

Module C: Constant coefficient linear ODEs

Standards for this Module

How can we solve and apply linear constant coefficient ODEs? At the end of this module, students will be able to...

- C1. Constant coefficient first order.** ...find the general solution to a first order constant coefficient ODE.
- C2. Modeling motion in viscous fluids.** ...model the motion of a falling object with linear drag
- C3. Homogeneous constant coefficient second order.** ...find the general solution to a homogeneous second order constant coefficient ODE.
- C4. IVPs.** ...solve initial value problems for constant coefficient ODEs
- C5. Non-homogenous constant coefficient second order.** ...find the general solution to a non-homogeneous second order constant coefficient ODE
- C6. Modeling oscillators.** ...model (free or forced, damped or undamped) mechanical oscillators with a second order ODE

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Describe Newton's laws in terms of differential equations.
- Find all roots of a quadratic polynomial.
- Use Euler's theorem to relate $\sin(t)$, $\cos(t)$, and e^t .
- Use Euler's theorem to simplify complex exponentials.
- Use substitution to compute indefinite integrals.
- Use integration by parts to compute indefinite integrals.
- Solve systems of two linear equations in two variables.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Describe Newton's laws in terms of differential equations. <https://youtu.be/cioi4lRrAzw>
- Find all roots of a quadratic polynomial. <https://youtu.be/2ZzuZvz33X0> <https://youtu.be/TV5kDqiJ10s>
- Use Euler's theorem to relate $\sin(t)$, $\cos(t)$, and e^t and to simplify complex exponentials. https://youtu.be/F_0yfvmOUoU <https://youtu.be/sn3orkHWqUQ>
- Use substitution to compute indefinite integrals. <https://youtu.be/b76wePnIBdU>
- Use integration by parts to compute indefinite integrals. <https://youtu.be/bZ8YAHDTFJ8>
- Solve systems of two linear equations in two variables. <https://youtu.be/Y6JsEja15Vk>

Section C.1

Activity C.1.1 (~ 5 min) Why don't clouds fall out of the sky?



- (a) They are lighter than air
- (b) Wind keeps them from falling
- (c) Electrostatic charge
- (d) They do fall, just very slowly

Activity C.1.2 (~ 5 min) List all of the forces acting on a tiny droplet of water falling from the sky.

Activity C.1.3 (~ 5 min) Tiny droplets of water obey **Stoke's law**, which says that air resistance is proportional to (the magnitude of) velocity.

- Let v be the velocity of a droplet of water (positive for upward, negative for downward).
- Let $g > 0$ be the magnitude of acceleration due to gravity and $b > 0$ be another positive constant.

Apply Newton's second law (force = mass \times acceleration) to determine which of the following **ordinary differential equations (ODEs)** models the velocity of a falling droplet of water.

- (a) $v' = g - v$
- (b) $v' = g + v$
- (c) $mv' = -mg - bv$
- (d) $mv' = -mg + bv$

Observation C.1.4 The modeling equation

$$mv' = -mg - bv$$

may be obtained by splitting the total force into gravity and air resistance:

$$F = F_g + F_r$$

Then $F = ma = mv'$ and $F_g = m(-g) = -mg$ are the result of Newton's second law, and $F_r = -bv$ holds because it should be (a) in the opposite direction of velocity and (b) a constant multiple of velocity. Note that this equation may be rearranged as follows to group v and its derivative v' together on the left-hand side:

$$v' + \left(\frac{b}{m}\right)v = -g$$

Definition C.1.5 A **first order constant coefficient** differential equation can be written in the form

$$y' + by = f(x),$$

or equivalently,

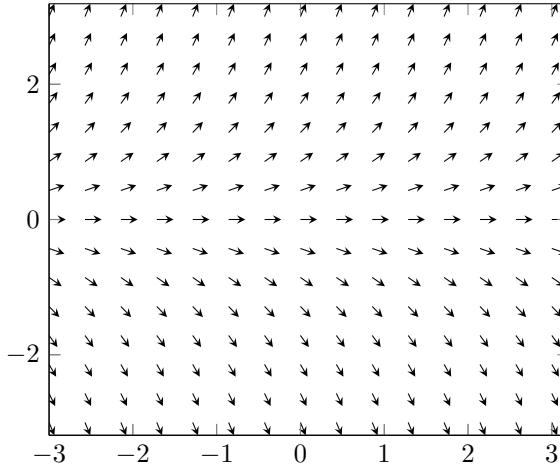
$$\frac{dy}{dx} + by = f(x).$$

We will use both notations interchangeably.

Here, **first order** refers to the fact that the highest derivative we see is the first derivative of y .

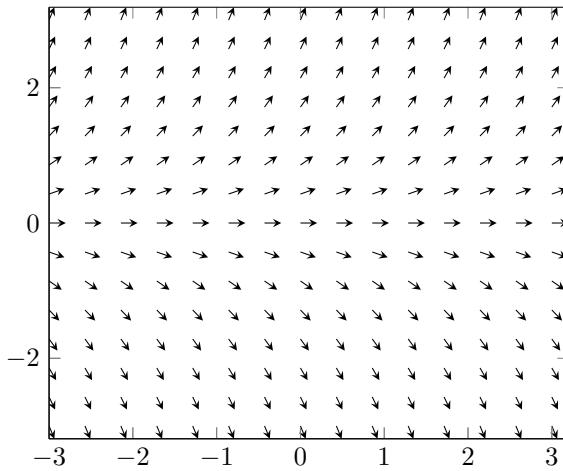
Observation C.1.6 Consider the differential equation $y' = y$.

A useful way to visualize a first order differential equation is by a **slope field**



Each arrow represents the slope of a solution **trajectory** through that point.

Activity C.1.7 (~ 5 min) Consider the differential equation $y' = y$ with slope field below.



Part 1: Draw a trajectory through the point $(0, 1)$.

Part 2: Draw a trajectory through the point $(-1, -1)$.

Activity C.1.8 (~ 15 min) Consider the differential equation $y' = y$.

Part 1: Find a solution to $y' = y$.

Part 2: Modify this solution to write an expression describing **all** solutions to $y' = y$.

Definition C.1.9 A differential equation will have many solutions. Each individual solution is said to be a **particular solution**, while the **general solution** encompasses **all** of these by using parameters such as C, k, c_0, c_1 and so on. For example:

- The general solution to the differential equation $y' = 2x - 3$ is $y = x^2 - 3x + C$ (as done in Calculus courses).
- The general solution for $y' = y$ is $y = ke^x$ (as done in the previous activity).

