

Module S

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Module S: Systems of ODEs

Module S

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How can we solve and apply systems of linear ODEs?

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At the end of this module, students will be able to...

- S1. Solving systems.** ...solve systems of constant coefficient ODEs
- S2. Modeling interacting populations.** ...model the populations of two interacting populations with a system of ODEs
- S3. Modeling coupled oscillators.** ...model systems of coupled mechanical oscillators using a system of ODEs

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Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve systems of two equations in two variables, even when coefficients are functions.
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equilibria of autonomous ODES **F4**

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The following resources will help you prepare for this module.

- Solve systems of two equations in two variables, even when coefficients are functions. <https://youtu.be/Y6JsEja15Vk>
- Solve second order constant coefficient equations, including non-homogeneous ones **C3,C5**.
- Model simple mechanical oscillators (e.g. spring-damper systems) **C6**.
- Find and classify the equilibria of autonomous ODES **F4**

Module S Section 1

Activity S.1.1 (~ 10 min)

Consider the countries of Transia and Wakanda: each year, 8% of people living in Transia move to Wakanda, and 3% of Wakandans move to Transia.

Let T be the population of Transia, and W the population of Wakanda (both are functions of time, t).

Which **system of differential equations** models the population changes $\frac{dT}{dt}$ and $\frac{dW}{dt}$?

(A)

$$\begin{aligned}\frac{dT}{dt} &= 0.03W + 0.08T \\ \frac{dW}{dt} &= 0.08T + 0.03W\end{aligned}$$

(C)

$$\begin{aligned}\frac{dT}{dt} &= 0.03W - 0.08T \\ \frac{dW}{dt} &= 0.08T - 0.03W\end{aligned}$$

(B)

$$\begin{aligned}\frac{dT}{dt} &= -0.03W + 0.08T \\ \frac{dW}{dt} &= -0.08T + 0.03W\end{aligned}$$

(D)

$$\begin{aligned}\frac{dT}{dt} &= -0.03W - 0.08T \\ \frac{dW}{dt} &= 0.08T + 0.03W\end{aligned}$$

Activity S.1.2 (*~5 min*)

This problem resulted in a **system of linear differential equations**, namely

$$T' = 0.03W - 0.08T$$

$$W' = 0.08T - 0.03W$$

Rewrite this system using differential operators.

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Activity S.1.3 (~ 15 min)

Solve the system

$$(D + 0.08)T - (0.03)W = 0$$

$$-0.08T + (D + 0.03)W = 0$$

Observation S.1.4

Because D is linear, $a(D + b) = (D + b)a$ for constants a, b . This is not true in general!

Thus, for any **constant coefficient linear systems of differential equations**, we can use our typical elimination technique.

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Activity S.1.5 (*~15 min*)

Solve the system

$$x' = 5x - 2y$$

$$y' = 6y - 3x$$

with initial conditions $x(0) = 2$, $y(0) = -1$.

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Activity S.1.6 (*~15 min*)

Solve the system

$$x' = -y + 3t^2$$

$$y' = x + 2y - t^3$$

with initial conditions $x(0) = 1$, $y(0) = 1$.

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Activity S.2.1 (*~20 min*)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

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Activity S.2.1 (*~20 min*)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

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Activity S.2.1 (*~20 min*)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators*Part 2:* Use elimination to eliminate a variable

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Activity S.2.1 (*~20 min*)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

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Activity S.2.1 (~ 20 min)

Solve the system

$$x' = 3x - 4y + 1$$

$$y' = 4x - 7y + 10t$$

Part 1: Rewrite the system using differential operators

Part 2: Use elimination to eliminate a variable

Part 3: Solve the resulting second order ODE in one variable.

Part 4: Find the solution for the other variable.

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Activity S.2.2 (*~20 min*)

Solve the system

$$x' = 2x + 6y - 2$$

$$y' = 5x + 3y + 5 - e^{-3t}$$

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Activity S.2.3 (*~15 min*)

Solve the system

$$x' = 3x - 2y + \sin(t)$$

$$y' = 4x - y - \cos(t)$$

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Activity S.3.1 (*~5 min*)

Consider a forest of bamboo that grows unimpeded by other organisms. Which ODE models the size of the population best (all constants are positive)?

