# Module C: Constant coefficient linear ODEs

# How can we solve and apply linear constant coefficient ODEs?

At the end of this module, students will be able to...

**C1.** **Constant coefficient first order.** ...find the general solution to a first order constant coefficient ODE.

**C2.** **Modeling motion in viscous fluids.** ...model the motion of a falling object with linear drag

**C3.** **Homogeneous constant coefficient second order.** ...find the general solution to a homogeneous second order constant coefficient ODE.

**C4.** **IVPs.** ...solve initial value problems for constant coefficient ODEs

**C5.** **Non-homogenous constant coefficient second order.** ...find the general solution to a non-homogeneous second order constant coefficient ODE

**C6.** **Modeling oscillators.** ...model (free or forced, damped or undamped) mechanical oscillators with a second order ODE

**Readiness Assurance Outcomes**

Before beginning this module, each student should be able to...

- Describe Newton's laws in terms of differential equations.
- Find all roots of a quadratic polynomial.
- Use Euler's theorem to relate $\sin(t)$, $\cos(t)$, and $e^t$.
- Use Euler's theorem to simplify complex exponentials.
- Use substitution to compute indefinite integrals.
- Use integration by parts to compute indefinite integrals.
- Solve systems of two linear equations in two variables.

The following resources will help you prepare for this module.

- Describe Newtons laws in terms of differential equations. `https://youtu.be/cioi4lRrAzw`
- Find all roots of a quadratic polynomial. `https://youtu.be/2ZzuZvz33X0` `https://youtu.be/TV5kDqiJ1Os`
- Use Eulers theorem to relate $\sin(t)$, $\cos(t)$, and $e^t$ and to simplify complex exponentials. `https://youtu.be/F_0yfvm0UoU` `https://youtu.be/sn3orkHWqUQ`
- Use substitution to compute indefinite integrals. `https://youtu.be/b76wePnIBdU`
- Use integration by parts to compute indefinite integrals. `https://youtu.be/bZ8YAHDTFJ8`
- Solve systems of two linear equations in two variables. `https://youtu.be/Y6JsEja15Vk`

# Module C Section 1

**Activity C.1.1** *($\sim$5 min)*

Why don't clouds fall out of the sky?

(a) They are lighter than air

(b) Wind keeps them from falling

(c) Electrostatic charge

(d) They do fall, just very slowly

**Activity C.1.2** *(∼5 min)*

List all of the forces acting on a tiny droplet of water falling from the sky.

## Activity C.1.3 *(∼5 min)*

Tiny droplets of water obey **Stoke's law**, which says that air resistance is proportional to (the magnitude of) velocity.

- Let $v$ be the velocity of a droplet of water (positive for upward, negative for downward).
- Let $g > 0$ be the magnitude of acceleration due to gravity and $b > 0$ be another positive constant.

Apply Newton's second law (force = mass × acceleration) to determine which of the following **ordinary differential equations (ODEs)** models the velocity of a falling droplet of water.

(a) $v' = g - v$

(b) $v' = g + v$

(c) $mv' = -mg - bv$

(d) $mv' = -mg + bv$

**Observation C.1.4**

The modeling equation

$$mv' = -mg - bv$$

may be obtained by splitting the total force into gravity and air resistance:

$$F = F_g + F_r$$

Then $F = ma = mv'$ and $F_g = m(-g) = -mg$ are the result of Newton's second law, and $F_r = -bv$ holds because it should be (a) in the opposite direction of velocity and (b) a constant multiple of velocity.

Note that this equation may be rearranged as follows to group $v$ and its derivative $v'$ together on the left-hand side:

$$v' + \left(\frac{b}{m}\right) v = -g$$

## Definition C.1.5

A **first order constant coefficient** differential equation can be written in the form

$$y' + by = f(x),$$

or equivalently,

$$\frac{dy}{dx} + by = f(x).$$

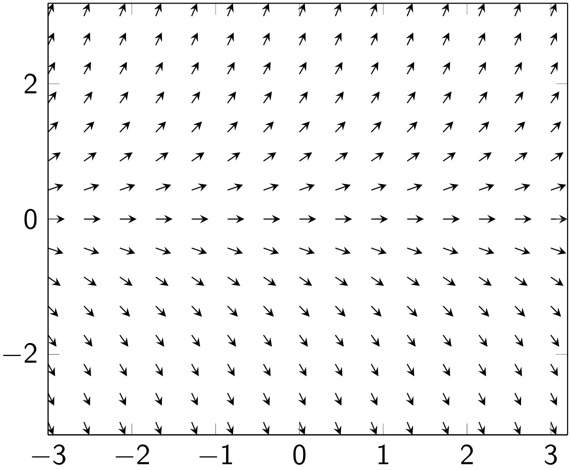
We will use both notations interchangeably.

