# Module C: Constant coefficient linear ODEs

## Standards for this Module

**How can we solve and apply linear constant coefficient ODEs?** At the end of this module, students will be able to...

**C1. Constant coefficient first order.** ...find the general solution to a first order constant coefficient ODE.

**C2. Modeling motion in viscous fluids.** ...model the motion of a falling object with linear drag

**C3. Homogeneous constant coefficient second order.** ...find the general solution to a homogeneous second order constant coefficient ODE.

**C4. IVPs.** ...solve initial value problems for constant coefficient ODEs

**C5. Non-homogenous constant coefficient second order.** ...find the general solution to a non-homogeneous second order constant coefficient ODE

**C6. Modeling oscillators.** ...model (free or forced, damped or undamped) mechanical oscillators with a second order ODE

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Describe Newton's laws in terms of differential equations.
- Find all roots of a quadratic polynomial.
- Use Euler's theorem to relate $\sin(t)$, $\cos(t)$, and $e^t$.
- Use Euler's theorem to simplify complex exponentials.
- Use substitution to compute indefinite integrals.
- Use integration by parts to compute indefinite integrals.
- Solve systems of two linear equations in two variables.

## Readiness Assurance Resources

The following resources will help you prepare for this module.

- Describe Newtons laws in terms of differential equations. `https://youtu.be/cioi4lRrAzw`
- Find all roots of a quadratic polynomial. `https://youtu.be/2ZzuZvz33X0` `https://youtu.be/TV5kDqiJ1Os`
- Use Eulers theorem to relate $\sin(t)$, $\cos(t)$, and $e^t$ and to simplify complex exponentials. `https://youtu.be/F_0yfvm0UoU` `https://youtu.be/sn3orkHWqUQ`
- Use substitution to compute indefinite integrals. `https://youtu.be/b76wePnIBdU`
- Use integration by parts to compute indefinite integrals. `https://youtu.be/bZ8YAHDTFJ8`
- Solve systems of two linear equations in two variables. `https://youtu.be/Y6JsEja15Vk`

## Section C.1

**Activity C.1.1** *(∼5 min)* Why don't clouds fall out of the sky?

(a) They are lighter than air

(b) Wind keeps them from falling

(c) Electrostatic charge

(d) They do fall, just very slowly

**Activity C.1.2** *(∼5 min)* List all of the forces acting on a tiny droplet of water falling from the sky.

**Activity C.1.3** *(∼5 min)* Tiny droplets of water obey **Stoke's law**, which says that air resistance is proportional to (the magnitude of) velocity.

- Let $v$ be the velocity of a droplet of water (positive for upward, negative for downward).
- Let $g > 0$ be the magnitude of acceleration due to gravity and $b > 0$ be another positive constant.

Apply Newton's second law (force = mass × acceleration) to determine which of the following **ordinary differential equations (ODEs)** models the velocity of a falling droplet of water.

(a) $v' = g - v$

(b) $v' = g + v$

(c) $mv' = -mg - bv$

(d) $mv' = -mg + bv$

**Observation C.1.4** The modeling equation

$$mv' = -mg - bv$$

may be obtained by splitting the total force into gravity and air resistance:

$$F = F_g + F_r$$

Then $F = ma = mv'$ and $F_g = m(-g) = -mg$ are the result of Newton's second law, and $F_r = -bv$ holds because it should be (a) in the opposite direction of velocity and (b) a constant multiple of velocity.
Note that this equation may be rearranged as follows to group $v$ and its derivative $v'$ together on the left-hand side:

$$v' + \left(\frac{b}{m}\right) v = -g$$

**Definition C.1.5** A **first order constant coefficient** differential equation can be written in the form

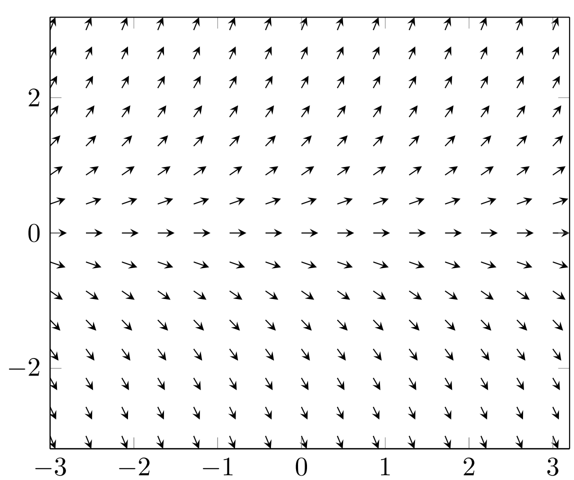
$$y' + by = f(x),$$

or equivalently,

$$\frac{dy}{dx} + by = f(x).$$

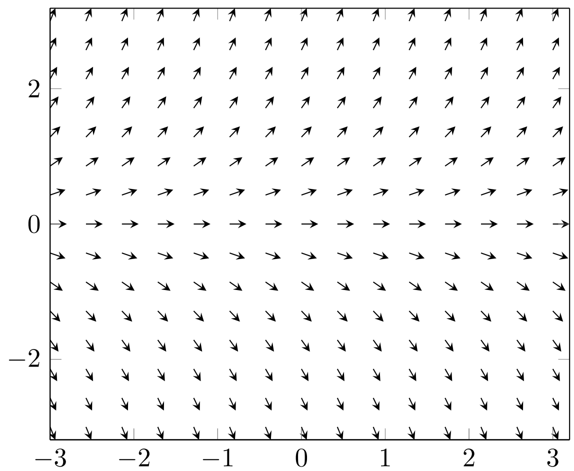
We will use both notations interchangeably.
Here, **first order** refers to the fact that the highest derivative we see is the first derivative of $y$.

**Observation C.1.6** Consider the differential equation $y' = y$.
A useful way to visualize a first order differential equation is by a **slope field**




Each arrow represents the slope of a solution **trajectory** through that point.

**Activity C.1.7** *(∼5 min)* Consider the differential equation $y' = y$ with slope field below.




*Part 1:* Draw a trajectory through the point $(0, 1)$.
*Part 2:* Draw a trajectory through the point $(-1, -1)$.

**Activity C.1.8** *(∼15 min)* Consider the differential equation $y' = y$.
*Part 1:* Find a solution to $y' = y$.
*Part 2:* Modify this solution to write an expression describing **all** solutions to $y' = y$.

**Definition C.1.9** A differential equation will have many solutions. Each individual solution is said to be a **particular solution**, while the **general solution** encompasses **all** of these by using parameters such as $C, k, c_0, c_1$ and so on. For example:

- The general solution to the differential equation $y' = 2x - 3$ is $y = x^2 - 3x + C$ (as done in Calculus courses).
- The general solution for $y' = y$ is $y = ke^x$ (as done in the previous activity).

**Activity C.1.10** *(∼15 min)* Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.
*Part 1:* Solve $y' = 2y$.
*Part 2:* Solve $y' = y + 2$.

## Section C.2

**Observation C.2.1** Recall the last activity from yesterday:
Solve $y' = y + 2$
This is very similar to the equation $y' = y$, which we were able to solve.

**Activity C.2.2** *(∼15 min)* Solve $y' = y + 2$
**Simple idea:** Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.
*Part 1:* Use the Product Rule to find $y_p' = \frac{d}{dx}[ve^x]$.
*Part 2:* Substitute $y_p$ and $y_p'$ into the equation $y' = y + 2$.
*Part 3:* Solve for $v$.
*Part 4:* Find $y_p$.

**Observation C.2.3** The technique outlined in the previous activity is called **variation of parameters**. If $y_0$ is a particular solution of the **homogeneous** equation, assume that a particular solution of the **non-homogeneous** equation has the form $y_p = vy_0$, and then determine what $v$ must be.
**Example:**

$$\begin{aligned} y' + 3y &= 0 && \text{homogeneous} \\ y' + 3y &= x && \text{non-homogeneous} \end{aligned}$$

Note that each term of the homogeneous equation includes $y$ or it derivatives.

**Activity C.2.4** *(∼20 min)* Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$\begin{aligned} y' + 3y &= 0 && \text{homogeneous} \\ y' + 3y &= x && \text{non-homogeneous} \end{aligned}$$

*Part 1:* Modify $e^x$ to find the general solution $y_h$ for the homogeneous equation.
*Part 2:* Choose a particular solution $y_0$ for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** $v$. Substitute $y_p$ into non-homogeneous equation and simplify.
*Part 3:* Determine $v$, and then determine $y_p$.

