Module C: Constant coefficient linear ODEs

Standards for this Module

How can we solve and apply linear constant coefficient ODEs? At the end of this module, students will be able to...

C1. Sketching trajectories. ...

C2. Constant coefficient first order. ...

C3. Homogeneous constant coefficient second order. ...

C4. Non-homogenous constant coefficient second order. ...

C5. IVPs. ...

C6. Modeling motion in viscous fluids. ...

C7. Modeling oscillators. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.
- Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): `http://bit.ly/2l21etm`
- Solving linear systems with substitution (Khan Academy): `http://bit.ly/1SlMpix`
- Set builder notation: `https://youtu.be/xnfUZ-NTsCE`

Section C.0

Definition C.0.1 A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

Remark C.0.2 In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition C.0.3 A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{array}{rcrcccrcl} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \middle| \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

Remark C.0.4 When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{array}{rcr} x_1 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ -x_2 + x_3 & = & -2 \end{array}$$

Verbose standard form:

$$\begin{array}{rcr} 1x_1 + 0x_2 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ 0x_1 - 1x_2 + 1x_3 & = & -2 \end{array}$$

Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Definition C.0.5 A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

Fact C.0.6 All linear systems are one of the following:

- **Consistent with one solution**: its solution set contains a single vector, e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$
- **Inconsistent**: its solution set is the empty set $\{\}=\emptyset$

Activity C.0.7 *(∼10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is $\emptyset$.

$$\begin{aligned}-x_1+2x_2&=5\\2x_1-4x_2&=6\end{aligned}$$

Activity C.0.8 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned}-x_1+2x_2&=-3\\2x_1-4x_2&=6\end{aligned}$$

Part 1: Find three different solutions for this system.
Part 2: Let $x_2=a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{\begin{bmatrix}?\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$ for the linear system.

Activity C.0.9 *(∼10 min)* Consider the following linear system.

$$\begin{alignedat}{4}x_1+2x_2&&&-&x_4&=&&3\\&&x_3&+&4x_4&=&&-2\end{alignedat}$$

Describe the solution set

$$\left\{\begin{bmatrix}?\\a\\?\\b\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

to the linear system by setting $x_2=a$ and $x_4=b$, and then solving for x_1 and x_3.

Observation C.0.10 Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{aligned}
-2x_1 - 4x_2 + x_3 - 4x_4 &= -8\\
x_1 + 2x_2 + 2x_3 + 12x_4 &= -1\\
x_1 + 2x_2 + x_3 + 8x_4 &= 1
\end{aligned}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Section C.1

Remark C.1.1 The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition C.1.2 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\;\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example C.1.3 The corresopnding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Definition C.1.4 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned}$$

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity C.1.5 *(~10 min)* Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

g) Replace a row with zeros.

(Instructor Note:) This activity could be ran as a card sort. Allow 5 additional minutes for intra team discussion.

Definition C.1.6 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity C.1.7 *(~10 min)* Consider the following (equivalent) linear systems.

(A)

$$\begin{aligned} -2x_1 + 4x_2 - 2x_3 &= -8 \\ x_1 - 2x_2 + 2x_3 &= 7 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$$

(B)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ -2x_1 + 4x_2 - 2x_3 &= -8 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$$

(C)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ 2x_3 &= 6 \\ -2x_3 &= -6 \end{aligned}$$

(D)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ x_3 &= 3 \\ -2x_3 &= -6 \end{aligned}$$

(E)

$$\begin{aligned} x_1 - 2x_2 \quad &= 1 \\ x_3 &= 3 \\ 0 &= 0 \end{aligned}$$

(F)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ 2x_3 &= 6 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$$

Part 1: Find a solution to one of these systems.
Part 2: Rank the six linear systems from most complicated to simplest.

Activity C.1.8 *(∼5 min)* We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} -2 & 4 & -2 & -8 \\ 1 & -2 & 2 & 7 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ -2 & 4 & -2 & -8 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 3 & -6 & 4 & 15 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Activity C.1.9 *(∼10 min)* A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Circle the leading terms in each example, and label it as RREF or not RREF.

