Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

C1. Find the general solution to

$$y' + y = -t^2.$$

Solution: First, we find a general solution to the homogeneous equation (removing all non-y terms)

$$y_h' + y_h = 0.$$

Since this is a first-order constant coefficient ODE, we know the answer is a modification of ke^t. We find that $y_h = ke^{-t}$ is valid, since

$$y_h' + y_h = \frac{d}{dt}[ke^{-t}] + ke^{-t} = -ke^{-t} + ke^t = 0.$$

The simplest particular solution for this homogeneous ODE is $y_0 = e^{-t}$, so we use variation of parameters by assuming there is a non-homogeneous particular solution $y_p = vy_0 = ve^{-t}$ for some function v (which we now need to find).

Using the product rule we find $y_p' = v'e^{-t} - ve^{-t}$, so we may substitute into the original ODE to find

$$y_p' + y_p = (v'e^{-t} - ve^{-t}) + ve^{-t} = v'e^{-t} = -t^2.$$

Thus $v' = -t^2e^t$, so (after integrating by parts twice) we conclude $v = \int -t^2e^t\, dt = -t^2e^t + 2te^t - 2e^t$. Thus $y_p = ve^{-t} = (-t^2 + 2t - 2)e^te^{-t} = -t^2 + 2t - 2$, so the general solution is

$$y = y_h + y_p = ke^{-t} - t^2 + 2t - 2.$$

$\square$

C2. Consider the following scenario: A water droplet with a radius of 10 μm has a mass of about 4×10^{-15}kg and a terminal velocity of 3 $\frac{\text{cm}}{\text{s}}$. The droplet is dropped from rest; assume that acceleration due to gravity is given by $9.8\frac{\text{m}}{\text{s}^2}$.

(a) Write down an IVP modelling the velocity.

(b) What is its velocity after 0.01 s?

Solution:

(a) The ODE modeling the velocity of a tiny mass under gravity and air resistence is $mv' = -mg - bv$ where m is the mass and g is the acceleration due to gravity, both given. The coefficient of air resistence b is not given. Since the object is dropped from rest, we know $v = 0$ when $t = 0$. So the initial value problem is given by

$$mv' = -mg - bv \qquad v(0) = 0$$

To find b, we note that the given terminal velocity v_t occurs when there is no acceleration: $v' = 0$. So we may solve

$$0 = -mg - bv_t$$

to get $b = -\frac{mg}{v_t}$. Thus the simplified IVP is given by

$$v' - av = -g \qquad v(0) = 0$$

where $a = \frac{g}{v_t} = \frac{9.8}{-0.03}\mathrm{s}^{-1} \approx -326.7\mathrm{s}^{-1}$.

(b) To solve for v, we first solve the homogeneous system

$$v_h' - av_h = 0$$

which has the general solution $v_h = ke^{at}$, and a simple particular solution $v_0 = e^{at}$. So we use varation of parameters to assert that $v_p = wv_0 = we^{at}$ is a particular solution for the original equation. If so, then $v_p' = w'e^{at} + awe^{at}$ and thus

$$v_p' - av_p = w'e^{at} + awe^{at} - awe^{at} = w'e^{at} = -g$$

and therefore $w' = -ge^{-at}$. By integration, $w = \frac{g}{a}e^{-at}$ and thus $v_p = \frac{g}{a}$.
Therefore the general solution is given by $v = v_h + v_p = ke^{at} + \frac{g}{a}$. Using the initial condition $v(0) = 0$, we have $0 = ke^0 + \frac{g}{a}$ and thus $k = -\frac{g}{a}$. So we have that $v = -\frac{g}{a}e^{at} + \frac{g}{a}$ or $v = \frac{g}{a}(1 - e^{at})$. Thus our final answer is given by plugging in

$$g = 9.8, \quad a = \frac{g}{v_t} = \frac{9.8}{-0.03} \approx -326.7, \quad t = 0.01$$

which yields the result $v \approx -0.029$. So after 1/100 of a second, the droplet is falling at a rate of about 29 millimeters per second.

