

Module S

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Module S: Systems of ODEs

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How can we solve and apply systems of linear ODEs?

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At the end of this module, students will be able to...

- S1. Solving systems.** ...solve systems of constant coefficient ODEs
- S2. Modeling interacting populations.** ...model the populations of two interacting populations with a system of ODEs
- S3. Modeling coupled oscillators.** ...model systems of coupled mechanical oscillators using a system of ODEs

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Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

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The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Academy):
<http://bit.ly/2y8A0wa>
- Linear combinations of Euclidean vectors (Khan Academy):
<http://bit.ly/2nK3wne>
- Adding and subtracting complex numbers (Khan Academy):
<http://bit.ly/1PE3ZMQ>
- Adding and subtracting polynomials (Khan Academy):
<http://bit.ly/2d5SLGZ>

Module S Section 1

Activity S.1.1 (*~10 min*)

Consider the countries of Transia and Wakanda: each year, 8% of people living in Transia move to Wakanda, and 3% of Wakandans move to Transia.

Let T be the population of Transia, and W the population of Wakanda (both are functions of time, t).

Write down two differential equations modelling the population changes $\frac{dT}{dt}$ and $\frac{dW}{dt}$.

Activity S.1.2 (*~5 min*)

This problem resulted in a **system of linear differential equations**, namely

$$T' = 0.03W - 0.08T$$

$$W' = 0.08T - 0.03W$$

Rewrite this system using differential operators.

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Activity S.1.3 (~ 15 min)

Solve the system

$$(D + 0.08)T - (0.03)W = 0$$

$$-0.08T + (D + 0.03)W = 0$$

Observation S.1.4

Because D is linear, $a(D + b) = (D + b)a$ for constants a, b . This is not true in general!

Thus, for any **constant coefficient linear systems of differential equations**, we can use our typical elimination technique.

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Activity S.1.5 (*~15 min*)

Solve the system

$$x' = 5x - 2y$$

$$y' = 6y - 3x$$

with initial conditions $x(0) = 2$, $y(0) = -1$.

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Activity S.2.1 (*~20 min*)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

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Activity S.2.1 (*~20 min*)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

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Activity S.2.1 (*~20 min*)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators*Part 2:* Use elimination to eliminate a variable

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Activity S.2.1 (*~20 min*)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

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Activity S.2.1 (~ 20 min)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

Part 4: Find the solution for the other variable.

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Activity S.2.2 (*~10 min*)

Solve the system

$$x' = 3x - 2y + \sin(t)$$

$$y' = 4x - y - \cos(t)$$

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Activity S.3.1 (~ 5 min)

Consider a forest of bamboo that grows unimpeded by other organisms. Which ODE models the size of the population best (all constants are positive)?

(a) $\frac{dB}{dt} = k$

(b) $\frac{dB}{dt} = kB$

(c) $\frac{dB}{dt} = kB - aB^2$

(d) $\frac{dB}{dt} = kB^2$

Activity S.3.2 (*~5 min*)

The model

$$\frac{dB}{dt} = kB$$

models an ideal growth, free from competition (e.g. if population is sparse).

The model

$$\frac{dB}{dt} = kB - aB^2$$

models competitive growth.

Observe that both models are autonomous. Draw a phase line for each model, and describe the possible long term behaviors.

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Activity S.3.3 (~ 10 min)

Which of the following best models the bamboo population in the presence of a panda population (P)?

(a) $\frac{dB}{dt} = kB - aB^2$

(b) $\frac{dB}{dt} = kB - aB^2 - cP$

(c) $\frac{dB}{dt} = kB - aB^2 - cP^2$

(d) $\frac{dB}{dt} = kB - aB^2 - cBP$

Activity S.3.4 (~ 5 min)

Which of the following best models the (sparse) Panda population in the bamboo forest?

(a) $\frac{dP}{dt} = -dP$

(b) $\frac{dP}{dt} = -dP + fBP$

(c) $\frac{dP}{dt} = -dP - fBP$

(d) $\frac{dP}{dt} = -dP - fBP - gP^2$

Observation S.3.5

The interacting bamboo and Panda populations are modelled by the **autonomous system**

$$\begin{aligned}\frac{dB}{dt} &= kb - aB^2 - cBP \\ \frac{dP}{dt} &= -dP + fBP\end{aligned}$$

These are referred to as **Lotka-Volterra equations**

Activity S.3.6 (*~10 min*)

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$

$$\frac{dP}{dt} = -dP + fBP$$

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Activity S.3.6 (~ 10 min)

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Part 1: When is $\frac{dB}{dt}$ zero?

