

Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

## Module C: Constant coefficient linear ODEs

Module C

Section C.1  
Section C.2  
Section C.3  
Section C.4  
Section C.5  
Section C.6  
Section C.7  
Section C.8  
Section C.9

# How can we solve and apply linear constant coefficient ODEs?

## Module C

Section C.1  
Section C.2  
Section C.3  
Section C.4  
Section C.5  
Section C.6  
Section C.7  
Section C.8  
Section C.9

At the end of this module, students will be able to...

- C1. Constant coefficient first order.** ...find the general solution to a first order constant coefficient ODE.
- C2. Modeling motion in viscous fluids.** ...model the motion of a falling object with linear drag
- C3. Homogeneous constant coefficient second order.** ...find the general solution to a homogeneous second order constant coefficient ODE.
- C4. IVPs.** ...solve initial value problems for constant coefficient ODEs
- C5. Non-homogenous constant coefficient second order.** ...find the general solution to a non-homogeneous second order constant coefficient ODE
- C6. Modeling oscillators.** ...model (free or forced, damped or undamped) mechanical oscillators with a second order ODE

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Describe Newton's laws in terms of differential equations.
- Find all roots of a quadratic polynomial.
- Use Euler's theorem to relate  $\sin(t)$ ,  $\cos(t)$ , and  $e^t$ .
- Use Euler's theorem to simplify complex exponentials.
- Use substitution to compute indefinite integrals.
- Use integration by parts to compute indefinite integrals.
- Solve systems of two linear equations in two variables.

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

The following resources will help you prepare for this module.

- Describe Newtons laws in terms of differential equations.  
<https://youtu.be/cioi4lRrAzw>
- Find all roots of a quadratic polynomial. <https://youtu.be/2ZzuZvz33X0>  
<https://youtu.be/TV5kDqiJ10s>
- Use Eulers theorem to relate  $\sin(t)$ ,  $\cos(t)$ , and  $e^t$  and to simplify complex exponentials. [https://youtu.be/F\\_0yfvm0UoU](https://youtu.be/F_0yfvm0UoU)  
<https://youtu.be/sn3orkHWqUQ>
- Use substitution to compute indefinite integrals.  
<https://youtu.be/b76wePnIBdU>
- Use integration by parts to compute indefinite integrals.  
<https://youtu.be/bZ8YAHDTFJ8>
- Solve systems of two linear equations in two variables.  
<https://youtu.be/Y6JsEja15Vk>

Module C

**Section C.1**

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

# Module C Section 1

**Activity C.1.1** (*~5 min*)

Why don't clouds fall out of the sky?



- (a) They are lighter than air
- (b) Wind keeps them from falling
- (c) Electostatic charge
- (d) They do fall, just very slowly

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.2** ( $\sim 5$  min)

List all of the forces acting on a tiny droplet of water falling from the sky.



**Activity C.1.3** ( $\sim 5$  min)

Tiny droplets of water obey **Stoke's law**, which says that air resistance is proportional to (the magnitude of) velocity.

- Let  $v$  be the velocity of a droplet of water (positive for upward, negative for downward).
- Let  $g > 0$  be the magnitude of acceleration due to gravity and  $b > 0$  be another positive constant.

Apply Newton's second law (force = mass  $\times$  acceleration) to determine which of the following **ordinary differential equations (ODEs)** models the velocity of a falling droplet of water.

(a)  $v' = g - v$

(b)  $v' = g + v$

(c)  $mv' = -mg - bv$

(d)  $mv' = -mg + bv$

## Observation C.1.4

The modeling equation

$$mv' = -mg - bv$$

may be obtained by splitting the total force into gravity and air resistance:

$$F = F_g + F_r$$

Then  $F = ma = mv'$  and  $F_g = m(-g) = -mg$  are the result of Newton's second law, and  $F_r = -bv$  holds because it should be (a) in the opposite direction of velocity and (b) a constant multiple of velocity.

Note that this equation may be rearranged as follows to group  $v$  and its derivative  $v'$  together on the left-hand side:

$$v' + \left(\frac{b}{m}\right)v = -g$$

## Definition C.1.5

A **first order constant coefficient** differential equation can be written in the form

$$y' + by = f(x),$$

or equivalently,

$$\frac{dy}{dx} + by = f(x).$$

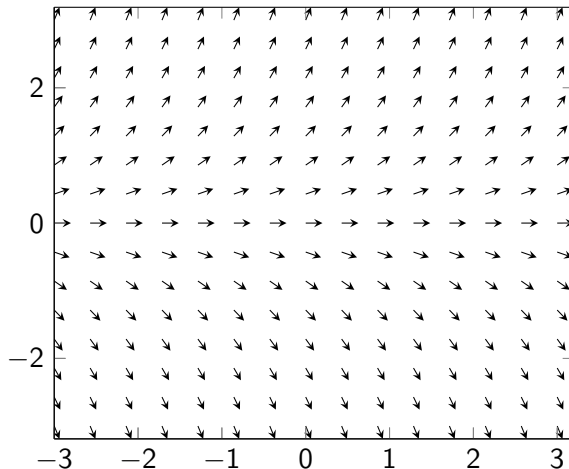
We will use both notations interchangeably.

