

Time-optimal Convexified Reeds-Shepp Path Generation on a Sphere

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1 Derivation

The spherical convexified Reeds-Shepp model:

$$\frac{d\mathbf{X}_{\mathbf{v}}}{dt} = v(t)\mathbf{T}_{\mathbf{v}}(t), \quad (1)$$

$$\frac{d\mathbf{T}_{\mathbf{v}}}{dt} = -v(t)\mathbf{X}_{\mathbf{v}}(t) + u_g(t)\mathbf{N}_{\mathbf{v}}(t), \quad (2)$$

$$\frac{d\mathbf{N}_{\mathbf{v}}}{dt} = -u_g(t)\mathbf{T}_{\mathbf{v}}(t), \quad (3)$$

$$\mathbf{R}(0) = I_3, \quad \mathbf{R}(T) = R_f, \quad (4)$$

where $v \in [-1, 1]$ and $u_g \in [-U_{max}, U_{max}]$, $\mathbf{R}(t) = [\mathbf{X}_{\mathbf{v}}(t), \mathbf{T}_{\mathbf{v}}(t), \mathbf{N}_{\mathbf{v}}(t)] \in SO(3)$ and R_f is the desired terminal configuration.

Note that the model is equivalent to:

$$\frac{d\mathbf{R}(t)}{dt} = \underbrace{\mathbf{R}(t) \begin{pmatrix} 0 & -v & 0 \\ v & 0 & -u_g \\ 0 & u_g & 0 \end{pmatrix}}_{\Omega}. \quad (5)$$

Since v and u_g remain constant on each segment, the solution of (5) on each segment is

$$\mathbf{R}(t) = \mathbf{R}(t_i) \left(e^{\Omega^T(t-t_i)} \right)^T, \quad (6)$$

where t_i denotes the initial time of the i^{th} segment. It is simpler to deal with arc angles instead of time, hence, we define $\phi = \omega(t - t_i) = \sqrt{v^2 + u_g^2}(t - t_i)$, where ϕ represents the arc angle, and ω denotes the angular frequency. Let $\hat{\Omega} = \frac{1}{\sqrt{v^2 + u_g^2}}\Omega$.

We define $\mathbf{M}(\phi) := \left(e^{\hat{\Omega}^T \phi} \right)^T = \left(e^{\Omega^T(t-t_i)} \right)^T$. Substituting specific values of v and u_g , $\mathbf{M}(\phi)$ for each type of segment can be calculated using the Euler-Rodriguez formula. Hence, we obtain

$$\mathbf{M}_{G^+}(\phi) = \begin{pmatrix} c(\phi) & -s(\phi) & 0 \\ s(\phi) & c(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (7)$$

$$\mathbf{M}_{L^+}(r, \phi) = \begin{pmatrix} \eta_{11} & -rs(\phi) & \eta_{13} \\ rs(\phi) & c(\phi) & -\eta_{23} \\ \eta_{13} & \eta_{23} & \eta_{33} \end{pmatrix}, \quad (8)$$

$$\mathbf{M}_{R^+}(r, \phi) = \begin{pmatrix} \eta_{11} & -rs(\phi) & -\eta_{13} \\ rs(\phi) & c(\phi) & \eta_{23} \\ -\eta_{13} & -\eta_{23} & \eta_{33} \end{pmatrix}, \quad (9)$$

$$\mathbf{M}_{L^0}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{pmatrix}, \quad (10)$$

$$\mathbf{M}_{G^-}(\phi) = \mathbf{M}_{G^+}^T(\phi), \quad (11)$$

$$\mathbf{M}_{L^-}(r, \phi) = \mathbf{M}_{R^+}^T(r, \phi), \quad (12)$$

$$\mathbf{M}_{R^-}(r, \phi) = \mathbf{M}_{L^+}^T(r, \phi), \quad (13)$$

$$\mathbf{M}_{R^0}(\phi) = \mathbf{M}_{L^0}^T(\phi), \quad (14)$$

where $\eta_{11} = 1 - (1 - c(\phi))r^2$, $\eta_{13} = (1 - c(\phi))r\sqrt{1 - r^2}$, $\eta_{23} = s(\phi)\sqrt{1 - r^2}$, $\eta_{33} = c(\phi) + (1 - c(\phi))r^2$, $c(\phi) = \cos(\phi)$, and $s(\phi) = \sin(\phi)$.

