Note on Generating Time-optimal Convexified Reeds–Shepp Paths on a Sphere

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This document serves as a supplementary note detailing the path generation of time-optimal convexified Reeds-Shepp paths on a sphere. In [1], a sufficient list of optimal path types was derived for the problem. Given this list, candidate solutions for each path need to be generated through inverse kinematics, given an initial configuration, a desired terminal configuration, and a maximum absolute turning rate U_{max}. In this note, we present the closed-form expressions for the segment angles associated with each path type in the sufficient list to facilitate path generation.

1 Derivation

The spherical convexified Reeds-Shepp model [1]:

$$\frac{d\mathbf{X_v}}{dt} = v(t)\mathbf{T_v}(t), \tag{1}$$

$$\frac{d\mathbf{T_v}}{dt} = -v(t)\mathbf{X_v}(t) + u_g(t)\mathbf{N_v}(t), \tag{2}$$

$$\frac{d\mathbf{N_v}}{dt} = -u_g(t)\mathbf{T_v}(t), \tag{3}$$

$$\mathbf{R}(0) = \mathbf{I_3},\ \mathbf{R}(T) = R_f, \tag{4}$$

where $v \in [-1,1]$ and $u_g \in [-U_{max}, U_{max}]$, $\mathbf{R}(t) = [\mathbf{X_v}(t), \mathbf{T_v}(t), \mathbf{N_v}(t)] \in SO(3)$ and R_f is the desired terminal configuration.

Note that the model is equivalent to:

$$\frac{d\mathbf{R}(t)}{dt} = \mathbf{R}(t) \underbrace{\begin{pmatrix} 0 & -v & 0 \\ v & 0 & -u_g \\ 0 & u_g & 0 \end{pmatrix}}_{\Omega}. \tag{5}$$

Since v and u_g remain constant on each segment, the solution of (5) on each segment is

$$\mathbf{R}(t) = \mathbf{R}(t_i)e^{(t-t_i)\Omega}, \tag{6}$$

where t_i denotes the initial time of the i^{th} segment. It is simpler to deal with arc angles instead of time; hence, we define $\phi = \omega(t-t_i) = \sqrt{v^2+u_g^2}(t-t_i)$, where ϕ represents the arc angle, and ω denotes the angular frequency. Let $\hat{\Omega} = \frac{1}{\sqrt{v^2+u_g^2}}\Omega$. We define $\mathbf{M}(\phi) := e^{\phi\hat{\Omega}} = e^{(t-t_i)\Omega}$. Substituting specific values of v and u_g, $\mathbf{M}(\phi)$ for each type of segment can be calculated using the Euler-Rodriguez formula. Hence, we obtain

$$\mathbf{M}_{G^+}(\phi) = \begin{pmatrix} c(\phi) & -s(\phi) & 0 \\ s(\phi) & c(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

$$\mathbf{M}_{L^+}(r,\phi) = \begin{pmatrix} \eta_{11} & -rs(\phi) & \eta_{13} \\ rs(\phi) & c(\phi) & -\eta_{23} \\ \eta_{13} & \eta_{23} & \eta_{33} \end{pmatrix}, \tag{8}$$

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$$\mathbf{M}_{R^+}(r,\phi) = \begin{pmatrix} \eta_{11} & -rs(\phi) & -\eta_{13} \\ rs(\phi) & c(\phi) & \eta_{23} \\ -\eta_{13} & -\eta_{23} & \eta_{33} \end{pmatrix}, \tag{9}$$

$$\mathbf{M}_{L^0}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{pmatrix}, \tag{10}$$

$$\mathbf{M}_{G^-}(\phi) = \mathbf{M}_{G^+}^T(\phi), \tag{11}$$

$$\mathbf{M}_{L^-}(r,\phi) = \mathbf{M}_{R^+}^T(r,\phi), \tag{12}$$

$$\mathbf{M}_{R^-}(r,\phi) = \mathbf{M}_{L^+}^T(r,\phi), \tag{13}$$

$$\mathbf{M}_{R^0}(\phi) = \mathbf{M}_{L^0}^T(\phi), \tag{14}$$

where $\eta_{11} = 1-(1-c(\phi))r^2, \quad \eta_{13} = (1-c(\phi))r\sqrt{1-r^2}, \eta_{23} = s(\phi)\sqrt{1-r^2}, \quad \eta_{33} = c(\phi)+(1-c(\phi))r^2, c(\phi) = \cos(\phi)$, and $s(\phi) = \sin(\phi)$.

The corresponding axial vectors are $\mathbf{u}_{G^+} := [0,0,1]^T$, $\mathbf{u}_{L^+} := [\sqrt{1-r^2},0,r]^T$, $\mathbf{u}_{R^+} := [-\sqrt{1-r^2},0,r]^T$, $\mathbf{u}_{L^0} := [1,0,0]^T$, $\mathbf{u}_{G^-} := [0,0,-1]^T$, $\mathbf{u}_{L^-} := [-\sqrt{1-r^2},0,-r]^T$, $\mathbf{u}_{R^-} := [\sqrt{1-r^2},0,-r]^T$, $\mathbf{u}_{R^0} := [-1,0,0]^T$.

The sufficient list of optimal paths is characterized as follows [1]:

Theorem 1. *For $U_{max} \geq 1$ (or $r \leq \frac{1}{\sqrt{2}}$), the optimal path may be restricted to the following types, together with their symmetric forms:*
C, G, T, CC, GC, $C|C$, TC,
$CC_\psi|C$, CGC, $C|C_\beta G$, CTC,
$C|C_\psi C_\psi|C$, $CGC_\beta|C$, $CC_\mu|C_\mu C$,
$C|C_\beta GC_\beta|C$, $C|C_\mu C_\mu|C_\mu C$, $CC_\mu|C_\mu C_\mu|C_\mu C$,
where $0 < \psi \leq \arctan(\frac{1}{\sqrt{U_{max}^4-1}}) + \frac{\pi}{2}$, $\beta = \arctan(\frac{1}{\sqrt{U_{max}^4-1}}) + \frac{\pi}{2}$, and $0 < \mu < \arctan(\frac{1}{\sqrt{U_{max}^4-1}}) + \frac{\pi}{2}$.

Here, C represents a tight turn with radius $r = \frac{1}{\sqrt{1+U_{max}^2}}$, G represents a great circular arc, and T represents a turn-in-place motion.

Given the sufficient list above, for each path, candidate solutions must be generated using inverse kinematics, based on an initial configuration, a desired terminal configuration, and a U_{max} (or r). In this note, we employ rotation matrices and their associated axial vectors to derive closed-form expressions for the angles of each path in the sufficient list.

1.1 C Paths

1.1.1 L^+ Paths

For a $L^+_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{15}$$

Pre-multiplying (15) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2-1\right)\cos(\phi_1) - r^2 = -\alpha_{33}, \tag{16}$$

which gives

$$\cos(\phi_1) = \frac{r^2-\alpha_{33}}{r^2-1}, \tag{17}$$

and yields two solutions of ϕ_1.

1.1.2 R^+ Paths

For a $R^+_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{18}$$

Pre-multiplying (18) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2-1\right)\cos(\phi_1) - r^2 = -\alpha_{33}, \tag{19}$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \tag{20}$$

and yields two solutions of ϕ_1.

1.1.3 R^- Paths

For a $R^-_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{21}$$

Pre-multiplying (21) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2 - 1\right)\cos(\phi_1) - r^2 = -\alpha_{33}, \tag{22}$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \tag{23}$$

and yields two solutions of ϕ_1.

1.1.4 L^- Paths

For a $L^-_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{24}$$

Pre-multiplying (24) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2 - 1\right)\cos(\phi_1) - r^2 = -\alpha_{33}, \tag{25}$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \tag{26}$$

and yields two solutions of ϕ_1.

1.2 G Paths

1.2.1 G^+ Paths

For a $G^+_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{27}$$

Pre-multiplying (27) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\left(1 - r^2\right)\cos(\phi_1) - r^2 = -\left(\alpha_{11}\left(r^2 - 1\right)\right) - r\left(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33} r\right), \tag{28}$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r\left(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + (\alpha_{33} - 1)\, r\right)}{r^2 - 1}, \tag{29}$$

and yields two solutions of ϕ_1.

1.2.2 G^- Paths

For a $G^-_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{30}$$

Pre-multiplying (30) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\left(1 - r^2\right) \cos(\phi_1) - r^2 = -\left(\alpha_{11}\left(r^2 - 1\right)\right) - r\left(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33} r\right), \tag{31}$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r\left(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + (\alpha_{33} - 1)\, r\right)}{r^2 - 1}, \tag{32}$$

and yields two solutions of ϕ_1.

1.3 T Paths

1.3.1 L^0 Paths

For a $L^0_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{33}$$

Pre-multiplying (33) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\cos(\phi_1) = \alpha_{33}, \tag{34}$$

and yields two solutions of ϕ_1.

1.3.2 R^0 Paths

For a $R^0_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{35}$$

Pre-multiplying (35) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\cos(\phi_1) = \alpha_{33}, \tag{36}$$

and yields two solutions of ϕ_1.

1.4 CC Paths

1.4.1 L^+R^+ Paths

For a $L^+_{\phi_1} R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1)\mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{37}$$

Pre-multiplying (37) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$-2r^3 + 2\left(r^2 - 1\right) r \cos(\phi_1) + r = \alpha_{31}\sqrt{1 - r^2} - \alpha_{33} r, \tag{38}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33} r + r}{2r - 2r^3}, \tag{39}$$

and yields two solutions of ϕ_1.

Pre-multiplying (37) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$-2r^3 + 2\left(r^2 - 1\right) r\cos(\phi_2) + r = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{40}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{41}$$

and yields two solutions of ϕ_2.

1.4.2 R^+L^+ Paths

For a $R^+_{\phi_1}L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{42}$$

Pre-multiplying (42) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$-2r^3 + 2\left(r^2 - 1\right) r\cos(\phi_1) + r = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{43}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{44}$$

and yields two solutions of ϕ_1.

Pre-multiplying (42) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$-2r^3 + 2\left(r^2 - 1\right) r\cos(\phi_2) + r = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r, \tag{45}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{46}$$

and yields two solutions of ϕ_2.

1.4.3 R^-L^- Paths

For a $R^-_{\phi_1}L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{47}$$

Pre-multiplying (47) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$-2r^3 + 2\left(r^2 - 1\right) r\cos(\phi_1) + r = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r, \tag{48}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{49}$$

and yields two solutions of ϕ_1.

Pre-multiplying (47) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{G^+}$:

$$-2r^3 + 2\left(r^2 - 1\right) r\cos(\phi_2) + r = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{50}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{51}$$

and yields two solutions of ϕ_2.

1.4.4 L^-R^- Paths

For a $L^-_{\phi_1}R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{52}$$

Pre-multiplying (52) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$-2r^3 + 2\left(r^2-1\right)r\cos(\phi_1) + r = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{53}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{54}$$

and yields two solutions of ϕ_1.

Pre-multiplying (52) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{G^+}$:

$$-2r^3 + 2\left(r^2-1\right)r\cos(\phi_2) + r = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r, \tag{55}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{56}$$

and yields two solutions of ϕ_2.

1.5 GC Paths

1.5.1 G^+L^+ Paths

For a $G^+_{\phi_1}L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{57}$$

Pre-multiplying (57) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\left(1-r^2\right)\cos(\phi_1) - r^2 = -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{58}$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \left(\alpha_{33}-1\right)r\right)}{r^2-1}, \tag{59}$$

and yields two solutions of ϕ_1.

Pre-multiplying (57) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2-1\right)\cos(\phi_2) - r^2 = -\alpha_{33}, \tag{60}$$

which gives

$$\cos(\phi_2) = \frac{r^2 - \alpha_{33}}{r^2-1}, \tag{61}$$

and yields two solutions of ϕ_2.

1.5.2 G^+R^+ Paths

For a $G^+_{\phi_1}R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{62}$$

Pre-multiplying (62) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\left(r^2-1\right)\cos(\phi_1)-r^2=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \tag{63}$$

which gives

$$\cos(\phi_1)=\alpha_{11}+\frac{r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r\right)}{r^2-1}, \tag{64}$$

and yields two solutions of ϕ_1.

Pre-multiplying (62) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2-1\right)\cos(\phi_2)-r^2=-\alpha_{33}, \tag{65}$$

which gives

$$\cos(\phi_2)=\frac{r^2-\alpha_{33}}{r^2-1}, \tag{66}$$

and yields two solutions of ϕ_2.

1.5.3 G^-R^- Paths

For a $G^-_{\phi_1}R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1)\mathbf{M}_{R^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{67}$$

Pre-multiplying (67) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\left(r^2-1\right)\cos(\phi_1)-r^2=\alpha_{11}\left(r^2-1\right)-r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{68}$$

which gives

$$\cos(\phi_1)=\alpha_{11}-\frac{r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\left(\alpha_{33}-1\right)r\right)}{r^2-1}, \tag{69}$$

and yields two solutions of ϕ_1.

Pre-multiplying (67) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{G^+}$:

$$\left(r^2-1\right)\cos(\phi_2)-r^2=-\alpha_{33}, \tag{70}$$

which gives

$$\cos(\phi_2)=\frac{r^2-\alpha_{33}}{r^2-1}, \tag{71}$$

and yields two solutions of ϕ_2.

1.5.4 G^-L^- Paths

For a $G^-_{\phi_1}L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1)\mathbf{M}_{L^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{72}$$

Pre-multiplying (72) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\left(1-r^2\right)\cos(\phi_1)-r^2=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{73}$$

which gives

$$\cos(\phi_1)=\alpha_{11}+\frac{r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\left(\alpha_{33}-1\right)r\right)}{r^2-1}, \tag{74}$$

and yields two solutions of ϕ_1.

Pre-multiplying (72) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{G^+}$:

$$\left(r^2-1\right)\cos(\phi_2)-r^2=-\alpha_{33}, \tag{75}$$

which gives

$$\cos(\phi_2)=\frac{r^2-\alpha_{33}}{r^2-1}, \tag{76}$$

and yields two solutions of ϕ_2.

1.6 $C|C$ Paths

1.6.1 $L^+|L^-$ Paths

For a $L^+_{\phi_1}|L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{77}$$

Pre-multiplying (77) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \tag{78}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{79}$$

and yields two solutions of ϕ_1.

Pre-multiplying (77) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$r\left(-2\left(r^2-1\right)\cos(\phi_2) + 2r^2 - 1\right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r, \tag{80}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{81}$$

and yields two solutions of ϕ_2.

1.6.2 $R^+|R^-$ Paths

For a $R^+_{\phi_1}|R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{82}$$

Pre-multiplying (82) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_1) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r, \tag{83}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{84}$$

and yields two solutions of ϕ_1.

