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Module C: Constant coefficient linear ODEs

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How can we solve and apply linear constant coefficient ODEs?

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At the end of this module, students will be able to...

- C1. Constant coefficient first order.** ...find the general solution to a first order constant coefficient ODE.
- C2. Modeling motion in viscous fluids.** ...model the motion of a falling object with linear drag
- C3. Homogeneous constant coefficient second order.** ...find the general solution to a homogeneous second order constant coefficient ODE.
- C4. IVPs.** ...solve initial value problems for constant coefficient ODEs
- C5. Non-homogenous constant coefficient second order.** ...find the general solution to a non-homogeneous second order constant coefficient ODE
- C6. Modeling oscillators.** ...model (free or forced, damped or undamped) mechanical oscillators with a second order ODE

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Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Describe Newton's laws in terms of differential equations.
- Find all roots of a quadratic polynomial.
- Use Euler's theorem to relate $\sin(t)$, $\cos(t)$, and e^t .
- Use Euler's theorem to simplify complex exponentials.
- Use substitution to compute indefinite integrals.
- Use integration by parts to compute indefinite integrals.
- Solve systems of two linear equations in two variables.

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The following resources will help you prepare for this module.

- Describe Newtons laws in terms of differential equations.
<https://youtu.be/cioi4lRrAzw>
- Find all roots of a quadratic polynomial. <https://youtu.be/2ZzuZvz33X0>
<https://youtu.be/TV5kDqiJ10s>
- Use Eulers theorem to relate $\sin(t)$, $\cos(t)$, and e^t and to simplify complex exponentials. https://youtu.be/F_0yfvm0UoU
<https://youtu.be/sn3orkHWqUQ>
- Use substitution to compute indefinite integrals.
<https://youtu.be/b76wePnIBdU>
- Use integration by parts to compute indefinite integrals.
<https://youtu.be/bZ8YAHDTFJ8>
- Solve systems of two linear equations in two variables.
<https://youtu.be/Y6JsEja15Vk>

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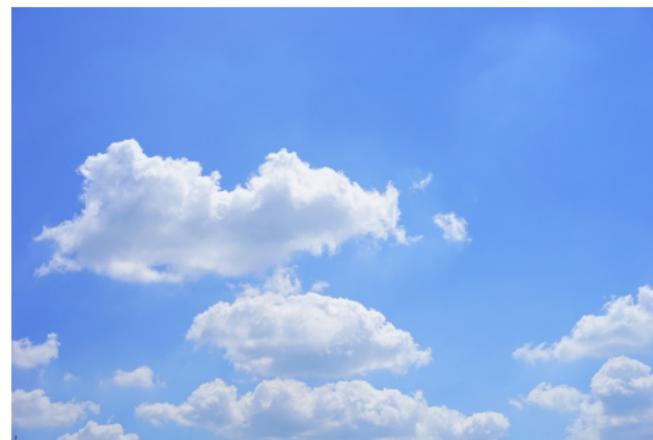
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Activity C.1.1 ($\sim 5 \text{ min}$)

Why don't clouds fall out of the sky?



- (a) They are lighter than air
- (b) Wind keeps them from falling
- (c) Electostatic charge
- (d) They do fall, just very slowly

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Activity C.1.2 ($\sim 5 \text{ min}$)

List all of the forces acting on a tiny droplet of water falling from the sky.

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Activity C.1.3 ($\sim 5 \text{ min}$)

Tiny droplets of water obey **Stoke's law**, which says that air resistance is proportional to (the magnitude of) velocity.

- Let v be the velocity of a droplet of water (positive for upward, negative for downward).
- Let $g > 0$ be the magnitude of acceleration due to gravity and $b > 0$ be another positive constant.

Apply Newton's second law (force = mass \times acceleration) to determine which of the following **ordinary differential equations (ODEs)** models the velocity of a falling droplet of water.

- (a) $v' = g - v$
(b) $v' = g + v$
(c) $mv' = -mg - bv$
(d) $mv' = -mg + bv$

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Observation C.1.4

The modeling equation

$$mv' = -mg - bv$$

may be obtained by splitting the total force into gravity and air resistance:

$$F = F_g + F_r$$

Then $F = ma = mv'$ and $F_g = m(-g) = -mg$ are the result of Newton's second law, and $F_r = -bv$ holds because it should be (a) in the opposite direction of velocity and (b) a constant multiple of velocity.

