

Module V

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# Module V: Vector Spaces

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# What is a vector space?

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At the end of this module, students will be able to...

- V1. Vector property verification.** ... show why an example satisfies a given vector space property, but does not satisfy another given property.
- V2. Vector space identification.** ... list the eight defining properties of a vector space, infer which of these properties a given example satisfies, and thus determine if the example is a vector space.
- V3. Linear combinations.** ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors.
- V4. Spanning sets.** ... determine if a set of Euclidean vectors spans  $\mathbb{R}^n$ .
- V5. Subspaces.** ... determine if a subset of  $\mathbb{R}^n$  is a subspace or not.

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## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems  
**E1,E2,E3.**

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The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Academy):  
<http://bit.ly/2y8A0wa>
- Linear combinations of Euclidean vectors (Khan Academy):  
<http://bit.ly/2nK3wne>
- Adding and subtracting complex numbers (Khan Academy):  
<http://bit.ly/1PE3ZMQ>
- Adding and subtracting polynomials (Khan Academy):  
<http://bit.ly/2d5SLGZ>

# Module V Section 0

**Activity V.0.1** ( $\sim 20$  min)

Consider each of the following vector properties. Label each property with  $\mathbb{R}^1$ ,  $\mathbb{R}^2$ , and/or  $\mathbb{R}^3$  if that property holds for Euclidean vectors/scalars  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  of that dimension.

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**1 Addition associativity.**

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

**2 Addition commutivity.**

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

**3 Addition identity.**

There exists some  $\mathbf{z}$  where  $\mathbf{v} + \mathbf{z} = \mathbf{v}$ .

**4 Addition inverse.**

There exists some  $-\mathbf{v}$  where  
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$ .

**5 Addition midpoint uniqueness.**

There exists a unique  $\mathbf{m}$  where the distance from  $\mathbf{u}$  to  $\mathbf{m}$  equals the distance from  $\mathbf{m}$  to  $\mathbf{v}$ .

**6 Scalar multiplication associativity.**

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$

**7 Scalar multiplication identity.**

$$1\mathbf{v} = \mathbf{v}.$$

**8 Scalar multiplication relativity.**

There exists some scalar  $c$  where either  $c\mathbf{v} = \mathbf{w}$  or  $c\mathbf{w} = \mathbf{v}$ .

**9 Scalar distribution.**

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

**10 Vector distribution.**

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

**11 Orthogonality.**

There exists a non-zero vector  $\mathbf{n}$  such that  $\mathbf{n}$  is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$ .

**12 Bidimensionality.**

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \text{ for some value of } a, b.$$

## Definition V.0.2

A **vector space**  $V$  is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  belong to  $V$ , and let  $a, b$  be scalar numbers.

- **Addition is associative.**  
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition is commutative.**  
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Additive identity exists.**  
There exists some  $\mathbf{z}$  where  
 $\mathbf{v} + \mathbf{z} = \mathbf{v}.$
- **Additive inverses exist.**  
There exists some  $-\mathbf{v}$  where  
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}.$
- **Scalar multiplication is associative.**  
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **1 is a scalar multiplicative identity.**  
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar multiplication distributes over vector addition.**  
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Scalar multiplication distributes over scalar addition.**  
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

Any **Euclidean vector space**  $\mathbb{R}^n$  satisfies all eight requirements regardless of the value of  $n$ , but we will also study other types of vector spaces.



# Module V Section 1

## Remark V.1.1

Last time, we defined a **vector space**  $V$  to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $V$ , and all scalars (i.e. real numbers)  $a, b$ .

- **Addition is associative.**  
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition is commutative.**  
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Additive identity exists.**  
There exists some  $\mathbf{z}$  where  
 $\mathbf{v} + \mathbf{z} = \mathbf{v}.$
- **Additive inverses exist.**  
There exists some  $-\mathbf{v}$  where  
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}.$
- **Scalar multiplication is associative.**  
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **1 is a scalar multiplicative identity.**  
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar multiplication distributes over vector addition.**  
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Scalar multiplication distributes over scalar addition.**  
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

## Remark V.1.2

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$ : Euclidean vectors with  $n$  components.
- $\mathbb{R}^\infty$ : Sequences of real numbers  $(v_1, v_2, \dots)$ .
- $M_{m,n}$ : Matrices of real numbers with  $m$  rows and  $n$  columns.
- $\mathbb{C}$ : Complex numbers.
- $\mathcal{P}^n$ : Polynomials of degree  $n$  or less.
- $\mathcal{P}$ : Polynomials of any degree.
- $C(\mathbb{R})$ : Real-valued continuous functions.