(A) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(B) $\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(C) $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

(D) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(E) $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(F) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Remark C.1.10 It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix into reduced row echelon form.

A video outlining how to perform the Gauss-Jordan Elimination algorithm by hand is available at `https://youtu.be/Cq0Nxk2dhhU`. Practicing several exercises outside of class using this method is recommended.

In the next section, we will learn to use technology to perform this operation for us, as will be expected when applying row-reduced matrices to solve other problems.

Section C.2

Activity C.2.1 *(∼10 min)* Free browser-based technologies for mathematical computation are available online.

- Go to `http://cocalc.com` and create an account.
- Create a project titled “Linear Algebra Team X” with your appropriate team number. Add all team members as collaborators.
- Open the project and click on “New”
- Give it an appropriate name such as “Class E.2 workbook”. Make a new Jupyter notebook.
- Click on “Kernel” and make sure “Octave” is selected.

Module S: Systems of ODEs

Standards for this Module

How can we solve and apply systems of linear ODEs? At the end of this module, students will be able to...

S1. Solving systems. ...

S2. Modeling interacting populations. ...

S3. Modeling coupled oscillators. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Section S.1

Activity S.1.1 *(~10 min)* Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad\qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

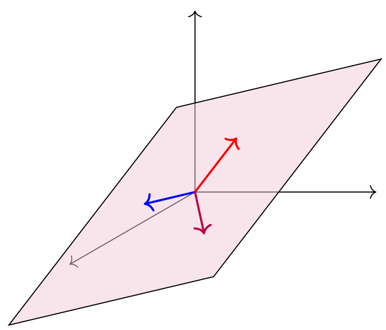
Which of the following is true?

(A) $\operatorname{span} S$ is bigger than $\operatorname{span} T$.

(B) $\operatorname{span} S$ and $\operatorname{span} T$ are the same size.

(C) $\operatorname{span} S$ is smaller than $\operatorname{span} T$.

Definition S.1.2 We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay on the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

Activity S.1.3 *(~10 min)* Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$. Suppose $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$, so the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Which of the following is true of the vector equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}$?

(A) It is consistent with one solution

(B) It is consistent with infinitely many solutions

(C) It is inconsistent.

Fact S.1.4 For any vector space, the set $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{z}$ is consistent with infinitely many solutions.

Activity S.1.5 *(~10 min)* Find

$$\operatorname{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Fact S.1.6 A set of Euclidean vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

Activity S.1.7 *(~5 min)* Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

Activity S.1.8 *(~10 min)* Is the set of polynomials $\{x^3 + 1, x^2 + 2x, x^2 + 7x + 4\}$ linearly dependent or linearly independent?

Activity S.1.9 *(~5 min)* What is the largest number of vectors in $\mathbb{R}^4$ that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.10 *(∼5 min)* What is the largest number of vectors in

$$\mathcal{P}^4 = \left\{ax^4 + bx^3 + cx^2 + dx + e \,\middle|\, a, b, c, d, e \in \mathbb{R}\right\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.11 *(∼5 min)* What is the largest number of vectors in

$$\mathcal{P} = \{f(x) \mid f(x) \text{ is any polynomial}\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Section S.2

Definition S.2.1 A **basis** is a linearly independent set that spans a vector space.

The **standard basis** of $\mathbb{R}^n$ is the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For $\mathbb{R}^3$, these are the vectors $\mathbf{e}_1 = \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{e}_3 = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Observation S.2.2 A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

For example, in many calculus courses, vectors in $\mathbb{R}^3$ are often expressed in their component form

$$(3, -2, 4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

or in their standard basic vector form

$$3\mathbf{e}_1 - 2\mathbf{e}_2 + 4\mathbf{e}_3 = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}.$$

Since every vector in $\mathbb{R}^3$ can be uniquely described as a linear combination of the vectors in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, this set is indeed a basis.

