Module N: Numerical

How can we use numerical approximation methods to apply and solve unsolvable ODEs?

At the end of this module, students will be able to...

N1. **First Order Existence and Uniqueness.** ...determine when a unique solution exists for a first order ODE

N2. **Second Order Linear Existence and Uniqueness.** ...determine when a unique solution exists for a second order linear ODE

N3. **Systems Existence and Uniqueness.** ...determine when a unique solution exists for a system of first order ODEs

N4. **Euler's method for first order ODES.** ...use Euler's method to find approximate solution to first order ODEs

N5. **Euler's method for systems.** ...use Euler's method to find approximate solutions to systems of first order ODEs

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Module N Section 1

Activity N.1.1 *(∼5 min)*

Solve the IVP

$$y' = \frac{3}{2}y^{\frac{1}{3}}, \qquad y(1) = 0.$$

(A) $y = t^{\frac{3}{2}}$

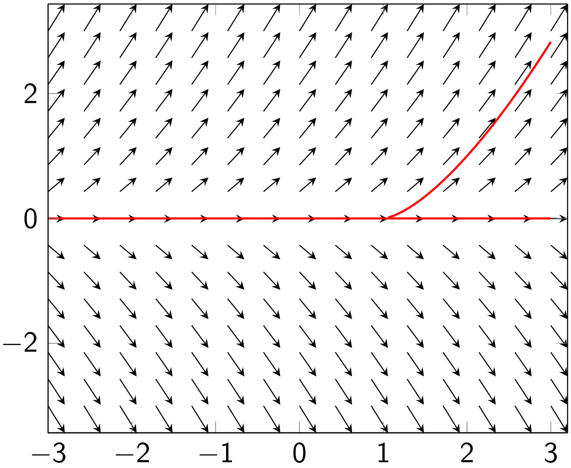
(B) $y = (t-1)^{\frac{3}{2}}$

(C) $y = t^{\frac{3}{2}} - 1$

(D) $y = 0$

Observation N.1.2

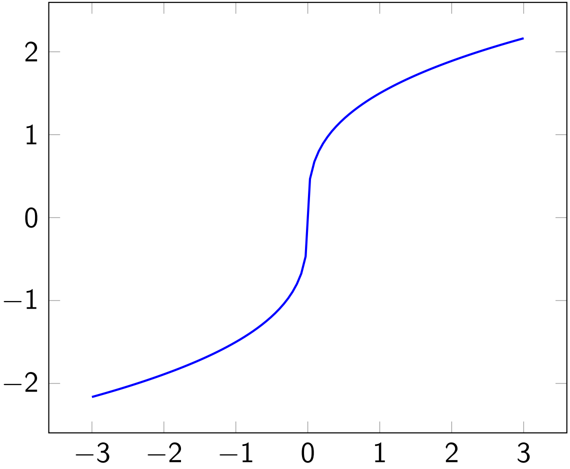
The ODE $y' = y^{\frac{1}{3}}$ has multiple solutions through the point $(1, 0)$.



How can we guarantee our ODEs have a unique solution?

Observation N.1.3

Let's plot the function $f(y) = \frac{3}{2}y^{\frac{1}{3}}$.



Observe: $f(y)$ is not differentiable at 0!

Observation N.1.4

If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are **continuous** on a rectangle containing x_0, y_0, then the IVP

$$y' = f(x, y), \qquad y(x_0) = y_0$$

has a unique solution.

The problem with our example

$$y' = \frac{3}{2}y^{\frac{1}{3}}, \qquad y(1) = 0.$$

is that, for $f(x, y) = \frac{3}{2}y^{\frac{1}{3}}$, the derivative

$$\frac{\partial f}{\partial y} = \frac{1}{2}y^{-\frac{2}{3}}$$

is not continuous at $(1, 0)$.

Activity N.1.5 *(∼5 min)*

Consider the IVP

$$y' = \sqrt{x^2 + y^2}, \qquad y(0) = 0.$$

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$$y' = \sqrt{x^2 + y^2}, \qquad y(0) = 0.$$

Part 1: Is $f(x, y) = \sqrt{x^2 + y^2}$ continuous at $0, 0$?