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Activity S.3.6 (~ 10 min)

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$

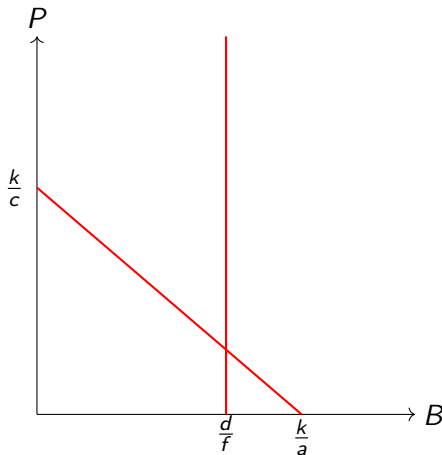
$$\frac{dP}{dt} = -dP + fBP$$

Part 1: When is $\frac{dB}{dt}$ zero?

Part 2: When is $\frac{dP}{dt}$ zero?

Observation S.3.7

These lines where the population of one species is unchanging are called **isoclines**



Activity S.3.8 (~ 15 min)

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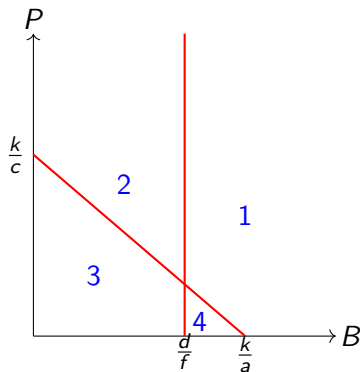
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For each of the four regions

Activity S.3.8 (~ 15 min)

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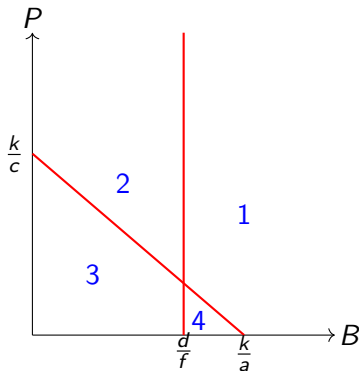
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For each of the four regions

Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Activity S.3.8 (~15 min)

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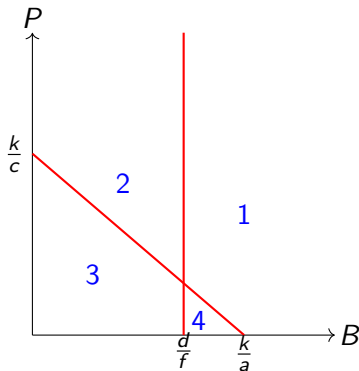
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For each of the four regions

Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

Activity S.3.8 (~ 15 min)

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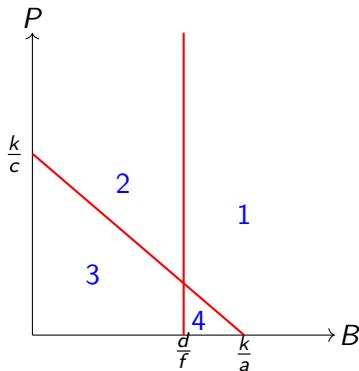
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For each of the four regions

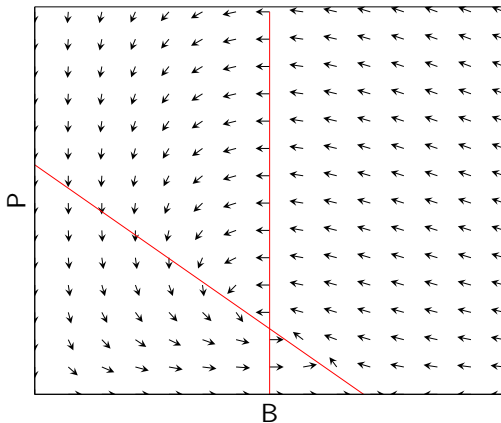
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

Part 3: Describe the general shape of the trajectories.

Observation S.3.9

Plotting the slope field with software makes it more clear that the trajectories are closed curves.



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Activity S.4.1 (*~20 min*)

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Activity S.5.1 (*~20 min*)