**Observation C.2.5** Since $y_h = ke^{-3x}$ was the general solution of $y' + 3y = 0$, and $y_p = \frac{x}{3} - \frac{1}{9}$ is a particular solution of $y' + 3y = x$,

$$y = y_h + y_p = \left(ke^{-3x}\right) + \left(\frac{x}{3} - \frac{1}{9}\right)$$

is a solution to $y' + 3y = x$:

$$\frac{d}{dx}[y_h + y_p] + 3(y_h + y_p) = (y_h' + 3y_h) + (y_p' + 3y_p) = 0 + x = x$$

**Fact C.2.6** Let $a$ be a constant real number. Every constant coefficient first order ODE

$$y' + ay = f(x)$$

has the general solution

$$y = y_h + y_p$$

where $y_h$ is the general solution to the homogeneous equation $y' + ay = 0$ and $y_p$ is a particular solution to $y' + ay = f(t)$.

**Activity C.2.7** *(~15 min)* Find the general solution to $y' = 2y + x + 1$ using variation of parameters:

- Write the homogeneous equation and find its general solution $y_h$.
- Use a particular solution $y_0$ for the homogeneous equation to find a particular solution $y_p = vy_0$ for the original equation.
- Then $y = y_h + y_p$ gives the general solution to the equation.

## Section C.3

**Observation C.3.1** Recall that we can model the velocity of a water droplet in a cloud by

$$mv' = -mg - bv$$

where negative numbers represent downward motion, $m > 0$ is the mass of the droplet, $g > 0$ is the magnitude of acceleration due to gravity, and $b > 0$ is the proportion of wind resistance to speed.

**Activity C.3.2** *(~20 min)* A water droplet with a radius of $10\,\mu$m has a mass of about $4 \times 10^{-15}$ kg. It is determined in a laboratory that for a droplet this size, the constant $b$ has a value of $3 \times 10^{-3}$ kg/s, and it is known that $g$ is approximately $9.8\,\mathrm{m/s}^2$.
Complete the following tasks to study the motion of this droplet.
*Part 1:* Rewrite $mv' = -mg - bv$ in the form of $v' + av = \,?\,$ for some value of $a$.
*Part 2:* Find the general solution of this ODE in terms of $a$ and $g$. (Let $v_p = wv_0$ when using variation of parameters to avoid confusion.)
*Part 3:* Due to wind resistence, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of $a$ and $g$?
*Part 4:* If the droplet starts from rest ($v = 0$ when $t = 0$), what is its velocity after $0.01$ s? Use a calculator to compute the answer in m/s.

**Definition C.3.3** The last part of the previous activity is an example of an **Initial Value Problem (IVP)**; we were given the initial value of the velocity in addition to our differential equation.
Physical scenarios often produce IVPs with a unique solution.

**Activity C.3.4** *(~10 min)* Solve the IVP

$$y' + 3y = 0, \qquad y(0) = 2.$$

**Activity C.3.5** *(~10 min)* Solve the IVP

$$y' - 2y = 2, \qquad y(0) = 1.$$

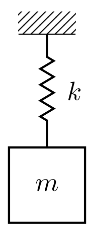
**Activity C.3.6** *(∼5 min)* Solve the IVP

$$y' - 2y = 2, \qquad y(2) = 1.$$

## Section C.4

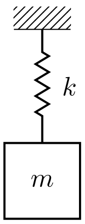
**Observation C.4.1** What happens when your tire hits a pothole?


**Activity C.4.2** *(∼5 min)* More abstractly, let's attach a mass (weighing $m$ kg) to a spring.




List all forces acting on the mass.

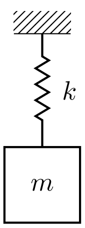
**Activity C.4.3** *(∼5 min)* **Hooke's law** says that the force exerted by the spring is proportional to the distance the spring is stretched.




Write a differential equation modeling the displacement of the mass.

**Observation C.4.4** There is an equillibrium point where the force of gravity balances the spring force. If we measure displacement from this point, we can model the mass-spring system by

$$my'' = -ky.$$

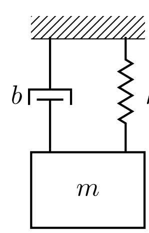



**Activity C.4.5** *(∼15 min)* Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

*Part 1:* Find a solution.
*Part 2:* Find the general solution.
*Part 3:* Describe the long term behavior of the mass-spring system.

**Activity C.4.6** *(∼5 min)* In applications, this infinitely oscillating behavior is often inappropriate. Thus, a damper (dashpot) is often incorporated. This provides a force proportional to the velocity.




Write a differential equation modeling the displacement of a mass in a **damped** mass-spring system.

**Observation C.4.7** The damped mass-spring system can be modelled by

$$my'' = -by' - ky.$$

Here $m$ is the mass, $k$ is the spring constant, and $b$ is the damping constant. We can rearrange this as

$$my'' + by' + ky = 0.$$

This is a **homogeneous second order constant coefficient** differential equation. Here, **homogeneous** refers to the 0 on the right hand side of the equation.

**Activity C.4.8** *(∼15 min)* Consider the second order constant coefficient equation

$$y'' = y.$$

*Part 1:* Find a solution.
*Part 2:* Find the general solution.
*Part 3:* Describe the long term behavior of the solutions.

## Section C.5

**Observation C.5.1** It is sometimes useful to think in terms of **differential operators**.

- We will use $D$ to represent a derivative; another common notation is $\frac{\partial}{\partial x}$. So for any function $y$,

$$D(y) = \frac{\partial y}{\partial x} = y'.$$

- $D^2$ will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use $I$ for the identity operator; it does nothing to a function. That is, $I(y) = y$. It can be thought of as $I = D^0$ (i.e. differentiate zero times).

In this language, the differential equation $y' + 3y = 0$ can be rewritten as $D(y) + 3I(y) = 0$, or $(D + 3I)(y) = 0$. Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator $D + 3I$.

**Activity C.5.2** *(∼5 min)* What is the kernel of $D - I$ ?
*Part 1:* Write a differential equation that corresponds to this question.
*Part 2:* Find the general solution of this differential equation.

**Activity C.5.3** *(∼5 min)* Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

**Activity C.5.4** *(∼10 min)* Consider the ODE

$$y'' + 5y' + 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.
*Part 2:* Factor this differential operator as a composition of two operators. (This works because $D$ and $I$ commute).
*Part 3:* Find the general solution of the ODE.

**Observation C.5.5** If we let $\mathcal{L} = D^2 + 5D + 6I$, we can write the ODE

$$y'' + 5y + 6y = 0$$

as

$$\mathcal{L}(y) = 0.$$

Note that such an $\mathcal{L}$ is always a **linear transformation**.

**Activity C.5.6** *(∼5 min)* Find the general solution to

$$y'' + y' - 12y = 0.$$

## Section C.6

**Activity C.6.1** *(~5 min)* Consider the ODE

$$y'' + 5y' - 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.
*Part 2:* Factor this differential operator as a composition of two operators. (This works because $D$ and $I$ commute).
*Part 3:* Find the general solution of the ODE.

**Activity C.6.2** *(~5 min)* Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

**Activity C.6.3** *(~5 min)* An **Initial Value Problem (IVP)** consists of an ODE along with some initial conditions that allow you to determine a single solution.
Solve the IVP

$$2y'' + 7y' + 6y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

.

**Activity C.6.4** *(~5 min)* Solve the ODE

$$y'' + y = 0.$$

**Activity C.6.5** *(~15 min)* Consider the ODE

$$y'' + 2y' + 5y = 0$$

.
*Part 1:* Find the general solution.
*Part 2:* Describe the long-term behavior of the solutions.