The corresponding axial vectors are $\mathbf{u}_{G^+} := [0, 0, 1]^T$, $\mathbf{u}_{L^+} := [\sqrt{1 - r^2}, 0, r]^T$, $\mathbf{u}_{R^+} := [-\sqrt{1 - r^2}, 0, r]^T$, $\mathbf{u}_{L^0} := [1, 0, 0]^T$, $\mathbf{u}_{G^-} := [0, 0, -1]^T$, $\mathbf{u}_{L^-} := [-\sqrt{1 - r^2}, 0, -r]^T$, $\mathbf{u}_{R^-} := [\sqrt{1 - r^2}, 0, -r]^T$, $\mathbf{u}_{R^0} := [-1, 0, 0]^T$.

The sufficient list of optimal paths is characterized as follows:

Theorem 1. For $U_{max} \geq 1$ (or $r \leq \frac{1}{\sqrt{2}}$), the optimal path may be restricted to the following types, together with their symmetric forms:

$$\begin{aligned} & C, G, T, CC, GC, C|C, TC, \\ & CC_\psi|C, CGC, C|C_\beta G, CTC, \\ & C|C_\psi C_\psi|C, CGC_\beta|C, CC_\mu|C_\mu C, \\ & C|C_\beta GC_\beta|C, C|C_\mu C_\mu|C_\mu C, CC_\mu|C_\mu C_\mu|C_\mu C, \\ & \text{where } 0 < \psi \leq \arctan\left(\frac{1}{\sqrt{U_{max}^4 - 1}}\right) + \frac{\pi}{2}, \beta = \arctan\left(\frac{1}{\sqrt{U_{max}^4 - 1}}\right) + \frac{\pi}{2}, \text{ and } 0 < \mu < \arctan\left(\frac{1}{\sqrt{U_{max}^4 - 1}}\right) + \frac{\pi}{2}. \end{aligned}$$

Given the sufficient list above, for each path, candidate solutions must be generated using inverse kinematics, based on an initial configuration, a desired terminal configuration, and a U_{max} (or r). In this note, we employ rotation matrices and their associated axial vectors to derive closed-form expressions for the angles of each path in the sufficient list.

1.1 C Paths

1.1.1 L^+ Paths

For a $L_{\phi_1}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (15)$$

Pre-multiplying (15) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$(r^2 - 1) \cos(\phi_1) - r^2 = -\alpha_{33}, \quad (16)$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \quad (17)$$

and yields two solutions of ϕ_1 .

1.1.2 R^+ Paths

For a $R_{\phi_1}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (18)$$

Pre-multiplying (18) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$(r^2 - 1) \cos(\phi_1) - r^2 = -\alpha_{33}, \quad (19)$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \quad (20)$$

and yields two solutions of ϕ_1 .

1.1.3 R^- Paths

For a $R_{\phi_1}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (21)$$

Pre-multiplying (21) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$(r^2 - 1) \cos(\phi_1) - r^2 = -\alpha_{33}, \quad (22)$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \quad (23)$$

and yields two solutions of ϕ_1 .

1.1.4 L^- Paths

For a $L_{\phi_1}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (24)$$

Pre-multiplying (24) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$(r^2 - 1) \cos(\phi_1) - r^2 = -\alpha_{33}, \quad (25)$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \quad (26)$$

and yields two solutions of ϕ_1 .

1.2 G Paths

1.2.1 G^+ Paths

For a $G_{\phi_1}^+$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (27)$$

Pre-multiplying (27) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$(1 - r^2) \cos(\phi_1) - r^2 = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r), \quad (28)$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + (\alpha_{33} - 1)r)}{r^2 - 1}, \quad (29)$$

and yields two solutions of ϕ_1 .

1.2.2 G^- Paths

For a $G_{\phi_1}^-$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (30)$$

Pre-multiplying (30) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$(1 - r^2) \cos(\phi_1) - r^2 = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r), \quad (31)$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + (\alpha_{33} - 1)r)}{r^2 - 1}, \quad (32)$$

and yields two solutions of ϕ_1 .