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**Activity V.1.3** ( $\sim 20$  min)

Consider the set  $V = \{(x, y) \mid y = e^x\}$  with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

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**Activity V.1.3** ( $\sim 20$  min)

Consider the set  $V = \{(x, y) \mid y = e^x\}$  with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

*Part 1:* Show that  $V$  satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting  $\mathbf{v} = (x, y)$  and showing both sides simplify to the same expression.

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**Activity V.1.3** ( $\sim 20$  min)

Consider the set  $V = \{(x, y) \mid y = e^x\}$  with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

*Part 1:* Show that  $V$  satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting  $\mathbf{v} = (x, y)$  and showing both sides simplify to the same expression.

*Part 2:* Show that  $V$  contains an additive identity element by choosing  $\mathbf{z} = (?, ?)$  such that  $\mathbf{v} \oplus \mathbf{z} = (x, y) \oplus (?, ?) = \mathbf{v}$  for any  $\mathbf{v} = (x, y) \in V$ .

**Remark V.1.4**

It turns out  $V = \{(x, y) \mid y = e^x\}$  with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

satisfies all eight properties.

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutivity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some  $\mathbf{z}$  where

$$\mathbf{v} \oplus \mathbf{z} = \mathbf{v}.$$

- **Addition inverse.**

There exists some  $-\mathbf{v}$  where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

Thus,  $V$  is a vector space.

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**Activity V.1.5** (*~15 min*)

Let  $V = \{(x, y) \mid x, y \in \mathbb{R}\}$  have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$



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**Activity V.1.5** (*~15 min*)

Let  $V = \{(x, y) \mid x, y \in \mathbb{R}\}$  have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

*Part 1:* Show that the scalar multiplication identity holds by simplifying  $1 \odot (x, y)$  to  $(x, y)$ .

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**Activity V.1.5** (*~15 min*)

Let  $V = \{(x, y) \mid x, y \in \mathbb{R}\}$  have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

*Part 1:* Show that the scalar multiplication identity holds by simplifying  $1 \odot (x, y)$  to  $(x, y)$ .

*Part 2:* Show that the addition identity property fails by showing that  $(0, -1) \oplus \mathbf{z} \neq (0, -1)$  no matter how  $\mathbf{z} = (z_1, z_2)$  is chosen.

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**Activity V.1.5** (*~15 min*)

Let  $V = \{(x, y) \mid x, y \in \mathbb{R}\}$  have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

*Part 1:* Show that the scalar multiplication identity holds by simplifying  $1 \odot (x, y)$  to  $(x, y)$ .

*Part 2:* Show that the addition identity property fails by showing that  $(0, -1) \oplus \mathbf{z} \neq (0, -1)$  no matter how  $\mathbf{z} = (z_1, z_2)$  is chosen.

*Part 3:* Can  $V$  be a vector space?

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**Definition V.1.6**

A **linear combination** of a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  is given by  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$  for any choice of scalar multiples  $c_1, c_2, \dots, c_m$ .

For example, we can say  $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

## Definition V.1.7

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \mid c_i \in \mathbb{R}\}.$$

For example:

$$\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R}\right\}$$

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**Activity V.1.8** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

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**Activity V.1.8** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , and  $-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane.

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**Activity V.1.8** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , and  $-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane.

*Part 2:* Sketch a representation of all the vectors belonging to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid a \in \mathbb{R} \right\}$  in the  $xy$  plane.



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**Activity V.1.9** (*~10 min*)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

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**Activity V.1.9** (*~10 min*)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane.

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

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**Activity V.1.9** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane.

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

*Part 2:* Sketch a representation of all the vectors belonging to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$  in the  $xy$  plane.

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**Activity V.1.10** ( $\sim 5$  min)

Sketch a representation of all the vectors belonging to  $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$  in the  $xy$  plane.

## Module V Section 2

## Remark V.2.1

Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

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**Activity V.2.2** (*~15 min*)

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when there exists a solution to the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ .

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**Activity V.2.2** (*~15 min*)

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when there exists a solution to the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.



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**Activity V.2.2** (*~15 min*)

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when there exists a

solution to the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

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**Activity V.2.2** (*~15 min*)

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when there exists a

solution to the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

*Part 3:* Given this solution set, does  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ ?

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**Fact V.2.3**

A vector  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  if and only if the linear system corresponding to  $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$  is consistent.

