Module F: First order ODEs

How can we solve and apply first order ODEs?

At the end of this module, students will be able to...

F1. **Sketching trajectories.** ...given a slope field, sketch a trajectory of a solution to a first order ODE

F2. **Separable ODEs.** ...find the general solution to a separable first order ODE

F3. **Modeling motion.** ...model the motion of an object with quadratic drag

F4. **Autonomous ODEs.** ...find and classify the equillibria of an autonomous first order ODE, and describe the long term behavior of solutions

F5. **First order linear ODEs.** ...find the general solution to a first order linear ODE

F6. **Exact ODES.** ...find the general solution to an exact first order ODE

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Use integration techniques like substitution and integration by parts to compute indefinite integrals.
- Determine the intervals on which a polynomial is positive, negative, or zero.
- Determine when a vector field is conservative.
- Find the potential function of a conservative vector field.
- Use variation of parameters to solve non-homogeneous first order constant coefficient ODEs (Standard C1)

The following resources will help you prepare for this module.

- Use integration techniques like substitution to compute indefinite integrals.
  https://youtu.be/b76wePnIBdU
- Determine the intervals on which a polynomial is positive, negative, or zero.
  https://youtu.be/jGa0GJjwQh8
- Determine when a vector field is conservative.
  https://youtu.be/gAb1ZTD41wo
- Find the potential function of a conservative vector field.
  https://youtu.be/nY4mW_R-T40
- Use variation of parameters to solve non-homogeneous ODEs when given the solution to the corresponding homogeneous ODE (Standard C5)

Module F Section 1

Definition F.1.1
A **first order ODE** is an equation involving (for a function $y(x)$) only y', y, and x.

We will most often deal with **explicit first order ODEs**, which can be written in the form

$$y' = f(y, x)$$

for some function $f(y, x)$.

Activity F.1.2 *(∼5 min)*
Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.

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Part 1: Compute y' at each of the points $(1, 1)$, $(2, 1)$, $(3, -2)$, and $(4, -7)$.

Activity F.1.2 *(~5 min)*

Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.

Part 1: Compute y' at each of the points $(1, 1)$, $(2, 1)$, $(3, -2)$, and $(4, -7)$.

Part 2:

Let $y_0(x)$ be a solution that passes through the point $(1, 1)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$?

(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

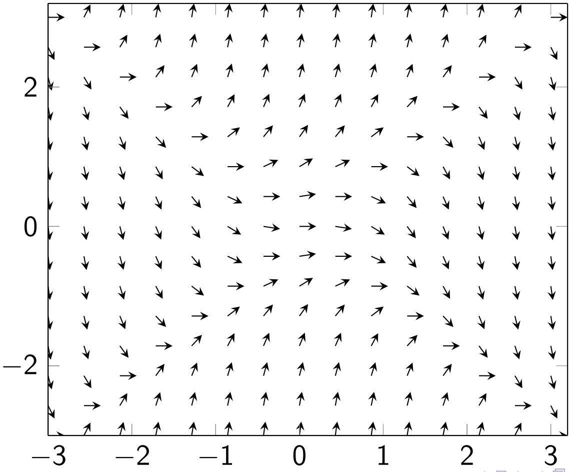
(C) $\lim_{x\to\infty} y_0(x) = \infty$

Definition F.1.3

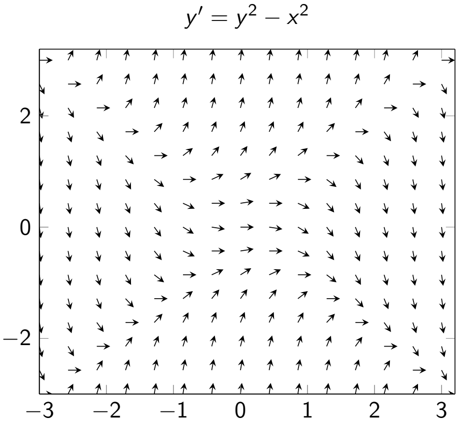
These kinds of questions are easier to answer if we draw a **slope field** (sometimes called a **direction field**.

