Module D: Discontinuous functions in ODEs

How can we solve and apply ODEs involving functions that are not continuous?

At the end of this module, students will be able to...

D1. **Laplace Transform.** ...compute the Laplace transform of a function

D2. **Discontinuous ODEs.** ...solve initial value problems for ODEs with discontinuous coefficients

D3. **Modeling non-smooth motion.** ...model the motion of an object undergoing discontinuous acceleration

D4. **Modeling non-smooth oscillators.** ...model mechanical oscillators undergoing discontinuous acceleration

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Partialf ractions

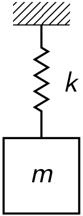
The following resources will help you prepare for this module.

- TODO

Module D Section 1

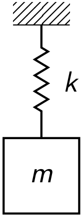
Activity D.1.1 *(∼10 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting 3J of energy.



Activity D.1.1 *(∼10 min)*

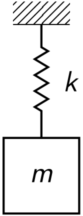
A 1 kg mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting 3J of energy.



Part 1: Draw a graph of the kinetic energy in the system with respect to time.

Activity D.1.1 *(∼10 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting 3J of energy.



Part 1: Draw a graph of the kinetic energy in the system with respect to time.
Part 2: Write an initial value problem modelling this system.

Definition D.1.2

The **Dirac delta distribution** $\delta(t)$ models the application of instantaneous force. **It is not a function**, but makes sense in definite integrals:

If a, b is any open interval containing 0, then

$$\int_a^b f(t)\delta(t)dt = f(0)$$

for any function $f(t)$ that is continuous around 0.

Definition D.1.3
The **unit impulse function** $u(t)$ is given by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0. \end{cases}$$

Note that $u(s) = \int_{-\infty}^{s} \delta(t)\, dt$; in this fuzzy sense, δ is the derivative of $u(t)$ (which is not differentiable everywhere!)

Activity D.1.4 *(∼10 min)*
Try to solve the IVP

$$y'' + y = \delta(t)$$

Where does our existing technique break down?

Observation D.1.5

To get around this difficulty, we will apply an **integral transform** called the **Laplace Transform** to our ODE.

- We want to use a definite integral to handle things like δ, which we can only understand via a definite integral.
- Since we are focused on IVPs, we can integrate starting at 0, but need to go to ∞
- But now we need to worry about convergence–thus we will multiply by a suitable function that decays fast enough to make most functions converge.

Activity D.1.6 *(∼5 min)*

Arrange the following functions in order of how fast they decay to zero in the limit at infinity:

(A) x^{-n} for a positive integer n

(B) e^{-ax} for a positive integer a

(C) $\frac{1}{\ln(ax)}$ for a positive integer a

(D) $\frac{1}{\ln(x^n)}$ for a positive integer n

Definition D.1.7
The **Laplace Transform** of a function $f(t)$ is the function

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

We will also use the notation $\mathcal{L}(f) = F$.

Note that the Laplace transform turns a function of t into a function of s.

Moreover, $\mathcal{L}$ is linear: $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$, and $\mathcal{L}(cf) = c\mathcal{L}(f)$ for constants c.

Activity D.1.8 *(∼5 min)*

Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\ dt.$$

Activity D.1.8 *(∼5 min)*

Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}(\delta(t))$

Activity D.1.8 *(~5 min)*
Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}(\delta(t))$
Part 2: If $a > 0$, compute $\mathcal{L}(\delta(t-a))$

Activity D.1.9 *(∼5 min)*

Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Activity D.1.9 *(∼5 min)*
Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\ dt.$$

Part 1: Compute $\mathcal{L}(e^t)$

Activity D.1.9 *(∼5 min)*
Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}(e^t)$
Part 2: If $a > 0$, compute $\mathcal{L}(e^{at})$

Activity D.1.10 *(∼15 min)*

Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\ dt.$$

Activity D.1.10 *(∼15 min)*
Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}(1)$

Activity D.1.10 *(~15 min)*

Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}(1)$

Part 2: Compute $\mathcal{L}(t)$

Activity D.1.10 *(∼15 min)*

Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}(1)$

Part 2: Compute $\mathcal{L}(t)$

Part 3: Compute $\mathcal{L}(t^2)$

Activity D.1.10 *(∼15 min)*
Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}(1)$
Part 2: Compute $\mathcal{L}(t)$
Part 3: Compute $\mathcal{L}(t^2)$
Part 4: Compute $\mathcal{L}(t^3)$

Activity D.1.10 *(∼15 min)*

Recall that

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}(1)$

Part 2: Compute $\mathcal{L}(t)$

Part 3: Compute $\mathcal{L}(t^2)$

Part 4: Compute $\mathcal{L}(t^3)$

Part 5: Compute $\mathcal{L}(t^4)$

Module D Section 2

Observation D.2.1

Last week, we encountered the **Laplace Transform** of a function $f(t)$:

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

We also use the notation $\mathcal{L}(f) = F$.

Recall that the Laplace transform turns a function of t into a function of s.

Moreover, $\mathcal{L}$ is linear: $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$, and $\mathcal{L}(cf) = c\mathcal{L}(f)$ for constants c.