Activity N.1.5 *(∼5 min)*

Consider the IVP

$$y' = \sqrt{x^2 + y^2}, \qquad y(0) = 0.$$

Part 1: Is $f(x, y) = \sqrt{x^2 + y^2}$ continuous at $0, 0$?

Part 2: Compute $\frac{\partial f}{\partial y}$. Is $\frac{\partial f}{\partial y}$ continuous at $0, 0$?

Activity N.1.5 *(∼5 min)*

Consider the IVP

$$y' = \sqrt{x^2 + y^2}, \qquad\qquad y(0) = 0.$$

Part 1: Is $f(x, y) = \sqrt{x^2 + y^2}$ continuous at $0, 0$?

Part 2: Compute $\frac{\partial f}{\partial y}$. Is $\frac{\partial f}{\partial y}$ continuous at $0, 0$?

Part 3: Can you conclude the IVP has a unique solution?

Activity N.1.6 *(∼10 min)*
Consider the ODE

$$y' = \sqrt{x^2 + y^2 - 1}.$$

This ODE is guaranteed to have a unique solution passing through which of the following points?

(A) $y(1) = 1$

(B) $y(1) = -1$

(C) $y(1) = 0$

(D) $y(0) = 1$

Activity N.1.7 *(∼10 min)*

Consider the ODE

$$y' = \sqrt[3]{x^2 - y^2}.$$

This ODE is guaranteed to have a unique solution passing through which of the following points?

(A) $y(1) = 1$

(B) $y(1) = -1$

(C) $y(1) = 0$

(D) $y(0) = 1$

Activity N.1.8 *(∼10 min)*

Describe all points (x_0, y_0) for which the IVP

$$y' = \ln(x^2 + y^2 - 1) - \sqrt[3]{4 - x^2 - y^2}, \qquad y(x_0) = y_0$$

is guaranteed to have a unique solution.

Module N Section 2

Observation N.2.1

We previously saw that the first order IVP

$$y' = f(x, y), \qquad y(x_0) = y_0$$

had a unique solution on some (possibly tiny!) interval containing x_0 when $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous at (x_0, y_0).

Activity N.2.2 *(∼10 min)*

Consider the second order ODE

$$(x^2-1)^2y''+4x=0$$

.

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Part 1: Solve for y'', and then integrate to find y'.

Activity N.2.2 *(∼10 min)*
Consider the second order ODE

$$(x^2-1)^2 y'' + 4x = 0$$

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Part 1: Solve for y'', and then integrate to find y'.
Part 2: Integrate again to find y. (**Hint:** $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$)

Activity N.2.2 *(~10 min)*
Consider the second order ODE

$$(x^2-1)^2y''+4x=0$$

.
Part 1: Solve for y'', and then integrate to find y'.
Part 2: Integrate again to find y. (**Hint:** $\frac{2}{x^2-1}=\frac{1}{x-1}-\frac{1}{x+1}$)
Part 3: For what values of x is your solution valid?

Observation N.2.3

The ODE $(x^2-1)^2y''+4x=0$ did not have a solution where the coefficient of y'' vanished, i.e. at $x=1$ and $x=-1$.

In general, if $x_1<x<x_2$ is an interval containing x_0 for which

- $(a(x), b(x), c(x),$ and $f(x)$ are continuous, and
- $a(x)$ does not vanish

Then the second order **linear** IVP

$$a(x)y''+b(x)y'+c(x)y=f(x), \qquad\qquad y(x_0)=y_0,\ y'(x_0)=y_1$$

will have a unique solution on $x_1<x<x_2$.

Observation N.2.4

Our uniqueness result for first order equations applied to all first order equations. Our second order result applies to only **linear** equations, but provides added information–a precise interval on which the unique solution exists.

For example, the IVP

$$(x^2 - 1)^2 y'' + 4x = 0, \qquad\qquad y(2) = 3,\ y'(2) = 4$$

will have a unique solution valid for $1 < x < \infty$.