To draw one, draw a small line segment or arrow with the correct slope at each point.

$$y' = y^2 - x^2$$



Activity F.1.4 *(∼5 min)*



Let $y_1(x)$ be a solution that passes through the point $(2, 1)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$?

(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

(C) $\lim_{x\to\infty} y_0(x) = \infty$

Activity F.1.5 *(∼15 min)*

Consider the ODE

$$y' = xy - x.$$

Activity F.1.5 *(∼15 min)*
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Part 1: Draw a slope field for this ODE.

Activity F.1.5 *(∼15 min)*

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$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.

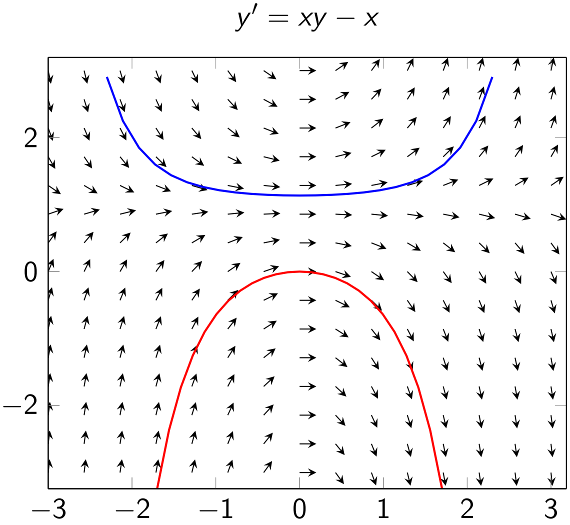
Part 2: Draw a solution that passes through the point (0,0).

Activity F.1.5 *(∼15 min)*
Consider the ODE

$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.
Part 2: Draw a solution that passes through the point (0,0).
Part 3: Draw a solution that passes through the point (-2,2).

Observation F.1.6



Observation F.1.7

How can we solve $y' = xy - x$ exactly?

Notice $xy - x = x(y-1)$, so we can write $y' = x(y-1)$.

Write

$$\frac{y'}{y-1} = x.$$

This is called a **separable** DE.

Observation F.1.8

Integrate both sides (and switch to Leibniz notation):

$$\int \frac{1}{y-1}\frac{dy}{dx}\ dx = \int x\ dx.$$

The substitution rule (i.e. chain rule) says this is equivalent to

$$\int \frac{1}{y-1} dy = \int x\ dx.$$

Thus, $\ln|y-1| = \frac{1}{2}x^2 + c$. Exponentiating, we have

$$|y-1| = e^{\frac{1}{2}x^2+c} = e^{\frac{1}{2}x^2}e^c = c_0 e^{\frac{1}{2}x^2}.$$

Allowing c_0 to take on negative values, we can drop the absolute value sign, and obtain

$$y = 1 + c_0 e^{\frac{1}{2}x^2}.$$

Activity F.1.9 *(∼10 min)*
Find the general solution to

$$y' = xy + y.$$

Activity F.1.10 *(∼10 min)*

Solve the IVP

$$y' = \frac{x}{y}, \qquad y(0) = -1.$$

Module F Section 2

Activity F.2.1 *(∼5 min)*

In Module C, we discussed that tiny spherical objects like droplets of water obey Stoke's law: drag is proportional to velocity (speed). But for larger objects, a better model incorporates **quadratic drag**, i.e. drag is proportional to the square of velocity.

Which of the following ODEs models the velocity of a falling object subject to quadratic drag?

(a) $mv' = -mg + bv$

(b) $mv' = -mg - bv$

(c) $mv' = -mg + bv^2$

(d) $mv' = -mg - bv^2$

Activity F.2.2 *(∼10 min)*

Consider our model of a falling object under quadratic drag

$$mv' = -mg + bv^2.$$

Activity F.2.2 *(∼10 min)*

Consider our model of a falling object under quadratic drag

$$mv' = -mg + bv^2.$$

Part 1: For what value of v will the change in velocity be 0?