(a) $\frac{dB}{dt} = k$

(b) $\frac{dB}{dt} = kB$

(c) $\frac{dB}{dt} = kB - aB^2$

(d) $\frac{dB}{dt} = kB^2$

Activity S.3.2 (*~5 min*)

The model

$$\frac{dB}{dt} = kB$$

models an ideal growth, free from competition (e.g. if population is sparse).

The model

$$\frac{dB}{dt} = kB - aB^2$$

models competitive growth.

Observe that both models are autonomous. Draw a phase line for each model, and describe the possible long term behaviors.

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Activity S.3.3 (~ 10 min)

Which of the following best models the bamboo population in the presence of a panda population (P)?

(a) $\frac{dB}{dt} = kB - aB^2$

(b) $\frac{dB}{dt} = kB - aB^2 - cP$

(c) $\frac{dB}{dt} = kB - aB^2 - cP^2$

(d) $\frac{dB}{dt} = kB - aB^2 - cBP$

Activity S.3.4 (~ 5 min)

Which of the following best models the (sparse) Panda population in the bamboo forest?

(a) $\frac{dP}{dt} = -dP$

(b) $\frac{dP}{dt} = -dP + fBP$

(c) $\frac{dP}{dt} = -dP - fBP$

(d) $\frac{dP}{dt} = -dP - fBP - gP^2$

Observation S.3.5

The interacting bamboo and Panda populations are modelled by the **autonomous system**

$$\begin{aligned}\frac{dB}{dt} &= kb - aB^2 - cBP \\ \frac{dP}{dt} &= -dP + fBP\end{aligned}$$

These are referred to as **Lotka-Volterra equations**

Activity S.3.6 (*~10 min*)

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$

$$\frac{dP}{dt} = -dP + fBP$$

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Activity S.3.6 (~ 10 min)

Consider our Panda-Bamboo system

$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Part 1: When is $\frac{dB}{dt}$ zero?

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Activity S.3.6 (~ 10 min)

Consider our Panda-Bamboo system

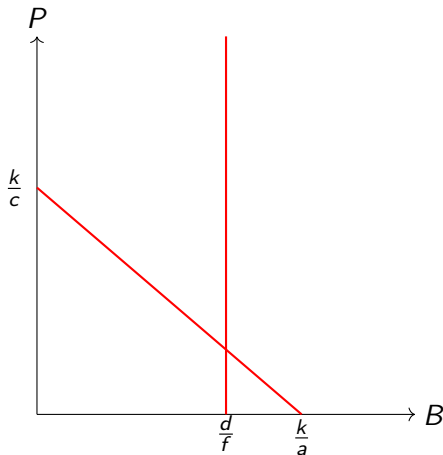
$$\frac{dB}{dt} = kB - aB^2 - cBP$$
$$\frac{dP}{dt} = -dP + fBP$$

Part 1: When is $\frac{dB}{dt}$ zero?

Part 2: When is $\frac{dP}{dt}$ zero?

Observation S.3.7

These lines where the population of one species is unchanging are called **isoclines**



Activity S.3.8 (~ 15 min)

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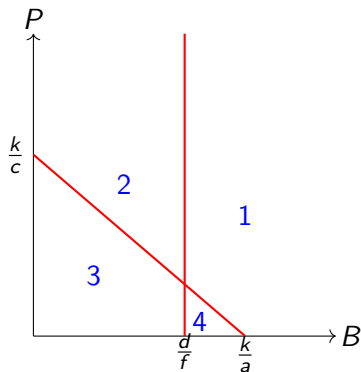
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For each of the four regions

Activity S.3.8 (~ 15 min)