Activity C.1.10 (~ 15 min) Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.

Part 1: Solve $y' = 2y$.

Part 2: Solve $y' = y + 2$.

Section C.2

Observation C.2.1 Recall the last activity from yesterday:

Solve $y' = y + 2$

This is very similar to the equation $y' = y$, which we were able to solve.

Activity C.2.2 ($\sim 15 \text{ min}$) Solve $y' = y + 2$

Simple idea: Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

Part 1: Use the Product Rule to find $y'_p = \frac{d}{dx}[ve^x]$.

Part 2: Substitute y_p and y'_p into the equation $y' = y + 2$.

Part 3: Solve for v .

Part 4: Find y_p .

Observation C.2.3 The technique outlined in the previous activity is called **variation of parameters**. If y_0 is a particular solution of the **homogeneous** equation, assume that a particular solution of the **non-homogeneous** equation has the form $y_p = vy_0$, and then determine what v must be.

Example:

$$\begin{array}{ll} y' + 3y = 0 & \text{homogeneous} \\ y' + 3y = x & \text{non-homogeneous} \end{array}$$

Note that each term of the homogeneous equation includes y or its derivatives.

Activity C.2.4 ($\sim 20 \text{ min}$) Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$\begin{array}{ll} y' + 3y = 0 & \text{homogeneous} \\ y' + 3y = x & \text{non-homogeneous} \end{array}$$

Part 1: Modify e^x to find the general solution y_h for the homogeneous equation.

Part 2: Choose a particular solution y_0 for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** v . Substitute y_p into non-homogeneous equation and simplify.

Part 3: Determine v , and then determine y_p .

Observation C.2.5 Since $y_h = ke^{-3x}$ was the general solution of $y' + 3y = 0$, and $y_p = \frac{x}{3} - \frac{1}{9}$ is a particular solution of $y' + 3y = x$,

$$y = y_h + y_p = \left(ke^{-3x}\right) + \left(\frac{x}{3} - \frac{1}{9}\right)$$

is a solution to $y' + 3y = x$:

$$\frac{d}{dx}[y_h + y_p] + 3(y_h + y_p) = (y'_h + 3y_h) + (y'_p + 3y_p) = 0 + x = x$$

Fact C.2.6 Let a be a constant real number. Every constant coefficient first order ODE

$$y' + ay = f(x)$$

has the general solution

$$y = y_h + y_p$$

where y_h is the general solution to the homogeneous equation $y' + ay = 0$ and y_p is a particular solution to $y' + ay = f(t)$.

Activity C.2.7 (~ 15 min) Find the general solution to $y' = 2y + x + 1$ using variation of parameters:

- Write the homogeneous equation and find its general solution y_h .
- Use a particular solution y_0 for the homogeneous equation to find a particular solution $y_p = vy_0$ for the original equation.
- Then $y = y_h + y_p$ gives the general solution to the equation.

Section C.3

Observation C.3.1 Recall that we can model the velocity of a water droplet in a cloud by

$$mv' = -mg - bv$$

where negative numbers represent downward motion, $m > 0$ is the mass of the droplet, $g > 0$ is the magnitude of acceleration due to gravity, and $b > 0$ is the proportion of wind resistance to speed.



Activity C.3.2 (~ 20 min) A water droplet with a radius of $10 \mu\text{m}$ has a mass of about $4 \times 10^{-15} \text{ kg}$. It is determined in a laboratory that for a droplet this size, the constant b has a value of $3 \times 10^{-3} \text{ kg/s}$, and it is known that g is approximately 9.8 m/s^2 .

Complete the following tasks to study the motion of this droplet.

Part 1: Rewrite $mv' = -mg - bv$ in the form of $v' + av = ?$ for some value of a .

Part 2: Find the general solution of this ODE in terms of a and g . (Let $v_p = wv_0$ when using variation of parameters to avoid confusion.)

Part 3: Due to wind resistance, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of a and g ?

Part 4: If the droplet starts from rest ($v = 0$ when $t = 0$), what is its velocity after 0.01 s ? Use a calculator to compute the answer in m/s.

Definition C.3.3 The last part of the previous activity is an example of an **Initial Value Problem (IVP)**; we were given the initial value of the velocity in addition to our differential equation.

Physical scenarios often produce IVPs with a unique solution.

Activity C.3.4 (~ 10 min) Solve the IVP

$$y' + 3y = 0, \quad y(0) = 2.$$

Activity C.3.5 (~ 10 min) Solve the IVP

$$y' - 2y = 2, \quad y(0) = 1.$$

Activity C.3.6 ($\sim 5 \text{ min}$) Solve the IVP

$$y' - 2y = 2, \quad y(2) = 1.$$

Section C.4

Observation C.4.1 What happens when your tire hits a pothole?

Activity C.4.2 ($\sim 5 \text{ min}$) More abstractly, let's attach a mass (weighing m kg) to a spring.



List all forces acting on the mass.

Activity C.4.3 ($\sim 5 \text{ min}$) **Hooke's law** says that the force exerted by the spring is proportional to the distance the spring is stretched.



Write a differential equation modeling the displacement of the mass.

Observation C.4.4 There is an equilibrium point where the force of gravity balances the spring force. If we measure displacement from this point, we can model the mass-spring system by

$$my'' = -ky.$$



Activity C.4.5 ($\sim 15 \text{ min}$) Consider the (numerically simplified) mass-spring equation

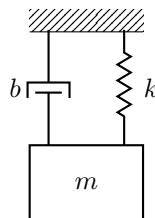
$$y'' = -y.$$

Part 1: Find a solution.

Part 2: Find the general solution.

Part 3: Describe the long term behavior of the mass-spring system.

Activity C.4.6 ($\sim 5 \text{ min}$) In applications, this infinitely oscillating behavior is often inappropriate. Thus, a damper (dashpot) is often incorporated. This provides a force proportional to the velocity.



Write a differential equation modeling the displacement of a mass in a **damped** mass-spring system.

Observation C.4.7 The damped mass-spring system can be modelled by

$$my'' = -by' - ky.$$

Here m is the mass, k is the spring constant, and b is the damping constant. We can rearrange this as

$$my'' + by' + ky = 0.$$

This is a **homogeneous second order constant coefficient** differential equation. Here, **homogeneous** refers to the 0 on the right hand side of the equation.

Activity C.4.8 ($\sim 15 \text{ min}$) Consider the second order constant coefficient equation

$$y'' = y.$$

Part 1: Find a solution.