□

C3. Find the general solution to

$$y'' + 6y' + 13y = 0.$$

Solution: Rewriting in terms of differential operators, we have $(D^2 + 6D + 13I)(y) = 0$, so we need to find the roots of the equation $r^2 + 6r + 13 = 0$, which are $r = -3 \pm 2i$. Thus the general solution using complex numbers is

$$y = k_1 e^{-3t+2it} + k_2 e^{-3t-2it}.$$

By applying Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we may obtain the real-valued general solution

$$y = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t).$$

□

C4. Find the solution to

$$y'' + 10y' + 24y = 0$$

when $y(0) = -3$ and $y'(0) = 2$.

Solution: Rewriting in terms of differential operators, we have $(D^2 + 10D + 24I)(y) = 0$, so we need to find the roots of the equation $r^2 + 10r + 24 = 0$, namely roots $r = -6$ and $r = -4$. Thus, the general solution is of the form $y = c_1 e^{-4t} + c_2 e^{-6t}$.

$$\begin{aligned} -3 &= y(0) = c_1 + c_2 \\ 2 &= y'(0) = -4c_1 - 6c_2 \end{aligned}$$

Solving this system yields $c_1 = -8$ and $c_2 = 5$, so the solution to the IVP is

$$y = -8e^{-4t} + 5e^{-6t}$$

□

C5. Find the general solution to

$$y'' + 10y' + 24y = e^{-4t}$$

Solution: First, we find a general solution to the homogenous equation $y'' + 10y' + 24y = 0$. We begin by writing the auxilliary equation $r^2 + 10r + 24 = 0$, which has roots $r = -6$ and $r = -4$. Thus, the general solution is of the form $y = c_1 e^{-4t} + c_2 e^{-6t}$.

We use variation of parameters to find a particular solution, $y_p = v_1 e^{-4t} + v_2 e^{-6t}$ for some functions v_1, v_2. We assume for convenience $v_1' e^{-4t} + v_2' e^{-6t} = 0$, so that $y_p' = -4v_1 e^{-4t} - 6v_2 e^{-6t}$, and $y_p'' = -4v_1' e^{-4t} + 16v_1 e^{[} - 4t] - 6v_2' e^{-6t} + 36v_2 e^{-6t}$. Substituting into the original ODE and simplifying yields

$$e^{-4t} = y_p'' + 10y_p' + 14y_p = -4v_1' e^{-4t} - 6v_2' e^{-6t}.$$

Combining with our earlier assumption we have the system

$$\begin{aligned} v_1' e^{-4t} + v_2' e^{-6t} &= 0 \\ -4v_1' e^{-4t} - 6v_2' e^{-6t} &= e^{-4t} \end{aligned}$$

Multipy both equations by e^{4t} for convenience:

$$\begin{aligned} v_1' + v_2' e^{-2t} &= 0 \\ -4v_1' - 6v_2' e^{-2t} &= 1 \end{aligned}$$

Then six times the first equation plus the second yields $2v_1' = 1$, so $v_1 = \int \frac{1}{2}\, dt = \frac{1}{2}t$. Substituting back into the first yields $v_2' = -\frac{1}{2}e^{2t}$, so $v_2 = \int -\frac{1}{2}e^{2t}\, dt = -\frac{1}{4}e^{2t}\, dt$, and thus

$$y_p = \frac{1}{2}te^{-4t} - \frac{1}{4}e^{-4t},$$

and thus the general solution is

$$y = \frac{1}{2}te^{-4t} + c_1 e^{-4t} + c_2 e^{-6t}.$$

□

C6. Consider the following scenario: A 2kg mass is suspended by a spring (with spring constant 8kg/s^2). The mass is pulled down 1m from its equillibrium position and released from rest.

(a) Write down an IVP modelling the position of the mass.

(b) How long does it take for the mass to return to its equillibrium point?