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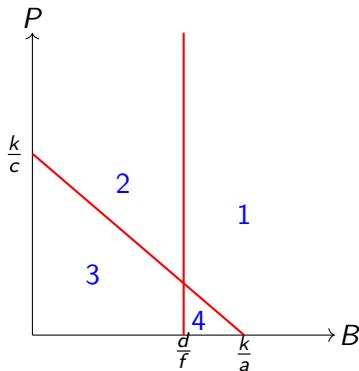
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For each of the four regions

Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Activity S.3.8 (~ 15 min)

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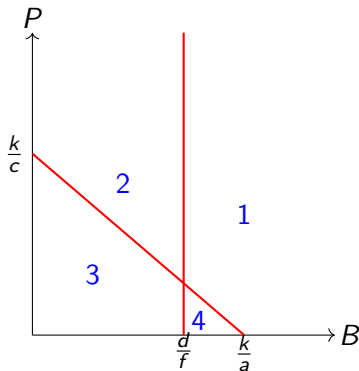
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For each of the four regions

Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

Activity S.3.8 (~ 15 min)

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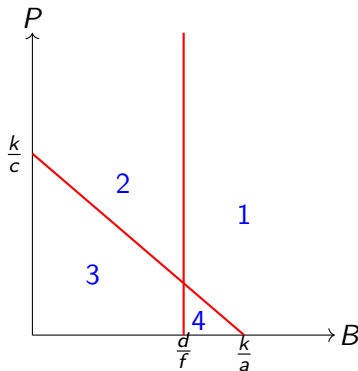
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For each of the four regions

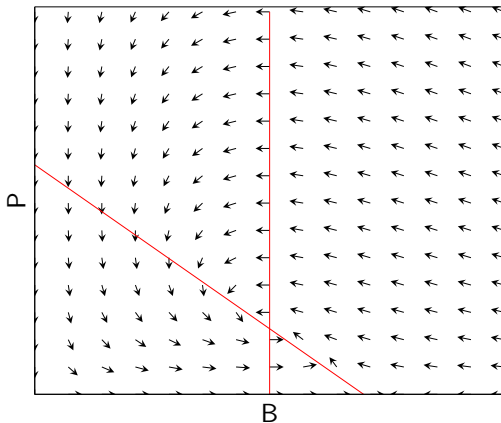
Part 1: Determine if each of $\frac{dP}{dt}$ and $\frac{dB}{dt}$ is positive or negative.

Part 2: Determine the general direction of a solution curve (**trajectory**) in that region (e.g. up and right).

Part 3: Describe the general shape of the trajectories.

Observation S.3.9

Plotting the slope field with software makes it more clear that the trajectories are closed curves.



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Activity S.4.1 (~ 10 min)

Consider populations of Green Sunfish (G) and Bluegills (B) in the same lake. They compete for the same food.

Which system of ODEs would model this interaction best?

(A)

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

(B)

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= -0.1B - 0.003B^2 + 0.005BG\end{aligned}$$

(C)

$$\begin{aligned}\frac{dG}{dt} &= -0.1G - 0.002G^2 + 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

(D)

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 + 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 + 0.005BG\end{aligned}$$

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Activity S.4.2 (*~15 min*)

Consider our Greenfish-Bluegill lake modeled by

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

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Activity S.4.2 (*~15 min*)

Consider our Greenfish-Bluegill lake modeled by

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

Part 1: Plot the isoclines for each species.

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Activity S.4.2 (*~15 min*)

Consider our Greenfish-Bluegill lake modeled by

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

Part 1: Plot the isoclines for each species.

Part 2: If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?

