Module C: Constant coefficient linear ODEs

How can we solve and apply linear constant coefficient ODEs?

At the end of this module, students will be able to...

C1. Constant coefficient first order. ...find the general solution to a first order constant coefficient ODE.

C2. Modeling motion in viscous fluids. ...model the motion of a falling object with linear drag

C3. Homogeneous constant coefficient second order. ...find the general solution to a homogeneous second order constant coefficient ODE.

C4. IVPs. ...solve initial value problems for constant coefficient ODEs

C5. Non-homogenous constant coefficient second order. ...find the general solution to a non-homogeneous second order constant coefficient ODE

C6. Modeling oscillators. ...model (free or forced, damped or undamped) mechanical oscillators with a second order ODE

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Describe Newton's laws in terms of differential equations.
- Find all roots of a quadratic polynomial.
- Use Euler's theorem to relate $\sin(t)$, $\cos(t)$, and e^t.
- Use Euler's theorem to simplify complex exponentials.
- Use substitution to compute indefinite integrals.
- Use integration by parts to compute indefinite integrals.
- Solve systems of two linear equations in two variables.

The following resources will help you prepare for this module.

- Describe Newtons laws in terms of differential equations. https://youtu.be/cioi4lRrAzw
- Find all roots of a quadratic polynomial. https://youtu.be/2ZzuZvz33X0 https://youtu.be/TV5kDqiJ1Os
- Use Eulers theorem to relate $\sin(t)$, $\cos(t)$, and e^t and to simplify complex exponentials. https://youtu.be/F_0yfvm0UoU https://youtu.be/sn3orkHWqUQ
- Use substitution to compute indefinite integrals. https://youtu.be/b76wePnIBdU
- Use integration by parts to compute indefinite integrals. https://youtu.be/bZ8YAHDTFJ8
- Solve systems of two linear equations in two variables. https://youtu.be/Y6JsEja15Vk

Module C Section 1

Activity C.1.1 *(∼5 min)*

Why don't clouds fall out of the sky?

(a) They are lighter than air

(b) Wind keeps them from falling

(c) Electrostatic charge

(d) They do fall, just very slowly

Activity C.1.2 *(∼5 min)*

List all of the forces acting on a tiny droplet of water falling from the sky.

Activity C.1.3 *(∼5 min)*

Tiny droplets of water obey **Stoke's law**, which says that air resistance is proportional to (the magnitude of) velocity.

- Let v be the velocity of a droplet of water (positive for upward, negative for downward).
- Let $g > 0$ be the magnitude of acceleration due to gravity and $b > 0$ be another positive constant.

Apply Newton's second law (force $=$ mass $\times$ acceleration) to determine which of the following **ordinary differential equations (ODEs)** models the velocity of a falling droplet of water.

(a) $v' = g - v$

(b) $v' = g + v$

(c) $mv' = -mg - bv$

(d) $mv' = -mg + bv$

Observation C.1.4

The modeling equation

$$mv' = -mg - bv$$

may be obtained by splitting the total force into gravity and air resistance:

$$F = F_g + F_r$$

Then $F = ma = mv'$ and $F_g = m(-g) = -mg$ are the result of Newton's second law, and $F_r = -bv$ holds because it should be (a) in the opposite direction of velocity and (b) a constant multiple of velocity.

Note that this equation may be rearranged as follows to group v and its derivative v' together on the left-hand side:

$$v' + \left(\frac{b}{m}\right) v = -g$$

Definition C.1.5

A **first order constant coefficient** differential equation can be written in the form

$$y' + by = f(x),$$

or equivalently,

$$\frac{dy}{dx} + by = f(x).$$

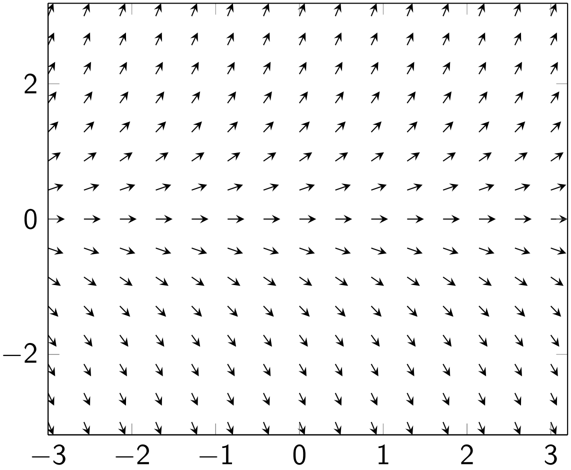
We will use both notations interchangeably.