1.3 T Paths

1.3.1 L^0 Paths

For a $L^0_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (33)$$

Pre-multiplying (33) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$\cos(\phi_1) = \alpha_{33}, \quad (34)$$

and yields two solutions of ϕ_1 .

1.3.2 R^0 Paths

For a $R^0_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (35)$$

Pre-multiplying (35) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$\cos(\phi_1) = \alpha_{33}, \quad (36)$$

and yields two solutions of ϕ_1 .

1.4 CC Paths

1.4.1 L^+R^+ Paths

For a $L^+_{\phi_1} R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (37)$$

Pre-multiplying (37) with $\mathbf{u}_{G^-}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$-2r^3 + 2(r^2 - 1)r \cos(\phi_1) + r = \alpha_{31}\sqrt{1 - r^2} - \alpha_{33}r, \quad (38)$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (39)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (37) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$-2r^3 + 2(r^2 - 1)r \cos(\phi_2) + r = \alpha_{13}(-\sqrt{1 - r^2}) - \alpha_{33}r, \quad (40)$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1 - r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (41)$$

and yields two solutions of ϕ_2 .

1.4.2 R^+L^+ Paths

For a $R_{\phi_1}^+L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (42)$$

Pre-multiplying (42) with \mathbf{u}_{G-}^T and post-multiplying \mathbf{u}_{L+} :

$$-2r^3 + 2(r^2 - 1)r \cos(\phi_1) + r = \alpha_{31}(-\sqrt{1-r^2}) - \alpha_{33}r, \quad (43)$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (44)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (42) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_{G-} :

$$-2r^3 + 2(r^2 - 1)r \cos(\phi_2) + r = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r, \quad (45)$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (46)$$

and yields two solutions of ϕ_2 .

1.4.3 R^-L^- Paths

For a $R_{\phi_1}^-L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1)\mathbf{M}_{L^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (47)$$

Pre-multiplying (47) with \mathbf{u}_{G+}^T and post-multiplying \mathbf{u}_{L-} :

$$-2r^3 + 2(r^2 - 1)r \cos(\phi_1) + r = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r, \quad (48)$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (49)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (47) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_{G+} :

$$-2r^3 + 2(r^2 - 1)r \cos(\phi_2) + r = \alpha_{13}(-\sqrt{1-r^2}) - \alpha_{33}r, \quad (50)$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (51)$$

and yields two solutions of ϕ_2 .

1.4.4 L^-R^- Paths

For a $L_{\phi_1}^-R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1)\mathbf{M}_{R^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (52)$$

Pre-multiplying (52) with \mathbf{u}_{G+}^T and post-multiplying \mathbf{u}_{R-} :

$$-2r^3 + 2(r^2 - 1)r \cos(\phi_1) + r = \alpha_{31}(-\sqrt{1-r^2}) - \alpha_{33}r, \quad (53)$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (54)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (52) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{G+} :

$$-2r^3 + 2(r^2 - 1)r \cos(\phi_2) + r = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r, \quad (55)$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (56)$$

and yields two solutions of ϕ_2 .

1.5 GC Paths

1.5.1 G^+L^+ Paths

For a $G_{\phi_1}^+ L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1)\mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (57)$$

Pre-multiplying (57) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{L+} :

$$(1 - r^2) \cos(\phi_1) - r^2 = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}(-\sqrt{1-r^2}) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \quad (58)$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r(\alpha_{13}(-\sqrt{1-r^2}) + \alpha_{31}\sqrt{1-r^2} + (\alpha_{33} - 1)r)}{r^2 - 1}, \quad (59)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (57) with \mathbf{u}_{G+}^T and post-multiplying \mathbf{u}_{G-} :

$$(r^2 - 1) \cos(\phi_2) - r^2 = -\alpha_{33}, \quad (60)$$

which gives

$$\cos(\phi_2) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \quad (61)$$

and yields two solutions of ϕ_2 .