Put another way,  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  exactly when  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$  doesn't have a row  $[0 \ \dots \ 0 \mid 1]$  representing the contradiction  $0 = 1$ .

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**Activity V.2.4** (*~10 min*)

Determine if  $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

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**Activity V.2.5** (*~5 min*)

Determine if  $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

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**Activity V.2.6** (*~10 min*)

Does the third-degree polynomial  $3y^3 - 2y^2 + y + 5$  in  $\mathcal{P}^3$  belong to  $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$ ?

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**Activity V.2.6** (*~10 min*)

Does the third-degree polynomial  $3y^3 - 2y^2 + y + 5$  in  $\mathcal{P}^3$  belong to  $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$ ?

*Part 1:* Reinterpret this question as an equivalent exercise involving Euclidean vectors in  $\mathbb{R}^4$ . (Hint: What four numbers must you know to write a  $\mathcal{P}^3$  polynomial?)

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**Activity V.2.6** (*~10 min*)

Does the third-degree polynomial  $3y^3 - 2y^2 + y + 5$  in  $\mathcal{P}^3$  belong to  $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$ ?

*Part 1:* Reinterpret this question as an equivalent exercise involving Euclidean vectors in  $\mathbb{R}^4$ . (Hint: What four numbers must you know to write a  $\mathcal{P}^3$  polynomial?)

*Part 2:* Solve this equivalent exercise, and use its solution to answer the original question.



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**Activity V.2.7** (*~5 min*)

Does the matrix  $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \right\}$ ?

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**Activity V.2.8** ( $\sim 5$  min)

Does the complex number  $2i$  belong to  $\text{span}\{-3 + i, 6 - 2i\}$ ?

## Module V Section 3

## Module V

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**Activity V.3.1** ( $\sim 5$  min)

How many vectors are required to span  $\mathbb{R}^2$ ? Sketch a drawing in the  $xy$  plane to support your answer.

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

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**Activity V.3.2** (*~5 min*)

How many vectors are required to span  $\mathbb{R}^3$ ?

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

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**Fact V.3.3**

At least  $n$  vectors are required to span  $\mathbb{R}^n$ .



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**Activity V.3.4** (*~15 min*)

Choose a vector  $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$  in  $\mathbb{R}^3$  that is not in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$  by using CoCalc

to verify that  $\text{RREF} \left[ \begin{array}{cc|c} 1 & -2 & ? \\ -1 & 0 & ? \\ 0 & 1 & ? \end{array} \right] = \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$ . (Why does this work?)

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**Fact V.3.5**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  fails to span all of  $\mathbb{R}^n$  exactly when  $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$  has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & \big| & a \\ -1 & 0 & \big| & b \\ 0 & 1 & \big| & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \big| & 0 \\ 0 & 1 & \big| & 0 \\ 0 & 0 & \big| & 1 \end{bmatrix} \text{ for some choice of vector } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



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**Activity V.3.6** (*~5 min*)

Consider the set of vectors  $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ . Does

$\mathbb{R}^4 = \text{span } S$ ?

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**Activity V.3.7** (*~10 min*)

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, \\ -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}.$$

Does  $\mathcal{P}^3 = \text{span } S$ ? (Hint: first rewrite the question so it is about Euclidean vectors.)

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**Activity V.3.8** (*~5 min*)

Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does  $M_{2,2} = \text{span } S$ ?

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**Activity V.3.9** ( $\sim 5$  min)

Let  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^7$  be three vectors, and suppose  $\mathbf{w}$  is another vector with  $\mathbf{w} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ . What can you conclude about  $\text{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  ?

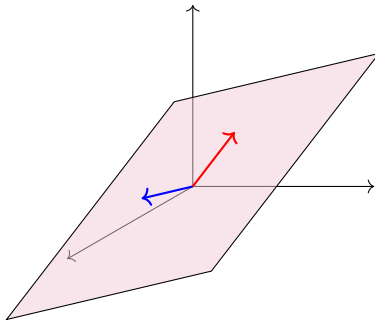
- (a)  $\text{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is larger than  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .
- (b)  $\text{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .
- (c)  $\text{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is smaller than  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .

## Module V Section 4

## Definition V.4.1

A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space  $\mathbb{R}^3$ .



## Fact V.4.2

Any **subset**  $S$  of a vector space  $V$  satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a **subspace**, we need to check that addition and multiplication still make sense using only vectors from  $S$ . So we need to check two things:

- The set is **closed under addition**: for any  $\mathbf{x}, \mathbf{y} \in S$ , the sum  $\mathbf{x} + \mathbf{y}$  is also in  $S$ .
- The set is **closed under scalar multiplication**: for any  $\mathbf{x} \in S$  and scalar  $c \in \mathbb{R}$ , the product  $c\mathbf{x}$  is also in  $S$ .

