Module N: Numerical

How can we use numerical approximation methods to apply and solve unsolvable ODEs?

At the end of this module, students will be able to...

N1. First Order Existence and Uniqueness. ...determine when a unique solution exists for a first order ODE

N2. Second Order Linear Existence and Uniqueness. ...determine when a unique solution exists for a second order linear ODE

N3. Systems Existence and Uniqueness. ...determine when a unique solution exists for a system of first order ODEs

N4. Euler's method for first order ODES. ...use Euler's method to find approximate solution to first order ODEs

N5. Euler's method for systems. ...use Euler's method to find approximate solutions to systems of first order ODEs

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compute partial derivatives.
- Determine where multivariate functions are continuous.
- Use a linear approximation to estimate the value of a function.
- Solve separable ODEs **F2**.

The following resources will help you prepare for this module.

- Compute partial derivatives `https://youtu.be/3itjTS2Y9oE`.
- Determine where multivariate functions are continuous `https://youtu.be/RGx-pmWlOpk`.
- Use a linear approximation to estimate the value of a function `https://youtu.be/oxwCRzQOCu8`.
- Solve separable ODEs **F2**.

Module N Section 1

Activity N.1.1 *(∼5 min)*

Solve the IVP

$$y' = \frac{3}{2}y^{\frac{1}{3}}, \qquad y(1) = 0.$$

(A) $y = t^{\frac{3}{2}}$

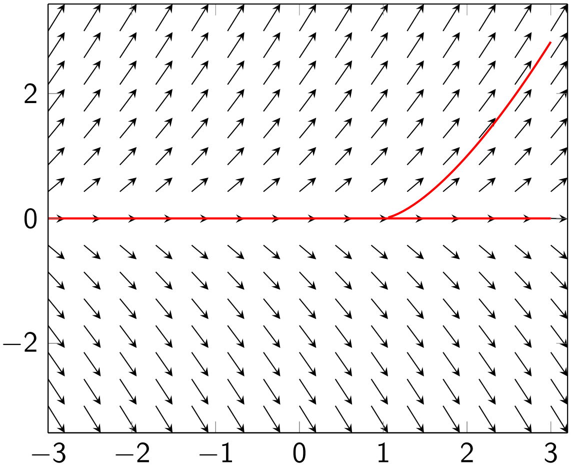
(B) $y = (t-1)^{\frac{3}{2}}$

(C) $y = t^{\frac{3}{2}} - 1$

(D) $y = 0$

Observation N.1.2

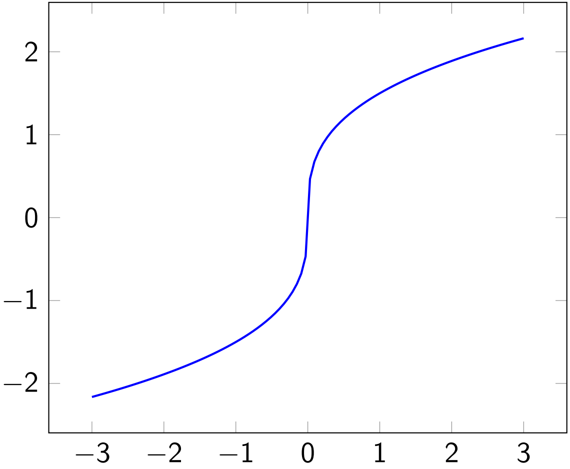
The ODE $y' = y^{\frac{1}{3}}$ has multiple solutions through the point $(1, 0)$.



How can we guarantee our ODEs have a unique solution?

Observation N.1.3

Let's plot the function $f(y) = \frac{3}{2}y^{\frac{1}{3}}$.



Observe: $f(y)$ is not differentiable at 0!

Observation N.1.4

If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are **continuous** on a rectangle containing x_0, y_0, then the IVP

$$y' = f(x, y), \qquad\qquad y(x_0) = y_0$$

has a unique solution.

The problem with our example

$$y' = \frac{3}{2}y^{\frac{1}{3}}, \qquad\qquad y(1) = 0.$$

is that, for $f(x, y) = \frac{3}{2}y^{\frac{1}{3}}$, the derivative

$$\frac{\partial f}{\partial y} = \frac{1}{2}y^{-\frac{2}{3}}$$

is not continuous at $(1, 0)$.