Note that this equation may be rearranged as follows to group v and its derivative v' together on the left-hand side:

$$v' + \left(\frac{b}{m}\right)v = -g$$

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Definition C.1.5

A **first order constant coefficient** differential equation can be written in the form

$$y' + by = f(x),$$

or equivalently,

$$\frac{dy}{dx} + by = f(x).$$

We will use both notations interchangeably.

Here, **first order** refers to the fact that the highest derivative we see is the first derivative of y .

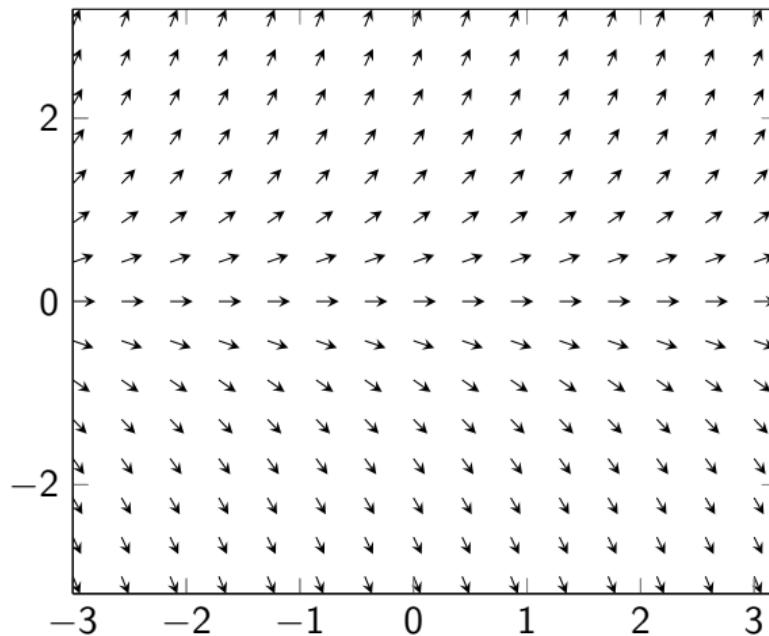
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Observation C.1.6

Consider the differential equation $y' = y$.

A useful way to visualize a first order differential equation is by a **slope field**



Each arrow represents the slope of a solution **trajectory** through that point.

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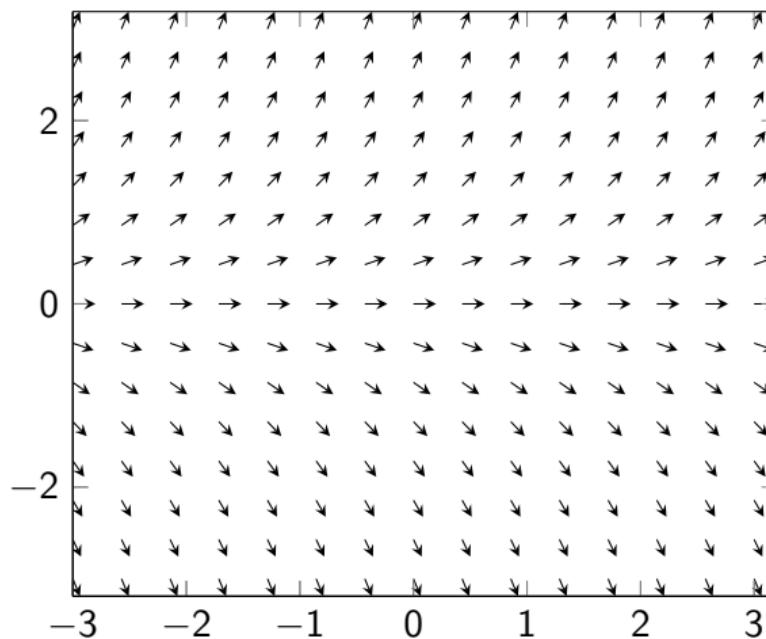
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Activity C.1.7 ($\sim 5 \text{ min}$)

Consider the differential equation $y' = y$ with slope field below.