Pre-multiplying (82) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_2) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}, \tag{85}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{86}$$

and yields two solutions of ϕ_2.

1.6.3 $R^-|R^+$ Paths

For a $R^-_{\phi_1}|R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{87}$$

Pre-multiplying (87) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \tag{88}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{89}$$

and yields two solutions of ϕ_1.

Pre-multiplying (87) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$r\left(2r^2 - 1\right) - 2r\left(r^2 - 1\right)\cos(\phi_2) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r, \tag{90}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{91}$$

and yields two solutions of ϕ_2.

1.6.4 $L^-|L^+$ Paths

For a $L^-_{\phi_1}|L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{92}$$

Pre-multiplying (92) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$r\left(2r^2 - 1\right) - 2r\left(r^2 - 1\right)\cos(\phi_1) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r, \tag{93}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{94}$$

and yields two solutions of ϕ_1.

Pre-multiplying (92) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$r\left(2r^2 - 1\right) - 2r\left(r^2 - 1\right)\cos(\phi_2) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}, \tag{95}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \tag{96}$$

and yields two solutions of ϕ_2.

1.7 TC Paths

1.7.1 L^0L^+ Paths

For a $L^0_{\phi_1}L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{97}$$

Pre-multiplying (97) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$-r\cos(\phi_1) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{98}$$

which gives

$$\cos(\phi_1) = \alpha_{33} + \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \tag{99}$$

and yields two solutions of ϕ_1.

Pre-multiplying (97) with $\mathbf{u}_{L^0}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$r\sqrt{1-r^2}(\cos(\phi_2) - 1) = -\alpha_{13}, \tag{100}$$

which gives

$$\cos(\phi_2) = 1 - \frac{\alpha_{13}}{r\sqrt{1-r^2}}, \tag{101}$$

and yields two solutions of ϕ_2.

1.7.2 L^0L^- Paths

For a $L^0_{\phi_1}L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{102}$$

Pre-multiplying (102) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \tag{103}$$

which gives

$$\cos(\phi_1) = \alpha_{33} - \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \tag{104}$$

and yields two solutions of ϕ_1.

Pre-multiplying (102) with $\mathbf{u}_{L^0}^T$ and post-multiplying $\mathbf{u}_{G^+}$:

$$r\sqrt{1-r^2}(\cos(\phi_2) - 1) = \alpha_{13}, \tag{105}$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{13}}{r\sqrt{1-r^2}} + 1, \tag{106}$$

and yields two solutions of ϕ_2.

1.7.3 R^0R^- Paths

For a $R^0_{\phi_1}R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{107}$$

Pre-multiplying (107) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$-r\cos(\phi_1) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{108}$$

which gives

$$\cos(\phi_1) = \alpha_{33} + \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \tag{109}$$

and yields two solutions of ϕ_1.

Pre-multiplying (107) with $\mathbf{u}_{R^0}^T$ and post-multiplying $\mathbf{u}_{G^+}$:

$$r\sqrt{1-r^2}(\cos(\phi_2) - 1) = -\alpha_{13}, \tag{110}$$

which gives

$$\cos(\phi_2) = 1 - \frac{\alpha_{13}}{r\sqrt{1-r^2}}, \tag{111}$$

and yields two solutions of ϕ_2.

1.7.4 R^0R^+ Paths

For a $R^0_{\phi_1}R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{112}$$

Pre-multiplying (112) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \tag{113}$$

which gives

$$\cos(\phi_1) = \alpha_{33} - \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \tag{114}$$

and yields two solutions of ϕ_1.

Pre-multiplying (112) with $\mathbf{u}_{R^0}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$r\sqrt{1-r^2}(\cos(\phi_2)-1) = \alpha_{13}, \tag{115}$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{13}}{r\sqrt{1-r^2}} + 1, \tag{116}$$

and yields two solutions of ϕ_2.

1.8 $CC_\psi|C$ Paths

1.8.1 $L^+R^+_\psi|R^-$ Paths

For a $L^+_{\phi_1}R^+_\psi|R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{R^+}(r,\psi)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{117}$$

Pre-multiplying (117) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$4r^2\left(r^2-1\right)\cos(\psi) - \left(1-2r^2\right)^2 = \alpha_{11}\left(r^2-1\right) - r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{118}$$

which gives

$$\cos(\psi) = \frac{-\alpha_{11} + 4r^4 + \alpha_{11}r^2 - \alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r - 4r^2 + 1}{4r^2\left(r^2-1\right)}, \tag{119}$$

and yields two solutions of ψ.

Pre-multiplying (117) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned} & r\sin(\phi_1)\left(2\sin(\psi) - 2r^2\sin(\psi)\right) + r\left(4r^4 - 4\left(r^2-1\right)r^2\cos(\psi) - 4r^2 + 1\right) - 4r\left(2r^4 - 3r^2 + 1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_1) \\ =& \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r. \end{aligned} \tag{120}$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2 + \left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2 + \left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\left(2\sin(\psi)-2r^2\sin(\psi)\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2 + \left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\gamma+\phi_1) = -\frac{r\left(\alpha_{33} + 4\left(r^2-1\right)r^2\cos(\psi) - \left(1-2r^2\right)^2\right) + \alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4 + 2\left(r^2-1\right)r^2\cos(\psi) + 2r^2 - 1\right)}} \tag{121}$$

$$\Longrightarrow \phi_1 = \cos^{-1}\left(-\frac{r\left(\alpha_{33} + 4\left(r^2-1\right)r^2\cos(\psi) - \left(1-2r^2\right)^2\right) + \alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4 + 2\left(r^2-1\right)r^2\cos(\psi) + 2r^2 - 1\right)}}\right) - \tan^{-1}\left(\frac{2r(\sin(\psi) - r^2\sin(\psi))}{4r\left(2r^4 - 3r^2 + 1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{122}$$

which yields two solutions for each value of ψ.

Pre-multiplying (117) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned} & r\sin(\phi_2)\left(2r^2\sin(\psi) - 2\sin(\psi)\right) + r\left(4r^4 - 4\left(r^2-1\right)r^2\cos(\psi) - 4r^2 + 1\right) - 4r\left(2r^4 - 3r^2 + 1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_2) \\ =& \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r. \end{aligned} \tag{123}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right) = -\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{124}$$

$$\Longrightarrow \phi_2 = \cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)+\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{125}$$

which yields two solutions for each value of ψ.

1.8.2 $L^-R^-_\psi|R^+$ Paths

For a $L^-_{\phi_1}R^-_\psi|R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{R^-}(r,\psi)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{126}$$

Pre-multiplying (126) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$4r^2\left(r^2-1\right)\cos(\psi)-\left(1-2r^2\right)^2 = \alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \tag{127}$$

which gives

$$\cos(\psi) = \frac{-\alpha_{11}+4r^4+\alpha_{11}r^2-\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-4r^2+1}{4r^2\left(r^2-1\right)}, \tag{128}$$

and yields two solutions of ψ.

Pre-multiplying (117) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}&r\sin(\phi_1)\left(2\sin(\psi)-2r^2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_1)\\&=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}.\end{aligned} \tag{129}$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\left(2\sin(\psi)-2r^2\sin(\psi)\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\gamma+\phi_1) = -\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{130}$$

$$\Longrightarrow \phi_1 = \cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)-\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{131}$$

which yields two solutions for each value of ψ.

Pre-multiplying (117) with $\mathbf{u}^T_{L^-}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&r\sin(\phi_2)\left(2r^2\sin(\psi)-2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_2)\\&=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}.\end{aligned} \tag{132}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{133}$$

$$\implies \phi_2=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)+\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{134}$$

which yields two solutions for each value of ψ.

1.8.3 $R^-L^-_\psi|L^+$ Paths

For a $R^-_{\phi_1}L^-_\psi|L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{L^-}(r,\psi)\mathbf{M}_{L^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{135}$$

Pre-multiplying (135) with $\mathbf{u}^T_{R^-}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$4r^2\left(r^2-1\right)\cos(\psi)-\left(1-2r^2\right)^2=\alpha_{11}\left(r^2-1\right)-r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{136}$$

which gives

$$\cos(\psi)=\frac{-\alpha_{11}+4r^4+\alpha_{11}r^2-\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r-4r^2+1}{4r^2\left(r^2-1\right)}, \tag{137}$$

and yields two solutions of ψ.

Pre-multiplying (135) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}&r\sin(\phi_1)\left(2\sin(\psi)-2r^2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_1)\\ =&\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{138}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r\left(2\sin(\psi)-2r^2\sin(\psi)\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\gamma+\phi_1)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{139}$$

$$\implies \phi_1=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)-\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{140}$$

which yields two solutions for each value of ψ.

Pre-multiplying (135) with $\mathbf{u}^T_{R^-}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&r\sin(\phi_2)\left(2r^2\sin(\psi)-2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_2)\\ =&\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{141}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right) = -\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{142}$$

$$\implies \phi_2 = \cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)+\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{143}$$

which yields two solutions for each value of ψ.

1.8.4 $R^+L^+_\psi|L^-$ Paths

For a $R^+_{\phi_1}L^+_\psi|L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{L^+}(r,\psi)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{144}$$

Pre-multiplying (144) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$4r^2\left(r^2-1\right)\cos(\psi)-\left(1-2r^2\right)^2 = \alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \tag{145}$$

which gives

$$\cos(\psi) = \frac{-\alpha_{11}+4r^4+\alpha_{11}r^2-\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-4r^2+1}{4r^2\left(r^2-1\right)}, \tag{146}$$

and yields two solutions of ψ.

Pre-multiplying (144) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}&r\sin(\phi_1)\left(2\sin(\psi)-2r^2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_1)\\ =&\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}.\end{aligned} \tag{147}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\left(2\sin(\psi)-2r^2\sin(\psi)\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\gamma+\phi_1) = -\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{148}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)-\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{149}$$

which yields two solutions for each value of ψ.

Pre-multiplying (144) with $\mathbf{u}^T_{R^+}$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&r\sin(\phi_2)\left(2r^2\sin(\psi)-2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_2)\\ =&\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}.\end{aligned} \tag{150}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, it is obtained that

$$\cos(\phi_2-\gamma) = -\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{151}$$

$$\implies \phi_2 = \cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)+\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{152}$$

which yields two solutions for each value of ψ.

1.9 CGC Paths

1.9.1 $L^+G^+L^+$ Paths

For a $L^+_{\phi_1}G^+_{\phi_2}L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{153}$$

Pre-multiplying (153) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$r^2(-\cos(\phi_2))+r^2+\cos(\phi_2) = r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \tag{154}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11}+\alpha_{11}r^2-\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+r^2}{r^2-1}, \tag{155}$$

and yields two solutions of ϕ_2.

Pre-multiplying (153) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$-r^3+\sin(\phi_1)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2) = \alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{156}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r^2\sin(\phi_2)-\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{157}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{158}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (153) with $\mathbf{u}^T_{L^+}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$-r^3+\sin(\phi_3)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_3)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2) = \alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{159}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{160}$$

$$\Longrightarrow\phi_3=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{161}$$

which yields two solutions for each value of ϕ_2.

1.9.2 $R^+G^+R^+$ Paths

For a $R^+_{\phi_1}G^+_{\phi_2}R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{162}$$

Pre-multiplying (162) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)=r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \tag{163}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2-\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+r^2}{r^2-1}, \tag{164}$$

and yields two solutions of ϕ_2.

Pre-multiplying (162) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$-r^3+\sin(\phi_1)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \tag{165}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{166}$$

$$\Longrightarrow\phi_1=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2)))+r^2+\cos(\phi_2)\right)-\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{167}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (162) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$-r^3+\sin(\phi_3)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_3)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \tag{168}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{169}$$

$$\Longrightarrow\phi_3=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{170}$$

which yields two solutions for each value of ϕ_2.

1.9.3 $R^-G^-R^-$ Paths

For a $R^-_{\phi_1}G^-_{\phi_2}R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{171}$$

Pre-multiplying (171) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2) = r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{172}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 - \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2}{r^2-1}, \tag{173}$$

and yields two solutions of ϕ_2.

Pre-multiplying (171) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$-r^3 + \sin(\phi_1)\left(r^2\sin(\phi_2) - \sin(\phi_2)\right) + 2\left(r^2-1\right)r\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + \left(r^2-1\right)r\cos(\phi_2) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \tag{174}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r^2\sin(\phi_2)-\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \tag{175}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{r^2\sin(\phi_2) - \sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{176}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (171) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$-r^3 + \sin(\phi_3)\left(r^2\sin(\phi_2) - \sin(\phi_2)\right) + 2\left(r^2-1\right)r\cos(\phi_3)\sin^2\left(\frac{\phi_2}{2}\right) + \left(r^2-1\right)r\cos(\phi_2) = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \tag{177}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \tag{178}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{r^2\sin(\phi_2) - \sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{179}$$

which yields two solutions for each value of ϕ_2.

1.9.4 $L^-G^-L^-$ Paths

For a $L^-_{\phi_1}G^-_{\phi_2}L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{180}$$

Pre-multiplying (180) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2) = r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{181}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 - \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2}{r^2-1}, \tag{182}$$

and yields two solutions of ϕ_2.

Pre-multiplying (180) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$-r^3 + \sin(\phi_1)\left(r^2\sin(\phi_2) - \sin(\phi_2)\right) + 2\left(r^2-1\right)r\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + \left(r^2-1\right)r\cos(\phi_2) = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \tag{183}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r^2\sin(\phi_2)-\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \tag{184}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{185}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (180) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$-r^3 + \sin(\phi_3)\left(r^2\sin(\phi_2) - \sin(\phi_2)\right) + 2\left(r^2-1\right)r\cos(\phi_3)\sin^2\left(\frac{\phi_2}{2}\right) + \left(r^2-1\right)r\cos(\phi_2) = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r. \tag{186}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \tag{187}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{188}$$

which yields two solutions for each value of ϕ_2.

1.9.5 $L^+G^+R^+$ Paths

For a $L_{\phi_1}^+G_{\phi_2}^+R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{189}$$

Pre-multiplying (189) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\left(r^2-1\right)\cos(\phi_2) + r^2 = \alpha_{11}\left(r^2-1\right) + r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{190}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r - r^2}{r^2-1}, \tag{191}$$

and yields two solutions of ϕ_2.

Pre-multiplying (189) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\left(r-r^3\right)\cos(\phi_2)-r^3+\sin(\phi_1)\left(\sin(\phi_2)-r^2\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \tag{192}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{\sin(\phi_2)-r^2\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r^3\cos(\phi_2)+r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{193}$$

$$\Longrightarrow \phi_1=\cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{194}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (189) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$-r^3-\left(r^2-1\right)\sin(\phi_2)\sin(\phi_3)+2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)-\left(r^2-1\right)r\cos(\phi_2)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{195}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{196}$$

$$\Longrightarrow \phi_3=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{197}$$

which yields two solutions for each value of ϕ_2.

