vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix}\mathbf{v}_1 & \dots & \mathbf{v}_n\end{bmatrix}$ has a column without a pivot position.

Activity S.1.7 *(∼5 min)*

Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

Activity S.1.8 *(∼10 min)*
Is the set of polynomials $\{x^3+1, x^2+2x, x^2+7x+4\}$ linearly dependent or linearly independent?

Activity S.1.9 *(∼5 min)*

What is the largest number of vectors in $\mathbb{R}^4$ that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.10 *(∼5 min)*

What is the largest number of vectors in

$$\mathcal{P}^4 = \left\{ ax^4 + bx^3 + cx^2 + dx + e \,\middle|\, a, b, c, d, e \in \mathbb{R} \right\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.11 *(∼5 min)*

What is the largest number of vectors in

$$\mathcal{P} = \{f(x) \mid f(x) \text{ is any polynomial}\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Module S Section 2

Definition S.2.1

A **basis** is a linearly independent set that spans a vector space.

The **standard basis** of $\mathbb{R}^n$ is the set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For $\mathbb{R}^3$, these are the vectors $\mathbf{e}_1 = \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{e}_3 = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Observation S.2.2

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

For example, in many calculus courses, vectors in $\mathbb{R}^3$ are often expressed in their component form

$$(3, -2, 4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

or in their standard basic vector form

$$3\mathbf{e}_1 - 2\mathbf{e}_2 + 4\mathbf{e}_3 = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}.$$

Since every vector in $\mathbb{R}^3$ can be uniquely described as a linear combination of the vectors in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, this set is indeed a basis.

Activity S.2.3 *(∼15 min)*

Label each of the sets A, B, C, D, E as

- SPANS $\mathbb{R}^4$ or DOES NOT SPAN $\mathbb{R}^4$
- LINEARLY INDEPENDENT or LINEARLY DEPENDENT
- BASIS FOR $\mathbb{R}^4$ or NOT A BASIS FOR $\mathbb{R}^4$

by finding RREF for their corresponding matrices.

$$A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Activity S.2.4 *(∼10 min)*

If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\text{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$ doesn't have a non-pivot column, and doesn't have a row of zeros. What is $\text{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$?

$$\text{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Fact S.2.5

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and

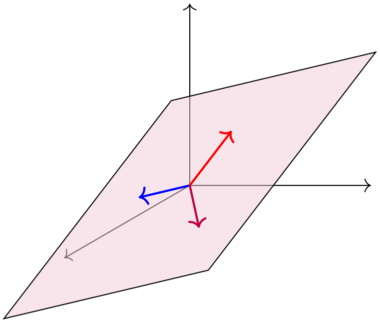
$$\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for $\mathbb{R}^n$ must have exactly n vectors and its square matrix must row-reduce to the so-called **identity matrix** containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)

Observation S.2.6

Recall that a **subspace** of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in the following image are needed to span the planar subspace.



Activity S.2.7 *(∼10 min)*

Consider the subspace $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

Activity S.2.7 *(∼10 min)*

Consider the subspace $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

Part 1: Mark the part of RREF $\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Activity S.2.7 *(∼10 min)*

Consider the subspace $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

Part 1: Mark the part of RREF $\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Part 2: Find a basis for W by removing a vector from its spanning set to make it linearly independent.

Fact S.2.8

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. The easiest basis describing span S is the set of vectors in S given by the pivot columns of $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

Put another way, to compute a basis for the subspace span S, simply remove the vectors corresponding to the non-pivot columns of $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

Activity S.2.9 *(~10 min)*

Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Find a basis for W.

Activity S.2.10 *(∼10 min)*

Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left\{x^3+3x^2+x-1, 2x^3-x^2+x+2, 4x^3+5x^2+3x, 3x^3+2x^2+2x+1\right\}$$

Find a basis for W.

Module S Section 3

Observation S.3.1

In the previous section, we learned that computing a basis for the subspace $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, is as simple as removing the vectors corresponding to the non-pivot columns of $\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

For example, since

$$\mathrm{RREF}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$ as a basis.

Activity S.3.2 *(∼10 min)*

Let

$$S = \left\{ \begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix} \right\} \text{ and } T = \left\{ \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix} \right\}$$

Activity S.3.2 *(∼10 min)*

Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Part 1: Find a basis for span S.

Activity S.3.2 *(∼10 min)*

Let

$$S = \left\{ \begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix} \right\} \text{ and } T = \left\{ \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix} \right\}$$

Part 1: Find a basis for span S.

Part 2: Find a basis for span T.

Observation S.3.3

Even though we found different bases for them, span S and span T are exactly the same subspace of $\mathbb{R}^4$, since

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} = T$$

Fact S.3.4

Any non-trivial vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.

For example,

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

are all valid bases for $\mathbb{R}^3$, and they all contain three vectors.

Definition S.3.5

The **dimension** of a vector space is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension n. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

contains exactly three vectors.

Activity S.3.6 *(∼10 min)*

Find the dimension of each subspace of $\mathbb{R}^4$ by finding RREF for each corresponding matrix.

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\} \quad \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\} \quad \operatorname{span}\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

Fact S.3.7

Every vector space with finite dimension, that is, every vector space V with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be **isomorphic** to a Euclidean space $\mathbb{R}^n$, since there exists a natural correspondance between vectors in V and vectors in $\mathbb{R}^n$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Observation S.3.8

We've already been taking advantage of the previous fact by converting polynomials and matrices into Euclidean vectors. Since $\mathcal{P}^3$ and $M_{2,2}$ are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 5 \leftrightarrow \begin{bmatrix} 4 \\ 0 \\ -1 \\ 5 \end{bmatrix} \leftrightarrow \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$$

Observation S.3.9

The space of polynomials $\mathcal{P}$ (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since $\mathcal{P}$ and other infinite-dimensional spaces cannot be treated as an isomorphic finite-dimensional Euclidean space $\mathbb{R}^n$, vectors in such spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

Definition S.3.10

A **homogeneous** system of linear equations is one of the form:

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n &= 0\\
a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n &= 0\\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\ \ \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}$$

This system is equivalent to the vector equation:

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & 0\\
a_{21} & a_{22} & \cdots & a_{2n} & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
a_{m1} & a_{m2} & \cdots & a_{mn} & 0
\end{array}\right]$$

Activity S.3.11 *(~5 min)*

Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ so is $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$, since

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0} \text{ and } b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = \mathbf{0}$$

implies

$$(a_1 + b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n = \mathbf{0}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is...

a) A basis for $\mathbb{R}^n$.

b) A subspace of $\mathbb{R}^n$.

c) The empty set.