Activity N.1.5 *(∼5 min)*

Consider the IVP

$$y' = \sqrt{x^2 + y^2}, \qquad\qquad y(0) = 0.$$

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$$y' = \sqrt{x^2 + y^2}, \qquad y(0) = 0.$$

Part 1: Is $f(x, y) = \sqrt{x^2 + y^2}$ continuous at $0, 0$?

Activity N.1.5 *(∼5 min)*

Consider the IVP

$$y' = \sqrt{x^2 + y^2}, \qquad\qquad y(0) = 0.$$

Part 1: Is $f(x, y) = \sqrt{x^2 + y^2}$ continuous at $0, 0$?

Part 2: Compute $\frac{\partial f}{\partial y}$. Is $\frac{\partial f}{\partial y}$ continuous at $0, 0$?

Activity N.1.5 *($\sim$5 min)*

Consider the IVP

$$y' = \sqrt{x^2 + y^2}, \qquad\qquad y(0) = 0.$$

Part 1: Is $f(x, y) = \sqrt{x^2 + y^2}$ continuous at $0, 0$?
Part 2: Compute $\frac{\partial f}{\partial y}$. Is $\frac{\partial f}{\partial y}$ continuous at $0, 0$?
Part 3: Can you conclude the IVP has a unique solution?

Activity N.1.6 *(∼10 min)*
Consider the ODE

$$y' = \sqrt{x^2 + y^2 - 1}.$$

This ODE is guaranteed to have a unique solution passing through which of the following points?

(A) $y(1) = 1$

(B) $y(1) = -1$

(C) $y(1) = 0$

(D) $y(0) = 1$

Activity N.1.7 *(∼10 min)*

Consider the ODE

$$y' = \sqrt[3]{x^2 - y^2}.$$

This ODE is guaranteed to have a unique solution passing through which of the following points?

(A) $y(1) = 1$

(B) $y(1) = -1$

(C) $y(1) = 0$

(D) $y(0) = 1$

Activity N.1.8 *($\sim$10 min)*
Describe all points (x_0, y_0) for which the IVP

$$y' = \ln(x^2 + y^2 - 1) - \sqrt[3]{4 - x^2 - y^2}, \qquad y(x_0) = y_0$$

is guaranteed to have a unique solution.

Module N Section 2

Observation N.2.1

We previously saw that the first order IVP

$$y' = f(x, y), \qquad y(x_0) = y_0$$

had a unique solution on some (possibly tiny!) interval containing x_0 when $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous at (x_0, y_0).

Activity N.2.2 *(∼10 min)*
Consider the second order ODE

$$(x^2 - 1)^2 y'' + 4x = 0$$

.

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Part 1: Solve for y'', and then integrate to find y'.

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Consider the second order ODE

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Part 1: Solve for y'', and then integrate to find y'.
Part 2: Integrate again to find y. (**Hint:** $\frac{2}{x^2-1}=\frac{1}{x-1}-\frac{1}{x+1}$)

Activity N.2.2 *(∼10 min)*
Consider the second order ODE

$$(x^2 - 1)^2 y'' + 4x = 0$$

.
Part 1: Solve for y'', and then integrate to find y'.
Part 2: Integrate again to find y. (**Hint:** $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$)
Part 3: For what values of x is your solution valid?

Observation N.2.3

The ODE $(x^2-1)^2y''+4x=0$ did not have a solution where the coefficient of y'' vanished, i.e. at $x=1$ and $x=-1$.

In general, if $x_1 < x < x_2$ is an interval containing x_0 for which

- $(a(x), b(x), c(x),$ and $f(x)$ are continuous, and
- $a(x)$ does not vanish

Then the second order **linear** IVP

$$a(x)y''+b(x)y'+c(x)y=f(x), \qquad y(x_0)=y_0,\ y'(x_0)=y_1$$

will have a unique solution on $x_1 < x < x_2$.

Observation N.2.4

Our uniqueness result for first order equations applied to all first order equations. Our second order result applies to only **linear** equations, but provides added information–a precise interval on which the unique solution exists.

For example, the IVP

$$(x^2 - 1)^2 y'' + 4x = 0, \qquad y(2) = 3,\ y'(2) = 4$$

will have a unique solution valid for $1 < x < \infty$.

