Module S: Systems of ODEs

How can we solve and apply systems of linear ODEs?

At the end of this module, students will be able to...

S1. Solving systems. ...solve systems of constant coefficient ODEs

S2. Modeling interacting populations. ...model the populations of two interacting populations with a system of ODEs

S3. Modeling coupled oscillators. ...model systems of coupled mechanical oscillators using a system of ODEs

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve systems of two equations in two variables, even when coefficients are functions.
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equillibria of autonomous ODES **F4**

The following resources will help you prepare for this module.

- Solve systems of two equations in two variables, even when coefficients are functions. `https://youtu.be/Y6JsEja15Vk`
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equillibria of autonomous ODES **F4**

Module S Section 1

Activity S.1.1 *(∼10 min)*

Consider the countries of Transia and Wakanda: each year, 8% of people living in Transia move to Wakanda, and 3% of Wakandans move to Transia.

Let T be the population of Transia, and W the population of Wakanda (both are functions of time, t.

Which **system of differential equations** models the population changes $\frac{dT}{dt}$ and $\frac{dW}{dt}$?

(A)

$$\frac{dT}{dt} = 0.03W + 0.08T$$
$$\frac{dW}{dt} = 0.08T + 0.03W$$

(B)

$$\frac{dT}{dt} = -0.03W + 0.08T$$
$$\frac{dW}{dt} = -0.08T + 0.03W$$

(C)

$$\frac{dT}{dt} = 0.03W - 0.08T$$
$$\frac{dW}{dt} = 0.08T - 0.03W$$

(D)

$$\frac{dT}{dt} = -0.03W - 0.08T$$
$$\frac{dW}{dt} = 0.08T + 0.03W$$

Activity S.1.2 *(∼5 min)*

This problem resulted in a **system of linear differential equations**, namely

$$\begin{aligned} T' &= 0.03W - 0.08T \\ W' &= 0.08T - 0.03W \end{aligned}$$

Rewrite this system using differential operators.

Activity S.1.3 *(∼15 min)*
Solve the system

$$\begin{aligned}(D+0.08)T-(0.03)W&=0\\-0.08T+(D+0.03)W&=0\end{aligned}$$

Observation S.1.4

Because D is linear, $a(D+b) = (D+b)a$ for constants a, b. This is not true in general!

Thus, for any **constant coefficient linear systems of differential equations**, we can use our typical elimination technique.

Activity S.1.5 *(∼15 min)*

Solve the system

$$\begin{aligned} x' &= 5x - 2y \\ y' &= 6y - 3x \end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = -1$.

Module S Section 2

Activity S.2.1 *(∼20 min)*

Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

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$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Activity S.2.1 *(∼20 min)*

Solve the system

$$\begin{aligned} x' &= 3x - 4y + 1 \\ y' &= 4x - 7y + 10t \end{aligned}$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Activity S.2.1 *(∼20 min)*

Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

Activity S.2.1 *(∼20 min)*

Solve the system

$$x' = 3x - 4y + 1$$
$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

Part 4: Find the solution for the other variable.

Activity S.2.2 *(∼10 min)*

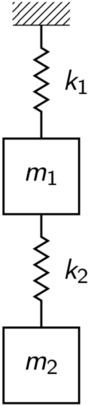
Solve the system

$$x' = 3x - 2y + \sin(t)$$
$$y' = 4x - y - \cos(t)$$

Module S Section 3

Activity S.3.1 *(∼10 min)*

Consider two coupled masses with two springs.



Let x_1 be the position of the upper mass, and x_2 the position of the lower mass (both measured from equillibrium). Which ODE models the forces acting on the **lower** mass?

(A) $m_2x_2'' + k_2x_2 = 0$

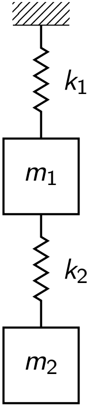
(B) $m_2x_2'' + k_2x_1 = 0$

(C) $m_2x_2'' + k_2(x_2 - x_1) = 0$

(D) $m_2x_2'' + k_2(x_1 - x_2) = 0$

Activity S.3.2 *(∼5 min)*

Consider two coupled masses with two springs.



Let x_1 be the position of the upper mass, and x_2 the position of the lower mass. Which ODE models the forces acting on the **upper** mass?