Activity S.3.12 *(∼10 min)*

Consider the homogeneous system of equations

$$
\begin{aligned}
x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 0\\
2x_1 + 4x_2 - x_3 - 2x_4 &= 0\\
3x_1 + 6x_2 - x_3 - x_4 &= 0
\end{aligned}
$$

Activity S.3.12 *(∼10 min)*

Consider the homogeneous system of equations

$$\begin{alignedat}{5}
x_1 &+ 2x_2 & & &+ \ & x_4 &= 0\\
2x_1 &+ 4x_2 &- x_3 & &- 2&x_4 &= 0\\
3x_1 &+ 6x_2 &- x_3 & &- \ &x_4 &= 0
\end{alignedat}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).

Activity S.3.12 *(~10 min)*

Consider the homogeneous system of equations

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & &+& x_4 &=0 \\ 2x_1 &+ 4x_2 &-& x_3 &-& 2x_4 &=0 \\ 3x_1 &+ 6x_2 &-& x_3 &-& x_4 &=0 \end{alignedat}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).

Part 2: Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Fact S.3.13

The coefficients of the free variables in the solution set of a linear system always yield linearly independent vectors.

Thus if

$$\left\{ a \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

is the solution space for a homoegeneous system, then

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the solution space.

Activity S.3.14 *(∼10 min)*

Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 \phantom{{}+3x_4} &= 0 \\ 2x_1 - 6x_2 + 4x_3 + 3x_4 &= 0 \\ -2x_1 + 6x_2 - 4x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for its solution space.

Activity S.3.15 *(∼5 min)*

Suppose W is a subspace of $\mathcal{P}^8$, and you know that it contains a **linearly independent** set of 3 vectors. What can you conclude about W?

(a) The dimension of W is at most 3.

(b) The dimension of W is exactly 3.

(c) The dimension of W is at least 3.

Activity S.3.16 *(∼5 min)*

Suppose W is a subspace of $\mathcal{P}^8$, and you know that it contains a **spanning set** of 3 vectors. What can you conclude about W?

(a) The dimension of W is at most 3.

(b) The dimension of W is exactly 3.

(c) The dimension of W is at least 3.

Module F: First order ODEs

How can we solve and apply first order ODEs?

At the end of this module, students will be able to...

F1. Separable ODEs. ...

F2. Autonomous ODEs. ...

F3. First order linear ODEs. ...

F4. Exact ODES. ...

F5. Modeling motion. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Module F Section 0

Activity F.0.1 (∼20 min)

Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**
   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
2. **Addition commutivity.**
   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Addition identity.**
   There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
4. **Addition inverse.**
   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
5. **Addition midpoint uniqueness.**
   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.
6. **Scalar multiplication associativity.**
   $a(b\mathbf{v}) = (ab)\mathbf{v}$.
7. **Scalar multiplication identity.**
   $1\mathbf{v} = \mathbf{v}$.
8. **Scalar multiplication relativity.**
   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.
9. **Scalar distribution.**
   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
10. **Vector distribution.**
    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
11. **Orthogonality.**
    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
12. **Bidimensionality.**
    $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some value of a, b.

Definition F.0.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition is associative.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition is commutative.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Additive identity exists.** There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
- **Additive inverses exist.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
- **Scalar multiplication is associative.** $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- 1 **is a scalar multiplicative identity.** $1\mathbf{v} = \mathbf{v}$.
- **Scalar multiplication distributes over vector addition.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Scalar multiplication distributes over scalar addition.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Any **Euclidean vector space** $\mathbb{R}^n$ satisfies all eight requirements regardless of the value of n, but we will also study other types of vector spaces.

Module F Section 1

Remark F.1.1

Last time, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, and all scalars (i.e. real numbers) a, b.

- **Addition is associative.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition is commutative.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Additive identity exists.** There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
- **Additive inverses exist.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
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- **Scalar multiplication distributes over scalar addition.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Remark F.1.2

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \dots)$.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity F.1.3 *(∼20 min)*

Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Activity F.1.3 *(∼20 min)*
Consider the set $V = \{(x, y) \,|\, y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting $\mathbf{v} = (x, y)$ and showing both sides simplify to the same expression.

Activity F.1.3 *(∼20 min)*
Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting $\mathbf{v} = (x, y)$ and showing both sides simplify to the same expression.
Part 2: Show that V contains an additive identity element by choosing $\mathbf{z} = (\,?\,,\,?\,)$ such that $\mathbf{v} \oplus \mathbf{z} = (x, y) \oplus (\,?\,,\,?\,) = \mathbf{v}$ for any $\mathbf{v} = (x, y) \in V$.

Remark F.1.4

It turns out $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

satisifes all eight properties.

- **Addition associativity.** $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{z}$ where $\mathbf{v} \oplus \mathbf{z} = \mathbf{v}$.
- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}$.
- **Scalar multiplication associativity.** $a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}$.
- **Scalar multiplication identity.** $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.** $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Thus, V is a vector space.

Activity F.1.5 *(∼15 min)*

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Activity F.1.5 *(∼15 min)*

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y).

Activity F.1.5 *(∼15 min)*

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y).