Solution: Let y denote the vertical distance from equillibrium; then the forces acting are gravity and the spring force, giving the ODE

$$my'' = -ky.$$

Substituting in the values of the constants $m = 2$ and $k = 8$, we have

$$2y'' + 8y = 0.$$

Note that the initial conditions are $y(0) = -1$ and $y'(0) = 0$, so an IVP model is

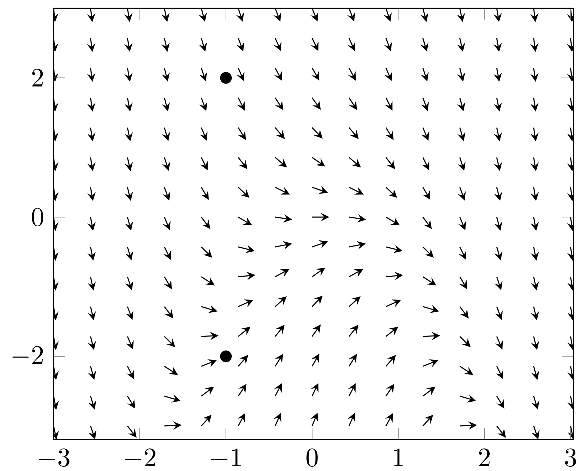
$$2y'' + 8y = 0 \qquad\qquad y(0) = 1,\ y'(0) = 0.$$

Simplifying, we have $y'' + 4y = 0$, so the general solution is $y = c_1 \cos(2t) + c_2 \sin(2t)$. The initial conditions are $y(0) = -1$ and $y'(0) = 0$, which imply $c_1 = -1$ and $c_2 = 0$, so the system is modelled by $y = -\cos(2t)$. This is first zero when $2t = \frac{\pi}{2}$, i.e. when $t = \frac{\pi}{4}$. Thus the mass takes $\frac{\pi}{4}$ s to return to its equillibrium point.

$\square$

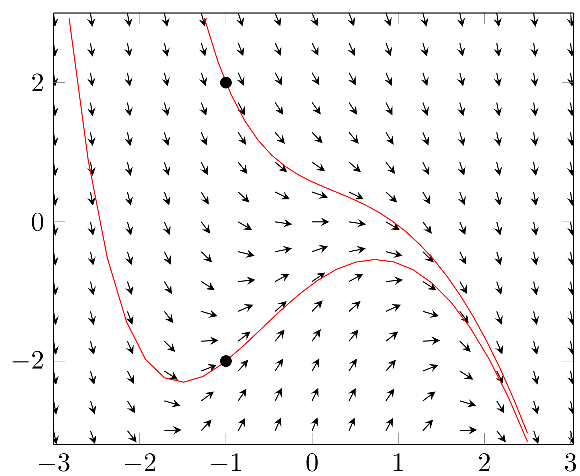
F1. Sketch a solution curve through each point marked in the slope field.

$y' = -y - x^2$



Solution:

$y' = -y - x^2$



$\square$

F2. Solve $y' + xy = x$.

Solution: Rearranging, we have $y' = x - xy = x(1 - y)$, so we see the equation is separable, and write

$$\frac{y'}{1-y} = x.$$

Thus we compute $\int \frac{y'}{1-y}\, dx = \int \frac{1}{1-y}\, dy = -\ln|1-y| + c_1$, and $\int x\, dx = \frac{1}{2}x^2 + c_2$. Thus, we have (letting $c_3 = c_2 - c_1$)

$$-\ln|1-y| = \frac{1}{2}x^2 + c_3.$$

Then exponentiating, we have (letting $c_4 = \pm e^{-c_3}$) $1 - y = c_4 e^{-\frac{1}{2}x^2}$, so (with $c = -c_4$) the general solution is

$$y = 1 + ce^{-\frac{1}{2}x^2}.$$

$\square$

F3. A ball has a mass of 0.1kg and a drag coefficient of 0.001kg/m. It is thrown with a velocity of 20m/s.