Part 2: Find the general solution.

Part 3: Describe the long term behavior of the solutions.

Section C.5

Observation C.5.1 It is sometimes useful to think in terms of **differential operators**.

- We will use D to represent a derivative; another common notation is $\frac{\partial}{\partial x}$. So for any function y ,

$$D(y) = \frac{\partial y}{\partial x} = y'.$$

- D^2 will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use I for the identity operator; it does nothing to a function. That is, $I(y) = y$. It can be thought of as $I = D^0$ (i.e. differentiate zero times).

In this language, the differential equation $y' + 3y = 0$ can be rewritten as $D(y) + 3I(y) = 0$, or $(D + 3I)(y) = 0$. Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator $D + 3I$.

Activity C.5.2 (~ 5 min) What is the kernel of $D - I$?

Part 1: Write a differential equation that corresponds to this question.

Part 2: Find the general solution of this differential equation.

Activity C.5.3 (~ 5 min) Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

Activity C.5.4 (~ 10 min) Consider the ODE

$$y'' + 5y' + 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

Part 3: Find the general solution of the ODE.

Observation C.5.5 If we let $\mathcal{L} = D^2 + 5D + 6I$, we can write the ODE

$$y'' + 5y' + 6y = 0$$

as

$$\mathcal{L}(y) = 0.$$

Note that such an \mathcal{L} is always a **linear transformation**.

Activity C.5.6 (~ 5 min) Find the general solution to

$$y'' + y' - 12y = 0.$$

Section C.6

Activity C.6.1 (~ 5 min) Consider the ODE

$$y'' + 5y' - 6y = 0.$$

Part 1: Find a differential operator whose kernel is the solution set of the above ODE.

Part 2: Factor this differential operator as a composition of two operators. (This works because D and I commute).

Part 3: Find the general solution of the ODE.

Activity C.6.2 (~ 5 min) Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

Activity C.6.3 (~ 5 min) An **Initial Value Problem (IVP)** consists of an ODE along with some initial conditions that allow you to determine a single solution.

Solve the IVP

$$2y'' + 7y' + 6y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

.

Activity C.6.4 (~ 5 min) Solve the ODE

$$y'' + y = 0.$$

Activity C.6.5 (~ 15 min) Consider the ODE

$$y'' + 2y' + 5y = 0$$

Part 1: Find the general solution.

Part 2: Describe the long-term behavior of the solutions.

Observation C.6.6 Solving $y'' + 2y' + 5y = 0$ produced a general solution

$$y = c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t}.$$

Applying Euler's formula yields

$$\begin{aligned} y &= c_1 e^{-t} (\cos(2t) + i \sin(2t)) + c_2 e^{-t} (\cos(2t) - i \sin(2t)) \\ &= (c_1 + c_2) e^{-t} \cos(2t) + i(c_1 - c_2) e^{-t} \sin(2t) \end{aligned}$$

which we can rewrite (letting $k_1 = c_1 + c_2$ and $k_2 = i(c_1 - c_2)$) as

$$y = k_1 e^{-t} \cos(2t) + k_2 e^{-t} \sin(2t).$$

Activity C.6.7 (~ 15 min) Solve the IVP

$$y'' + 6y' + 34y = 0, \quad y(0) = 2, \quad y'(0) = 4.$$

Section C.7

Activity C.7.1 ($\sim 10 \text{ min}$) Solve the ODE

$$y'' - 4y' + 4y = 0.$$

Observation C.7.2 To solve this, we need to find the kernel of $(D - 2I)(D - 2I)$.

- The kernel of $D - 2I$ is $\{ce^{2t} \mid c \in \mathbb{R}\}$.
- However, if $(D - 2I)(y) = Ae^{2t}$, then applying $D - 2I$ twice will yield zero!
- So we must solve the ODE

$$y' - 2y = e^{2t}.$$

Activity C.7.3 ($\sim 15 \text{ min}$) Solve $y' - 2y = e^{2t}$.

Observation C.7.4 Thus, we have shown that the general solution of

$$y'' - 4y' + 4y = 0$$

is

$$y = c_0e^{2t} + c_1te^{2t}.$$

Activity C.7.5 ($\sim 15 \text{ min}$) Solve $y'' - 6y' + 9y = 0$.

Activity C.7.6 ($\sim 10 \text{ min}$) Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

If r is a number such that $ar^2 + br + c = 0$, what can you conclude?

- e^{rt} is a solution.
- e^{-rt} is a solution.
- te^{rt} is a solution.
- There are no solutions.

Activity C.7.7 ($\sim 5 \text{ min}$) Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

When does the general solution have the form $c_0 e^{rt} + t e^{rt}$?

- (a) When the polynomial $ax^2 + bx + c$ has two distinct real roots.
- (b) When the polynomial $ax^2 + bx + c$ has a repeated real root.
- (c) When the polynomial $ax^2 + bx + c$ has two distinct non-real roots.
- (d) When the polynomial $ax^2 + bx + c$ has a repeated non-real root.

Observation C.7.8 Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ax^2 + bx + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that $t e^{rt}$ is also a solution.
- If r is not real, Euler's formula allows us to express the complex exponential part of the solution in terms of $\sin(rt)$ and $\cos(rt)$.

Section C.8

Observation C.8.1 Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If r is a root of $ax^2 + bx + c = 0$, then e^{rt} is a solution of the ODE.
- If r is a double root, variation of parameters shows that te^{rt} is also a solution.
- If $r = a + bi$ is not real, Euler's formula allows us to express the complex exponential part of the solution in terms of e^{at} , $\sin(bt)$, and $\cos(bt)$.

Activity C.8.2 ($\sim 15 \text{ min}$) Consider the following scenario: a mass of 4 kg suspended from a damped spring with spring constant $k = 2 \text{ kg/s}^2$ and damping constant $b = 6 \text{ kg/s}$.

The mass is pulled down 0.3 m and released from rest.

Part 1: Write down an ODE modelling this scenario, and find the general solution.

Part 2: Use the initial conditions $y(0) = -0.3$ and $y'(0) = 0$ to find particular values of the constants.

Definition C.8.3 In the previous problem, we needed to solve

$$4y'' + 6y' + 2y = 0, \quad y(0) = -0.3, \quad y'(0) = 0.$$

This is called an **Initial Value Problem (IVP)** since we are provided with initial values of y and y' .

To solve an IVP, find a general solution of the ODE, and use the initial conditions to find the values of the constants.

