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Module F: First order ODEs

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How can we solve and apply first order ODEs?

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At the end of this module, students will be able to...

F1. **Sketching trajectories.** ...given a slope field, sketch a trajectory of a solution to a first order ODE

F2. **Separable ODEs.** ...find the general solution to a separable first order ODE

F3. **Modeling motion.** ...model the motion of an object with quadratic drag

F4. **Autonomous ODEs.** ...find and classify the equillibria of an autonomous first order ODE, and describe the long term behavior of solutions

F5. **First order linear ODEs.** ...find the general solution to a first order linear ODE

F6. **Exact ODES.** ...find the general solution to an exact first order ODE

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Use integration techniques like substitution to compute indefinite integrals.
- Determine the intervals on which a polynomial is positive, negative, or zero.
- Determine when a vector field is conservative.
- Find the potential function of a conservative vector field.
- Use variation of parameters to solve non-homogeneous ODEs when given the solution to the corresponding homogeneous ODE (Standard C5)

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The following resources will help you prepare for this module.

- Use integration techniques like substitution to compute indefinite integrals.
  `https://youtu.be/b76wePnIBdU`
- Determine the intervals on which a polynomial is positive, negative, or zero.
  `https://youtu.be/jGa0GJjwQh8`
- Determine when a vector field is conservative.
  `https://youtu.be/gAb1ZTD41wo`
- Find the potential function of a conservative vector field.
  `https://youtu.be/nY4mW_R-T40`
- Use variation of parameters to solve non-homogeneous ODEs when given the solution to the corresponding homogeneous ODE (Standard C5)

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Module F Section 1

Definition F.1.1

A **first order ODE** is an equation involving (for a function $y(x)$) only y', y, and x.

We will most often deal with **explicit first order ODEs**, which can be written in the form

$$y' = f(y, x)$$

for some function $f(y, x)$.

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Activity F.1.2 *(∼5 min)*

Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.

Activity F.1.2 *(∼5 min)*
Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.

Part 1: Compute y' at each of the points $(1, 1)$, $(2, 1)$, $(3, -2)$, and $(4, -7)$.

Activity F.1.2 *(∼5 min)*
Consider the (explicit) first order ODE

$$y' = y^2 - x^2$$

.

Part 1: Compute y' at each of the points $(1, 1)$, $(2, 1)$, $(3, -2)$, and $(4, -7)$.
Part 2:
Let $y_0(x)$ be a solution that passes through the point $(1, 1)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$?

(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

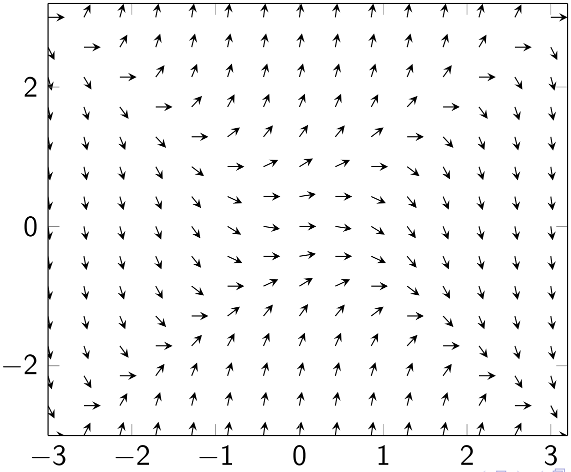
(C) $\lim_{x\to\infty} y_0(x) = \infty$

Definition F.1.3

These kinds of questions are easier to answer if we draw a **slope field** (sometimes called a **direction field**.

To draw one, draw a small line segment or arrow with the correct slope at each point.

$$y' = y^2 - x^2$$



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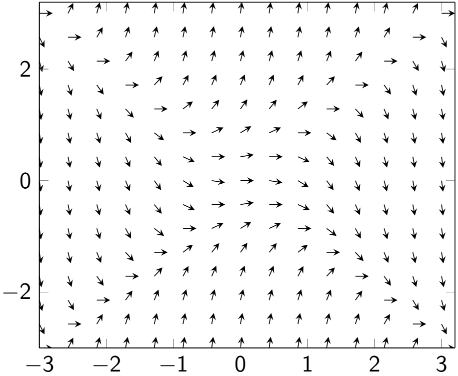
Activity F.1.4 *(∼5 min)*

$$y' = y^2 - x^2$$



Let $y_1(x)$ be a solution that passes through the point $(1, 3)$. What can you conclude about $\lim_{x\to\infty} y_0(x)$?