Here, **first order** refers to the fact that the highest derivative we see is the first derivative of  $y$ .

## Observation C.1.6

Consider the differential equation  $y' = y$ .

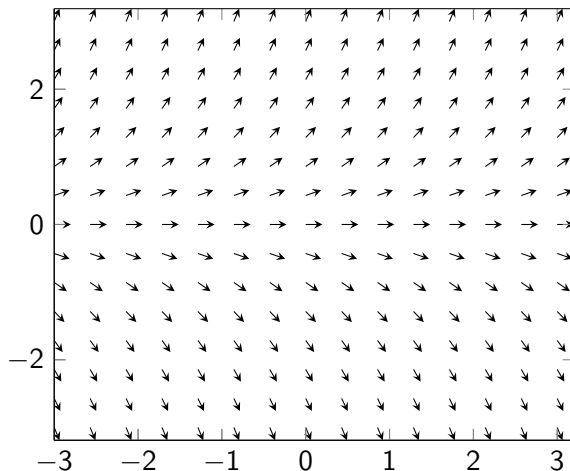
A useful way to visualize a first order differential equation is by a **slope field**



Each arrow represents the slope of a solution **trajectory** through that point.

**Activity C.1.7** ( $\sim 5$  min)

Consider the differential equation  $y' = y$  with slope field below.



## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

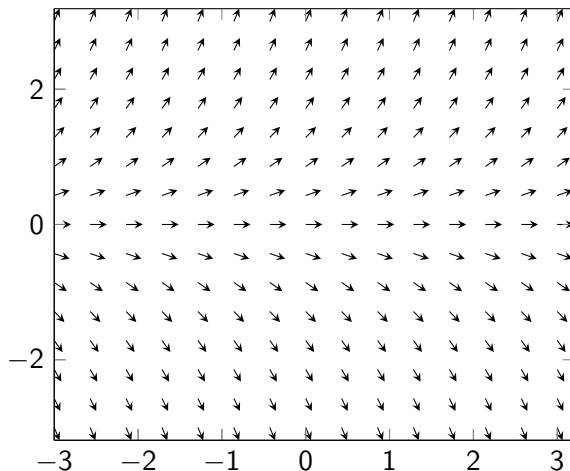
## Section C.7

## Section C.8

## Section C.9

**Activity C.1.7** ( $\sim 5$  min)

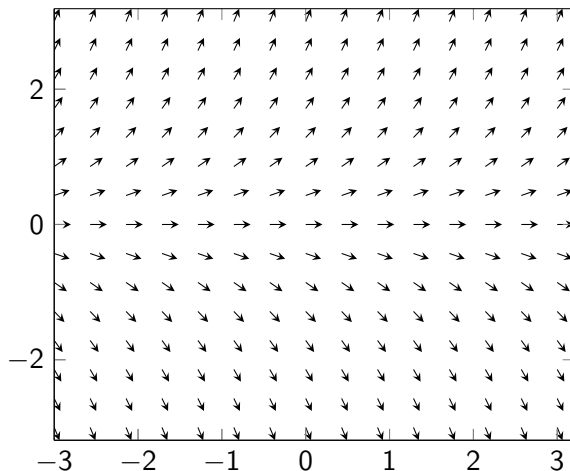
Consider the differential equation  $y' = y$  with slope field below.



*Part 1:* Draw a trajectory through the point  $(0, 1)$ .

**Activity C.1.7** ( $\sim 5$  min)

Consider the differential equation  $y' = y$  with slope field below.



*Part 1:* Draw a trajectory through the point  $(0, 1)$ .

*Part 2:* Draw a trajectory through the point  $(-1, -1)$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.8** (*~15 min*)

Consider the differential equation  $y' = y$ .



## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.8** ( $\sim 15$  min)

Consider the differential equation  $y' = y$ .

*Part 1:* Find a solution to  $y' = y$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.8** ( $\sim 15$  min)

Consider the differential equation  $y' = y$ .

*Part 1:* Find a solution to  $y' = y$ .