(A) $m_1 x_1'' + k_1 x_1 = 0$

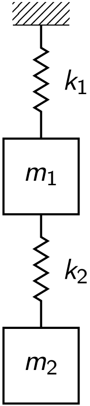
(B) $m_1 x_1'' + k_1 x_1 - k_2 x_2 = 0$

(C) $m_1 x_1'' + k_1 x_1 + k_2(x_2 - x_1) = 0$

(D) $m_1 x_1'' + k_1 x_1 + k_2(x_1 - x_2) = 0$

Activity S.3.3 *(∼30 min)*

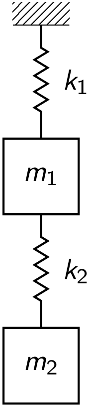
Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s}^2$, and $k_2 = 2\mathrm{kg/s}^2$. Then our model is



$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

Activity S.3.3 *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s^2}$, and $k_2 = 2\mathrm{kg/s^2}$. Then our model is

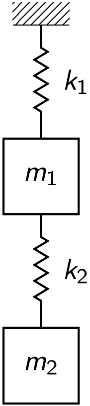


$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

Part 1: Rewrite the system using differential operators.

Activity S.3.3 *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg}/\mathrm{s}^2$, and $k_2 = 2\mathrm{kg}/\mathrm{s}^2$. Then our model is

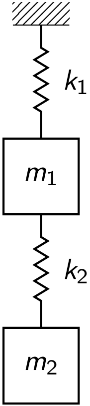


$$x_1'' + 6x_1 - 2x_2 = 0$$
$$2x_2'' + 2x_2 - 2x_1 = 0$$

Part 1: Rewrite the system using differential operators.
Part 2: Use elimination to write a single fourth order ODE for x_1.

Activity S.3.3 *(∼30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s^2}$, and $k_2 = 2\mathrm{kg/s^2}$. Then our model is



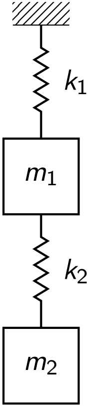
$$
\begin{aligned}
x_1'' + 6x_1 - 2x_2 &= 0\\
2x_2'' + 2x_2 - 2x_1 &= 0
\end{aligned}
$$

Part 1: Rewrite the system using differential operators.
Part 2: Use elimination to write a single fourth order ODE for x_1.
Part 3: Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

Activity S.3.3 *(~30 min)*

Suppose we are given $m_1 = 2\mathrm{kg}$, $m_2 = 1\mathrm{kg}$, $k_1 = 4\mathrm{kg/s^2}$, and $k_2 = 2\mathrm{kg/s^2}$. Then our model is



$$\begin{aligned} x_1'' + 6x_1 - 2x_2 &= 0 \\ 2x_2'' + 2x_2 - 2x_1 &= 0 \end{aligned}$$

Part 1: Rewrite the system using differential operators.
Part 2: Use elimination to write a single fourth order ODE for x_1.
Part 3: Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

Part 4: Determine a function for x_2 as well.

Module S Section 4

Activity S.4.1 *(∼5 min)*

Consider a forest of bamboo that grows unimpeded by other organisms. Which ODE models the size of the population best (all constants are positive)?

(a) $\frac{dB}{dt} = k$

(b) $\frac{dB}{dt} = kB$

(c) $\frac{dB}{dt} = kB - aB^2$

(d) $\frac{dB}{dt} = kB^2$

Activity S.4.2 *(∼5 min)*

The model

$$\frac{dB}{dt} = kB$$

models an ideal growth, free from competition (e.g. if population is sparse).

The model

$$\frac{dB}{dt} = kB - aB^2$$

models competitive growth.

Observe that both models are autonomous. Draw a phase line for each model, and describe the possible long term behaviors.

Activity S.4.3 *(∼10 min)*

Which of the following best models the bamboo population in the presence of a panda population (P)?

(a) $\frac{dB}{dt} = kB - aB^2$

(b) $\frac{dB}{dt} = kB - aB^2 - cP$

(c) $\frac{dB}{dt} = kB - aB^2 - cP^2$

(d) $\frac{dB}{dt} = kB - aB^2 - cBP$

Activity S.4.4 *(∼5 min)*

Which of the following best models the (sparse) Panda population in the bamboo forest?