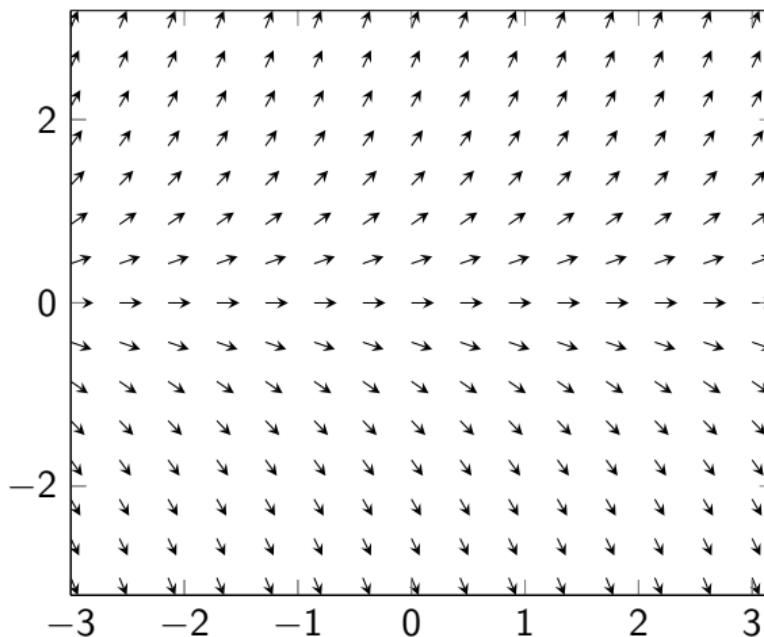
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Activity C.1.7 ($\sim 5 \text{ min}$)

Consider the differential equation $y' = y$ with slope field below.



Part 1: Draw a trajectory through the point $(0, 1)$.

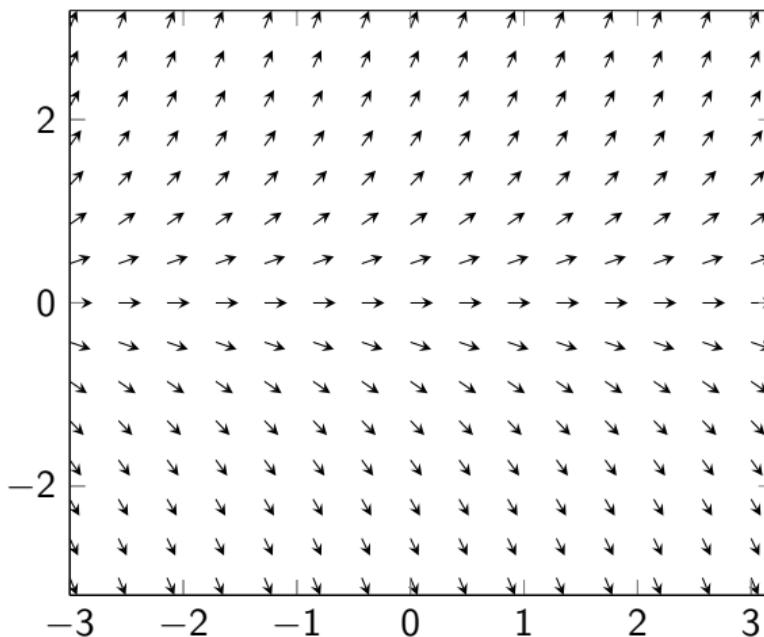
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Activity C.1.7 ($\sim 5 \text{ min}$)

Consider the differential equation $y' = y$ with slope field below.



Part 1: Draw a trajectory through the point $(0, 1)$.

Part 2: Draw a trajectory through the point $(-1, -1)$.

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Activity C.1.8 ($\sim 15 \text{ min}$)

Consider the differential equation $y' = y$.

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Consider the differential equation $y' = y$.

Part 1: Find a solution to $y' = y$.

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Activity C.1.8 ($\sim 15 \text{ min}$)

Consider the differential equation $y' = y$.

Part 1: Find a solution to $y' = y$.

Part 2: Modify this solution to write an expression describing **all** solutions to $y' = y$.

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Definition C.1.9

A differential equation will have many solutions. Each individual solution is said to be a **particular solution**, while the **general solution** encompasses **all** of these by using parameters such as C, k, c_0, c_1 and so on. For example:

- The general solution to the differential equation $y' = 2x - 3$ is $y = x^2 - 3x + C$ (as done in Calculus courses).
- The general solution for $y' = y$ is $y = ke^x$ (as done in the previous activity).

