Module D: Discontinuous functions in ODEs

How can we solve and apply ODEs involving functions that are not continuous?

At the end of this module, students will be able to...

D1. Laplace Transform. ...compute the Laplace transform of a function

D2. Discontinuous ODEs. ...solve initial value problems for ODEs with discontinuous coefficients

D3. Modeling non-smooth motion. ...model the motion of an object undergoing discontinuous acceleration

D4. Modeling non-smooth oscillators. ...model mechanical oscillators undergoing discontinuous acceleration

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compute integrals by using integration by parts
- Evaluate improper integrals
- Use a partial fraction decomposition to rewrite a rational expression
- Model a mass-spring system (Standard C6)

The following resources will help you prepare for this module.

- Compute integrals by using integration by parts
  https://youtu.be/S1Bp9hZBqaQ, https://youtu.be/bZ8YAHDTFJ8
- Evaluate improper integrals https://youtu.be/qv7DM5Ph0vU
- Use a partial fraction decomposition to rewrite a rational expression
  https://youtu.be/HZTv4zCgEnA, https://youtu.be/ofQt2NOUpCg
- Model a mass-spring system (Standard C6)

Module D Section 1

Observation D.1.1

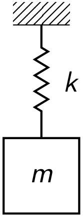
In this module, we want to learn how to model (and solve) situtations with **discontinuous** force, such as

- Collisions
- Thrust that can be turned on and off instantly
- Applied voltages that can be turned on and off instantly

Today we will learn how to model these forces, and introduce a tool called them **Laplace Transform** that we will use to solve the resulting IVPs.

Activity D.1.2 *(∼5 min)*

A 4 kg mass is hung from a spring with spring constant $k = 16\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting $3\mathrm{Ns}$ of upward impulse.



Write an initial value problem modelling this system.

Definition D.1.3

The **Dirac delta distribution** $\delta(t)$ models the application of instantaneous force. **It is not a function**, but makes sense in definite integrals:

If a, b is any open interval containing 0, then

$$\int_a^b f(t)\delta(t)dt = f(0)$$

for any function $f(t)$ that is continuous around 0.

Thus, we can model the situation in the previous activity by

$$4y'' + 16y = 3\delta(t), \qquad y(0) = 0,\ y'(0) = 0$$

Definition D.1.4

We can make sense of δ in another way: as the "derivative" of a non-differentiable function.

The **unit impulse function** $u(t)$ is given by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0. \end{cases}$$

Note that $u(s) = \int_{-\infty}^{s} \delta(t)\, dt$; in this fuzzy sense, δ is the "derivative" of $u(t)$ (which is not differentiable everywhere!)

Activity D.1.5 *(∼10 min)*
Try to solve the IVP

$$y'' + y = \delta(t)$$

Where does our existing technique break down?

Observation D.1.6

To get around this difficulty, we will apply an **integral transform** called the **Laplace Transform** to our ODE.

- We want to use a definite integral to handle things like δ, which we can only understand via a definite integral.
- Since we are focused on IVPs, we can integrate starting at 0, but need to go to ∞
- But now we need to worry about convergence–thus we will multiply by a suitable function that decays fast enough to make most functions converge.

Activity D.1.7 *(∼5 min)*

Arrange the following functions in order of how fast they decay to zero in the limit at infinity:

(A) x^{-n} for a positive integer n

(B) e^{-ax} for a positive integer a

(C) $\frac{1}{\ln(ax)}$ for a positive integer a

(D) $\frac{1}{\ln(x^n)}$ for a positive integer n

Definition D.1.8

The **Laplace Transform** of a function $f(t)$ is the function

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

Note that the Laplace transform turns a function of t into a function of s.

Moreover, $\mathcal{L}$ is linear: $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$, and $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for constants c.

Activity D.1.9 *(∼5 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

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$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{\delta(t)\}$

Activity D.1.9 *(∼5 min)*
Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{\delta(t)\}$
Part 2: If $a > 0$, compute $\mathcal{L}\{\delta(t-a)\}$

Activity D.1.10 *(∼5 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Activity D.1.10 *(∼5 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{e^t\}$

Activity D.1.10 *(∼5 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{e^t\}$

Part 2: If $a > 0$, compute $\mathcal{L}\{e^{at}\}$

Module D Section 2

Observation D.2.1

Last week, we encountered the **Laplace Transform** of a function $f(t)$:

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

Recall that the Laplace transform turns a function of t into a function of s.

Moreover, $\mathcal{L}$ is linear: $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$, and $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for constants c.

Our goal for today is to develop a few more properties of $\mathcal{L}$, and then see how to use it to solve IVPs (standards D1,D2).

