Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

C1. Find the general solution to

$$y' + y = -t^2.$$

Solution: First, we find a general solution to the homogeneous equation

$$y' + y = 0.$$

This has auxilliary equation $r + 1 = 0$, which has a single root at $r = 1$, so ce^{-t} is a solution.
We can use variation of parameters to find a particular solution y_p; suppose $y_p = ve^{-t}$ for some function v.
Then $y_p' = v'e^{-t} - ve^{-t}$, so we have

$$-t^2 = y_p' + y_p = v'e^{-t} - ve^{-t} + ve^{-t} = v'e^{-t}.$$

Thus $v' = -t^2e^t$, so (integrating by parts twice), we conclude $v = \int -t^2e^t \, dt = -t^2e^t + 2te^t - 2e^t$. Thus $y_p = ve^{-t} = -t^2 + 2t - 2$, so the general solution is

$$y = -t^2 + 2t - 2 + ce^{-t}.$$

$\square$

C2. Consider the following scenario: A water droplet with a radius of 10 μm has a mass of about 4×10^{-15}kg and a terminal velocity of 3 $\frac{\text{cm}}{\text{s}}$. The droplet is dropped from rest.

(a) Write down an IVP modelling the velocity of the water droplet.

(b) What is its velocity after 0.01 s?

Solution: The forces acting on the water droplet are gravity (mg downwards) and air resistance; in a water droplet this size, this is proportional to its velocity. Let b denote this drag coefficient; then by Newton's second law (if we let up be the positive direction and down be negative), we have

$$m\frac{dv}{dt} = -mg - bv.$$

Note that we have $-mg$ since gravity always acts downwards, and $-bv$ because the drag acts in the opposite direction of v. It is actually convenient to let $a = \frac{b}{m}$, and divide this equation by the mass, yielding

$$\frac{dv}{dt} = -g - av.$$

The intial condition is that it is dropped from rest, i.e. $v(0) = 0$, so an IVP modelling velocity is

$$\frac{dv}{dt} = -g - av \qquad\qquad v(0) = 0.$$

This is a separable differential equation; we thus have

$$\int \frac{1}{g + av} \, dv = \int (-1) dt.$$

Thus, $\frac{1}{a} \ln |g + av| = -t + c_1$, or $|g + av| = e^{-at+ac_1} = c_2e^{-at}$ (where $c_2 = e^{ac_1}$). To avoid the pesky absolute value sign, we can absorb a potential negative into the constant to write $g + av = c_3e^{-at}$, or (letting $c_0 = \frac{c_3}{a}$)

$$v = -\frac{g}{a} + c_0e^{-at}.$$

Note that $\lim_{t\to\infty} v = -\frac{g}{a}$, so since we are given the terminal velocity, so after converting to meters per second we can solve $-0.03 = -\frac{9.8}{a}$ to obtain $a = \frac{9.8}{0.03}$. We also know that it is dropped from rest, so we compute $0 = v(0) = -0.03 + c_0$, so $c_0 = 0.03$. Thus our model for velocity is

$$v = -0.03 + 0.03e^{-\frac{9.8}{0.03}t}.$$

Then we simply compute $v(0.01) = -0.03 + 0.03e^{-\frac{9.8}{0.03}(0.01)} \approx -0.029$. So after 0.01 s, the drop is falling approximately 2.9 $\frac{\text{cm}}{\text{s}}$.

□

C3. Find the general solution to

$$y'' + 6y' + 13y = 0.$$

Solution: We begin by writing the auxilliary equation $r^2 + 6r + 13 = 0$ and finding the roots. There are many ways to do this; here, we complete the square:

$$0 = r^2 + 6r + 13 = r^2 + 6r + 9 + 4 = (r + 3)^2 + 4.$$

Thus, we can easily solve to obtain $r = -3 \pm 2i$. Thus the general solution is

$$y = c_1e^{-3t}\cos(2t) + c_2e^{-3t}\sin(2t).$$

□

C4. Find the solution to

$$y'' + 10y' + 24y = 0$$

when $y(0) = -3$ and $y'(0) = 2$.