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Activity C.1.10 ($\sim 15 \text{ min}$)

Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.

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Activity C.1.10 ($\sim 15 \text{ min}$)

Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.

Part 1: Solve $y' = 2y$.

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Activity C.1.10 ($\sim 15 \text{ min}$)

Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.

Part 1: Solve $y' = 2y$.

Part 2: Solve $y' = y + 2$.

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Observation C.2.1

Recall the last activity from yesterday:

Solve $y' = y + 2$

This is very similar to the equation $y' = y$, which we were able to solve.

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Activity C.2.2 ($\sim 15 \text{ min}$)Solve $y' = y + 2$

Simple idea: Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

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Activity C.2.2 ($\sim 15 \text{ min}$)

$$\text{Solve } y' = y + 2$$

Simple idea: Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

Part 1: Use the Product Rule to find $y'_p = \frac{d}{dx}[ve^x]$.

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Activity C.2.2 ($\sim 15 \text{ min}$)

$$\text{Solve } y' = y + 2$$

Simple idea: Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

Part 1: Use the Product Rule to find $y'_p = \frac{d}{dx}[ve^x]$.

Part 2: Substitute y_p and y'_p into the equation $y' = y + 2$.

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Activity C.2.2 ($\sim 15 \text{ min}$)Solve $y' = y + 2$

Simple idea: Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

Part 1: Use the Product Rule to find $y'_p = \frac{d}{dx}[ve^x]$.

Part 2: Substitute y_p and y'_p into the equation $y' = y + 2$.

Part 3: Solve for v .

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Activity C.2.2 ($\sim 15 \text{ min}$)Solve $y' = y + 2$

Simple idea: Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

Part 1: Use the Product Rule to find $y'_p = \frac{d}{dx}[ve^x]$.

Part 2: Substitute y_p and y'_p into the equation $y' = y + 2$.

Part 3: Solve for v .

Part 4: Find y_p .

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Observation C.2.3

The technique outlined in the previous activity is called **variation of parameters**.

If y_0 is a particular solution of the **homogeneous** equation, assume that a particular solution of the **non-homogeneous** equation has the form $y_p = vy_0$, and then determine what v must be.

Example:

$$\begin{array}{ll} y' + 3y = 0 & \text{homogeneous} \\ y' + 3y = x & \text{non-homogeneous} \end{array}$$

Note that each term of the homogeneous equation includes y or its derivatives.

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Activity C.2.4 ($\sim 20\text{ min}$)

Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0 \quad \text{homogeneous}$$

$$y' + 3y = x \quad \text{non-homogeneous}$$

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Activity C.2.4 ($\sim 20 \text{ min}$)

Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0 \quad \text{homogeneous}$$

$$y' + 3y = x \quad \text{non-homogeneous}$$

Part 1: Modify e^x to find the general solution y_h for the homogeneous equation.

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Activity C.2.4 ($\sim 20\text{ min}$)

Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0 \quad \text{homogeneous}$$

$$y' + 3y = x \quad \text{non-homogeneous}$$

Part 1: Modify e^x to find the general solution y_h for the homogeneous equation.

Part 2: Choose a particular solution y_0 for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** v . Substitute y_p into non-homogeneous equation and simplify.

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Activity C.2.4 ($\sim 20 \text{ min}$)

Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0$$

homogeneous

$$y' + 3y = x$$

non-homogeneous

Part 1: Modify e^x to find the general solution y_h for the homogeneous equation.

Part 2: Choose a particular solution y_0 for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** v . Substitute y_p into non-homogeneous equation and simplify.

Part 3: Determine v , and then determine y_p .

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Observation C.2.5

Since $y_h = ke^{-3x}$ was the general solution of $y' + 3y = 0$, and $y_p = \frac{x}{3} - \frac{1}{9}$ is a particular solution of $y' + 3y = x$,

$$y = y_h + y_p = (ke^{-3x}) + \left(\frac{x}{3} - \frac{1}{9}\right)$$

is a solution to $y' + 3y = x$:

$$\frac{d}{dx}[y_h + y_p] + 3(y_h + y_p) = (y'_h + 3y_h) + (y'_p + 3y_p) = 0 + x = x$$

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[Section C.9](#)**Fact C.2.6**

Let a be a constant real number. Every constant coefficient first order ODE

$$y' + ay = f(x)$$

has the general solution

$$y = y_h + y_p$$

where y_h is the general solution to the homogeneous equation $y' + ay = 0$ and y_p is a particular solution to $y' + ay = f(t)$.