Observation D.2.2

We computed a few Laplace Transforms:

- $\mathcal{L}\{\delta(t)\} = 1$
- $\mathcal{L}\{\delta(t-a)\} = e^{-as}$ for any $a > 0$.
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

Activity D.2.3 *(∼10 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Activity D.2.3 *(∼10 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$

Activity D.2.3 *(∼10 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$

Part 2: Compute $\mathcal{L}\{t\}$

Activity D.2.3 *(∼10 min)*

Recall that

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Part 1: Compute $\mathcal{L}\{1\}$

Part 2: Compute $\mathcal{L}\{t\}$

Part 3: Compute $\mathcal{L}\{t^2\}$

Activity D.2.4 *(∼10 min)*

Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)dt.$$

Activity D.2.4 *(∼10 min)*
Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)dt.$$

Part 1: Compute $\mathcal{L}\{\sin(t)\}$.

Activity D.2.4 *(∼10 min)*
Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)dt.$$

Part 1: Compute $\mathcal{L}\{\sin(t)\}$.
Part 2: Compute $\mathcal{L}\{\cos(t)\}$.

Observation D.2.5

So now our list of Laplace transforms is:

- $\mathcal{L}\{\delta(t-a)\} = e^{-as}$ for any $a > 0$.
- $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
- $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ for any positive integer n.
- $\mathcal{L}\{1\} = \frac{1}{s}$
- $\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$
- $\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$

Activity D.2.6 *(∼10 min)*

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt.$$

Activity D.2.6 *(∼10 min)*
Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt.$$

Part 1: Use integration by parts to relate $\mathcal{L}\{y'\}$ to $\mathcal{L}\{y\}$.

Activity D.2.6 *(∼10 min)*

Suppose we want to apply the Laplace transform to an IVP: we will need to know how $\mathcal{L}\{y'\}$ is related to $\mathcal{L}\{y\}$. Recall

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt.$$

Part 1: Use integration by parts to relate $\mathcal{L}\{y'\}$ to $\mathcal{L}\{y\}$.
Part 2: Use integration by parts (and the fact that $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$) to relate $\mathcal{L}\{y''\}$ to $\mathcal{L}\{y\}$.

Observation D.2.7

We have

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$
$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

This allows us to easily rewrite expressions like $ay'' + by' + cy$ in terms of $\mathcal{L}\{y\}$.

Activity D.2.8 *(∼10 min)*
Consider the simple IVP

$$y'' + y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$

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Part 1: Apply the Laplace transform to this IVP, and simplify. Solve for $\mathcal{L}\{y\}$.

Activity D.2.8 *(∼10 min)*
Consider the simple IVP

$$y'' + y = \delta(t), \qquad y(0) = 0, y'(0) = 0.$$

Part 1: Apply the Laplace transform to this IVP, and simplify. Solve for $\mathcal{L}\{y\}$.
Part 2: Find a function y satisfying $\mathcal{L}\{y\} = \frac{1}{s^2+1}$. We write $y = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$.

Activity D.2.9 *(∼15 min)*
Solve the IVP

$$y'' + y = \delta(t), \qquad y(0) = 1, y'(0) = 2.$$

Module D Section 3

Observation D.3.1

To solve a linear IVP using Laplace transforms:

1) Apply $\mathcal{L}$ to the ODE. Use the initial condition(s) in computing $\mathcal{L}\{y'\}$, $\mathcal{L}\{y''\}$, etc.
2) Solve for $\mathcal{L}\{y\}$.
3) Take the inverse transform (using a table) to find the solution y.

Today our goal is to practice the last step (taking the inverse transform), and then practice solving IVPs this way.

Activity D.3.2 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$.

(a) $u(t-5)$

(b) $\delta(t-5)$

(c) e^{5t}

(d) e^{-5t}

Activity D.3.3 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{s}\right\}$.

(a) $u(t-10)$

(b) $\delta(t-10)$

(c) $u(t-10)e^{-t}$

(d) $\delta(t-10)e^{-t}$

Activity D.3.4 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2+4}\right\}$.

(a) $u(t)\sin(2t)$

(b) $u(t-2)\sin(2t)$

(c) $u(t-2)\sin(2t-2)$

(d) $u(t-2)\sin(2t-4)$

Activity D.3.5 *(∼5 min)*

Compute $\mathcal{L}^{-1}\left\{\frac{e^{-100s}}{s^2}\right\}$.

(a) $u(t)t$

(b) $u(t)(t-100)$

(c) $u(t-100)t$

(d) $u(t-100)(t-100)$

Activity D.3.6 *(∼15 min)*
Solve the IVP

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad v(0) = 50.$$

Activity D.3.7 *(∼15 min)*

Solve the IVP

$$y'' + 4y = \delta(t-2), \qquad y(0) = 0,\ y'(0) = 1$$

Module D Section 4

Observation D.4.1

Today we will practice modeling an object undergoing discontinuous acceleration (standard D3).