Here, **first order** refers to the fact that the highest derivative we see is the first derivative of $y$.

# Observation C.1.6

Consider the differential equation $y' = y$.

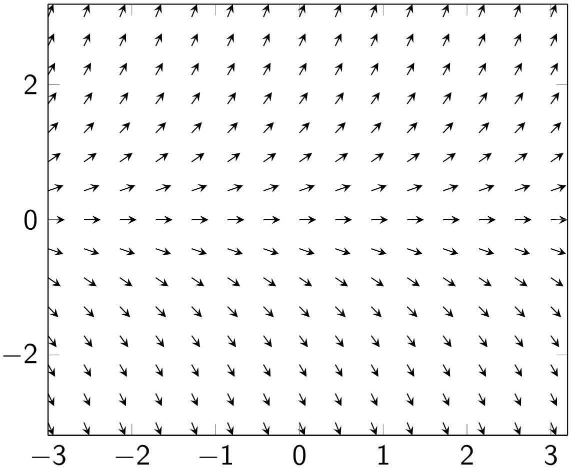
A useful way to visualize a first order differential equation is by a **slope field**




Each arrow represents the slope of a solution **trajectory** through that point.

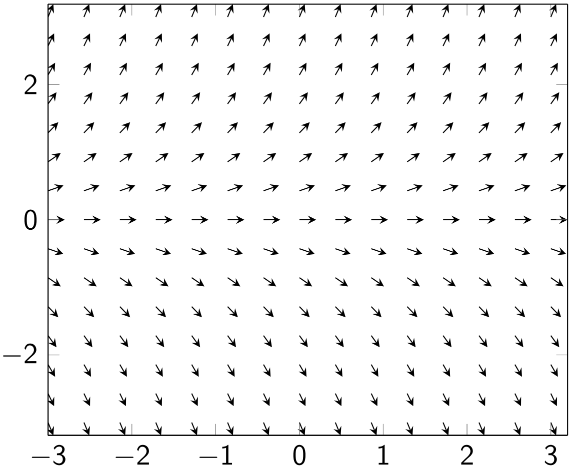
## **Activity C.1.7** *(∼5 min)*

Consider the differential equation $y' = y$ with slope field below.



**Activity C.1.7** *(∼5 min)*

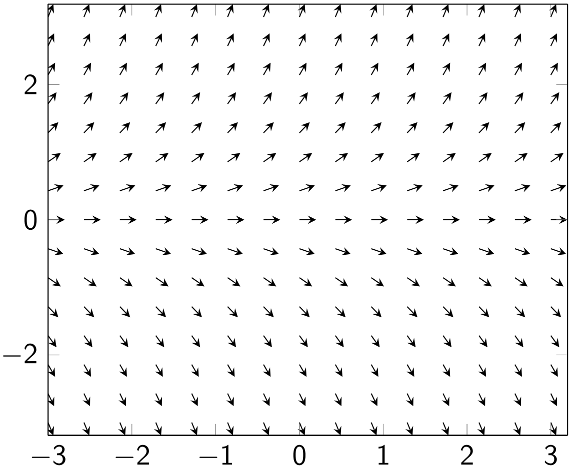
Consider the differential equation $y' = y$ with slope field below.




*Part 1:* Draw a trajectory through the point $(0, 1)$.

**Activity C.1.7** *(∼5 min)*

Consider the differential equation $y' = y$ with slope field below.




*Part 1:* Draw a trajectory through the point $(0, 1)$.

*Part 2:* Draw a trajectory through the point $(-1, -1)$.

**Activity C.1.8** *(∼15 min)*

Consider the differential equation $y' = y$.

**Activity C.1.8** *(~15 min)*
Consider the differential equation $y' = y$.
*Part 1:* Find a solution to $y' = y$.

**Activity C.1.8** *(∼15 min)*

Consider the differential equation $y' = y$.

*Part 1:* Find a solution to $y' = y$.

*Part 2:* Modify this solution to write an expression describing **all** solutions to $y' = y$.

