

Uniform Turán densities of 3-uniform hypergraphs

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Turán Problems

Question

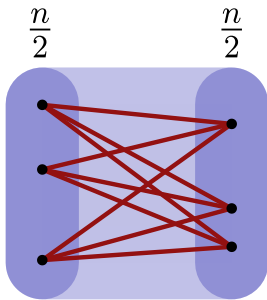
How many edges can a graph have if we forbid a fixed subgraph?

Question

How many edges can a graph have if it is triangle-free?

Theorem (Mantel 1907)

Suppose that for given $n \in \mathbb{N}$, graph G of order n is triangle-free. Then, the number of edges in G is at most $\lfloor n^2/4 \rfloor$.



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Theorem (Turán 1941)

Suppose that for given $n, k \in \mathbb{N}$, graph G of order n is K_k -free. Then, the number of edges in G is at most

$$\left(\frac{k-2}{k-1} + o(1) \right) \binom{n}{2}.$$

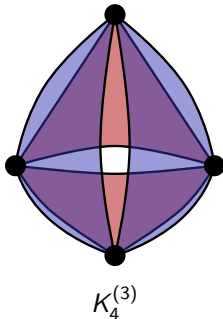
Theorem (Erdős-Stone-Simonovits 1946)

Suppose that graph H has chromatic number $\chi > 2$. Then, the number of edges in any graph G which is H -free is at most

$$\left(\frac{\chi-2}{\chi-1} + o(1) \right) \binom{n}{2}.$$

Question

How dense can a 3-uniform hypergraph be if it does not contain a tetrahedron ($K_4^{(3)}$)?

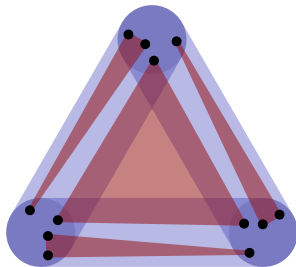


Question (Turán)

How dense can a 3-uniform hypergraph be if it does not contain a tetrahedron ($K_4^{(3)}$)?

Known

$\frac{5}{9} \leq \pi(K_4^{(3)}) \leq 0.5615$ (Razborov; Baber, Talbot)



Turán's construction with density $\frac{5}{9}$

Question (Turán)

How dense can a 3-uniform hypergraph be if it does not contain a tetrahedron ($K_4^{(3)}$)?

Question (Erdős and Sós)

How dense can a 3-uniform hypergraph be if it does not contain a tetrahedron ($K_4^{(3)}$) and we require the edges to be uniformly distributed?

Definition

For $d \in [0, 1]$ and $\eta > 0$ we say that hypergraph F is (d, η) -dense if for all $U \subseteq V$ the following inequality holds:

$$\left| \binom{U}{3} \cap E \right| \geq d \binom{|U|}{3} - \eta |V|^3.$$

Definition

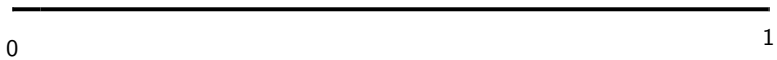
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Definition

Let F be a hypergraph; its uniform Turán density is the following:

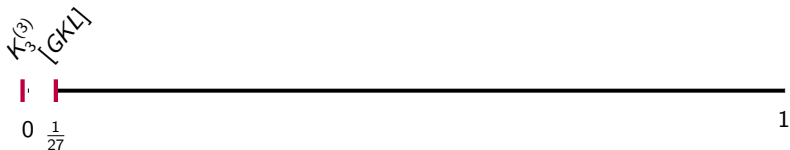
$$\pi_u(F) = \sup \left\{ d \in [0, 1] : \text{for every } \eta > 0 \text{ and } n \in \mathbb{N}, \text{ there exists} \right. \\ \left. \text{an } F\text{-free } (d, \eta)\text{-dense hypergraph } H \right. \\ \left. \text{of order at least } n \right\}.$$



$\tau_3^{(3)}$  $0 \quad \frac{1}{27}$ 1

Proposition (Reiher, Rödl, Schacht 2018)

The uniform Turán density of any graph F is zero or at least $1/27$.



Theorem (Garbe, Král', Lamaison 2022)

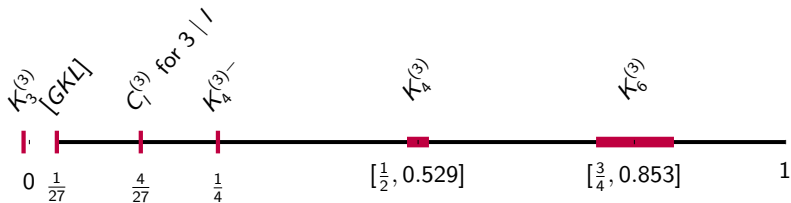
There exists a hypergraph F whose uniform Turán density is $1/27$.