Activity N.2.5 *(~5 min)*
Consider the IVP

$$\sin(x)y'' + \cos(x)y = x^2 - 4, \qquad y\left(\frac{\pi}{4}\right) = 1,\ y'\left(\frac{\pi}{4}\right) = 0.$$

Determine the largest interval on which a unique solution is guaranteed to exist.

Activity N.2.6 *(∼5 min)*
Consider the IVP

$$y'' + \frac{1}{x}y' - \frac{1}{x-4}y = 0 \qquad\qquad y(2) = 5,\ \ y'(2) = -1.$$

Determine the largest interval on which a unique solution is guaranteed to exist.

Activity N.2.7 *(∼5 min)*
Consider the ODE

$$(x^2-1)y'' + \frac{1}{x}y' + e^x y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

Activity N.2.8 *(∼5 min)*

Consider the ODE

$$\frac{x}{x-1}y'' + \frac{x+2}{x+1}y' + e^{-x}y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

Activity N.2.9 *(∼5 min)*

Consider the ODE

$$\sqrt{x^2-1}y'' + y' + \frac{1}{x}y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

Module N Section 3

Activity N.3.1 *(∼10 min)*

Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.

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Part 1: Compute the slope of the solution at the point $(2, 4)$.

Activity N.3.1 *(∼10 min)*
Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.
Part 1: Compute the slope of the solution at the point $(2, 4)$.
Part 2: Use a linear approximation to estimate the value of $y(2.1)$.

Activity N.3.1 *($\sim$10 min)*
Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.
Part 1: Compute the slope of the solution at the point $(2, 4)$.
Part 2: Use a linear approximation to estimate the value of $y(2.1)$.
Part 3: Calculate the slope at the point $(2.1, 4.4)$.

Activity N.3.1 *(∼10 min)*
Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.
Part 1: Compute the slope of the solution at the point $(2, 4)$.
Part 2: Use a linear approximation to estimate the value of $y(2.1)$.
Part 3: Calculate the slope at the point $(2.1, 4.4)$.
Part 4: Use a linear approximation at $(2.1, 4.4)$ to estimate the value of $y(2.2)$.

Observation N.3.2

This technique is called **Euler's method** (with step size $h = 0.1$) for the IVP

$$y' = x + \sqrt{y}, \qquad y(2) = 4.$$

It is often convenient to organize this information in a table

x_n	y_n	$y'(x_n, y_n)$	$x_{n+1} = x_n + h$	$y_{n+1} = y_n + hy'(x_n, y_n)$
2	4	4	2.1	4.4
2.1	4.4	4.19762	2.2	4.81976

Activity N.3.3 *(∼10 min)*

Use Euler's method with stepsize $h = 0.2$ to estimate $y(3)$, where y is a solution of the IVP

$$y' = x - 3y^2, \qquad y(2.2) = 1.$$

Activity N.3.4 *(∼10 min)*

Use Euler's method with stepsize $h = 0.2$ to estimate $y(4)$, where y is a solution of the IVP

$$y' = \sqrt{x - y}, \qquad y(3) = 1.$$

Module N Section 4

Observation N.4.1

The same problem we saw with a first order ODE failing to have a unique solution can also occur in systems of first order ODEs.

Thus, to ensure that the system

$$\begin{aligned} x' &= f(t,x,y) \\ y' &= g(t,x,y) \end{aligned}$$

has a unique solution, you must check that $f(t,x,y)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $g(t,x,y)$, $\frac{\partial g}{\partial x}$, and $\frac{\partial g}{\partial y}$ are all continuous near the initial point.

Observation N.4.2

If the system is linear, we can say more!

Suppose $a(t), b(t), c(t), d(t), f(t), g(t)$ are all continous on the interval $h < t < k$. Then for any $h < t_0 < k$, the IVP

$$\begin{aligned} x' &= a(t)x + b(t)y + f(t) & \qquad x(t_0) &= x_0 \\ y' &= c(t)x + d(t)y + g(t) & \qquad y(t_0) &= y_0 \end{aligned}$$

has a unique solution on the (time) interval $h < t < k$.