**Definition C.1.9**

A differential equation will have many solutions. Each individual solution is said to be a **particular solution**, while the **general solution** encompasses **all** of these by using parameters such as $C, k, c_0, c_1$ and so on. For example:

- The general solution to the differential equation $y' = 2x - 3$ is $y = x^2 - 3x + C$ (as done in Calculus courses).
- The general solution for $y' = y$ is $y = ke^x$ (as done in the previous activity).

**Activity C.1.10** *(∼15 min)*
Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.

**Activity C.1.10** *(∼15 min)*
Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.
*Part 1:* Solve $y' = 2y$.

**Activity C.1.10** *($\sim$15 min)*
Adapt the general solution $y = ke^x$ for $y' = y$ to find general solutions for the following differential equations.
*Part 1:* Solve $y' = 2y$.
*Part 2:* Solve $y' = y + 2$.

# Module C Section 2

## Observation C.2.1

Recall the last activity from yesterday:

Solve $y' = y + 2$

This is very similar to the equation $y' = y$, which we were able to solve.

**Activity C.2.2** *(~15 min)*
Solve $y' = y + 2$

**Simple idea:** Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

**Activity C.2.2** *(∼15 min)*
Solve $y' = y + 2$

**Simple idea:** Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

*Part 1:* Use the Product Rule to find $y_p' = \frac{d}{dx}[ve^x]$.

**Activity C.2.2** *(∼15 min)*
Solve $y' = y + 2$

**Simple idea:** Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

*Part 1:* Use the Product Rule to find $y_p' = \frac{d}{dx}[ve^x]$.
*Part 2:* Substitute $y_p$ and $y_p'$ into the equation $y' = y + 2$.

**Activity C.2.2** *(∼15 min)*
Solve $y' = y + 2$

**Simple idea:** Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

*Part 1:* Use the Product Rule to find $y_p' = \frac{d}{dx}[ve^x]$.
*Part 2:* Substitute $y_p$ and $y_p'$ into the equation $y' = y + 2$.
*Part 3:* Solve for $v$.

**Activity C.2.2** *(∼15 min)*
Solve $y' = y + 2$

**Simple idea:** Since $y_0 = e^x$ was a particular solution of $y' = y$, we guess that a particular solution for $y' = y + 2$ is of the form $y_p = ve^x$ for some **function** $v(x)$.

*Part 1:* Use the Product Rule to find $y_p' = \frac{d}{dx}[ve^x]$.
*Part 2:* Substitute $y_p$ and $y_p'$ into the equation $y' = y + 2$.
*Part 3:* Solve for $v$.
*Part 4:* Find $y_p$.

## Observation C.2.3

The technique outlined in the previous activity is called **variation of parameters**. If $y_0$ is a particular solution of the **homogeneous** equation, assume that a particular solution of the **non-homogeneous** equation has the form $y_p = vy_0$, and then determine what $v$ must be.

**Example:**

$$y' + 3y = 0 \qquad \text{homogeneous}$$
$$y' + 3y = x \qquad \text{non-homogeneous}$$

Note that each term of the homogeneous equation includes $y$ or it derivatives.

**Activity C.2.4** *(∼20 min)*

Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

| | |
|---|---|
| $y' + 3y = 0$ | homogeneous |
| $y' + 3y = x$ | non-homogeneous |

**Activity C.2.4** *($\sim$20 min)*

Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

| | |
|---|---|
| $y' + 3y = 0$ | homogeneous |
| $y' + 3y = x$ | non-homogeneous |

*Part 1:* Modify $e^x$ to find the general solution $y_h$ for the homogeneous equation.

**Activity C.2.4** *(∼20 min)*

Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

$$y' + 3y = 0 \qquad \text{homogeneous}$$
$$y' + 3y = x \qquad \text{non-homogeneous}$$

*Part 1:* Modify $e^x$ to find the general solution $y_h$ for the homogeneous equation.

*Part 2:* Choose a particular solution $y_0$ for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** $v$. Substitute $y_p$ into non-homogeneous equation and simplify.

**Activity C.2.4** *(∼20 min)*

Solve $y' = x - 3y$ by first solving its corresponding homogeneous equation and using variation of parameters:

| | |
|---|---|
| $y' + 3y = 0$ | homogeneous |
| $y' + 3y = x$ | non-homogeneous |

*Part 1:* Modify $e^x$ to find the general solution $y_h$ for the homogeneous equation.
*Part 2:* Choose a particular solution $y_0$ for the homogeneous equation, and assume $y_p = vy_0$ is a particular solution of the non-homogeneous equation for some **function** $v$. Substitute $y_p$ into non-homogeneous equation and simplify.
*Part 3:* Determine $v$, and then determine $y_p$.