1.9.6 $R^+G^+L^+$ Paths

For a $R^+_{\phi_1}G^+_{\phi_2}L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{198}$$

Pre-multiplying (198) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\left(r^2-1\right)\cos(\phi_2)+r^2=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{199}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-r^2}{r^2-1}, \tag{200}$$

and yields two solutions of ϕ_2.

Pre-multiplying (198) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\left(r-r^3\right)\cos(\phi_2)-r^3+\sin(\phi_1)\left(\sin(\phi_2)-r^2\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{201}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{\sin(\phi_2)-r^2\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3\cos(\phi_2)+r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{202}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{203}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (198) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$-r^3-\left(r^2-1\right)\sin(\phi_2)\sin(\phi_3)+2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)-\left(r^2-1\right)r\cos(\phi_2) = \alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \tag{204}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r^3\cos(\phi_2)+r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{205}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{206}$$

which yields two solutions for each value of ϕ_2.

1.9.7 $R^-G^-L^-$ Paths

For a $R^-_{\phi_1}G^-_{\phi_2}L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{207}$$

Pre-multiplying (207) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\left(r^2-1\right)\cos(\phi_2)+r^2 = \alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{208}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r-r^2}{r^2-1}, \tag{209}$$

and yields two solutions of ϕ_2.

Pre-multiplying (207) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\left(r-r^3\right)\cos(\phi_2)-r^3+\sin(\phi_1)\left(\sin(\phi_2)-r^2\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \tag{210}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{\sin(\phi_2)-r^2\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r^3\cos(\phi_2)+r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{211}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{212}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (207) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$-r^3-\left(r^2-1\right)\sin(\phi_2)\sin(\phi_3)+2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)-\left(r^2-1\right)r\cos(\phi_2)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{213}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r^3\cos(\phi_2)+r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{214}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{215}$$

which yields two solutions for each value of ϕ_2.

1.9.8 $L^-G^-R^-$ Paths

For a $L_{\phi_1}^-G_{\phi_2}^-R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{216}$$

Pre-multiplying (216) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\left(r^2-1\right)\cos(\phi_2)+r^2=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{217}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-r^2}{r^2-1}, \tag{218}$$

and yields two solutions of ϕ_2.

Pre-multiplying (216) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\left(r-r^3\right)\cos(\phi_2)-r^3+\sin(\phi_1)\left(\sin(\phi_2)-r^2\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{219}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r^3\cos(\phi_2)+r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{220}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{221}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (216) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$-r^3 - \left(r^2-1\right)\sin(\phi_2)\sin(\phi_3) + 2\left(r^2-1\right) r\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) - \left(r^2-1\right) r\cos(\phi_2) = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33} r. \tag{222}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r^3\cos(\phi_2)+r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r - r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{223}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r - r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{224}$$

which yields two solutions for each value of ϕ_2.

1.10 $C|C_\beta G$ Paths

1.10.1 $L^+|L_\beta^- G^-$ Paths

For a $L_{\phi_1}^+|L_\beta^- G_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{225}$$

Pre-multiplying (225) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$2r^4 + \sin(\phi_1)\left(r^2\sin(\beta)-\sin(\beta)\right) - r^2 + \cos(\phi_1)\left(-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)\right) + \left(2r^2-2r^4\right)\cos(\beta) = \alpha_{33}, \tag{226}$$

This equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$ and defining $\cos\gamma := \frac{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$, $\sin\gamma := \frac{r^2\sin(\beta)-\sin(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}} \tag{227}$$

$$\begin{aligned}\implies \phi_1 =& \cos^{-1}\left(\frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}}\right)\\ &+\tan^{-1}\left(\frac{r^2\sin(\beta)-\sin(\beta)}{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}\right),\end{aligned} \tag{228}$$

which yields two solutions.

Pre-multiplying (225) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}&2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2) - 2\left(r^2-1\right)r^2\cos(\beta) + \left(2r^2-1\right)r^2 + \cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\ =&r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right).\end{aligned} \tag{229}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, and defining $\cos\theta := \frac{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, $\sin\theta := \frac{2\left(r^2-1\right)r\sin(\beta)}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$,

it is obtained that

$$\cos\left(\phi_2-\theta\right)=\frac{r\left(2r^3\cos(\beta)-2r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}} \tag{230}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{r\left(2r^3\cos(\beta)-2r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}}\right) \tag{231}$$

$$+\tan^{-1}\left(\frac{2\left(r^2-1\right)r\sin(\beta)}{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}\right), \tag{232}$$

which yields two solutions.

1.10.2 $R^+|R_\beta^-G^-$ Paths

For a $R^+_{\phi_1}|R^-_\beta G^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{233}$$

Pre-multiplying (233) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$2r^4+\sin(\phi_1)\left(r^2\sin(\beta)-\sin(\beta)\right)-r^2+\cos(\phi_1)\left(-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)\right)+\left(2r^2-2r^4\right)\cos(\beta)=\alpha_{33}, \tag{234}$$

This equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$ and defining $\cos\gamma := \frac{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$, $\sin\gamma := \frac{r^2\sin(\beta)-\sin(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}} \tag{235}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}}\right)$$
$$+\tan^{-1}\left(\frac{r^2\sin(\beta)-\sin(\beta)}{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}\right), \tag{236}$$

which yields two solutions.

Pre-multiplying (233) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)-2\left(r^2-1\right)r^2\cos(\beta)+\left(2r^2-1\right)r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)$$
$$=r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right). \tag{237}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, and defining $\cos\theta := \frac{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, $\sin\theta := \frac{2\left(r^2-1\right)r\sin(\beta)}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, it is obtained that

$$\cos\left(\phi_2-\theta\right)=\frac{r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}} \tag{238}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}}\right) \tag{239}$$

$$+\tan^{-1}\left(\frac{2\left(r^2-1\right)r\sin(\beta)}{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}\right), \tag{240}$$

which yields two solutions.

1.10.3 $R^-|R^+_\beta G^+$ Paths

For a $R^-_{\phi_1}|R^+_\beta G^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{241}$$

Pre-multiplying (241) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$2r^4 + \sin(\phi_1)\left(r^2\sin(\beta) - \sin(\beta)\right) - r^2 + \cos(\phi_1)\left(-2r^4 + 2r^2 + \left(2r^4 - 3r^2 + 1\right)\cos(\beta)\right) + \left(2r^2 - 2r^4\right)\cos(\beta) = \alpha_{33}, \tag{242}$$

This equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$ and defining $\cos\gamma := \frac{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$, $\sin\gamma := \frac{r^2\sin(\beta)-\sin(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{33} + r^2\left(2\left(r^2-1\right)\cos(\beta) - 2r^2 + 1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4 + 2\left(r^2-1\right)r^2\cos(2\beta) - 2r^2 + \left(4r^2 - 8r^4\right)\cos(\beta) + 1\right)}} \tag{243}$$

$$\begin{aligned}\implies \phi_1 =& \cos^{-1}\left(\frac{\alpha_{33} + r^2\left(2\left(r^2-1\right)\cos(\beta) - 2r^2 + 1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4 + 2\left(r^2-1\right)r^2\cos(2\beta) - 2r^2 + \left(4r^2 - 8r^4\right)\cos(\beta) + 1\right)}}\right) \\ &+ \tan^{-1}\left(\frac{r^2\sin(\beta) - \sin(\beta)}{-2r^4 + 2r^2 + \left(2r^4 - 3r^2 + 1\right)\cos(\beta)}\right),\end{aligned} \tag{244}$$

which yields two solutions.

Pre-multiplying (241) with $\mathbf{u}^T_{R^-}$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2) - 2\left(r^2-1\right)r^2\cos(\beta) + \left(2r^2-1\right)r^2 + \cos(\phi_2)\left(2r^4 - 2\left(r^2-1\right)r^2\cos(\beta) - 3r^2 + 1\right) \\ =& r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right).\end{aligned} \tag{245}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, and defining $\cos\theta := \frac{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, $\sin\theta := \frac{2\left(r^2-1\right)r\sin(\beta)}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, it is obtained that

$$\cos\left(\phi_2 - \theta\right) = \frac{r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right) - \alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta) + 4r^2\sin^2(\beta) + \left(1-2r^2\right)^2 + \left(4r^2-8r^4\right)\cos(\beta)\right)}} \tag{246}$$

$$\implies \phi_2 = \cos^{-1}\left(\frac{r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right) - \alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta) + 4r^2\sin^2(\beta) + \left(1-2r^2\right)^2 + \left(4r^2-8r^4\right)\cos(\beta)\right)}}\right) \tag{247}$$

$$+ \tan^{-1}\left(\frac{2\left(r^2-1\right)r\sin(\beta)}{2r^4 - 2\left(r^2-1\right)r^2\cos(\beta) - 3r^2 + 1}\right), \tag{248}$$

which yields two solutions.

1.10.4 $L^-|L^+_\beta G^+$ Paths

For a $L^-_{\phi_1}|L^+_\beta G^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{249}$$

Pre-multiplying (249) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$2r^4 + \sin(\phi_1)\left(r^2\sin(\beta) - \sin(\beta)\right) - r^2 + \cos(\phi_1)\left(-2r^4 + 2r^2 + \left(2r^4 - 3r^2 + 1\right)\cos(\beta)\right) + \left(2r^2 - 2r^4\right)\cos(\beta) = \alpha_{33}, \tag{250}$$

This equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$ and defining $\cos\gamma := \frac{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$, $\sin\gamma := \frac{r^2\sin(\beta)-\sin(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}} \tag{251}$$

$$\begin{aligned}\implies \phi_1 =& \cos^{-1}\left(\frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}}\right)\\ &+\tan^{-1}\left(\frac{r^2\sin(\beta)-\sin(\beta)}{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}\right),\end{aligned} \tag{252}$$

which yields two solutions.

Pre-multiplying (249) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}&2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)-2\left(r^2-1\right)r^2\cos(\beta)+\left(2r^2-1\right)r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\ =&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right).\end{aligned} \tag{253}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, and defining $\cos\theta := \frac{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, $\sin\theta := \frac{2\left(r^2-1\right)r\sin(\beta)}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, it is obtained that

$$\cos\left(\phi_2-\theta\right) = \frac{r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}} \tag{254}$$

$$\implies \phi_2 = \cos^{-1}\left(\frac{r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}}\right) \tag{255}$$

$$+\tan^{-1}\left(\frac{2\left(r^2-1\right)r\sin(\beta)}{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}\right), \tag{256}$$

which yields two solutions.

1.11 *CTC* Paths

1.11.1 $L^+L^0L^-$ Paths

For a $L^+_{\phi_1}L^0_{\phi_2}L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^-}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{257}$$

Pre-multiplying (257) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r^2(-\cos(\phi_2))-r^2+1 = -\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{258}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r-r^2+1}{r^2}, \tag{259}$$

and yields two solutions of ϕ_2.

Pre-multiplying (257) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$-r\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)+r\cos(\phi_3)\left(r^2(-\cos(\phi_2))-r^2+\cos(\phi_2)+1\right)+r\left(r^2\cos(\phi_2)+r^2-1\right) = \alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \tag{260}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = -\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{261}$$

$$\implies \phi_1 = \cos^{-1}\left(-\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{262}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (257) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2\cos(\phi_2)+r^2-1\right) = \alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \tag{263}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r^3(-\cos(\phi_2))-r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{264}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3(-\cos(\phi_2))-r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{265}$$

which yields two solutions for each value of ϕ_2.

1.11.2 $R^+R^0R^-$ Paths

For a $R^+_{\phi_1}R^0_{\phi_2}R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{266}$$

Pre-multiplying (266) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$r^2(-\cos(\phi_2))-r^2+1 = -\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{267}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-r^2+1}{r^2}, \tag{268}$$

and yields two solutions of ϕ_2.

Pre-multiplying (266) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2)-2r\left(r^2-1\right)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right)+r\left(r^2\cos(\phi_2)+r^2-1\right) = \alpha_{31}\sqrt{1-r^2}+\alpha_{33}r. \tag{269}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3(-\cos(\phi_2))-r^3+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{270}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3(-\cos(\phi_2))-r^3+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{271}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (266) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2\cos(\phi_2)+r^2-1\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}. \tag{272}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=-\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{273}$$

$$\implies\phi_3=\cos^{-1}\left(-\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{274}$$

which yields two solutions for each value of ϕ_2.

1.11.3 $R^-R^0R^+$ Paths

For a $R^-_{\phi_1}R^0_{\phi_2}R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{275}$$

Pre-multiplying (275) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r^2(-\cos(\phi_2))-r^2+1=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{276}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r-r^2+1}{r^2}, \tag{277}$$

and yields two solutions of ϕ_2.

Pre-multiplying (275) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2)-2r\left(r^2-1\right)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right)+r\left(r^2\cos(\phi_2)+r^2-1\right)=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \tag{278}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=-\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{279}$$

$$\implies\phi_1=\cos^{-1}\left(-\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{280}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (275) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2\cos(\phi_2)+r^2-1\right)=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \tag{281}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r^3(-\cos(\phi_2))-r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{282}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r^3(-\cos(\phi_2))-r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{283}$$

which yields two solutions for each value of ϕ_2.

1.11.4 $L^-L^0L^+$ Paths

For a $L^-_{\phi_1}L^0_{\phi_2}L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{284}$$

Pre-multiplying (284) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$r^2(-\cos(\phi_2))-r^2+1=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{285}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-r^2+1}{r^2}, \tag{286}$$

and yields two solutions of ϕ_2.

Pre-multiplying (284) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2)-2r\left(r^2-1\right)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right)+r\left(r^2\cos(\phi_2)+r^2-1\right)=\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r. \tag{287}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r^3(-\cos(\phi|2))-r^3+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi|2}{2}\right)\left(r^2\cos(\phi|2)+r^2-2\right)}} \tag{288}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{r^3(-\cos(\phi|2))-r^3+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi|2}{2}\right)\left(r^2\cos(\phi|2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{289}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (284) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2\cos(\phi_2)+r^2-1\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}. \tag{290}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=-\frac{r\left(-\alpha_{33}+r^2\cos(\phi|2)+r^2-1\right)+\alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{291}$$

$$\implies \phi_3=\cos^{-1}\left(-\frac{r\left(-\alpha_{33}+r^2\cos(\phi|2)+r^2-1\right)+\alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{292}$$

which yields two solutions for each value of ϕ_2.

1.11.5 $L^+L^0L^+$ Paths

For a $L^+_{\phi_1}L^0_{\phi_2}L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{293}$$

Pre-multiplying (293) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$r^2\cos(\phi_2) - r^2 + 1 = r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{294}$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \tag{295}$$

and yields two solutions of ϕ_2.