Activity C.8.4 ($\sim 15 \text{ min}$) Consider a mass of 5 kg suspended from a damped spring with spring constant $k = 2 \text{ kg/s}^2$ and damping constant $b = 6 \text{ kg/s}$.

The mass is pulled down 0.3m and released from rest. How many times does it pass back through its equilibrium state?

- (a) 0
- (b) 1
- (c) 2
- (d) Infinitely many

Observation C.8.5 It can be shown that in the **overdamped** situation, the spring might pass through the equilibrium position once (e.g. if given an initial push), but never more than once.

Section C.9

Activity C.9.1 ($\sim 10 \text{ min}$) A 1 kg mass is suspended from a spring with spring constant $k = 9 \text{ kg/s}^2$. An external force is applied by an electromagnet and is modeled by the function $F(t) = \sin(t)$. Write an ODE modeling the displacement of the spring.

Observation C.9.2 In the previous activity, we encountered a **nonhomogeneous** second order constant coefficient ODE, i.e. of the form

$$ay'' + by' + cy = f(x)$$

where a, b, c are constants, and $f(x)$ is a function.

We will again use **variation of parameters** to find a particular solution.

Activity C.9.3 ($\sim 15 \text{ min}$) Suppose y_1 and y_2 are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1y_1 + v_2y_2$ for some TBD functions v_1, v_2 .

Part 1: Use the product rule (twice) to compute y'_p .

Part 2: To simplify calculations, we will **assume** $v'_1y_1 + v'_2y_2 = 0$. Assuming this, compute y''_p .

Part 3: Compute $\mathcal{L}(y_p)$; simplify the ODE $\mathcal{L}(y_p) = f(x)$.

Observation C.9.4 If we can find v_1 and v_2 that satisfy

$$\begin{aligned} y_1v'_1 + y_2v'_2 &= 0 \\ y'_1v'_1 + y'_2v'_2 &= \frac{f}{a} \end{aligned}$$

then we have a solution. So we just need to solve this system of equations for v'_1 and v'_2 .

Activity C.9.5 ($\sim 15 \text{ min}$) Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

Part 1: Find y_1 and y_2 , two independent solutions of $y'' + 9y = 0$.

Part 2: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v'_1 + \sin(3t)v'_2 &= 0 \\ -3\sin(3t)v'_1 + 3\cos(3t)v'_2 &= \sin(t) \end{aligned}$$

Part 3: Write the general solution of the original nonhomogeneous ODE.

Activity C.9.6 ($\sim 10 \text{ min}$) Consider the nonhomogeneous ODE $y'' + 9y = \sin(3t)$.

Part 1: Find v_1 and v_2 by solving

$$\begin{aligned} \cos(3t)v'_1 + \sin(3t)v'_2 &= 0 \\ -3\sin(3t)v'_1 + 3\cos(3t)v'_2 &= \sin(3t) \end{aligned}$$

Part 2: Write the general solution of the original nonhomogeneous ODE.

Module F: First order ODEs

Standards for this Module

How can we solve and apply first order ODEs? At the end of this module, students will be able to...

F1. Sketching trajectories. ...given a slope field, sketch a trajectory of a solution to a first order ODE

F2. Separable ODEs. ...find the general solution to a separable first order ODE

F3. Modeling motion. ...model the motion of an object with quadratic drag

F4. Autonomous ODEs. ...find and classify the equilibria of an autonomous first order ODE, and describe the long term behavior of solutions

F5. First order linear ODEs. ...find the general solution to a first order linear ODE

F6. Exact ODES. ...find the general solution to an exact first order ODE

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine the intervals on which a polynomial is positive, negative, or zero.
- Use integration techniques like substitution, integration by parts, and partial fraction decomposition to compute indefinite integrals.
- Determine when a vector field is conservative.
- Find the potential function of a conservative vector field.
- Use variation of parameters to solve non-homogeneous first order constant coefficient ODEs (Standard C1)

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Determine the intervals on which a polynomial is positive, negative, or zero. <https://youtu.be/jGa0GJjwQh8>
- Use integration techniques like substitution, integration by parts, and partial fraction decomposition to compute indefinite integrals. <https://youtu.be/b76wePnIBdU> <https://youtu.be/bZ8YAHDTFJ8> <https://youtu.be/qMX4vRhXB0E>
- Determine when a vector field is conservative. <https://youtu.be/gAb1ZTD41wo>
- Find the potential function of a conservative vector field. https://youtu.be/nY4mW_R-T40
- Use variation of parameters to solve non-homogeneous ODEs when given the solution to the corresponding homogeneous ODE (Standard C5)

Section F.1

Definition F.1.1 A **first order ODE** is an equation involving (for a function $y(x)$) only y' , y , and x . We will most often deal with **explicit first order ODEs**, which can be written in the form

$$y' = f(y, x)$$

for some function $f(y, x)$.

Activity F.1.2 ($\sim 5 \text{ min}$) Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

Part 1: Compute y' at each of the points $(1, 1)$, $(2, 1)$, $(3, -2)$, and $(4, -7)$.

Part 2:

Let $y_0(x)$ be a solution that passes through the point $(1, 1)$. What can you conclude about $\lim_{x \rightarrow \infty} y_0(x)$?

- (A) $\lim_{x \rightarrow \infty} y_0(x) = -\infty$
- (B) $\lim_{x \rightarrow \infty} y_0(x)$ is a finite number
- (C) $\lim_{x \rightarrow \infty} y_0(x) = \infty$

Definition F.1.3 These kinds of questions are easier to answer if we draw a **slope field** (sometimes called a **direction field**).

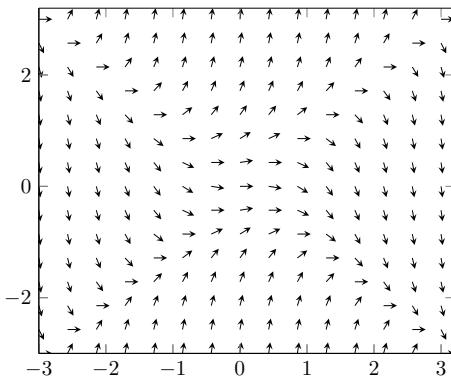
To draw one, draw a small line segment or arrow with the correct slope at each point.

$$y' = y^2 - x^2$$



Activity F.1.4 ($\sim 5 \text{ min}$)

$$y' = y^2 - x^2$$



Let $y_1(x)$ be a solution that passes through the point $(2, 1)$. What can you conclude about $\lim_{x \rightarrow \infty} y_0(x)$?