Activity S.2.3 *($\sim$15 min)* Label each of the sets A, B, C, D, E as

- SPANS $\mathbb{R}^4$ or DOES NOT SPAN $\mathbb{R}^4$
- LINEARLY INDEPENDENT or LINEARLY DEPENDENT
- BASIS FOR $\mathbb{R}^4$ or NOT A BASIS FOR $\mathbb{R}^4$

by finding RREF for their corresponding matrices.

$$A = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix}5\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\\3\end{bmatrix} \right\}$$

Activity S.2.4 *($\sim$10 min)* If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$, that means RREF$[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$ doesn't have a non-pivot column, and doesn't have a row of zeros. What is RREF$[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$?

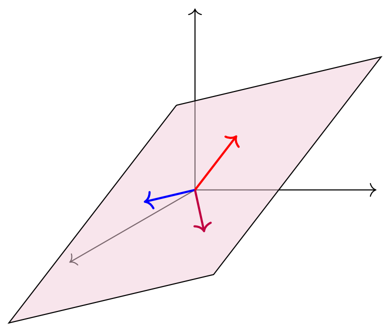
$$\text{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Fact S.2.5 The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and $\text{RREF}[\mathbf{v}_1\,\dots\,\mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$.

That is, a basis for $\mathbb{R}^n$ must have exactly n vectors and its square matrix must row-reduce to the so-called **identity matrix** containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)

Observation S.2.6 Recall that a **subspace** of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in the following image are needed to span the planar subspace.



Activity S.2.7 *(~10 min)* Consider the subspace $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

Part 1: Mark the part of RREF $\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Part 2: Find a basis for W by removing a vector from its spanning set to make it linearly independent.

Fact S.2.8 Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. The easiest basis describing $\operatorname{span} S$ is the set of vectors in S given by the pivot columns of RREF$[\mathbf{v}_1 \ldots \mathbf{v}_m]$.

Put another way, to compute a basis for the subspace $\operatorname{span} S$, simply remove the vectors corresponding to the non-pivot columns of RREF$[\mathbf{v}_1 \ldots \mathbf{v}_m]$.

Activity S.2.9 *(~10 min)* Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}4\\5\\3\\0\end{bmatrix}, \begin{bmatrix}3\\2\\2\\1\end{bmatrix}\right\}$$

Find a basis for W.

Activity S.2.10 *(~10 min)* Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left\{x^3 + 3x^2 + x - 1, 2x^3 - x^2 + x + 2, 4x^3 + 5x^2 + 3x, 3x^3 + 2x^2 + 2x + 1\right\}$$

Find a basis for W.

Section S.3

Observation S.3.1 In the previous section, we learned that computing a basis for the subspace $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, is as simple as removing the vectors corresponding to the non-pivot columns of $\mathrm{RREF}[\mathbf{v}_1 \ \ldots \ \mathbf{v}_m]$.

For example, since

$$\mathrm{RREF}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$ as a basis.

Activity S.3.2 *(∼10 min)* Let

$$S = \left\{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}\right\} \qquad \text{and} \qquad T = \left\{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Part 1: Find a basis for $\operatorname{span} S$.
Part 2: Find a basis for $\operatorname{span} T$.

Observation S.3.3 Even though we found different bases for them, $\operatorname{span} S$ and $\operatorname{span} T$ are exactly the same subspace of $\mathbb{R}^4$, since

$$S = \left\{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\} = T$$

Fact S.3.4 Any non-trivial vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.

For example,

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} \text{ and } \left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}\right\}$$

are all valid bases for $\mathbb{R}^3$, and they all contain three vectors.

Definition S.3.5 The **dimension** of a vector space is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension n. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\-2\\5\end{bmatrix} \right\}$$

contains exactly three vectors.