1.5.2 G^+R^+ Paths

For a $G_{\phi_1}^+ R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1)\mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (62)$$

Pre-multiplying (62) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{R+} :

$$(r^2 - 1) \cos(\phi_1) - r^2 = \alpha_{11}(r^2 - 1) + r(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r), \quad (63)$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r)}{r^2 - 1}, \quad (64)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (62) with \mathbf{u}_{G+}^T and post-multiplying \mathbf{u}_{G-} :

$$(r^2 - 1) \cos(\phi_2) - r^2 = -\alpha_{33}, \quad (65)$$

which gives

$$\cos(\phi_2) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \quad (66)$$

and yields two solutions of ϕ_2 .

1.5.3 G^-R^- Paths

For a $G_{\phi_1}^- R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1)\mathbf{M}_{R^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (67)$$

Pre-multiplying (67) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{R^-} :

$$(r^2 - 1) \cos(\phi_1) - r^2 = \alpha_{11} (r^2 - 1) - r (\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r), \quad (68)$$

which gives

$$\cos(\phi_1) = \alpha_{11} - \frac{r (\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + (\alpha_{33} - 1) r)}{r^2 - 1}, \quad (69)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (67) with $\mathbf{u}_{G^-}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$(r^2 - 1) \cos(\phi_2) - r^2 = -\alpha_{33}, \quad (70)$$

which gives

$$\cos(\phi_2) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \quad (71)$$

and yields two solutions of ϕ_2 .

1.5.4 G^-L^- Paths

For a $G_{\phi_1}^- L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1)\mathbf{M}_{L^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (72)$$

Pre-multiplying (72) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$(1 - r^2) \cos(\phi_1) - r^2 = -(\alpha_{11} (r^2 - 1)) - r (\alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r), \quad (73)$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r (\alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + (\alpha_{33} - 1) r)}{r^2 - 1}, \quad (74)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (72) with $\mathbf{u}_{G^-}^T$ and post-multiplying \mathbf{u}_{G^+} :

$$(r^2 - 1) \cos(\phi_2) - r^2 = -\alpha_{33}, \quad (75)$$

which gives

$$\cos(\phi_2) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \quad (76)$$

and yields two solutions of ϕ_2 .

1.6 $C|C$ Paths

1.6.1 $L^+|L^-$ Paths

For a $L_{\phi_1}^+ | L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1)\mathbf{M}_{L^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (77)$$

Pre-multiplying (77) with $\mathbf{u}_{G^-}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$r (2r^2 - 1) - 2r (r^2 - 1) \cos(\phi_1) = \alpha_{33} r - \alpha_{31} \sqrt{1 - r^2}, \quad (78)$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (79)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (77) with \mathbf{u}_{L+}^T and post-multiplying \mathbf{u}_{G+} :

$$r(-2(r^2 - 1)\cos(\phi_2) + 2r^2 - 1) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r, \quad (80)$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (81)$$

and yields two solutions of ϕ_2 .

1.6.2 $R^+|R^-$ Paths

For a $R_{\phi_1}^+|R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{R^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (82)$$

Pre-multiplying (82) with \mathbf{u}_{G-}^T and post-multiplying \mathbf{u}_{R-} :

$$r(2r^2 - 1) - 2r(r^2 - 1)\cos(\phi_1) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r, \quad (83)$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (84)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (82) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_{G+} :

$$r(2r^2 - 1) - 2r(r^2 - 1)\cos(\phi_2) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}, \quad (85)$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (86)$$

and yields two solutions of ϕ_2 .

1.6.3 $R^-|R^+$ Paths

For a $R_{\phi_1}^-|R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1)\mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (87)$$

Pre-multiplying (87) with \mathbf{u}_{G+}^T and post-multiplying \mathbf{u}_{R+} :

$$r(2r^2 - 1) - 2r(r^2 - 1)\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \quad (88)$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (89)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (87) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_{G-} :

$$r(2r^2 - 1) - 2r(r^2 - 1)\cos(\phi_2) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r, \quad (90)$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (91)$$

and yields two solutions of ϕ_2 .