Theorem (Bucić, Cooper, Král', Mohr, Munhá-Correia 2023)

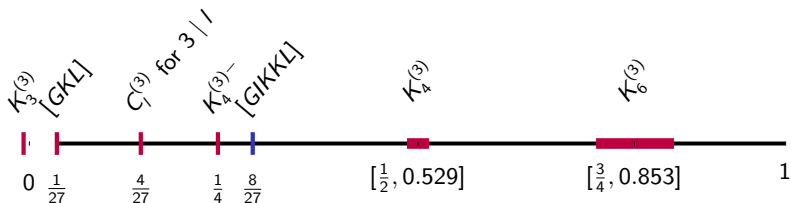
For integer $l \geq 5$, the uniform Turán density of a tight cycle C_l is

1. zero if l is divisible by three and
2. $4/27$ otherwise.



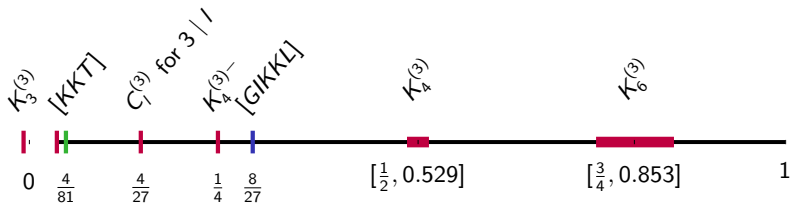
Theorem (Glebov, Král', Volec 2016; Reiher, Rödl, Schacht 2017)

The uniform Turán density of $K_4^{(3)-}$ (tetrahedron without one edge) is $1/4$.



Theorem (Garbe, Il'kovič, Král', K., Lamaison 2023+)

There exists a hypergraph F whose uniform Turán density is $8/27$.



Theorem (Garbe, Il'kovič, Král', K., Lamaison 2023+)

There exists a hypergraph F whose uniform Turán density is $8/27$.

Theorem (Král', K., Tardos 2024++)

There exists a hypergraph F whose uniform Turán density is $4/81$.

Definition

Let Φ be a finite set of colors. We call a subset $\mathcal{P} \subseteq \Phi^3$ a coloring palette.

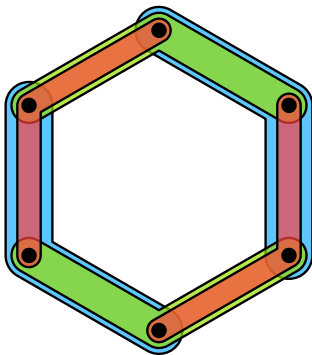
Example

Let $\Phi = \{ \text{red}, \text{green}, \text{blue} \}$ and $\mathcal{P} = \{ (\text{red}, \text{green}, \text{blue}) \}$

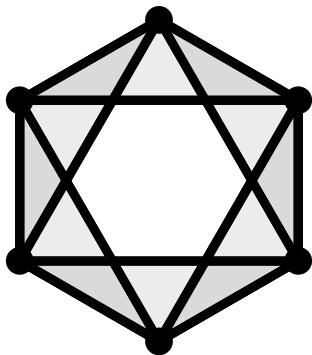
Definition

We say that hypergraph F is \mathcal{P} -colorable if there exists an ordering \prec of the vertex set of F and assignment $\varphi : \partial F \rightarrow \Phi$ with the property that for all $uvw \in E(F)$ where $u \prec v \prec w$ we have

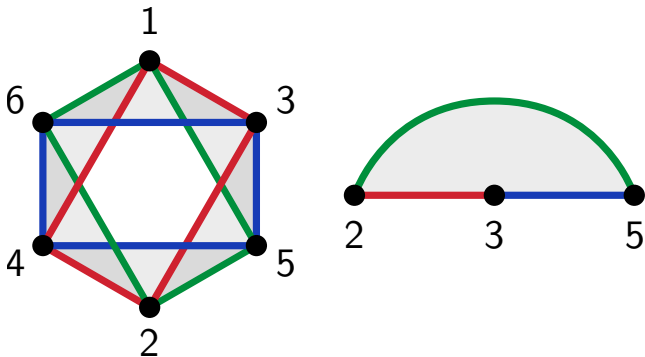
$$(\varphi(uv), \varphi(uw), \varphi(vw)) \in \mathcal{P}.$$



$$C_6^{(3)}$$



$$\Phi = \{ \text{red}, \text{green}, \text{blue} \} \text{ and } \mathcal{P} = \{ (\text{red}, \text{green}, \text{blue}) \}$$



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Theorem (Reiher, Rödl, Schacht 2018)

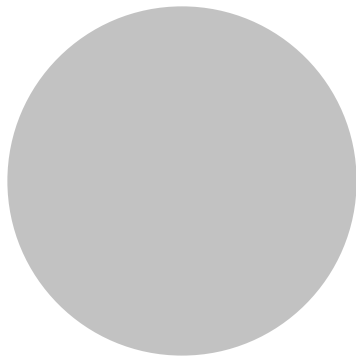
Let the set of colors be $\Phi = \{ \text{red}, \text{green}, \text{blue} \}$ and let the Φ -palette \mathcal{P} contain only the pattern $(\text{red}, \text{green}, \text{blue})$. Let F be a 3-uniform hypergraph, then $\pi_u(F) = 0$ if and only if it is \mathcal{P} -colorable.