## Observation C.2.5

Since $y_h = ke^{-3x}$ was the general solution of $y' + 3y = 0$, and $y_p = \frac{x}{3} - \frac{1}{9}$ is a particular solution of $y' + 3y = x$,

$$y = y_h + y_p = \left(ke^{-3x}\right) + \left(\frac{x}{3} - \frac{1}{9}\right)$$

is a solution to $y' + 3y = x$:

$$\frac{d}{dx}[y_h + y_p] + 3(y_h + y_p) = (y_h' + 3y_h) + (y_p' + 3y_p) = 0 + x = x$$

**Fact C.2.6**

Let $a$ be a constant real number. Every constant coefficient first order ODE

$$y' + ay = f(x)$$

has the general solution

$$y = y_h + y_p$$

where $y_h$ is the general solution to the homogeneous equation $y' + ay = 0$ and $y_p$ is a particular solution to $y' + ay = f(t)$.

**Activity C.2.7** *(∼15 min)*
Find the general solution to $y' = 2y + x + 1$ using variation of parameters:

- Write the homogeneous equation and find its general solution $y_h$.
- Use a particular solution $y_0$ for the homogeneous equation to find a particular solution $y_p = vy_0$ for the original equation.
- Then $y = y_h + y_p$ gives the general solution to the equation.

# Module C Section 3

## Observation C.3.1

Recall that we can model the velocity of a water droplet in a cloud by

$$mv' = -mg - bv$$

where negative numbers represent downward motion, $m > 0$ is the mass of the droplet, $g > 0$ is the magnitude of acceleration due to gravity, and $b > 0$ is the proportion of wind resistance to speed.

**Activity C.3.2** *(∼20 min)*
A water droplet with a radius of $10\,\mu$m has a mass of about $4 \times 10^{-15}$ kg. It is determined in a laboratory that for a droplet this size, the constant $b$ has a value of $3 \times 10^{-3}$ kg/s, and it is known that $g$ is approximately $9.8\,\mathrm{m}/\mathrm{s}^2$.

Complete the following tasks to study the motion of this droplet.

**Activity C.3.2** *(∼20 min)*
A water droplet with a radius of $10\,\mu$m has a mass of about $4 \times 10^{-15}$ kg. It is determined in a laboratory that for a droplet this size, the constant $b$ has a value of $3 \times 10^{-3}$ kg/s, and it is known that $g$ is approximately $9.8\,\mathrm{m/s^2}$.

Complete the following tasks to study the motion of this droplet.
*Part 1:* Rewrite $mv' = -mg - bv$ in the form of $v' + av = \ ?$ for some value of $a$.

**Activity C.3.2** *(∼20 min)*

A water droplet with a radius of $10\,\mu$m has a mass of about $4 \times 10^{-15}$ kg. It is determined in a laboratory that for a droplet this size, the constant $b$ has a value of $3 \times 10^{-3}$ kg/s, and it is known that $g$ is approximately $9.8\,\mathrm{m/s^2}$.

Complete the following tasks to study the motion of this droplet.

*Part 1:* Rewrite $mv' = -mg - bv$ in the form of $v' + av = \ ?$ for some value of $a$.

*Part 2:* Find the general solution of this ODE in terms of $a$ and $g$. (Let $v_p = wv_0$ when using variation of parameters to avoid confusion.)

**Activity C.3.2** *(∼20 min)*
A water droplet with a radius of $10\,\mu$m has a mass of about $4 \times 10^{-15}$ kg. It is determined in a laboratory that for a droplet this size, the constant $b$ has a value of $3 \times 10^{-3}$ kg/s, and it is known that $g$ is approximately $9.8\,\mathrm{m/s^2}$.

Complete the following tasks to study the motion of this droplet.
*Part 1:* Rewrite $mv' = -mg - bv$ in the form of $v' + av = ?$ for some value of $a$.
*Part 2:* Find the general solution of this ODE in terms of $a$ and $g$. (Let $v_p = wv_0$ when using variation of parameters to avoid confusion.)
*Part 3:* Due to wind resistence, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of $a$ and $g$?