Solution: The auxilliary equation is $r^2 + 10r + 24 = 0$, which has roots $r = -6$ and $r = -4$. Thus, the general solution is of the form $y = c_1e^{-4t} + c_2e^{-6t}$.

$$\begin{aligned} -3 &= y(0) = c_1 + c_2 \\ 2 &= y'(0) = -4c_1 - 6c_2 \end{aligned}$$

Solving this system yields $c_1 = -8$ and $c_2 = 5$, so the solution to the IVP is

$$y = -8e^{-4t} + 5e^{-6t}$$

□

C5. Find the general solution to

$$y'' + 10y' + 24y = e^{-4t}$$

Solution: First, we find a general solution to the homogenous equation $y'' + 10y' + 24y = 0$. We begin by writing the auxilliary equation $r^2 + 10r + 24 = 0$, which has roots $r = -6$ and $r = -4$. Thus, the general solution is of the form $y = c_1e^{-4t} + c_2e^{-6t}$.
We use variation of parameters to find a particular solution, $y_p = v_1e^{-4t} + v_2e^{-6t}$ for some functions v_1, v_2. We assume for convenience $v_1'e^{-4t} + v_2'e^{-6t} = 0$, so that $y_p' = -4v_1e^{-4t} - 6v_2e^{-6t}$, and $y_p'' = -4v_1'e^{-4t} + 16v_1e^[-4t] - 6v_2'e^{-6t} + 36v_2e^{-6t}$. Substituting into the original ODE and simplifying yields

$$e^{-4t} = y_p'' + 10y_p' + 14y_p = -4v_1'e^{-4t} - 6v_2'e^{-6t}.$$

Combining with our earlier assumption we have the system

$$\begin{aligned} v_1'e^{-4t} + v_2'e^{-6t} &= 0 \\ -4v_1'e^{-4t} - 6v_2'e^{-6t} &= e^{-4t} \end{aligned}$$

Multipy both equations by e^{4t} for convenience:

$$
\begin{aligned}
v_1' + v_2' e^{-2t} &= 0 \\
-4v_1' - 6v_2' e^{-2t} &= 1
\end{aligned}
$$

Then six times the first equation plus the second yields $2v_1' = 1$, so $v_1 = \int \frac{1}{2}\, dt = \frac{1}{2}t$. Substituting back into the first yields $v_2' = -\frac{1}{2}e^{2t}$, so $v_2 = \int -\frac{1}{2}e^{2t}\, dt = -\frac{1}{4}e^{2t}\, dt$, and thus

$$y_p = \frac{1}{2}te^{-4t} - \frac{1}{4}e^{-4t},$$

and thus the general solution is

$$y = \frac{1}{2}te^{-4t} + c_1e^{-4t} + c_2e^{-6t}.$$

$\square$

C6. Consider the following scenario: A 2kg mass is suspended by a spring (with spring constant 8kg/s^2). The mass is pulled down 1m from its equillibrium position and released from rest.

(a) Write down an IVP modelling the position of the mass.

(b) How long does it take for the mass to return to its equillibrium point?

Solution: Let y denote the vertical distance from equillibrium; then the forces acting are gravity and the spring force, giving the ODE

$$my'' = -ky.$$

Substituting in the values of the constants $m = 2$ and $k = 8$, we have

$$2y'' + 8y = 0.$$

Note that the initial conditions are $y(0) = -1$ and $y'(0) = 0$, so an IVP model is

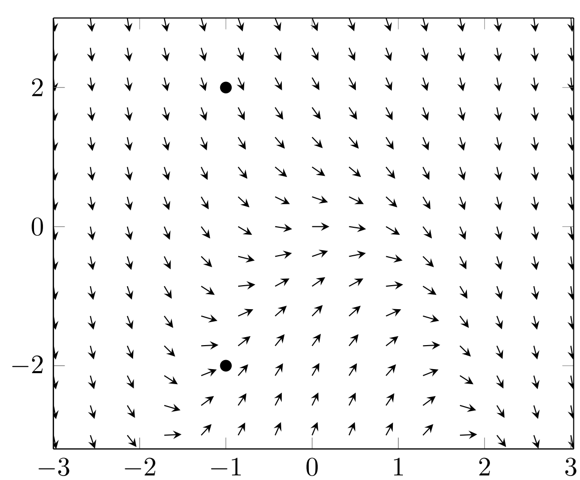
$$2y'' + 8y = 0 \qquad\qquad y(0) = 1,\ y'(0) = 0.$$