Observation D.2.2

We computed a few Laplace Transforms:

- $\mathcal{L}(\delta(t-a)) = e^{-as}$ for any $a > 0$.
- $\mathcal{L}(e^{at}) = \frac{1}{s-a}$
- $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ for any positive integer n.
- $\mathcal{L}(1) = \frac{1}{s}$

Activity D.2.3 *(∼10 min)*

Recall

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)dt.$$

Activity D.2.3 *(∼10 min)*
Recall

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)dt.$$

Part 1: Compute $\mathcal{L}(\sin(t))$.

Activity D.2.3 *(∼10 min)*
Recall

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}(\sin(t))$.
Part 2: Compute $\mathcal{L}(\cos(t))$.

Observation D.2.4

So now our list of Laplace transforms is:

- $\mathcal{L}(\delta(t-a)) = e^{-as}$ for any $a > 0$.
- $\mathcal{L}(e^{at}) = \frac{1}{s-a}$
- $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ for any positive integer n.
- $\mathcal{L}(1) = \frac{1}{s}$
- $\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$
- $\mathcal{L}(\cos(t)) = \frac{s}{s^2+1}$

Activity D.2.5 *(∼10 min)*

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}(y')$ is related to $\mathcal{L}(y)$. Recall

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t) dt.$$

Activity D.2.5 *(∼10 min)*
Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}(y')$ is related to $\mathcal{L}(y)$. Recall

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t) dt.$$

Part 1: Use integration by parts to relate $\mathcal{L}(y')$ to $\mathcal{L}(y)$.

Activity D.2.5 *(∼10 min)*

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}(y')$ is related to $\mathcal{L}(y)$. Recall

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)dt.$$

Part 1: Use integration by parts to relate $\mathcal{L}(y')$ to $\mathcal{L}(y)$.
Part 2: Use integration by parts (and the fact that $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$) to relate $\mathcal{L}(y'')$ to $\mathcal{L}(y)$.

Observation D.2.6

We have

$$\mathcal{L}(y') = sL(y) - y(0)$$
$$\mathcal{L}(y'') = s^2 L(y) - sy(0) - y'(0)$$

This allows us to easily rewrite expressions like $ay'' + by' + cy$ in terms of $\mathcal{L}(y)$.

Activity D.2.7 *(∼10 min)*
Consider the simple IVP

$$y'' + y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$

Activity D.2.7 *(∼10 min)*
Consider the simple IVP

$$y'' + y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$

Part 1: Apply the Laplace transform to this IVP, and simplify. Solve for $\mathcal{L}(y)$.

Activity D.2.7 *(∼10 min)*
Consider the simple IVP

$$y'' + y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$

Part 1: Apply the Laplace transform to this IVP, and simplify. Solve for $\mathcal{L}(y)$.
Part 2: Find a function y satisfying $\mathcal{L}(y) = \frac{1}{s^2+1}$. We write $y = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$.

Activity D.2.8 *(∼15 min)*
Solve the IVP

$$y'' + y = \delta(t), \qquad y(0) = 1, y'(0) = 2.$$

Module D Section 3

Observation D.3.1

To solve an IVP using Laplace transforms:

1) Apply $\mathcal{L}$ to the ODE. Use the initial condition(s) in computing $\mathcal{L}(y')$, $\mathcal{L}(y'')$, etc.
2) Solve for $\mathcal{L}(y)$.
3) Take the inverse transform (using a table) to find the solution y.

Activity D.3.2 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left(\frac{e^{-10s}}{s}\right)$.

(a) $u(t-10)$

(b) $\delta(t-10)$

(c) $u(t-10)e^{-t}$

(d) $\delta(t-10)e^{-t}$

Activity D.3.3 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)$.

(a) $u(t-5)$

(b) $\delta(t-5)$

(c) e^{5t}

(d) e^{-5t}

Activity D.3.4 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left(\frac{2e^{-2s}}{s^2+4}\right)$.

(a) $u(t)\sin(2t)$

(b) $u(t-2)\sin(2t)$

(c) $u(t-2)\sin(2t-2)$

(d) $u(t-2)\sin(2t-4)$

Activity D.3.5 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left(\frac{e^{-100s}}{s^2}\right)$.

(a) $u(t)t$

(b) $u(t)(t-100)$

(c) $u(t-100)t$

(d) $u(t-100)(t-100)$

Activity D.3.6 *(∼15 min)*
Solve the IVP

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad v(0) = 50.$$

Activity D.3.7 *(∼15 min)*

Solve the IVP

$$y'' + 4y = \delta(t-2), \qquad y(0) = 0,\ y'(0) = 1$$

Module D Section 4

Activity D.4.1 *(∼30 min)*

A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Activity D.4.1 *(∼30 min)*

A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

Activity D.4.1 *(∼30 min)*

A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

Part 2: Write down an IVP modelling the velocity of the spacecraft.

Activity D.4.1 *(∼30 min)*
A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.
Part 1: Write down a function modelling the thrust force on the spacecraft.
Part 2: Write down an IVP modelling the velocity of the spacecraft.
Part 3: Solve the IVP:

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad v(0) = 50.$$

Activity D.4.1 *(∼30 min)*
A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.
Part 1: Write down a function modelling the thrust force on the spacecraft.
Part 2: Write down an IVP modelling the velocity of the spacecraft.
Part 3: Solve the IVP:

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad v(0) = 50.$$

Part 4: What is its velocity after 200 s?

Module D Section 5

Module D Section 6