Here, **first order** refers to the fact that the highest derivative we see is the first derivative of y.

Observation C.1.6

Consider the differential equation $y' = y$.

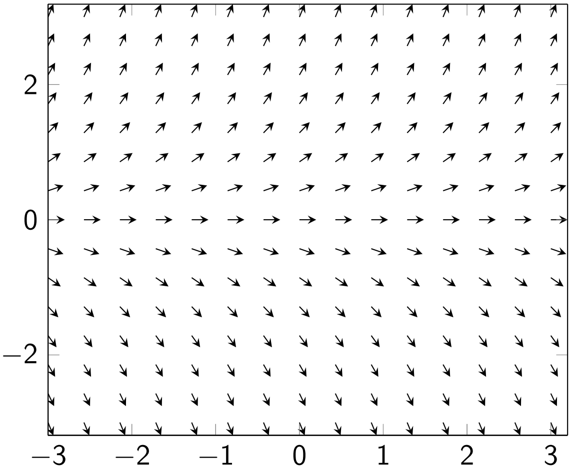
A useful way to visualize a first order differential equation is by a **slope field**



Each arrow represents the slope of a solution **trajectory** through that point.

Activity C.1.7 *(∼5 min)*

Consider the differential equation $y' = y$ with slope field below.



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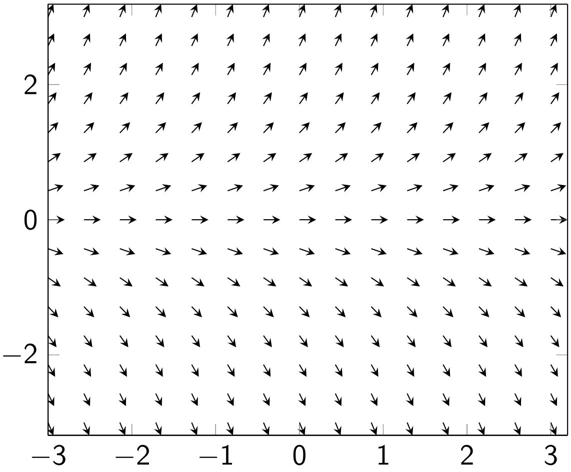

Part 1: Draw a trajectory through the point $(0, 1)$.

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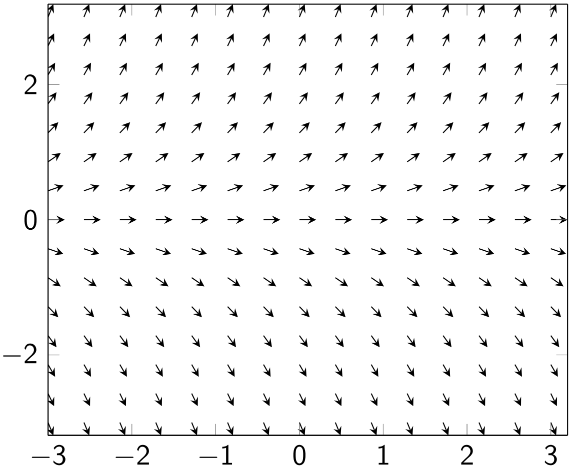

Part 1: Draw a trajectory through the point $(0, 1)$.

Part 2: Draw a trajectory through the point $(-1, -1)$.

Activity C.1.8 *(∼15 min)*

Consider the differential equation $y' = y$.

Activity C.1.8 *(∼15 min)*
Consider the differential equation $y' = y$.
Part 1: Find a solution to $y' = y$.

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Consider the differential equation $y' = y$.
Part 1: Find a solution to $y' = y$.
Part 2: Modify this solution to write an expression describing **all** solutions to $y' = y$.

Definition C.1.9

A differential equation will have many solutions. Each individual solution is said to be a **particular solution**, while the **general solution** encompasses **all** of these by using parameters such as C, k, c_0, c_1 and so on. For example:

- The general solution to the differential equation $y' = 2x - 3$ is $y = x^2 - 3x + C$ (as done in Calculus courses).
- The general solution for $y' = y$ is $y = ke^x$ (as done in the previous activity).