Pre-multiplying (293) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r\left(r^2-1\right)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + r\left(r^2(-\cos(\phi_2)) + r^2 - 1\right) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \tag{296}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3\cos(\phi_2) - r^3 - \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}} \tag{297}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 - \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{298}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (293) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2(-\cos(\phi_2)) + r^2 - 1\right) = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \tag{299}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3 - \gamma\right) = \frac{r^3\cos(\phi_2) - r^3 - \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}} \tag{300}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 - \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{301}$$

which yields two solutions for each value of ϕ_2.

1.11.6 $R^+R^0R^+$ Paths

For a $R^+_{\phi_1}R^0_{\phi_2}R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{302}$$

Pre-multiplying (302) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r^2\cos(\phi_2) - r^2 + 1 = r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{303}$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \tag{304}$$

and yields two solutions of ϕ_2.

Pre-multiplying (302) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r\left(r^2-1\right)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + r\left(r^2(-\cos(\phi_2)) + r^2 - 1\right) = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \tag{305}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3\cos(\phi_2) - r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}} \tag{306}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{307}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (302) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2(-\cos(\phi_2)) + r^2 - 1\right) = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r. \tag{308}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3 - \gamma\right) = \frac{r^3\cos(\phi_2) - r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}} \tag{309}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{310}$$

which yields two solutions for each value of ϕ_2.

1.11.7 $R^-R^0R^-$ Paths

For a $R^-_{\phi_1}R^0_{\phi_2}R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{311}$$

Pre-multiplying (311) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$r^2\cos(\phi_2) - r^2 + 1 = r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{312}$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \tag{313}$$

and yields two solutions of ϕ_2.

Pre-multiplying (311) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2)-2r\left(r^2-1\right)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right)+r\left(r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{314}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r^3\cos(\phi_2)-r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{315}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{316}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (311) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{317}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r^3\cos(\phi_2)-r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{318}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{319}$$

which yields two solutions for each value of ϕ_2.

1.11.8 $L^-L^0L^-$ Paths

For a $L^-_{\phi_1}L^0_{\phi_2}L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{320}$$

Pre-multiplying (320) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r^2\cos(\phi_2)-r^2+1=r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \tag{321}$$

which gives

$$\cos(\phi_2)=\frac{\alpha_{11}-\alpha_{11}r^2+\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+r^2-1}{r^2}, \tag{322}$$

and yields two solutions of ϕ_2.

Pre-multiplying (320) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2)-2r\left(r^2-1\right)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right)+r\left(r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \tag{323}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3\cos(\phi_2)-r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{324}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{325}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (320) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \tag{326}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r^3\cos(\phi_2)-r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{327}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{328}$$

which yields two solutions for each value of ϕ_2.

1.12 $C|C_\psi C_\psi|C$ Paths

1.12.1 $L^+|L_\psi^- R_\psi^-|R^+$ Paths

For a $L_{\phi_1}^+|L_\psi^- R_\psi^-|R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\psi)\mathbf{M}_{R^-}(r,\psi)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{329}$$

Pre-multiplying (329) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned}&12r^6-20r^4+10r^2+8\left(r^2-1\right)r^4\cos^2(\psi)-4\left(r^2-1\right)r^4-8\left(2r^6-3r^4+r^2\right)\cos(\psi)-1\\ =&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned} \tag{330}$$

which gives

$$\cos(\psi) = \frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^4\left(r^2-1\right)\left(\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)-1\right)}}{4r^4\left(r^2-1\right)}, \tag{331}$$

and yields four solutions of ψ.

Pre-multiplying (329) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_1)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\ &-2\left(r^2-1\right)r\cos(\phi_1)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\ =&\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r.\end{aligned} \tag{332}$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2\left(r^2-1\right) r \sin(\psi)\left(2r^2\cos(\psi) - 2r^2 + 1\right) \tag{333}$$

$$B = -2\left(r^2-1\right) r \left(\left(8r^4-8r^2+1\right)\cos(\psi) - \left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right), \tag{334}$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{31}\sqrt{1-r^2} - r\left(\alpha_{33} - 4r^6\cos(2\psi) - 12r^6 + 4r^4\cos(2\psi) + 20r^4 - 10r^2 + 8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}} \tag{335}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{\alpha_{31}\sqrt{1-r^2} - r\left(\alpha_{33} - 4r^6\cos(2\psi) - 12r^6 + 4r^4\cos(2\psi) + 20r^4 - 10r^2 + 8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)$$
$$+ \tan^{-1}\left(\frac{A}{B}\right), \tag{336}$$

which yields two solutions for each value of ψ.

Pre-multiplying (329) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&-12r^7 + 20r^5 - 10r^3 + 2\left(r^2-1\right) r\sin(\psi)\sin(\phi_2)\left(2r^2\cos(\psi) - 2r^2+1\right) - 4\left(r^2-1\right) r^5\cos(2\psi)\\ &-2\left(r^2-1\right) r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\psi) - \left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right) + 8\left(2r^4-3r^2+1\right) r^3\cos(\psi) + r\\ =&\alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33} r\end{aligned} \tag{337}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2 - \gamma\right) = \frac{\alpha_{13}\left(-\sqrt{1-r^2}\right) - r\left(\alpha_{33} - \left(4r^6-4r^4\right)\cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}} \tag{338}$$

$$\implies \phi_2 = \cos^{-1}\left(\frac{\alpha_{13}\left(-\sqrt{1-r^2}\right) - r\left(\alpha_{33} - \left(4r^6-4r^4\right)\cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)$$
$$+ \tan^{-1}\left(\frac{A}{B}\right), \tag{339}$$

which yields two solutions for each value of ψ.

1.12.2 $R^+|R_\psi^- L_\psi^-|L^+$ Paths

For a $R_{\phi_1}^+|R_\psi^- L_\psi^-|L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\psi)\mathbf{M}_{L^-}(r,\psi)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{340}$$

Pre-multiplying (340) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned}&12r^6 - 20r^4 + 10r^2 + 8\left(r^2-1\right) r^4\cos^2(\psi) - 4\left(r^2-1\right) r^4 - 8\left(2r^6-3r^4+r^2\right)\cos(\psi) - 1\\ =&\alpha_{11}\left(r^2-1\right) + r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r\right),\end{aligned} \tag{341}$$

which gives

$$\cos(\psi) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2}\sqrt{r^4\left(r^2-1\right)\left(\alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + \alpha_{11}\left(r^2-1\right)-1\right)}}{4r^4\left(r^2-1\right)}, \tag{342}$$

and yields four solutions of ψ.

Pre-multiplying (340) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}
&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_1)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\
&-2\left(r^2-1\right)r\cos(\phi_1)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\
=&\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (343)
\end{aligned}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2\left(r^2-1\right)r\sin(\psi)\left(2r^2\cos(\psi)-2r^2+1\right) \qquad (344)$$

$$B=-2\left(r^2-1\right)r\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right), \qquad (345)$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{\alpha_{31}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}} \qquad (346)$$

$$\begin{aligned}
\Longrightarrow \phi_1=&\cos^{-1}\left(\frac{\alpha_{31}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (347)
\end{aligned}$$

which yields two solutions for each value of ψ.

Pre-multiplying (340) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}
&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_2)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\
&-2\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\
=&\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r \qquad (348)
\end{aligned}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{\alpha_{13}\sqrt{1-r^2}-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}} \qquad (349)$$

$$\begin{aligned}
\Longrightarrow \phi_2=&\cos^{-1}\left(\frac{\alpha_{13}\sqrt{1-r^2}-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (350)
\end{aligned}$$

which yields two solutions for each value of ψ.

1.12.3 $R^-|R_\psi^+L_\psi^+|L^-$ Paths

For a $R_{\phi_1}^-|R_\psi^+L_\psi^+|L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\psi)\mathbf{M}_{L^+}(r,\psi)\mathbf{M}_{L^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\ \alpha_{21}&\alpha_{22}&\alpha_{23}\\ \alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \qquad (351)$$

Pre-multiplying (351) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&12r^6-20r^4+10r^2+8\left(r^2-1\right)r^4\cos^2(\psi)-4\left(r^2-1\right)r^4-8\left(2r^6-3r^4+r^2\right)\cos(\psi)-1\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (352)
\end{aligned}$$

which gives

$$\cos(\psi)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^4\left(r^2-1\right)\left(\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)-1\right)}}{4r^4\left(r^2-1\right)}, \qquad (353)$$

and yields four solutions of ψ.

Pre-multiplying (351) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_1)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\
&-2\left(r^2-1\right)r\cos(\phi_1)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\
=&\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \qquad (354)
\end{aligned}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2\left(r^2-1\right)r\sin(\psi)\left(2r^2\cos(\psi)-2r^2+1\right) \qquad (355)$$

$$B=-2\left(r^2-1\right)r\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right), \qquad (356)$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{\alpha_{31}\sqrt{1-r^2}-r\left(\alpha_{33}-(4r^6-4r^4)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}} \qquad (357)$$

$$\begin{aligned}
\implies\phi_1=&\cos^{-1}\left(\frac{\alpha_{31}\sqrt{1-r^2}-r\left(\alpha_{33}-(4r^6-4r^4)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (358)
\end{aligned}$$

which yields two solutions for each value of ψ.

Pre-multiplying (351) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_2)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\
&-2\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\
=&\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (359)
\end{aligned}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{\alpha_{13}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-(4r^6-4r^4)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}} \qquad (360)$$

$$\begin{aligned}
\implies\phi_2=&\cos^{-1}\left(\frac{\alpha_{13}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-(4r^6-4r^4)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (361)
\end{aligned}$$

which yields two solutions for each value of ψ.

1.12.4 $L^-|L_\psi^+R_\psi^+|R^-$ Paths

For a $L_{\phi_1}^-|L_\psi^+R_\psi^+|R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\psi)\mathbf{M}_{R^+}(r,\psi)\mathbf{M}_{R^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \qquad (362)$$

Pre-multiplying (362) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}
&12r^6-20r^4+10r^2+8\left(r^2-1\right)r^4\cos^2(\psi)-4\left(r^2-1\right)r^4-8\left(2r^6-3r^4+r^2\right)\cos(\psi)-1\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (363)
\end{aligned}$$

which gives

$$\cos(\psi)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^4\left(r^2-1\right)\left(\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)-1\right)}}{4r^4\left(r^2-1\right)}, \qquad (364)$$

and yields four solutions of ψ.

Pre-multiplying (362) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}
&12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_1)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\
&-2\left(r^2-1\right)r\cos(\phi_1)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\
=&\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}\tag{365}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2\left(r^2-1\right)r\sin(\psi)\left(2r^2\cos(\psi)-2r^2+1\right)\tag{366}$$

$$B=-2\left(r^2-1\right)r\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right),\tag{367}$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{\alpha_{31}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\tag{368}$$

$$\begin{aligned}
\Longrightarrow\phi_1=&\cos^{-1}\left(\frac{\alpha_{31}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}\tag{369}$$

which yields two solutions for each value of ψ.

Pre-multiplying (362) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_2)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\
&-2\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\
=&\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}\tag{370}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{\alpha_{13}\sqrt{1-r^2}-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\tag{371}$$

$$\begin{aligned}
\Longrightarrow\phi_2=&\cos^{-1}\left(\frac{\alpha_{13}\sqrt{1-r^2}-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}\tag{372}$$

which yields two solutions for each value of ψ.

1.13 $CGC_\beta|C$ Paths

1.13.1 $L^+G^+L^+_\beta|L^-$ Paths

For a $L^+_{\phi_1}G^+_{\phi_2}L^+_\beta|L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{L^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}\tag{373}$$

Pre-multiplying (373) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&-2r^4+2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\
=&-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),
\end{aligned}\tag{374}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2\left(r^2-1\right) r\sin(\beta) \tag{375}$$
$$B = 2r^4 - 2\left(r^2-1\right) r^2\cos(\beta) - 3r^2 + 1, \tag{376}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{-\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}} \tag{377}$$

$$\implies \phi_2 = \cos^{-1}\left(\frac{-\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{378}$$

and yields two solutions of ϕ_2.

Pre-multiplying (373) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}&2r^5 - r^3 - \left(r^2-1\right) r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2) - r^2 + 1\right) - 2r^2\cos(\phi_2) + 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) - 1\right)\\ &+ \left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) - 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2)\\ &- \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) - 4\left(r^2-1\right) r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}.\end{aligned} \tag{379}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = \left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) \tag{380}$$
$$D = -\left(r^2-1\right) r\left(2\cos(\beta)\left(r^2\cos(\phi_2) - r^2 + 1\right) - 2r^2\cos(\phi_2) + 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) - 1\right), \tag{381}$$

it is obtained that

$$\cos(\phi_1-\theta) = \frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2-1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right) r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{C^2+D^2}} - \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \tag{382}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2-1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right) r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{C^2+D^2}} - \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right) + \tan^{-1}\left(\frac{C}{D}\right), \tag{383}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (373) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&2r^5 - r^3 - \left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)-1)) - 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2) - \left(2r^5 - 3r^3 + r\right)\cos(\phi_2)\\ &- \left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2-1\right) r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right) r\cos(\phi_2)\right)\\ &- 4\left(r^2-1\right) r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r.\end{aligned} \tag{384}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = -\left(r^2-1\right)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)-1)) \tag{385}$$
$$F = -\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2-1\right) r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right) r\cos(\phi_2)\right), \tag{386}$$

it is obtained that

$$\cos(\phi_3-\sigma)=\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \tag{387}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \tag{388}$$

which yields two solutions for each value of ϕ_2.

1.13.2 $R^+G^+R^+_\beta|R^-$ Paths

For a $R^+_{\phi_1}G^+_{\phi_2}R^+_\beta|R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{R^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{389}$$

Pre-multiplying (389) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}&-2r^4+2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\ &=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned} \tag{390}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2\left(r^2-1\right)r\sin(\beta) \tag{391}$$
$$B=2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1, \tag{392}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}} \tag{393}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}}\right)+\tan^{-1}\left(\frac{A}{B}\right), \tag{394}$$

and yields two solutions of ϕ_2.