*Part 2:* Modify this solution to write an expression describing **all** solutions to  $y' = y$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

## Definition C.1.9

A differential equation will have many solutions. Each individual solution is said to be a **particular solution**, while the **general solution** encompasses **all** of these by using parameters such as  $C$ ,  $k$ ,  $c_0$ ,  $c_1$  and so on. For example:

- The general solution to the differential equation  $y' = 2x - 3$  is  $y = x^2 - 3x + C$  (as done in Calculus courses).
- The general solution for  $y' = y$  is  $y = ke^x$  (as done in the previous activity).

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.10** (*~15 min*)

Adapt the general solution  $y = ke^x$  for  $y' = y$  to find general solutions for the following differential equations.

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.10** (*~15 min*)

Adapt the general solution  $y = ke^x$  for  $y' = y$  to find general solutions for the following differential equations.

*Part 1:* Solve  $y' = 2y$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.10** (*~15 min*)

Adapt the general solution  $y = ke^x$  for  $y' = y$  to find general solutions for the following differential equations.

*Part 1:* Solve  $y' = 2y$ .

*Part 2:* Solve  $y' = y + 2$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.11** (*~15 min*)

Find the solution for  $y' = y + 2$  directly.

**Simple idea:** Since  $y_0 = e^x$  was a particular solution of  $y' = y$ , we guess that a particular solution for  $y' = y + 2$  is of the form  $y_p = ve^x$  for some **function**  $v(x)$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.11** (*~15 min*)

Find the solution for  $y' = y + 2$  directly.

**Simple idea:** Since  $y_0 = e^x$  was a particular solution of  $y' = y$ , we guess that a particular solution for  $y' = y + 2$  is of the form  $y_p = ve^x$  for some **function**  $v(x)$ .

*Part 1:* Use the Product Rule to find  $y'_p = \frac{d}{dx}[ve^x]$ .



## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.11** (*~15 min*)

Find the solution for  $y' = y + 2$  directly.

**Simple idea:** Since  $y_0 = e^x$  was a particular solution of  $y' = y$ , we guess that a particular solution for  $y' = y + 2$  is of the form  $y_p = ve^x$  for some **function**  $v(x)$ .

*Part 1:* Use the Product Rule to find  $y'_p = \frac{d}{dx}[ve^x]$ .

*Part 2:* Substitute  $y_p$  and  $y'_p$  into the equation  $y' = y + 2$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.11** (*~15 min*)

Find the solution for  $y' = y + 2$  directly.

**Simple idea:** Since  $y_0 = e^x$  was a particular solution of  $y' = y$ , we guess that a particular solution for  $y' = y + 2$  is of the form  $y_p = ve^x$  for some **function**  $v(x)$ .

*Part 1:* Use the Product Rule to find  $y'_p = \frac{d}{dx}[ve^x]$ .

*Part 2:* Substitute  $y_p$  and  $y'_p$  into the equation  $y' = y + 2$ .

*Part 3:* Solve for  $v'$ , and integrate to find  $v$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.11** (*~15 min*)

Find the solution for  $y' = y + 2$  directly.

**Simple idea:** Since  $y_0 = e^x$  was a particular solution of  $y' = y$ , we guess that a particular solution for  $y' = y + 2$  is of the form  $y_p = ve^x$  for some **function**  $v(x)$ .

*Part 1:* Use the Product Rule to find  $y'_p = \frac{d}{dx}[ve^x]$ .

*Part 2:* Substitute  $y_p$  and  $y'_p$  into the equation  $y' = y + 2$ .

*Part 3:* Solve for  $v'$ , and integrate to find  $v$ .

*Part 4:* Find  $y_p$ .

## Module C

- Section C.1
- Section C.2
- Section C.3
- Section C.4
- Section C.5
- Section C.6
- Section C.7
- Section C.8
- Section C.9

## Observation C.1.12

The technique outlined in the previous activity is called **variation of parameters**. If  $y_0$  is a particular solution of the **homogeneous** equation, assume that a particular solution of the **non-homogeneous** equation has the form  $y_p = v y_0$ , and then determine what  $v$  must be.

### Example:

$$y' + 3y = 0$$

homogeneous

$$y' + 3y = x$$

non-homogeneous

Note that each term of the homogeneous equation includes  $y$  or its derivatives.

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.1.13** (*~20 min*)

Solve  $y' = x - 3y$  by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0$$

homogeneous

$$y' + 3y = x$$

non-homogeneous

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.13** ( $\sim 20$  min)

Solve  $y' = x - 3y$  by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0$$

homogeneous

$$y' + 3y = x$$

non-homogeneous

*Part 1:* Modify  $e^x$  to find the general solution  $y_h$  for the homogeneous equation.