Part 2: Show that the addition identity property fails by showing that $(0, -1) \oplus \mathbf{z} \neq (0, -1)$ no matter how $\mathbf{z} = (z_1, z_2)$ is chosen.

Activity F.1.5 *(∼15 min)*

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y).

Part 2: Show that the addition identity property fails by showing that $(0, -1) \oplus \mathbf{z} \neq (0, -1)$ no matter how $\mathbf{z} = (z_1, z_2)$ is chosen.

Part 3: Can V be a vector space?

Definition F.1.6

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \ldots, c_m$.

For example, we can say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition F.1.7

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \,|\, c_i \in \mathbb{R}\}.$$

For example:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity F.1.8 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Activity F.1.8 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $1\begin{bmatrix}1\\2\end{bmatrix}$, $3\begin{bmatrix}1\\2\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix}$, and $-2\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane.

Activity F.1.8 *(~10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $1\begin{bmatrix}1\\2\end{bmatrix}$, $3\begin{bmatrix}1\\2\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix}$, and $-2\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\2\end{bmatrix} \middle| a \in \mathbb{R}\right\}$ in the xy plane.

Activity F.1.9 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Activity F.1.9 *(~10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix} \qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}$$

Activity F.1.9 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}\qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}\qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}\qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}$$

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity F.1.10 *(~5 min)*

Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$ in the xy plane.

Module F Section 2

Remark F.2.1

Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity F.2.2 *(∼15 min)*

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Activity F.2.2 *(∼15 min)*

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Activity F.2.2 *(∼15 min)*

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

Activity F.2.2 *(∼15 min)*

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

Part 3: Given this solution set, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Fact F.2.3

A vector $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \, \dots \, \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Put another way, $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ exactly when $\mathrm{RREF}[\mathbf{v}_1 \, \dots \, \mathbf{v}_n \,|\, \mathbf{b}]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Activity F.2.4 *(∼10 min)*

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity F.2.5 *(∼5 min)*

Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity F.2.6 *(∼10 min)*

Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Activity F.2.6 *(∼10 min)*
Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?
Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)

Activity F.2.6 *(∼10 min)*
Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?
Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)
Part 2: Solve this equivalent exercise, and use its solution to answer the original question.

Activity F.2.7 *(∼5 min)*

Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \right\}$?

Activity F.2.8 *(∼5 min)*

Does the complex number $2i$ belong to $\operatorname{span}\{-3+i, 6-2i\}$?

Module F Section 3

Activity F.3.1 *($\sim$5 min)*

How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the *xy* plane to support your answer.

(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Activity F.3.2 *(∼5 min)*

How many vectors are required to span $\mathbb{R}^3$?

(a) 1

(b) 2

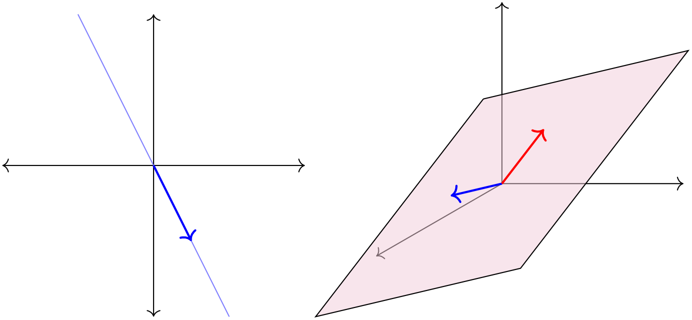
(c) 3

(d) 4

(e) Infinitely Many

Fact F.3.3

At least n vectors are required to span $\mathbb{R}^n$.



Activity F.3.4 *(∼15 min)*

Choose a vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by using CoCalc to verify that RREF $\left[\begin{array}{cc|c} 1 & -2 & ? \\ -1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. (Why does this work?)

Fact F.3.5

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when RREF$[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ for some choice of vector } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Activity F.3.6 *(∼5 min)*

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does $\mathbb{R}^4 = \operatorname{span} S$?

Activity F.3.7 *(∼10 min)*
Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, \\ -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}.$$

Does $\mathcal{P}^3 = \operatorname{span} S$? (Hint: first rewrite the question so it is about Euclidean vectors.)

Activity F.3.8 *(∼5 min)*

Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$?

Activity F.3.9 *(∼5 min)*

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^7$ be three vectors, and suppose $\mathbf{w}$ is another vector with $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What can you conclude about $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is larger than $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

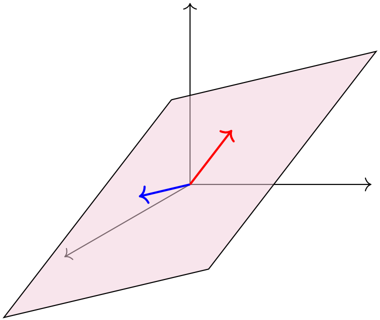
(c) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is smaller than $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Module F Section 4

Definition F.4.1

A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space $\mathbb{R}^3$.



Fact F.4.2

Any sub**set** S of a vector space V satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a sub**space**, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is **closed under addition**: for any $\mathbf{x}, \mathbf{y} \in S$, the sum $\mathbf{x} + \mathbf{y}$ is also in S.
- The set is **closed under scalar multiplication**: for any $\mathbf{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\mathbf{x}$ is also in S.

Activity F.4.3 *(∼15 min)*

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Activity F.4.3 *(∼15 min)*

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Activity F.4.3 *(∼15 min)*

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\mathbf{v}$ also belongs to S for any $c \in \mathbb{R}$.

Activity F.4.3 *(∼15 min)*

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\mathbf{v}$ also belongs to S for any $c \in \mathbb{R}$.

Part 3: Is S is a subspace of $\mathbb{R}^3$?

Activity F.4.4 *(∼10 min)*

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 4 \right\}$. Choose a vector $\mathbf{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = ?$, and show that $c\mathbf{v}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Remark F.4.5
Since 0 is a scalar and $0\mathbf{v} = \mathbf{z}$ for any vector $\mathbf{v}$, a set that is closed under scalar multiplication must contain the zero vector $\mathbf{z}$ for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain $\mathbf{0}$.