(a) $\frac{dP}{dt} = -dP$

(b) $\frac{dP}{dt} = -dP + fBP$

(c) $\frac{dP}{dt} = -dP - fBP$

(d) $\frac{dP}{dt} = -dP - fBP - gP^2$

Observation S.4.5

The interacting bamboo and Panda populations are modelled by the **autonomous system**

$$\frac{dB}{dt} = kb - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

These are referred to as **Lotka-Volterra equations**

Activity S.4.6 *(∼10 min)*

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Activity S.4.6 *(∼10 min)*

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Part 1: When is $\frac{dB}{dt}$ zero?

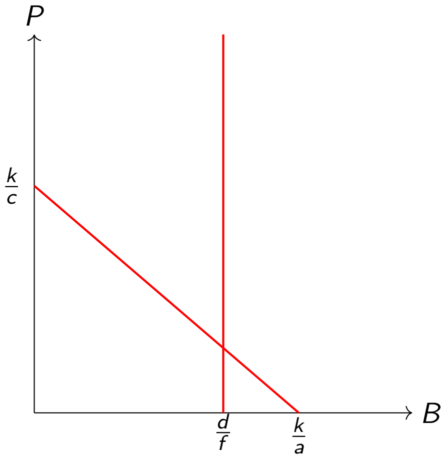
Activity S.4.6 *(∼10 min)*
Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

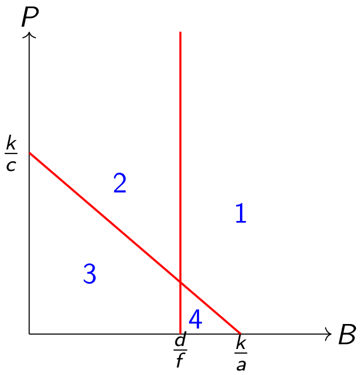
Part 1: When is $\frac{dB}{dt}$ zero?
Part 2: When is $\frac{dP}{dt}$ zero?

Observation S.4.7

These lines where the population of one species is unchanging are called **isoclines**

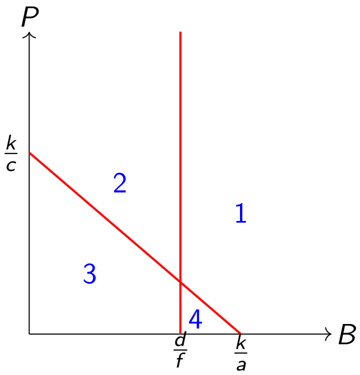


Activity S.4.8 *(∼15 min)*



For each of the four regions

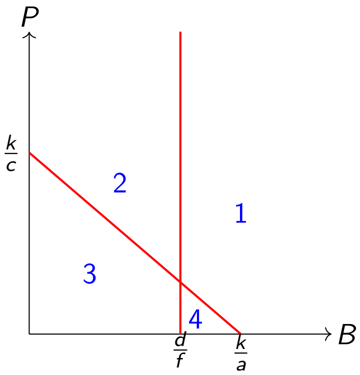
Activity S.4.8 *(∼15 min)*



For each of the four regions

Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Activity S.4.8 *(~15 min)*

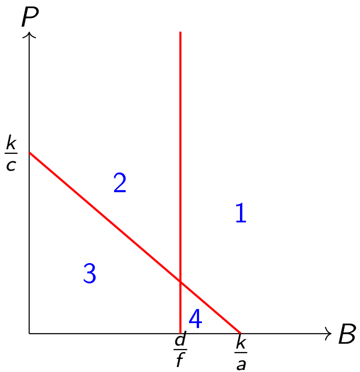


For each of the four regions

Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

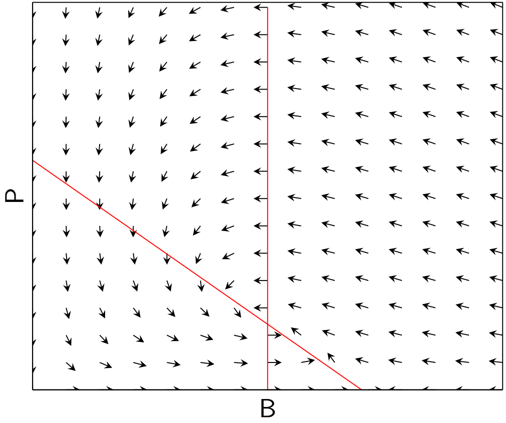
Activity S.4.8 *(∼15 min)*



For each of the four regions
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.
Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).
Part 3: Describe the general shape of the trajectories.

Observation S.4.9

Plotting the slope field with software makes it more clear that the trajectories are closed curves.