**Activity C.3.2** *(~20 min)*

A water droplet with a radius of $10\,\mu$m has a mass of about $4 \times 10^{-15}$ kg. It is determined in a laboratory that for a droplet this size, the constant $b$ has a value of $3 \times 10^{-3}$ kg/s, and it is known that $g$ is approximately $9.8\,\mathrm{m/s^2}$.

Complete the following tasks to study the motion of this droplet.

*Part 1:* Rewrite $mv' = -mg - bv$ in the form of $v' + av = \ ?$ for some value of $a$.

*Part 2:* Find the general solution of this ODE in terms of $a$ and $g$. (Let $v_p = wv_0$ when using variation of parameters to avoid confusion.)

*Part 3:* Due to wind resistence, eventually the droplet will effectively stop accelerating upon reaching a certain velocity. What is this **terminal velocity** of the droplet in terms of $a$ and $g$?

*Part 4:* If the droplet starts from rest ($v = 0$ when $t = 0$), what is its velocity after $0.01$ s? Use a calculator to compute the answer in m/s.

**Definition C.3.3**

The last part of the previous activity is an example of an **Initial Value Problem (IVP)**; we were given the initial value of the velocity in addition to our differential equation.

Physical scenarios often produce IVPs with a unique solution.

**Activity C.3.4** *(∼10 min)*
Solve the IVP

$$y' + 3y = 0, \qquad y(0) = 2.$$

**Activity C.3.5** *(∼10 min)*
Solve the IVP

$$y' - 2y = 2, \qquad y(0) = 1.$$

**Activity C.3.6** *(∼5 min)*

Solve the IVP

$$y' - 2y = 2, \qquad y(2) = 1.$$

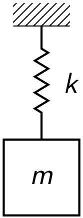
# Module C Section 4

## Observation C.4.1

What happens when your tire hits a pothole?

**Activity C.4.2** *(~5 min)*

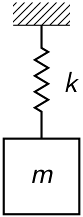
More abstractly, let's attach a mass (weighing $m$ $\mathrm{kg}$) to a spring.




List all forces acting on the mass.

**Activity C.4.3** *(∼5 min)*

**Hooke's law** says that the force exerted by the spring is proportional to the distance the spring is stretched.

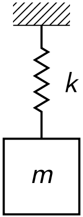



Write a differential equation modeling the displacement of the mass.

## Observation C.4.4

There is an equillibrium point where the force of gravity balances the spring force. If we measure displacement from this point, we can model the mass-spring system by

$$my'' = -ky.$$



**Activity C.4.5** *(∼15 min)*

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

**Activity C.4.5** *(∼15 min)*
Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

*Part 1:* Find a solution.

**Activity C.4.5** *($\sim$15 min)*

Consider the (numerically simplified) mass-spring equation

$$y'' = -y.$$

*Part 1:* Find a solution.

*Part 2:* Find the general solution.

**Activity C.4.5** *(∼15 min)*
Consider the (numerically simplified) mass-spring equation

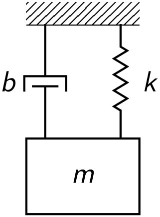
$$y'' = -y.$$

*Part 1:* Find a solution.
*Part 2:* Find the general solution.
*Part 3:* Describe the long term behavior of the mass-spring system.

## Activity C.4.6 *(∼5 min)*

In applications, this infinitely oscillating behavior is often inappropriate.

Thus, a damper (dashpot) is often incorporated. This provides a force proportional to the velocity.




Write a differential equation modeling the displacement of a mass in a **damped** mass-spring system.

## Observation C.4.7

The damped mass-spring system can be modelled by

$$my'' = -by' - ky.$$

Here $m$ is the mass, $k$ is the spring constant, and $b$ is the damping constant. We can rearrange this as

$$my'' + by' + ky = 0.$$

This is a **homogeneous second order constant coefficient** differential equation. Here, **homogeneous** refers to the 0 on the right hand side of the equation.

**Activity C.4.8** *($\sim$15 min)*

Consider the second order constant coefficient equation

$$y'' = y.$$

**Activity C.4.8** *(∼15 min)*
Consider the second order constant coefficient equation

$$y'' = y.$$

*Part 1:* Find a solution.

**Activity C.4.8** *(∼15 min)*
Consider the second order constant coefficient equation

$$y'' = y.$$

*Part 1:* Find a solution.
*Part 2:* Find the general solution.