**Observation C.6.6** Solving $y'' + 2y' + 5y = 0$ produced a general solution

$$y = c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t}.$$

Applying Euler's formula yields

$$\begin{aligned} y &= c_1 e^{-t} \left(\cos(2t) + i \sin(2t)\right) + c_2 e^{-t} \left(\cos(2t) - i \sin(2t)\right) \\ &= (c_1 + c_2) e^{-t} \cos(2t) i (c_1 - c_2) e^{-t} \sin(2t) \end{aligned}$$

which we can rewrite (letting $k_1 = c_1 + c_2$ and $k_2 = i(c_1 - c_2)$) as

$$y = k_1 e^{-t} \cos(2t) + k_2 e^{-t} \sin(2t).$$

**Activity C.6.7** *(∼15 min)* Solve the IVP

$$y'' + 6y' + 34y = 0, \qquad y(0) = 2, \qquad y'(0) = 4.$$

## Section C.7

**Activity C.7.1** *(∼10 min)* Solve the ODE

$$y'' - 4y' + 4y = 0.$$

**Observation C.7.2** To solve this, we need to find the kernel of $(D-2I)(D-2I)$.

- The kernel of $D-2I$ is $\left\{ce^{2t} \mid c \in \mathbb{R}\right\}$.
- However, if $(D-2I)(y) = Ae^{2t}$, then applying $D-2I$ twice will yield zero!
- So we must solve the ODE
$$y' - 2y = e^{2t}.$$

**Activity C.7.3** *(∼15 min)* Solve $y' - 2y = e^{2t}$.

**Observation C.7.4** Thus, we have shown that the general solution of

$$y'' - 4y' + 4y = 0$$

is

$$y = c_0 e^{2t} + c_1 t e^{2t}.$$

**Activity C.7.5** *(∼15 min)* Solve $y'' - 6y' + 9y = 0$.

**Activity C.7.6** *(∼10 min)* Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

If $r$ is a number such that $ar^2 + br + c = 0$, what can you conclude?

(a) $e^{rt}$ is a solution.

(b) $e^{-rt}$ is a solution.

(c) $te^{rt}$ is a solution.

(d) There are no solutions.

**Activity C.7.7** *(∼5 min)* Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

When does the general solution have the form $c_0e^{rt} + te^{rt}$ ?

(a) When the polynomial $ax^2 + bx + c$ has two distinct real roots.

(b) When the polynomial $ax^2 + bx + c$ has a repeated real root.

(c) When the polynomial $ax^2 + bx + c$ has two distinct non-real roots.

(d) When the polynomial $ax^2 + bx + c$ has a repeated non-real root.

**Observation C.7.8** Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If $r$ is a root of $ax^2 + bx + c = 0$, then $e^{rt}$ is a solution of the ODE.
- If $r$ is a double root, variation of parameters shows that $te^{rt}$ is also a solution.
- If $r$ is not real, Euler's formula allows us to express the complex exponential part of the solution in terms of $\sin(rt)$ and $\cos(rt)$.

## Section C.8

**Observation C.8.1** Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If $r$ is a root of $ax^2 + bx + c = 0$, then $e^{rt}$ is a solution of the ODE.
- If $r$ is a double root, variation of parameters shows that $te^{rt}$ is also a solution.
- If $r = a + bi$ is not real, Euler's formula allows us to express the complex exponential part of the solution in terms of $e^{at}$, $\sin(bt)$, and $\cos(bt)$.

**Activity C.8.2** *(∼15 min)* Consider the following scenario: a mass of 4 kg suspended from a damped spring with spring constant $k = 2$ kg/s$^2$ and damping constant $b = 6$ kg/s.
The mass is pulled down 0.3 m and released from rest.
*Part 1:* Write down an ODE modelling this scenario, and find the general solution.
*Part 2:* Use the initial conditions $y(0) = -0.3$ and $y'(0) = 0$ to find particular values of the constants.

**Definition C.8.3** In the previous problem, we needed to solve

$$4y'' + 6y' + 2y = 0, \qquad y(0) = -0.3, \qquad y'(0) = 0.$$

This is called an **Initial Value Problem (IVP)** since we are provided with initial values of $y$ and $y'$.
To solve an IVP, find a general solution of the ODE, and use the initial conditions to find the values of the constants.

**Activity C.8.4** *(∼15 min)* Consider a mass of 5 kg suspended from a damped spring with spring constant $k = 2$ kg/s$^2$ and damping constant $b =$ 6kg/s.
The mass is pulled down 0.3m and released from rest. How many times does it pass back through its equillibrium state?

(a) 0

(b) 1

(c) 2

(d) Infinitely many

**Observation C.8.5** It can be shown that in the **overdamped** situation, the spring might pass through the equillibrium position once (e.g. if given an initial push), but never more than once.

## Section C.9

**Activity C.9.1** *(~10 min)* A 1 kg mass is suspended from a spring with spring constant $k = 9$ kg/s$^2$. An external force is applied by an electromagnet and is modeled by the function $F(t) = \sin(t)$. Write an ODE modeling the displacement of the spring.

**Observation C.9.2** In the previous activity, we encountered a **nonhomogeneous** second order constant coefficient ODE, i.e. of the form

$$ay'' + by' + cy = f(x)$$

where $a, b, c$ are constants, and $f(x)$ is a function.
We will again use **variation of parameters** to find a particular solution.

**Activity C.9.3** *(~15 min)* Suppose $y_1$ and $y_2$ are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.
Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions $v_1, v_2$.
*Part 1:* Use the product rule (twice) to compute $y_p'$.
*Part 2:* To simplify calculations, we will **assume** $v_1' y_1 + v_2' y_2 = 0$. Assuming this, compute $y_p''$.
*Part 3:* Compute $\mathcal{L}(y_p)$; simplify the ODE $\mathcal{L}(y_p) = f(x)$.

**Observation C.9.4** If we can find $v_1$ and $v_2$ that satisfy

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= \frac{f}{a} \end{aligned}$$

then we have a solution. So we just need to solve this system of equations for $v_1'$ and $v_2'$.

**Activity C.9.5** *(~15 min)* Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.
*Part 1:* Find $y_1$ and $y_2$, two independent solutions of $y'' + 9y = 0$.
*Part 2:* Find $v_1$ and $v_2$ by solving

$$\begin{aligned} \cos(3t) v_1' + \sin(3t) v_2' &= 0 \\ -3\sin(3t) v_1' + 3\cos(3t) v_2' &= \sin(t) \end{aligned}$$

*Part 3:* Write the general solution of the original nonhomogeneous ODE.

**Activity C.9.6** *(~10 min)* Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.
*Part 1:* Find $v_1$ and $v_2$ by solving

$$\begin{aligned} \cos(3t) v_1' + \sin(3t) v_2' &= 0 \\ -3\sin(3t) v_1' + 3\cos(3t) v_2' &= \sin(3t) \end{aligned}$$

*Part 2:* Write the general solution of the original nonhomogeneous ODE.

# Module F: First order ODEs

## Standards for this Module

**How can we solve and apply first order ODEs?** At the end of this module, students will be able to...

**F1. Sketching trajectories.** ...given a slope field, sketch a trajectory of a solution to a first order ODE

**F2. Separable ODEs.** ...find the general solution to a separable first order ODE

**F3. Modeling motion.** ...model the motion of an object with quadratic drag

**F4. Autonomous ODEs.** ...find and classify the equillibria of an autonomous first order ODE, and describe the long term behavior of solutions

**F5. First order linear ODEs.** ...find the general solution to a first order linear ODE

**F6. Exact ODES.** ...find the general solution to an exact first order ODE

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine the intervals on which a polynomial is positive, negative, or zero.
- Use integration techniques like substitution, integration by parts, and partial fraction decomposition to compute indefinite integrals.
- Determine when a vector field is conservative.
- Find the potential function of a conservative vector field.
- Use variation of parameters to solve non-homogeneous first order constant coefficient ODEs (Standard C1)

## Readiness Assurance Resources

The following resources will help you prepare for this module.

- Determine the intervals on which a polynomial is positive, negative, or zero. `https://youtu.be/jGa0GJjwQh8`
- Use integration techniques like substitution, integration by parts, and partial fraction decomposition to compute indefinite integrals. `https://youtu.be/b76wePnIBdU` `https://youtu.be/bZ8YAHDTFJ8` `https://youtu.be/qMX4vRhXBOE`
- Determine when a vector field is conservative. `https://youtu.be/gAb1ZTD41wo`
- Find the potential function of a conservative vector field. `https://youtu.be/nY4mW_R-T40`
- Use variation of parameters to solve non-homogeneous ODEs when given the solution to the corresponding homogeneous ODE (Standard C5)

## Section F.1

**Definition F.1.1** A **first order ODE** is an equation involving (for a function $y(x)$) only $y'$, $y$, and $x$.
We will most often deal with **explicit first order ODEs**, which can be written in the form

$$y' = f(y, x)$$

for some function $f(y, x)$.