Activity C.1.10 *($\sim$15 min)*
Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.

Activity C.1.10 *(∼15 min)*
Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.
Part 1: Solve $y' = 2y$.

Activity C.1.10 *(∼15 min)*
Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.
Part 1: Solve $y' = 2y$.
Part 2: Solve $y' = y + 2$.

Activity C.1.11 *(∼15 min)*
Find the solution for $y' = y + 2$ directly.

Simple idea: Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

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Part 1: Use the Product Rule to find $y_p' = \frac{d}{dx}[ve^x]$.
Part 2: Substitute y_p and y_p' into the equation $y' = y + 2$.

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Part 1: Use the Product Rule to find $y_p' = \frac{d}{dx}[ve^x]$.
Part 2: Substitute y_p and y_p' into the equation $y' = y + 2$.
Part 3: Solve for v', and integrate to find v.

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Part 2: Substitute y_p and y_p' into the equation $y' = y + 2$.
Part 3: Solve for v', and integrate to find v.
Part 4: Find y_p.

Observation C.1.12

The technique outlined in the previous activity is called **variation of parameters**. If y_0 is a particular solution of the **homogeneous** equation, assume that a particular solution of the **non-homogeneous** equation has the form $y_p = vy_0$, and then determine what v must be.

Example:

$$y' + 3y = 0 \qquad \text{homogeneous}$$

$$y' + 3y = x \qquad \text{non-homogeneous}$$

Note that each term of the homogeneous equation includes y or it derivatives.

Activity C.1.13 *(∼20 min)*
Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0 \qquad \text{homogeneous}$$
$$y' + 3y = x \qquad \text{non-homogeneous}$$

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Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$y' + 3y = 0$	homogeneous
$y' + 3y = x$	non-homogeneous

Part 1: Modify e^x to find the general solution y_h for the homogeneous equation.

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$$y' + 3y = x \qquad \text{non-homogeneous}$$

Part 1: Modify e^x to find the general solution y_h for the homogeneous equation.
Part 2: Choose a particular solution y_0 for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** v. Substitute y_p into non-homogeneous equation and simplify.

Activity C.1.13 *(∼20 min)*

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Part 1: Modify e^x to find the general solution y_h for the homogeneous equation.
Part 2: Choose a particular solution y_0 for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** v. Substitute y_p into non-homogeneous equation and simplify.
Part 3: Determine v, and then determine y_p.

Observation C.1.14

Since $y_h = ke^{-3x}$ was the general solution of $y' + 3y = 0$, and $y_p = \frac{x}{3} - \frac{1}{9}$ is a particular solution of $y' + 3y = x$,

$$y = y_h + y_p = \left(ke^{-3x}\right) + \left(\frac{x}{3} - \frac{1}{9}\right)$$

is a solution to $y' + 3y = x$:

$$\frac{d}{dx}[y_h + y_p] + 3(y_h + y_p) = (y_h' + 3y_h) + (y_p' + 3y_p) = 0 + x = x$$

Fact C.1.15

Let a be a constant real number. Every constant coefficient first order ODE

$$y' + ay = f(x)$$

has the general solution

$$y = y_h + y_p$$

where y_h is the general solution to the homogeneous equation $y' + ay = 0$ and y_p is a particular solution to $y' + ay = f(t)$.

Activity C.1.16 *(∼15 min)*
Find the general solution to $y' = 2y + x + 1$ using variation of parameters:

- Write the homogeneous equation and find its general solution y_h.
- Use a particular solution y_0 for the homogeneous equation to find a particular solution $y_p = vy_0$ for the original equation.
- Then $y = y_h + y_p$ gives the general solution to the equation.

Module C Section 2

Observation C.2.1

Recall that we can model the velocity of a water droplet in a cloud by

$$mv' = -mg - bv$$

where negative numbers represent downward motion, $m > 0$ is the mass of the droplet, $g > 0$ is the magnitude of acceleration due to gravity, and $b > 0$ is the proportion of wind resistance to speed.

Activity C.2.2 *(∼20 min)*
A water droplet with a radius of $10\,\mu$m has a mass of about 4×10^{-15} kg. It is determined in a laboratory that for a droplet this size, the constant b has a value of 3×10^{-3} kg/s, and it is known that g is approximately $9.8\,\mathrm{m/s}^2$.

Complete the following tasks to study the motion of this droplet.