**Activity C.4.8** *(∼15 min)*
Consider the second order constant coefficient equation

$$y'' = y.$$

*Part 1:* Find a solution.
*Part 2:* Find the general solution.
*Part 3:* Describe the long term behavior of the solutions.

# Module C Section 5

## Observation C.5.1

It is sometimes useful to think in terms of **differential operators**.

- We will use $D$ to represent a derivative; another common notation is $\frac{\partial}{\partial x}$. So for any function $y$,
$$D(y) = \frac{\partial y}{\partial x} = y'.$$
- $D^2$ will denote the second derivative operator (i.e. differentiate twice, or apply D twice).
- We will use $I$ for the identity operator; it does nothing to a function. That is, $I(y) = y$. It can be thought of as $I = D^0$ (i.e. differentiate zero times).

In this language, the differential equation $y' + 3y = 0$ can be rewritten as $D(y) + 3I(y) = 0$, or $(D + 3I)(y) = 0$.

Thus, the question of solving the homogeneous differential equation is the question of finding the **kernel** of the differential operator $D + 3I$.

**Activity C.5.2** *(∼5 min)*

What is the kernel of $D - I$ ?

**Activity C.5.2** *($\sim$5 min)*
What is the kernel of $D - I$ ?

*Part 1:* Write a differential equation that corresponds to this question.

**Activity C.5.2** *($\sim$5 min)*
What is the kernel of $D - I$ ?

*Part 1:* Write a differential equation that corresponds to this question.
*Part 2:* Find the general solution of this differential equation.

**Activity C.5.3** *(~5 min)*
Find a differential operator whose kernel is the solution set of the ODE $y' = 4y$.

**Activity C.5.4** *(∼10 min)*

Consider the ODE

$$y'' + 5y' + 6y = 0.$$

**Activity C.5.4** *(∼10 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.

**Activity C.5.4** *(∼10 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.
*Part 2:* Factor this differential operator as a composition of two operators. (This works because $D$ and $I$ commute).

**Activity C.5.4** *(∼10 min)*
Consider the ODE

$$y'' + 5y' + 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.
*Part 2:* Factor this differential operator as a composition of two operators. (This works because $D$ and $I$ commute).
*Part 3:* Find the general solution of the ODE.

**Observation C.5.5**

If we let $\mathcal{L} = D^2 + 5D + 6I$, we can write the ODE

$$y'' + 5y + 6y = 0$$

as

$$\mathcal{L}(y) = 0.$$

Note that such an $\mathcal{L}$ is always a **linear transformation**.

**Activity C.5.6** *(∼5 min)*
Find the general solution to

$$y'' + y' - 12y = 0.$$

# Module C Section 6

**Activity C.6.1** *(∼5 min)*
Consider the ODE

$$y'' + 5y' - 6y = 0.$$

**Activity C.6.1** *(∼5 min)*
Consider the ODE

$$y'' + 5y' - 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.

**Activity C.6.1** *($\sim$5 min)*
Consider the ODE

$$y'' + 5y' - 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.
*Part 2:* Factor this differential operator as a composition of two operators. (This works because $D$ and $I$ commute).

**Activity C.6.1** *(∼5 min)*
Consider the ODE

$$y'' + 5y' - 6y = 0.$$

*Part 1:* Find a differential operator whose kernel is the solution set of the above ODE.
*Part 2:* Factor this differential operator as a composition of two operators. (This works because $D$ and $I$ commute).
*Part 3:* Find the general solution of the ODE.

**Activity C.6.2** *(∼5 min)*

Solve the ODE

$$2y'' + 7y' + 6y = 0.$$

**Activity C.6.3** *(∼5 min)*
An **Initial Value Problem (IVP)** consists of an ODE along with some initial conditions that allow you to determine a single solution.

Solve the IVP

$$2y'' + 7y' + 6y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

.

**Activity C.6.4** *(∼5 min)*
Solve the ODE

$$y'' + y = 0.$$

**Activity C.6.5** *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

**Activity C.6.5** *(∼15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

*Part 1:* Find the general solution.

**Activity C.6.5** *($\sim$15 min)*
Consider the ODE

$$y'' + 2y' + 5y = 0$$

.

*Part 1:* Find the general solution.
*Part 2:* Describe the long-term behavior of the solutions.