Pre-multiplying (389) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&2r^5-r^3-\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right)\\ &+\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)-2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\\ &-\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)=\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{395}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C=\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) \tag{396}$$
$$D=-\left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right), \tag{397}$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2 - 1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2 - 1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{C^2 + D^2}}$$
$$+ \frac{\alpha_{31}\sqrt{1 - r^2}}{\sqrt{C^2 + D^2}} \tag{398}$$
$$\implies \phi_1 = \cos^{-1}\left(\frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2 - 1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2 - 1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{C^2 + D^2}}\right.$$
$$\left. + \frac{\alpha_{31}\sqrt{1 - r^2}}{\sqrt{C^2 + D^2}}\right) + \tan^{-1}\left(\frac{C}{D}\right), \tag{399}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (389) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$2r^5 - r^3 - \left(r^2 - 1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2) - 1)) - 2\left(r^2 - 1\right)r^2\sin(\beta)\sin(\phi_2) - \left(2r^5 - 3r^3 + r\right)\cos(\phi_2)$$
$$- \left(r^2 - 1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2 - 1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2 - 1\right)r\cos(\phi_2)\right)$$
$$- 4\left(r^2 - 1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{13}\sqrt{1 - r^2}. \tag{400}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = -\left(r^2 - 1\right)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2) - 1)) \tag{401}$$
$$F = -\left(r^2 - 1\right)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2 - 1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2 - 1\right)r\cos(\phi_2)\right), \tag{402}$$

it is obtained that

$$\cos\left(\phi_3 - \sigma\right) = \frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2 - 1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2 - 1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{E^2 + F^2}}$$
$$+ \frac{-\alpha_{13}\sqrt{1 - r^2}}{\sqrt{E^2 + F^2}} \tag{403}$$
$$\implies \phi_3 = \cos^{-1}\left(\frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2 - 1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2 - 1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{E^2 + F^2}}\right.$$
$$\left. + \frac{-\alpha_{13}\sqrt{1 - r^2}}{\sqrt{E^2 + F^2}}\right) + \tan^{-1}\left(\frac{E}{F}\right), \tag{404}$$

which yields two solutions for each value of ϕ_2.

1.13.3 $R^-G^-R_\beta^-|R^+$ Paths

For a $R_{\phi_1}^-G_{\phi_2}^-R_\beta^-|R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{R^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{405}$$

Pre-multiplying (405) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$-2r^4 + 2\left(r^2 - 1\right)r\sin(\beta)\sin(\phi_2) + 2\left(r^2 - 1\right)r^2\cos(\beta) + r^2 + \cos(\phi_2)\left(2r^4 - 2\left(r^2 - 1\right)r^2\cos(\beta) - 3r^2 + 1\right)$$
$$= -\left(\alpha_{11}\left(r^2 - 1\right)\right) - r\left(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r\right), \tag{406}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2\left(r^2-1\right) r\sin(\beta) \tag{407}$$
$$B = 2r^4 - 2\left(r^2-1\right) r^2\cos(\beta) - 3r^2 + 1, \tag{408}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{-\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}} \tag{409}$$

$$\Longrightarrow \phi_2 = \cos^{-1}\left(\frac{-\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{410}$$

and yields two solutions of ϕ_2.

Pre-multiplying (405) with $\mathbf{u}_{G+}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$2r^5 - r^3 - \left(r^2-1\right) r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2) - r^2 + 1\right) - 2r^2\cos(\phi_2) + 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) - 1\right) + \left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) - 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2) - \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) - 4\left(r^2-1\right) r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}. \tag{411}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = \left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) \tag{412}$$
$$D = -\left(r^2-1\right) r\left(2\cos(\beta)\left(r^2\cos(\phi_2) - r^2 + 1\right) - 2r^2\cos(\phi_2) + 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) - 1\right), \tag{413}$$

it is obtained that

$$\cos(\phi_1-\theta) = \frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2-1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right) r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{C^2+D^2}} + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \tag{414}$$

$$\Longrightarrow \phi_1 = \cos^{-1}\left(\frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2-1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right) r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{C^2+D^2}} + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right) + \tan^{-1}\left(\frac{C}{D}\right), \tag{415}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (405) with $\mathbf{u}_{R-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$2r^5 - r^3 - \left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2) - 1)) - 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2) - \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) - \left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2-1\right) r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right) r\cos(\phi_2)\right) - 4\left(r^2-1\right) r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \tag{416}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = -\left(r^2-1\right)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2) - 1)) \tag{417}$$
$$F = -\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2-1\right) r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right) r\cos(\phi_2)\right), \tag{418}$$

it is obtained that

$$\cos(\phi_3-\sigma) = \frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \tag{419}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \tag{420}$$

which yields two solutions for each value of ϕ_2.

1.13.4 $L^-G^-L^-_\beta|L^+$ Paths

For a $L^-_{\phi_1}G^-_{\phi_2}L^-_\beta|L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{L^+}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{421}$$

Pre-multiplying (421) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned}&-2r^4+2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\ &=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned} \tag{422}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2\left(r^2-1\right)r\sin(\beta) \tag{423}$$

$$B = 2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1, \tag{424}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}} \tag{425}$$

$$\implies \phi_2 = \cos^{-1}\left(\frac{-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}}\right)+\tan^{-1}\left(\frac{A}{B}\right), \tag{426}$$

and yields two solutions of ϕ_2.

Pre-multiplying (421) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}&2r^5-r^3-\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right)\\ &+\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)-2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\\ &-\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)=\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{427}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = \left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) \tag{428}$$

$$D = -\left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right), \tag{429}$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_1-\theta) =& \frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{C^2+D^2}} \\
&+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \qquad (430)\\
\implies \phi_1 =& \cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{C^2+D^2}}\right. \\
&\left.+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (431)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (421) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$
\begin{aligned}
&2r^5-r^3-\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)-1))-2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)-\left(2r^5-3r^3+r\right)\cos(\phi_2) \\
&-\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)+2r^3-2r^2\sin(\beta)\sin(\phi_2)-2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right) \\
&-4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}. \qquad (432)
\end{aligned}
$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$
\begin{aligned}
E &= -\left(r^2-1\right)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)-1)) \qquad (433)\\
F &= -\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)+2r^3-2r^2\sin(\beta)\sin(\phi_2)-2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), \qquad (434)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_3-\sigma) =& \frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}} \\
&+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \qquad (435)\\
\implies \phi_3 =& \cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}}\right. \\
&\left.+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \qquad (436)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

1.13.5 $L^+G^+R^+_\beta|R^-$ Paths

For a $L^+_{\phi_1}G^+_{\phi_2}R^+_\beta|R^-_{\phi_3}$ path, the equation to be solved is:

$$
\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{R^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \qquad (437)
$$

Pre-multiplying (437) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$
\begin{aligned}
&-2r^4-2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\beta)+3r^2-1\right) \\
=&\alpha_{11}\left(r^2-1\right)-r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (438)
\end{aligned}
$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = -2\left(r^2-1\right) r\sin(\beta) \tag{439}$$
$$B = -\left(2r^4 - 2\left(r^2-1\right) r^2\cos(\beta) - 3r^2 + 1\right), \tag{440}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{\alpha_{11}\left(r^2-1\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}} \tag{441}$$
$$\implies \phi_2 = \cos^{-1}\left(\frac{\alpha_{11}\left(r^2-1\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}}\right)$$
$$+ \tan^{-1}\left(\frac{A}{B}\right), \tag{442}$$

and yields two solutions of ϕ_2.

Pre-multiplying (437) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$2r^5 - r^3 + \left(r^2-1\right) r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2) + r^2 - 1\right) - 2r^2\cos(\phi_2) - 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) + 1\right)$$
$$- \left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) + 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2)$$
$$+ \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) - 4\left(r^2-1\right) r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r. \tag{443}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) \tag{444}$$
$$D = \left(r^2-1\right) r\left(2\cos(\beta)\left(r^2\cos(\phi_2) + r^2 - 1\right) - 2r^2\cos(\phi_2) - 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) + 1\right), \tag{445}$$

it is obtained that

$$\cos(\phi_1-\theta) = \frac{-2r^5 + r^3 + \left(-2r^5 + 3r^3 - r\right)\cos(\phi_2) + \left(2r^2 - 2r^4\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right) r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{C^2+D^2}}$$
$$+ \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \tag{446}$$
$$\implies \phi_1 = \cos^{-1}\left(\frac{-2r^5 + r^3 + \left(-2r^5 + 3r^3 - r\right)\cos(\phi_2) + \left(2r^2 - 2r^4\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right) r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{C^2+D^2}}\right.$$
$$\left. + \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right) + \tan^{-1}\left(\frac{C}{D}\right), \tag{447}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (437) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$2r^5 - r^3 + \left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)+1)) + 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2) + \left(2r^5 - 3r^3 + r\right)\cos(\phi_2)$$
$$+ \left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2) - 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) + 2\left(2r^2-1\right) r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right) r\cos(\phi_2)\right)$$
$$- 4\left(r^2-1\right) r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r. \tag{448}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = \left(r^2-1\right)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)+1)) \tag{449}$$
$$F = \left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2) - 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) + 2\left(2r^2-1\right) r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right) r\cos(\phi_2)\right), \tag{450}$$

it is obtained that

$$
\begin{aligned}
\cos\left(\phi_3-\sigma\right) =& \frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}} \\
&+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \qquad (451) \\
\implies \phi_3 = \cos^{-1}&\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\right. \\
&\left.+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \qquad (452)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

1.13.6 $R^+G^+L^+_\beta|L^-$ Paths

For a $R^+_{\phi_1}G^+_{\phi_2}L^+_\beta|L^-_{\phi_3}$ path, the equation to be solved is:

$$
\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{L^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \qquad (453)
$$

Pre-multiplying (453) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$
\begin{aligned}
&-2r^4-2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\beta)+3r^2-1\right) \\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \qquad (454)
\end{aligned}
$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$
\begin{aligned}
A &= -2\left(r^2-1\right)r\sin(\beta) \qquad (455) \\
B &= -\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right), \qquad (456)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_2-\gamma) =& \frac{r\left(-r\left(\alpha_{33}+2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \qquad (457) \\
\implies \phi_2 = \cos^{-1}&\left(\frac{r\left(-r\left(\alpha_{33}+2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right) \\
&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (458)
\end{aligned}
$$

and yields two solutions of ϕ_2.

Pre-multiplying (453) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$
\begin{aligned}
&2r^5-r^3+\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right) \\
&-\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2) \\
&+\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \qquad (459)
\end{aligned}
$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C &= -\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) \qquad (460) \\
D &= \left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right), \qquad (461)
\end{aligned}
$$

it is obtained that

$$\cos(\phi_1-\theta)=\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}$$
$$+\frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \tag{462}$$
$$\implies \phi_1=\cos^{-1}\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\right.$$
$$\left.+\frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \tag{463}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (453) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$2r^5-r^3+\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1))+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)+\left(2r^5-3r^3+r\right)\cos(\phi_2)$$
$$+\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right)$$
$$-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}. \tag{464}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma:=\frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma:=\frac{F}{\sqrt{E^2+F^2}}$, where

$$E=\left(r^2-1\right)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1)) \tag{465}$$
$$F=\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), \tag{466}$$

it is obtained that

$$\cos\left(\phi_3-\sigma\right)=\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}$$
$$+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \tag{467}$$
$$\implies \phi_3=\cos^{-1}\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\right.$$
$$\left.+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \tag{468}$$

which yields two solutions for each value of ϕ_2.

1.13.7 $R^-G^-L_\beta^-|L^+$ Paths

For a $R_{\phi_1}^-G_{\phi_2}^-L_\beta^-|L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{469}$$

Pre-multiplying (469) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$-2r^4-2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\beta)+3r^2-1\right)$$
$$=\alpha_{11}\left(r^2-1\right)-r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{470}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A &= -2\left(r^2-1\right)r\sin(\beta) && (471)\\ B &= -\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right), && (472) \end{aligned}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{\alpha_{11}\left(r^2-1\right)-r\left(2r^3\cos(\beta)-2r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}} \tag{473}$$

$$\begin{aligned} \implies \phi_2 = \cos^{-1}&\left(\frac{\alpha_{11}\left(r^2-1\right)-r\left(2r^3\cos(\beta)-2r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}}\right)\\ &+\tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{474}$$

and yields two solutions of ϕ_2.

Pre-multiplying (469) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned} &2r^5-r^3+\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right)\\ &-\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\\ &+\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \end{aligned} \tag{475}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C &= -\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) && (476)\\ D &= \left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right), && (477) \end{aligned}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) = &\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\\ &+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \end{aligned} \tag{478}$$

$$\begin{aligned} \implies \phi_1 = \cos^{-1}&\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\right.\\ &\left.+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{479}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (469) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned} &2r^5-r^3+\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1))+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)+\left(2r^5-3r^3+r\right)\cos(\phi_2)\\ &+\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right)\\ &-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \end{aligned} \tag{480}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$\begin{aligned} E &= \left(r^2-1\right)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1)) && (481)\\ F &= \left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), && (482) \end{aligned}$$

it is obtained that

$$\cos\left(\phi_3-\sigma\right)=\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}$$
$$+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \tag{483}$$
$$\implies \phi_3=\cos^{-1}\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\right.$$
$$\left.+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \tag{484}$$

which yields two solutions for each value of ϕ_2.

1.13.8 $L^-G^-R^-_\beta|R^+$ Paths

For a $L^-_{\phi_1}G^-_{\phi_2}R^-_\beta|R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{R^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{485}$$

Pre-multiplying (485) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$-2r^4-2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\beta)+3r^2-1\right)$$
$$=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \tag{486}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=-2\left(r^2-1\right)r\sin(\beta) \tag{487}$$
$$B=-\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right), \tag{488}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(-r\left(\alpha_{33}+2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{489}$$
$$\implies \phi_2=\cos^{-1}\left(\frac{r\left(-r\left(\alpha_{33}+2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)$$
$$+\tan^{-1}\left(\frac{A}{B}\right), \tag{490}$$

and yields two solutions of ϕ_2.

Pre-multiplying (485) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$2r^5-r^3+\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right)$$
$$-\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)$$
$$+\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \tag{491}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$C=-\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) \tag{492}$$
$$D=\left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right), \tag{493}$$

it is obtained that

$$\cos(\phi_1-\theta) = \frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}} + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \tag{494}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}} + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \tag{495}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (485) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&2r^5-r^3+\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1))+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)+\left(2r^5-3r^3+r\right)\cos(\phi_2)\\&+\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right)\\&-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}.\end{aligned} \tag{496}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = \left(r^2-1\right)\left(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1)\right) \tag{497}$$

$$F = \left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), \tag{498}$$

it is obtained that

$$\cos\left(\phi_3-\sigma\right) = \frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}} + \frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \tag{499}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}} + \frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \tag{500}$$

which yields two solutions for each value of ϕ_2.

1.14 $CC_\mu|C_\mu C$ Paths

1.14.1 $L^+R^+_\mu|R^-_\mu L^-$ Paths

For a $L^+_{\phi_1}R^+_\mu|R^-_\mu L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{501}$$

Pre-multiplying (501) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}&-12r^6+16r^4-8\left(r^2-1\right)^2r^2\cos^2(\mu)+4\left(r^2-1\right)^2r^2-6r^2+8\left(2r^6-3r^4+r^2\right)\cos(\mu)+1\\&=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned} \tag{502}$$

which gives

$$\cos(\mu) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2}\sqrt{r^2\left(r^2-1\right)^2\left(\alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + \alpha_{11}\left(r^2-1\right)+1\right)}}{4r^2\left(r^2-1\right)^2}, \tag{503}$$

and yields four solutions of μ.