Activity C.2.2 *(∼20 min)*
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Part 1: Rewrite $mv' = -mg - bv$ in the form of $v' + av = \ ?$ for some value of a.

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Complete the following tasks to study the motion of this droplet.
Part 1: Rewrite $mv' = -mg - bv$ in the form of $v' + av = \;?$ for some value of a.
Part 2: Find the general solution of this ODE in terms of a and g. (Let $v_p = wv_0$ when using variation of parameters to avoid confusion.)

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Part 3: Due to wind resistence, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of a and g?

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Complete the following tasks to study the motion of this droplet.

Part 1: Rewrite $mv' = -mg - bv$ in the form of $v' + av = ?$ for some value of a.

Part 2: Find the general solution of this ODE in terms of a and g. (Let $v_p = wv_0$ when using variation of parameters to avoid confusion.)

Part 3: Due to wind resistence, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of a and g?

Part 4: If the droplet starts from rest ($v = 0$ when $t = 0$), what is its velocity after 0.01 s? Use a calculator to compute the answer in m/s.

Definition C.2.3
The last part of the previous activity is an example of an **Initial Value Problem (IVP)**; we were given the initial value of the velocity in addition to our differential equation.

$$v' + (b/m)v = -g \qquad v(0) = 0$$

Physical scenarios often produce IVPs with a unique solution.

Module C Section 3

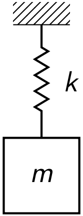
Observation C.3.1

What happens when your tire hits a pothole?

`https://prof.clontz.org/assets/img/good-bad-shocks.gif`

Activity C.3.2 *($\sim$5 min)*

Hooke's law says that the force exerted by the spring is proportional to the distance the spring is stretched from its natural length, given by a spring coefficient $k > 0$.

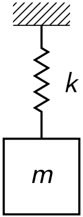


Let y measure the displacement of the mass from the spring's natural length. Write a differential equation modeling the displacement of the m kg mass, assuming that the only force acting on the mass comes from the spring.

Observation C.3.3

Since the spring acts on the mass in the opposite direction of displacement, we may model the mass-spring system with

$$my'' = -ky.$$



Activity C.3.4 *(∼15 min)*

Consider the mass-spring equation $my'' = -ky$ where $m = k = 1$:

$$y'' = -y.$$

Activity C.3.4 *(∼15 min)*

Consider the mass-spring equation $my'' = -ky$ where $m = k = 1$:

$$y'' = -y.$$

Part 1: Find a solution.

Activity C.3.4 *(∼15 min)*
Consider the mass-spring equation $my'' = -ky$ where $m = k = 1$:

$$y'' = -y.$$

Part 1: Find a solution.
Part 2: Find the general solution.

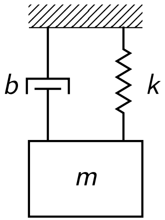
Activity C.3.4 *(∼15 min)*
Consider the mass-spring equation $my'' = -ky$ where $m = k = 1$:

$$y'' = -y.$$

Part 1: Find a solution.
Part 2: Find the general solution.
Part 3: Describe the long term behavior of the mass-spring system.

Activity C.3.5 *(~5 min)*
The general solution $y = c_1 \cos(t) + c_2 \sin(t)$ models infinitely oscillating behavior, but in applications this does not occur.

Thus, a damper (a.k.a. dashpot) is often considered, which provides a force proportional to velocity, given by the coefficient $b > 0$. For example, friction may act as a damper to a mass-spring system.



Write a differential equation modeling the displacement of a mass in a **damped** mass-spring system.

Observation C.3.6

The damped mass-spring system can be modelled by

$$my'' = -by' - ky.$$

Here m is the mass, k is the spring constant, and b is the damping constant. We can rearrange this as

$$y'' + By' + Ky = 0$$

where $B = \frac{b}{m}$ and $K = \frac{k}{m}$.

This is a **homogeneous second order constant coefficient** differential equation. Here, **homogeneous** refers to the 0 on the right hand side of the equation.

Activity C.3.7 *(∼15 min)*

Consider the second order constant coefficient equation

$$y'' = y.$$

Activity C.3.7 *(∼15 min)*

Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.

Activity C.3.7 *(∼15 min)*
Consider the second order constant coefficient equation

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Part 1: Find a solution.
Part 2: Find the general solution.

