

## Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

**C1.** Find the general solution to

$$y' + y = -t^2.$$

**Solution:** First, we find a general solution to the homogeneous equation

$$y' + y = 0.$$

This has auxiliary equation  $r + 1 = 0$ , which has a single root at  $r = -1$ , so  $ce^{-t}$  is a solution. We can find a particular solution  $y_p$  to the given equation by using undetermined coefficients; since  $-t^2$  is a polynomial, we let  $y_p = At^2 + Bt + D$  and determine the coefficients  $A$ ,  $B$ , and  $D$ .

$$\begin{aligned} y_p' + y_p &= (2At + B) + (At^2 + Bt + D) \\ &= At^2 + (2A + B)t + (B + D) \end{aligned}$$

So if  $y_p$  is a solution, we must have  $y_p' + y_p = -t^2$ , giving us the system of equations

$$\begin{aligned} A &= -1 \\ 2A + B &= 0 \\ B + D &= 0 \end{aligned}$$

Thus we easily deduce that  $A = -1$ ,  $B = 2$ , and  $D = -2$ , giving  $y_p = -t^2 + 2t - 2$ . Thus, the general solution is

$$y = -t^2 + 2t - 2 + ce^{-t}.$$

□

**C2.** Consider the following scenario: A water droplet with a radius of  $10 \mu\text{m}$  has a mass of about  $4 \times 10^{-15}\text{kg}$  and a terminal velocity of  $3 \frac{\text{cm}}{\text{s}}$ . The droplet is dropped from rest.

(a) Write down an IVP modelling the velocity of the water droplet.

(b) What is its velocity after  $0.01 \text{ s}$ ?

**Solution:** The forces acting on the water droplet are gravity ( $mg$  downwards) and air resistance; in a water droplet this size, this is proportional to its velocity. Let  $b$  denote this drag coefficient; then by Newton's second law (if we let up be the positive direction and down be negative), we have

$$m \frac{dv}{dt} = -mg - bv.$$

Note that we have  $-mg$  since gravity always acts downwards, and  $-bv$  because the drag acts in the opposite direction of  $v$ . It is actually convenient to let  $a = \frac{b}{m}$ , and divide this equation by the mass, yielding

$$\frac{dv}{dt} = -g - av.$$

This is a separable differential equation; we thus have

$$\int \frac{1}{g + av} dv = \int (-1) dt.$$

Thus,  $\frac{1}{a} \ln |g + av| = -t + c_1$ , or  $|g + av| = e^{-at+ac_1} = c_2 e^{-at}$  (where  $c_2 = e^{ac_1}$ ). To avoid the pesky absolute value sign, we can absorb a potential negative into the constant to write  $g + av = c_3 e^{-at}$ , or (letting  $c_0 = \frac{c_3}{a}$ )

$$v = -\frac{g}{a} + c_0 e^{-at}.$$

Note that  $\lim_{t \rightarrow \infty} v = -\frac{g}{a}$ , so since we are given the terminal velocity, so after converting to meters per second we can solve  $-0.03 = -\frac{9.8}{a}$  to obtain  $a = \frac{9.8}{0.03}$ . We also know that it is dropped from rest, so we compute  $0 = v(0) = -0.03 + c_0$ , so  $c_0 = 0.03$ . Thus our model for velocity is

$$v = -0.03 + 0.03e^{-\frac{9.8}{0.03}t}.$$

Then we simply compute  $v(0.01) = -0.03 + 0.03e^{-\frac{9.8}{0.03}(0.01)} \approx -0.029$ . So after 0.01 s, the drop is falling approximately  $2.9 \frac{\text{cm}}{\text{s}}$ . □

**C3.** Find the general solution to

$$y'' + 6y' + 13y = 0.$$

**Solution:** We begin by writing the auxilliary equation  $r^2 + 6r + 13 = 0$  and finding the roots. There are many ways to do this; here, we complete the square:

$$0 = r^2 + 6r + 13 = r^2 + 6r + 9 + 4 = (r + 3)^2 + 4.$$

Thus, we can easily solve to obtain  $r = -3 \pm 2i$ . Thus the general solution is

$$y = c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t).$$
□

**C4.** Find the solution to

$$y'' + 10y' + 24y = 0$$

when  $y(0) = -3$  and  $y'(0) = 2$ .