(a) Write down an IVP modelling the horizontal velocity of the ball (ignore any vertical movement in the ball).

(b) How long does it take the ball to travel 20m?

Solution: The only force acting on the ball is air resistance (drag), so we have

$$mv' = -bv^2.$$

Substituting in our constants (and dividing through by m), we have

$$v' = -0.1v^2.$$

The initial condition is that it is travelling 20m/s, i.e. $v(0) = 20$, giving the IVP

$$v' = -0.1v^2 \qquad\qquad v(0) = 20.$$

Solving this separable ODE, we obtain $v = \frac{20}{1+2t}$. To model the position of the ball, we integrate again, obtaining $x = \int \frac{20}{1+2t}\, dt = 10\ln|1+2t| + C$; using $x(0) = 0$, we see $C = 0$. Then we simply need to solve $x(t) = 20$, i.e. $20 = 10\ln|1+2t|$, so $t = \frac{1}{2}(e^2 - 1) \approx 3.19$s.

$\square$

F4. Consider the autonomous equation

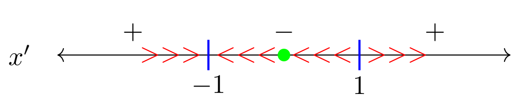
$$\frac{dx}{dt} = x^2 - 1.$$

(a) Find and classify the critical points.

(b) Describe the long term behavior of the solution passing through the point $x(4) = 0$.

Solution:

Note that $x^2 - 1 = (x-1)(x+1)$, so there are equillibria solutions at $x = 1$ and $x = -1$. We can thus compute a number line for x':



We see that -1 is a sink (stable), while 1 is a source (unstable). A trajectory passing through $x(4) = 0$ will approach $x = -1$ in the limit.

$\square$

F5. Solve $y' + xy = x^3$.

Solution: First, we solve the homogeneous equation $y' + xy = 0$, which is separable: $\frac{y'}{y} = -x$, so $\ln |y| = -\frac{1}{2}x^2 + c$, or $y = c_1 e^{-\frac{1}{2}x^2}$. We then use variation of parameters to find a particular solution, writing $y_p = ve^{-\frac{1}{2}x^2}$; substituting in and simplifying yields

$$x^3 = y_p' + xy_p = v'e^{-\frac{1}{2}x^2} - vxe^{-\frac{1}{2}x^2} + xve^{-\frac{1}{2}x^2} = v'e^{-\frac{1}{2}x^2}.$$

Thus, $v' = x^3 e^{\frac{1}{2}x^2}$. This can be integrated by parts to obtain $v = x^2 e^{\frac{1}{2}x^2} - 2e^{\frac{1}{2}x^2}$, so $y_p = ve^{-\frac{1}{2}x^2} = x^2 - 2$. Thus the general solution is

$$y = c_1 e^{-\frac{1}{2}x^2} + x^2 - 2.$$

$\square$

F6. One of the two ODEs below is exact. Identify which one, and solve it.

$$(x + 3y)y' + y = 3x$$
$$(x + 3y)y' - y = -3x$$

Solution: Rewrite the first equation as $y' = \frac{3x-y}{x+3y}$; if there is a function with $\nabla F = \langle -(3x - y), x + 3y\rangle$, then

$$\frac{d}{dt}F(x, y) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} = -(3x - y)(x + 3y) + (x + 3y)(3x - y) = 0,$$

so $F(x, y) = c$ is a solution of the autonomous system.