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Activity S.4.2 (~ 15 min)

Consider our Greenfish-Bluegill lake modeled by

$$\begin{aligned}\frac{dG}{dt} &= 0.1G - 0.002G^2 - 0.005BG \\ \frac{dB}{dt} &= 0.1B - 0.003B^2 - 0.005BG\end{aligned}$$

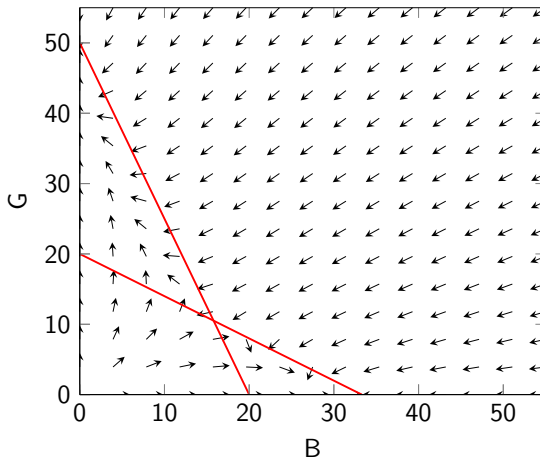
Part 1: Plot the isoclines for each species.

Part 2: If the lake is stocked with 10 Bluegills and 20 Greenfish, what will happen?

Part 3: If the lake is stocked with 25 Bluegills and 5 Greenfish, what will happen?

Activity S.4.3 (~ 5 min)

Plotting the slope field along with the isoclines makes the unstable behavior more clear.



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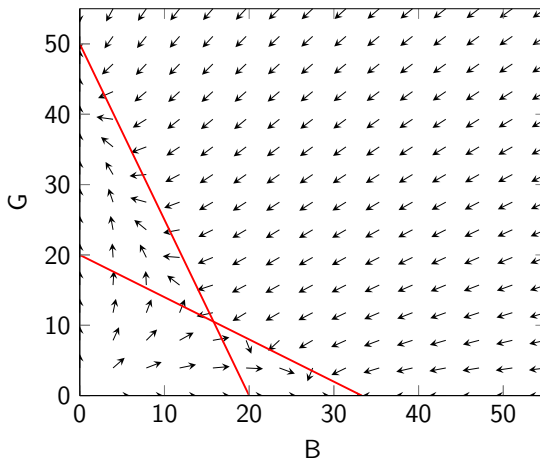
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Activity S.4.3 (~ 5 min)

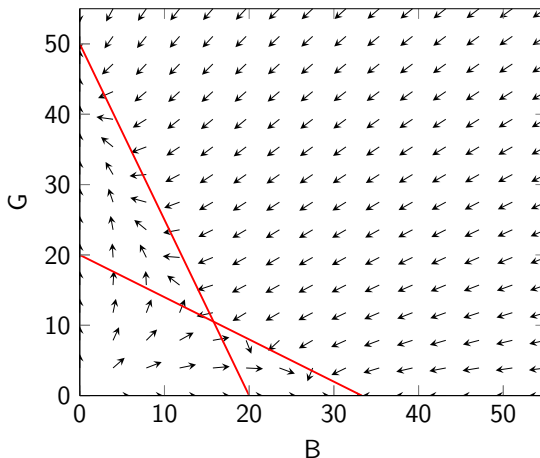
Plotting the slope field along with the isoclines makes the unstable behavior more clear.



Part 1: If the lake is stocked with 20 of each species, what will happen?

Activity S.4.3 (~ 5 min)

Plotting the slope field along with the isoclines makes the unstable behavior more clear.



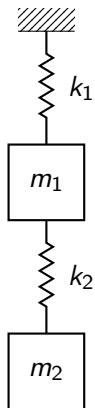
Part 1: If the lake is stocked with 20 of each species, what will happen?

Part 2: If the lake is stocked with 30 Bluegills and 10 Greenfish, what will happen?

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Activity S.5.1 (~ 10 min)

Consider two coupled masses with two springs.

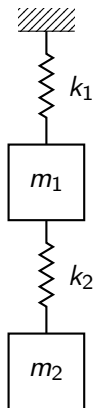


Let x_1 be the position of the upper mass, and x_2 the position of the lower mass (both measured from equilibrium). Which ODE models the forces acting on the **lower** mass?

- (A) $m_2 x_2'' + k_2 x_2 = 0$
- (B) $m_2 x_2'' + k_2 x_1 = 0$
- (C) $m_2 x_2'' + k_2(x_2 - x_1) = 0$
- (D) $m_2 x_2'' + k_2(x_1 - x_2) = 0$

Activity S.5.2 (~ 5 min)

Consider two coupled masses with two springs.

