Module D: Discontinuous functions in ODEs

How can we solve and apply ODEs involving functions that are not continuous?

At the end of this module, students will be able to...

D1. **Laplace Transform.** ...compute the Laplace transform of a function

D2. **Discontinuous ODEs.** ...solve initial value problems for ODEs with discontinuous coefficients

D3. **Modeling non-smooth motion.** ...model the motion of an object undergoing discontinuous acceleration

D4. **Modeling non-smooth oscillators.** ...model mechanical oscillators undergoing discontinuous acceleration

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compute integrals by using integration by parts
- Evaluate improper integrals
- Use a partial fraction decomposition to rewrite a rational expression
- Model a mass-spring system (Standard C6)

The following resources will help you prepare for this module.

- Compute integrals by using integration by parts
  `https://youtu.be/S1Bp9hZBqaQ`, `https://youtu.be/bZ8YAHDTFJ8`
- Evaluate improper integrals `https://youtu.be/qv7DM5Ph0vU`
- Use a partial fraction decomposition to rewrite a rational expression
  `https://youtu.be/HZTv4zCgEnA`, `https://youtu.be/ofQt2N0UpCg`
- Model a mass-spring system (Standard C6)

Module D Section 1

Observation D.1.1

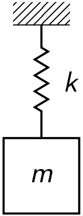
In this module, we want to learn how to model (and solve) situtations with **discontinuous** force, such as

- Collisions
- Thrust that can be turned on and off instantly
- Applied voltages that can be turned on and off instantly

Today we will learn how to model these forces, and introduce a tool called them **Laplace Transform** that we will use to solve the resulting IVPs.

Activity D.1.2 *(∼5 min)*

A 4 kg mass is hung from a spring with spring constant $k = 16\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting $3\mathrm{Ns}$ of upward impulse.



Write an initial value problem modelling this system.

Definition D.1.3
The **Dirac delta distribution** $\delta(t)$ models the application of instantaneous force. **It is not a function**, but makes sense in definite integrals:

If a, b is any open interval containing 0, then

$$\int_a^b f(t)\delta(t)dt = f(0)$$

for any function $f(t)$ that is continuous around 0.

Thus, we can model the situation in the previous activity by

$$4y'' + 16y = 3\delta(t), \qquad y(0) = 0,\ y'(0) = 0$$

Definition D.1.4

We can make sense of δ in another way: as the "derivative" of a non-differentiable function.

The **unit impulse function** $u(t)$ is given by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0. \end{cases}$$

Note that $u(s) = \int_{-\infty}^{s} \delta(t)\ dt$; in this fuzzy sense, δ is the "derivative" of $u(t)$ (which is not differentiable everywhere!)

Activity D.1.5 *(∼10 min)*
Try to solve the IVP

$$y'' + y = \delta(t)$$

Where does our existing technique break down?

Observation D.1.6

To get around this difficulty, we will apply an **integral transform** called the **Laplace Transform** to our ODE.

- We want to use a definite integral to handle things like δ, which we can only understand via a definite integral.
- Since we are focused on IVPs, we can integrate starting at 0, but need to go to ∞
- But now we need to worry about convergence–thus we will multiply by a suitable function that decays fast enough to make most functions converge.

Activity D.1.7 *($\sim$5 min)*

Arrange the following functions in order of how fast they decay to zero in the limit at infinity:

(A) x^{-n} for a positive integer n

(B) e^{-ax} for a positive integer a

(C) $\frac{1}{\ln(ax)}$ for a positive integer a

(D) $\frac{1}{\ln(x^n)}$ for a positive integer n

Definition D.1.8
The **Laplace Transform** of a function $f(t)$ is the function

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

Note that the Laplace transform turns a function of t into a function of s.

Moreover, $\mathcal{L}$ is linear: $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$, and $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for constants c.