(A) $\lim_{x\to\infty} y_0(x) = -\infty$

(B) $\lim_{x\to\infty} y_0(x)$ is a finite number

(C) $\lim_{x\to\infty} y_0(x) = \infty$

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Activity F.1.5 *(∼15 min)*

Consider the ODE

$$y' = xy - x.$$

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Activity F.1.5 *(∼15 min)*
Consider the ODE

$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.

Activity F.1.5 *(∼15 min)*
Consider the ODE

$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.
Part 2: Draw a solution that passes through the point (0,0).

Activity F.1.5 *(∼15 min)*
Consider the ODE

$$y' = xy - x.$$

Part 1: Draw a slope field for this ODE.
Part 2: Draw a solution that passes through the point (0,0).
Part 3: Draw a solution that passes through the point (-2,2).

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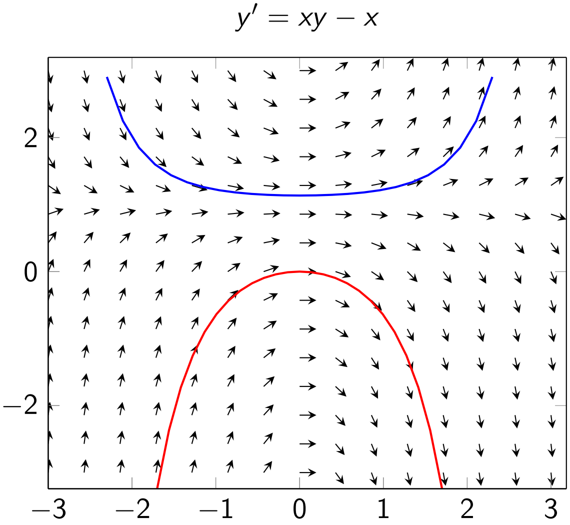
Observation F.1.6



Observation F.1.7

How can we solve $y' = xy - x$ exactly?

Notice $xy - x = x(y-1)$, so we can write $y' = x(y-1)$.

Write

$$\frac{y'}{y-1} = x.$$

This is called a **separable** DE.

Observation F.1.8

Integrate both sides (and switch to Leibniz notation):

$$\int \frac{1}{y-1}\frac{dy}{dx}\;dx = \int x\;dx.$$

The substitution rule (i.e. chain rule) says this is equivalent to

$$\int \frac{1}{y-1}dy = \int x\;dx.$$

Thus, $\ln|y-1| = \frac{1}{2}x^2 + c$. Exponentiating, we have

$$|y-1| = e^{\frac{1}{2}x^2+c} = e^{\frac{1}{2}x^2}e^c = c_0e^{\frac{1}{2}x^2}.$$

Allowing c_0 to take on negative values, we can drop the absolute value sign, and obtain

$$y = 1 + c_0e^{\frac{1}{2}x^2}.$$

Activity F.1.9 *(∼10 min)*
Find the general solution to

$$y' = xy + y.$$

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Activity F.1.10 *(∼10 min)*

Solve the IVP

$$y' = \frac{x}{y}, y(0) = -1.$$

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Activity F.2.1 *(∼5 min)*
In Module C, we discussed that tiny spherical objects like droplets of water obey Stoke's law: drag is proportional to velocity (speed). But for larger objects, a better model incorporates **quadratic drag**, i.e. drag is proportional to the square of velocity.

Which of the following ODEs models the velocity of a falling object subject subject to quadratic drag?

(a) $mv' = mg + bv$
(b) $mv' = mg - bv$
(c) $mv' = mg + bv^2$
(d) $mv' = mg - bv^2$

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Activity F.2.2 *(∼10 min)*

Consider our model of a falling object under quadratic drag

$$mv' = mg - bv^2.$$

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Activity F.2.2 *(∼10 min)*
Consider our model of a falling object under quadratic drag

$$mv' = mg - bv^2.$$

Part 1: For what value of v will the change in velocity be 0?

Activity F.2.2 *(∼10 min)*
Consider our model of a falling object under quadratic drag

$$mv' = mg - bv^2.$$

Part 1: For what value of v will the change in velocity be 0?
Part 2: Suppose the object is currently falling at a rate slower than this speed. What will happen?

(a) It will slow down

(b) It will keep falling at the same speed.