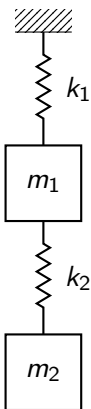


Let x_1 be the position of the upper mass, and x_2 the position of the lower mass. Which ODE models the forces acting on the **upper** mass?

- (A) $m_1 x_1'' + k_1 x_1 = 0$
- (B) $m_1 x_1'' + k_1 x_1 - k_2 x_2 = 0$
- (C) $m_1 x_1'' + k_1 x_1 + k_2(x_2 - x_1) = 0$
- (D) $m_1 x_1'' + k_1 x_1 + k_2(x_1 - x_2) = 0$

Activity S.5.3 (*~30 min*)

Suppose we are given $m_1 = 2\text{kg}$, $m_2 = 1\text{kg}$, $k_1 = 4\text{kg/s}^2$, and $k_2 = 2\text{kg/s}^2$. Then our model is

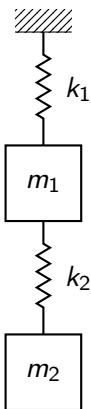


$$x_1'' + 6x_1 - 2x_2 = 0$$

$$2x_2'' + 2x_2 - 2x_1 = 0$$

Activity S.5.3 (*~30 min*)

Suppose we are given $m_1 = 2\text{kg}$, $m_2 = 1\text{kg}$, $k_1 = 4\text{kg/s}^2$, and $k_2 = 2\text{kg/s}^2$. Then our model is

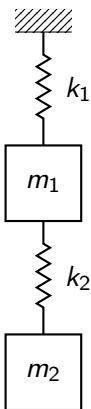


$$\begin{aligned}x_1'' + 6x_1 - 2x_2 &= 0 \\2x_2'' + 2x_2 - 2x_1 &= 0\end{aligned}$$

Part 1: Rewrite the system using differential operators.

Activity S.5.3 (~ 30 min)

Suppose we are given $m_1 = 2\text{kg}$, $m_2 = 1\text{kg}$, $k_1 = 4\text{kg/s}^2$, and $k_2 = 2\text{kg/s}^2$. Then our model is



$$x_1'' + 6x_1 - 2x_2 = 0$$

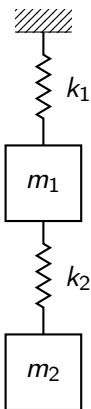
$$2x_2'' + 2x_2 - 2x_1 = 0$$

Part 1: Rewrite the system using differential operators.

Part 2: Use elimination to write a single fourth order ODE for x_1 .

Activity S.5.3 (~ 30 min)

Suppose we are given $m_1 = 2\text{kg}$, $m_2 = 1\text{kg}$, $k_1 = 4\text{kg/s}^2$, and $k_2 = 2\text{kg/s}^2$. Then our model is



$$x_1'' + 6x_1 - 2x_2 = 0$$

$$2x_2'' + 2x_2 - 2x_1 = 0$$

Part 1: Rewrite the system using differential operators.

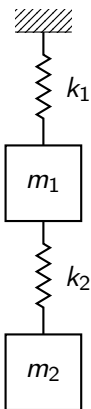
Part 2: Use elimination to write a single fourth order ODE for x_1 .

Part 3: Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

Activity S.5.3 (~ 30 min)

Suppose we are given $m_1 = 2\text{kg}$, $m_2 = 1\text{kg}$, $k_1 = 4\text{kg/s}^2$, and $k_2 = 2\text{kg/s}^2$. Then our model is



$$x_1'' + 6x_1 - 2x_2 = 0$$

$$2x_2'' + 2x_2 - 2x_1 = 0$$

Part 1: Rewrite the system using differential operators.

Part 2: Use elimination to write a single fourth order ODE for x_1 .

Part 3: Solve the ODE

$$2x_1'''' + 10x_1'' + 8x_1 = 0.$$

Part 4: Determine a function for x_2 as well.