**Activity F.1.2** *(∼5 min)* Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.
*Part 1:* Compute $y'$ at each of the points $(1, 1)$ , $(2, 1)$, $(3, -2)$, and $(4, -7)$.
*Part 2:*
Let $y_0(x)$ be a solution that passes through the point $(1, 1)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$ ?

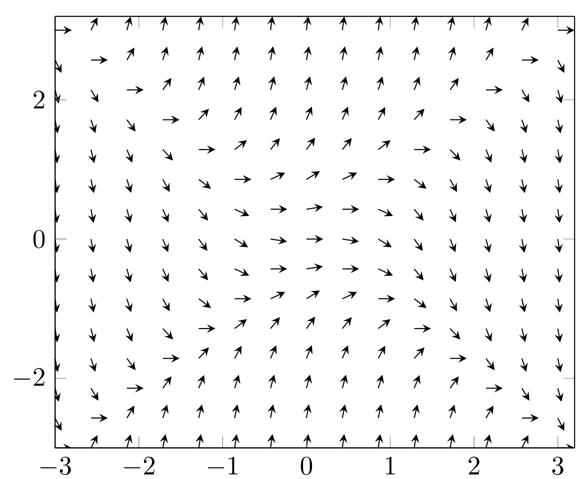
(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

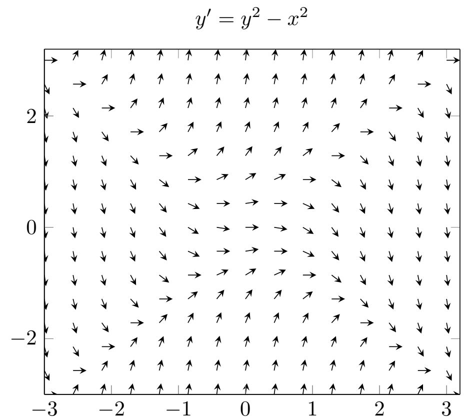
(C) $\lim_{x\to\infty} y_0(x) = \infty$

**Definition F.1.3** These kinds of questions are easier to answer if we draw a **slope field** (sometimes called a **direction field**.
To draw one, draw a small line segment or arrow with the correct slope at each point.

$$y' = y^2 - x^2$$



**Activity F.1.4** *(∼5 min)*




Let $y_1(x)$ be a solution that passes through the point $(2, 1)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$ ?

(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

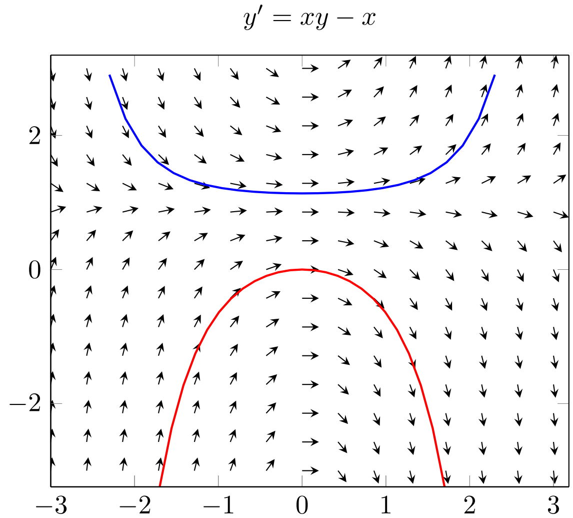
(C) $\lim_{x\to\infty} y_0(x) = \infty$

**Activity F.1.5** *(∼15 min)* Consider the ODE

$$y' = xy - x.$$

*Part 1:* Draw a slope field for this ODE.
*Part 2:* Draw a solution that passes through the point (0,0).
*Part 3:* Draw a solution that passes through the point (-2,2).

**Observation F.1.6**




**Observation F.1.7** How can we solve $y' = xy - x$ exactly?
Notice $xy - x = x(y - 1)$, so we can write $y' = x(y - 1)$.
Write

$$\frac{y'}{y-1} = x.$$

This is called a **separable** DE.

**Observation F.1.8** Integrate both sides (and switch to Leibniz notation):

$$\int \frac{1}{y-1}\frac{dy}{dx}\ dx = \int x\ dx.$$

The substitution rule (i.e. chain rule) says this is equivalent to

$$\int \frac{1}{y-1} dy = \int x\ dx.$$

Thus, $\ln|y-1| = \frac{1}{2}x^2 + c$. Exponentiating, we have

$$|y-1| = e^{\frac{1}{2}x^2 + c} = e^{\frac{1}{2}x^2} e^c = c_0 e^{\frac{1}{2}x^2}.$$

Allowing $c_0$ to take on negative values, we can drop the absolute value sign, and obtain

$$y = 1 + c_0 e^{\frac{1}{2}x^2}.$$

**Activity F.1.9** *(∼10 min)* Find the general solution to

$$y' = xy + y.$$

**Activity F.1.10** *(∼10 min)* Solve the IVP

$$y' = \frac{x}{y}, \qquad y(0) = -1.$$

## Section F.2

**Activity F.2.1** *(~5 min)* In Module C, we discussed that tiny spherical objects like droplets of water obey Stoke's law: drag is proportional to velocity (speed). But for larger objects, a better model incorporates **quadratic drag**, i.e. drag is proportional to the square of velocity.
Which of the following ODEs models the velocity of a falling object subject to quadratic drag?

(a) $mv' = -mg + bv$

(b) $mv' = -mg - bv$

(c) $mv' = -mg + bv^2$

(d) $mv' = -mg - bv^2$

**Activity F.2.2** *(~10 min)* Consider our model of a falling object under quadratic drag

$$mv' = -mg + bv^2.$$

*Part 1:* For what value of $v$ will the change in velocity be 0?
*Part 2:* Suppose the object is currently falling at a rate slower than this speed. What will happen?

(a) It will slow down

(b) It will keep falling at the same speed.

(c) It will speed up

**Observation F.2.3** This equillibrium speed is called the **terminal velocity**.

**Activity F.2.4** *(~5 min)* Consider the following question:
A penny is dropped off the top of the Empire State Building. How fast will it be going when it hits the ground?
What information do we need to answer this question?

**Observation F.2.5** The mass of a penny is 2.5g. The Empire State Building is (roughly) 400m tall.
The terminal velocity of a penny is about 25m/s.

**Activity F.2.6** *(~20 min)* We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.
*Part 1:* Solve for $b$ in terms of $v_t, m, g$, and substitute this in to our model $v' = -g + \frac{b}{m}v^2$.
*Part 2:* Solve this separable ODE
**Hint:** $\frac{1}{v_t^2 - v^2} = \frac{1}{2v_t}\left(\frac{1}{v_t - v} + \frac{1}{v_t + v}\right)$
*Part 3:* How fast is the penny going after 10 seconds?

## Section F.3

**Observation F.3.1** There are two very simple kinds of separable ODEs.
Equations of the form $y' = f(x)$ can be solved immediately by integrating and produce explicit solutions.
Equations of the form $y' = f(y)$ are often impossible or difficult to solve explicitly. They are called **autonomous** equations.

**Activity F.3.2** *(∼10 min)* Consider the autonomous equation

$$y' = y^2.$$

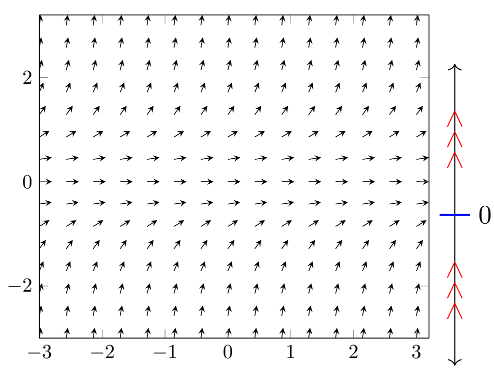
*Part 1:* Draw a slope field
*Part 2:* Suppose a solution goes through the point $y(10) = 50$. What can you say about $y(11)$?

(a) $y(10) < y(11)$

(b) $y(10) = y(11)$

(c) $y(10) > y(11)$

**Observation F.3.3** Since the slopes do not change when moving horizontally (i.e. in the $x$ direction), we often collapse the slope field onto the $y$-axis.




This is called a **phase line**.