1.6.4 $L^-|L^+$ Paths

For a $L_{\phi_1}^-|L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1)\mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (92)$$

Pre-multiplying (92) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{L^+} :

$$r(2r^2 - 1) - 2r(r^2 - 1)\cos(\phi_1) = \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r, \quad (93)$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (94)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (92) with $\mathbf{u}_{L^-}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$r(2r^2 - 1) - 2r(r^2 - 1)\cos(\phi_2) = \alpha_{33}r - \alpha_{13}\sqrt{1 - r^2}, \quad (95)$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1 - r^2} + \alpha_{33}r + r}{2r - 2r^3}, \quad (96)$$

and yields two solutions of ϕ_2 .

1.7 TC Paths

1.7.1 L^0L^+ Paths

For a $L_{\phi_1}^0L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1)\mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (97)$$

Pre-multiplying (97) with $\mathbf{u}_{G^-}^T$ and post-multiplying \mathbf{u}_{L^+} :

$$-r\cos(\phi_1) = \alpha_{31}(-\sqrt{1 - r^2}) - \alpha_{33}r, \quad (98)$$

which gives

$$\cos(\phi_1) = \alpha_{33} + \frac{\alpha_{31}\sqrt{1 - r^2}}{r}, \quad (99)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (97) with $\mathbf{u}_{L^0}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$r\sqrt{1 - r^2}(\cos(\phi_2) - 1) = -\alpha_{13}, \quad (100)$$

which gives

$$\cos(\phi_2) = 1 - \frac{\alpha_{13}}{r\sqrt{1 - r^2}}, \quad (101)$$

and yields two solutions of ϕ_2 .

1.7.2 L^0L^- Paths

For a $L_{\phi_1}^0L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1)\mathbf{M}_{L^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (102)$$

Pre-multiplying (102) with $\mathbf{u}_{G^-}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$r\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1 - r^2}, \quad (103)$$

which gives

$$\cos(\phi_1) = \alpha_{33} - \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \quad (104)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (102) with $\mathbf{u}_{L^0}^T$ and post-multiplying \mathbf{u}_{G^+} :

$$r\sqrt{1-r^2}(\cos(\phi_2) - 1) = \alpha_{13}, \quad (105)$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{13}}{r\sqrt{1-r^2}} + 1, \quad (106)$$

and yields two solutions of ϕ_2 .

1.7.3 $R^0 R^-$ Paths

For a $R_{\phi_1}^0 R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1)\mathbf{M}_{R^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (107)$$

Pre-multiplying (107) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{R^-} :

$$-r \cos(\phi_1) = \alpha_{31} \left(-\sqrt{1-r^2} \right) - \alpha_{33}r, \quad (108)$$

which gives

$$\cos(\phi_1) = \alpha_{33} + \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \quad (109)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (107) with $\mathbf{u}_{R^0}^T$ and post-multiplying \mathbf{u}_{G^+} :

$$r\sqrt{1-r^2}(\cos(\phi_2) - 1) = -\alpha_{13}, \quad (110)$$

which gives

$$\cos(\phi_2) = 1 - \frac{\alpha_{13}}{r\sqrt{1-r^2}}, \quad (111)$$

and yields two solutions of ϕ_2 .

1.7.4 $R^0 R^+$ Paths

For a $R_{\phi_1}^0 R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1)\mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (112)$$

Pre-multiplying (112) with $\mathbf{u}_{G^+}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$r \cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \quad (113)$$

which gives

$$\cos(\phi_1) = \alpha_{33} - \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \quad (114)$$

and yields two solutions of ϕ_1 .

Pre-multiplying (112) with $\mathbf{u}_{R^0}^T$ and post-multiplying \mathbf{u}_{G^+} :

$$r\sqrt{1-r^2}(\cos(\phi_2) - 1) = \alpha_{13}, \quad (115)$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{13}}{r\sqrt{1-r^2}} + 1, \quad (116)$$

and yields two solutions of ϕ_2 .

1.8 $CC_\psi|C$ Paths

1.8.1 $L^+R_\psi^+|R^-$ Paths

For a $L_{\phi_1}^+R_\psi^+|R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1)\mathbf{M}_{R^+}(r, \psi)\mathbf{M}_{R^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (117)$$

Pre-multiplying (117) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{R^-} :

$$4r^2(r^2 - 1)\cos(\psi) - (1 - 2r^2)^2 = \alpha_{11}(r^2 - 1) - r(\alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r), \quad (118)$$

which gives

$$\cos(\psi) = \frac{-\alpha_{11} + 4r^4 + \alpha_{11}r^2 - \alpha_{33}r^2 - \alpha_{13}\sqrt{1 - r^2}r - \alpha_{31}\sqrt{1 - r^2}r - 4r^2 + 1}{4r^2(r^2 - 1)}, \quad (119)$$

and yields two solutions of ψ .

Pre-multiplying (117) with $\mathbf{u}_{G^-}^T$ and post-multiplying with \mathbf{u}_{R^-} :

$$\begin{aligned} & r \sin(\phi_1) (2 \sin(\psi) - 2r^2 \sin(\psi)) + r (4r^4 - 4(r^2 - 1)r^2 \cos(\psi) - 4r^2 + 1) - 4r (2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right) \cos(\phi_1) \\ & = \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r. \end{aligned} \quad (120)$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$ and defining $\cos \gamma := \frac{4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2})}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$, $\sin \gamma := \frac{r(2 \sin(\psi) - 2r^2 \sin(\psi))}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$. It is obtained that

$$\cos(\gamma + \phi_1) = -\frac{r(\alpha_{33} + 4(r^2 - 1)r^2 \cos(\psi) - (1 - 2r^2)^2) + \alpha_{31}\sqrt{1 - r^2}}{4\sqrt{-r^2(r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2(r^2 - 1)r^2 \cos(\psi) + 2r^2 - 1)}} \quad (121)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(-\frac{r(\alpha_{33} + 4(r^2 - 1)r^2 \cos(\psi) - (1 - 2r^2)^2) + \alpha_{31}\sqrt{1 - r^2}}{4\sqrt{-r^2(r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2(r^2 - 1)r^2 \cos(\psi) + 2r^2 - 1)}} \right) - \tan^{-1} \left(\frac{2r(\sin(\psi) - r^2 \sin(\psi))}{4r(2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right)} \right), \quad (122)$$

which yields two solutions for each value of ψ .

Pre-multiplying (117) with $\mathbf{u}_{L^+}^T$ and post-multiplying with \mathbf{u}_{G^+} :

$$\begin{aligned} & r \sin(\phi_2) (2r^2 \sin(\psi) - 2 \sin(\psi)) + r (4r^4 - 4(r^2 - 1)r^2 \cos(\psi) - 4r^2 + 1) - 4r (2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right) \cos(\phi_2) \\ & = \alpha_{13}\sqrt{1 - r^2} + \alpha_{33}r. \end{aligned} \quad (123)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$, it is obtained that

$$\cos(\phi_2 - \gamma) = -\frac{r(\alpha_{33} + 4(r^2 - 1)r^2 \cos(\psi) - (1 - 2r^2)^2) + \alpha_{13}\sqrt{1 - r^2}}{4\sqrt{-r^2(r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2(r^2 - 1)r^2 \cos(\psi) + 2r^2 - 1)}} \quad (124)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(-\frac{r(\alpha_{33} + 4(r^2 - 1)r^2 \cos(\psi) - (1 - 2r^2)^2) + \alpha_{13}\sqrt{1 - r^2}}{4\sqrt{-r^2(r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2(r^2 - 1)r^2 \cos(\psi) + 2r^2 - 1)}} \right) + \tan^{-1} \left(\frac{2r(\sin(\psi) - r^2 \sin(\psi))}{4r(2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right)} \right), \quad (125)$$

which yields two solutions for each value of ψ .

1.8.2 $L^- R_\psi^- | R^+$ Paths

For a $L_{\phi_1}^- R_\psi^- | R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1) \mathbf{M}_{R^-}(r, \psi) \mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (126)$$

Pre-multiplying (126) with $\mathbf{u}_{L^-}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$4r^2 (r^2 - 1) \cos(\psi) - (1 - 2r^2)^2 = \alpha_{11} (r^2 - 1) + r (\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r), \quad (127)$$

which gives

$$\cos(\psi) = \frac{-\alpha_{11} + 4r^4 + \alpha_{11}r^2 - \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r - 4r^2 + 1}{4r^2 (r^2 - 1)}, \quad (128)$$

and yields two solutions of ψ .