Activity C.3.7 *(∼15 min)*
Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.
Part 2: Find the general solution.
Part 3: Describe the long term behavior of the solutions.

Observation C.3.8

It is sometimes useful to think in terms of **differential operators**.

- We will use D to represent a derivative.So for any function y,

$$D(y) = \frac{\partial y}{\partial x} = y'.$$

- D^2 will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use I for the identity operator, so $I(y) = y$. (It can be thought of as $I = D^0$, take the derivative zero times.)

In this language, the differential equation $y' + 3y = 0$ can be rewritten as $D(y) + 3I(y) = 0$, or more simply $(D + 3I)(y) = 0$.

Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator $D + 3I$: all the functions y that the transformation $D + 3I$ turns into the zero function.

Activity C.3.9 *($\sim$5 min)*

Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

a) $D - 4I$

b) $D + 4I$

c) $D^2 - 4I$

d) $D^2 + 4D$

Activity C.3.10 *(∼5 min)*

The kernel of the differential operator $D - 4I$ whose kernel is the general solution of the ODE $y' = 4y$. What is its general solution?

a) $y = ke^{-4x}$

b) $y = ke^{4x}$

c) $y = 4x + k$

d) $y = 4$

Activity C.3.11 *(∼5 min)*

What are ODE and general solution given by the kernel of the differential operator $D - aI$ for a real number a?

a) $y' - ay = 0$ and $y = ke^{ax}$.

b) $y' + ay = 0$ and $y = ke^{-ax}$.

c) $y' - a = 0$ and $y = ax + k$.

d) $y'' + a = 0$ and $y = -\frac{a}{2}x^2 + kx + l$.

Observation C.3.12

The kernel of the differential operator $D - aI$ is given by the general solution $y = ke^{ax}$.

Activity C.3.13 *(∼15 min)*

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Activity C.3.13 *(∼15 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Use I, D, D^2 to write a differential operator whose kernel is the solution set of the above ODE.

Activity C.3.13 *(∼15 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Use I, D, D^2 to write a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two simpler operators, as you would a polynomial. (This works because the order of applying the transformations D and I doesn't matter).

Activity C.3.13 *(∼15 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Use I, D, D^2 to write a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two simpler operators, as you would a polynomial. (This works because the order of applying the transformations D and I doesn't matter).
Part 3: Find the general solution for each factor, and then combine to find the general solution to the overall ODE.

Activity C.3.13 *(∼15 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Use I, D, D^2 to write a differential operator whose kernel is the solution set of the above ODE.
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Part 3: Find the general solution for each factor, and then combine to find the general solution to the overall ODE.
Part 4: Check that your general solution is valid by computing y', y'' and plugging into $y'' + 5y' + 6y = 0$.

Observation C.3.14

The kernel of $(D+3I)(D+2I)$ is given by $y = k_1 e^{-3t} + k_2 e^{-2t}$.

In general for $\alpha \neq \beta$, the kernel of $(D-\alpha I)(D-\beta I)$ is given by $y = k_1 e^{at} + k_2 e^{bt}$.

Activity C.3.15 *(~10 min)*
Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

Activity C.3.16 *(∼15 min)*
Recall that the general solution to $y'' + y = 0$ is given by $y = c_1 \sin(x) + c_2 \cos(x)$. Show how to find this solution using the differential operator $D^2 + 1$.

Activity C.3.17 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Activity C.3.17 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Part 1: Find its general solution using complex numbers.

Activity C.3.17 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Part 1: Find its general solution using complex numbers.
Part 2: Describe the general solution only involving real numbers.

Activity C.3.18 *(∼5 min)*

Which of these are solutions to the following ODE?

$$y'' - 4y' + 4y = 0$$

a) $y = e^{2t}$, where $y' = 2e^{2t}$ and $y'' = 4e^{2t}$

b) $y = te^{2t}$, where $y' = e^{2t} + 2te^{2t}$ and $y'' = 4e^{2t} + 4e^{2t}$

c) $y = e^{2t} + te^{2t}$, where $y' = 3e^{2t} + 2te^{2t}$ and $y'' = 8e^{2t} + 4e^{2t}$

d) All of the above

Observation C.3.19

To solve $y'' - 4y' + 4y = 0$, we need to find the kernel of $(D-2I)(D-2I) = (D-2I)^2$.