Pre-multiplying (501) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&r\Bigg(12r^6 - 16r^4 + 2\left(r^2-1\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu) - 2r^2+1\right) + 4\left(r^2-1\right)^2 r^2\cos(2\mu) + 6r^2 \\
&\quad - 2\left(r^2-1\right)\cos(\phi_1)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right) + \left(-8r^4+8r^2-1\right)\cos(\mu)\right) - 8\left(2r^4-3r^2+1\right)r^2\cos(\mu) - 1\Bigg) \\
&= \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}.
\end{aligned} \tag{504}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2r\left(r^2-1\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu) - 2r^2+1\right) \tag{505}$$

$$B = -2r\left(r^2-1\right)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right) + \left(-8r^4+8r^2-1\right)\cos(\mu)\right), \tag{506}$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{31}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^3\cos(2\mu) + 8\left(2r^7-3r^5+r^3\right)\cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}} \tag{507}$$

$$\begin{aligned}
\implies \phi_1 = {} & \cos^{-1}\left(\frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{31}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^3\cos(2\mu) + 8\left(2r^7-3r^5+r^3\right)\cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}}\right) \\
& + \tan^{-1}\left(\frac{A}{B}\right),
\end{aligned} \tag{508}$$

which yields two solutions for each value of μ.

Pre-multiplying (501) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&r\Bigg(12r^6 - 16r^4 + 2\left(r^2-1\right)\sin(\mu)\sin(\phi_2)\left(2\left(r^2-1\right)\cos(\mu) - 2r^2+1\right) + 4\left(r^2-1\right)^2 r^2\cos(2\mu) + 6r^2 \\
&\quad - 2\left(r^2-1\right)\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right) + \left(-8r^4+8r^2-1\right)\cos(\mu)\right) - 8\left(2r^6-3r^4+r^2\right)\cos(\mu) - 1\Bigg) \\
&= \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r.
\end{aligned} \tag{509}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2 - \gamma\right) = \frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{13}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^3\cos(2\mu) + 8\left(2r^7-3r^5+r^3\right)\cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}} \tag{510}$$

$$\begin{aligned}
\implies \phi_2 = {} & \cos^{-1}\left(\frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{13}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^3\cos(2\mu) + 8\left(2r^7-3r^5+r^3\right)\cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}}\right) \\
& + \tan^{-1}\left(\frac{A}{B}\right),
\end{aligned} \tag{511}$$

which yields two solutions for each value of μ.

1.14.2 $R^+L_\mu^+|L_\mu^-R^-$ Paths

For a $R_{\phi_1}^+L_\mu^+|L_\mu^-R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{512}$$

Pre-multiplying (512) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}&-12r^6+16r^4-8\left(r^2-1\right)^2r^2\cos^2(\mu)+4\left(r^2-1\right)^2r^2-6r^2+8\left(2r^6-3r^4+r^2\right)\cos(\mu)+1\\ =&-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned} \tag{513}$$

which gives

$$\cos(\mu)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^2\left(r^2-1\right)^2\left(\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)+1\right)}}{4r^2\left(r^2-1\right)^2}, \tag{514}$$

and yields four solutions of μ.

Pre-multiplying (512) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&r\Bigg(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2\\&\quad-2\left(r^2-1\right)\cos(\phi_1)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^4-3r^2+1\right)r^2\cos(\mu)-1\Bigg)\\=&\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{515}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2r\left(r^2-1\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right) \tag{516}$$

$$B=-2r\left(r^2-1\right)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right), \tag{517}$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{-12r^7+16r^5-6r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{518}$$

$$\begin{aligned}\Longrightarrow\phi_1=&\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right)\\&+\tan^{-1}\left(\frac{A}{B}\right),\end{aligned} \tag{519}$$

which yields two solutions for each value of μ.

Pre-multiplying (512) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&r\Bigg(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_2)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2\\&\quad-2\left(r^2-1\right)\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^6-3r^4+r^2\right)\cos(\mu)-1\Bigg)\\=&\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}.\end{aligned} \tag{520}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{-12r^7+16r^5-6r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{521}$$

$$\begin{aligned}\Longrightarrow\phi_2=&\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right)\\&+\tan^{-1}\left(\frac{A}{B}\right),\end{aligned} \tag{522}$$

which yields two solutions for each value of μ.

1.14.3 $R^-L_\mu^-|L_\mu^+R^+$ Paths

For a $R_{\phi_1}^-L_\mu^-|L_\mu^+R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{523}$$

Pre-multiplying (523) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned} &-12r^6+16r^4-8\left(r^2-1\right)^2r^2\cos^2(\mu)+4\left(r^2-1\right)^2r^2-6r^2+8\left(2r^6-3r^4+r^2\right)\cos(\mu)+1 \\ &=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \end{aligned} \tag{524}$$

which gives

$$\cos(\mu) = \frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^2\left(r^2-1\right)^2\left(\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)+1\right)}}{4r^2\left(r^2-1\right)^2}, \tag{525}$$

and yields four solutions of μ.

Pre-multiplying (523) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned} &r\Big(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2 \\ &\quad-2\left(r^2-1\right)\cos(\phi_1)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^4-3r^2+1\right)r^2\cos(\mu)-1\Big) \\ &=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \end{aligned} \tag{526}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2r\left(r^2-1\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right) \tag{527}$$

$$B = -2r\left(r^2-1\right)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right), \tag{528}$$

it is obtained that

$$\cos(\phi_1-\gamma) = \frac{-12r^7+16r^5-6r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{529}$$

$$\begin{aligned} \implies \phi_1 = \cos^{-1}&\left(\frac{-12r^7+16r^5-6r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right) \\ &+\tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{530}$$

which yields two solutions for each value of μ.

Pre-multiplying (523) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned} &r\Big(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_2)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2 \\ &\quad-2\left(r^2-1\right)\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^6-3r^4+r^2\right)\cos(\mu)-1\Big) \\ &=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \end{aligned} \tag{531}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right) = \frac{-12r^7+16r^5-6r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{532}$$

$$\begin{aligned} \implies \phi_2 = \cos^{-1}&\left(\frac{-12r^7+16r^5-6r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right) \\ &+\tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{533}$$

which yields two solutions for each value of μ.

1.14.4 $L^- R_\mu^- | R_\mu^+ L^+$ Paths

For a $L_{\phi_1}^- R_\mu^- | R_\mu^+ L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{534}$$

Pre-multiplying (534) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned} &-12r^6 + 16r^4 - 8\left(r^2-1\right)^2 r^2\cos^2(\mu) + 4\left(r^2-1\right)^2 r^2 - 6r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\mu) + 1 \\ &= -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \end{aligned} \tag{535}$$

which gives

$$\cos(\mu) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2}\sqrt{r^2\left(r^2-1\right)^2\left(\alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + \alpha_{11}\left(r^2-1\right) + 1\right)}}{4r^2\left(r^2-1\right)^2}, \tag{536}$$

and yields four solutions of μ.

Pre-multiplying (534) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned} &r\Bigg(12r^6 - 16r^4 + 2\left(r^2-1\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu) - 2r^2 + 1\right) + 4\left(r^2-1\right)^2 r^2\cos(2\mu) + 6r^2 \\ &\quad - 2\left(r^2-1\right)\cos(\phi_1)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu) + 3r^2 - 1\right) + \left(-8r^4 + 8r^2 - 1\right)\cos(\mu)\right) - 8\left(2r^4 - 3r^2 + 1\right)r^2\cos(\mu) - 1\Bigg) \\ &= \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r. \end{aligned} \tag{537}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2r\left(r^2-1\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu) - 2r^2 + 1\right) \tag{538}$$

$$B = -2r\left(r^2-1\right)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu) + 3r^2 - 1\right) + \left(-8r^4 + 8r^2 - 1\right)\cos(\mu)\right), \tag{539}$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{31}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^3\cos(2\mu) + 8\left(2r^7 - 3r^5 + r^3\right)\cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}} \tag{540}$$

$$\begin{aligned} \Longrightarrow \phi_1 = \cos^{-1}&\left(\frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{31}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^3\cos(2\mu) + 8\left(2r^7 - 3r^5 + r^3\right)\cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}}\right) \\ &+ \tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{541}$$

which yields two solutions for each value of μ.

Pre-multiplying (534) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned} &r\Bigg(12r^6 - 16r^4 + 2\left(r^2-1\right)\sin(\mu)\sin(\phi_2)\left(2\left(r^2-1\right)\cos(\mu) - 2r^2 + 1\right) + 4\left(r^2-1\right)^2 r^2\cos(2\mu) + 6r^2 \\ &\quad - 2\left(r^2-1\right)\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu) + 3r^2 - 1\right) + \left(-8r^4 + 8r^2 - 1\right)\cos(\mu)\right) - 8\left(2r^6 - 3r^4 + r^2\right)\cos(\mu) - 1\Bigg) \\ &= \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \end{aligned} \tag{542}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{-12r^7+16r^5-6r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{543}$$

$$\Longrightarrow \phi_2=\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right)$$
$$+\tan^{-1}\left(\frac{A}{B}\right), \tag{544}$$

which yields two solutions for each value of μ.

1.15 $C|C_\beta G C_\beta|C$ Paths

1.15.1 $L^+|L^-_\beta G^- L^-_\beta|L^+$ Paths

For a $L^+_{\phi_1}|L^-_\beta G^-_{\phi_2} L^-_\beta|L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{545}$$

Pre-multiplying (545) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned}&4\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos^2(\beta)+4\left(1-2r^2\right)\left(r^2-1\right)r^2\cos(\beta)+\left(1-2r^2\right)^2r^2\\&-\left(r^2-1\right)\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)\\=&r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),\end{aligned} \tag{546}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4\left(r^2-1\right)r\sin(\beta)\left(2r^2\cos(\beta)-2r^2+1\right) \tag{547}$$
$$B=-\left(r^2-1\right)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right), \tag{548}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{549}$$

$$\Longrightarrow \phi_2=\cos^{-1}\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)$$
$$+\tan^{-1}\left(\frac{A}{B}\right), \tag{550}$$

and yields two solutions of ϕ_2.

Pre-multiplying (545) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)+4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=-\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}.\end{aligned} \tag{551}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C =& \left(r^5+r\right)\sin(2\beta) - \left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta) - 2r^2\cos(\beta) + 2r^2\cos(2\beta)\right) \\
&+ r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right) - 2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right) \qquad (552)\\
D =& 4r^7 - 6r^5 - 6r^4\sin(2\beta)\sin(\phi_2) + 2r^3 + 2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta) \\
&+ \left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta) - \left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\
&+ \left(-8r^7+16r^5-9r^3+r\right)\cos(\beta) + \sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right), \qquad (553)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta) =& \frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\
&+ \frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
&+ \frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (554)\\
\implies \phi_1 = \cos^{-1}\Bigg(& \frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\
&+ \frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
&+ \frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg) + \tan^{-1}\left(\frac{C}{D}\right), \qquad (555)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (545) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right) \\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta) \\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2) \\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (556)
\end{aligned}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right) =& \frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\
&+ \frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
&+ \frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (557)\\
\implies \phi_3 = \cos^{-1}\Bigg(& \frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\
&+ \frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
&+ \frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg) + \tan^{-1}\left(\frac{C}{D}\right), \qquad (558)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.15.2 $R^+|R^-_\beta G^- R^-_\beta|R^+$ Paths

For a $R^+_{\phi_1}|R^-_\beta G^-_{\phi_2} R^-_\beta|R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{R^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{559}$$

Pre-multiplying (559) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned}
&4\left(r^2-1\right) r\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+4\left(r^2-1\right)^2 r^2\cos^2(\beta)+4\left(1-2r^2\right)\left(r^2-1\right) r^2\cos(\beta)+\left(1-2r^2\right)^2 r^2 \\
&-\left(r^2-1\right)\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right) \\
=&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),
\end{aligned} \tag{560}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 4\left(r^2-1\right) r\sin(\beta)\left(2r^2\cos(\beta)-2r^2+1\right) \tag{561}$$

$$B = -\left(r^2-1\right)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right), \tag{562}$$

it is obtained that

$$\cos(\phi_2-\gamma) = -\frac{r\left(r\left(\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2-\alpha_{33}\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{563}$$

$$\begin{aligned}
\implies \phi_2 = \cos^{-1}&\left(-\frac{r\left(r\left(\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2-\alpha_{33}\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right) \\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned} \tag{564}$$

and yields two solutions of ϕ_2.

Pre-multiplying (559) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right) r^4\sin(2\beta)\sin(\phi_2)+4\left(2r^4-3r^2+1\right) r^2\sin(\beta)\sin(\phi_2) \\
&+\left(r^2-1\right) r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2 r^3\cos^2(\beta) \\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2) \\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right) r\cos^2(\beta)+\left(r^2-1\right) r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r.
\end{aligned} \tag{565}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C = &\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)
\end{aligned} \tag{566}$$

$$\begin{aligned}
D = &4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right) r\cos^2(\beta) \\
&+\left(r^2-1\right) r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right),
\end{aligned} \tag{567}$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_1-\theta) &= \frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (568)
\end{aligned}
$$

$$
\begin{aligned}
\implies \phi_1 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (569)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (559) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$
\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right) \\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta) \\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2) \\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \\
&\qquad (570)
\end{aligned}
$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$
\begin{aligned}
\cos\left(\phi_3-\theta\right) &= \frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (571)
\end{aligned}
$$

$$
\begin{aligned}
\implies \phi_3 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (572)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

1.15.3 $R^-|R^+_\beta G^+ R^+_\beta|R^-$ Paths

For a $R^-_{\phi_1}|R^+_\beta G^+_{\phi_2} R^+_\beta|R^-_{\phi_3}$ path, the equation to be solved is:

$$
\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{R^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \qquad (573)
$$

Pre-multiplying (573) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$
\begin{aligned}
&4\left(r^2-1\right) r \sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+4\left(r^2-1\right)^2 r^2\cos^2(\beta)+4\left(1-2r^2\right)\left(r^2-1\right) r^2\cos(\beta)+\left(1-2r^2\right)^2 r^2 \\
&-\left(r^2-1\right)\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right) \\
=&r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),
\end{aligned}
\tag{574}
$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4\left(r^2-1\right) r\sin(\beta)\left(2r^2\cos(\beta)-2r^2+1\right) \tag{575}$$

$$B=-\left(r^2-1\right)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right), \tag{576}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{577}$$

$$
\begin{aligned}
\implies \phi_2=\cos^{-1}&\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right) \\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}
\tag{578}
$$

and yields two solutions of ϕ_2.