(c) It will speed up

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Observation F.2.3

This equillibrium speed is called the **terminal velocity**.

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Activity F.2.4 *(∼5 min)*
Consider the following question:

A penny is dropped off the top of the Empire State Building. How fast will it be going when it hits the ground? // // What information do we need to answer this question?

Observation F.2.5

The mass of a penny is $2.5\mathrm{g}$. The Empire State Building is (roughly) $400\mathrm{m}$ tall.

The terminal velocity of a penny is about $25\mathrm{m}/\mathrm{s}$.

Activity F.2.6 *(~20 min)*

We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.

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Activity F.2.6 *(∼20 min)*

We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.

Part 1: Solve for g in terms of v_t, m, b, and substitute this in to our model $v' = g - \frac{b}{m}v^2$.

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Activity F.2.6 *(∼20 min)*

We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.

Part 1: Solve for g in terms of v_t, m, b, and substitute this in to our model $v' = g - \frac{b}{m}v^2$.

Part 2: Solve this separable ODE

Hint: $\frac{1}{v_t^2 - v^2} = \frac{2}{v_t}\left(\frac{1}{v_t - v} + \frac{1}{v_t + v}\right)$

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Activity F.2.6 *(∼20 min)*

We calculated earlier that the terminal velocity is $v_t = \sqrt{\frac{mg}{b}}$.

Part 1: Solve for g in terms of v_t, m, b, and substitute this in to our model $v' = g - \frac{b}{m}v^2$.

Part 2: Solve this separable ODE

Hint: $\frac{1}{v_t^2 - v^2} = \frac{2}{v_t}\left(\frac{1}{v_t - v} + \frac{1}{v_t + v}\right)$

Part 3: How fast is the penny going after 10 seconds?

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Observation F.3.1

There are two very simple kinds of separable ODEs.

Equations of the form $y' = f(x)$ can be solved immediately by integrating and produce explicit solutions.

Equations of the form $y' = f(y)$ are often impossible or difficult to solve explicitly. They are called **autonomous** equations.

Activity F.3.2 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2.$$

Activity F.3.2 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2.$$

Part 1: Draw a slope field

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Activity F.3.2 *(∼10 min)*
Consider the autonomous equation

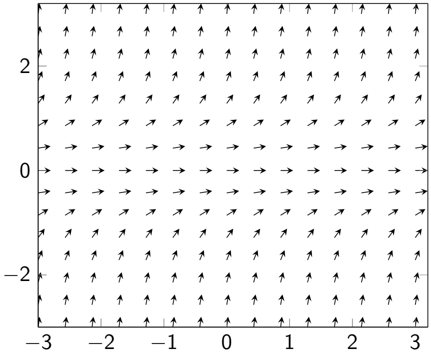
$$y' = y^2.$$

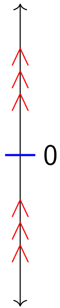
Part 1: Draw a slope field
Part 2: Suppose a solution goes through the point $y(10) = 50$. What can you say about $y(11)$?

(a) $y(10) < y(11)$
(b) $y(10) = y(11)$
(c) $y(10) > y(11)$

Observation F.3.3

Since the slopes do not change when moving horizontally (i.e. in the x direction), we often collapse the slope field onto the y-axis.





This is called a **phase line**.

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Activity F.3.4 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2(y-2).$$

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Activity F.3.4 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2(y-2).$$

Part 1: Draw a number line for y', indicating where it is positive or negative.

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Activity F.3.4 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2(y - 2).$$

Part 1: Draw a number line for y', indicating where it is positive or negative.
Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?

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Activity F.3.4 *(∼10 min)*

Consider the autonomous equation

$$y' = y^2(y-2).$$

Part 1: Draw a number line for y', indicating where it is positive or negative.

Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?

Part 3: What can you say about the long term behavior of a solution passing through $y(2) = 0.001$?

Activity F.3.4 *(∼10 min)*
Consider the autonomous equation

$$y' = y^2(y-2).$$

Part 1: Draw a number line for y', indicating where it is positive or negative.
Part 2: What can you say about the long term behavior of a solution passing through $y(4) = 1$?
Part 3: What can you say about the long term behavior of a solution passing through $y(2) = 0.001$?
Part 4: What can you say about the long term behavior of a solution passing through $y(2) = -0.001$?