Activity N.2.5 *(∼5 min)*

Consider the IVP

$$\sin(x)y'' + \cos(x)y = x^2 - 4, \qquad y\left(\frac{\pi}{4}\right) = 1,\ y'\left(\frac{\pi}{4}\right) = 0.$$

Determine the largest interval on which a unique solution is guaranteed to exist.

Activity N.2.6 *(∼5 min)*
Consider the IVP

$$y'' + \frac{1}{x}y' - \frac{1}{x-4}y = 0 \qquad\qquad y(2) = 5,\ \ y'(2) = -1.$$

Determine the largest interval on which a unique solution is guaranteed to exist.

Activity N.2.7 *(∼5 min)*

Consider the ODE

$$(x^2 - 1)y'' + \frac{1}{x}y' + e^x y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

Activity N.2.8 *(∼5 min)*

Consider the ODE

$$\frac{x}{x-1}y'' + \frac{x+2}{x+1}y' + e^{-x}y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

Activity N.2.9 *($\sim$5 min)*

Consider the ODE

$$\sqrt{x^2-1}y'' + y' + \frac{1}{x}y = 0.$$

Determine **all** intervals on which a unique solution is guaranteed to exist.

Module N Section 3

Activity N.3.1 *(∼10 min)*
Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.

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Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.
Part 1: Compute the slope of the solution at the point $(2, 4)$.

Activity N.3.1 *(~10 min)*
Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.
Part 1: Compute the slope of the solution at the point $(2, 4)$.
Part 2: Use a linear approximation to estimate the value of $y(2.1)$.

Activity N.3.1 *(∼10 min)*
Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.
Part 1: Compute the slope of the solution at the point $(2, 4)$.
Part 2: Use a linear approximation to estimate the value of $y(2.1)$.
Part 3: Calculate the slope at the point $(2.1, 4.4)$.

Activity N.3.1 *(~10 min)*
Consider the first order ODE $y' = x + \sqrt{y}$. Suppose $y(x)$ is a solution with $y(2) = 4$.
Part 1: Compute the slope of the solution at the point $(2, 4)$.
Part 2: Use a linear approximation to estimate the value of $y(2.1)$.
Part 3: Calculate the slope at the point $(2.1, 4.4)$.
Part 4: Use a linear approximation at $(2.1, 4.4)$ to estimate the value of $y(2.2)$.

Observation N.3.2

This technique is called **Euler's method** (with step size $h = 0.1$) for the IVP

$$y' = x + \sqrt{y}, \qquad y(2) = 4.$$

It is often convenient to organize this information in a table

x_n	y_n	$y'(x_n, y_n)$	$x_{n+1} = x_n + h$	$y_{n+1} = y_n + hy'(x_n, y_n)$
2	4	4	2.1	4.4
2.1	4.4	4.19762	2.2	4.81976

Activity N.3.3 *(∼10 min)*

Use Euler's method with stepsize $h = 0.2$ to estimate $y(3)$, where y is a solution of the IVP

$$y' = x - 3y^2, \qquad y(2.2) = 1.$$

Activity N.3.4 *(∼10 min)*

Use Euler's method with stepsize $h = 0.2$ to estimate $y(4)$, where y is a solution of the IVP

$$y' = \sqrt{x - y}, \qquad y(3) = 1.$$

Module N Section 4

Observation N.4.1

The same problem we saw with a first order ODE failing to have a unique solution can also occur in systems of first order ODEs.

Thus, to ensure that the system

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned}$$

has a unique solution, you must check that $f(t, x, y)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $g(t, x, y)$, $\frac{\partial g}{\partial x}$, and $\frac{\partial g}{\partial y}$ are all continuous near the initial point.

Observation N.4.2

If the system is linear, we can say more!

Suppose $a(t), b(t), c(t), d(t), f(t), g(t)$ are all continous on the interval $h < t < k$. Then for any $h < t_0 < k$, the IVP

$$\begin{aligned} x' &= a(t)x + b(t)y + f(t) & \qquad x(t_0) &= x_0 \\ y' &= c(t)x + d(t)y + g(t) & \qquad y(t_0) &= y_0 \end{aligned}$$

has a unique solution on the (time) interval $h < t < k$.