- (A) $\lim_{x \rightarrow \infty} y_0(x) = -\infty$
- (B) $\lim_{x \rightarrow \infty} y_0(x)$ is a finite number
- (C) $\lim_{x \rightarrow \infty} y_0(x) = \infty$

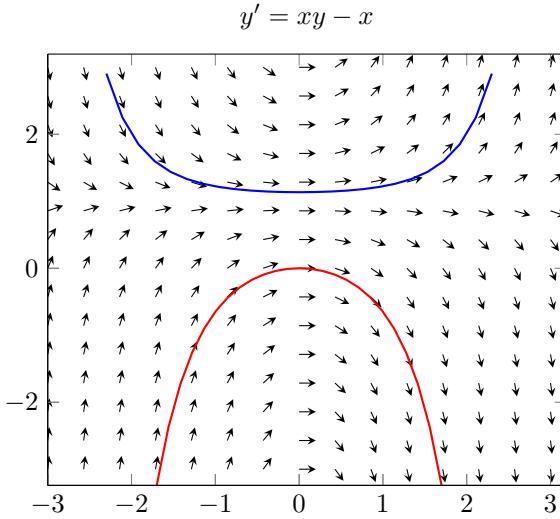
Activity F.1.5 ($\sim 15 \text{ min}$) Consider the ODE

$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.

Part 2: Draw a solution that passes through the point $(0,0)$.

Part 3: Draw a solution that passes through the point $(-2,2)$.

Observation F.1.6


Observation F.1.7 How can we solve $y' = xy - x$ exactly?

Notice $xy - x = x(y - 1)$, so we can write $y' = x(y - 1)$.

Write

$$\frac{y'}{y-1} = x.$$

This is called a **separable** DE.

Observation F.1.8 Integrate both sides (and switch to Leibniz notation):

$$\int \frac{1}{y-1} \frac{dy}{dx} dx = \int x dx.$$

The substitution rule (i.e. chain rule) says this is equivalent to

$$\int \frac{1}{y-1} dy = \int x dx.$$

Thus, $\ln|y-1| = \frac{1}{2}x^2 + c$. Exponentiating, we have

$$|y-1| = e^{\frac{1}{2}x^2+c} = e^{\frac{1}{2}x^2} e^c = c_0 e^{\frac{1}{2}x^2}.$$

Allowing c_0 to take on negative values, we can drop the absolute value sign, and obtain

$$y = 1 + c_0 e^{\frac{1}{2}x^2}.$$

Activity F.1.9 ($\sim 10 \text{ min}$) Find the general solution to

$$y' = xy + y.$$

Activity F.1.10 ($\sim 10 \text{ min}$) Solve the IVP

$$y' = \frac{x}{y}, \quad y(0) = -1.$$

Section F.2

Activity F.2.1 ($\sim 5 \text{ min}$) In Module C, we discussed that tiny spherical objects like droplets of water obey Stoke's law: drag is proportional to velocity (speed). But for larger objects, a better model incorporates **quadratic drag**, i.e. drag is proportional to the square of velocity.

Which of the following ODEs models the velocity of a falling object subject to quadratic drag?

- (a) $mv' = -mg + bv$
- (b) $mv' = -mg - bv$
- (c) $mv' = -mg + bv^2$
- (d) $mv' = -mg - bv^2$

Activity F.2.2 ($\sim 10 \text{ min}$) Consider our model of a falling object under quadratic drag

$$mv' = -mg + bv^2.$$

Part 1: For what value of v will the change in velocity be 0?

Part 2: Suppose the object is currently falling at a rate slower than this speed. What will happen?

- (a) It will slow down
- (b) It will keep falling at the same speed.
- (c) It will speed up

Observation F.2.3 This equilibrium speed is called the **terminal velocity**.

Activity F.2.4 ($\sim 5 \text{ min}$) Consider the following question:

A penny is dropped off the top of the Empire State Building. How fast will it be going when it hits the ground?

What information do we need to answer this question?

Observation F.2.5 The mass of a penny is 2.5g. The Empire State Building is (roughly) 400m tall. The terminal velocity of a penny is about 25m/s.

Activity F.2.6 ($\sim 20 \text{ min}$) We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.

Part 1: Solve for b in terms of v_t, m, g , and substitute this in to our model $v' = -g + \frac{b}{m}v^2$.

Part 2: Solve this separable ODE

Hint: $\frac{1}{v_t^2 - v^2} = \frac{1}{2v_t} \left(\frac{1}{v_t - v} + \frac{1}{v_t + v} \right)$

Part 3: How fast is the penny going after 10 seconds?

Section F.3

Observation F.3.1 There are two very simple kinds of separable ODEs.

Equations of the form $y' = f(x)$ can be solved immediately by integrating and produce explicit solutions. Equations of the form $y' = f(y)$ are often impossible or difficult to solve explicitly. They are called **autonomous** equations.

Activity F.3.2 (~ 10 min) Consider the autonomous equation

$$y' = y^2.$$

Part 1: Draw a slope field

Part 2: Suppose a solution goes through the point $y(10) = 50$. What can you say about $y(11)$?

- (a) $y(10) < y(11)$
- (b) $y(10) = y(11)$
- (c) $y(10) > y(11)$

Observation F.3.3 Since the slopes do not change when moving horizontally (i.e. in the x direction), we often collapse the slope field onto the y -axis.



This is called a **phase line**.

Activity F.3.4 (~ 10 min) Consider the autonomous equation

$$y' = y^2(y - 2).$$

Part 1: Draw a number line for y' , indicating where it is positive or negative.

Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?

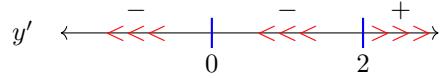
Part 3: What can you say about the long term behavior of a solution passing through $y(2) = 0.001$?

Part 4: What can you say about the long term behavior of a solution passing through $y(2) = -0.001$?

Section F.4

Definition F.4.1 Recall from last week: the **phase line** is a useful way to visualize the long term behavior of an autonomous DE.

For example, here is a phase line for the autonomous DE $y' = y^2(y - 2)$.



Activity F.4.2 (~ 15 min) Consider the autonomous equation

$$y' = y(y + 1)^2(y - 2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.

Part 5: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

Part 6: Describe the long term behavior of solutions passing near the point $y(11) = 2$.

Definition F.4.3 The **critical points** of an autonomous DE are the numbers that give rise to equilibrium solutions (e.g. 0, -1, 2 in the previous problem).

A **source** is an unstable equilibrium in which all nearby trajectories move away in the limit.