Module S Section 5

Activity S.5.1 *(∼10 min)*

Consider populations of Green Sunfish (G) and Bluegills (B) in the same lake. They compete for the same food.

Which system of ODEs would model this interaction best?

(A)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

(B)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = -0.1B - 0.003B^2 + 0.005BG$$

(C)

$$\frac{dG}{dt} = -0.1G - 0.002G^2 + 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

(D)

$$\frac{dG}{dt} = 0.1G - 0.002G^2 + 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 + 0.005BG$$

Activity S.5.2 *(∼15 min)*

Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

Activity S.5.2 *(∼15 min)*

Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

Part 1: Plot the isoclines for each species.

Activity S.5.2 *(∼15 min)*
Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

Part 1: Plot the isoclines for each species.
Part 2: If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?

Activity S.5.2 *(~15 min)*

Consider our Greenfish-Bluegill lake modeled by

$$\frac{dG}{dt} = 0.1G - 0.002G^2 - 0.005BG$$
$$\frac{dB}{dt} = 0.1B - 0.003B^2 - 0.005BG$$

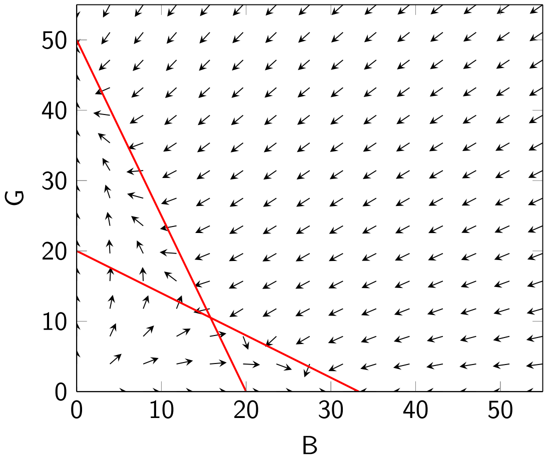
Part 1: Plot the isoclines for each species.

Part 2: If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?

Part 3: If the lake is stocked with 25 Bluegills and 5 Greenfish, what will happen?

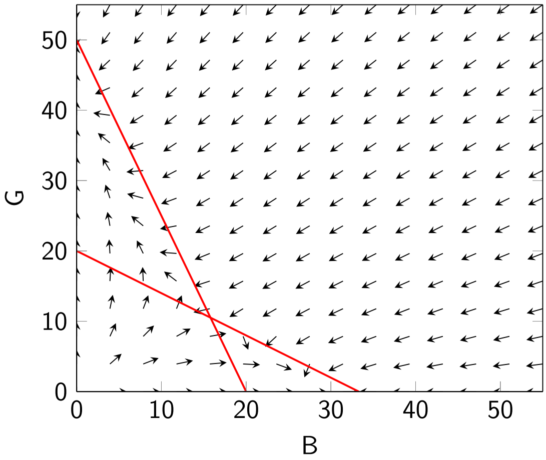
Activity S.5.3 *(~5 min)*

Plotting the slope field along with the isoclines makes the unstable behavior more clear.



Activity S.5.3 *(∼5 min)*

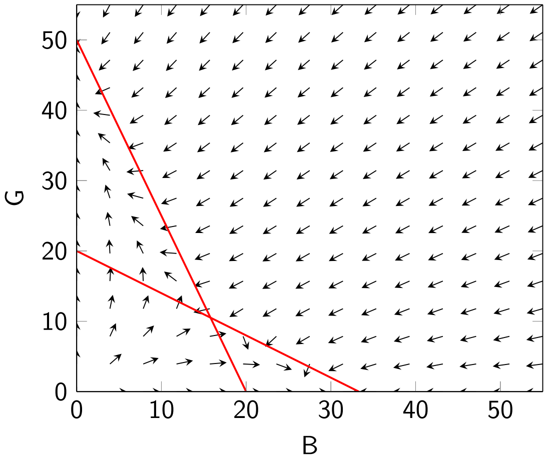
Plotting the slope field along with the isoclines makes the unstable behavior more clear.



Part 1: If the lake is stocked with 20 of each species, what will happen?

Activity S.5.3 *(∼5 min)*

Plotting the slope field along with the isoclines makes the unstable behavior more clear.



Part 1: If the lake is stocked with 20 of each species, what will happen?

Part 2: If the lake is stocked with 30 Bluegills and 10 Greenfish, what will happen?