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.1.13** ( $\sim 20$  min)

Solve  $y' = x - 3y$  by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0$$

homogeneous

$$y' + 3y = x$$

non-homogeneous

*Part 1:* Modify  $e^x$  to find the general solution  $y_h$  for the homogeneous equation.

*Part 2:* Choose a particular solution  $y_0$  for the homogeneous equation, and assume  $y_p = v y_0$  is a particular solution of the non-homogeneous equation for some **function**  $v$ . Substitute  $y_p$  into non-homogeneous equation and simplify.

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.13** ( $\sim 20$  min)

Solve  $y' = x - 3y$  by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0$$

homogeneous

$$y' + 3y = x$$

non-homogeneous

*Part 1:* Modify  $e^x$  to find the general solution  $y_h$  for the homogeneous equation.

*Part 2:* Choose a particular solution  $y_0$  for the homogeneous equation, and assume  $y_p = v y_0$  is a particular solution of the non-homogeneous equation for some **function**  $v$ . Substitute  $y_p$  into non-homogeneous equation and simplify.

*Part 3:* Determine  $v$ , and then determine  $y_p$ .



## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Observation C.1.14**

Since  $y_h = ke^{-3x}$  was the general solution of  $y' + 3y = 0$ , and  $y_p = \frac{x}{3} - \frac{1}{9}$  is a particular solution of  $y' + 3y = x$ ,

$$y = y_h + y_p = (ke^{-3x}) + \left(\frac{x}{3} - \frac{1}{9}\right)$$

is a solution to  $y' + 3y = x$ :

$$\frac{d}{dx}[y_h + y_p] + 3(y_h + y_p) = (y'_h + 3y_h) + (y'_p + 3y_p) = 0 + x = x$$

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Fact C.1.15**

Let  $a$  be a constant real number. Every constant coefficient first order ODE

$$y' + ay = f(x)$$

has the general solution

$$y = y_h + y_p$$

where  $y_h$  is the general solution to the homogeneous equation  $y' + ay = 0$  and  $y_p$  is a particular solution to  $y' + ay = f(t)$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.1.16** (*~15 min*)

Find the general solution to  $y' = 2y + x + 1$  using variation of parameters:

- Write the homogeneous equation and find its general solution  $y_h$ .
- Use a particular solution  $y_0$  for the homogeneous equation to find a particular solution  $y_p = v y_0$  for the original equation.
- Then  $y = y_h + y_p$  gives the general solution to the equation.

Module C

Section C.1

**Section C.2**

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

## Module C Section 2

## Observation C.2.1

Recall that we can model the velocity of a water droplet in a cloud by

$$mv' = -mg - bv$$

where negative numbers represent downward motion,  $m > 0$  is the mass of the droplet,  $g > 0$  is the magnitude of acceleration due to gravity, and  $b > 0$  is the proportion of wind resistance to speed.



**Activity C.2.2** (*~20 min*)

A water droplet with a radius of  $10\text{ }\mu\text{m}$  has a mass of about  $4 \times 10^{-15}\text{ kg}$ . It is determined in a laboratory that for a droplet this size, the constant  $b$  has a value of  $3 \times 10^{-3}\text{ kg/s}$ , and it is known that  $g$  is approximately  $9.8\text{ m/s}^2$ .

Complete the following tasks to study the motion of this droplet.

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.2.2** (*~20 min*)

A water droplet with a radius of  $10\text{ }\mu\text{m}$  has a mass of about  $4 \times 10^{-15}\text{ kg}$ . It is determined in a laboratory that for a droplet this size, the constant  $b$  has a value of  $3 \times 10^{-3}\text{ kg/s}$ , and it is known that  $g$  is approximately  $9.8\text{ m/s}^2$ .

Complete the following tasks to study the motion of this droplet.

*Part 1:* Rewrite  $mv' = -mg - bv$  in the form of  $v' + av = ?$  for some value of  $a$ .

**Activity C.2.2** ( $\sim 20$  min)

A water droplet with a radius of  $10\text{ }\mu\text{m}$  has a mass of about  $4 \times 10^{-15}$  kg. It is determined in a laboratory that for a droplet this size, the constant  $b$  has a value of  $3 \times 10^{-3}$  kg/s, and it is known that  $g$  is approximately  $9.8\text{ m/s}^2$ .

Complete the following tasks to study the motion of this droplet.

*Part 1:* Rewrite  $mv' = -mg - bv$  in the form of  $v' + av = ?$  for some value of  $a$ .