Activity S.3.6 *(~10 min)* Find the dimension of each subspace of $\mathbb{R}^4$ by finding RREF for each corresponding matrix.

$$\operatorname{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix} \right\} \quad \operatorname{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix} \right\}$$

$$\operatorname{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix} \right\} \quad \operatorname{span}\left\{ \begin{bmatrix}5\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\\3\end{bmatrix} \right\}$$

Fact S.3.7 Every vector space with finite dimension, that is, every vector space V with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is said to be **isomorphic** to a Euclidean space $\mathbb{R}^n$, since there exists a natural correspondance between vectors in V and vectors in $\mathbb{R}^n$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

Observation S.3.8 We've already been taking advantage of the previous fact by converting polynomials and matrices into Euclidean vectors. Since $\mathcal{P}^3$ and $M_{2,2}$ are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 5 \leftrightarrow \begin{bmatrix}4\\0\\-1\\5\end{bmatrix} \leftrightarrow \begin{bmatrix}4 & 0\\-1 & 5\end{bmatrix}$$

Observation S.3.9 The space of polynomials $\mathcal{P}$ (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since $\mathcal{P}$ and other infinite-dimensional spaces cannot be treated as an isomorphic finite-dimensional Euclidean space $\mathbb{R}^n$, vectors in such spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

Definition S.3.10 A **homogeneous** system of linear equations is one of the form:

$$\begin{aligned}
a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n &= 0 \\
a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\;\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}$$

This system is equivalent to the vector equation:

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

Activity S.3.11 *(~5 min)* Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ so is $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$, since

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0} \text{ and } b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = \mathbf{0}$$

implies

$$(a_1 + b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n = \mathbf{0}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is...

a) A basis for $\mathbb{R}^n$. b) A subspace of $\mathbb{R}^n$. c) The empty set.

Activity S.3.12 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).
Part 2: Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}.$$

Fact S.3.13 The coefficients of the free variables in the solution set of a linear system always yield linearly independent vectors.

Thus if

$$\left\{ a \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

is the solution space for a homoegeneous system, then

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the solution space.

Activity S.3.14 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 \phantom{{}+ 3x_4} &= 0 \\ 2x_1 - 6x_2 + 4x_3 + 3x_4 &= 0 \\ -2x_1 + 6x_2 - 4x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for its solution space.

Activity S.3.15 *(∼5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that it contains a **linearly independent** set of 3 vectors. What can you conclude about W?

(a) The dimension of W is at most 3.

(b) The dimension of W is exactly 3.

(c) The dimension of W is at least 3.

Activity S.3.16 *(∼5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that it contains a **spanning set** of 3 vectors. What can you conclude about W?

(a) The dimension of W is at most 3.

(b) The dimension of W is exactly 3.

(c) The dimension of W is at least 3.

Module F: First order ODEs

Standards for this Module

How can we solve and apply first order ODEs? At the end of this module, students will be able to...

F1. Separable ODEs. ...

F2. Autonomous ODEs. ...

F3. First order linear ODEs. ...

F4. Exact ODES. ...

F5. Modeling motion. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Section F.0

Activity F.0.1 *(~20 min)* Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**
   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
2. **Addition commutivity.**
   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Addition identity.**
   There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
4. **Addition inverse.**
   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
5. **Addition midpoint uniqueness.**
   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.
6. **Scalar multiplication associativity.**
   $a(b\mathbf{v}) = (ab)\mathbf{v}$.
7. **Scalar multiplication identity.**
   $1\mathbf{v} = \mathbf{v}$.
8. **Scalar multiplication relativity.**
   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.
9. **Scalar distribution.**
   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
10. **Vector distribution.**
    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
11. **Orthogonality.**
    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
12. **Bidimensionality.**
    $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some value of a, b.

Definition F.0.2 A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition is associative.**
  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition is commutative.**
  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Additive identity exists.**
  There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
- **Additive inverses exist.**
  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
- **Scalar multiplication is associative.**
  $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **1 is a scalar multiplicative identity.**
  $1\mathbf{v} = \mathbf{v}$.
- **Scalar multiplication distributes over vector addition.**
  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Scalar multiplication distributes over scalar addition.**
  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Any **Euclidean vector space** $\mathbb{R}^n$ satisfies all eight requirements regardless of the value of n, but we will also study other types of vector spaces.

Section F.1

Remark F.1.1 Last time, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, and all scalars (i.e. real numbers) a, b.