Activity F.2.2 *(∼10 min)*
Consider our model of a falling object under quadratic drag

$$mv' = -mg + bv^2.$$

Part 1: For what value of v will the change in velocity be 0?
Part 2: Suppose the object is currently falling at a rate slower than this speed. What will happen?

(a) It will slow down
(b) It will keep falling at the same speed.
(c) It will speed up

Observation F.2.3

This equillibrium speed is called the **terminal velocity**.

Activity F.2.4 *(~5 min)*
Consider the following question:

A penny is dropped off the top of the Empire State Building. How fast will it be going when it hits the ground?

What information do we need to answer this question?

Observation F.2.5

The mass of a penny is $2.5\mathrm{g}$. The Empire State Building is (roughly) $400\mathrm{m}$ tall.

The terminal velocity of a penny is about $25\mathrm{m}/\mathrm{s}$.

Activity F.2.6 *($\sim$20 min)*

We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.

Activity F.2.6 *(∼20 min)*

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Part 1: Solve for b in terms of v_t, m, g, and substitute this in to our model $v' = -g + \frac{b}{m}v^2$.

Activity F.2.6 *(∼20 min)*

We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.

Part 1: Solve for b in terms of v_t, m, g, and substitute this in to our model $v' = -g + \frac{b}{m}v^2$.

Part 2: Solve this separable ODE

Hint: $\frac{1}{v_t^2 - v^2} = \frac{2}{v_t}\left(\frac{1}{v_t - v} + \frac{1}{v_t + v}\right)$

Activity F.2.6 *(∼20 min)*

We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.

Part 1: Solve for b in terms of v_t, m, g, and substitute this in to our model $v' = -g + \frac{b}{m}v^2$.

Part 2: Solve this separable ODE

Hint: $\frac{1}{v_t^2 - v^2} = \frac{2}{v_t}\left(\frac{1}{v_t - v} + \frac{1}{v_t + v}\right)$

Part 3: How fast is the penny going after 10 seconds?

Module F Section 3

Observation F.3.1

There are two very simple kinds of separable ODEs.

Equations of the form $y' = f(x)$ can be solved immediately by integrating and produce explicit solutions.

Equations of the form $y' = f(y)$ are often impossible or difficult to solve explicitly. They are called **autonomous** equations.

Activity F.3.2 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2.$$

Activity F.3.2 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2.$$

Part 1: Draw a slope field

Activity F.3.2 *(∼10 min)*
Consider the autonomous equation

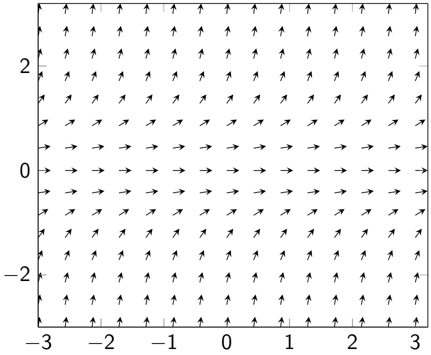
$$y' = y^2.$$

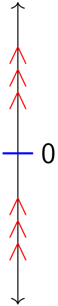
Part 1: Draw a slope field
Part 2: Suppose a solution goes through the point $y(10) = 50$. What can you say about $y(11)$?

(a) $y(10) < y(11)$
(b) $y(10) = y(11)$
(c) $y(10) > y(11)$

Observation F.3.3

Since the slopes do not change when moving horizontally (i.e. in the x direction), we often collapse the slope field onto the y-axis.





This is called a **phase line**.

Activity F.3.4 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2(y-2).$$

Activity F.3.4 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2(y - 2).$$

Part 1: Draw a number line for y', indicating where it is positive or negative.

Activity F.3.4 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2(y-2).$$

Part 1: Draw a number line for y', indicating where it is positive or negative.
Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?

Activity F.3.4 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2(y-2).$$

Part 1: Draw a number line for y', indicating where it is positive or negative.
Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?
Part 3: What can you say about the long term behavior of a solution passing through $y(2) = 0.001$?