*Part 2:* Find the general solution of this ODE in terms of  $a$  and  $g$ . (Let  $v_p = wv_0$  when using variation of parameters to avoid confusion.)



**Activity C.2.2** ( $\sim 20$  min)

A water droplet with a radius of  $10\text{ }\mu\text{m}$  has a mass of about  $4 \times 10^{-15}$  kg. It is determined in a laboratory that for a droplet this size, the constant  $b$  has a value of  $3 \times 10^{-3}$  kg/s, and it is known that  $g$  is approximately  $9.8\text{ m/s}^2$ .

Complete the following tasks to study the motion of this droplet.

*Part 1:* Rewrite  $mv' = -mg - bv$  in the form of  $v' + av = ?$  for some value of  $a$ .

*Part 2:* Find the general solution of this ODE in terms of  $a$  and  $g$ . (Let  $v_p = wv_0$  when using variation of parameters to avoid confusion.)

*Part 3:* Due to wind resistance, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of  $a$  and  $g$ ?

**Activity C.2.2** ( $\sim 20$  min)

A water droplet with a radius of  $10\ \mu\text{m}$  has a mass of about  $4 \times 10^{-15}$  kg. It is determined in a laboratory that for a droplet this size, the constant  $b$  has a value of  $3 \times 10^{-3}$  kg/s, and it is known that  $g$  is approximately  $9.8\ \text{m/s}^2$ .

Complete the following tasks to study the motion of this droplet.

*Part 1:* Rewrite  $mv' = -mg - bv$  in the form of  $v' + av = ?$  for some value of  $a$ .

*Part 2:* Find the general solution of this ODE in terms of  $a$  and  $g$ . (Let  $v_p = wv_0$  when using variation of parameters to avoid confusion.)

*Part 3:* Due to wind resistance, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of  $a$  and  $g$ ?

*Part 4:* If the droplet starts from rest ( $v = 0$  when  $t = 0$ ), what is its velocity after  $0.01$  s? Use a calculator to compute the answer in m/s.

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Definition C.2.3**

The last part of the previous activity is an example of an **Initial Value Problem (IVP)**; we were given the initial value of the velocity in addition to our differential equation.

$$v' + (b/m)v = -g \quad v(0) = 0$$

Physical scenarios often produce IVPs with a unique solution.

Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

## Module C Section 3

Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

## Observation C.3.1

What happens when your tire hits a pothole?

<https://prof.clontz.org/assets/img/good-bad-shocks.gif>

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

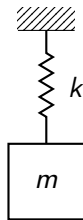
Section C.7

Section C.8

Section C.9

**Activity C.3.2** ( $\sim 5$  min)

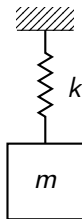
More abstractly, let's attach a mass (weighing  $m$  kg) to a spring.



List all forces acting on the mass.

**Activity C.3.3** ( $\sim 5$  min)

**Hooke's law** says that the force exerted by the spring is proportional to the distance the spring is stretched.

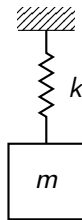


Write a differential equation modeling the displacement of the mass.

### Observation C.3.4

There is an equilibrium point where the force of gravity balances the spring force. If we measure displacement from this point, we can model the mass-spring system by

$$my'' = ky.$$





## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.5** (*~15 min*)

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.5** (*~15 min*)

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

*Part 1:* Find a solution.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.5** (*~15 min*)

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

*Part 1:* Find a solution.

*Part 2:* Find the general solution.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.5** (*~15 min*)

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

*Part 1:* Find a solution.

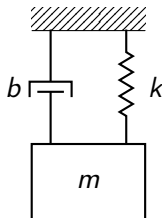
*Part 2:* Find the general solution.

*Part 3:* Describe the long term behavior of the mass-spring system.

**Activity C.3.6** ( $\sim 5$  min)

In applications, this infinitely oscillating behavior is often inappropriate.

Thus, a damper (dashpot) is often incorporated. This provides a force proportional to the velocity.



Write a differential equation modeling the displacement of a mass in a **damped** mass-spring system.

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

## Observation C.3.7

The damped mass-spring system can be modelled by

$$my'' = -by' - ky.$$

Here  $m$  is the mass,  $k$  is the spring constant, and  $b$  is the damping constant. We can rearrange this as

$$my'' + by' + ky = 0.$$

This is a **homogeneous second order constant coefficient** differential equation. Here, **homogeneous** refers to the 0 on the right hand side of the equation.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.8** (*~15 min*)

Consider the second order constant coefficient equation

$$y'' = y.$$

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.8** (*~15 min*)

Consider the second order constant coefficient equation

$$y'' = y.$$

*Part 1:* Find a solution.



## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.8** (*~15 min*)

Consider the second order constant coefficient equation

$$y'' = y.$$

*Part 1:* Find a solution.

*Part 2:* Find the general solution.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.8** (*~15 min*)

Consider the second order constant coefficient equation

$$y'' = y.$$

*Part 1:* Find a solution.

*Part 2:* Find the general solution.

*Part 3:* Describe the long term behavior of the solutions.

## Observation C.3.9

It is sometimes useful to think in terms of **differential operators**.

- We will use  $D$  to represent a derivative; another common notation is  $\frac{\partial}{\partial x}$ . So for any function  $y$ ,

$$D(y) = \frac{\partial y}{\partial x} = y'.$$

- $D^2$  will denote the second derivative operator (i.e. differentiate twice, or apply  $D$  twice).
- We will use  $I$  for the identity operator; it does nothing to a function. That is,  $I(y) = y$ . It can be thought of as  $I = D^0$  (i.e. differentiate zero times).

In this language, the differential equation  $y' + 3y = 0$  can be rewritten as  $D(y) + 3I(y) = 0$ , or  $(D + 3I)(y) = 0$ .

Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator  $D + 3I$ .

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.10** (*~5 min*)What is the kernel of  $D - I$  ?

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.10** (*~5 min*)What is the kernel of  $D - I$  ?

*Part 1:* Write a differential equation that corresponds to this question.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.10** (*~5 min*)What is the kernel of  $D - I$  ?

*Part 1:* Write a differential equation that corresponds to this question.

*Part 2:* Find the general solution of this differential equation.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.11** (*~5 min*)

Find a differential operator whose kernel is the solution set of the ODE  $y' = 4y$ .

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.12** (*~10 min*)

Consider the ODE

$$y'' + 5y' + 6y = 0.$$



## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.12** (*~10 min*)

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.12** (*~10 min*)

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.

*Part 2:* Factor this differential operator as a composition of two operators. (This works because  $D$  and  $I$  commute).

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.12** (*~10 min*)

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.

*Part 2:* Factor this differential operator as a composition of two operators. (This works because  $D$  and  $I$  commute).

*Part 3:* Find the general solution of the ODE.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Observation C.3.13**

If we let  $\mathcal{L} = D^2 + 5D + 6I$ , we can write the ODE

$$y'' + 5y' + 6y = 0$$

as

$$\mathcal{L}(y) = 0.$$

Note that such an  $\mathcal{L}$  is always a **linear transformation**.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.14** (*~10 min*)

Consider the ODE

$$y'' + 5y - 6y = 0.$$

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.14** (*~10 min*)

Consider the ODE

$$y'' + 5y - 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.14** (*~10 min*)

Consider the ODE

$$y'' + 5y - 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.

*Part 2:* Factor this differential operator as a composition of two operators. (This works because  $D$  and  $I$  commute).

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.14** (*~10 min*)

Consider the ODE

$$y'' + 5y - 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.

*Part 2:* Factor this differential operator as a composition of two operators. (This works because  $D$  and  $I$  commute).

*Part 3:* Find the general solution of the ODE.



## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.15** (*~10 min*)

Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.16** (*~15 min*)

Solve the ODE

$$y'' + y = 0.$$

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.17** (*~15 min*)

Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.17** (*~15 min*)

Consider the ODE

$$y'' + 2y' + 5y = 0$$

*Part 1:* Find the general solution.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.17** (*~15 min*)

Consider the ODE

$$y'' + 2y' + 5y = 0$$

*Part 1:* Find the general solution.

*Part 2:* Describe the long-term behavior of the solutions.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.18** (*~10 min*)

Solve the ODE

$$y'' - 4y' + 4y = 0.$$

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Observation C.3.19**

To solve this, we need to find the kernel of  $(D - 2I)(D - 2I)$ .

- The kernel of  $D - 2I$  is  $\{ce^{2t} \mid c \in \mathbb{R}\}$ .
- However, if  $(D - 2I)(y) = Ae^{2t}$ , then applying  $D - 2I$  twice will yield zero!
- So we must solve the ODE

$$y' - 2y = e^{2t}.$$

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.20** ( $\sim 15$  min)Solve  $y' - 2y = e^{2t}$ .



## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Observation C.3.21**

Thus, we have shown that the general solution of

$$y'' - 4y' + 4y = 0$$

is

$$y = c_0 e^{2t} + c_1 t e^{2t}.$$

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.22** (*~15 min*)Solve  $y'' - 6y' + 9y = 0$ .