Note that $\langle -(3x - y), x + 3y\rangle$ is conservative, as $\frac{\partial}{\partial x}(x + 3y) = 1 = \frac{\partial}{\partial y}(-3x + y)$; so a potential function F is found by integrating: $F = \int x + 3y \, dy = xy + \frac{3}{2}y^2 + f(x)$, and differentiating with respect to x, we have $-3x + y = y + f'(x)$, so $f'(x) = -3x$, in which case $f(x) = \int -3x \, dx = \frac{-3}{2}x^2$, so a potential function is $F(x, y) = xy + \frac{3}{2}y^2 - \frac{3}{2}x^2$. So the general solution to the ODE is

$$xy + \frac{3}{2}y^2 - \frac{3}{2}x^2 = c.$$

$\square$

S1. Find the general solution of the system

$$x' = 2x + y,$$
$$y' = x + 2y$$

Solution:

We begin by moving all x and y terms to one side, and rewriting in terms of differential operators

$$(D - 2I)x - Iy = 0,$$
$$(-I)x + (D - 2I)y = 0$$

Applying $D-2I$ to the top equation and adding, we obtain $(D-2I)^2x - Ix = 0$, or $(D^2-4D+3I)x = 0$, which factors as $(D-3I)(D-I)x = 0$. Thus the general solution for x is $x = c_1e^{3t} + c_2e^t$. But rearranging the first equation, we have $y = x' - 2x$, so $y = \left(c_1e^{3t} + c_2e^t\right)' - 2\left(c_1e^{3t} + c_2e^t\right) = c_1e^{3t} - c_2e^t$. Thus the general solution is

$$x = c_1e^{3t} + c_2e^t$$
$$y = c_1e^{3t} - c_2e^t$$

□

S2. Two populations of competing species of fish, bluegills and greenfish, are modelled by the system

$$\frac{dG}{dt} = 0.1G - 0.005G^2 - 0.004BG$$
$$\frac{dB}{dt} = 0.2B - 0.01B^2 - 0.004BG.$$

(a) Identify all equillibrium points for the system.

(b) If a lake is stocked with 25 bluegills and 35 greenfish , what will happen to the two populations in the long term?

Solution:

The greenfish isocline is given by

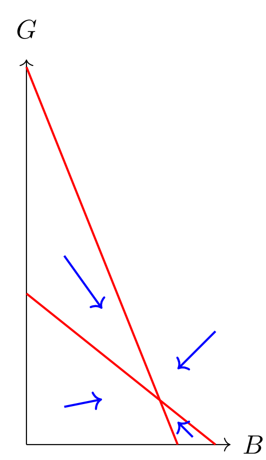
$$0 = 0.1G - 0.005G^2 - 0.004BG = 0.001G(100 - 5G - 4B)$$

, while the bluegill isocline is given by

$$0 = 0.2B - 0.01B^2 - 0.004BG = 0.001B(200 - 10B - 4G).$$

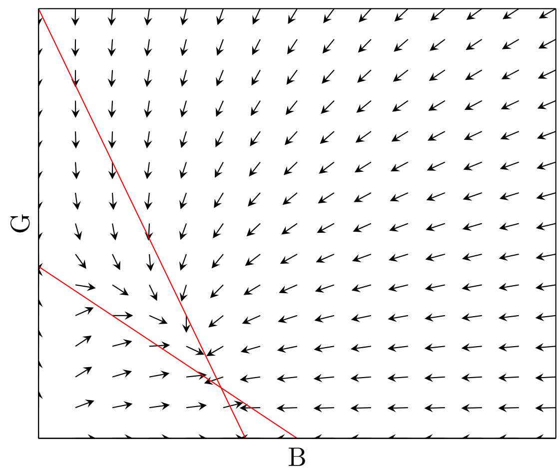
If $G = 0$, then the bluegill isocline implies $B = 0$ or $200 - 10B = 0$, i.e. $B = 20$, giving two equillibrium points ($B = 0, G = 0$ and $B = 20, G = 0$. On the other hand, if $100 - 5G - 4B = 0$ and $B = 0$, then we obtain an equillibrum point at $B = 0, G = 20$; but if $100 - 5G - 4B = 0$ and $200 - 10B - 4G = 0$, solving this system yields $B = 17.65, G = 5.88$ as an equillibrium point.

Now that we have identified the four equillbrium points, to answer part (b), we plot the isoclines, and the general direction of the slopes in each of the four regions



Analyzing the slopes in each of the four regions on the graph shows that the central equillibrium point is a sink. Thus, in the long term the populations will approach the central equillbirum of 17.65 bluegills and 5.88 greenfish.