**Activity V.4.3** ( $\sim 15$  min)

Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$

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**Activity V.4.3** ( $\sim 15$  min)

Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$

*Part 1:* Let  $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  be vectors in  $S$ , so  $x + 2y + z = 0$  and

$a + 2b + c = 0$ . Show that  $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$  also belongs to  $S$  by verifying that

$$(x + a) + 2(y + b) + (z + c) = 0.$$

**Activity V.4.3** ( $\sim 15$  min)

Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$

*Part 1:* Let  $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  be vectors in  $S$ , so  $x + 2y + z = 0$  and

$a + 2b + c = 0$ . Show that  $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$  also belongs to  $S$  by verifying that

$$(x + a) + 2(y + b) + (z + c) = 0.$$

*Part 2:* Let  $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$ , so  $x + 2y + z = 0$ . Show that  $c\mathbf{v}$  also belongs to  $S$  for any  $c \in \mathbb{R}$ .

**Activity V.4.3** ( $\sim 15$  min)

Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$

*Part 1:* Let  $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  be vectors in  $S$ , so  $x + 2y + z = 0$  and

$a + 2b + c = 0$ . Show that  $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$  also belongs to  $S$  by verifying that

$$(x + a) + 2(y + b) + (z + c) = 0.$$

*Part 2:* Let  $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$ , so  $x + 2y + z = 0$ . Show that  $c\mathbf{v}$  also belongs to  $S$  for

any  $c \in \mathbb{R}$ .

*Part 3:* Is  $S$  is a subspace of  $\mathbb{R}^3$ ?

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**Activity V.4.4** (*~10 min*)

Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 4 \right\}$ . Choose a vector  $\mathbf{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$  in  $S$  and a real number  $c = ?$ , and show that  $c\mathbf{v}$  isn't in  $S$ . Is  $S$  a subspace of  $\mathbb{R}^3$ ?

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**Remark V.4.5**

Since  $0$  is a scalar and  $0\mathbf{v} = \mathbf{z}$  for any vector  $\mathbf{v}$ , a set that is closed under scalar multiplication must contain the zero vector  $\mathbf{z}$  for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain  $\mathbf{0}$ .

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**Activity V.4.6** (*~10 min*)Consider these two subsets of  $\mathbb{R}^4$ :

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

$$T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

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**Activity V.4.6** (*~10 min*)Consider these two subsets of  $\mathbb{R}^4$ :

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

$$T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

*Part 1:* Which set is not a subspace of  $\mathbb{R}^4$ ?

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**Activity V.4.6** ( $\sim 10$  min)Consider these two subsets of  $\mathbb{R}^4$ :

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \mid a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

*Part 1:* Which set is not a subspace of  $\mathbb{R}^4$ ?*Part 2:* Is the set of polynomials

$$S = \{ ax^3 + bx^2 + (b-1)x + (a-1) \mid a, b \text{ are real numbers} \}$$

a subspace of  $\mathcal{P}^3$ ?



## Module V

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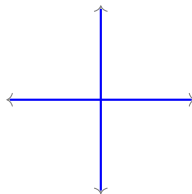
Section V.2

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**Activity V.4.7** (*~10 min*)

Consider the subset  $A$  of  $\mathbb{R}^2$  where at least one coordinate of each vector is 0.



This set contains  $\mathbf{0}$ , and it's not hard to show that for every  $\mathbf{v}$  in  $A$  and scalar  $c \in \mathbb{R}$ ,  $c\mathbf{v}$  is also in  $A$ . Is  $A$  a subspace of  $\mathbb{R}^2$ ? Why?

## Module V

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**Activity V.4.8** ( $\sim 5$  min)

Let  $W$  be a subspace of a vector space  $V$ . How are  $\text{span } W$  and  $W$  related?

- (a)  $\text{span } W$  is bigger than  $W$
- (b)  $\text{span } W$  is the same as  $W$
- (c)  $\text{span } W$  is smaller than  $W$

## Module V

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**Fact V.4.9**

If  $S$  is any subset of a vector space  $V$ , then since  $\text{span } S$  collects all possible linear combinations,  $\text{span } S$  is automatically a subspace of  $V$ .

In fact,  $\text{span } S$  is always the smallest subspace of  $V$  that contains all the vectors in  $S$ .