Corollary

Let F be a hypergraph, then either $\pi_u(F) = 0$ or $\pi_u(F) \geq 1/27$.

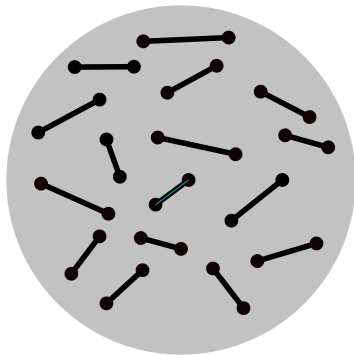
Proposition

Let the set of colors be $\Phi = \{ \text{red}, \text{green}, \text{blue} \}$ and let the Φ -palette \mathcal{P} contain only the pattern $(\text{red}, \text{green}, \text{blue})$. Let F be a graph such that it is not colorable by \mathcal{P} , then its density is at least $1/27$.



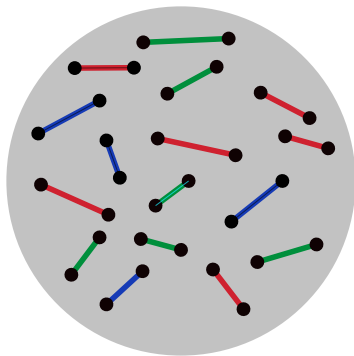
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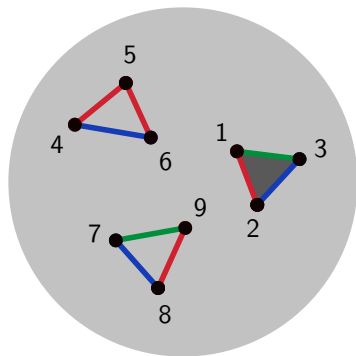
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Theorem (Garbe, Il'kovič, Král', K., Lamaison 2023+)

There exists a hypergraph F with uniform Turán density $\pi_u(F) = 8/27$.

Definition

1. Let $\Psi = \{ \zeta, \alpha_1, \alpha_2, \beta_1, \beta_3, \gamma_2, \gamma_3 \}$, and let \mathcal{R} be a Ψ -palette containing patterns $\{ (\alpha_1, \beta_1, \zeta), (\alpha_2, \zeta, \gamma_2), (\zeta, \beta_3, \gamma_3) \}$. We will call this palette the *required palette*.
2. Let $\Phi = \{ \alpha, \beta, \gamma \}$ and let \mathcal{F} be a Φ -palette which we define as $\mathcal{F} = \{ \alpha, \beta \} \times \{ \beta, \gamma \} \times \{ \alpha, \gamma \}$. We will call palette \mathcal{F} the *forbidden palette*.

Proposition

Any hypergraph F which is \mathcal{R} -colorable but is not \mathcal{F} -colorable has uniform Turán density $8/27$.

Proposition

There exists a hypergraph F such that it is \mathcal{R} -colorable but is not \mathcal{F} -colorable.

Proving upper bound

$(\frac{8}{27} + \delta, \eta)$ -dense hypergraph of order N

↓ hypergraph regularity lemma

hypergraph regularity partition

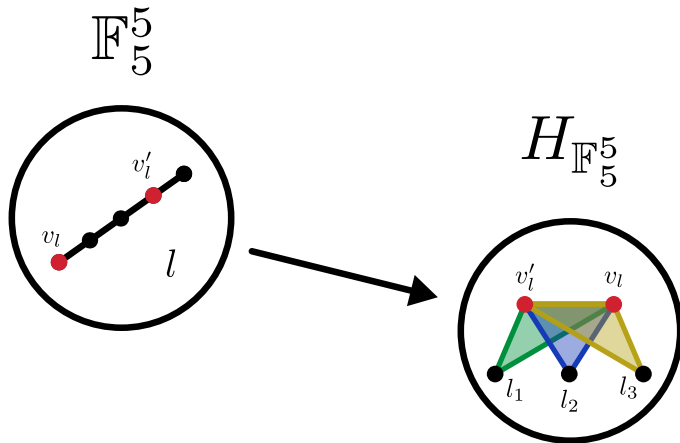
↓ cleaning

$(\frac{8}{27} + \delta')$ -dense *reduced hypergraph*

↓ applications of Ramsey theorems

reduced subhypergraph which “essentially behaves like \mathcal{R} ”

Construction



Open Problems

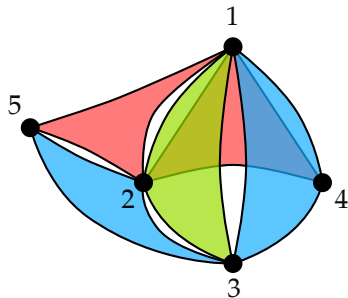
Question

Determine for which $d \in [0, 1]$ there exists an F such that $\pi_u(F) = d$.

Question

Determine uniform Turán density of graph

$$F = ([5], \{123, 124, 125, 145, 234\}).$$



Proposition

$$\pi_u(F) \geq \sup_{\mathcal{P}: F \text{ not } \mathcal{P}\text{-colorable}} d(\mathcal{P})$$