- **Addition is associative.**

  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Addition is commutative.**

  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **Additive identity exists.**

  There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.

- **Additive inverses exist.**

  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.

- **Scalar multiplication is associative.**

  $a(b\mathbf{v}) = (ab)\mathbf{v}$.

- **1 is a scalar multiplicative identity.**

  $1\mathbf{v} = \mathbf{v}$.

- **Scalar multiplication distributes over vector addition.**

  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

- **Scalar multiplication distributes over scalar addition.**

  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Section F.2

Remark F.2.1 Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity F.2.2 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.
Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.
Part 3: Given this solution set, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Fact F.2.3 A vector $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \, \dots \, \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Put another way, $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ exactly when $\operatorname{RREF}[\mathbf{v}_1 \, \dots \, \mathbf{v}_n \,|\, \mathbf{b}]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Activity F.2.4 *(∼10 min)* Determine if $\begin{bmatrix}3\\-2\\1\\5\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity F.2.5 *(∼5 min)* Determine if $\begin{bmatrix}-1\\-9\\0\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity F.2.6 *(∼10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?
Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)
Part 2: Solve this equivalent exercise, and use its solution to answer the original question.

Activity F.2.7 *(∼5 min)* Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\right\}$?

Activity F.2.8 *(∼5 min)* Does the complex number $2i$ belong to $\operatorname{span}\{-3 + i, 6 - 2i\}$?

Section F.3

Fact F.3.1 The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when RREF$[\mathbf{v}_1 \ \ldots \ \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ for some choice of vector } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Activity F.3.2 *(~5 min)* Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does $\mathbb{R}^4 = \operatorname{span} S$?

Activity F.3.3 *(~10 min)* Consider the set of third-degree polynomials

$$\begin{aligned} S = \{&2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, \\ &- x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}. \end{aligned}$$

Does $\mathcal{P}^3 = \operatorname{span} S$? (Hint: first rewrite the question so it is about Euclidean vectors.)

Activity F.3.4 *(~5 min)* Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$?

Activity F.3.5 *(~5 min)* Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^7$ be three vectors, and suppose $\mathbf{w}$ is another vector with $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What can you conclude about $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is larger than $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

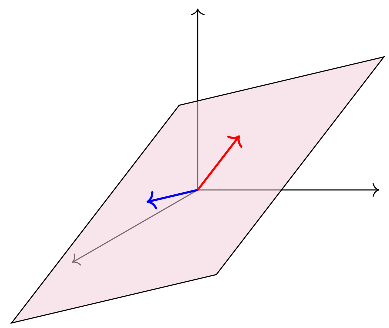
(b) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(c) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is smaller than $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Section F.4

Definition F.4.1 A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space $\mathbb{R}^3$.



Fact F.4.2 Any sub**set** S of a vector space V satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a sub**space**, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is **closed under addition**: for any $\mathbf{x}, \mathbf{y} \in S$, the sum $\mathbf{x}+\mathbf{y}$ is also in S.
- The set is **closed under scalar multiplication**: for any $\mathbf{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\mathbf{x}$ is also in S.

Activity F.4.3 *(∼15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x+2y+z=0 \right\}$.

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x+2y+z=0$ and $a+2b+c=0$. Show that $\mathbf{v}+\mathbf{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to S by verifying that $(x+a)+2(y+b)+(z+c)=0$.

Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x+2y+z=0$. Show that $c\mathbf{v}$ also belongs to S for any $c \in \mathbb{R}$.

Part 3: Is S is a subspace of $\mathbb{R}^3$?

Activity F.4.4 *(∼10 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x+2y+z=4 \right\}$. Choose a vector $\mathbf{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = \;?$, and show that $c\mathbf{v}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Remark F.4.5 Since 0 is a scalar and $0\mathbf{v} = \mathbf{z}$ for any vector $\mathbf{v}$, a set that is closed under scalar multiplication must contain the zero vector $\mathbf{z}$ for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain $\mathbf{0}$.