A **sink** is a stable equilibrium in which all nearby trajectories approach the equilibrium in the limit.

There are also unstable equilibria in which some nearby trajectories return, while others diverge, analogous to a saddle point.

Activity F.4.4 (~ 15 min) Consider the autonomous equation

$$y' = y^3(y - 2)^2(y + 1)(y - 1).$$

Part 1: Find and classify all of the critical points.

Part 2: Describe the long term behavior of solutions passing near the point $y(1) = 1.5$.

Activity F.4.5 (~ 15 min) Consider the autonomous equation

$$y' = y^4(y + 3)^2(y - 1)(y + 2).$$

Part 1: Find and classify all of the critical points.

Part 2: Describe the long term behavior of solutions passing near the point $y(0) = 0.5$.

Part 3: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

Section F.5

Observation F.5.1 In module C, we solved **constant coefficient linear ODEs**.

Today we will observe that our existing techniques allow us to solve all **first order linear ODES**, i.e. ODEs of the form

$$a(x)y' + b(x)y + c(x) = 0.$$

Such equations can always be rewritten (by rearranging and dividing by $a(x)$) in **standard form**:

$$y' + P(x)y = Q(x).$$

Activity F.5.2 ($\sim 20 \text{ min}$) Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

Part 1: Solve the **homogeneous** first order linear ODE

$$y' + \frac{1}{x}y = 0.$$

Part 2: Use variation of parameters to the solve the original ODE

Activity F.5.3 ($\sim 15 \text{ min}$) Solve the first order linear ODE

$$\frac{1}{x}y' - \frac{2}{x^2}y - x\cos(x) = 0.$$

Activity F.5.4 ($\sim 15 \text{ min}$) Solve

$$(x+1)y' + y = x.$$

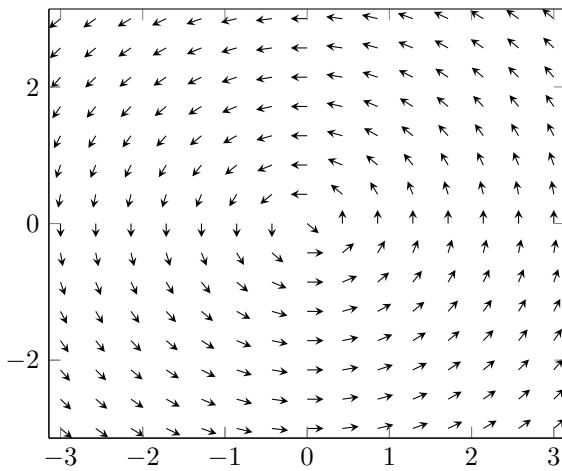
Remark F.5.5 The book provides a different technique (multiplying by an integrating factor); however, the method presented here does not require memorizing anything new.

Section F.6

Observation F.6.1 A vector field $\langle P, Q \rangle$ corresponds to the slope field of the differential equation

$$\frac{dy}{dx} = \frac{Q}{P}.$$

Thus, a solution to this ODE describes the path taken by the particle in this fluid flow.



Activity F.6.2 (~ 10 min) Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2y \frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2y^2 + x$.

Part 1: Compute $\nabla\phi$.

Part 2: Differentiate the equation $\phi(x, y) = c$ with respect to x .

Part 3: Solve the ODE $(2xy^2 + 1) + 2x^2y \frac{dy}{dx} = 0$.

Definition F.6.3 If $\langle M, N \rangle$ is a conservative vector field, the ODE

$$M + N \frac{dy}{dx} = 0$$

is called **exact**. This ODE can also be written

$$\frac{dy}{dx} = \frac{-M}{N}.$$

If $\phi(x, y)$ is a potential function of $\langle M, N \rangle$, the general solution to the ODE is $\phi(x, y) = c$.

Careful: The slope field of the ODE $\frac{dy}{dx} = \frac{-M}{N}$ is the vector field $\langle -N, M \rangle$!

Activity F.6.4 (~ 10 min) Determine which of the following ODEs are exact.

(a) $2xy + (x^2 - 2y) \frac{dy}{dx} = 0$

(b) $\frac{dy}{dx} = \frac{2xy}{x^2 + 2y}$

(c) $\frac{dy}{dx} = -\frac{2xy}{x^2 + 2y}$

Activity F.6.5 (~ 10 min) Solve the exact ODE $2xy + (x^2 - 2y) \frac{dy}{dx} = 0$.

Section F.7

Activity F.7.1 ($\sim 10 \text{ min}$) Determine which of the following ODEs are exact.

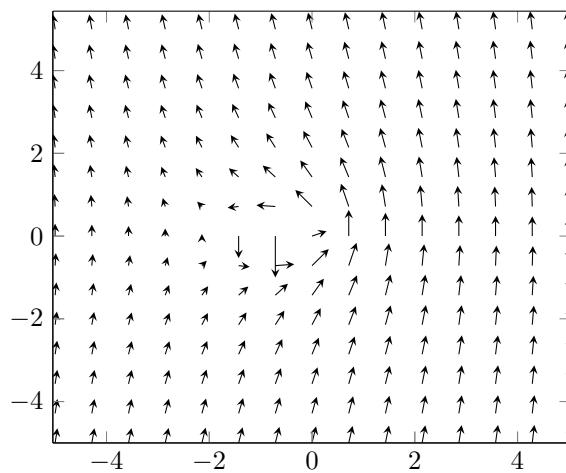
(a) $\frac{dy}{dx} = \frac{-y}{x^2+y^2+x}$

(b) $1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2} \right) \frac{dy}{dx} = 0$

Activity F.7.2 ($\sim 15 \text{ min}$) Solve the exact ODE

$$1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2} \right) \frac{dy}{dx} = 0.$$

These solutions describe the trajectories taken by particles in the fluid flow below



Activity F.7.3 ($\sim 20 \text{ min}$) Find solutions for the ODE

$$1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2} \right) \frac{dy}{dx} = 0$$

for each of the following initial conditions

(a) $y(0) = -1$.

(b) $y(-2) = -2$.

(c) $y(-4) = -4$.

Plot each of the solution curves.

Module S: Systems of ODEs

Standards for this Module

How can we solve and apply systems of linear ODEs? At the end of this module, students will be able to...