Activity F.4.6 *(∼10 min)*

Consider these two subsets of $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\}$$

Activity F.4.6 *(∼10 min)*

Consider these two subsets of $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of $\mathbb{R}^4$?

Activity F.4.6 *(~10 min)*

Consider these two subsets of $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of $\mathbb{R}^4$?

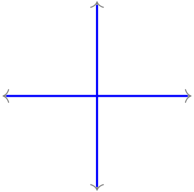
Part 2: Is the set of polynomials

$$S = \left\{ ax^3 + bx^2 + (b-1)x + (a-1) \,\middle|\, a, b \text{ are real numbers} \right\}$$

a subspace of $\mathcal{P}^3$?

Activity F.4.7 *(∼10 min)*

Consider the subset A of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



This set contains $\mathbf{0}$, and it's not hard to show that for every $\mathbf{v}$ in A and scalar $c \in \mathbb{R}$, $c\mathbf{v}$ is also in A. Is A a subspace of $\mathbb{R}^2$? Why?

Activity F.4.8 *(∼5 min)*

Let W be a subspace of a vector space V. How are $\operatorname{span} W$ and W related?

(a) $\operatorname{span} W$ is bigger than W

(b) $\operatorname{span} W$ is the same as W

(c) $\operatorname{span} W$ is smaller than W

Fact F.4.9

If S is any subset of a vector space V, then since span S collects all possible linear combinations, span S is automatically a subspace of V.

In fact, span S is always the smallest subspace of V that contains all the vectors in S.

Module N: Numerical

How can we use numerical approximation methods to apply and solve unsolvable ODEs?

At the end of this module, students will be able to...

N1. **First Order Existence and Uniqueness.** ...

N2. **Second Order Linear Existence and Uniqueness.** ...

N3. **Systems Existence and Uniqueness.** ...

N4. **Euler's method for first order ODES.** ...

N5. **Euler's method for systems.** ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Module N Section 1

Definition N.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

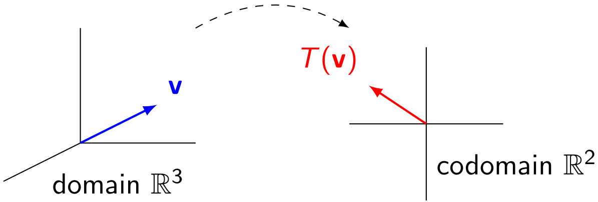
1. $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition N.1.2

Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

Linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$



Example N.1.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x-z\\3y\end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}u\\v\\w\end{bmatrix}\right) = T\left(\begin{bmatrix}x+u\\y+v\\z+w\end{bmatrix}\right) = \begin{bmatrix}(x+u)-(z+w)\\3(y+v)\end{bmatrix}$$

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)+T\left(\begin{bmatrix}u\\v\\w\end{bmatrix}\right) = \begin{bmatrix}x-z\\3y\end{bmatrix}+\begin{bmatrix}u-w\\3v\end{bmatrix} = \begin{bmatrix}(x+u)-(z+w)\\3(y+v)\end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = T\left(\begin{bmatrix}cx\\cy\\cz\end{bmatrix}\right) = \begin{bmatrix}cx-cz\\3cy\end{bmatrix} \text{ and } cT\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = c\begin{bmatrix}x-z\\3y\end{bmatrix} = \begin{bmatrix}cx-cz\\3cy\end{bmatrix}$$

Therefore T is a linear transformation.

Example N.1.4

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact N.1.5

A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2, $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity N.1.6 *($\sim$5 min)*

Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space

b) D is a linear map

c) D is not a linear map

Activity N.1.7 *(∼10 min)*

Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact N.1.8

If $L : V \to W$ is linear, then $L(\mathbf{z}) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{z}$ where $\mathbf{z}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Activity N.1.9 *($\sim$15 min)*

Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Activity N.1.9 *(~15 min)*
Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.

Activity N.1.9 *(~15 min)*
Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f. Is S linear?

Activity N.1.10 *($\sim$20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Activity N.1.10 *(∼20 min)*

Let the polynomial maps $S:\mathcal{P}\to\mathcal{P}$ and $T:\mathcal{P}\to\mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1)\neq S(x)+S(1)$ to verify that S is not linear.

Activity N.1.10 *(∼20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Part 2: Prove that T is linear by verifying that $T(f(x)+g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Observation N.1.11

Note that S in the previous activity is not linear, even though $S(0) = (0)^2 = 0$. So showing $S(0) = 0$ isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain $\mathbf{z}$, then the subset isn't a subspace. But if the subset contains $\mathbf{z}$, you cannot conclude anything.

Module N Section 2

Remark N.2.1

Recall that a linear map $T : V \to W$ satisfies

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vecor space operations can be applied before or after the transformation without affecting the result.

Activity N.2.2 *(∼5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity N.2.3 *(∼3 min)*

Suppose $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity N.2.4 *(~2 min)*

Suppose $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity N.2.5 *(∼5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Do you have enough information to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^3$?

(a) Yes.

(b) No, exactly one more piece of information is needed.

(c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

Fact N.2.6

Consider any basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for V. Since every vector $\mathbf{v}$ can be written *uniquely* as a linear combination of basis vectors, $x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n$, we may compute $T(\mathbf{v})$ as follows:

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1 T(\mathbf{b}_1) + \cdots + x_n T(\mathbf{b}_n).$$

Therefore any linear transformation $T : V \to W$ can be defined by just describing the values of $T(\mathbf{b}_i)$.

Put another way, the images of the basis vectors **determine** the transformation T.

Definition N.2.7

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T(\mathbf{e}_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T(\mathbf{e}_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3)\end{bmatrix} = \begin{bmatrix}3 & -1 & 5\\ 2 & 4 & 0\end{bmatrix}.$$

Activity N.2.8 *(~3 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ for T.