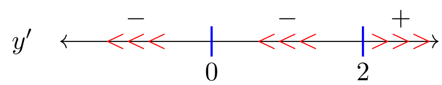
**Activity F.3.4** *(∼10 min)* Consider the autonomous equation

$$y' = y^2(y - 2).$$

*Part 1:* Draw a number line for $y'$, indicating where it is positive or negative.
*Part 2:* What can you say about the long term behavior of a solution passing through $y(4) = 1$?
*Part 3:* What can you say about the long term behavior of a solution passing through $y(2) = 0.001$?
*Part 4:* What can you say about the long term behavior of a solution passing through $y(2) = -0.001$?

## Section F.4

**Definition F.4.1** Recall from last week: the **phase line** is a useful way to visualize the long term behavior of an autonomous DE.
For example, here is a phase line for the autonomous DE $y' = y^2(y-2)$.




**Activity F.4.2** *(∼15 min)* Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

*Part 1:* Draw a phase line.
*Part 2:* Describe the long term behavior of a solution passing through $y(2) = -0.9999$.
*Part 3:* Describe the long term behavior of a solution passing through $y(7) = -1.0001$.
*Part 4:* Describe the long term behavior of a solution passing through $y(4) = -1$.
*Part 5:* Describe the long term behavior of solutions passing near the point $y(3) = 0$.
*Part 6:* Describe the long term behavior of solutions passing near the point $y(11) = 2$.

**Definition F.4.3** The **critical points** of an autonomous DE are the numbers that give rise to equillibrium solutions (e.g. $0, -1, 2$ in the previous problem).
A **source** is an unstable equillibrium in which all nearby trajectories move away in the limit.
A **sink** is a stable equillibrium in which all nearby trajectories approach the equillibrium in the limit.
There are also unstable equillibria in which some nearby trajectories return, while others diverge, analagous to a saddle point.

**Activity F.4.4** *(∼15 min)* Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

*Part 1:* Find and classify all of the critical points.
*Part 2:* Describe the long term behavior of solutions passing near the point $y(1) = 1.5$.

**Activity F.4.5** *(∼15 min)* Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

*Part 1:* Find and classify all of the critical points.
*Part 2:* Describe the long term behavior of solutions passing near the point $y(0) = 0.5$.
*Part 3:* Describe the long term behavior of solutions passing near the point $y(3) = 0$.

## Section F.5

**Observation F.5.1** In module C, we solved **constant coefficient linear ODEs**.
Today we will observe that our existing techniques allow us to solve all **first order linear ODES**, i.e. ODEs of the form

$$a(x)y' + b(x)y + c(x) = 0.$$

Such equations can always be rewritten (by rearranging and dividing by $a(x)$) in **standard form**:

$$y' + P(x)y = Q(x).$$

**Activity F.5.2** *(∼20 min)* Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

*Part 1:* Solve the **homogeneous** first order linear ODE

$$y' + \frac{1}{x}y = 0.$$

*Part 2:* Use variation of parameters to the solve the original ODE

**Activity F.5.3** *(∼15 min)* Solve the first order linear ODE

$$\frac{1}{x}y' - \frac{2}{x^2}y - x\cos(x) = 0.$$

**Activity F.5.4** *(∼15 min)* Solve

$$(x+1)y' + y = x.$$

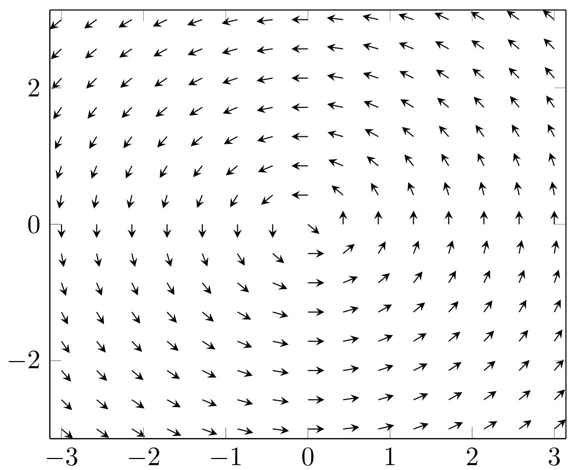
**Remark F.5.5** The book provides a different technique (multiplying by an integrating factor); however, the method presented here does not require memorizing anything new.

## Section F.6

**Observation F.6.1** A vector field $\langle P, Q\rangle$ corresponds to the slope field of the differential equation

$$\frac{dy}{dx} = \frac{Q}{P}.$$

Thus, a solution to this ODE describes the path taken by the particle in this fluid flow.




**Activity F.6.2** *(∼10 min)* Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2y^2 + x$.
*Part 1:* Compute $\nabla\phi$.
*Part 2:* Differentiate the equation $\phi(x, y) = c$ with respect to $x$.
*Part 3:* Solve the ODE $(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0$.

**Definition F.6.3** If $\langle M, N\rangle$ is a conservative vector field, the ODE

$$M + N\frac{dy}{dx} = 0$$

is called **exact**. This ODE can also be written

$$\frac{dy}{dx} = \frac{-M}{N}.$$

If $\phi(x, y)$ is a potential function of $\langle M, N\rangle$, the general solution to the ODE is $\phi(x, y) = c$.
**Careful:** The slope field of the ODE $\frac{dy}{dx} = \frac{-M}{N}$ is the vector field $\langle -N, M\rangle$ !

**Activity F.6.4** *(∼10 min)* Determine which of the following ODEs are exact.

(a) $2xy + (x^2 - 2y)\frac{dy}{dx} = 0$

(b) $\frac{dy}{dx} = \frac{2xy}{x^2+2y}$

(c) $\frac{dy}{dx} = -\frac{2xy}{x^2+2y}$

**Activity F.6.5** *(∼10 min)* Solve the exact ODE $2xy + (x^2 - 2y)\frac{dy}{dx} = 0$.

## Section F.7

**Activity F.7.1** *(∼10 min)* Determine which of the following ODEs are exact.

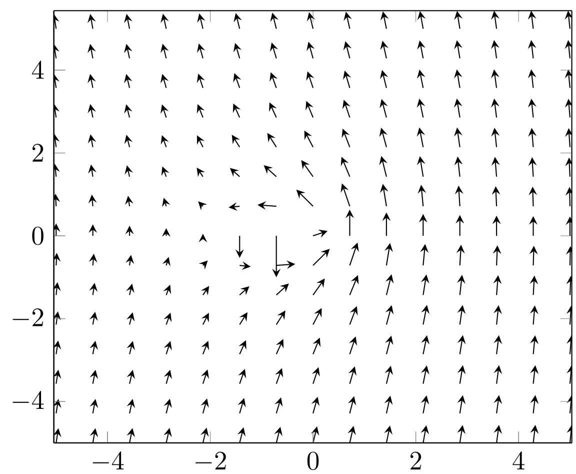
(a) $\frac{dy}{dx} = \frac{-y}{x^2+y^2+x}$

(b) $1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx} = 0$

**Activity F.7.2** *(∼15 min)* Solve the exact ODE

$$1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx} = 0.$$

These solutions describe the trajectories taken by particles in the fluid flow below




**Activity F.7.3** *(∼20 min)* Find solutions for the ODE

$$1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx} = 0$$

for each of the following initial conditions

(a) $y(0) = -1$.

(b) $y(-2) = -2$.

(c) $y(-4) = -4$.

Plot each of the solution curves.

# Module S: Systems of ODEs

## Standards for this Module

**How can we solve and apply systems of linear ODEs?** At the end of this module, students will be able to...

**S1. Solving systems.** ...solve systems of constant coefficient ODEs

**S2. Modeling interacting populations.** ...model the populations of two interacting populations with a system of ODEs

**S3. Modeling coupled oscillators.** ...model systems of coupled mechanical oscillators using a system of ODEs

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve systems of two equations in two variables, even when coefficients are functions.
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equillibria of autonomous ODES **F4**

## Readiness Assurance Resources

The following resources will help you prepare for this module.

- Solve systems of two equations in two variables, even when coefficients are functions. `https://youtu.be/Y6JsEja15Vk`
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equillibria of autonomous ODES **F4**

## Section S.1

**Activity S.1.1** *(~10 min)* Consider the countries of Transia and Wakanda: each year, 8% of people living in Transia move to Wakanda, and 3% of Wakandans move to Transia.
Let $T$ be the population of Transia, and $W$ the population of Wakanda (both are functions of time, $t$. Which **system of differential equations** models the population changes $\frac{dT}{dt}$ and $\frac{dW}{dt}$?