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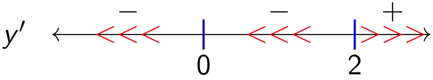
Definition F.4.1

Recall from last week: the **phase line** is a useful way to visualize the long term behavior of an autonomous DE.

For example, here is a phase line for the autonomous DE $y' = y^2(y-2)$.



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Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

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Activity F.4.2 *(∼15 min)*
Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

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Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

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Activity F.4.2 *(∼15 min)*

Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.

Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.

Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.

Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.

Activity F.4.2 *(∼15 min)*
Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.
Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.
Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.
Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.
Part 5: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

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Activity F.4.2 *(∼15 min)*
Consider the autonomous equation

$$y' = y(y+1)^2(y-2).$$

Part 1: Draw a phase line.
Part 2: Describe the long term behavior of a solution passing through $y(2) = -0.9999$.
Part 3: Describe the long term behavior of a solution passing through $y(7) = -1.0001$.
Part 4: Describe the long term behavior of a solution passing through $y(4) = -1$.
Part 5: Describe the long term behavior of solutions passing near the point $y(3) = 0$.
Part 6: Describe the long term behavior of solutions passing near the point $y(11) = 2$.

Definition F.4.3

The **critical points** of an autonomous DE are the numbers that give rise to equillibrium solutions (e.g. $0, -1, 2$ in the previous problem).

A **source** is an unstable equillibrium in which all nearby trajectories move away in the limit.

A **sink** is a stable equillibrium in which all nearby trajectories approach the equillibrium in the limit.

There are also unstable equillibria in which some nearby trajectories return, while others diverge, analagous to a saddle point.

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Activity F.4.4 *(∼15 min)*

Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

Activity F.4.4 *(∼15 min)*
Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

Part 1: Find and classify all of the critical points.

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Activity F.4.4 *(∼15 min)*
Consider the autonomous equation

$$y' = y^3(y-2)^2(y+1)(y-1).$$

Part 1: Find and classify all of the critical points.
Part 2: Describe the long term behavior of solutions passing near the point $y(1) = 1.5$.

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Activity F.4.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Activity F.4.5 *(∼15 min)*
Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Part 1: Find and classify all of the critical points.

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Activity F.4.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Part 1: Find and classify all of the critical points.

Part 2: Describe the long term behavior of solutions passing near the point $y(0) = 0.5$.

Activity F.4.5 *(∼15 min)*

Consider the autonomous equation

$$y' = y^4(y+3)^2(y-1)(y+2).$$

Part 1: Find and classify all of the critical points.

Part 2: Describe the long term behavior of solutions passing near the point $y(0) = 0.5$.

Part 3: Describe the long term behavior of solutions passing near the point $y(3) = 0$.

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Observation F.5.1

In module C, we solved **constant coefficient linear ODEs**.

Today we will observe that our existing techniques allow us to solve all **first order linear ODES**, i.e. ODEs of the form

$$a(x)y' + b(x)y + c(x) = 0.$$

Such equations can always be rewritten (by rearranging and dividing by $a(x)$ in **standard form**:

$$y' + P(x)y = Q(x).$$

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Activity F.5.2 *(∼20 min)*

Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

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Activity F.5.2 *(∼20 min)*
Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

Part 1: Solve the **homogeneous** first order linear ODE

$$y' + \frac{1}{x}y = 0.$$

Activity F.5.2 *(∼20 min)*
Consider the first order linear ODE

$$y' + \frac{1}{x}y = 1.$$

Part 1: Solve the **homogeneous** first order linear ODE

$$y' + \frac{1}{x}y = 0.$$

Part 2: Use variation of parameters to the solve the original ODE

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Activity F.5.3 *(∼15 min)*

Solve the first order linear ODE

$$\frac{1}{x}y' - \frac{2}{x^2}y - x\cos(x) = 0.$$

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Activity F.5.4 *($\sim$15 min)*
Solve the first order linear ODE

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Remark F.5.5

The book provides a different technique; however, the method presented here does not require memorizing anything new.