Pre-multiplying (573) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$
\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right) r^4\sin(2\beta)\sin(\phi_2)+4\left(2r^4-3r^2+1\right) r^2\sin(\beta)\sin(\phi_2) \\
&+\left(r^2-1\right) r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2 r^3\cos^2(\beta) \\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2) \\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right) r\cos^2(\beta)+\left(r^2-1\right) r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}
\tag{579}
$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)
\end{aligned}
\tag{580}
$$

$$
\begin{aligned}
D=&4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right) r\cos^2(\beta) \\
&+\left(r^2-1\right) r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right),
\end{aligned}
\tag{581}
$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta) =& \frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\end{aligned} \tag{582}$$

$$\begin{aligned}\implies \phi_1 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),\end{aligned} \tag{583}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (573) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.\end{aligned} \tag{584}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}\cos\left(\phi_3-\theta\right) =& \frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\end{aligned} \tag{585}$$

$$\begin{aligned}\implies \phi_3 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),\end{aligned} \tag{586}$$

which yields two solutions for each value of ϕ_2.

1.15.4 $L^-|L_\beta^+G^+L_\beta^+|L^-$ Paths

For a $L_{\phi_1}^-|L_\beta^+G_{\phi_2}^+L_\beta^+|L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{L^-}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{587}$$

Pre-multiplying (587) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$
\begin{aligned}
&4\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos^2(\beta)+4\left(1-2r^2\right)\left(r^2-1\right)r^2\cos(\beta)+\left(1-2r^2\right)^2r^2\\
&-\left(r^2-1\right)\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)\\
=&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \qquad (588)
\end{aligned}
$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$
A=4\left(r^2-1\right)r\sin(\beta)\left(2r^2\cos(\beta)-2r^2+1\right) \qquad (589)
$$

$$
B=-\left(r^2-1\right)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right), \qquad (590)
$$

it is obtained that

$$
\cos(\phi_2-\gamma)=-\frac{r\left(r\left(\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2-\alpha_{33}\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \qquad (591)
$$

$$
\begin{aligned}
\implies \phi_2=\cos^{-1}&\left(-\frac{r\left(r\left(\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2-\alpha_{33}\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (592)
\end{aligned}
$$

and yields two solutions of ϕ_2.

Pre-multiplying (587) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$
\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)+4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \qquad (593)
\end{aligned}
$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right) \qquad (594)\\
D=&4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right), \qquad (595)
\end{aligned}
$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta) = & \frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\ & +\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & +\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{596}$$

$$\begin{aligned}\implies \phi_1 = \cos^{-1}\Bigg(& \frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\ & +\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & +\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{597}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (587) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned} & -4r^7+4r^5-r^3-4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right) \\ & +\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta) \\ & +4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\ & +r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2) \\ & +2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\ & +\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \end{aligned} \tag{598}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}\cos\left(\phi_3-\theta\right) = & \frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\ & +\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & +\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{599}$$

$$\begin{aligned}\implies \phi_3 = \cos^{-1}\Bigg(& \frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\ & +\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & +\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{600}$$

which yields two solutions for each value of ϕ_2.

1.15.5 $L^+|L_\beta^-G^-R_\beta^-|R^+$ Paths

For a $L_{\phi_1}^+|L_\beta^-G_{\phi_2}^-R_\beta^-|R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{R^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{601}$$

Pre-multiplying (601) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$
\begin{aligned}
&r^2\left(2r^2-1\right)^2+\cos(\phi_2)\left(-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right)\right)\\
&+\left(4r^4\left(r^2-1\right)-4r^2\left(r^2-1\right)\right)\cos^2(\beta)+\left(4r^2\left(r^2-1\right)-8r^4\left(r^2-1\right)\right)\cos(\beta)\\
&+\sin(\phi_2)\left(4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\right)\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),
\end{aligned}
\tag{602}
$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$
A=4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta) \tag{603}
$$

$$
B=-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right), \tag{604}
$$

it is obtained that

$$
\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{605}
$$

$$
\begin{aligned}
\implies \phi_2=&\cos^{-1}\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}
\tag{606}
$$

and yields two solutions of ϕ_2.

Pre-multiplying (601) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$
\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)-4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r.
\end{aligned}
\tag{607}
$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)
\end{aligned}
\tag{608}
$$

$$
\begin{aligned}
D=&4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right),
\end{aligned}
\tag{609}
$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta) &= \frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{610}$$

$$\begin{aligned}\implies \phi_1 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{611}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (601) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\ &-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\ &+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\ &+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\ &+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \end{aligned} \tag{612}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}\cos\left(\phi_3-\theta\right) &= \frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{613}$$

$$\begin{aligned}\implies \phi_3 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{614}$$

which yields two solutions for each value of ϕ_2.

1.15.6 $R^+|R_\beta^-G^-L_\beta^-|L^+$ Paths

For a $R_{\phi_1}^+|R_\beta^-G_{\phi_2}^-L_\beta^-|L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{615}$$

Pre-multiplying (615) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$
\begin{aligned}
&r^2\left(2r^2-1\right)^2+\cos(\phi_2)\left(-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right)\right)\\
&+\left(4r^4\left(r^2-1\right)-4r^2\left(r^2-1\right)\right)\cos^2(\beta)+\left(4r^2\left(r^2-1\right)-8r^4\left(r^2-1\right)\right)\cos(\beta)\\
&+\sin(\phi_2)\left(4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\right)\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),
\end{aligned}
\tag{616}
$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta) \tag{617}$$

$$B=-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right), \tag{618}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{619}$$

$$
\begin{aligned}
\implies \phi_2 = \cos^{-1}&\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}
\tag{620}
$$

and yields two solutions of ϕ_2.

Pre-multiplying (615) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$
\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)-4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}
\tag{621}
$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)
\end{aligned}
\tag{622}
$$

$$
\begin{aligned}
D=&4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right),
\end{aligned}
\tag{623}
$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta) &= \frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\end{aligned} \tag{624}$$

$$\begin{aligned}\implies \phi_1 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),\end{aligned} \tag{625}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (615) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right) \\ &-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta) \\ &+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\ &+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2) \\ &+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right) \\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r.\end{aligned} \tag{626}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}\cos\left(\phi_3-\theta\right) &= \frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\end{aligned} \tag{627}$$

$$\begin{aligned}\implies \phi_3 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),\end{aligned} \tag{628}$$

which yields two solutions for each value of ϕ_2.

1.15.7 $R^-|R_\beta^+G^+L_\beta^+|L^-$ Paths

For a $R_{\phi_1}^-|R_\beta^+G_{\phi_2}^+L_\beta^+|L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{L^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{629}$$

Pre-multiplying (629) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&r^2\left(2r^2-1\right)^2+\cos(\phi_2)\left(-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right)\right)\\
&+\left(4r^4\left(r^2-1\right)-4r^2\left(r^2-1\right)\right)\cos^2(\beta)+\left(4r^2\left(r^2-1\right)-8r^4\left(r^2-1\right)\right)\cos(\beta)\\
&+\sin(\phi_2)\left(4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\right)\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),
\end{aligned}\tag{630}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\tag{631}$$

$$B=-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right),\tag{632}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\tag{633}$$

$$\begin{aligned}
\implies\phi_2=\cos^{-1}&\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}\tag{634}$$

and yields two solutions of ϕ_2.

Pre-multiplying (629) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)-4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r.
\end{aligned}\tag{635}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)
\end{aligned}\tag{636}$$

$$\begin{aligned}
D=&4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right),
\end{aligned}\tag{637}$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta) &= \frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{638}$$

$$\begin{aligned}\implies \phi_1 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{639}$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (629) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\ &-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\ &+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\ &+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\ &+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \end{aligned} \tag{640}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}\cos\left(\phi_3-\theta\right) &= \frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{641}$$

$$\begin{aligned}\implies \phi_3 = \cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{642}$$

which yields two solutions for each value of ϕ_2.

1.15.8 $L^-|L_\beta^+G^+R_\beta^+|R^-$ Paths

For a $L_{\phi_1}^-|L_\beta^+G_{\phi_2}^+R_\beta^+|R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{R^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\ \alpha_{21}&\alpha_{22}&\alpha_{23}\\ \alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{643}$$

Pre-multiplying (643) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}
&r^2\left(2r^2-1\right)^2+\cos(\phi_2)\left(-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right)\right)\\
&+\left(4r^4\left(r^2-1\right)-4r^2\left(r^2-1\right)\right)\cos^2(\beta)+\left(4r^2\left(r^2-1\right)-8r^4\left(r^2-1\right)\right)\cos(\beta)\\
&+\sin(\phi_2)\left(4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\right)\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (644)
\end{aligned}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta) \qquad (645)$$

$$B=-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right), \qquad (646)$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \qquad (647)$$

$$\begin{aligned}
\Longrightarrow \phi_2=&\cos^{-1}\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (648)
\end{aligned}$$

and yields two solutions of ϕ_2.

Pre-multiplying (643) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)-4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (649)
\end{aligned}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right) \qquad (650)\\
D=&4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right), \qquad (651)
\end{aligned}$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_1-\theta) = {} & \frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\
& + \frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
& + \frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (652)\\
\implies \phi_1 = {} & \cos^{-1}\left(\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\right.\\
& + \frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
& \left.+ \frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (653)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

Pre-multiplying (643) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$
\begin{aligned}
& -4r^7+4r^5-r^3+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right) \\
& -\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta) \\
& +4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\
& +r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2) \\
& +2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right) \\
& +\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r.
\end{aligned} \tag{654}
$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$
\begin{aligned}
\cos\left(\phi_3-\theta\right) = {} & \frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\
& + \frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
& + \frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (655)\\
\implies \phi_3 = {} & \cos^{-1}\left(\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\right.\\
& + \frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\
& \left.+ \frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (656)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

1.16 $C|C_\mu C_\mu|C_\mu C$ Paths

1.16.1 $L^+|L^-_\mu R^-_\mu|R^+_\mu L^+$ Paths

For a $L^+_{\phi_1}|L^-_\mu R^-_\mu|R^+_\mu L^+_{\phi_2}$ path, the equation to be solved is:

$$
\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{657}
$$

Pre-multiplying (657) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned}
&16r^8-32r^6+24r^4-8r^2+1+\left(-16r^8+32r^6-16r^4\right)\cos^3(\mu)+\left(48r^8-96r^6+56r^4-8r^2\right)\cos^2(\mu)\\
&+\left(-48r^8+96r^6-64r^4+16r^2\right)\cos(\mu)\\
=&r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),
\end{aligned}\tag{658}$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ.

Pre-multiplying (657) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}
&r\Big(-40r^8+80r^6-52r^4+8\left(r^2-1\right)r^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+12r^2\\
&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_1)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right)\\
&+4\left(r^2-1\right)^2r^4\cos(3\mu)+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)\\
=&\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}\tag{659}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}
A=&8\left(r^2-1\right)r^3\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)\\
B=&8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right),
\end{aligned}\tag{660}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta)=&\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}
\end{aligned}\tag{661}$$

$$\begin{aligned}
\Longrightarrow\phi_1=\cos^{-1}\Bigg(&\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}\tag{662}$$

which yields two solutions for each value of μ.

Pre-multiplying (657) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}
&r\Big(-40r^8+80r^6-52r^4-8\left(r^2-1\right)^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_2)\left(2r^2\cos(\mu)-2r^2+1\right)+12r^2+4\left(r^2-1\right)^2r^4\cos(3\mu)\\
&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_2)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right)\\
&+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}\tag{663}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$C=-8\left(r^2-1\right)^2r\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2r^2\cos(\mu)-2r^2+1\right)\tag{664}$$

$$D=8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right),\tag{665}$$

it is obtained that

$$\cos\left(\phi_2 - \sigma\right) = \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{13}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^5\cos(3\mu) - 4\left(15r^6 - 30r^4 + 19r^2 - 4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} + \frac{4\left(6r^6 - 12r^4 + 7r^2 - 1\right)r^3\cos(2\mu) - \alpha_{33}r + r}{\sqrt{C^2+D^2}} \tag{666}$$

$$\implies \phi_2 = \cos^{-1}\left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{13}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^5\cos(3\mu) - 4\left(15r^6 - 30r^4 + 19r^2 - 4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} + \frac{4\left(6r^6 - 12r^4 + 7r^2 - 1\right)r^3\cos(2\mu) - \alpha_{33}r + r}{\sqrt{C^2+D^2}}\right) + \tan^{-1}\left(\frac{C}{D}\right), \tag{667}$$

which yields two solutions for each value of μ.

1.16.2 $R^+|R^-_\mu L^-_\mu|L^+_\mu R^+$ Paths

For a $R^+_{\phi_1}|R^-_\mu L^-_\mu|L^+_\mu R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{668}$$

Pre-multiplying (668) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned}&16r^8 - 32r^6 + 24r^4 - 8r^2 + 1 + \left(-16r^8 + 32r^6 - 16r^4\right)\cos^3(\mu) + \left(48r^8 - 96r^6 + 56r^4 - 8r^2\right)\cos^2(\mu)\\ &+ \left(-48r^8 + 96r^6 - 64r^4 + 16r^2\right)\cos(\mu)\\ =&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right),\end{aligned} \tag{669}$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ.

Pre-multiplying (668) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}&r\Big(-40r^8 + 80r^6 - 52r^4 + 8\left(r^2-1\right)r^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu) - 2r^2 + 1\right) + 12r^2\\ &+ 8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_1)\left(\left(r^2-1\right)\left(6r^4 + \left(2r^2-1\right)r^2\cos(2\mu) - 3r^2 + 1\right) - r^2\left(8r^4 - 12r^2 + 5\right)\cos(\mu)\right)\\ &+ 4\left(r^2-1\right)^2 r^4\cos(3\mu) + 4\left(15r^6 - 30r^4 + 19r^2 - 4\right)r^2\cos(\mu) - 4\left(6r^6 - 12r^4 + 7r^2 - 1\right)r^2\cos(2\mu) - 1\Big)\\ =&\alpha_{31}\sqrt{1-r^2} - \alpha_{33}r.\end{aligned} \tag{670}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}A =&8\left(r^2-1\right)r^3\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu) - 2r^2 + 1\right)\\ B =&8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(\left(r^2-1\right)\left(6r^4 + \left(2r^2-1\right)r^2\cos(2\mu) - 3r^2 + 1\right) - r^2\left(8r^4 - 12r^2 + 5\right)\cos(\mu)\right),\end{aligned} \tag{671}$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{31}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^5\cos(3\mu) - 4\left(15r^6 - 30r^4 + 19r^2 - 4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} + \frac{4\left(6r^6 - 12r^4 + 7r^2 - 1\right)r^3\cos(2\mu) - \alpha_{33}r + r}{\sqrt{A^2+B^2}} \tag{672}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{31}\sqrt{1-r^2} - 4\left(r^2-1\right)^2 r^5\cos(3\mu) - 4\left(15r^6 - 30r^4 + 19r^2 - 4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} + \frac{4\left(6r^6 - 12r^4 + 7r^2 - 1\right)r^3\cos(2\mu) - \alpha_{33}r + r}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{673}$$

which yields two solutions for each value of μ.

