Module S: Systems of ODEs

How can we solve and apply systems of linear ODEs?

At the end of this module, students will be able to...

S1. **Solving systems.** ...solve systems of constant coefficient ODEs

S2. **Modeling interacting populations.** ...model the populations of two interacting populations with a system of ODEs

S3. **Modeling coupled oscillators.** ...model systems of coupled mechanical oscillators using a system of ODEs

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve systems of two equations in two variables, even when coefficients are functions.
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equillibria of autonomous ODES **F4**

The following resources will help you prepare for this module.

- Solve systems of two equations in two variables, even when coefficients are functions. `https://youtu.be/Y6JsEja15Vk`
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equillibria of autonomous ODES **F4**

Module S Section 1

Activity S.1.1 *(∼10 min)*

Consider the countries of Transia and Wakanda: each year, 8% of people living in Transia move to Wakanda, and 3% of Wakandans move to Transia.

Let T be the population of Transia, and W the population of Wakanda (both are functions of time, t.

Which **system of differential equations** models the population changes $\frac{dT}{dt}$ and $\frac{dW}{dt}$?

(A)

$$\frac{dT}{dt} = 0.03W + 0.08T$$
$$\frac{dW}{dt} = 0.08T + 0.03W$$

(B)

$$\frac{dT}{dt} = -0.03W + 0.08T$$
$$\frac{dW}{dt} = -0.08T + 0.03W$$

(C)

$$\frac{dT}{dt} = 0.03W - 0.08T$$
$$\frac{dW}{dt} = 0.08T - 0.03W$$

(D)

$$\frac{dT}{dt} = -0.03W - 0.08T$$
$$\frac{dW}{dt} = 0.08T + 0.03W$$

Activity S.1.2 *(∼5 min)*
This problem resulted in a **system of linear differential equations**, namely

$$
\begin{aligned}
T' &= 0.03W - 0.08T \\
W' &= 0.08T - 0.03W
\end{aligned}
$$

Rewrite this system using differential operators.

Activity S.1.3 *(∼15 min)*

Solve the system

$$\begin{aligned} (D+0.08)T-(0.03)W &= 0 \\ -0.08T+(D+0.03)W &= 0 \end{aligned}$$

Observation S.1.4
Because D is linear, $a(D+b) = (D+b)a$ for constants a, b. This is not true in general!

Thus, for any **constant coefficient linear systems of differential equations**, we can use our typical elimination technique.

Activity S.1.5 *(~15 min)*
Solve the system

$$\begin{aligned} x' &= 5x - 2y \\ y' &= 6y - 3x \end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = -1$.

Activity S.1.6 *(∼15 min)*

Solve the system

$$x' = -y + 3t^2$$
$$y' = x + 2y - t^3$$

with initial conditions $x(0) = 1$, $y(0) = 1$.

Module S Section 2

Activity S.2.1 *(∼20 min)*

Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

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Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Activity S.2.1 *(∼20 min)*
Solve the system

$$\begin{aligned} x' &= 3x - 4y + 1 \\ y' &= 4x - 7y + 10t \end{aligned}$$

Part 1: Rewrite the system using differential operators
Part 2: Use elimination to eliminate a variable

Activity S.2.1 *(∼20 min)*

Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

Activity S.2.1 *(∼20 min)*

Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

Part 4: Find the solution for the other variable.

Activity S.2.2 *(∼20 min)*

Solve the system

$$\begin{aligned} x' &= 2x + 6y - 2 \\ y' &= 5x + 3y + 5 - e^{-3t} \end{aligned}$$

Activity S.2.3 *(∼15 min)*

Solve the system

$$\begin{aligned} x' &= 3x - 2y + \sin(t) \\ y' &= 4x - y - \cos(t) \end{aligned}$$

Module S Section 3

Activity S.3.1 *(∼5 min)*
Consider a forest of bamboo that grows unimpeded by other organisms. Which ODE models the size of the population best (all constants are positive)?

(a) $\frac{dB}{dt} = k$
(b) $\frac{dB}{dt} = kB$
(c) $\frac{dB}{dt} = kB - aB^2$
(d) $\frac{dB}{dt} = kB^2$

Activity S.3.2 *(∼5 min)*
The model

$$\frac{dB}{dt} = kB$$

models an ideal growth, free from competition (e.g. if population is sparse).

The model

$$\frac{dB}{dt} = kB - aB^2$$

models competitive growth.

Observe that both models are autonomous. Draw a phase line for each model, and describe the possible long term behaviors.

Activity S.3.3 *(∼10 min)*

Which of the following best models the bamboo population in the presence of a panda population (P)?

(a) $\frac{dB}{dt} = kB - aB^2$

(b) $\frac{dB}{dt} = kB - aB^2 - cP$

(c) $\frac{dB}{dt} = kB - aB^2 - cP^2$

(d) $\frac{dB}{dt} = kB - aB^2 - cBP$

Activity S.3.4 *(∼5 min)*

Which of the following best models the (sparse) Panda population in the bamboo forest?

(a) $\frac{dP}{dt} = -dP$

(b) $\frac{dP}{dt} = -dP + fBP$

(c) $\frac{dP}{dt} = -dP - fBP$

(d) $\frac{dP}{dt} = -dP - fBP - gP^2$

Observation S.3.5

The interacting bamboo and Panda populations are modelled by the **autonomous system**

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

These are referred to as **Lotka-Volterra equations**

Activity S.3.6 *(∼10 min)*

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Activity S.3.6 *(∼10 min)*
Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Part 1: When is $\frac{dB}{dt}$ zero?

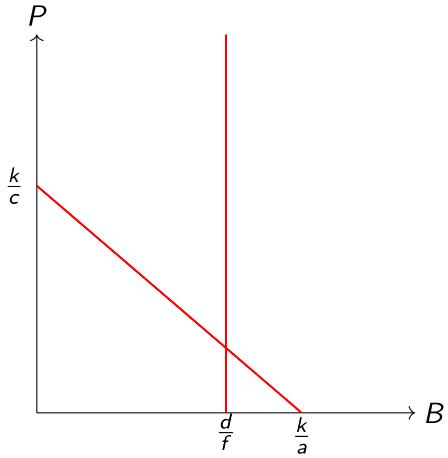
Activity S.3.6 *(∼10 min)*
Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

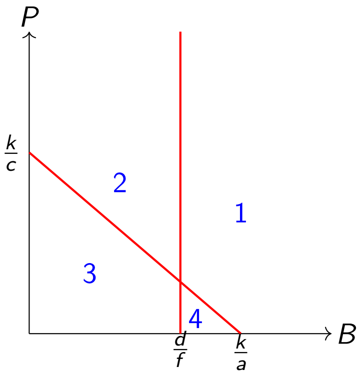
Part 1: When is $\frac{dB}{dt}$ zero?
Part 2: When is $\frac{dP}{dt}$ zero?

Observation S.3.7

These lines where the population of one species is unchanging are called **isoclines**

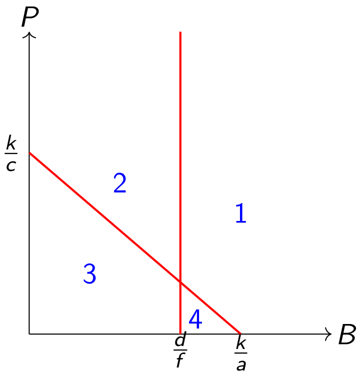


Activity S.3.8 *(∼15 min)*



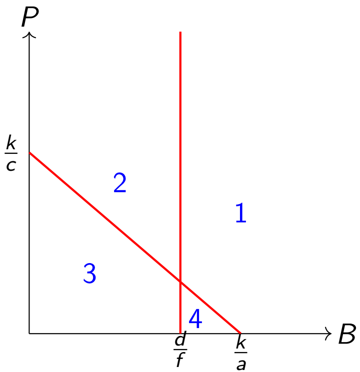
For each of the four regions

Activity S.3.8 *(∼15 min)*



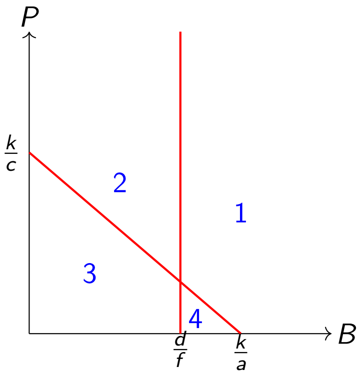
For each of the four regions
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Activity S.3.8 *(∼15 min)*



For each of the four regions
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.
Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