Activity D.1.9 *(∼5 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Activity D.1.9 *(∼5 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

Part 1: Compute $\mathcal{L}\{\delta(t)\}$

Activity D.1.9 *(∼5 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{\delta(t)\}$

Part 2: If $a > 0$, compute $\mathcal{L}\{\delta(t-a)\}$

Activity D.1.10 *(∼5 min)*
Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Activity D.1.10 *(∼5 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{e^t\}$

Activity D.1.10 *(∼5 min)*
Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{e^t\}$
Part 2: If $a > 0$, compute $\mathcal{L}\{e^{at}\}$

Module D Section 2

Observation D.2.1

Last week, we encountered the **Laplace Transform** of a function $f(t)$:

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

Recall that the Laplace transform turns a function of t into a function of s.

Moreover, $\mathcal{L}$ is linear: $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$, and $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for constants c.

Our goal for today is to develop a few more properties of $\mathcal{L}$, and then see how to use it to solve IVPs (standards D1,D2).

Observation D.2.2

We computed a few Laplace Transforms:

- $\mathcal{L}\{\delta(t)\} = 1$
- $\mathcal{L}\{\delta(t-a)\} = e^{-as}$ for any $a > 0$.
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

Activity D.2.3 *(∼15 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Activity D.2.3 *(∼15 min)*
Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$

Activity D.2.3 *(∼15 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

Part 1: Compute $\mathcal{L}\{1\}$

Part 2: Compute $\mathcal{L}\{t\}$

Activity D.2.3 *(∼15 min)*
Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$
Part 2: Compute $\mathcal{L}\{t\}$
Part 3: Compute $\mathcal{L}\{t^2\}$

Activity D.2.4 *(∼10 min)*

Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt.$$

Activity D.2.4 *(∼10 min)*

Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}\{\sin(t)\}$ (**Hint:** $\int e^(ax) \sin(bx)\ dx = \frac{1}{a^2+b^2} e^{ax} (a \sin(bx) - b \cos(bx)) + C$.)

Activity D.2.4 *(∼10 min)*
Recall
$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt.$$

Part 1: Compute $\mathcal{L}\{\sin(t)\}$ (**Hint:** $\int e^(ax) \sin(bx)\ dx = \frac{1}{a^2+b^2} e^{ax} \left(a \sin(bx) - b \cos(bx)\right) + C$.)
Part 2: Compute $\mathcal{L}\{\cos(t)\}$.

Observation D.2.5

So now our list of Laplace transforms is:

- $\mathcal{L}\{\delta(t-a)\} = e^{-as}$ for any $a > 0$.
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ for any positive integer n.
- $\mathcal{L}\{1\} = \frac{1}{s}$
- $\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$
- $\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$

Activity D.2.6 *(∼10 min)*
Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)dt.$$

Activity D.2.6 *(∼10 min)*

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)dt.$$

Part 1: Use integration by parts to relate $\mathcal{L}\{y'\}$ to $\mathcal{L}\{y\}$.

Activity D.2.6 *(∼10 min)*

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt.$$

Part 1: Use integration by parts to relate $\mathcal{L}\{y'\}$ to $\mathcal{L}\{y\}$.
Part 2: Use the fact that $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$ to relate $\mathcal{L}\{y''\}$ to $\mathcal{L}\{y\}$.

Observation D.2.7

We have

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$
$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

This allows us to easily rewrite expressions like $ay'' + by' + cy$ in terms of $\mathcal{L}\{y\}$.

Activity D.2.8 *(∼15 min)*
Consider the simple IVP

$$y'' + y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$

Activity D.2.8 *(∼15 min)*
Consider the simple IVP

$$y'' + y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$

Part 1: Apply the Laplace transform to this IVP, and simplify. Solve for $\mathcal{L}\{y\}$.

Activity D.2.8 *(∼15 min)*
Consider the simple IVP

$$y'' + y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$

Part 1: Apply the Laplace transform to this IVP, and simplify. Solve for $\mathcal{L}\{y\}$.
Part 2: Find a function y satisfying $\mathcal{L}\{y\} = \frac{1}{s^2+1}$. We write $y = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$.