Activity F.3.4 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2(y-2).$$

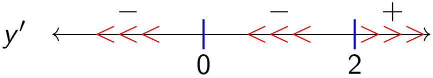
Part 1: Draw a number line for y', indicating where it is positive or negative.
Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?
Part 3: What can you say about the long term behavior of a solution passing through $y(2) = 0.001$?
Part 4: What can you say about the long term behavior of a solution passing through $y(2) = -0.001$?

Module F Section 4

Definition F.4.1

Recall from last week: the **phase line** is a useful way to visualize the long term behavior of an autonomous DE.

For example, here is a phase line for the autonomous DE $y' = y^2(y - 2)$.



Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Activity F.4.2 *(∼15 min)*
Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.
Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.
Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.
Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.

Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.

Part 5: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.

Part 5: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

Part 6: Describe the long term behavior of solutions passing near the point $y(11) = 2$.

Definition F.4.3
The **critical points** of an autonomous DE are the numbers that give rise to equillibrium solutions (e.g. $0, -1, 2$ in the previous problem).

A **source** is an unstable equillibrium in which all nearby trajectories move away in the limit.

A **sink** is a stable equillibrium in which all nearby trajectories approach the equillibrium in the limit.

There are also unstable equillibria in which some nearby trajectories return, while others diverge, analagous to a saddle point.

Activity F.4.4 *(∼15 min)*

Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

Activity F.4.4 *(∼15 min)*
Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

Part 1: Find and classify all of the critical points.

Activity F.4.4 *(∼15 min)*
Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

Part 1: Find and classify all of the critical points.
Part 2: Describe the long term behavior of solutions passing near the point $y(1) = 1.5$.

Activity F.4.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Activity F.4.5 *(∼15 min)*
Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Part 1: Find and classify all of the critical points.

Activity F.4.5 *(~15 min)*

Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Part 1: Find and classify all of the critical points.

Part 2: Describe the long term behavior of solutions passing near the point $y(0) = 0.5$.

Activity F.4.5 *(∼15 min)*
Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Part 1: Find and classify all of the critical points.
Part 2: Describe the long term behavior of solutions passing near the point $y(0) = 0.5$.
Part 3: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

Module F Section 5

Observation F.5.1

In module C, we solved **constant coefficient linear ODEs**.

Today we will observe that our existing techniques allow us to solve all **first order linear ODES**, i.e. ODEs of the form

$$a(x)y' + b(x)y + c(x) = 0.$$

Such equations can always be rewritten (by rearranging and dividing by $a(x)$) in **standard form**:

$$y' + P(x)y = Q(x).$$

Activity F.5.2 *(∼20 min)*
Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

Activity F.5.2 *(∼20 min)*
Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

Part 1: Solve the **homogeneous** first order linear ODE

$$y' + \frac{1}{x}y = 0.$$

Activity F.5.2 *(∼20 min)*
Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

Part 1: Solve the **homogeneous** first order linear ODE

$$y' + \frac{1}{x}y = 0.$$

Part 2: Use variation of parameters to the solve the original ODE

Activity F.5.3 *(∼15 min)*

Solve the first order linear ODE

$$\frac{1}{x}y' - \frac{2}{x^2}y - x\cos(x) = 0.$$

Activity F.5.4 *(∼15 min)*

Solve

$$(x+1)y' + y = x.$$

Remark F.5.5

The book provides a different technique (multiplying by an integrating factor); however, the method presented here does not require memorizing anything new.

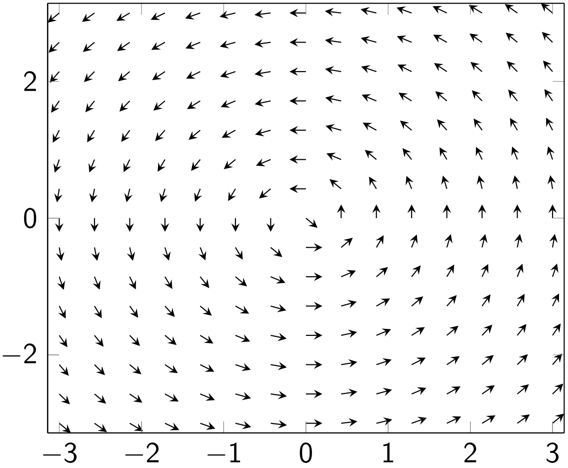
Module F Section 6

Observation F.6.1

A vector field $\langle P, Q\rangle$ corresponds to the slope field of the differential equation

$$\frac{dy}{dx} = \frac{Q}{P}.$$

Thus, a solution to this ODE describes the path taken by the particle in this fluid flow.