(A)

$$\begin{aligned}\frac{dT}{dt} &= 0.03W + 0.08T\\ \frac{dW}{dt} &= 0.08T + 0.03W\end{aligned}$$

(B)

$$\begin{aligned}\frac{dT}{dt} &= -0.03W + 0.08T\\ \frac{dW}{dt} &= -0.08T + 0.03W\end{aligned}$$

(C)

$$\begin{aligned}\frac{dT}{dt} &= 0.03W - 0.08T\\ \frac{dW}{dt} &= 0.08T - 0.03W\end{aligned}$$

(D)

$$\begin{aligned}\frac{dT}{dt} &= -0.03W - 0.08T\\ \frac{dW}{dt} &= 0.08T + 0.03W\end{aligned}$$

**Activity S.1.2** *(~5 min)* This problem resulted in a **system of linear differential equations**, namely

$$\begin{aligned}T' &= 0.03W - 0.08T\\ W' &= 0.08T - 0.03W\end{aligned}$$

Rewrite this system using differential operators.

**Activity S.1.3** *(~15 min)* Solve the system

$$\begin{aligned}(D + 0.08)T - (0.03)W &= 0\\ -0.08T + (D + 0.03)W &= 0\end{aligned}$$

**Observation S.1.4** Because $D$ is linear, $a(D + b) = (D + b)a$ for constants $a, b$. This is not true in general! Thus, for any **constant coefficient linear systems of differential equations**, we can use our typical elimination technique.

**Activity S.1.5** *(~15 min)* Solve the system

$$\begin{aligned}x' &= 5x - 2y\\ y' &= 6y - 3x\end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = -1$.

**Activity S.1.6** *(∼15 min)* Solve the system

$$x' = -y + 3t^2$$
$$y' = x + 2y - t^3$$

with initial conditions $x(0) = 1$, $y(0) = 1$.

## Section S.2

**Activity S.2.1** *(∼20 min)* Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

*Part 1:* Rewrite the system using differential operators
*Part 2:* Use elimination to eliminate a variable
*Part 3:* Solve the resulting second order ODE in one variable.
*Part 4:* Find the solution for the other variable.

**Activity S.2.2** *(∼20 min)* Solve the system

$$x' = 2x + 6y - 2$$
$$y' = 5x + 3y + 5 - e^{-3t}$$

**Activity S.2.3** *(∼15 min)* Solve the system

$$x' = 3x - 2y + \sin(t)$$
$$y' = 4x - y - \cos(t)$$

## Section S.3

**Activity S.3.1** *(∼5 min)* Consider a forest of bamboo that grows unimpeded by other organisms. Which ODE models the size of the population best (all constants are positive)?

(a) $\frac{dB}{dt} = k$

(b) $\frac{dB}{dt} = kB$

(c) $\frac{dB}{dt} = kB - aB^2$

(d) $\frac{dB}{dt} = kB^2$

**Activity S.3.2** *(∼5 min)* The model

$$\frac{dB}{dt} = kB$$

models an ideal growth, free from competition (e.g. if population is sparse).
The model

$$\frac{dB}{dt} = kB - aB^2$$

models competitive growth.
Observe that both models are autonomous. Draw a phase line for each model, and describe the possible long term behaviors.

**Activity S.3.3** *(∼10 min)* Which of the following best models the bamboo population in the presence of a panda population ($P$)?

(a) $\frac{dB}{dt} = kB - aB^2$

(b) $\frac{dB}{dt} = kB - aB^2 - cP$

(c) $\frac{dB}{dt} = kB - aB^2 - cP^2$

(d) $\frac{dB}{dt} = kB - aB^2 - cBP$

**Activity S.3.4** *(∼5 min)* Which of the following best models the (sparse) Panda population in the bamboo forest?

(a) $\frac{dP}{dt} = -dP$

(b) $\frac{dP}{dt} = -dP + fBP$

(c) $\frac{dP}{dt} = -dP - fBP$

(d) $\frac{dP}{dt} = -dP - fBP - gP^2$

**Observation S.3.5** The interacting bamboo and Panda populations are modelled by the **autonomous system**

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

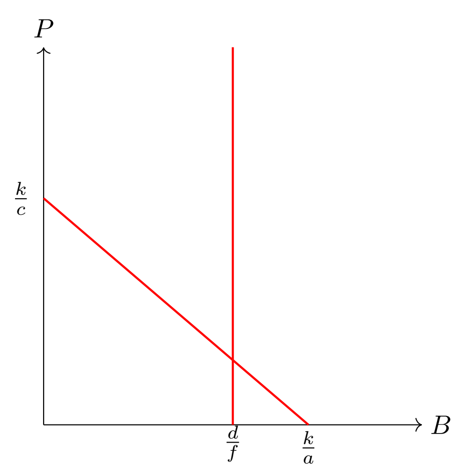
These are referred to as **Lotka-Volterra equations**

**Activity S.3.6** *(∼10 min)* Consider our Panda-Bamboo system

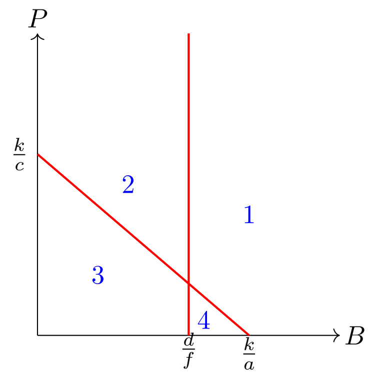
$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

*Part 1:* When is $\frac{dB}{dt}$ zero?
*Part 2:* When is $\frac{dP}{dt}$ zero?

**Observation S.3.7** These lines where the population of one species is unchanging are called **isoclines**

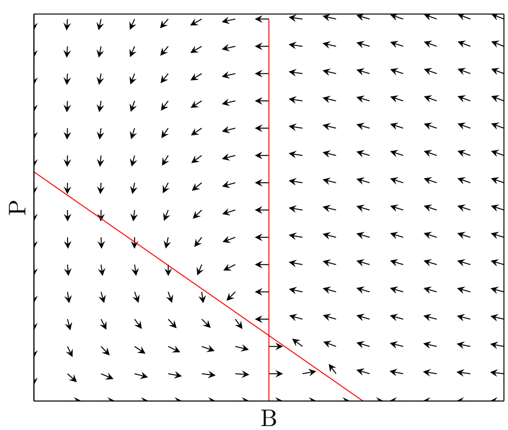


**Activity S.3.8** *(∼15 min)*




For each of the four regions
*Part 1:* Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.
*Part 2:* Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).
*Part 3:* Describe the general shape of the trajectories.

**Observation S.3.9** Plotting the slope field with software makes it more clear that the trajectories are closed curves.



## Section S.4

**Activity S.4.1** *(∼10 min)* Consider populations of Green Sunfish ($G$) and Bluegills ($B$) in the same lake. They compete for the same food.
Which system of ODEs would model this interaction best?

(A)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

(C)

$$\frac{dG}{dt} = -0.1G - 0.002G^2 + 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

(B)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = -0.1B - 0.003B^2 + 0.005BG$$

(D)

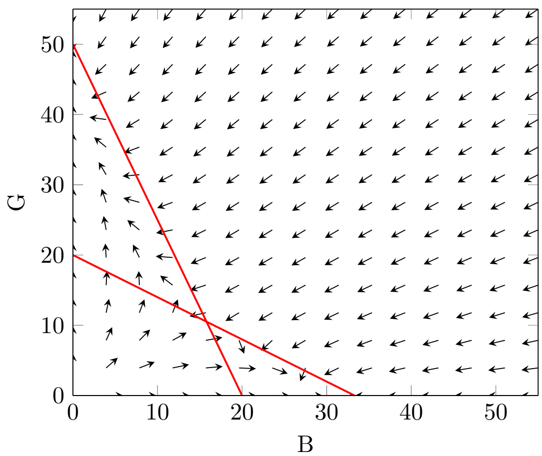
$$\frac{dG}{dt} = 0.1G - 0.002G^2 + 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 + 0.005BG$$

**Activity S.4.2** *(∼15 min)* Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

*Part 1:* Plot the isoclines for each species.
*Part 2:* If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?
*Part 3:* If the lake is stocked with 25 Bluegills and 5 Greenfish, what will happen?

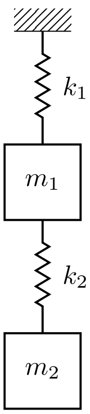
**Activity S.4.3** *(∼5 min)* Plotting the slope field along with the isoclines makes the unstable behavior more clear.