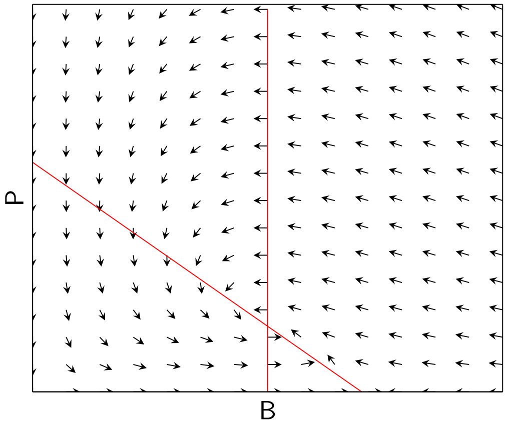
Activity S.3.8 *(∼15 min)*



For each of the four regions
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.
Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).
Part 3: Describe the general shape of the trajectories.

Observation S.3.9

Plotting the slope field with software makes it more clear that the trajectories are closed curves.



Module S Section 4

Activity S.4.1 *(∼10 min)*

Consider populations of Green Sunfish (G) and Bluegills (B) in the same lake. They compete for the same food.

Which system of ODEs would model this interaction best?

(A)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

(B)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = -0.1B - 0.003B^2 + 0.005BG$$

(C)

$$\frac{dG}{dt} = -0.1G - 0.002G^2 + 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

(D)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 + 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 + 0.005BG$$

Activity S.4.2 *(∼15 min)*

Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

Activity S.4.2 *(∼15 min)*

Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

Part 1: Plot the isoclines for each species.

Activity S.4.2 *(∼15 min)*

Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

Part 1: Plot the isoclines for each species.

Part 2: If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?

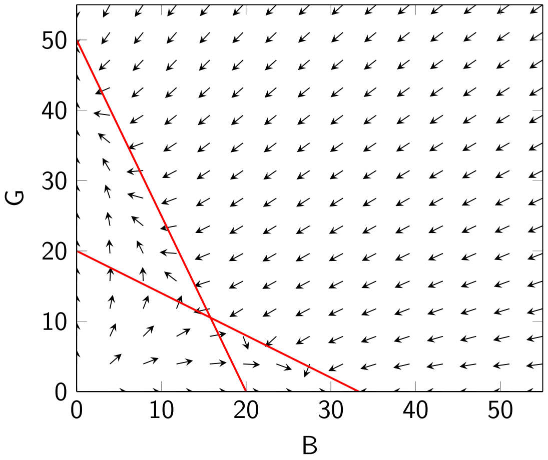
Activity S.4.2 *(∼15 min)*
Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

Part 1: Plot the isoclines for each species.
Part 2: If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?
Part 3: If the lake is stocked with 25 Bluegills and 5 Greenfish, what will happen?

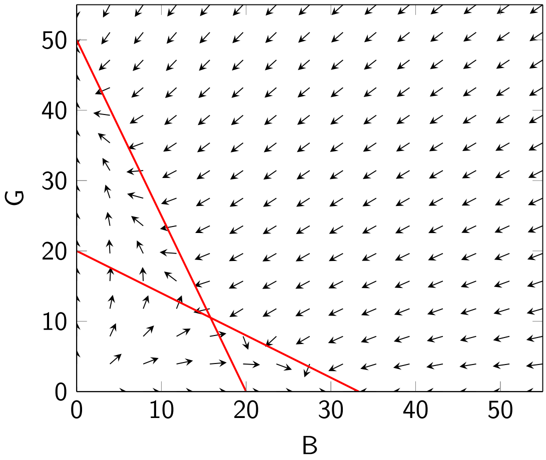
Activity S.4.3 *(∼5 min)*

Plotting the slope field along with the isoclines makes the unstable behavior more clear.



Activity S.4.3 *(~5 min)*

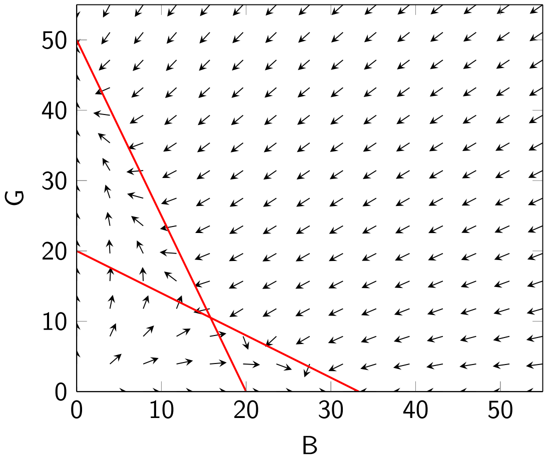
Plotting the slope field along with the isoclines makes the unstable behavior more clear.