Activity F.4.6 *(~10 min)* Consider these two subsets of $\mathbb{R}^4$:

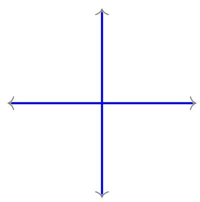
$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of $\mathbb{R}^4$?
Part 2: Is the set of polynomials

$$S = \left\{ ax^3 + bx^2 + (b-1)x + (a-1) \,\middle|\, a, b \text{ are real numbers} \right\}$$

a subspace of $\mathcal{P}^3$?

Activity F.4.7 *(~10 min)* Consider the subset A of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



This set contains $\mathbf{0}$, and it's not hard to show that for every $\mathbf{v}$ in A and scalar $c \in \mathbb{R}$, $c\mathbf{v}$ is also in A. Is A a subspace of $\mathbb{R}^2$? Why?
(Instructor Note:) Sketch the sum of two vectors on different axes to give a geometrical argument.

Activity F.4.8 *(~5 min)* Let W be a subspace of a vector space V. How are $\operatorname{span} W$ and W related?

(a) $\operatorname{span} W$ is bigger than W

(b) $\operatorname{span} W$ is the same as W

(c) $\operatorname{span} W$ is smaller than W

Fact F.4.9 If S is any subset of a vector space V, then since $\operatorname{span} S$ collects all possible linear combinations, $\operatorname{span} S$ is automatically a subspace of V.

In fact, $\operatorname{span} S$ is always the smallest subspace of V that contains all the vectors in S.

Module N: Numerical

Standards for this Module

How can we use numerical approximation methods to apply and solve unsolvable ODEs? At the end of this module, students will be able to...

N1. First Order Existence and Uniqueness. ...

N2. Second Order Linear Existence and Uniqueness. ...

N3. Systems Existence and Uniqueness. ...

N4. Euler's method for first order ODES. ...

N5. Euler's method for systems. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Section N.1

Definition N.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Section N.2

Remark N.2.1 Recall that a linear map $T : V \to W$ satisfies

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vecor space operations can be applied before or after the transformation without affecting the result.

Section N.3

Observation N.3.1 As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Section N.4

Observation N.4.1 Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all vectors in V that are mapped to $\mathbf{z} \in W$. It is a subspace of V.
- The **image** of T is the set of all vectors in W that are mapped to by something in V. It is a subspace of W.

Activity N.4.2 *(~5 min)* Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

Module D: Discontinuous functions in ODEs

Standards for this Module

How can we solve and apply ODEs involving functions that are not continuous? At the end of this module, students will be able to...

D1. Laplace Transform. ...

D2. Discontinuous ODEs. ...

D3. Modeling non-smooth motion. ...

D4. Modeling non-smooth oscillators. ...

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Section D.1

Definition D.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Section D.2

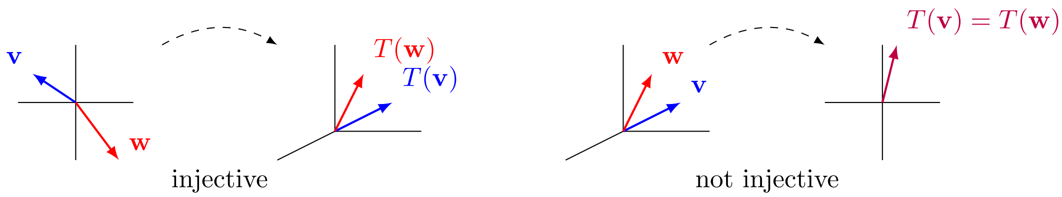
Remark D.2.1 Recall that a linear map $T : V \to W$ satisfies

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vecor space operations can be applied before or after the transformation without affecting the result.

Section D.3

Definition D.3.1 Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.



Section D.4

Observation D.4.1 Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all vectors in V that are mapped to $\mathbf{z} \in W$. It is a subspace of V.
- The **image** of T is the set of all vectors in W that are mapped to by something in V. It is a subspace of W.

Activity D.4.2 *(~5 min)* Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective