Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

C1. Set up and solve a differential equation to answer the following question:
A water droplet with a radius of 10 μm has a mass of about 4×10^{-15}kg and a terminal velocity of 3 $\frac{\text{cm}}{\text{s}}$. Such a droplet is dropped from rest. What is its velocity after 0.01 s?

Solution: The forces acting on the water droplet are gravity (mg downwards) and air resistance; in a water droplet this size, this is proportional to its velocity. Let b denote this drag coefficient; then by Newton's second law (if we let up be the positive direction and down be negative), we have

$$m\frac{dv}{dt} = -mg - bv.$$

Note that we have $-mg$ since gravity always acts downwards, and $-bv$ because the drag acts in the opposite direction of v. It is actually convenient to let $a = \frac{b}{m}$, and divide this equation by the mass, yielding

$$\frac{dv}{dt} = -g - av.$$

This is a separable differential equation; we thus have

$$\int \frac{1}{g + av}\ dv = \int (-1)dt.$$

Thus, $\frac{1}{a}\ln|g+av| = -t+c_1$, or $|g+av| = e^{-at+ac_1} = c_2e^{-at}$ (where $c_2 = e^{ac_1}$). To avoid the pesky absolute value sign, we can absorb a potential negative into the constant to write $g+av = c_3e^{-at}$, or (letting $c_0 = \frac{c_3}{a}$)

$$v = -\frac{g}{a} + c_0e^{-at}.$$

Note that $\lim_{t\to\infty} v = -\frac{g}{a}$, so since we are given the terminal velocity, so after converting to meters per second we can solve $-0.03 = -\frac{9.8}{a}$ to obtain $a = \frac{9.8}{0.03}$. We also know that it is dropped from rest, so we compute $0 = v(0) = -0.03 + c_0$, so $c_0 = 0.03$. Thus our model for velocity is

$$v = -0.03 + 0.03e^{-\frac{9.8}{0.03}t}.$$

Then we simply compute $v(0.01) = -0.03 + 0.03e^{-\frac{9.8}{0.03}(0.01)} \approx -0.029$. So after 0.01 s, the drop is falling approximately 2.9 $\frac{\text{cm}}{\text{s}}$.

$\square$

C2. Find the general solution to

$$y' + y = -t^2.$$

Solution: First, we find a general solution to the homogeneous equation

$$y' + y = 0.$$

This has auxilliary equation $r + 1 = 0$, which has a single root at $r = 1$, so ce^{-t} is a solution. We can find a particular solution y_p to the given equation by using undetermined coefficients; since $-t^2$ is a polynomial, we let $y_p = At^2 + Bt + D$ and determine the coefficients A, B, and D.

$$\begin{aligned} y_p' + y_p &= = (2At + B) + (At^2 + Bt + D) \\ &= At^2 + (2A + B) + (B + D) \end{aligned}$$

So if y_p is a solution, we must have $y_p' + y_p = -t^2$, giving us the system of equations

$$\begin{aligned} A &= -1 \\ 2A + B &= 0 \\ B + D &= 0 \end{aligned}$$

Thus we easily deduce that $A = -1$, $B = 2$, and $D = -2$, giving $y_p = -t^2 + 2t - 2$. Thus, the general solution is

$$y = -t^2 + 2t - 2 + ce^{-t}.$$

□

C3. A 2kg mass is suspended by a spring (with spring constant 8kg/s^2). The mass is pulled down 1m from its equillibrium position and released from rest. How long does it take to return to its equillibrium point?

Solution: Let y denote the vertical distance from equillibrium; then the forces acting are gravity and the spring force, giving the ODE

$$2y'' + 8y = 0.$$

Simplifying, we have $y'' + 4y = 0$, so the general solution is $y = c_1 \cos(2t) + c_2 \sin(2t)$. The initial conditions are $y(0) = -1$ and $y'(0) = 0$, which imply $c_1 = -1$ and $c_2 = 0$, so the system is modelled by $y = -\cos(2t)$. This is first zero when $2t = \frac{\pi}{2}$, i.e. when $t = \frac{\pi}{4}$. Thus the mass takes $\frac{\pi}{4}$ s to return to its equillibrium point.

□

C4. Find the general solution to

$$y'' + 6y' + 13y = 0.$$

Solution: We begin by writing the auxilliary equation $r^2 + 6r + 13 = 0$ and finding the roots. There are many ways to do this; here, we complete the square:

$$0 = r^2 + 6r + 13 = r^2 + 6r + 9 + 4 = (r+3)^2 + 4.$$

Thus, we can easily solve to obtain $r = -3 \pm 2i$. Thus the general solution is

$$y = c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t).$$

□

C5. Find the general solution to

$$y'' + 6y' + 13y = 13t^2 - t - 4.$$

Solution: First, we find a general solution to the homogenous equation $y'' + 6y' + 13y = 0$. We begin by writing the auxilliary equation $r^2 + 6r + 13 = 0$ and finding the roots. There are many ways to do this; here, we complete the square:

$$0 = r^2 + 6r + 13 = r^2 + 6r + 9 + 4 = (r+3)^2 + 4.$$

Thus, we can easily solve to obtain $r = -3 \pm 2i$. Thus the general solution to the homogeneous equation is

$$y = c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t).$$

To find a particular solution to $y'' + 6y' + 13y = 13t^2 - t - 4$, we set $y_p = At^2 + Bt + D$ and determine the coefficients A, B, D.

$$\begin{aligned} y_p'' + 6y_p' + 13y_p &= (2A) + 6(2At + B) + 13(At^2 + Bt + D) \\ &= (13A)t^2 + (12A + 13B)t + (2A + 6B + 13D) \end{aligned}$$

This gives us the system of equations

$$\begin{aligned} 13A &= 13 \\ 12A + 13B &= -1 \\ 2A + 6B + 13D &= -4 \end{aligned}$$

Then we easily deduce $A = 1$, $B = -1$, and $D = 0$, so that $y_p = t^2 - t$. Thus, the general solution to the nonhomogeneous equation is

$$y = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t) + t^2 - t.$$

□

C6. Find the solution to

$$y'' + 10y' + 24y = 0$$

when $y(0) = -3$ and $y'(0) = 2$.