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.23** (*~10 min*)

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

If  $r$  is a number such that  $ar^2 + br + c = 0$ , what can you conclude?

- (a)  $e^{rt}$  is a solution.
- (b)  $e^{-rt}$  is a solution.
- (c)  $te^{rt}$  is a solution.
- (d) There are no solutions.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.3.24** (*~5 min*)

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

When does the general solution have the form  $c_0e^{rt} + te^{rt}$  ?

- (a) When the polynomial  $ax^2 + bx + c$  has two distinct real roots.
- (b) When the polynomial  $ax^2 + bx + c$  has a repeated real root.
- (c) When the polynomial  $ax^2 + bx + c$  has two distinct non-real roots.
- (d) When the polynomial  $ax^2 + bx + c$  has a repeated non-real root.

## Module C

Section C.1

Section C.2

**Section C.3**

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Observation C.3.25**

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If  $r$  is a root of  $ax^2 + bx + c = 0$ , then  $e^{rt}$  is a solution of the ODE.
- If  $r$  is a double root, variation of parameters shows that  $te^{rt}$  is also a solution.
- if  $r$  is not real, Euler's formula allows us to express the solution in terms of  $\sin(rt)$  and  $\cos(rt)$ .

Module C

Section C.1

Section C.2

Section C.3

**Section C.4**

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

# Module C Section 4

Module C

Section C.1

Section C.2

Section C.3

**Section C.4**

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

## Remark C.4.1

TODO: 2nd-order IVPS

Module C

Section C.1

Section C.2

Section C.3

Section C.4

**Section C.5**

Section C.6

Section C.7

Section C.8

Section C.9

## Module C Section 5



Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

**Section C.6**

Section C.7

Section C.8

Section C.9

## Module C Section 6

Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

**Section C.7**

Section C.8

Section C.9

# Module C Section 7

Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

**Section C.8**

Section C.9

## Module C Section 8

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

**Section C.8**

Section C.9

**Observation C.8.1**

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If  $r$  is a root of  $ax^2 + bx + c = 0$ , then  $e^{rt}$  is a solution of the ODE.
- If  $r$  is a double root, variation of parameters shows that  $te^{rt}$  is also a solution.
- if  $r$  is not real, Euler's formula allows us to express the solution in terms of  $\sin(rt)$  and  $\cos(rt)$ .

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.8.2** (*~15 min*)

Consider a mass of 4 kg suspended from a damped spring with spring constant  $k = 2 \text{ kg/s}^2$  and damping constant  $b = 6 \text{ kg/s}$ .

The mass is pulled down 0.3 m and released from rest. How many times does it pass back through its equilibrium state?

- (a) 0
- (b) 1
- (c) 2
- (d) Infinitely many

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.8.3** ( $\sim 15$  min)

Consider a mass of 5 kg suspended from a damped spring with spring constant  $k = 2 \text{ kg/s}^2$  and damping constant  $b = 6 \text{ kg/s}$ .

The mass is pulled down  $0.3\text{m}$  and released from rest. How many times does it pass back through its equilibrium state?

- (a) 0
- (b) 1
- (c) 2
- (d) Infinitely many

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

**Section C.8**

Section C.9

## Observation C.8.4

It can be shown that in the **overdamped** situation, the spring might pass through the equilibrium position once (e.g. if given an initial push), but never more than once.

Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

**Section C.9**

## Module C Section 9



## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.9.1** ( $\sim 10$  min)

A 1 kg mass is suspended from a spring with spring constant  $k = 9 \text{ kg/s}^2$ . An external force is applied by an electromagnet and is modeled by the function  $F(t) = \sin(t)$ . Write an ODE modeling the displacement of the spring.

## Module C

Section C.1  
Section C.2  
Section C.3  
Section C.4  
Section C.5  
Section C.6  
Section C.7  
Section C.8  
Section C.9

## Observation C.9.2

In the previous activity, we encountered a **nonhomogeneous** second order constant coefficient ODE, i.e. of the form

$$ay'' + by' + cy = f(x)$$

where  $a, b, c$  are constants, and  $f(x)$  is a function.

We will again use **variation of parameters** to find a particular solution.

**Activity C.9.3** (*~15 min*)

Suppose  $y_1$  and  $y_2$  are two independent particular solutions of  $\mathcal{L}(y) = 0$ , where  $\mathcal{L}(y) = ay'' + by' + cy$ .

Our goal is to find a particular solution of  $\mathcal{L}(y) = f(x)$  of the form  $y_p = v_1y_1 + v_2y_2$  for some TBD functions  $v_1, v_2$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.9.3** (*~15 min*)

Suppose  $y_1$  and  $y_2$  are two independent particular solutions of  $\mathcal{L}(y) = 0$ , where  $\mathcal{L}(y) = ay'' + by' + cy$ .