**Observation C.6.6**

Solving $y'' + 2y' + 5y = 0$ produced a general solution

$$y = c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t}.$$

Applying Euler's formula yields

$$\begin{aligned} y &= c_1 e^{-t}(\cos(2t) + i\sin(2t)) + c_2 e^{-t}(\cos(2t) - i\sin(2t)) \\ &= (c_1 + c_2)e^{-t}\cos(2t) i(c_1 - c_2)e^{-t}\sin(2t) \end{aligned}$$

which we can rewrite (letting $k_1 = c_1 + c_2$ and $k_2 = i(c_1 - c_2)$) as

$$y = k_1 e^{-t}\cos(2t) + k_2 e^{-t}\sin(2t).$$

**Activity C.6.7** *(∼15 min)*

Solve the IVP

$$y'' + 6y' + 34y = 0, \qquad y(0) = 2, \qquad y'(0) = 4.$$

# Module C Section 7

**Activity C.7.1** *(∼10 min)*
Solve the ODE

$$y'' - 4y' + 4y = 0.$$

**Observation C.7.2**

To solve this, we need to find the kernel of $(D - 2I)(D - 2I)$.

- The kernel of $D - 2I$ is $\left\{ce^{2t} \mid c \in \mathbb{R}\right\}$.
- However, if $(D - 2I)(y) = Ae^{2t}$, then applying $D - 2I$ twice will yield zero!
- So we must solve the ODE

$$y' - 2y = e^{2t}.$$

**Activity C.7.3** *(∼15 min)*
Solve $y' - 2y = e^{2t}$.

**Observation C.7.4**

Thus, we have shown that the general solution of

$$y'' - 4y' + 4y = 0$$

is

$$y = c_0 e^{2t} + c_1 t e^{2t}.$$

**Activity C.7.5** *(∼15 min)*

Solve $y'' - 6y' + 9y = 0$.

**Activity C.7.6** *(∼10 min)*

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

If $r$ is a number such that $ar^2 + br + c = 0$, what can you conclude?

(a) $e^{rt}$ is a solution.

(b) $e^{-rt}$ is a solution.

(c) $te^{rt}$ is a solution.

(d) There are no solutions.

**Activity C.7.7** *(∼5 min)*

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

When does the general solution have the form $c_0 e^{rt} + te^{rt}$ ?

(a) When the polynomial $ax^2 + bx + c$ has two distinct real roots.

(b) When the polynomial $ax^2 + bx + c$ has a repeated real root.

(c) When the polynomial $ax^2 + bx + c$ has two distinct non-real roots.

(d) When the polynomial $ax^2 + bx + c$ has a repeated non-real root.

## Observation C.7.8

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If $r$ is a root of $ax^2 + bx + c = 0$, then $e^{rt}$ is a solution of the ODE.
- If $r$ is a double root, variation of parameters shows that $te^{rt}$ is also a solution.
- If $r$ is not real, Euler's formula allows us to express the complex exponential part of the solution in terms of $\sin(rt)$ and $\cos(rt)$.

# Module C Section 8

## Observation C.8.1

Consider the homogeneous second order constant coefficient ODE

$$ay'' + by' + cy = 0.$$

- If $r$ is a root of $ax^2 + bx + c = 0$, then $e^{rt}$ is a solution of the ODE.
- If $r$ is a double root, variation of parameters shows that $te^{rt}$ is also a solution.
- If $r = a + bi$ is not real, Euler's formula allows us to express the complex exponential part of the solution in terms of $e^{at}$, $\sin(bt)$, and $\cos(bt)$.

**Activity C.8.2** *(∼15 min)*
Consider the following scenario: a mass of $4\ \mathrm{kg}$ suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\ \mathrm{kg/s}$.

The mass is pulled down $0.3\ \mathrm{m}$ and released from rest.

**Activity C.8.2** *(∼15 min)*
Consider the following scenario: a mass of 4 $\mathrm{kg}$ suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\ \mathrm{kg/s}$.

The mass is pulled down 0.3 $\mathrm{m}$ and released from rest.
*Part 1:* Write down an ODE modelling this scenario, and find the general solution.

**Activity C.8.2** *(∼15 min)*
Consider the following scenario: a mass of 4 $\mathrm{kg}$ suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\ \mathrm{kg/s}$.

The mass is pulled down 0.3 $\mathrm{m}$ and released from rest.
*Part 1:* Write down an ODE modelling this scenario, and find the general solution.
*Part 2:* Use the initial conditions $y(0) = -0.3$ and $y'(0) = 0$ to find particular values of the constants.