- S1. Solving systems.** ...solve systems of constant coefficient ODEs
- S2. Modeling interacting populations.** ...model the populations of two interacting populations with a system of ODEs
- S3. Modeling coupled oscillators.** ...model systems of coupled mechanical oscillators using a system of ODEs

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve systems of two equations in two variables, even when coefficients are functions.
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equilibria of autonomous ODES **F4**

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Academy): <http://bit.ly/2y8AOwa>
- Linear combinations of Euclidean vectors (Khan Academy): <http://bit.ly/2nK3wne>
- Adding and subtracting complex numbers (Khan Academy): <http://bit.ly/1PE3ZMQ>
- Adding and subtracting polynomials (Khan Academy): <http://bit.ly/2d5SLGZ>

Section S.1

Activity S.1.1 ($\sim 10 \text{ min}$) Consider the countries of Transia and Wakanda: each year, 8% of people living in Transia move to Wakanda, and 3% of Wakandans move to Transia.

Let T be the population of Transia, and W the population of Wakanda (both are functions of time, t). Which **system of differential equations** models the population changes $\frac{dT}{dt}$ and $\frac{dW}{dt}$?

(A)

$$\begin{aligned}\frac{dT}{dt} &= 0.03W + 0.08T \\ \frac{dW}{dt} &= 0.08T + 0.03W\end{aligned}$$

(B)

$$\begin{aligned}\frac{dT}{dt} &= -0.03W + 0.08T \\ \frac{dW}{dt} &= -0.08T + 0.03W\end{aligned}$$

(C)

$$\begin{aligned}\frac{dT}{dt} &= 0.03W - 0.08T \\ \frac{dW}{dt} &= 0.08T - 0.03W\end{aligned}$$

(D)

$$\begin{aligned}\frac{dT}{dt} &= -0.03W - 0.08T \\ \frac{dW}{dt} &= 0.08T + 0.03W\end{aligned}$$

Activity S.1.2 ($\sim 5 \text{ min}$) This problem resulted in a **system of linear differential equations**, namely

$$\begin{aligned}T' &= 0.03W - 0.08T \\ W' &= 0.08T - 0.03W\end{aligned}$$

Rewrite this system using differential operators.

Activity S.1.3 ($\sim 15 \text{ min}$) Solve the system

$$\begin{aligned}(D + 0.08)T - (0.03)W &= 0 \\ -0.08T + (D + 0.03)W &= 0\end{aligned}$$

Observation S.1.4 Because D is linear, $a(D + b) = (D + b)a$ for constants a, b . This is not true in general! Thus, for any **constant coefficient linear systems of differential equations**, we can use our typical elimination technique.

Activity S.1.5 ($\sim 15 \text{ min}$) Solve the system

$$\begin{aligned}x' &= 5x - 2y \\ y' &= 6y - 3x\end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = -1$.

Section S.2

Activity S.2.1 (~ 20 min) Solve the system

$$\begin{aligned}x' &= 3x - 4y + 1 \\y' &= 4x - 7y + 10t\end{aligned}$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

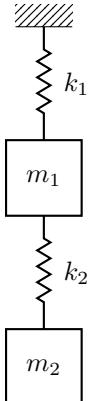
Part 4: Find the solution for the other variable.

Activity S.2.2 (~ 10 min) Solve the system

$$\begin{aligned}x' &= 3x - 2y + \sin(t) \\y' &= 4x - y - \cos(t)\end{aligned}$$

Section S.3

Activity S.3.1 ($\sim 10 \text{ min}$) Consider two coupled masses with two springs.

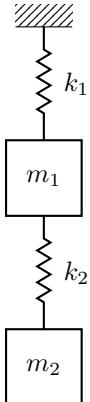


Let x_1 be the position of the upper mass, and x_2 the position of the lower mass (both measured

from equilibrium). Which ODE models the forces acting on the **lower** mass?

- (A) $m_2 x_2'' + k_2 x_2 = 0$
- (B) $m_2 x_2'' + k_2 x_1 = 0$
- (C) $m_2 x_2'' + k_2(x_2 - x_1) = 0$
- (D) $m_2 x_2'' + k_2(x_1 - x_2) = 0$

Activity S.3.2 ($\sim 5 \text{ min}$) Consider two coupled masses with two springs.

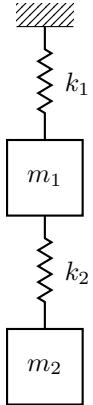


Let x_1 be the position of the upper mass,

and x_2 the position of the lower mass. Which ODE models the forces acting on the **upper** mass?

- (A) $m_1 x_1'' + k_1 x_1 = 0$
- (B) $m_1 x_1'' + k_1 x_1 - k_2 x_2 = 0$
- (C) $m_1 x_1'' + k_1 x_1 + k_2(x_2 - x_1) = 0$
- (D) $m_1 x_1'' + k_1 x_1 + k_2(x_1 - x_2) = 0$

Activity S.3.3 ($\sim 30 \text{ min}$) Suppose we are given $m_1 = 2\text{kg}$, $m_2 = 1\text{kg}$, $k_1 = 4\text{kg/s}^2$, and $k_2 = 2\text{kg/s}^2$. Then our model is



Part 1: Rewrite the system using differential operators.

Part 2: Use elimination to write a single fourth order ODE for x_1 .

Part 3: Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

Part 4: Determine a function for x_2 as well.

$$x_1'' + 6x_1 - 2x_2 = 0$$

$$2x_2'' + 2x_2 - 2x_1 = 0$$

Section S.4

Activity S.4.1 ($\sim 5 \text{ min}$) Consider a forest of bamboo that grows unimpeded by other organisms. Which ODE models the size of the population best (all constants are positive)?

- (a) $\frac{dB}{dt} = k$
- (b) $\frac{dB}{dt} = kB$
- (c) $\frac{dB}{dt} = kB - aB^2$
- (d) $\frac{dB}{dt} = kB^2$

Activity S.4.2 ($\sim 5 \text{ min}$) The model

$$\frac{dB}{dt} = kB$$

models an ideal growth, free from competition (e.g. if population is sparse).

The model

$$\frac{dB}{dt} = kB - aB^2$$

models competitive growth.

Observe that both models are autonomous. Draw a phase line for each model, and describe the possible long term behaviors.

Activity S.4.3 ($\sim 10 \text{ min}$) Which of the following best models the bamboo population in the presence of a panda population (P)?

- (a) $\frac{dB}{dt} = kB - aB^2$
- (b) $\frac{dB}{dt} = kB - aB^2 - cP$
- (c) $\frac{dB}{dt} = kB - aB^2 - cP^2$
- (d) $\frac{dB}{dt} = kB - aB^2 - cBP$

Activity S.4.4 ($\sim 5 \text{ min}$) Which of the following best models the (sparse) Panda population in the bamboo forest?