**Solution:** The auxilliary equation is  $r^2 + 10r + 24 = 0$ , which has roots  $r = -6$  and  $r = -4$ . Thus, the general solution is of the form  $y = c_1 e^{-4t} + c_2 e^{-6t}$ .

$$\begin{aligned} -3 &= y(0) = c_1 + c_2 \\ 2 &= y'(0) = -4c_1 - 6c_2 \end{aligned}$$

Solving this system yields  $c_1 = -8$  and  $c_2 = 5$ , so the solution to the IVP is

$$y = -8e^{-4t} + 5e^{-6t}$$
□

**C5.** Find the general solution to

$$y'' + 6y' + 13y = 13t^2 - t - 4.$$

**Solution:** First, we find a general solution to the homogenous equation  $y'' + 6y' + 13y = 0$ . We begin by writing the auxilliary equation  $r^2 + 6r + 13 = 0$  and finding the roots. There are many ways to do this; here, we complete the square:

$$0 = r^2 + 6r + 13 = r^2 + 6r + 9 + 4 = (r + 3)^2 + 4.$$

Thus, we can easily solve to obtain  $r = -3 \pm 2i$ . Thus the general solution to the homogeneous equation is

$$y = c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t).$$

To find a particular solution to  $y'' + 6y' + 13y = 13t^2 - t - 4$ , we set  $y_p = At^2 + Bt + D$  and determine the coefficients  $A, B, D$ .

$$\begin{aligned} y_p'' + 6y_p' + 13y_p &= (2A) + 6(2At + B) + 13(At^2 + Bt + D) \\ &= (13A)t^2 + (12A + 13B)t + (2A + 6B + 13D) \end{aligned}$$

This gives us the system of equations

$$\begin{aligned} 13A &= 13 \\ 12A + 13B &= -1 \\ 2A + 6B + 13D &= -4 \end{aligned}$$

Then we easily deduce  $A = 1$ ,  $B = -1$ , and  $D = 0$ , so that  $y_p = t^2 - t$ . Thus, the general solution to the nonhomogeneous equation is

$$y = c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t) + t^2 - t.$$

□

**C6.** Consider the following scenario: A 2kg mass is suspended by a spring (with spring constant  $8\text{kg/s}^2$ ). The mass is pulled down 1m from its equilibrium position and released from rest.

- (a) Write down an IVP modelling the position of the mass.
- (b) How long does it take for the mass to return to its equilibrium point?

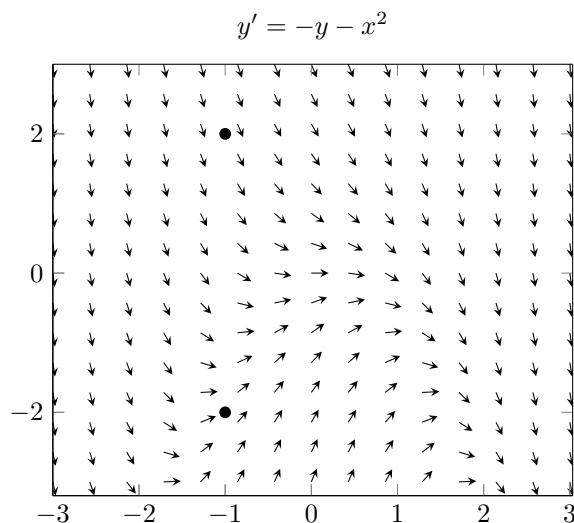
**Solution:** Let  $y$  denote the vertical distance from equilibrium; then the forces acting are gravity and the spring force, giving the ODE

$$2y'' + 8y = 0.$$

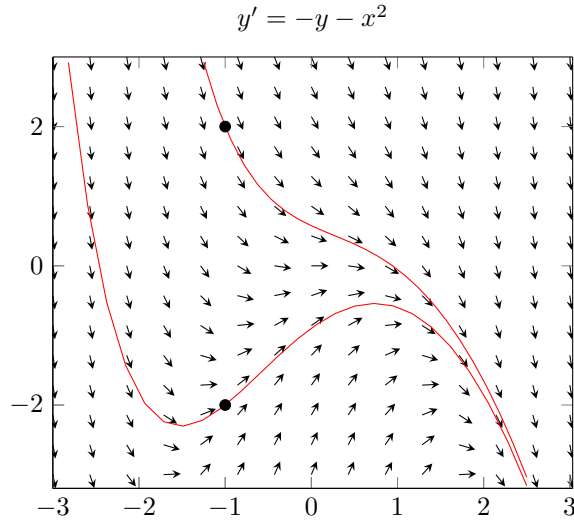
Simplifying, we have  $y'' + 4y = 0$ , so the general solution is  $y = c_1 \cos(2t) + c_2 \sin(2t)$ . The initial conditions are  $y(0) = -1$  and  $y'(0) = 0$ , which imply  $c_1 = -1$  and  $c_2 = 0$ , so the system is modelled by  $y = -\cos(2t)$ . This is first zero when  $2t = \frac{\pi}{2}$ , i.e. when  $t = \frac{\pi}{4}$ . Thus the mass takes  $\frac{\pi}{4}$  s to return to its equilibrium point.