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Activity C.2.7 ($\sim 15 \text{ min}$)

Find the general solution to $y' = 2y + x + 1$ using variation of parameters:

- Write the homogeneous equation and find its general solution y_h .
- Use a particular solution y_0 for the homogeneous equation to find a particular solution $y_p = vy_0$ for the original equation.
- Then $y = y_h + y_p$ gives the general solution to the equation.

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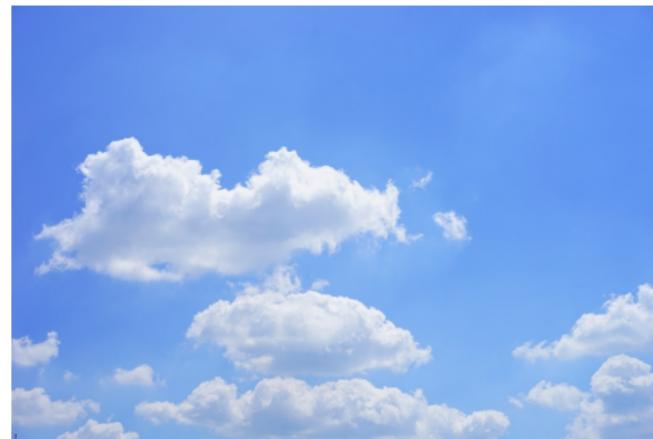
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Observation C.3.1

Recall that we can model the velocity of a water droplet in a cloud by

$$mv' = -mg + bv$$

where here, negative denotes downward velocity. m is the mass, g is Newton's gravitational constant, and b is a physical constant (like a coefficient of friction).



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Activity C.3.2 ($\sim 25 \text{ min}$)

A water droplet with a radius of $10 \mu\text{m}$ has a mass of about $4 \times 10^{-15}\text{kg}$. It is determined in a laboratory that for a droplet this size, the constant b has a value of 0.003kg/s .

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Activity C.3.2 ($\sim 25 \text{ min}$)

A water droplet with a radius of $10 \mu\text{m}$ has a mass of about $4 \times 10^{-15}\text{kg}$. It is determined in a laboratory that for a droplet this size, the constant b has a value of 0.003kg/s .

Part 1: Write down the differential equation modelling this scenario.

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Activity C.3.2 ($\sim 25 \text{ min}$)

A water droplet with a radius of $10 \mu\text{m}$ has a mass of about $4 \times 10^{-15}\text{kg}$. It is determined in a laboratory that for a droplet this size, the constant b has a value of 0.003kg/s .

Part 1: Write down the differential equation modelling this scenario.

Part 2: Find the general solution of this ODE.

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Activity C.3.2 ($\sim 25 \text{ min}$)

A water droplet with a radius of $10 \mu\text{m}$ has a mass of about $4 \times 10^{-15} \text{ kg}$. It is determined in a laboratory that for a droplet this size, the constant b has a value of 0.003 kg/s .

Part 1: Write down the differential equation modelling this scenario.

Part 2: Find the general solution of this ODE.

Part 3: What is the **terminal velocity** of the droplet?

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Activity C.3.2 ($\sim 25 \text{ min}$)

A water droplet with a radius of $10 \mu\text{m}$ has a mass of about $4 \times 10^{-15} \text{ kg}$. It is determined in a laboratory that for a droplet this size, the constant b has a value of 0.003 kg/s .

Part 1: Write down the differential equation modelling this scenario.

Part 2: Find the general solution of this ODE.

Part 3: What is the **terminal velocity** of the droplet?

Part 4: If the droplet starts from rest ($v = 0$), what is its velocity after 0.01 s?

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Definition C.3.3

The last part of the previous activity is an example of an **Initial Value Problem (IVP)**; we were given the initial value of the velocity in addition to our differential equation.

Physical scenarios often produce IVPs with a unique solution.

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Activity C.3.4 ($\sim 10 \text{ min}$)

Solve the IVP

$$y' + 3y = 0, \quad y(0) = 2.$$

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Activity C.3.5 ($\sim 10 \text{ min}$)

Solve the IVP

$$y' - 2y = 2, \quad y(0) = 1.$$

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Activity C.3.6 ($\sim 5 \text{ min}$)

Solve the IVP

$$y' - 2y = 2, \quad y(2) = 1.$$

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Observation C.4.1

What happens when your tire hits a pothole?