Activity F.6.2 *(∼10 min)*
Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2y^2 + x$.

Activity F.6.2 *(∼10 min)*
Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2y^2 + x$.
Part 1: Compute $\nabla\phi$.

Activity F.6.2 *($\sim$10 min)*
Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2 y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2 y \frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2 y^2 + x$.
Part 1: Compute $\nabla\phi$.
Part 2: Differentiate the equation $\phi(x, y) = c$ with respect to x.

Activity F.6.2 *(∼10 min)*
Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2y^2 + x$.
Part 1: Compute $\nabla\phi$.
Part 2: Differentiate the equation $\phi(x, y) = c$ with respect to x.
Part 3: Solve the ODE $(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0$.

Definition F.6.3

If $\langle M, N\rangle$ is a conservative vector field, the ODE

$$M + N\frac{dy}{dx} = 0$$

is called **exact**. This ODE can also be written

$$\frac{dy}{dx} = \frac{-M}{N}.$$

If $\phi(x, y)$ is a potential function of $\langle M, N\rangle$, the general solution to the ODE is $\phi(x, y) = c$.

Careful: The slope field of the ODE $\frac{dy}{dx} = \frac{-M}{N}$ is the vector field $\langle -N, M\rangle$!

Activity F.6.4 *(∼10 min)*

Determine which of the following ODEs are exact.

(a) $2xy + (x^2 - 2y)\frac{dy}{dx} = 0$

(b) $\frac{dy}{dx} = \frac{2xy}{x^2+2y}$

(c) $\frac{dy}{dx} = -\frac{2xy}{x^2+2y}$

Activity F.6.5 *(∼10 min)*
Solve the exact ODE $2xy + (x^2 - 2y)\frac{dy}{dx} = 0$.

Module F Section 7

Activity F.7.1 *(∼10 min)*

Determine which of the following ODEs are exact.

(a) $\frac{dy}{dx} = \frac{-y}{x^2+y^2+x}$

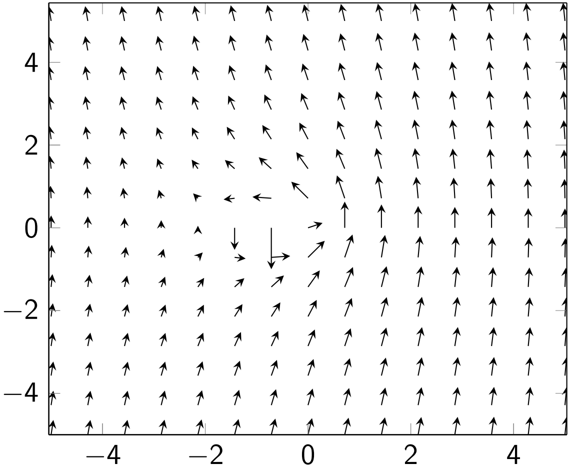
(b) $1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2}\right) \frac{dy}{dx} = 0$

Activity F.7.2 *(∼15 min)*

Solve the exact ODE

$$1 + \frac{x}{x^2 + y^2} + \left(\frac{y}{x^2 + y^2}\right)\frac{dy}{dx} = 0.$$

These solutions describe the trajectories taken by particles in the fluid flow below



Activity F.7.3 *(∼20 min)*

Find solutions for the ODE

$$1+\frac{x}{x^2+y^2}+\left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx}=0$$

for each of the following initial conditions

(a) $y(0)=-1$.

(b) $y(-2)=-2$.

(c) $y(-4)=-4$.

Plot each of the solution curves.