Module D Section 3

Observation D.3.1

To solve a linear IVP using Laplace transforms:

1) Apply $\mathcal{L}$ to the ODE. Use the initial condition(s) in computing $\mathcal{L}\{y'\}$, $\mathcal{L}\{y''\}$, etc.
2) Solve for $\mathcal{L}\{y\}$.
3) Take the inverse transform (using a table) to find the solution y.

Today our goal is to practice the last step (taking the inverse transform), and then practice solving IVPs this way.

Activity D.3.2 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$.

(a) $u(t-5)$

(b) $\delta(t-5)$

(c) e^{5t}

(d) e^{-5t}

Activity D.3.3 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s}\right\}$.

(a) $u(t-10)$

(b) $\delta(t-10)$

(c) $u(t-10)e^{-t}$

(d) $\delta(t-10)e^{-t}$

Activity D.3.4 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2+4}\right\}$.

(a) $u(t)\sin(2t)$

(b) $u(t-2)\sin(2t)$

(c) $u(t-2)\sin(2t-2)$

(d) $u(t-2)\sin(2t-4)$

Activity D.3.5 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left\{\frac{e^{-100s}}{s^2}\right\}$.

(a) $u(t)t$

(b) $u(t)(t-100)$

(c) $u(t-100)t$

(d) $u(t-100)(t-100)$

Activity D.3.6 *(∼10 min)*
Solve the IVP

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad v(0) = 50.$$

Activity D.3.7 *(∼10 min)*
Solve the IVP

$$y'' + 4y = \delta(t-2), \qquad y(0) = 0,\ y'(0) = 1$$

Activity D.3.8 *(∼15 min)*
Solve the IVP

$$y'' + 5y' + 6y = \delta(t), \qquad y(0) = 0,\ y'(0) = 1$$

Module D Section 4

Observation D.4.1

Today we will practice modeling an object undergoing discontinuous acceleration (standard D3).

Activity D.4.2 *(∼30 min)*

A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Activity D.4.2 *($\sim$30 min)*
A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.
Part 1: Write down a function modelling the thrust force on the spacecraft.

Activity D.4.2 *(∼30 min)*

A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

Part 2: Write down an IVP modelling the velocity of the spacecraft.

Activity D.4.2 *(∼30 min)*
A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.
Part 1: Write down a function modelling the thrust force on the spacecraft.
Part 2: Write down an IVP modelling the velocity of the spacecraft.
Part 3: Solve the IVP:

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad v(0) = 50.$$

Activity D.4.2 *(∼30 min)*
A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.
Part 1: Write down a function modelling the thrust force on the spacecraft.
Part 2: Write down an IVP modelling the velocity of the spacecraft.
Part 3: Solve the IVP:

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad v(0) = 50.$$

Part 4: What is its velocity after 200 s?

Module D Section 5

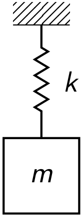
Observation D.5.1

Last week we saw how to use the Laplace transform to model a spacecraft undergoing discontinuous acceleration.

Today we will model springs undergoing discontinuous acceleration (standard D4).

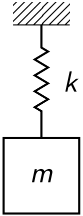
Activity D.5.2 *(∼10 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting $3\mathrm{Ns}$ of upward impulse.



Activity D.5.2 *(∼10 min)*

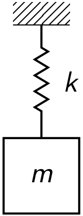
A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting $3\mathrm{Ns}$ of upward impulse.



Part 1: Write an initial value problem modelling this system.

Activity D.5.2 *(∼10 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting $3\mathrm{Ns}$ of upward impulse.

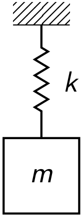


Part 1: Write an initial value problem modelling this system.

Part 2: Use the Laplace transform to solve this IVP.