Our goal is to find a particular solution of  $\mathcal{L}(y) = f(x)$  of the form  $y_p = v_1y_1 + v_2y_2$  for some TBD functions  $v_1, v_2$ .

*Part 1:* Use the product rule (twice) to compute  $y_p'$ .

## Module C

## Section C.1

## Section C.2

## Section C.3

## Section C.4

## Section C.5

## Section C.6

## Section C.7

## Section C.8

## Section C.9

**Activity C.9.3** ( $\sim 15$  min)

Suppose  $y_1$  and  $y_2$  are two independent particular solutions of  $\mathcal{L}(y) = 0$ , where  $\mathcal{L}(y) = ay'' + by' + cy$ .

Our goal is to find a particular solution of  $\mathcal{L}(y) = f(x)$  of the form  $y_p = v_1y_1 + v_2y_2$  for some TBD functions  $v_1, v_2$ .

*Part 1:* Use the product rule (twice) to compute  $y_p'$ .

*Part 2:* To simplify calculations, we will **assume**  $v_1'y_1 + v_2'y_2 = 0$ . Assuming this, compute  $y_p''$ .

**Activity C.9.3** ( $\sim 15$  min)

Suppose  $y_1$  and  $y_2$  are two independent particular solutions of  $\mathcal{L}(y) = 0$ , where  $\mathcal{L}(y) = ay'' + by' + cy$ .

Our goal is to find a particular solution of  $\mathcal{L}(y) = f(x)$  of the form  $y_p = v_1y_1 + v_2y_2$  for some TBD functions  $v_1, v_2$ .

*Part 1:* Use the product rule (twice) to compute  $y_p'$ .

*Part 2:* To simplify calculations, we will **assume**  $v_1'y_1 + v_2'y_2 = 0$ . Assuming this, compute  $y_p''$ .

*Part 3:* Compute  $\mathcal{L}(y_p)$ ; simplify the ODE  $\mathcal{L}(y_p) = f(x)$ .

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Observation C.9.4**

If we can find  $v_1$  and  $v_2$  that satisfy

$$y_1 v'_1 + y_2 v'_2 = 0$$

$$y'_1 v'_1 + y'_2 v'_2 = \frac{f}{a}$$

then we have a solution. So we just need to solve this system of equations for  $v'_1$  and  $v'_2$ .

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.9.5** (*~15 min*)

Consider the nonhomogeneous ODE  $y'' + 9y = \sin(t)$ .



## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.9.5** (*~15 min*)

Consider the nonhomogeneous ODE  $y'' + 9y = \sin(t)$ .

*Part 1:* Find  $y_1$  and  $y_2$ , two independent solutions of  $y'' + 9y = 0$ .

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.9.5** (*~15 min*)

Consider the nonhomogeneous ODE  $y'' + 9y = \sin(t)$ .

*Part 1:* Find  $y_1$  and  $y_2$ , two independent solutions of  $y'' + 9y = 0$ .

*Part 2:* Find  $v_1$  and  $v_2$  by solving

$$\begin{aligned}\cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(t)\end{aligned}$$

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.9.5** (*~15 min*)

Consider the nonhomogeneous ODE  $y'' + 9y = \sin(t)$ .

*Part 1:* Find  $y_1$  and  $y_2$ , two independent solutions of  $y'' + 9y = 0$ .

*Part 2:* Find  $v_1$  and  $v_2$  by solving

$$\begin{aligned}\cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(t)\end{aligned}$$

*Part 3:* Write the general solution of the original nonhomogeneous ODE.

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.9.6** (*~10 min*)

Consider the nonhomogeneous ODE  $y'' + 9y = \sin(3t)$ .

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.9.6** ( $\sim 10$  min)

Consider the nonhomogeneous ODE  $y'' + 9y = \sin(3t)$ .

*Part 1:* Find  $v_1$  and  $v_2$  by solving

$$\begin{aligned}\cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(3t)\end{aligned}$$

## Module C

Section C.1

Section C.2

Section C.3

Section C.4

Section C.5

Section C.6

Section C.7

Section C.8

Section C.9

**Activity C.9.6** ( $\sim 10$  min)

Consider the nonhomogeneous ODE  $y'' + 9y = \sin(3t)$ .

*Part 1:* Find  $v_1$  and  $v_2$  by solving

$$\begin{aligned}\cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(3t)\end{aligned}$$

*Part 2:* Write the general solution of the original nonhomogeneous ODE.