Simplifying, we have $y'' + 4y = 0$, so the general solution is $y = c_1\cos(2t) + c_2\sin(2t)$. The initial conditions are $y(0) = -1$ and $y'(0) = 0$, which imply $c_1 = -1$ and $c_2 = 0$, so the system is modelled by $y = -\cos(2t)$. This is first zero when $2t = \frac{\pi}{2}$, i.e. when $t = \frac{\pi}{4}$. Thus the mass takes $\frac{\pi}{4}$ s to return to its equillibrium point.

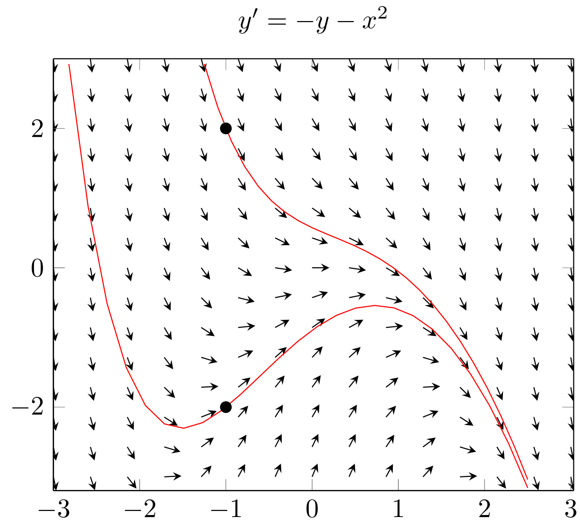
$\square$

F1. Sketch a solution curve through each point marked in the slope field.

$$y' = -y - x^2$$



Solution:



$\square$

F2. Solve $y' + xy = x$.

Solution: Rearranging, we have $y' = x - xy = x(1-y)$, so we see the equation is separable, and write

$$\frac{y'}{1-y} = x.$$

Thus we compute $\int \frac{y'}{1-y}\ dx = \int \frac{1}{1-y}\ dy = -\ln|1-y| + c_1$, and $\int x\ dx = \frac{1}{2}x^2 + c_2$. Thus, we have (letting $c_3 = c_2 - c_1$)

$$-\ln|1-y| = \frac{1}{2}x^2 + c_3.$$

Then exponentiating, we have (letting $c_4 = \pm e^{-c_3}$) $1 - y = c_4 e^{-\frac{1}{2}x^2}$, so (with $c = -c_4$) the general solution is

$$y = 1 + ce^{-\frac{1}{2}x^2}.$$

$\square$

F3. A 0.5kg ball is dropped from rest from the top of a 30m tall building. Its drag coefficient is 3kg s/m.

(a) Write down an IVP modelling this scenario.

(b) What is the terminal velocity of the ball?

Solution: The forces acting on the ball are gravity and air resistance (drag), so letting down be the positive direction, we have

$$mv' = mg - bv^2.$$

Substituting in our constants (and dividing through by m), we have

$$v' = 9.81 - 6v^2.$$

The initial condition is that it is dropped from rest, i.e. $v(0) = 0$, giving the IVP

$$v' = 9.81 - 6v^2 v(0) = 0.$$

The terminal velocity is when $v' = 0$, i.e. when $0 = 9.81 - 6v^2$, so when $v = \sqrt{\frac{9.81}{6}}$.

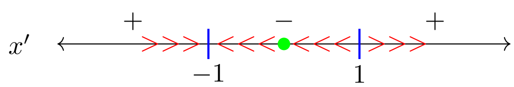
$\square$

F4. Consider the autonomous equation
$$\frac{dx}{dt} = x^2 - 1.$$

(a) Find and classify the critical points.

(b) Describe the long term behavior of the solution passing through the point $x(4) = 0$.