- The kernel of $D - 2I$ is given by ke^{2x}.
- But if $(D-2I)(y) = e^{2t}$, then $(D-2I)(D-2I)(y) = (D-2I)(e^{2t}) = 0$ also.
- That means the kernel of $(D-2I)^2$ is given by both $(D-2I)(y) = 0$ and $(D-2I)(y) = e^{2t}$.

Activity C.3.20 *(∼15 min)*
Solve $(D - 2I)(y) = e^{2x}$.

Observation C.3.21

Since $(D-2I)(y)=0$ solves to ke^{2t} and $(D-2I)(y)=e^{2t}$ solves to kte^{2t}, we have shown that the general solution of

$$y''-4y'+4y=0$$

is

$$y=c_0e^{2t}+c_1te^{2t}.$$

Activity C.3.22 *(∼10 min)*

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

If r is a number such that $ar^2 + br + c = 0$, what can you conclude?

(a) e^{rt} is a solution.

(b) e^{-rt} is a solution.

(c) te^{rt} is a solution.

(d) There are no solutions.

Activity C.3.23 *(∼5 min)*

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

When does the general solution have the form $c_0 e^{rt} + c_1 t e^{rt}$?

(a) When the polynomial $ax^2 + bx + c$ has two distinct real roots.

(b) When the polynomial $ax^2 + bx + c$ has a repeated real root.

(c) When the polynomial $ax^2 + bx + c$ has two distinct non-real roots.

(d) When the polynomial $ax^2 + bx + c$ has a repeated non-real root.

Observation C.3.24

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0$$

given by the differential operator $aD^2 + bD + cI$. Let r be a (possibly non-real) solution to $ax^2 + bx + c = 0$:

- e^{rt} is a particular solution of the ODE.
- If r is a double root, te^{rt} is also a particular solution.
- if $r = \alpha + \beta i$ is not real, Euler's formula allows us to express the real-valued solutions in terms of $\sin(\beta t)$ and $\cos(\beta t)$.

Due to the usefulness of its solutions, $ax^2 + bx + c = 0$ is called the **auxiliary equation** for this ODE.

Module C Section 4

Remark C.4.1

While first or second-order constant-coefficient ODEs usually solve to general solutions such as $y = c_1 e^t + c_2 e^{-2t}$, the values of the parameters c_1, c_2 may be determined when given additional information.

Activity C.4.2 *(∼10 min)*

Solve the IVP

$$y' + 3y = 0, \qquad y(0) = 2.$$

Activity C.4.3 *(∼15 min)*
Solve $y'' - 6y' + 9y = 0$ where $y(0) = 2$ and $y(1) = \frac{3}{e^3}$.

Activity C.4.4 *(∼15 min)*

Solve $y'' - 6y' + 8y = 0$ where $y(0) = 1$ and $y'(0) = -2$.

Module C Section 5

¡¡¡¡¡¡¡ HEAD =======

Observation C.5.1

It is sometimes useful to think in terms of **differential operators**.

- We will use D to represent a derivative; another common notation is $\frac{\partial}{\partial x}$. So for any function y,
$$D(y) = \frac{\partial y}{\partial x} = y'.$$
- D^2 will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use I for the identity operator; it does nothing to a function. That is, $I(y) = y$. It can be thought of as $I = D^0$ (i.e. differentiate zero times).

In this language, the differential equation $y' + 3y = 0$ can be rewritten as $D(y) + 3I(y) = 0$, or $(D + 3I)(y) = 0$.

Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator $D + 3I$.

Activity C.5.2 *($\sim$5 min)*

What is the kernel of $D - I$?

Activity C.5.2 *($\sim$5 min)*
What is the kernel of $D - I$?

Part 1: Write a differential equation that corresponds to this question.

Activity C.5.2 *(∼5 min)*
What is the kernel of $D - I$?

Part 1: Write a differential equation that corresponds to this question.
Part 2: Find the general solution of this differential equation.

Activity C.5.3 *(∼5 min)*

Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

Activity C.5.4 *(∼10 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Activity C.5.4 *(∼10 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Activity C.5.4 *(∼10 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

Activity C.5.4 *(∼10 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).
Part 3: Find the general solution of the ODE.

Observation C.5.5

If we let $\mathcal{L} = D^2 + 5D + 6I$, we can write the ODE

$$y'' + 5y + 6y = 0$$

as

$$\mathcal{L}(y) = 0.$$

Note that such an $\mathcal{L}$ is always a **linear transformation**.