*Part 1:* If the lake is stocked with 20 of each species, what will happen?
*Part 2:* If the lake is stocked with 30 Bluegills and 10 Greenfish, what will happen?

## Section S.5

**Activity S.5.1** *(∼10 min)* Consider two coupled masses with two springs.




Let $x_1$ be the position of the upper mass, and $x_2$ the position of the lower mass (both measured from equillibrium). Which ODE models the forces acting on the **lower** mass?

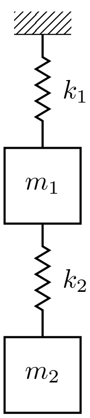
(A) $m_2x_2'' + k_2x_2 = 0$

(B) $m_2x_2'' + k_2x_1 = 0$

(C) $m_2x_2'' + k_2(x_2 - x_1) = 0$

(D) $m_2x_2'' + k_2(x_1 - x_2) = 0$

**Activity S.5.2** *(∼5 min)* Consider two coupled masses with two springs.




Let $x_1$ be the position of the upper mass, and $x_2$ the position of the lower mass. Which ODE models the forces acting on the **upper** mass?

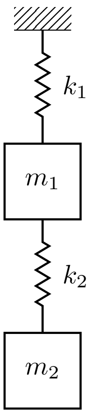
(A) $m_1x_1'' + k_1x_1 = 0$

(B) $m_1x_1'' + k_1x_1 - k_2x_2 = 0$

(C) $m_1x_1'' + k_1x_1 + k_2(x_2 - x_1) = 0$

(D) $m_1x_1'' + k_1x_1 + k_2(x_1 - x_2) = 0$

**Activity S.5.3** *(∼30 min)* Suppose we are given $m_1 = 2\text{kg}$, $m_2 = 1\text{kg}$, $k_1 = 4\text{kg/s}^2$, and $k_2 = 2\text{kg/s}^2$. Then our model is




$$\begin{aligned} 2x_1'' + 6x_1 - 2x_2 &= 0 \\ x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

*Part 1:* Rewrite the system using differential operators.
*Part 2:* Use elimination to write a single fourth order ODE for $x_1$.
*Part 3:* Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

*Part 4:* Determine a function for $x_2$ as well.

# Module N: Numerical

## Standards for this Module

**How can we use numerical approximation methods to apply and solve unsolvable ODEs?** At the end of this module, students will be able to...

**N1. First Order Existence and Uniqueness.** ...determine when a unique solution exists for a first order ODE

**N2. Second Order Linear Existence and Uniqueness.** ...determine when a unique solution exists for a second order linear ODE

**N3. Systems Existence and Uniqueness.** ...determine when a unique solution exists for a system of first order ODEs

**N4. Euler's method for first order ODES.** ...use Euler's method to find approximate solution to first order ODEs

**N5. Euler's method for systems.** ...use Euler's method to find approximate solutions to systems of first order ODEs

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compute partial derivatives.
- Determine where multivariate functions are continuous.
- Use a linear approximation to estimate the value of a function.
- Solve separable ODEs **F2**.

## Readiness Assurance Resources

The following resources will help you prepare for this module.

- Compute partial derivatives `https://youtu.be/3itjTS2Y9oE`.
- Determine where multivariate functions are continuous `https://youtu.be/RGx-pmWlOpk`.
- Use a linear approximation to estimate the value of a function `https://youtu.be/oxwCRzQOCu8`.
- Solve separable ODEs **F2**.

## Section N.1

**Activity N.1.1** *(∼5 min)* Solve the IVP

$$y' = \frac{3}{2} y^{\frac{1}{3}}, \qquad y(1) = 0.$$

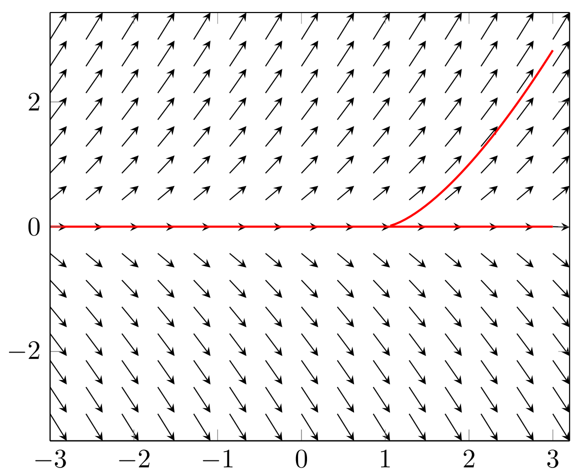
(A) $y = t^{\frac{3}{2}}$

(B) $y = (t-1)^{\frac{3}{2}}$

(C) $y = t^{\frac{3}{2}} - 1$

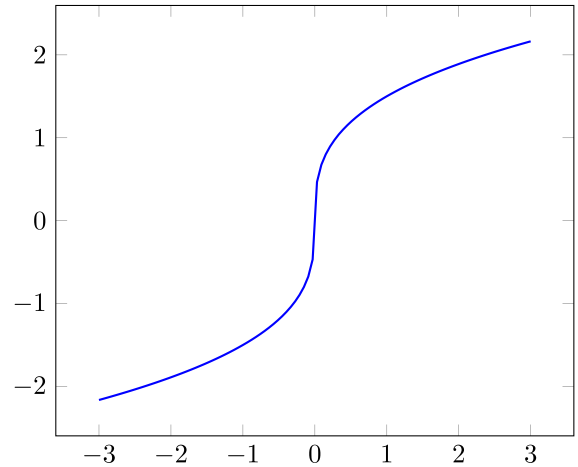
(D) $y = 0$

**Observation N.1.2** The ODE $y' = y^{\frac{1}{3}}$ has multiple solutions through the point $(1, 0)$.




How can we guarantee our ODEs have a unique solution?

**Observation N.1.3** Let's plot the function $f(y) = \frac{3}{2}y^{\frac{1}{3}}$.




Observe: $f(y)$ is not differentiable at 0!

**Observation N.1.4** If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are **continuous** on a rectangle containing $x_0, y_0$, then the IVP

$$y' = f(x, y), \qquad\qquad y(x_0) = y_0$$

has a unique solution.
The problem with our example

$$y' = \frac{3}{2}y^{\frac{1}{3}}, \qquad\qquad y(1) = 0.$$

is that, for $f(x, y) = \frac{3}{2}y^{\frac{1}{3}}$, the derivative

$$\frac{\partial f}{\partial y} = \frac{1}{2}y^{-\frac{2}{3}}$$

is not continuous at $(1, 0)$.

**Activity N.1.5** *(~5 min)* Consider the IVP

$$y' = \sqrt{x^2 + y^2}, \qquad\qquad y(0) = 0.$$

*Part 1:* Is $f(x, y) = \sqrt{x^2 + y^2}$ continuous at $0, 0$?
*Part 2:* Compute $\frac{\partial f}{\partial y}$. Is $\frac{\partial f}{\partial y}$ continuous at $0, 0$?
*Part 3:* Can you conclude the IVP has a unique solution?

**Activity N.1.6** *(∼10 min)* Consider the ODE

$$y' = \sqrt{x^2 + y^2 - 1}.$$

This ODE is guaranteed to have a unique solution passing through which of the following points?

(A) $y(1) = 1$

(B) $y(1) = -1$

(C) $y(1) = 0$

(D) $y(0) = 1$

**Activity N.1.7** *(∼10 min)* Consider the ODE

$$y' = \sqrt[3]{x^2 - y^2}.$$

This ODE is guaranteed to have a unique solution passing through which of the following points?

(A) $y(1) = 1$

(B) $y(1) = -1$

(C) $y(1) = 0$

(D) $y(0) = 1$

**Activity N.1.8** *(∼10 min)* Describe all points $(x_0, y_0)$ for which the IVP

$$y' = \ln(x^2 + y^2 - 1) - \sqrt[3]{4 - x^2 - y^2}, \qquad y(x_0) = y_0$$

is guaranteed to have a unique solution.

## Section N.2

**Observation N.2.1** We previously saw that the first order IVP

$$y' = f(x, y), \qquad y(x_0) = y_0$$

had a unique solution on some (possibly tiny!) interval containing $x_0$ when $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous at $(x_0, y_0)$.

**Activity N.2.2** *(∼10 min)* Consider the second order ODE

$$(x^2 - 1)^2 y'' + 4x = 0$$

.
*Part 1:* Solve for $y''$, and then integrate to find $y'$.
*Part 2:* Integrate again to find $y$. (**Hint:** $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$)
*Part 3:* For what values of $x$ is your solution valid?