Activity D.4.2 *(∼30 min)*

A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

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Part 1: Write down a function modelling the thrust force on the spacecraft.

Activity D.4.2 *(∼30 min)*

A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

Part 2: Write down an IVP modelling the velocity of the spacecraft.

Activity D.4.2 *($\sim$30 min)*

A spacecraft weighing 500 kg is travelling 50 $\mathrm{m/s}$. At time $t = 0$, its thrusters (which provide 20 N of force) are turned on and burn for 100 s.

Part 1: Write down a function modelling the thrust force on the spacecraft.

Part 2: Write down an IVP modelling the velocity of the spacecraft.

Part 3: Solve the IVP:

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad\qquad v(0) = 50.$$

Activity D.4.2 *(∼30 min)*

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Part 1: Write down a function modelling the thrust force on the spacecraft.

Part 2: Write down an IVP modelling the velocity of the spacecraft.

Part 3: Solve the IVP:

$$500v' = 20\left(u(t) - u(t-100)\right), \qquad v(0) = 50.$$

Part 4: What is its velocity after 200 s?

Module D Section 5

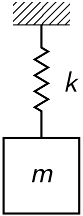
Observation D.5.1

Last week we saw how to use the Laplace transform to model a spacecraft undergoing discontinuous acceleration.

Today we will model springs undergoing discontinuous acceleration (standard D4).

Activity D.5.2 *(∼10 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting $3\mathrm{Ns}$ of upward impulse.



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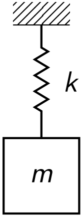

Part 1: Write an initial value problem modelling this system.

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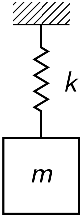
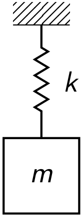


Part 1: Write an initial value problem modelling this system.

Part 2: Use the Laplace transform to solve this IVP.

Activity D.5.2 *(∼10 min)*

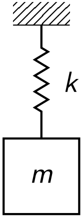
A 1 kg mass is hung from a spring with spring constant $k = 1\ \mathrm{N/m}$. The mass is at rest, when it is hit with a hammer imparting $3\mathrm{Ns}$ of upward impulse.



Part 1: Write an initial value problem modelling this system.
Part 2: Use the Laplace transform to solve this IVP.
Part 3: When will the mass first return to equillibrium?

Activity D.5.3 *(~15 min)*

A $1\ \mathrm{kg}$ mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$. The mass is pulled down $1\ \mathrm{m}$ and released from rest. 10 seconds later, it is hit with a hammer imparting $5\mathrm{Ns}$ of upward impulse.



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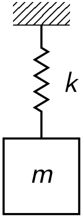

Part 1: Write an initial value problem modelling this system.

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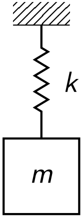

Part 1: Write an initial value problem modelling this system.

Part 2: Use the Laplace transform to solve this IVP.

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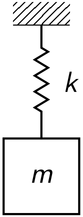
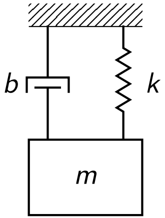


Part 1: Write an initial value problem modelling this system.
Part 2: Use the Laplace transform to solve this IVP.
Part 3: Where is the mass after 15 s?

Activity D.5.4 *(∼15 min)*

A 1 kg mass is hung from a spring with spring constant $k = 4\ \mathrm{N/m}$ and a $b = 4\ \mathrm{kg/s^2}$ linear damper. The mass is pulled down 1 m and released from rest. 5 seconds later, it is hit with a hammer imparting $10\mathrm{Ns}$ of upward impulse.



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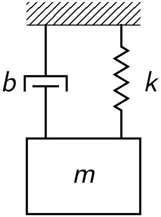

Part 1: Write an initial value problem modelling this system.

Activity D.5.4 *(∼15 min)*

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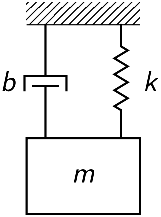

Part 1: Write an initial value problem modelling this system.

Part 2: Use the Laplace transform to solve this IVP.

Activity D.5.4 *(∼15 min)*

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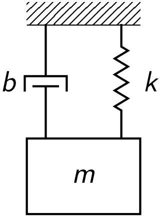

Part 1: Write an initial value problem modelling this system.
Part 2: Use the Laplace transform to solve this IVP.
Part 3: Where is the mass after $15\ \mathrm{s}$?

Module D Section 6