Activity N.2.9 *(∼5 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+3z\\2x-y-4z\end{bmatrix}$$

Find the standard matrix for T.

Fact N.2.10

Because every linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\mathbf{e}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an ordered list of the coefficients of the x_i:

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Activity N.2.11 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity N.2.12 *($\sim$5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Fact N.2.13

To quickly compute $T(\mathbf{v})$ from its standard matrix A, compute the **dot product** (defined in Calculus 3) of each matrix row with the vector. For example, if T has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

Activity N.2.14 *(∼15 min)*

Compute the following linear transformations of vectors given their standard matrices.

$$T_1\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \text{ for the standard matrix } A_1 = \begin{bmatrix}4 & 3\\0 & -1\\1 & 1\\3 & 0\end{bmatrix}$$

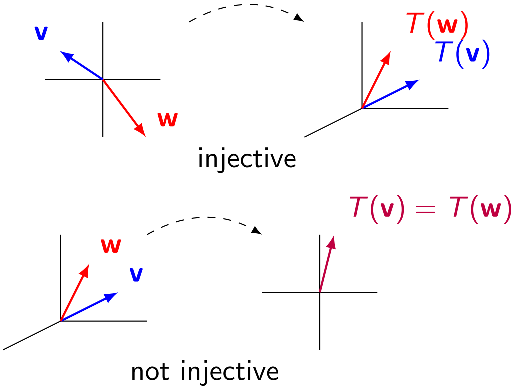
$$T_2\left(\begin{bmatrix}1\\1\\0\\-3\end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix}4 & 3 & 0 & -1\\1 & 1 & 3 & 0\end{bmatrix}$$

$$T_3\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix}4 & 3 & 0\\0 & -1 & 3\\5 & 1 & 1\\3 & 0 & 0\end{bmatrix}$$

Module N Section 3

Definition N.3.1

Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.



Activity N.3.2 *(~3 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Show that T is not injective by finding two different vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

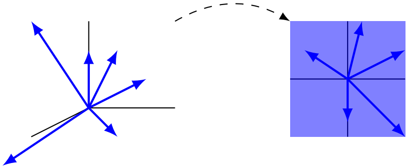
Activity N.3.3 *(∼2 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

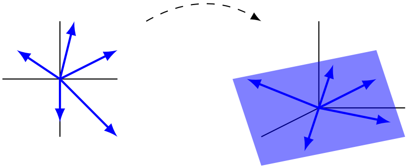
Is T injective? If not, find two different vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

Definition N.3.4

Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.



surjective



not surjective

Activity N.3.5 *(∼3 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Show that T is not surjective by finding a vector in $\mathbb{R}^3$ that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal.

Activity N.3.6 *(∼2 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective? If not, find a vector in $\mathbb{R}^2$ that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal.

Observation N.3.7

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

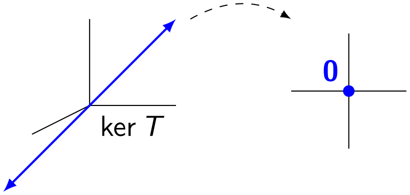
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Definition N.3.8

Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{ \mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \mathbf{z} \right\}$$



Activity N.3.9 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\ker T$, the set of all vectors that transform into $\mathbf{0}$?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2$

Activity N.3.10 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\mathbf{0}$?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3$

Activity N.3.11 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Activity N.3.11 *(∼10 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Activity N.3.11 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

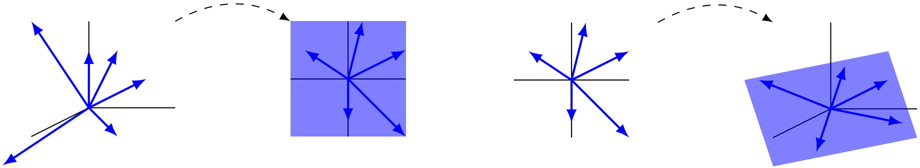
Part 2: Use RREF(A) to solve this homogeneous system of equations and find a basis for the kernel of T.

Definition N.3.12

Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \mathbf{w} \in W \;\middle|\; \text{there is some } \mathbf{v} \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \right\}$$

In the examples below, the left example's image is all of $\mathbb{R}^2$, but the right example's image is a planar subspace of $\mathbb{R}^3$.



Activity N.3.13 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3$

Activity N.3.14 *(∼5 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2$

Activity N.3.15 *(∼5 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & T(\mathbf{e}_4) \end{bmatrix}.$$

Since $T(\mathbf{v}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4)$, the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

a) spans $\operatorname{Im} T$

b) is a linearly independent subset of $\operatorname{Im} T$

c) is a basis for $\operatorname{Im} T$

Observation N.3.16

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ spans $\operatorname{Im} T$, we can obtain a basis for $\operatorname{Im} T$ by finding $\operatorname{RREF} A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and only using the vectors corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Fact N.3.17

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

- The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$. Use the coefficients of its free variables to get a basis for the kernel.
- The image of T is the span of the columns of A. Remove the vectors creating non-pivot columns in $\operatorname{RREF} A$ to get a basis for the image.

Activity N.3.18 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Module N Section 4

Observation N.4.1

Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all vectors in V that are mapped to $\mathbf{z} \in W$. It is a subspace of V.
- The **image** of T is the set of all vectors in W that are mapped to by something in V. It is a subspace of W.

Activity N.4.2 *(∼5 min)*

Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

(a) T is injective

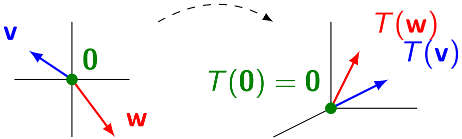
(b) T is not injective

(c) T is surjective

(d) T is not surjective

Fact N.4.3

A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.



Activity N.4.4 *(∼5 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be a linear transformation where $\operatorname{Im} T$ is spanned by four vectors. What can you conclude?