Here is a slope field to make the intuition of the above rough sketch more clear:

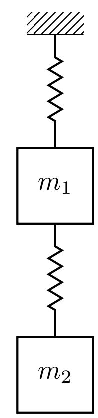


□

S3. Consider the dual mass-spring system at the right. The upper mass is 2kg while the lower is 1kg. The upper spring has a spring constant of 2N/m while the lower spring constant is 1N/m. The lower mass is pulled down 1m without disturbing the upper mass, and released from rest.

(a) Write down an IVP modelling the motion of the two masses.

(b) Where will the two masses be after 2s?



Solution: Let y_1, y_2 represent the positions of the two masses respectively, as measured from equillibrium. The lower mass is only affected by the lower spring, so its motion is modelled by $y_2'' = -1(y_2 - y_1)$, since the length the lower string is stretched is $y_2 - y_1$. The upper mass is affected by both springs, so it has a force of $1(y_2 - y_1)$ from the lower spring, and $-2y_1$ for the upper spring. Thus, the upper mass's motion is modelled by $2y_1'' = -2(y_1) + 1(y_2 - y_1)$. Simplifying and adding in the initial conditions $y_1(0) = 0, y_1'(0) = 0, y_2(0) = -1, y_2'(0) = 0$, we have the IVP

$$\begin{aligned} 2y_1'' &= -3y_1 + y_2 \qquad & y_1(0) = 0, y_1'(0) = 0 \\ y_2'' &= y_1 - y_2 \qquad & y_2(0) = -1, y_2'(0) = 0 \end{aligned}$$

To answer part (b), we need to solve this system. First, we rewrite in terms of differential operators:

$$\begin{aligned} (2D^2 + 3)y_1 - y_2 &= 0 \\ -y_1 + (D^2 + 1)y_2 &= 0 \end{aligned}$$

Applying D^2+1 to the top equation and adding to the bottom yields

$$0 = \left((D^2+1)(2D^2+3)-1\right) y_1 = (2D^4+5D^2+2)y_1 = (2D^2+1)(D^2+2)y_1$$

Note that the kernel of $2D^2+1$ is $c_1 \cos\left(\frac{t}{\sqrt{2}}\right) + c_2 \sin\left(\frac{t}{\sqrt{2}}\right)$ and the kernel of D^2+2 is $c_3 \cos\left(t\sqrt{2}\right) + c_4 \sin\left(t\sqrt{2}\right)$. Thus, the general solution for y_1 is

$$y_1 = c_1 \cos\left(\frac{t}{\sqrt{2}}\right) + c_2 \sin\left(\frac{t}{\sqrt{2}}\right) + c_3 \cos\left(t\sqrt{2}\right) + c_4 \sin\left(t\sqrt{2}\right).$$

Since $y_2 = 2y_1'' + 3y_1$, we thus obtain

$$y_2 = \frac{1}{2}c_1 \cos\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{2}c_2 \sin\left(\frac{t}{\sqrt{2}}\right) - 2c_3 \cos\left(t\sqrt{2}\right) - 2c_4 \sin\left(t\sqrt{2}\right).$$

Now, using our initial conditions, we have

$$\begin{aligned}
0 &= y_1(0) = c_1 + c_3 \\
-1 &= y_2(0) = \frac{1}{2}c_1 - 2c_3 \\
0 &= y_1'(0) = \frac{1}{\sqrt{2}}c_2 + \sqrt{2}c_4 \\
0 &= y_2'(0) = \frac{1}{2\sqrt{2}}c_2 - 2\sqrt{2}c_4
\end{aligned}$$

The last two equations imply $c_2 = c_4 = 0$, while the first two yield $c_1 = -\frac{2}{5}$ and $c_3 = \frac{2}{5}$. Thus, the solution to the IVP is

$$\begin{aligned}
y_1 &= -\frac{2}{5}\cos\left(\frac{t}{\sqrt{2}}\right) + \frac{2}{5}\cos\left(t\sqrt{2}\right) \\
y_2 &= \frac{1}{5}\cos\left(\frac{t}{\sqrt{2}}\right) - \frac{4}{5}\cos\left(t\sqrt{2}\right)
\end{aligned}$$

Thus, after two seconds, the upper mass is at $y_1(2) \approx -0.443$ (i.e. 0.443m down) and the lower mass is at $y_2(2) \approx 0.792$ (i.e. 0.792m up).