□

**F1.** Sketch a solution curve through each point marked in the slope field.



**Solution:**



□

**F2.** Solve  $y' + xy = x$ .

**Solution:** Rearranging, we have  $y' = x - xy = x(1 - y)$ , so we see the equation is separable, and write

$$\frac{y'}{1 - y} = x.$$

Thus we compute  $\int \frac{y'}{1 - y} dx = \int \frac{1}{1 - y} dy = -\ln|1 - y| + c_1$ , and  $\int x dx = \frac{1}{2}x^2 + c_2$ . Thus, we have (letting  $c_3 = c_2 - c_1$ )

$$-\ln|1 - y| = \frac{1}{2}x^2 + c_3.$$

Then exponentiating, we have (letting  $c_4 = \pm e^{-c_3}$ )  $1 - y = c_4 e^{-\frac{1}{2}x^2}$ , so (with  $c = -c_4$ ) the general solution is

$$y = 1 + ce^{-\frac{1}{2}x^2}.$$

□

**F3.**

**Solution:**

□

**F4.** Consider the autonomous equation

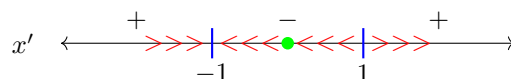
$$\frac{dx}{dt} = x^2 - 1.$$

(a) Find and classify the critical points.

(b) Describe the long term behavior of the solution passing through the point  $x(4) = 0$ .

**Solution:**

Note that  $x^2 - 1 = (x - 1)(x + 1)$ , so there are equilibria solutions at  $x = 1$  and  $x = -1$ . We can thus compute a number line for  $x'$ :



We see that  $-1$  is a sink (stable), while  $1$  is a source (unstable). A trajectory passing through  $x(4) = 0$  will approach  $x = -1$  in the limit.

□

**F5.** Solve  $y' + xy = x^3$ .

**Solution:** Solve homog; then variation of parameters.

□

**F6.** One of the two ODEs below is exact. Identify which one, and solve it.

$$\begin{aligned}(x + 3y)y' + y &= 3x \\ (x + 3y)y' - y &= -3x\end{aligned}$$

**Solution:** Rewrite the first equation as  $y' = \frac{3x-y}{x+3y}$ ; this corresponds to the autonomous system

$$\begin{aligned}\frac{dy}{dt} &= 3x - y \\ \frac{dx}{dt} &= x + 3y\end{aligned}$$

If there is a function with  $\nabla F = \langle -(3x - y), x + 3y \rangle$ , then

$$\frac{d}{dt}F(x, y) = \frac{\partial F}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial t} = -(3x - y)(x + 3y) + (x + 3y)(3x - y) = 0$$

, so  $F(x, y) = c$  is a solution of the autonomous system.

Note that  $\langle -(3x - y), x + 3y \rangle$  is conservative, as  $\frac{\partial}{\partial x}(x + 3y) = 1 = \frac{\partial}{\partial y}(-3x + y)$ ; so a potential function  $F$  is found by integrating:  $F = \int x + 3y \, dy = xy + \frac{3}{2}y^2 + f(x)$ , and differentiating with respect to  $x$ , we have  $-3x + y = y + f'(x)$ , so  $f'(x) = -3x$ , in which case  $f(x) = \int -3x \, dx = -\frac{3}{2}x^2$ , so a potential function is  $F(x, y) = xy + \frac{3}{2}y^2 - \frac{3}{2}x^2$ . So the general solution to the ODE is

$$xy + \frac{3}{2}y^2 - \frac{3}{2}x^2 = c.$$

□

**S1.**

**Solution:**

□

**S2.** Consider the following scenario: Two competing fish species (red fish and blue fish) live in a lake. In the absence of one species, the lake will support 30,000 of the other fish (as the species competes among itself for resources). It is calculated that the competition coefficients are  $\alpha_{1,2} = \alpha_{2,1} = 0.5$ .

(a) Write down a system of ODEs modelling the interaction of the two species.

(b) If the lake is stocked with 12,000 blue fish and 40,000 red fish, what will happen to the populations in the long run?

**Solution:**

$$\begin{aligned}x' &= \frac{r_1}{30000}x(30000 - x - 0.5y) \\ y' &= \frac{r_2}{30000}y(30000 - y - 0.5x)\end{aligned}$$

Then the nontrivial isoclines are  $x + 0.5y = 30000$  and  $0.5x + y = 30000$  which intersect at the (equilibrium) point 20000, 20000. Plotting the isoclines (TODO) shows that this a stable equilibrium (sink), so regardless of the initial populations, they will approach 20,000 of each species in the long run.

☐

**S3.**

**Solution:**

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**N1.**

**Solution:**

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**N2.**

**Solution:**

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**N3.**

**Solution:**

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**N4.**

**Solution:**

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**N5.**

**Solution:**

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**D1.**

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