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Module D: Discontinuous functions in ODEs

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How can we solve and apply ODEs involving functions that are not continuous?

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At the end of this module, students will be able to...

D1. Laplace Transform. ...compute the Laplace transform of a function

D2. Discontinuous ODEs. ...solve initial value problems for ODEs with discontinuous coefficients

D3. Modeling non-smooth motion. ...model the motion of an object undergoing discontinuous acceleration

D4. Modeling non-smooth oscillators. ...model mechanical oscillators undergoing discontinuous acceleration

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Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Partial fractions

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The following resources will help you prepare for this module.

- TODO

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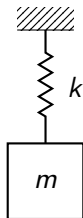
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Activity D.1.1 (~ 10 min)

A 1 kg mass is hung from a spring with spring constant $k = 1$ N/m. The mass is at rest, when it is hit with a hammer imparting 3J of energy.



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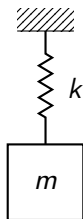
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Activity D.1.1 (~ 10 min)

A 1 kg mass is hung from a spring with spring constant $k = 1$ N/m. The mass is at rest, when it is hit with a hammer imparting 3J of energy.



Part 1: Draw a graph of the kinetic energy in the system with respect to time.

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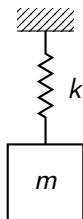
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Activity D.1.1 (~ 10 min)

A 1 kg mass is hung from a spring with spring constant $k = 1$ N/m. The mass is at rest, when it is hit with a hammer imparting 3J of energy.



Part 1: Draw a graph of the kinetic energy in the system with respect to time.

Part 2: Write an initial value problem modelling this system.

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Definition D.1.2

The **Dirac delta distribution** $\delta(t)$ models the application of instantaneous force. **It is not a function**, but makes sense in definite integrals:

If a, b is any open interval containing 0, then

$$\int_a^b f(t)\delta(t)dt = f(0)$$

for any function $f(t)$ that is continuous around 0.

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Definition D.1.3

The **unit impulse function** $u(t)$ is given by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0. \end{cases}$$

Note that $u(s) = \int_{-\infty}^s \delta(t) dt$; in this fuzzy sense, δ is the derivative of $u(t)$ (which is not differentiable everywhere!)

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Activity D.1.4 (*~10 min*)

Try to solve the IVP

$$y'' + y = \delta(t)$$

Where does our existing technique break down?

Observation D.1.5

To get around this difficulty, we will apply an **integral transform** called the **Laplace Transform** to our ODE.

- We want to use a definite integral to handle things like δ , which we can only understand via a definite integral.
- Since we are focused on IVPs, we can integrate starting at 0, but need to go to ∞
- But now we need to worry about convergence—thus we will multiply by a suitable function that decays fast enough to make most functions converge.

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Activity D.1.6 (~ 5 min)

Arrange the following functions in order of how fast they decay to zero in the limit at infinity:

- (A) x^{-n} for a positive integer n
- (B) e^{-ax} for a positive integer a
- (C) $\frac{1}{\ln(ax)}$ for a positive integer a
- (D) $\frac{1}{\ln(x^n)}$ for a positive integer n

Definition D.1.7

The **Laplace Transform** of a function $f(t)$ is the function

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Note that the Laplace transform turns a function of t into a function of s .

Moreover, \mathcal{L} is linear: $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$, and $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for constants c .

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Activity D.1.8 (*~5 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$

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Activity D.1.8 (~ 5 min)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}(\delta(t))$

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Activity D.1.8 (~ 5 min)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}(\delta(t))$

Part 2: If $a > 0$, compute $\mathcal{L}\{\delta(t - a)\}$

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Activity D.1.9 (~ 5 min)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$

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Activity D.1.9 (*~5 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}\{e^t\}$

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Activity D.1.9 (*~5 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}\{e^t\}$

Part 2: If $a > 0$, compute $\mathcal{L}\{e^{at}\}$

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Activity D.1.10 (*~15 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$

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Activity D.1.10 (*~15 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$

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Activity D.1.10 (*~15 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$ *Part 2:* Compute $\mathcal{L}\{t\}$

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Activity D.1.10 (*~15 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$ *Part 2:* Compute $\mathcal{L}\{t\}$ *Part 3:* Compute $\mathcal{L}\{t^2\}$

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Activity D.1.10 (*~15 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$ *Part 2:* Compute $\mathcal{L}\{t\}$ *Part 3:* Compute $\mathcal{L}\{t^2\}$ *Part 4:* Compute $\mathcal{L}\{t^3\}$

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Activity D.1.10 (*~15 min*)

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}\{1\}$ *Part 2:* Compute $\mathcal{L}\{t\}$ *Part 3:* Compute $\mathcal{L}\{t^2\}$ *Part 4:* Compute $\mathcal{L}\{t^3\}$ *Part 5:* Compute $\mathcal{L}\{t^4\}$

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Observation D.2.1

Last week, we encountered the **Laplace Transform** of a function $f(t)$:

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Recall that the Laplace transform turns a function of t into a function of s .

Moreover, \mathcal{L} is linear: $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$, and $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for constants c .

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Observation D.2.2

We computed a few Laplace Transforms:

- $\mathcal{L}\{\delta(t - a)\} = e^{-as}$ for any $a > 0$.
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ for any positive integer n .
- $\mathcal{L}\{1\} = \frac{1}{s}$

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Activity D.2.3 (*~10 min*)

Recall

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

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Activity D.2.3 (*~10 min*)

Recall

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}\{\sin(t)\}$.