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Activity C.5.6 *(∼5 min)*
Find the general solution to

$$y'' + y' - 12y = 0.$$

Module C Section 6

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Activity C.6.1 *(∼5 min)*
Consider the ODE

$$y'' + 5y' - 6y = 0.$$

Activity C.6.1 *(∼5 min)*
Consider the ODE

$$y'' + 5y' - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Activity C.6.1 *(∼5 min)*
Consider the ODE

$$y'' + 5y' - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

Activity C.6.1 *(∼5 min)*
Consider the ODE

$$y'' + 5y' - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.
Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).
Part 3: Find the general solution of the ODE.

Activity C.6.2 *(∼5 min)*
Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

Activity C.6.3 *(∼5 min)*
An **Initial Value Problem (IVP)** consists of an ODE along with some initial conditions that allow you to determine a single solution.

Solve the IVP

$$2y'' + 7y' + 6y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

.

Activity C.6.4 *($\sim$5 min)*

Solve the ODE

$$y'' + y = 0.$$

Activity C.6.5 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Activity C.6.5 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Part 1: Find the general solution.

Activity C.6.5 *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Part 1: Find the general solution.
Part 2: Describe the long-term behavior of the solutions.

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Observation C.6.6

Solving $y'' + 2y' + 5y = 0$ produced a general solution

$$y = c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t}.$$

Applying Euler's formula yields

$$\begin{aligned} y &= c_1 e^{-t}\left(\cos(2t) + i\sin(2t)\right) + c_2 e^{-t}\left(\cos(2t) - i\sin(2t)\right) \\ &= (c_1 + c_2)e^{-t}\cos(2t) i(c_1 - c_2)e^{-t}\sin(2t) \end{aligned}$$

which we can rewrite (letting $k_1 = c_1 + c_2$ and $k_2 = i(c_1 - c_2)$) as

$$y = k_1 e^{-t}\cos(2t) + k_2 e^{-t}\sin(2t).$$

Activity C.6.7 *(∼15 min)*
Solve the IVP

$$y'' + 6y' + 34y = 0, \qquad y(0) = 2, \qquad y'(0) = 4.$$

Module C Section 7

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Activity C.7.1 *(∼10 min)*
Solve the ODE

$$y'' - 4y' + 4y = 0.$$

Observation C.7.2

To solve this, we need to find the kernel of $(D - 2I)(D - 2I)$.

- The kernel of $D - 2I$ is $\left\{ce^{2t} \mid c \in \mathbb{R}\right\}$.
- However, if $(D - 2I)(y) = Ae^{2t}$, then applying $D - 2I$ twice will yield zero!
- So we must solve the ODE
$$y' - 2y = e^{2t}.$$

Activity C.7.3 *(∼15 min)*
Solve $y' - 2y = e^{2t}$.

Observation C.7.4

Thus, we have shown that the general solution of

$$y'' - 4y' + 4y = 0$$

is

$$y = c_0 e^{2t} + c_1 t e^{2t}.$$

Activity C.7.5 *(∼15 min)*

Solve $y'' - 6y' + 9y = 0$.

Activity C.7.6 *(~10 min)*

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

If r is a number such that $ar^2 + br + c = 0$, what can you conclude?

(a) e^{rt} is a solution.

(b) e^{-rt} is a solution.

(c) te^{rt} is a solution.

(d) There are no solutions.

Activity C.7.7 *(∼5 min)*

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

When does the general solution have the form $c_0e^{rt} + te^{rt}$?

(a) When the polynomial $ax^2 + bx + c$ has two distinct real roots.

(b) When the polynomial $ax^2 + bx + c$ has a repeated real root.

(c) When the polynomial $ax^2 + bx + c$ has two distinct non-real roots.

(d) When the polynomial $ax^2 + bx + c$ has a repeated non-real root.

Observation C.7.8

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ax^2 + bx + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that te^{rt} is also a solution.
- If r is not real, Euler's formula allows us to express the complex exponential part of the solution in terms of $\sin(rt)$ and $\cos(rt)$.

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Module C Section 8

Observation C.8.1

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ax^2 + bx + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that te^{rt} is also a solution.
- If $r = a + bi$ is not real, Euler's formula allows us to express the complex exponential part of the solution in terms of e^{at}, $\sin(bt)$, and $\cos(bt)$.

Activity C.8.2 *(∼15 min)*
Consider the following scenario: a mass of $4\ \mathrm{kg}$ suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\ \mathrm{kg/s}$.

The mass is pulled down $0.3\ \mathrm{m}$ and released from rest.

Activity C.8.2 *(∼15 min)*
Consider the following scenario: a mass of 4 kg suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\ \mathrm{kg/s}$.

The mass is pulled down 0.3 m and released from rest.
Part 1: Write down an ODE modelling this scenario, and find the general solution.

Activity C.8.2 *(∼15 min)*
Consider the following scenario: a mass of 4 kg suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\ \mathrm{kg/s}$.

The mass is pulled down 0.3 m and released from rest.
Part 1: Write down an ODE modelling this scenario, and find the general solution.
Part 2: Use the initial conditions $y(0) = -0.3$ and $y'(0) = 0$ to find particular values of the constants.

Definition C.8.3
In the previous problem, we needed to solve

$$4y'' + 6y' + 2y = 0, \qquad y(0) = -0.3, \qquad y'(0) = 0.$$

This is called an **Initial Value Problem (IVP)** since we are provided with initial values of y and y'.

To solve an IVP, find a general solution of the ODE, and use the initial conditions to find the values of the constants.

Activity C.8.4 *(∼15 min)*

Consider a mass of $5\ \mathrm{kg}$ suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\mathrm{kg/s}$.

The mass is pulled down $0.3\mathrm{m}$ and released from rest. How many times does it pass back through its equillibrium state?

(a) 0
(b) 1
(c) 2
(d) Infinitely many

Observation C.8.5

It can be shown that in the **overdamped** situation, the spring might pass through the equillibrium position once (e.g. if given an initial push), but never more than once.

Module C Section 9

Activity C.9.1 *(∼10 min)*

A $1\ \mathrm{kg}$ mass is suspended from a spring with spring constant $k = 9\ \mathrm{kg/s^2}$. An external force is applied by an electromagnet and is modeled by the function $F(t) = \sin(t)$. Write an ODE modeling the displacement of the spring.

Observation C.9.2

In the previous activity, we encountered a **nonhomogeneous** second order constant coefficient ODE, i.e. of the form

$$ay'' + by' + cy = f(x)$$

where a, b, c are constants, and $f(x)$ is a function.

We will again use **variation of parameters** to find a particular solution.

Activity C.9.3 *(∼15 min)*

Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions v_1, v_2.

Activity C.9.3 *(∼15 min)*

Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions v_1, v_2.

Part 1: Use the product rule (twice) to compute y_p'.

Activity C.9.3 *(∼15 min)*
Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions v_1, v_2.

Part 1: Use the product rule (twice) to compute y_p'.
Part 2: To simplify calculations, we will **assume** $v_1' y_1 + v_2' y_2 = 0$. Assuming this, compute y_p''.

Activity C.9.3 *(∼15 min)*

Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions v_1, v_2.

Part 1: Use the product rule (twice) to compute y_p'.
Part 2: To simplify calculations, we will **assume** $v_1' y_1 + v_2' y_2 = 0$. Assuming this, compute y_p''.
Part 3: Compute $\mathcal{L}(y_p)$; simplify the ODE $\mathcal{L}(y_p) = f(x)$.

Observation C.9.4

If we can find v_1 and v_2 that satisfy

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= \frac{f}{a} \end{aligned}$$

then we have a solution. So we just need to solve this system of equations for v_1' and v_2'.

Activity C.9.5 *(∼15 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Activity C.9.5 *(~15 min)*
Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2, two independent solutions of $y'' + 9y = 0$.

Activity C.9.5 *(∼15 min)*
Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2, two independent solutions of $y'' + 9y = 0$.
Part 2: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(t) \end{aligned}$$

Activity C.9.5 *(∼15 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2, two independent solutions of $y'' + 9y = 0$.

Part 2: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(t) \end{aligned}$$

Part 3: Write the general solution of the original nonhomogeneous ODE.

Activity C.9.6 *(∼10 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.

Activity C.9.6 *(∼10 min)*
Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.

Part 1: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(3t) \end{aligned}$$

Activity C.9.6 *(∼10 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.

Part 1: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(3t) \end{aligned}$$

Part 2: Write the general solution of the original nonhomogeneous ODE.