Activity N.4.3 *(∼5 min)*

Consider the IVP

$$x' = \frac{1}{t-1}x + \sqrt{t}y + t^2 \qquad x(2) = 5$$
$$y' = \frac{1}{t}x + \sqrt{t+1}y + t^2 \qquad y(2) = 7$$

What is the largest interval on which this IVP has a unique solution?

Activity N.4.4 *($\sim$5 min)*

Determine **all intervals** on which a unique solution is guaranteed to exist for the below system.

$$x' = \frac{1}{t-1}x + \sqrt{t}y + t^2$$
$$y' = \frac{1}{t}x + \sqrt{t+1}y + t^2$$

Activity N.4.5 *(~5 min)*

Determine **all intervals** on which a unique solution is guaranteed to exist for the below system.

$$\begin{aligned} x' &= \ln(t-2)x + \sqrt{t}y + \frac{1}{t-1} \\ y' &= \cos(t)x + y \end{aligned}$$

Activity N.4.6 *(∼10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$\begin{aligned} x' &= 3x + 4y - t & \qquad x(1) &= 2 \\ y' &= x - y + t & \qquad y(1) &= 3. \end{aligned}$$

Activity N.4.6 *($\sim$10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$x' = 3x + 4y - t \qquad x(1) = 2$$
$$y' = x - y + t \qquad y(1) = 3.$$

Part 1: Compute x' and y' when $t = 1$.

Activity N.4.6 *(∼10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$\begin{aligned} x' &= 3x + 4y - t \qquad & x(1) &= 2 \\ y' &= x - y + t \qquad & y(1) &= 3. \end{aligned}$$

Part 1: Compute x' and y' when $t = 1$.
Part 2: Use linear approximations to estimate the values of $x(1.1)$ and $y(1.1)$.

Activity N.4.6 *(∼10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$\begin{aligned} x' &= 3x + 4y - t & \qquad x(1) &= 2 \\ y' &= x - y + t & \qquad y(1) &= 3. \end{aligned}$$

Part 1: Compute x' and y' when $t = 1$.
Part 2: Use linear approximations to estimate the values of $x(1.1)$ and $y(1.1)$.
Part 3: Calculate the slopes x' and y' when $t = 1.1$ (and as just calculated, $x(1.1) =$ and $y(1.1) =$).

Activity N.4.6 *(∼10 min)*

Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$\begin{aligned} x' &= 3x + 4y - t & \qquad x(1) &= 2 \\ y' &= x - y + t & \qquad y(1) &= 3. \end{aligned}$$

Part 1: Compute x' and y' when $t = 1$.
Part 2: Use linear approximations to estimate the values of $x(1.1)$ and $y(1.1)$.
Part 3: Calculate the slopes x' and y' when $t = 1.1$ (and as just calculated, $x(1.1) =$ and $y(1.1) =$).
Part 4: Use linear approximations to estimate the values of $x(1.2)$ and $y(1.2)$.

Observation N.4.7

$$x' = 3x + 4y - t \qquad x(1) = 2$$
$$y' = x - y + t \qquad y(1) = 3.$$

It is often convenient to organize this information in a table

t_n	x_n	y_n	$x'(t_n, x_n, y_n)$	$y'(t_n, x_n, y_n)$	t_{n+1}	x_{n+1}	y_{n+1}
1	2	3	17	0	1.1	2.17	3
1.1	2.17	3	17.41	0.27	1.2	3.911	3.027
1.2	3.911	3.027					

Thus $x(1.2) \approx 3.911$ and $y(1.2) \approx 3.027$.

Activity N.4.8 *(∼10 min)*
Use Euler's method to estimate $x(3.3)$ and $y(3.3)$.

$$x' = 3x - ty \qquad x(3) = 2$$
$$y' = x - y^2 \qquad y(3) = 1.$$

Activity N.4.9 *(∼10 min)*

Use Euler's method to estimate $x(4.6)$ and $y(4.6)$.

$$\begin{aligned} x' &= xy - t & \qquad x(4) &= 2 \\ y' &= x + t & \qquad y(4) &= 0. \end{aligned}$$