Part 1: If the lake is stocked with 20 of each species, what will happen?

Activity S.4.3 *(∼5 min)*

Plotting the slope field along with the isoclines makes the unstable behavior more clear.



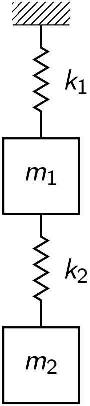
Part 1: If the lake is stocked with 20 of each species, what will happen?

Part 2: If the lake is stocked with 30 Bluegills and 10 Greenfish, what will happen?

Module S Section 5

Activity S.5.1 *(∼10 min)*

Consider two coupled masses with two springs.



Let x_1 be the position of the upper mass, and x_2 the position of the lower mass (both measured from equillibrium). Which ODE models the forces acting on the **lower** mass?

(A) $m_2x_2'' + k_2x_2 = 0$

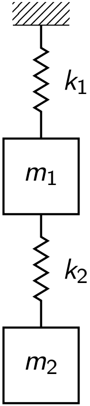
(B) $m_2x_2'' + k_2x_1 = 0$

(C) $m_2x_2'' + k_2(x_2 - x_1) = 0$

(D) $m_2x_2'' + k_2(x_1 - x_2) = 0$

Activity S.5.2 *(∼5 min)*

Consider two coupled masses with two springs.



Let x_1 be the position of the upper mass, and x_2 the position of the lower mass. Which ODE models the forces acting on the **upper** mass?

(A) $m_1 x_1'' + k_1 x_1 = 0$

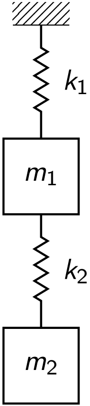
(B) $m_1 x_1'' + k_1 x_1 - k_2 x_2 = 0$

(C) $m_1 x_1'' + k_1 x_1 + k_2(x_2 - x_1) = 0$

(D) $m_1 x_1'' + k_1 x_1 + k_2(x_1 - x_2) = 0$

Activity S.5.3 *(∼30 min)*

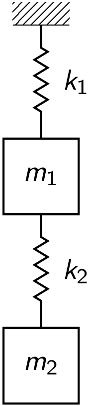
Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s}^2$, and $k_2 = 2\mathrm{kg/s}^2$. Then our model is



$$
\begin{aligned}
x_1'' + 6x_1 - 2x_2 &= 0\\
2x_2'' + 2x_2 - 2x_1 &= 0
\end{aligned}
$$

Activity S.5.3 *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg}/\mathrm{s}^2$, and $k_2 = 2\mathrm{kg}/\mathrm{s}^2$. Then our model is

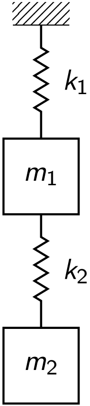


$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

Part 1: Rewrite the system using differential operators.

Activity S.5.3 *($\sim$30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg}/\mathrm{s}^2$, and $k_2 = 2\mathrm{kg}/\mathrm{s}^2$. Then our model is

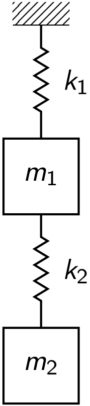


$$x_1'' + 6x_1 - 2x_2 = 0$$
$$2x_2'' + 2x_2 - 2x_1 = 0$$

Part 1: Rewrite the system using differential operators.
Part 2: Use elimination to write a single fourth order ODE for x_1.

Activity S.5.3 *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s^2}$, and $k_2 = 2\mathrm{kg/s^2}$. Then our model is



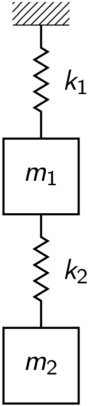
$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

Part 1: Rewrite the system using differential operators.
Part 2: Use elimination to write a single fourth order ODE for x_1.
Part 3: Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

Activity S.5.3 *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s}^2$, and $k_2 = 2\mathrm{kg/s}^2$. Then our model is



$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

Part 1: Rewrite the system using differential operators.
Part 2: Use elimination to write a single fourth order ODE for x_1.
Part 3: Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

Part 4: Determine a function for x_2 as well.