## Definition C.8.3

In the previous problem, we needed to solve

$$4y'' + 6y' + 2y = 0, \qquad y(0) = -0.3, \qquad y'(0) = 0.$$

This is called an **Initial Value Problem (IVP)** since we are provided with initial values of $y$ and $y'$.

To solve an IVP, find a general solution of the ODE, and use the initial conditions to find the values of the constants.

**Activity C.8.4** *(∼15 min)*

Consider a mass of 5 $\mathrm{kg}$ suspended from a damped spring with spring constant $k = 2\ \mathrm{kg/s^2}$ and damping constant $b = 6\mathrm{kg/s}$.

The mass is pulled down $0.3\mathrm{m}$ and released from rest. How many times does it pass back through its equillibrium state?

(a) 0

(b) 1

(c) 2

(d) Infinitely many

**Observation C.8.5**

It can be shown that in the **overdamped** situation, the spring might pass through the equillibrium position once (e.g. if given an initial push), but never more than once.

# Module C Section 9

**Activity C.9.1** *(∼10 min)*
A 1 $\mathrm{kg}$ mass is suspended from a spring with spring constant $k = 9\ \mathrm{kg/s^2}$. An external force is applied by an electromagnet and is modeled by the function $F(t) = \sin(t)$. Write an ODE modeling the displacement of the spring.

## Observation C.9.2

In the previous activity, we encountered a **nonhomogeneous** second order constant coefficient ODE, i.e. of the form

$$ay'' + by' + cy = f(x)$$

where $a, b, c$ are constants, and $f(x)$ is a function.

We will again use **variation of parameters** to find a particular solution.

**Activity C.9.3** *(~15 min)*
Suppose $y_1$ and $y_2$ are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions $v_1, v_2$.

**Activity C.9.3** *(∼15 min)*

Suppose $y_1$ and $y_2$ are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions $v_1, v_2$.

*Part 1:* Use the product rule (twice) to compute $y_p'$.

**Activity C.9.3** *(∼15 min)*
Suppose $y_1$ and $y_2$ are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions $v_1, v_2$.

*Part 1:* Use the product rule (twice) to compute $y_p'$.
*Part 2:* To simplify calculations, we will **assume** $v_1' y_1 + v_2' y_2 = 0$. Assuming this, compute $y_p''$.

**Activity C.9.3** *(∼15 min)*

Suppose $y_1$ and $y_2$ are two independent particular solutions of $\mathcal{L}(y) = 0$, where $\mathcal{L} = aD^2 + bD + cI$.

Our goal is to find a particular solution of $\mathcal{L}(y) = f(x)$ of the form $y_p = v_1 y_1 + v_2 y_2$ for some TBD functions $v_1, v_2$.

*Part 1:* Use the product rule (twice) to compute $y_p'$.
*Part 2:* To simplify calculations, we will **assume** $v_1' y_1 + v_2' y_2 = 0$. Assuming this, compute $y_p''$.
*Part 3:* Compute $\mathcal{L}(y_p)$; simplify the ODE $\mathcal{L}(y_p) = f(x)$.

**Observation C.9.4**

If we can find $v_1$ and $v_2$ that satisfy

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= \frac{f}{a} \end{aligned}$$

then we have a solution. So we just need to solve this system of equations for $v_1'$ and $v_2'$.

**Activity C.9.5** *(∼15 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

**Activity C.9.5** *(~15 min)*
Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

*Part 1:* Find $y_1$ and $y_2$, two independent solutions of $y'' + 9y = 0$.

**Activity C.9.5** *(∼15 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

*Part 1:* Find $y_1$ and $y_2$, two independent solutions of $y'' + 9y = 0$.

*Part 2:* Find $v_1$ and $v_2$ by solving

$$\begin{aligned} \cos(3t)v_1' + \sin(3t)v_2' &= 0 \\ -3\sin(3t)v_1' + 3\cos(3t)v_2' &= \sin(t) \end{aligned}$$

**Activity C.9.5** *(∼15 min)*

Consider the nonhomogeneous ODE $y'' + 9y = \sin(t)$.

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*Part 3:* Write the general solution of the original nonhomogeneous ODE.

**Activity C.9.6** *(∼10 min)*

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*Part 2:* Write the general solution of the original nonhomogeneous ODE.