(a) T is injective

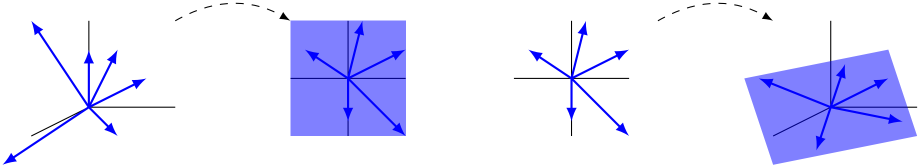
(b) T is not injective

(c) T is surjective

(d) T is not surjective

Fact N.4.5

A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective, $\operatorname{Im} T = \mathbb{R}^2$ not surjective, $\operatorname{Im} T \neq \mathbb{R}^3$

Activity N.4.6 *(∼15 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

(a) The kernel of T is trivial: $\ker T = \{\mathbf{0}\}$.

(b) The columns of A span $\mathbb{R}^m$.

(c) The columns of A are linearly independent.

(d) Every column of $\mathrm{RREF}(A)$ has a pivot.

(e) Every row of $\mathrm{RREF}(A)$ has a pivot.

(f) The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.

(g) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$.

(h) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ has exactly one solution.

Observation N.4.7

The easiest way to show that the linear map with standard matrix A is injective is to show that $\mathrm{RREF}(A)$ has a pivot in each column.

The easiest way to show that the linear map with standard matrix A is surjective is to show that $\mathrm{RREF}(A)$ has a pivot in each row.

Activity N.4.8 *(~3 min)*

What can you immediately conclude (i.e. without computing a RREF) about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -3 & 3 \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

Activity N.4.9 *(∼2 min)*

What can you immediately conclude (i.e. without computing a RREF) about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 4 \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

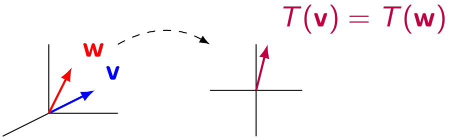
d) Its standard matrix has more rows than columns, so T is surjective.

Fact N.4.10

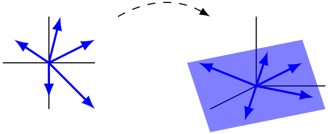
The following are true for any linear map $T : V \to W$:

- If $\dim(V) > \dim(W)$, then T is not injective.
- If $\dim(V) < \dim(W)$, then T is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase the dimension of its image.



not injective, $3 > 2$



not surjective, $2 < 3$

But dimension arguments **cannot** be used to prove a map **is** injective or surjective.

Activity N.4.11 *(∼5 min)*

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).

Activity N.4.11 *(∼5 min)*
Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).
Part 1: How many pivot columns must A have?

Activity N.4.11 *($\sim$5 min)*
Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).
Part 1: How many pivot columns must A have?
Part 2: How many pivot rows must A have?

Activity N.4.11 *(∼5 min)*

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot columns must A have?

Part 2: How many pivot rows must A have?

Part 3: What can you conclude about m and n?

Activity N.4.12 *($\sim$5 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^n$

(b) $\text{RREF}(A)$ is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \,\middle|\, \mathbf{b}\right]$ has exactly one solution for each $\mathbf{b} \in \mathbb{R}^n$.

Observation N.4.13

The easiest way to show that the linear map with standard matrix A is bijective is to show that $\mathrm{RREF}(A)$ is the identity matrix.

Activity N.4.14 *($\sim$3 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be given by the standard matrix

$$A=\begin{bmatrix}2 & 1 & -1\\ 4 & 1 & 1\\ 6 & 2 & 1\end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity N.4.15 *(~3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity N.4.16 *(∼3 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity N.4.17 *($\sim$3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Module D: Discontinuous functions in ODEs

How can we solve and apply ODEs involving functions that are not continuous?

At the end of this module, students will be able to...

D1. **Laplace Transform.** ...

D2. **Discontinuous ODEs.** ...

D3. **Modeling non-smooth motion.** ...

D4. **Modeling non-smooth oscillators.** ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2**,**S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Module D Section 1

Definition D.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

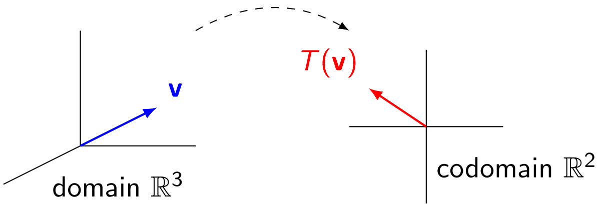
1. $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition D.1.2

Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

Linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$



Example D.1.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x-z\\3y\end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}u\\v\\w\end{bmatrix}\right)=T\left(\begin{bmatrix}x+u\\y+v\\z+w\end{bmatrix}\right)=\begin{bmatrix}(x+u)-(z+w)\\3(y+v)\end{bmatrix}$$

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)+T\left(\begin{bmatrix}u\\v\\w\end{bmatrix}\right)=\begin{bmatrix}x-z\\3y\end{bmatrix}+\begin{bmatrix}u-w\\3v\end{bmatrix}=\begin{bmatrix}(x+u)-(z+w)\\3(y+v)\end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=T\left(\begin{bmatrix}cx\\cy\\cz\end{bmatrix}\right)=\begin{bmatrix}cx-cz\\3cy\end{bmatrix} \text{ and } cT\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=c\begin{bmatrix}x-z\\3y\end{bmatrix}=\begin{bmatrix}cx-cz\\3cy\end{bmatrix}$$

Therefore T is a linear transformation.

Example D.1.4

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact D.1.5

A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2, $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity D.1.6 *(∼5 min)*

Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f+g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space

b) D is a linear map

c) D is not a linear map

Activity D.1.7 *(∼10 min)*
Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact D.1.8

If $L : V \to W$ is linear, then $L(\mathbf{z}) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{z}$ where $\mathbf{z}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Activity D.1.9 *(∼15 min)*

Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Activity D.1.9 *(∼15 min)*
Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.