Activity N.4.3 *(∼5 min)*

Consider the IVP

$$x' = \frac{1}{t-1}x + \sqrt{t}y + t^2 \qquad x(2) = 5$$
$$y' = \frac{1}{t}x + \sqrt{t+1}y + t^2 \qquad y(2) = 7$$

What is the largest interval on which this IVP has a unique solution?

Activity N.4.4 *(∼5 min)*
Determine **all intervals** on which a unique solution is guaranteed to exist for the below system.

$$x' = \frac{1}{t-1}x + \sqrt{t}y + t^2$$
$$y' = \frac{1}{t}x + \sqrt{t+1}y + t^2$$

Activity N.4.5 *(∼5 min)*
Determine **all intervals** on which a unique solution is guaranteed to exist for the below system.

$$x' = \ln(t-2)x + \sqrt{t}y + \frac{1}{t-1}$$
$$y' = \cos(t)x + y$$

Activity N.4.6 *(∼10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$\begin{aligned} x' &= 3x + 4y - t & \qquad x(1) &= 2 \\ y' &= x - y + t & \qquad y(1) &= 3. \end{aligned}$$

Activity N.4.6 *(∼10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$\begin{aligned} x' &= 3x + 4y - t & \qquad x(1) &= 2 \\ y' &= x - y + t & \qquad y(1) &= 3. \end{aligned}$$

Part 1: Compute x' and y' when $t = 1$.

Activity N.4.6 *(∼10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$\begin{aligned} x' &= 3x + 4y - t & \qquad x(1) &= 2 \\ y' &= x - y + t & \qquad y(1) &= 3. \end{aligned}$$

Part 1: Compute x' and y' when $t = 1$.
Part 2: Use linear approximations to estimate the values of $x(1.1)$ and $y(1.1)$.

Activity N.4.6 *(∼10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$
\begin{aligned}
x' &= 3x + 4y - t & \qquad x(1) &= 2 \\
y' &= x - y + t & \qquad y(1) &= 3.
\end{aligned}
$$

Part 1: Compute x' and y' when $t = 1$.
Part 2: Use linear approximations to estimate the values of $x(1.1)$ and $y(1.1)$.
Part 3: Calculate the slopes x' and y' when $t = 1.1$ (and as just calculated, $x(1.1) = 2.17$ and $y(1.1) = 3$).

Activity N.4.6 *(~10 min)*
Euler's method can be extended to systems in a straightforward way.

Consider the system IVP

$$\begin{aligned} x' &= 3x + 4y - t & x(1) &= 2 \\ y' &= x - y + t & y(1) &= 3. \end{aligned}$$

Part 1: Compute x' and y' when $t = 1$.
Part 2: Use linear approximations to estimate the values of $x(1.1)$ and $y(1.1)$.
Part 3: Calculate the slopes x' and y' when $t = 1.1$ (and as just calculated, $x(1.1) = 2.17$ and $y(1.1) = 3$).
Part 4: Use linear approximations to estimate the values of $x(1.2)$ and $y(1.2)$.

Observation N.4.7

$$
\begin{aligned}
x' &= 3x + 4y - t & \qquad x(1) &= 2 \\
y' &= x - y + t & \qquad y(1) &= 3.
\end{aligned}
$$

It is often convenient to organize this information in a table

t_n	x_n	y_n	$x'(t_n, x_n, y_n)$	$y'(t_n, x_n, y_n)$	t_{n+1}	x_{n+1}	y_{n+1}
1	2	3	17	0	1.1	2.17	3
1.1	2.17	3	17.41	0.27	1.2	3.911	3.027
1.2	3.911	3.027					

Thus $x(1.2) \approx 3.911$ and $y(1.2) \approx 3.027$.

Activity N.4.8 *(~10 min)*
Use Euler's method to estimate $x(3.3)$ and $y(3.3)$.

$$\begin{aligned} x' &= 3x - ty & \qquad x(3) &= 2 \\ y' &= x - y^2 & \qquad y(3) &= 1. \end{aligned}$$

Activity N.4.9 *(∼10 min)*

Use Euler's method to estimate $x(4.6)$ and $y(4.6)$.

$$\begin{aligned} x' &= xy - t & x(4) &= 2 \\ y' &= x + t & y(4) &= 0. \end{aligned}$$