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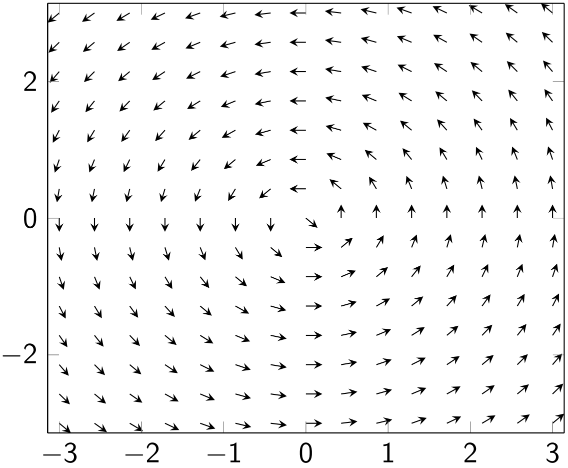
Observation F.6.1

A vector field $\langle P, Q\rangle$ corresponds to the slope field of the differential equation

$$\frac{dy}{dx} = \frac{Q}{P}.$$

Thus, a solution to this ODE describes the path taken by the particle in this fluid flow.



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Activity F.6.2 *(~10 min)*
Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2y^2 + x$.

Activity F.6.2 *(∼10 min)*
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Part 1: Compute $\nabla\phi$.

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Now, consider $\phi(x, y) = x^2 y^2 + x$.
Part 1: Compute $\nabla\phi$.
Part 2: Differentiate the equation $\phi(x, y) = c$ with respect to x.

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Activity F.6.2 *(∼10 min)*
Consider the ODE

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2x^2y}.$$

This can be rewritten as

$$(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0.$$

Now, consider $\phi(x, y) = x^2y^2 + x$.
Part 1: Compute $\nabla\phi$.
Part 2: Differentiate the equation $\phi(x, y) = c$ with respect to x.
Part 3: Solve the ODE $(2xy^2 + 1) + 2x^2y\frac{dy}{dx} = 0$.

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Definition F.6.3
If $\langle M, N \rangle$ is a conservative vector field, the ODE

$$M + N\frac{dy}{dx} = 0$$

is called **exact**. This ODE can also be written

$$\frac{dy}{dx} = \frac{-M}{N}.$$

If $\phi(x, y)$ is a potential function of $\langle M, N \rangle$, the general solution to the ODE is $\phi(x, y) = c$.

Careful: The slope field of the ODE $\frac{dy}{dx} = \frac{-M}{N}$ is the vector field $\langle -N, M \rangle$!

Activity F.6.4 *(~10 min)*

Determine which of the following ODEs are exact.

(a) $2xy + (x^2 - 2y)\frac{dy}{dx} = 0$

(b) $\frac{dy}{dx} = \frac{2xy}{x^2+2y}$

(c) $\frac{dy}{dx} = -\frac{2xy}{x^2+2y}$

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Activity F.6.5 *(∼10 min)*
Solve the exact ODE $2xy + (x^2 - 2y)\frac{dy}{dx} = 0$.

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Activity F.7.1 *(∼10 min)*

Determine which of the following ODEs are exact.

(a) $\frac{dy}{dx} = \frac{-y}{x^2+y^2+x}$

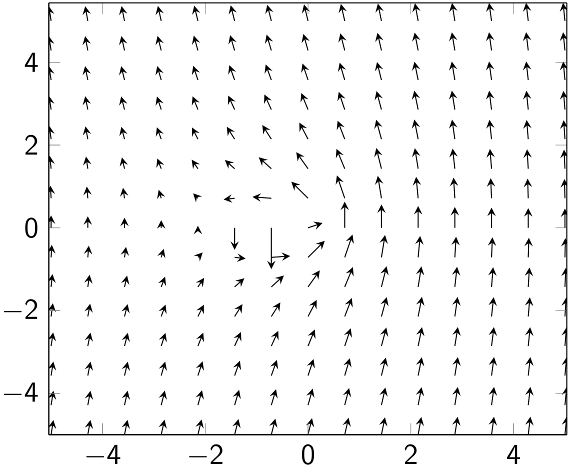
(b) $1 + \frac{x}{x^2+y^2} + \left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx} = 0$

Activity F.7.2 *(∼15 min)*

Solve the exact ODE

$$1 + \frac{x}{x^2 + y^2} + \left(\frac{y}{x^2 + y^2} \right) \frac{dy}{dx} = 0.$$

These solutions describe the trajectories taken by particles in the fluid flow below



Activity F.7.3 *(∼20 min)*
Find solutions for the ODE

$$1+\frac{x}{x^2+y^2}+\left(\frac{y}{x^2+y^2}\right)\frac{dy}{dx}=0$$

for each of the following initial conditions

(a) $y(0)=-1$.

(b) $y(-2)=-2$.

(c) $y(-4)=-4$.

Plot each of the solution curves.