**Observation N.2.3** The ODE $(x^2 - 1)^2 y'' + 4x = 0$ did not have a solution where the coefficient of $y''$ vanished, i.e. at $x = 1$ and $x = -1$.
In general, if $x_1 < x < x_2$ is an interval containing $x_0$ for which

- $(a(x), b(x), c(x),$ and $f(x)$ are continuous, and
- $a(x)$ does not vanish

Then the second order **linear** IVP

$$a(x)y'' + b(x)y' + c(x)y = f(x), \qquad y(x_0) = y_0,\ y'(x_0) = y_1$$

will have a unique solution on $x_1 < x < x_2$.

**Observation N.2.4** Our uniqueness result for first order equations applied to all first order equations. Our second order result applies to only **linear** equations, but provides added information–a precise interval on which the unique solution exists.
For example, the IVP

$$(x^2 - 1)^2 y'' + 4x = 0, \qquad y(2) = 3,\ y'(2) = 4$$

will have a unique solution valid for $1 < x < \infty$.

**Activity N.2.5** *(∼5 min)* Consider the IVP

$$\sin(x)y'' + \cos(x)y = x^2 - 4, \qquad y\left(\frac{\pi}{4}\right) = 1,\ y'\left(\frac{\pi}{4}\right) = 0.$$

Determine the largest interval on which a unique solution is guaranteed to exist.

**Activity N.2.6** *(∼5 min)* Consider the IVP

$$y'' + \frac{1}{x}y' - \frac{1}{x-4}y = 0 \qquad\qquad y(2) = 5,\ y'(2) = -1.$$

Determine the largest interval on which a unique solution is guaranteed to exist.

**Activity N.2.7** *(∼5 min)* Consider the ODE

$$(x^2-1)y'' + \frac{1}{x}y' + e^x y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

**Activity N.2.8** *(∼5 min)* Consider the ODE

$$\frac{x}{x-1}y'' + \frac{x+2}{x+1}y' + e^{-x}y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

**Activity N.2.9** *(∼5 min)* Consider the ODE

$$\sqrt{x^2-1}y'' + y' + \frac{1}{x}y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

## Section N.3

**Activity N.3.1** *(∼10 min)* Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.
*Part 1:* Compute the slope of the solution at the point $(2, 4)$.
*Part 2:* Use a linear approximation to estimate the value of $y(2.1)$.
*Part 3:* Calculate the slope at the point $(2.1, 4.4)$.
*Part 4:* Use a linear approximation at $(2.1, 4.4)$ to estimate the value of $y(2.2)$.

**Observation N.3.2** This technique is called **Euler's method** (with step size $h = 0.1$) for the IVP

$$y' = x + \sqrt{y}, \qquad y(2) = 4.$$

It is often convenient to organize this information in a table

| $x_n$ | $y_n$ | $y'(x_n, y_n)$ | $x_{n+1} = x_n + h$ | $y_{n+1} = y_n + hy'(x_n, y_n)$ |
|---|---|---|---|---|
| 2 | 4 | 4 | 2.1 | 4.4 |
| 2.1 | 4.4 | 4.19762 | 2.2 | 4.81976 |

**Activity N.3.3** *(∼10 min)* Use Euler's method with stepsize $h = 0.2$ to estimate $y(3)$, where $y$ is a solution of the IVP

$$y' = x - 3y^2, \qquad y(2.2) = 1.$$

**Activity N.3.4** *(∼10 min)* Use Euler's method with stepsize $h = 0.2$ to estimate $y(4)$, where $y$ is a solution of the IVP

$$y' = \sqrt{x - y}, \qquad y(3) = 1.$$

## Section N.4

**Observation N.4.1** The same problem we saw with a first order ODE failing to have a unique solution can also occur in systems of first order ODEs.
Thus, to ensure that the system

$$\begin{aligned} x' &= f(t,x,y) \\ y' &= g(t,x,y) \end{aligned}$$

has a unique solution, you must check that $f(t,x,y)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $g(t,x,y)$, $\frac{\partial g}{\partial x}$, and $\frac{\partial g}{\partial y}$ are all continuous near the initial point.

**Observation N.4.2** If the system is linear, we can say more!
Suppose $a(t), b(t), c(t), d(t), f(t), g(t)$ are all continous on the interval $h < t < k$. Then for any $h < t_0 < k$, the IVP

$$\begin{aligned} x' &= a(t)x + b(t)y + f(t) & \qquad x(t_0) &= x_0 \\ y' &= c(t)x + d(t)y + g(t) & \qquad y(t_0) &= y_0 \end{aligned}$$

has a unique solution on the (time) interval $h < t < k$.

**Activity N.4.3** *(∼5 min)* Consider the IVP

$$\begin{aligned} x' &= \frac{1}{t-1}x + \sqrt{t}y + t^2 & \qquad x(2) &= 5 \\ y' &= \frac{1}{t}x + \sqrt{t+1}y + t^2 & \qquad y(2) &= 7 \end{aligned}$$

What is the largest interval on which this IVP has a unique solution?

**Activity N.4.4** *(∼5 min)* Determine **all intervals** on which a unique solution is guaranteed to exist for the below system.

$$\begin{aligned} x' &= \frac{1}{t-1}x + \sqrt{t}y + t^2 \\ y' &= \frac{1}{t}x + \sqrt{t+1}y + t^2 \end{aligned}$$

**Activity N.4.5** *(∼5 min)* Determine **all intervals** on which a unique solution is guaranteed to exist for the below system.

$$\begin{aligned} x' &= \ln(t-2)x + \sqrt{t}y + \frac{1}{t-1} \\ y' &= \cos(t)x + y \end{aligned}$$

**Activity N.4.6** *(∼10 min)* Euler's method can be extended to systems in a straightforward way. Consider the system IVP

$$x' = 3x + 4y - t \qquad x(1) = 2$$
$$y' = x - y + t \qquad y(1) = 3.$$

*Part 1:* Compute $x'$ and $y'$ when $t = 1$.
*Part 2:* Use linear approximations to estimate the values of $x(1.1)$ and $y(1.1)$.
*Part 3:* Calculate the slopes $x'$ and $y'$ when $t = 1.1$ (and as just calculated, $x(1.1) =$ and $y(1.1) =$).
*Part 4:* Use linear approximations to estimate the values of $x(1.2)$ and $y(1.2)$.

**Observation N.4.7**

$$x' = 3x + 4y - t \qquad x(1) = 2$$
$$y' = x - y + t \qquad y(1) = 3.$$

It is often convenient to organize this information in a table

| $t_n$ | $x_n$ | $y_n$ | $x'(t_n, x_n, y_n)$ | $y'(t_n, x_n, y_n)$ | $t_{n+1}$ | $x_{n+1}$ | $y_{n+1}$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 17 | 0 | 1.1 | 2.17 | 3 |
| 1.1 | 2.17 | 3 | 17.41 | 0.27 | 1.2 | 3.911 | 3.027 |
| 1.2 | 3.911 | 3.027 | | | | | |

Thus $x(1.2) \approx 3.911$ and $y(1.2) \approx 3.027$.

**Activity N.4.8** *(∼10 min)* Use Euler's method to estimate $x(3.3)$ and $y(3.3)$.

$$x' = 3x - ty \qquad x(3) = 2$$
$$y' = x - y^2 \qquad y(3) = 1.$$

**Activity N.4.9** *(∼10 min)* Use Euler's method to estimate $x(4.6)$ and $y(4.6)$.

$$x' = xy - t \qquad x(4) = 2$$
$$y' = x + t \qquad y(4) = 0.$$

# Module D: Discontinuous functions in ODEs

## Standards for this Module

**How can we solve and apply ODEs involving functions that are not continuous?** At the end of this module, students will be able to...

**D1. Laplace Transform.** ...compute the Laplace transform of a function

**D2. Discontinuous ODEs.** ...solve initial value problems for ODEs with discontinuous coefficients

**D3. Modeling non-smooth motion.** ...model the motion of an object undergoing discontinuous acceleration

**D4. Modeling non-smooth oscillators.** ...model mechanical oscillators undergoing discontinuous acceleration

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

## Readiness Assurance Resources

The following resources will help you prepare for this module.

- TODO

## Section D.1

**Definition D.1.1** A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if $V$ and $W$ are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.