Pre-multiplying (668) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$
\begin{aligned}
&r\Big(-40r^8+80r^6-52r^4-8\left(r^2-1\right)^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_2)\left(2r^2\cos(\mu)-2r^2+1\right)+12r^2+4\left(r^2-1\right)^2r^4\cos(3\mu)\\
&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_2)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right)\\
&+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r.
\end{aligned}
\tag{674}
$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
C=-8\left(r^2-1\right)^2r\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2r^2\cos(\mu)-2r^2+1\right)
\tag{675}
$$

$$
D=8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right),
\tag{676}
$$

it is obtained that

$$
\begin{aligned}
\cos\left(\phi_2-\sigma\right)=&\frac{40r^9-80r^7+52r^5-12r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}
\end{aligned}
\tag{677}
$$

$$
\begin{aligned}
\implies \phi_2=\cos^{-1}\Bigg(&\frac{40r^9-80r^7+52r^5-12r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),
\end{aligned}
\tag{678}
$$

which yields two solutions for each value of μ.

1.16.3 $R^-|R_\mu^+L_\mu^+|L_\mu^-R^-$ Paths

For a $R_{\phi_1}^-|R_\mu^+L_\mu^+|L_\mu^-R_{\phi_2}^-$ path, the equation to be solved is:

$$
\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\ \alpha_{21}&\alpha_{22}&\alpha_{23}\\ \alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}
\tag{679}
$$

Pre-multiplying (679) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$
\begin{aligned}
&16r^8-32r^6+24r^4-8r^2+1+\left(-16r^8+32r^6-16r^4\right)\cos^3(\mu)+\left(48r^8-96r^6+56r^4-8r^2\right)\cos^2(\mu)\\
&+\left(-48r^8+96r^6-64r^4+16r^2\right)\cos(\mu)\\
=&r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),
\end{aligned}
\tag{680}
$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ.

Pre-multiplying (679) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$
\begin{aligned}
&r\Big(-40r^8+80r^6-52r^4+8\left(r^2-1\right)r^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+12r^2\\
&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_1)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right)\\
&+4\left(r^2-1\right)^2r^4\cos(3\mu)+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)\\
=&\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}
\tag{681}
$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$
\begin{aligned}
A=&8\left(r^2-1\right)r^3\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)\\
B=&8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right),
\end{aligned}
\tag{682}
$$

it is obtained that

$$\cos(\phi_1-\theta) = \frac{40r^9-80r^7+52r^5-12r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} + \frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{683}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} + \frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right)+\tan^{-1}\left(\frac{A}{B}\right), \tag{684}$$

which yields two solutions for each value of μ.

Pre-multiplying (679) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned} &r\Big(-40r^8+80r^6-52r^4-8\left(r^2-1\right)^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_2)\left(2r^2\cos(\mu)-2r^2+1\right)+12r^2+4\left(r^2-1\right)^2 r^4\cos(3\mu) \\ &+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_2)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right) \\ &+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \end{aligned} \tag{685}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -8\left(r^2-1\right)^2 r\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2r^2\cos(\mu)-2r^2+1\right) \tag{686}$$

$$D = 8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right), \tag{687}$$

it is obtained that

$$\cos(\phi_2-\sigma) = \frac{40r^9-80r^7+52r^5-12r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} + \frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}} \tag{688}$$

$$\implies \phi_2 = \cos^{-1}\left(\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} + \frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \tag{689}$$

which yields two solutions for each value of μ.

1.16.4 $L^-|L_\mu^+R_\mu^+|R_\mu^-L^-$ Paths

For a $L_{\phi_1}^-|L_\mu^+R_\mu^+|R_\mu^-L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{690}$$

Pre-multiplying (690) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned} &16r^8-32r^6+24r^4-8r^2+1+\left(-16r^8+32r^6-16r^4\right)\cos^3(\mu)+\left(48r^8-96r^6+56r^4-8r^2\right)\cos^2(\mu) \\ &+\left(-48r^8+96r^6-64r^4+16r^2\right)\cos(\mu) \\ =&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \end{aligned} \tag{691}$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ.

Pre-multiplying (690) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$
\begin{aligned}
& r\Big(-40r^8+80r^6-52r^4+8\left(r^2-1\right)r^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+12r^2 \\
& +8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_1)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right) \\
& +4\left(r^2-1\right)^2r^4\cos(3\mu)+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big) \\
=& \alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \qquad (692)
\end{aligned}
$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$
\begin{aligned}
A =& 8\left(r^2-1\right)r^3\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right) \\
B =& 8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right), \qquad (693)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_1-\theta) =& \frac{40r^9-80r^7+52r^5-12r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}} \qquad (694) \\
\implies \phi_1 = \cos^{-1}\Bigg(& \frac{40r^9-80r^7+52r^5-12r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right), \qquad (695)
\end{aligned}
$$

which yields two solutions for each value of μ.

Pre-multiplying (690) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$
\begin{aligned}
& r\Big(-40r^8+80r^6-52r^4-8\left(r^2-1\right)^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_2)\left(2r^2\cos(\mu)-2r^2+1\right)+12r^2+4\left(r^2-1\right)^2r^4\cos(3\mu) \\
& +8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_2)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right) \\
& +4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (696)
\end{aligned}
$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C =& -8\left(r^2-1\right)^2r\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2r^2\cos(\mu)-2r^2+1\right) \qquad (697) \\
D =& 8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right), \qquad (698)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos\left(\phi_2-\sigma\right) =& \frac{40r^9-80r^7+52r^5-12r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}} \qquad (699) \\
\implies \phi_2 = \cos^{-1}\Bigg(& \frac{40r^9-80r^7+52r^5-12r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (700)
\end{aligned}
$$

which yields two solutions for each value of μ.

1.17 $CC_\mu|C_\mu C_\mu|C_\mu C$ Paths

1.17.1 $L^+R^+_\mu|R^-_\mu L^-_\mu|L^+_\mu R^+$ Paths

For a $L^+_{\phi_1}R^+_\mu|R^-_\mu L^-_\mu|L^+_\mu R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{701}$$

Pre-multiplying (701) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned}&32r^{10}-64r^8+56r^6-32r^4+10r^2+\left(32r^{10}-96r^8+96r^6-32r^4\right)\cos^4(\mu)+\left(-128r^{10}+352r^8-320r^6+96r^4\right)\cos^3(\mu)\\&+\left(192r^{10}-480r^8+408r^6-136r^4+16r^2\right)\cos^2(\mu)+\left(-128r^{10}+288r^8-240r^6+104r^4-24r^2\right)\cos(\mu)-1\\=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned} \tag{702}$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ.

Pre-multiplying (701) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)\\&-4\left(r^2-1\right)^3r^5\cos(4\mu)+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2\\&+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)+2\left(r^2-1\right)r\cos(\phi_1)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)\\&-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2\\&+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)\\&+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r\\=&\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r.\end{aligned} \tag{703}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}A=&16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\right)\\B=&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big),\end{aligned} \tag{704}$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta)=&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\\&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}}\end{aligned} \tag{705}$$

$$\begin{aligned}\implies\phi_1=\cos^{-1}\Bigg(&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\\&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right),\end{aligned} \tag{706}$$

which yields two solutions for each value of μ.

Pre-multiplying (701) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$
\begin{aligned}
&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3r^5\cos(4\mu)\\
&+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\
&+2\left(r^2-1\right)r\cos(\phi_2)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\
&-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (707)
\end{aligned}
$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C=&16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big) \qquad (708)\\
D=&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \qquad (709)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos\left(\phi_2-\sigma\right)=&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}} \qquad (710)\\
\implies \phi_2=\cos^{-1}\Bigg(&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (711)
\end{aligned}
$$

which yields two solutions for each value of μ.

1.17.2 $R^+L_\mu^+|L_\mu^-R_\mu^-|R_\mu^+L^+$ Paths

For a $R_{\phi_1}^+L_\mu^+|L_\mu^-R_\mu^-|R_\mu^+L_{\phi_2}^+$ path, the equation to be solved is:

$$
\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \qquad (712)
$$

Pre-multiplying (712) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$
\begin{aligned}
&32r^{10}-64r^8+56r^6-32r^4+10r^2+\left(32r^{10}-96r^8+96r^6-32r^4\right)\cos^4(\mu)+\left(-128r^{10}+352r^8-320r^6+96r^4\right)\cos^3(\mu)\\
&+\left(192r^{10}-480r^8+408r^6-136r^4+16r^2\right)\cos^2(\mu)+\left(-128r^{10}+288r^8-240r^6+104r^4-24r^2\right)\cos(\mu)-1\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (713)
\end{aligned}
$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ.

Pre-multiplying (712) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$
\begin{aligned}
&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3r^5\cos(4\mu)\\
&+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\
&+2\left(r^2-1\right)r\cos(\phi_1)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\
&-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (714)
\end{aligned}
$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$
\begin{aligned}
A =&16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\
B =&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \qquad (715)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_1-\theta)=&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}} \qquad (716)\\
\implies \phi_1=\cos^{-1}&\Bigg(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right), \qquad (717)
\end{aligned}
$$

which yields two solutions for each value of μ.

Pre-multiplying (712) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$
\begin{aligned}
&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3r^5\cos(4\mu)\\
&+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\
&+2\left(r^2-1\right)r\cos(\phi_2)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\
&-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \qquad (718)
\end{aligned}
$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$,

$\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C =& 16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\right) && (719)\\
D =& 2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), && (720)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos\left(\phi_2-\sigma\right) =& \frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}} && (721)\\
\implies \phi_2 =& \cos^{-1}\Bigg(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), && (722)
\end{aligned}$$

which yields two solutions for each value of μ.

1.17.3 $R^-L^-_\mu|L^+_\mu R^+_\mu|R^-_\mu L^-$ Paths

For a $R^-_{\phi_1}L^-_\mu|L^+_\mu R^+_\mu|R^-_\mu L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \quad (723)$$

Pre-multiplying (723) with $\mathbf{u}^T_{R^-}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&32r^{10}-64r^8+56r^6-32r^4+10r^2+\left(32r^{10}-96r^8+96r^6-32r^4\right)\cos^4(\mu)+\left(-128r^{10}+352r^8-320r^6+96r^4\right)\cos^3(\mu)\\
&+\left(192r^{10}-480r^8+408r^6-136r^4+16r^2\right)\cos^2(\mu)+\left(-128r^{10}+288r^8-240r^6+104r^4-24r^2\right)\cos(\mu)-1\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), && (724)
\end{aligned}$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ.

Pre-multiplying (723) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)\\
&-4\left(r^2-1\right)^3 r^5\cos(4\mu)+16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2\\
&+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)+2\left(r^2-1\right)r\cos(\phi_1)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)\\
&-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2\\
&+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)\\
&+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r\\
=&\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. && (725)
\end{aligned}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}
A =& 16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\
B =& 2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big),
\end{aligned} \tag{726}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta) =& \frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}}
\end{aligned} \tag{727}$$

$$\begin{aligned}
\implies \phi_1 = \cos^{-1}\Bigg(& \frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned} \tag{728}$$

which yields two solutions for each value of μ.

Pre-multiplying (723) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3 r^5\cos(4\mu)\\
&+16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\
&+2\left(r^2-1\right)r\cos(\phi_2)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\
&-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned} \tag{729}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = 16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big) \tag{730}$$

$$\begin{aligned}
D =& 2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big),
\end{aligned} \tag{731}$$

it is obtained that

$$\begin{aligned}\cos\left(\phi_2-\sigma\right)=&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}}\end{aligned}\tag{732}$$

$$\begin{aligned}\implies\phi_2=\cos^{-1}&\left(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\right.\\&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\&\left.+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right),\end{aligned}\tag{733}$$

which yields two solutions for each value of μ.

1.17.4 $L^-R^-_\mu|R^+_\mu L^+_\mu|L^-_\mu R^-$ Paths

For a $L^-_{\phi_1}R^-_\mu|R^+_\mu L^+_\mu|L^-_\mu R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}\tag{734}$$

Pre-multiplying (734) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}&32r^{10}-64r^8+56r^6-32r^4+10r^2+\left(32r^{10}-96r^8+96r^6-32r^4\right)\cos^4(\mu)+\left(-128r^{10}+352r^8-320r^6+96r^4\right)\cos^3(\mu)\\&+\left(192r^{10}-480r^8+408r^6-136r^4+16r^2\right)\cos^2(\mu)+\left(-128r^{10}+288r^8-240r^6+104r^4-24r^2\right)\cos(\mu)-1\\=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned}\tag{735}$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ.

Pre-multiplying (734) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3r^5\cos(4\mu)\\&+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\&+2\left(r^2-1\right)r\cos(\phi_1)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\&-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.\end{aligned}\tag{736}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}A=&16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\B=&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big),\end{aligned}\tag{737}$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_1-\theta) = & \frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}} \\
& + \frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\
& + \frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}} \qquad (738) \\
\implies \phi_1 = \cos^{-1} & \Bigg(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}} \\
& + \frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\
& + \frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}} \Bigg) + \tan^{-1}\left(\frac{A}{B}\right), \qquad (739)
\end{aligned}
$$

which yields two solutions for each value of μ.

Pre-multiplying (734) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$
\begin{aligned}
& -140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3 r^5\cos(4\mu) \\
& +16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big) \\
& +2\left(r^2-1\right)r\cos(\phi_2)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu) \\
& +4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu) \\
& +\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu) \\
& -4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \qquad (740)
\end{aligned}
$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C = & 16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big) \qquad (741) \\
D = & 2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu) \\
& +4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu) \\
& +\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \qquad (742)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos\left(\phi_2-\sigma\right) = & \frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}} \\
& + \frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \\
& + \frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}} \qquad (743) \\
\implies \phi_2 = \cos^{-1} & \Bigg(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}} \\
& + \frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \\
& + \frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}} \Bigg) + \tan^{-1}\left(\frac{C}{D}\right), \qquad (744)
\end{aligned}
$$

which yields two solutions for each value of μ.

References

[1] S. Li, D. P. Kumar, S. Darbha, and Y. Zhou, "Time-optimal convexified reeds-shepp paths on a sphere," *arXiv preprint arXiv:2504.00966*, 2025.