□

N1. Determine whether a unique solution to the IVP below is guaranteed to exist. Be sure to explain your reasoning.

$$y' = |x+y|; \qquad y(2) = -2$$

Solution: Let $f(x, y) = |x+y|$. This is continuous around $(2, -2)$, but $\frac{\partial f}{\partial y}$ does not exist when $x + y = 0$. Since $\frac{\partial f}{\partial y}$ is not continuous around $(2, -2)$, the IVP is not guaranteed to have a unique solution.

□

N2. Consider the differential equation

$$xy'' - \frac{1}{1+x}y' + \frac{1}{(1+x)^2}y = 0.$$

Determine all intervals on which a unique solution is guaranteed to exist.

Solution: Since $\frac{1}{1+x}$ and $\frac{1}{(1+x)^2}$ are not continuous at $x = -1$, and the coefficient of y'' vanishes at $x = 0$, the ODE is only guaranteed to have a solution on the intervals $-\infty < x < -1$, $-1 < x < 0$, and $0 < x < \infty$.

□

N3. Determine all intervals on which a unique solution is guaranteed to exist.

$$x' = -tx + y + \sqrt{t-1},$$
$$y' = 2x + \sqrt{1-t}y + \sqrt[3]{t}$$

Solution: Note that the only coefficient functions that are not continous everywhere are $\sqrt{t+1}$ and $\sqrt{1-t}$. The former is continuous on $-1 < t < \infty$, while the latter is continuous on $-\infty < t < 1$. Thus, a unique solution exists on the intersection of these intervals, namely $-1 < t < 1$.

□

N4. Use Euler's method with stepsize $h = 0.25$ to estimate $y(2)$, where y is a solution to the IVP

$$y' + y^2 = x, \qquad y(1) = 1$$

.

Solution: First, we rewrite the ODE as $y' = x - y^2$. Then we construct a table

x_n	y_n	$y'(x_n, y_n)$	$x_{n+1} = x_n + h$	$y_{n+1} = y_n + hy'(x_n, y_n)$
1	1	0	1.25	1
1.25	1	0.25	1.5	1.0625
1.5	1.0625	0.3711	1.75	1.1553
1.75	1.1553	0.4153	2	1.2591
2	1.2591			

Thus, $y(2) \approx 1.2591$.

□

N5. Use Euler's method with stepsize $h = 0.1$ to estimate $x(2.3)$ and $y(2.3)$, where x and y are solutions to the IVP

$$x' = 3x - ty \qquad x(2) = 1$$
$$y' = x - y^2 \qquad y(2) = 0$$

Solution:

t_n	x_n	y_n	$x'(t_n, x_n, y_n)$	$y'(t_n, x_n, y_n)$	t_{n+1}	$x_{n+1} = x_n + hx'(t_n, x_n, y_n)$	$y_{n+1} = y_n + hy'(t_n, x_n, y_n)$
2	1	0	3	1	2.1	1.3	0.1
2.1	1.3	0.1	3.69	1.29	2.2	1.669	0.229
2.2	1.669	0.229	4.5032	1.6166	2.3	2.1193	0.3907
2.3	2.1993	0.3907					

Thus, $x(2.3) \approx 2.1993$ and $y(2.3) \approx 0.3907$.

$\square$

D1.

Solution:

$\square$

D2.

Solution:

$\square$

D3.

Solution:

$\square$

D4.

Solution:

$\square$