Solution:
Note that $x^2 - 1 = (x-1)(x+1)$, so there are equillibria solutions at $x = 1$ and $x = -1$. We can thus compute a number line for x':



We see that -1 is a sink (stable), while 1 is a source (unstable). A trajectory passing through $x(4) = 0$ will approach $x = -1$ in the limit.

$\square$

F5. Solve $y' + xy = x^3$.

Solution: First, we solve the homogeneous equation $y' + xy = 0$, which is separable: $\frac{y'}{y} = -x$, so $\ln|y| = -\frac{1}{2}x^2 + c$, or $y = c_1 e^{-\frac{1}{2}x^2}$. We then use variation of parameters to find a particular solution, writing $y_p = ve^{-\frac{1}{2}x^2}$; substituting in and simplifying yields
$$x^3 = y_p' + xy_p = v'e^{-\frac{1}{2}x^2} - vxe^{-\frac{1}{2}x^2} + xve^{-\frac{1}{2}x^2} = v'e^{-\frac{1}{2}x^2}.$$
Thus, $v' = x^3 e^{\frac{1}{2}x^2}$. This can be integrated by parts to obtain $v = x^2 e^{\frac{1}{2}x^2} - 2e^{\frac{1}{2}x^2}$, so $y_p = ve^{-\frac{1}{2}x^2} = x^2 - 2$. Thus the general solution is
$$y = c_1 e^{-\frac{1}{2}x^2} + x^2 - 2.$$

$\square$

F6. One of the two ODEs below is exact. Identify which one, and solve it.
$$(x + 3y)y' + y = 3x$$
$$(x + 3y)y' - y = -3x$$

Solution: Rewrite the first equation as $y' = \frac{3x-y}{x+3y}$; if there is a function with $\nabla F = \langle -(3x-y), x+3y \rangle$, then
$$\frac{d}{dt}F(x,y) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} = -(3x-y)(x+3y) + (x+3y)(3x-y) = 0,$$
so $F(x,y) = c$ is a solution of the autonomous system.
Note that $\langle -(3x-y), x+3y \rangle$ is conservative, as $\frac{\partial}{\partial x}(x+3y) = 1 = \frac{\partial}{\partial y}(-3x+y)$; so a potential function F is found by integrating: $F = \int x + 3y\, dy = xy + \frac{3}{2}y^2 + f(x)$, and differentiating with respect to x, we have $-3x + y = y + f'(x)$, so $f'(x) = -3x$, in which case $f(x) = \int -3x\, dx = \frac{-3}{2}x^2$, so a potential function is $F(x,y) = xy + \frac{3}{2}y^2 - \frac{3}{2}x^2$. So the general solution to the ODE is
$$xy + \frac{3}{2}y^2 - \frac{3}{2}x^2 = c.$$

$\square$

S1.

Solution:

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S2. Consider the following scenario: Two competing fish species (red fish and blue fish) live in a lake. In the absence of one species, the lake will support 30,000 of the other fish (as the species competes among itself for resources). It is calculated that the competition coefficients are $\alpha_{1,2} = \alpha_{2,1} = 0.5$.

(a) Write down a system of ODEs modelling the interaction of the two species.

(b) If the lake is stocked with 12,000 blue fish and 40,000 red fish, what will happen to the populations in the long run?

Solution:

$$
\begin{aligned}
x' &= \frac{r_1}{30000} x\,(30000 - x - 0.5y) \\
y' &= \frac{r_2}{30000} y\,(30000 - y - 0.5x)
\end{aligned}
$$

Then the nontrivial isoclines are $x + 0.5y = 30000$ and $0.5x + y = 30000$ which intersect at the (equillibrium) point $20000, 20000$. Plotting the isoclines (TODO) shows that this a stable equillibrium (sink), so regardless of the intial populations, they will approach 20,000 of each species in the long run.

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S3.

Solution:

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N1.

Solution:

☐

N2.

Solution:

☐

N3.

Solution:

☐

N4.

Solution:

☐

N5.

Solution:

□

D1.

Solution:

□

D2.

Solution:

□

D3.

Solution:

□

D4.

Solution:

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