Solution: The auxilliary equation is $r^2 + 10r + 24 = 0$, which has roots $r = -6$ and $r = -4$. Thus, the general solution is of the form $y = c_1 e^{-4t} + c_2 e^{-6t}$.

$$\begin{aligned} -3 &= y(0) = c_1 + c_2 \\ 2 &= y'(0) = -4c_1 - 6c_2 \end{aligned}$$

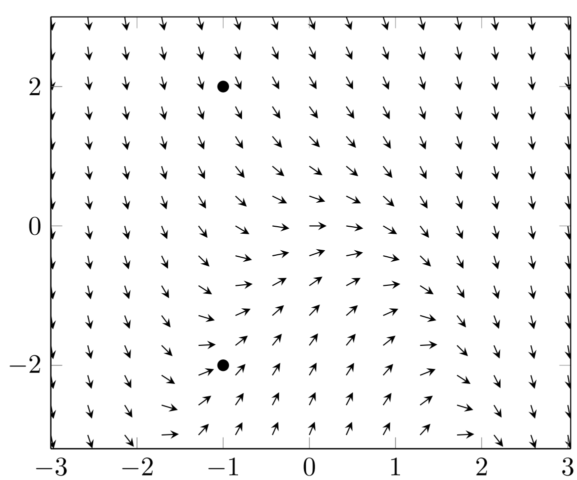
Solving this system yields $c_1 = -8$ and $c_2 = 5$, so the solution to the IVP is

$$y = -8e^{-4t} + 5e^{-6t}$$

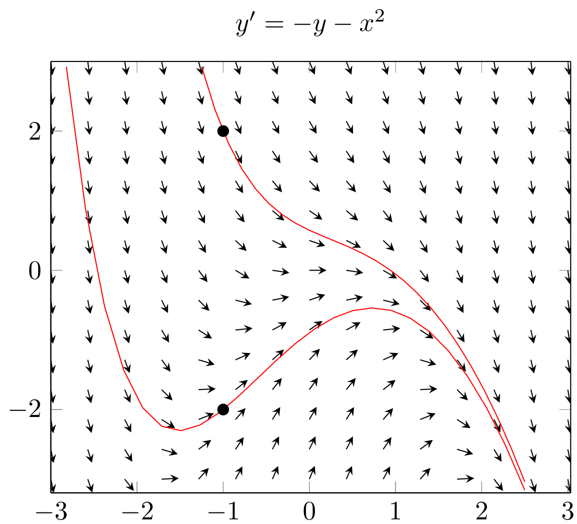
□

F1. Sketch a solution curve through each point marked in the slope field.

$$y' = -y - x^2$$



Solution:



$\square$

F2. Solve $y' + xy = x$.

Solution: Rearranging, we have $y' = x - xy = x(1-y)$, so we see the equation is separable, and write

$$\frac{y'}{1-y} = x.$$

Thus we compute $\int \frac{y'}{1-y}\ dx = \int \frac{1}{1-y}\ dy = -\ln|1-y| + c_1$, and $\int x\ dx = \frac{1}{2}x^2 + c_2$. Thus, we have (letting $c_3 = c_2 - c_1$)

$$-\ln|1-y| = \frac{1}{2}x^2 + c_3.$$

Then exponentiating, we have (letting $c_4 = \pm e^{-c_3}$) $1 - y = c_4 e^{-\frac{1}{2}x^2}$, so (with $c = -c_4$) the general solution is

$$y = 1 + ce^{-\frac{1}{2}x^2}.$$

$\square$

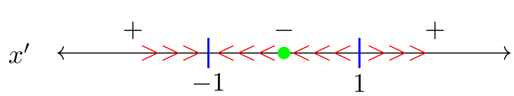
F3. Consider the autonomous equation

$$\frac{dx}{dt} = x^2 - 1.$$

(a) Find and classify the critical points.

(b) Describe the long term behavior of the solution passing through the point $x(4) = 0$.

Solution:
Note that $x^2 - 1 = (x-1)(x+1)$, so there are equillibria solutions at $x = 1$ and $x = -1$. We can thus compute a number line for x':



We see that -1 is a sink (stable), while 1 is a source (unstable). A trajectory passing through $x(4) = 0$ will approach $x = -1$ in the limit.

$\square$

F4. Solve $y' + xy = x^3$.

Solution: Solve homog; then variation of parameters.

□

F5. One of the two ODEs below is exact. Identify which one, and solve it.

$$(x + 3y)y' + y = 3x$$
$$(x + 3y)y' - y = -3x$$

Solution:

□

F6.

Solution:

□

S1.

Solution:

□

S2.

Solution:

□

S3.

Solution:

□

N1.

Solution:

□

N2.

Solution:

□

N3.

Solution:

□

N4.

Solution:

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N5.

Solution:

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D1.

Solution:

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D2.

Solution:

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D3.

Solution:

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D4.

Solution:

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