Activity D.5.2 *(∼10 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting $3\mathrm{Ns}$ of upward impulse.



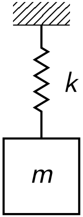
Part 1: Write an initial value problem modelling this system.

Part 2: Use the Laplace transform to solve this IVP.

Part 3: When will the mass first return to equillibrium?

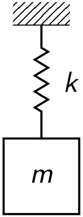
Activity D.5.3 *(∼15 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$. The mass is pulled down $1\ \mathrm{m}$ and released from rest. 10 seconds later, it is hit with a hammer imparting $5\mathrm{Ns}$ of upward impulse.



Activity D.5.3 *(∼15 min)*

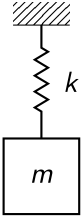
A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$. The mass is pulled down $1\ \mathrm{m}$ and released from rest. 10 seconds later, it is hit with a hammer imparting $5\mathrm{Ns}$ of upward impulse.



Part 1: Write an initial value problem modelling this system.

Activity D.5.3 *(∼15 min)*

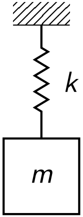
A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$. The mass is pulled down $1\ \mathrm{m}$ and released from rest. 10 seconds later, it is hit with a hammer imparting $5\mathrm{Ns}$ of upward impulse.



Part 1: Write an initial value problem modelling this system.

Part 2: Use the Laplace transform to solve this IVP.

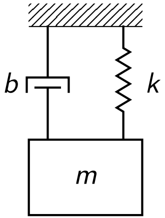
Activity D.5.3 *(∼15 min)*
A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 4\ \mathrm{N}/\mathrm{m}$. The mass is pulled down $1\ \mathrm{m}$ and released from rest. 10 seconds later, it is hit with a hammer imparting $5\mathrm{Ns}$ of upward impulse.



Part 1: Write an initial value problem modelling this system.
Part 2: Use the Laplace transform to solve this IVP.
Part 3: Where is the mass after $15\ \mathrm{s}$?

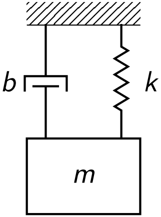
Activity D.5.4 *(~15 min)*

A 1 kg mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$ and a $b = 4\ \mathrm{kg/s^2}$ linear damper. The mass is pulled down 1 m and released from rest. 5 seconds later, it is hit with a hammer imparting $10\mathrm{Ns}$ of upward impulse.



Activity D.5.4 *(∼15 min)*

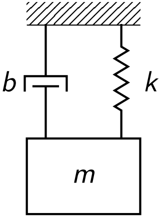
A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$ and a $b = 4\ \mathrm{kg/s^2}$ linear damper. The mass is pulled down $1\ \mathrm{m}$ and released from rest. 5 seconds later, it is hit with a hammer imparting $10\mathrm{Ns}$ of upward impulse.



Part 1: Write an initial value problem modelling this system.

Activity D.5.4 *(∼15 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$ and a $b = 4\ \mathrm{kg/s^2}$ linear damper. The mass is pulled down $1\ \mathrm{m}$ and released from rest. 5 seconds later, it is hit with a hammer imparting $10\mathrm{Ns}$ of upward impulse.

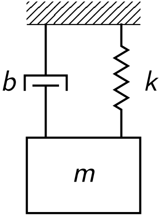


Part 1: Write an initial value problem modelling this system.

Part 2: Use the Laplace transform to solve this IVP.

Activity D.5.4 *(~15 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$ and a $b = 4\ \mathrm{kg/s^2}$ linear damper. The mass is pulled down $1\ \mathrm{m}$ and released from rest. 5 seconds later, it is hit with a hammer imparting $10\mathrm{Ns}$ of upward impulse.



Part 1: Write an initial value problem modelling this system.

Part 2: Use the Laplace transform to solve this IVP.

Part 3: Where is the mass after 15 s?

Module D Section 6