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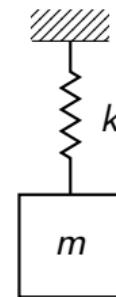
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Activity C.4.2 ($\sim 5 \text{ min}$)

More abstractly, let's attach a mass (weighing $m \text{ kg}$) to a spring.



List all forces acting on the mass.

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Activity C.4.3 ($\sim 5 \text{ min}$)

Hooke's law says that the force exerted by the spring is proportional to the distance the spring is stretched.

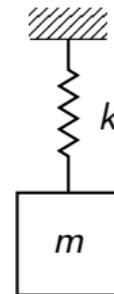


Write a differential equation modeling the displacement of the mass.

Observation C.4.4

There is an equilibrium point where the force of gravity balances the spring force. If we measure displacement from this point, we can model the mass-spring system by

$$my'' = ky.$$



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Activity C.4.5 ($\sim 15 \text{ min}$)

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

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Activity C.4.5 ($\sim 15 \text{ min}$)

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

Part 1: Find a solution.

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Activity C.4.5 ($\sim 15 \text{ min}$)

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

Part 1: Find a solution.

Part 2: Find the general solution.

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Activity C.4.5 ($\sim 15 \text{ min}$)

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

Part 1: Find a solution.

Part 2: Find the general solution.

Part 3: Describe the long term behavior of the mass-spring system.

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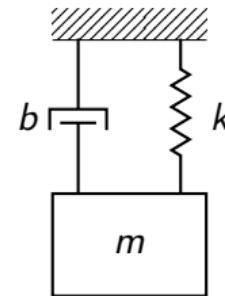
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Activity C.4.6 ($\sim 5 \text{ min}$)

In applications, this infinitely oscillating behavior is often inappropriate.

Thus, a damper (dashpot) is often incorporated. This provides a force proportional to the velocity.



Write a differential equation modeling the displacement of a mass in a **damped** mass-spring system.

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Observation C.4.7

The damped mass-spring system can be modelled by

$$my'' = -by' - ky.$$

Here m is the mass, k is the spring constant, and b is the damping constant. We can rearrange this as

$$my'' + by' + ky = 0.$$

This is a **homogeneous second order constant coefficient** differential equation. Here, **homogeneous** refers to the 0 on the right hand side of the equation.

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Activity C.4.8 ($\sim 15 \text{ min}$)

Consider the second order constant coefficient equation

$$y'' = y.$$

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Activity C.4.8 ($\sim 15 \text{ min}$)

Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.

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Activity C.4.8 ($\sim 15 \text{ min}$)

Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.

Part 2: Find the general solution.

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Activity C.4.8 ($\sim 15 \text{ min}$)

Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.

Part 2: Find the general solution.

Part 3: Describe the long term behavior of the solutions.

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Observation C.5.1

It is sometimes useful to think in terms of **differential operators**.

- We will use D to represent a derivative; another common notation is $\frac{\partial}{\partial x}$. So for any function y ,

$$D(y) = \frac{\partial y}{\partial x} = y'.$$

- D^2 will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use I for the identity operator; it does nothing to a function. That is, $I(y) = y$. It can be thought of as $I = D^0$ (i.e. differentiate zero times).

In this language, the differential equation $y' + 3y = 0$ can be rewritten as $D(y) + 3I(y) = 0$, or $(D + 3I)(y) = 0$.

Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator $D + 3I$.

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Activity C.5.2 ($\sim 5 \text{ min}$)

What is the kernel of $D - I$?

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What is the kernel of $D - I$?

Part 1: Write a differential equation that corresponds to this question.

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Activity C.5.2 ($\sim 5 \text{ min}$)What is the kernel of $D - I$?*Part 1:* Write a differential equation that corresponds to this question.*Part 2:* Find the general solution of this differential equation.

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Activity C.5.3 ($\sim 5 \text{ min}$)

Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

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Activity C.5.4 ($\sim 10 \text{ min}$)

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

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Activity C.5.4 ($\sim 10 \text{ min}$)

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

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Activity C.5.4 ($\sim 10 \text{ min}$)

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

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Activity C.5.4 ($\sim 10 \text{ min}$)

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

Part 3: Find the general solution of the ODE.