Activity D.1.9 *($\sim$15 min)*
Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f. Is S linear?

Activity D.1.10 *($\sim$20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Activity D.1.10 *(∼20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Activity D.1.10 *(∼20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Part 2: Prove that T is linear by verifying that $T(f(x)+g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Observation D.1.11

Note that S in the previous activity is not linear, even though $S(0) = (0)^2 = 0$. So showing $S(0) = 0$ isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain $\mathbf{z}$, then the subset isn't a subspace. But if the subset contains $\mathbf{z}$, you cannot conclude anything.

Module D Section 2

Remark D.2.1

Recall that a linear map $T : V \to W$ satisfies

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vecor space operations can be applied before or after the transformation without affecting the result.

Activity D.2.2 *(∼5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity D.2.3 *(∼3 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity D.2.4 *(∼2 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity D.2.5 *($\sim$5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Do you have enough information to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^3$?

(a) Yes.

(b) No, exactly one more piece of information is needed.

(c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

Fact D.2.6

Consider any basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for V. Since every vector $\mathbf{v}$ can be written *uniquely* as a linear combination of basis vectors, $x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n$, we may compute $T(\mathbf{v})$ as follows:

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1 T(\mathbf{b}_1) + \cdots + x_n T(\mathbf{b}_n).$$

Therefore any linear transformation $T : V \to W$ can be defined by just describing the values of $T(\mathbf{b}_i)$.

Put another way, the images of the basis vectors **determine** the transformation T.

Definition D.2.7

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\mathbf{e}_1\right) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\mathbf{e}_2\right) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\mathbf{e}_3\right) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

Activity D.2.8 *(∼3 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ for T.

Activity D.2.9 *(~5 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Find the standard matrix for T.

Fact D.2.10

Because every linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\mathbf{e}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an ordered list of the coefficients of the x_i:

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Activity D.2.11 *(∼5 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix}3 & -2 & -1\\ 4 & 5 & 2\\ 0 & -2 & 1\end{bmatrix}.$$

Compute $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$.

Activity D.2.12 *($\sim$5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Fact D.2.13

To quickly compute $T(\mathbf{v})$ from its standard matrix A, compute the **dot product** (defined in Calculus 3) of each matrix row with the vector. For example, if T has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

Activity D.2.14 *(∼15 min)*

Compute the following linear transformations of vectors given their standard matrices.

$$T_1\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \text{ for the standard matrix } A_1 = \begin{bmatrix}4 & 3\\0 & -1\\1 & 1\\3 & 0\end{bmatrix}$$

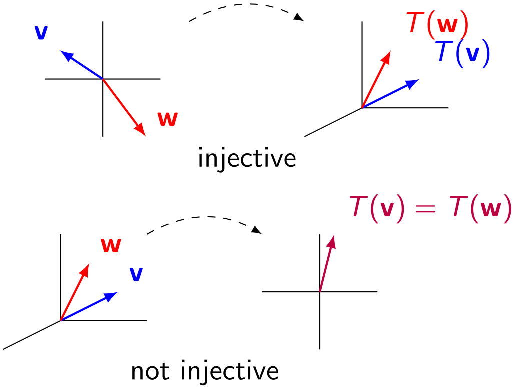
$$T_2\left(\begin{bmatrix}1\\1\\0\\-3\end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix}4 & 3 & 0 & -1\\1 & 1 & 3 & 0\end{bmatrix}$$

$$T_3\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix}4 & 3 & 0\\0 & -1 & 3\\5 & 1 & 1\\3 & 0 & 0\end{bmatrix}$$

Module D Section 3

Definition D.3.1

Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.



Activity D.3.2 *(~3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Show that T is not injective by finding two different vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

Activity D.3.3 *($\sim$2 min)*

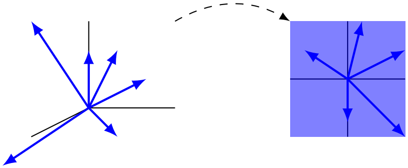
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

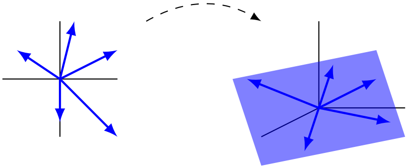
Is T injective? If not, find two different vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

Definition D.3.4

Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.



surjective



not surjective

Activity D.3.5 *(∼3 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Show that T is not surjective by finding a vector in $\mathbb{R}^3$ that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal.

Activity D.3.6 *(∼2 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective? If not, find a vector in $\mathbb{R}^2$ that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal.

Observation D.3.7

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

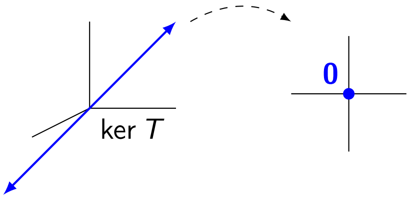
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Definition D.3.8

Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{ \mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \mathbf{z} \right\}$$



Activity D.3.9 *($\sim$5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\ker T$, the set of all vectors that transform into $\mathbf{0}$?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2$

Activity D.3.10 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\mathbf{0}$?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3$

Activity D.3.11 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Activity D.3.11 *(~10 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A=\begin{bmatrix}3 & 4 & -1\\ 1 & 2 & 1\end{bmatrix}.$$

Part 1: Set $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}? + ? + ?\\ ? + ? + ?\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Activity D.3.11 *(∼10 min)*
Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A=\begin{bmatrix}3 & 4 & -1\\ 1 & 2 & 1\end{bmatrix}.$$

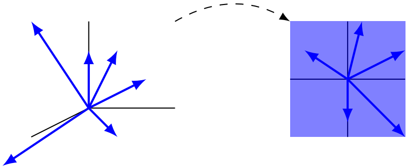
Part 1: Set $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}? + ? + ?\\ ? + ? + ?\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.
Part 2: Use RREF(A) to solve this homogeneous system of equations and find a basis for the kernel of T.