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Activity D.2.3 (*~10 min*)

Recall

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}\{\sin(t)\}$.*Part 2:* Compute $\mathcal{L}\{\cos(t)\}$.

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Observation D.2.4

So now our list of Laplace transforms is:

- $\mathcal{L}\{\delta(t - a)\} = e^{-as}$ for any $a > 0$.
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ for any positive integer n .
- $\mathcal{L}\{1\} = \frac{1}{s}$
- $\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$
- $\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$

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Activity D.2.5 (*~10 min*)

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

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Activity D.2.5 (*~10 min*)

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Use integration by parts to relate $\mathcal{L}\{y'\}$ to $\mathcal{L}\{y\}$.

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Activity D.2.5 (*~10 min*)

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Part 1: Use integration by parts to relate $\mathcal{L}\{y'\}$ to $\mathcal{L}\{y\}$.

Part 2: Use integration by parts (and the fact that $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$) to relate $\mathcal{L}\{y''\}$ to $\mathcal{L}\{y\}$.

Observation D.2.6

We have

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

This allows us to easily rewrite expressions like $ay'' + by' + cy$ in terms of $\mathcal{L}\{y\}$.

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Activity D.2.7 (*~10 min*)

Consider the simple IVP

$$y'' + y = \delta(t), \quad y(0) = 0, y'(0) = 0.$$

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Activity D.2.7 (*~10 min*)

Consider the simple IVP

$$y'' + y = \delta(t), \quad y(0) = 0, y'(0) = 0.$$

Part 1: Apply the Laplace transform to this IVP, and simplify. Solve for $\mathcal{L}\{y\}$.

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Activity D.2.7 (*~10 min*)

Consider the simple IVP

$$y'' + y = \delta(t), \quad y(0) = 0, y'(0) = 0.$$

Part 1: Apply the Laplace transform to this IVP, and simplify. Solve for $\mathcal{L}\{y\}$.

Part 2: Find a function y satisfying $\mathcal{L}\{y\} = \frac{1}{s^2+1}$. We write $y = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$.

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Activity D.2.8 (*~15 min*)

Solve the IVP

$$y'' + y = \delta(t), \quad y(0) = 1, y'(0) = 2.$$

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Observation D.3.1

To solve an IVP using Laplace transforms:

- 1) Apply \mathcal{L} to the ODE. Use the initial condition(s) in computing $\mathcal{L}\{y'\}$, $\mathcal{L}\{y''\}$, etc.
- 2) Solve for $\mathcal{L}\{y\}$.
- 3) Take the inverse transform (using a table) to find the solution y .

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Activity D.3.2 (~ 5 min)Compute $\mathcal{L}^{-1} \left\{ \frac{1}{s+5} \right\}$.

(a) $u(t-5)$

(b) $\delta(t-5)$

(c) e^{5t}

(d) e^{-5t}

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Activity D.3.3 (~ 5 min)Compute $\mathcal{L}^{-1} \left\{ \frac{e^{-10s}}{s} \right\}$.

(a) $u(t - 10)$

(b) $\delta(t - 10)$

(c) $u(t - 10)e^{-t}$

(d) $\delta(t - 10)e^{-t}$

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Activity D.3.4 (~ 5 min)Compute $\mathcal{L}^{-1} \left\{ \frac{2e^{-2s}}{s^2+4} \right\}$.

(a) $u(t) \sin(2t)$

(b) $u(t-2) \sin(2t)$

(c) $u(t-2) \sin(2t-2)$

(d) $u(t-2) \sin(2t-4)$

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Activity D.3.5 (~ 5 min)Compute $\mathcal{L}^{-1} \left\{ \frac{e^{-100s}}{s^2} \right\}$.

(a) $u(t)t$

(b) $u(t)(t - 100)$

(c) $u(t - 100)t$

(d) $u(t - 100)(t - 100)$

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Activity D.3.6 (*~15 min*)

Solve the IVP

$$500v' = 20(u(t) - u(t - 100)), \quad v(0) = 50.$$

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Activity D.3.7 (*~15 min*)

Solve the IVP

$$y'' + 4y = \delta(t - 2),$$

$$y(0) = 0, \quad y'(0) = 1$$

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Activity D.4.1 (~ 30 min)

A spacecraft weighing 500 kg is travelling 50 m/s. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

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Activity D.4.1 (~ 30 min)

A spacecraft weighing 500 kg is travelling 50 m/s. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

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Activity D.4.1 (~ 30 min)

A spacecraft weighing 500 kg is travelling 50 m/s. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

Part 2: Write down an IVP modelling the velocity of the spacecraft.

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Activity D.4.1 (~ 30 min)

A spacecraft weighing 500 kg is travelling 50 m/s. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

Part 2: Write down an IVP modelling the velocity of the spacecraft.

Part 3: Solve the IVP:

$$500v' = 20(u(t) - u(t - 100)), \quad v(0) = 50.$$

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Activity D.4.1 (~ 30 min)

A spacecraft weighing 500 kg is travelling 50 m/s. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

Part 2: Write down an IVP modelling the velocity of the spacecraft.

Part 3: Solve the IVP:

$$500v' = 20(u(t) - u(t - 100)), \quad v(0) = 50.$$

Part 4: What is its velocity after 200 s?

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