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Observation C.5.5

If we let $\mathcal{L} = D^2 + 5D + 6I$, we can write the ODE

$$y'' + 5y' + 6y = 0$$

as

$$\mathcal{L}(y) = 0.$$

Note that such an \mathcal{L} is always a **linear transformation**.

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Activity C.6.1 ($\sim 10 \text{ min}$)

Consider the ODE

$$y'' + 5y - 6y = 0.$$

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Activity C.6.1 ($\sim 10 \text{ min}$)

Consider the ODE

$$y'' + 5y - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

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Consider the ODE

$$y'' + 5y - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

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Activity C.6.1 ($\sim 10 \text{ min}$)

Consider the ODE

$$y'' + 5y - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

Part 3: Find the general solution of the ODE.

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Activity C.6.2 ($\sim 10 \text{ min}$)

Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

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Activity C.6.3 ($\sim 15 \text{ min}$)

Solve the ODE

$$y'' + y = 0.$$

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Activity C.6.4 ($\sim 15 \text{ min}$)

Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

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Activity C.6.4 ($\sim 15 \text{ min}$)

Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Part 1: Find the general solution.

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Activity C.6.4 ($\sim 15 \text{ min}$)

Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

Part 1: Find the general solution.*Part 2:* Describe the long-term behavior of the solutions.

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Activity C.7.1 ($\sim 10 \text{ min}$)

Solve the ODE

$$y'' - 4y' + 4y = 0.$$

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Observation C.7.2

To solve this, we need to find the kernel of $(D - 2I)(D - 2I)$.

- The kernel of $D - 2I$ is $\{ce^{2t} \mid c \in \mathbb{R}\}$.
- However, if $(D - 2I)(y) = Ae^{2t}$, then applying $D - 2I$ twice will yield zero!
- So we must solve the ODE

$$y' - 2y = e^{2t}.$$

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Activity C.7.3 ($\sim 15 \text{ min}$)

Solve $y' - 2y = e^{2t}$.

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Observation C.7.4

Thus, we have shown that the general solution of

$$y'' - 4y' + 4y = 0$$

is

$$y = c_0 e^{2t} + c_1 t e^{2t}.$$

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Solve $y'' - 6y' + 9y = 0$.

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Activity C.7.6 ($\sim 10 \text{ min}$)

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

If r is a number such that $ar^2 + br + c = 0$, what can you conclude?

- (a) e^{rt} is a solution.
- (b) e^{-rt} is a solution.
- (c) te^{rt} is a solution.
- (d) There are no solutions.

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Activity C.7.7 ($\sim 5 \text{ min}$)

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

When does the general solution have the form $c_0 e^{rt} + t e^{rt}$?

- (a) When the polynomial $ax^2 + bx + c$ has two distinct real roots.
- (b) When the polynomial $ax^2 + bx + c$ has a repeated real root.
- (c) When the polynomial $ax^2 + bx + c$ has two distinct non-real roots.
- (d) When the polynomial $ax^2 + bx + c$ has a repeated non-real root.

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Observation C.7.8

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ar^2 + br + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that te^{rt} is also a solution.
- if r is not real, Euler's formula allows us to express the solution in terms of $\sin(rt)$ and $\cos(rt)$.

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Observation C.8.1

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ar^2 + br + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that te^{rt} is also a solution.
- if r is not real, Euler's formula allows us to express the solution in terms of $\sin(rt)$ and $\cos(rt)$.

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Activity C.8.2 ($\sim 15 \text{ min}$)

Consider a mass of 4 kg suspended from a damped spring with spring constant $k = 2 \text{ kg/s}^2$ and damping constant $b = 6 \text{ kg/s}$.

The mass is pulled down 0.3 m and released from rest. How many times does it pass back through its equilibrium state?

- (a) 0
- (b) 1
- (c) 2
- (d) Infinitely many

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Activity C.8.3 ($\sim 15 \text{ min}$)

Consider a mass of 5 kg suspended from a damped spring with spring constant $k = 2 \text{ kg/s}^2$ and damping constant $b = 6\text{kg/s}$.

The mass is pulled down 0.3m and released from rest. How many times does it pass back through its equilibrium state?

- (a) 0
- (b) 1
- (c) 2
- (d) Infinitely many

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Observation C.8.4

It can be shown that in the **overdamped** situation, the spring might pass through the equilibrium position once (e.g. if given an initial push), but never more than once.

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[Section C.1](#)[Section C.2](#)[Section C.3](#)[Section C.4](#)[Section C.5](#)[Section C.6](#)[Section C.7](#)[Section C.8](#)[Section C.9](#)**Activity C.9.1 ($\sim 10 \text{ min}$)**

A 1 kg mass is suspended from a spring with spring constant $k = 9 \text{ kg/s}^2$. An external force is applied by an electromagnet and is modeled by the function $F(t) = \sin(t)$. Write an ODE modeling the displacement of the spring.

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Observation C.9.2

In the previous activity, we encountered a **nonhomogeneous** second order constant coefficient ODE, i.e. of the form

$$ay'' + by' + cy = f(x)$$

where a, b, c are constants, and $f(x)$ is a function.

We will again use **variation of parameters** to find a particular solution.

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Activity C.9.3 ($\sim 15 \text{ min}$)

Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L}(y) = ay'' + by' + cy$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form

$$y_p = v_1 y_1 + v_2 y_2 \text{ for some TBD functions } v_1, v_2.$$

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Activity C.9.3 ($\sim 15 \text{ min}$)

Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L}(y) = ay'' + by' + cy$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1y_1 + v_2y_2$ for some TBD functions v_1, v_2 .

Part 1: Use the product rule (twice) to compute y'_p .

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Activity C.9.3 ($\sim 15 \text{ min}$)

Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L}(y) = ay'' + by' + cy$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1y_1 + v_2y_2$ for some TBD functions v_1, v_2 .

Part 1: Use the product rule (twice) to compute y'_p .

Part 2: To simplify calculations, we will **assume** $v'_1y_1 + v'_2y_2 = 0$. Assuming this, compute y''_p .

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Activity C.9.3 ($\sim 15 \text{ min}$)

Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L}(y) = ay'' + by' + cy$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1y_1 + v_2y_2$ for some TBD functions v_1, v_2 .

Part 1: Use the product rule (twice) to compute y'_p .

Part 2: To simplify calculations, we will **assume** $v'_1y_1 + v'_2y_2 = 0$. Assuming this, compute y''_p .

Part 3: Compute $\mathcal{L}(y_p)$; simplify the ODE $\mathcal{L}(y_p) = f(x)$.

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Observation C.9.4

If we can find v_1 and v_2 that satisfy

$$y_1 v'_1 + y_2 v'_2 = 0$$

$$y'_1 v'_1 + y'_2 v'_2 = \frac{f}{a}$$

then we have a solution. So we just need to solve this system of equations for v'_1 and v'_2 .

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Section C.9**Activity C.9.5 ($\sim 15 \text{ min}$)**

Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

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Activity C.9.5 ($\sim 15 \text{ min}$)

Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2 , two independent solutions of $y'' + 9y = 0$.

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Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2 , two independent solutions of $y'' + 9y = 0$.

Part 2: Find v_1 and v_2 by solving

$$\cos(3t)v'_1 + \sin(3t)v'_2 = 0$$

$$-3\sin(3t)v'_1 + 3\cos(3t)v'_2 = \sin(t)$$

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Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2 , two independent solutions of $y'' + 9y = 0$.

Part 2: Find v_1 and v_2 by solving

$$\cos(3t)v'_1 + \sin(3t)v'_2 = 0$$

$$-3\sin(3t)v'_1 + 3\cos(3t)v'_2 = \sin(t)$$

Part 3: Write the general solution of the original nonhomogeneous ODE.

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Section C.9**Activity C.9.6 ($\sim 10 \text{ min}$)**

Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.

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Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.

Part 1: Find v_1 and v_2 by solving

$$\cos(3t)v'_1 + \sin(3t)v'_2 = 0$$

$$-3\sin(3t)v'_1 + 3\cos(3t)v'_2 = \sin(3t)$$

Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

Activity C.9.6 (~ 10 min)

Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.

Part 1: Find v_1 and v_2 by solving

$$\cos(3t)v'_1 + \sin(3t)v'_2 = 0$$

$$-3\sin(3t)v'_1 + 3\cos(3t)v'_2 = \sin(3t)$$

Part 2: Write the general solution of the original nonhomogeneous ODE.