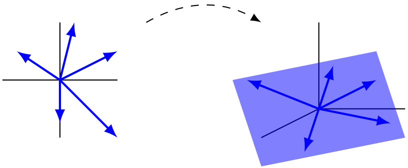
Definition D.3.12

Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{\mathbf{w} \in W \;\middle|\; \text{there is some } \mathbf{v} \in V \text{ with } T(\mathbf{v}) = \mathbf{w}\right\}$$

In the examples below, the left example's image is all of $\mathbb{R}^2$, but the right example's image is a planar subspace of $\mathbb{R}^3$.





Activity D.3.13 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\y\\0\end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

a) $\left\{ \begin{bmatrix}0\\0\\a\end{bmatrix} \middle|\, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix}a\\b\\0\end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix}0\\0\\0\end{bmatrix} \right\}$

d) $\mathbb{R}^3$

Activity D.3.14 *(∼5 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2$

Activity D.3.15 *(~5 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & T(\mathbf{e}_4) \end{bmatrix}.$$

Since $T(\mathbf{v}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4)$, the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

a) spans Im T

b) is a linearly independent subset of Im T

c) is a basis for Im T

Observation D.3.16

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ spans $\operatorname{Im} T$, we can obtain a basis for $\operatorname{Im} T$ by finding $\operatorname{RREF} A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and only using the vectors corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Fact D.3.17

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

- The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$. Use the coefficients of its free variables to get a basis for the kernel.
- The image of T is the span of the columns of A. Remove the vectors creating non-pivot columns in RREF A to get a basis for the image.

Activity D.3.18 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Module D Section 4

Observation D.4.1

Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all vectors in V that are mapped to $\mathbf{z} \in W$. It is a subspace of V.
- The **image** of T is the set of all vectors in W that are mapped to by something in V. It is a subspace of W.

Activity D.4.2 *($\sim$5 min)*

Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

(a) T is injective

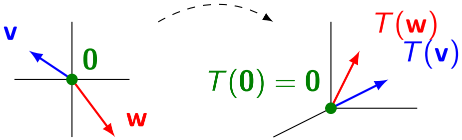
(b) T is not injective

(c) T is surjective

(d) T is not surjective

Fact D.4.3

A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.



Activity D.4.4 *(∼5 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be a linear transformation where $\operatorname{Im} T$ is spanned by four vectors. What can you conclude?

(a) T is injective

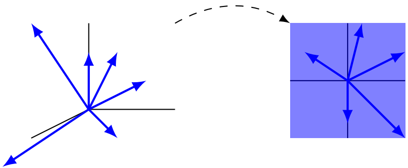
(b) T is not injective

(c) T is surjective

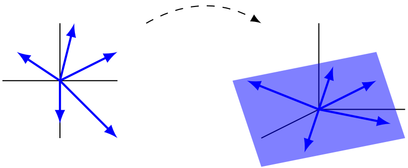
(d) T is not surjective

Fact D.4.5

A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective, $\operatorname{Im} T = \mathbb{R}^2$



not surjective, $\operatorname{Im} T \neq \mathbb{R}^3$

Activity D.4.6 *(∼15 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

(a) The kernel of T is trivial: $\ker T = \{\mathbf{0}\}$.

(b) The columns of A span $\mathbb{R}^m$.

(c) The columns of A are linearly independent.

(d) Every column of $\operatorname{RREF}(A)$ has a pivot.

(e) Every row of $\operatorname{RREF}(A)$ has a pivot.

(f) The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.

(g) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$.

(h) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ has exactly one solution.

Observation D.4.7

The easiest way to show that the linear map with standard matrix A is injective is to show that $\mathrm{RREF}(A)$ has a pivot in each column.

The easiest way to show that the linear map with standard matrix A is surjective is to show that $\mathrm{RREF}(A)$ has a pivot in each row.

Activity D.4.8 *(∼3 min)*

What can you immediately conclude (i.e. without computing a RREF) about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -3 & 3 \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

Activity D.4.9 *($\sim$2 min)*

What can you immediately conclude (i.e. without computing a RREF) about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 4 \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

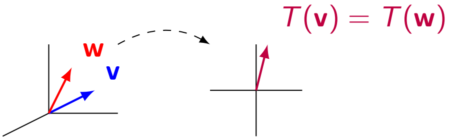
d) Its standard matrix has more rows than columns, so T is surjective.

Fact D.4.10

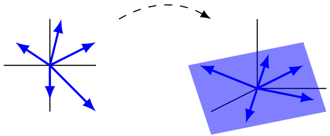
The following are true for any linear map $T : V \to W$:

- If $\dim(V) > \dim(W)$, then T is not injective.
- If $\dim(V) < \dim(W)$, then T is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase the dimension of its image.



not injective, $3 > 2$



not surjective, $2 < 3$

But dimension arguments **cannot** be used to prove a map **is** injective or surjective.

Activity D.4.11 *(∼5 min)*

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).

Activity D.4.11 *(∼5 min)*

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot columns must A have?

Activity D.4.11 *(∼5 min)*
Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).
Part 1: How many pivot columns must A have?
Part 2: How many pivot rows must A have?

Activity D.4.11 *(∼5 min)*

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot columns must A have?

Part 2: How many pivot rows must A have?

Part 3: What can you conclude about m and n?

Activity D.4.12 *(~5 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^n$

(b) $\mathrm{RREF}(A)$ is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \,\middle|\, \mathbf{b}\right]$ has exactly one solution for each $\mathbf{b} \in \mathbb{R}^n$.

Observation D.4.13

The easiest way to show that the linear map with standard matrix A is bijective is to show that RREF(A) is the identity matrix.

Activity D.4.14 *(∼3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity D.4.15 *(∼3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x+y-z\\4x+y+z\\6x+2y\end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity D.4.16 *(~3 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity D.4.17 *(∼3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.