- (a) $\frac{dP}{dt} = -dP$
- (b) $\frac{dP}{dt} = -dP + fBP$
- (c) $\frac{dP}{dt} = -dP - fBP$
- (d) $\frac{dP}{dt} = -dP - fBP - gP^2$

Observation S.4.5 The interacting bamboo and Panda populations are modelled by the **autonomous system**

$$\begin{aligned}\frac{dB}{dt} &= kb - aB^2 - cBP \\ \frac{dP}{dt} &= -dP + fBP\end{aligned}$$

These are referred to as **Lotka-Volterra equations**

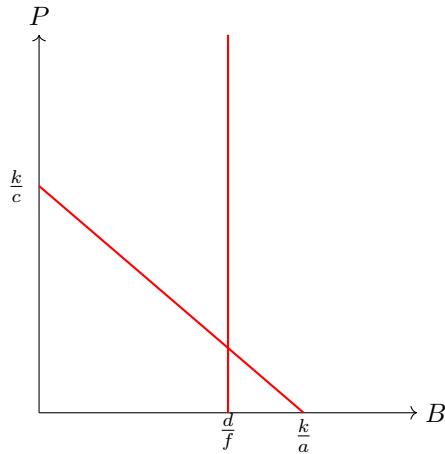
Activity S.4.6 (~ 10 min) Consider our Panda-Bamboo system

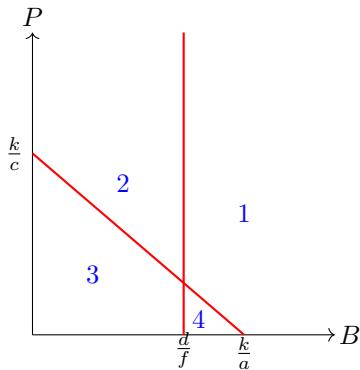
$$\begin{aligned}\frac{dB}{dt} &= kB - aB^2 - cBP \\ \frac{dP}{dt} &= -dP + fBP\end{aligned}$$

Part 1: When is $\frac{dB}{dt}$ zero?

Part 2: When is $\frac{dP}{dt}$ zero?

Observation S.4.7 These lines where the population of one species is unchanging are called **isoclines**



Activity S.4.8 ($\sim 15 \text{ min}$)


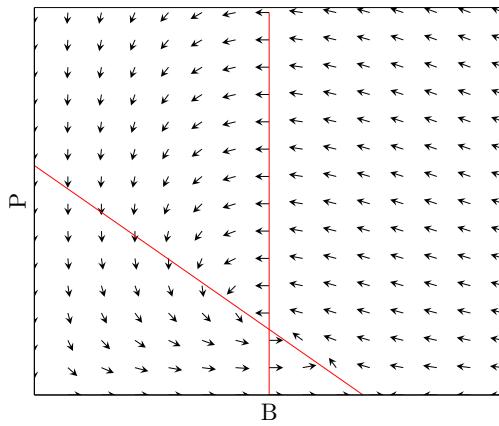
For each of the four regions

Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

Part 3: Describe the general shape of the trajectories.

Observation S.4.9 Plotting the slope field with software makes it more clear that the trajectories are closed curves.



Section S.5

Activity S.5.1 ($\sim 10 \text{ min}$) Consider populations of Green Sunfish (G) and Bluegills (B) in the same lake. They compete for the same food.

Which system of ODEs would model this interaction best?

(A)

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

(B)

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= -0.1B - 0.003B^2 + 0.005BG\end{aligned}$$

(C)

$$\begin{aligned}\frac{dG}{dt} &= -0.1G - 0.002G^2 + 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

(D)

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 + 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 + 0.005BG\end{aligned}$$

Activity S.5.2 ($\sim 15 \text{ min}$) Consider our Greenfish-Bluegill lake modeled by

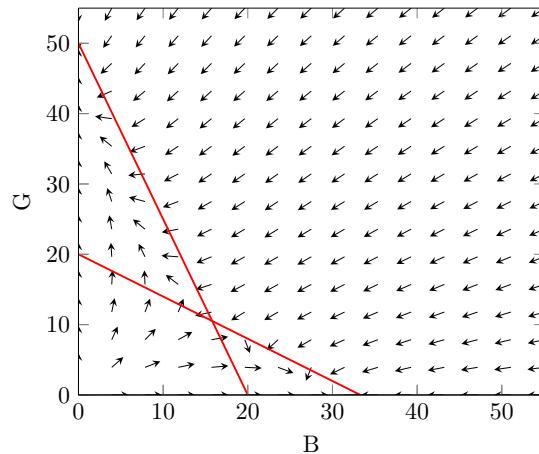
$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

Part 1: Plot the isoclines for each species.

Part 2: If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?

Part 3: If the lake is stocked with 25 Bluegills and 5 Greenfish, what will happen?

Activity S.5.3 ($\sim 5 \text{ min}$) Plotting the slope field along with the isoclines makes the unstable behavior more clear.



Part 1: If the lake is stocked with 20 of each species, what will happen?

Part 2: If the lake is stocked with 30 Bluegills and 10 Greenfish, what will happen?

Module N: Numerical

Standards for this Module

How can we use numerical approximation methods to apply and solve unsolvable ODEs? At the end of this module, students will be able to...

- N1. First Order Existence and Uniqueness.** ...determine when a unique solution exists for a first order ODE
- N2. Second Order Linear Existence and Uniqueness.** ...determine when a unique solution exists for a second order linear ODE
- N3. Systems Existence and Uniqueness.** ...determine when a unique solution exists for a system of first order ODEs
- N4. Euler's method for first order ODES.** ...use Euler's method to find approximate solution to first order ODEs
- N5. Euler's method for systems.** ...use Euler's method to find approximate solutions to systems of first order ODEs

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans \mathbb{R}^n **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Section N.1

Definition N.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \rightarrow W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Module D: Discontinuous functions in ODEs

Standards for this Module

How can we solve and apply ODEs involving functions that are not continuous? At the end of this module, students will be able to...

D1. Laplace Transform. ...compute the Laplace transform of a function

D2. Discontinuous ODEs. ...solve initial value problems for ODEs with discontinuous coefficients

D3. Modeling non-smooth motion. ...model the motion of an object undergoing discontinuous acceleration

D4. Modeling non-smooth oscillators. ...model mechanical oscillators undergoing discontinuous acceleration

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans \mathbb{R}^n **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Section D.1

Definition D.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \rightarrow W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.