Pre-multiplying (117) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{R^+} :

$$\begin{aligned} & r \sin(\phi_1) (2 \sin(\psi) - 2r^2 \sin(\psi)) + r (4r^4 - 4 (r^2 - 1) r^2 \cos(\psi) - 4r^2 + 1) - 4r (2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right) \cos(\phi_1) \\ &= \alpha_{33}r - \alpha_{31} \sqrt{1 - r^2}. \end{aligned} \quad (129)$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$ and defining $\cos \gamma := \frac{4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2})}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$, $\sin \gamma := \frac{r(2 \sin(\psi) - 2r^2 \sin(\psi))}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$. It is obtained that

$$\begin{aligned} \cos(\gamma + \phi_1) &= -\frac{r (\alpha_{33} + 4 (r^2 - 1) r^2 \cos(\psi) - (1 - 2r^2)^2) - \alpha_{31} \sqrt{1 - r^2}}{4 \sqrt{-r^2 (r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2 (r^2 - 1) r^2 \cos(\psi) + 2r^2 - 1)}} \\ \Rightarrow \phi_1 &= \cos^{-1} \left(-\frac{r (\alpha_{33} + 4 (r^2 - 1) r^2 \cos(\psi) - (1 - 2r^2)^2) - \alpha_{31} \sqrt{1 - r^2}}{4 \sqrt{-r^2 (r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2 (r^2 - 1) r^2 \cos(\psi) + 2r^2 - 1)}} \right) - \tan^{-1} \left(\frac{2r(\sin(\psi) - r^2 \sin(\psi))}{4r (2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right)} \right), \end{aligned} \quad (130)$$

which yields two solutions for each value of ψ .

Pre-multiplying (117) with $\mathbf{u}_{L^-}^T$ and post-multiplying with \mathbf{u}_{G^-} :

$$\begin{aligned} & r \sin(\phi_2) (2r^2 \sin(\psi) - 2 \sin(\psi)) + r (4r^4 - 4 (r^2 - 1) r^2 \cos(\psi) - 4r^2 + 1) - 4r (2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right) \cos(\phi_2) \\ &= \alpha_{33}r - \alpha_{13} \sqrt{1 - r^2}. \end{aligned} \quad (131)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_2 - \gamma) &= -\frac{r (\alpha_{33} + 4 (r^2 - 1) r^2 \cos(\psi) - (1 - 2r^2)^2) - \alpha_{13} \sqrt{1 - r^2}}{4 \sqrt{-r^2 (r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2 (r^2 - 1) r^2 \cos(\psi) + 2r^2 - 1)}} \\ \Rightarrow \phi_2 &= \cos^{-1} \left(-\frac{r (\alpha_{33} + 4 (r^2 - 1) r^2 \cos(\psi) - (1 - 2r^2)^2) - \alpha_{13} \sqrt{1 - r^2}}{4 \sqrt{-r^2 (r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2 (r^2 - 1) r^2 \cos(\psi) + 2r^2 - 1)}} \right) + \tan^{-1} \left(\frac{2r(\sin(\psi) - r^2 \sin(\psi))}{4r (2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right)} \right), \end{aligned} \quad (132)$$

which yields two solutions for each value of ψ .

1.8.3 $R^-L_\psi^-|L^+$ Paths

For a $R_{\phi_1}^-L_\psi^-|L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1)\mathbf{M}_{L^-}(r, \psi)\mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (135)$$

Pre-multiplying (135) with $\mathbf{u}_{R^-}^T$ and post-multiplying \mathbf{u}_{L^+} :

$$4r^2 (r^2 - 1) \cos(\psi) - (1 - 2r^2)^2 = \alpha_{11} (r^2 - 1) - r \left(\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right), \quad (136)$$

which gives

$$\cos(\psi) = \frac{-\alpha_{11} + 4r^4 + \alpha_{11}r^2 - \alpha_{33}r^2 - \alpha_{13}\sqrt{1 - r^2}r - \alpha_{31}\sqrt{1 - r^2}r - 4r^2 + 1}{4r^2 (r^2 - 1)}, \quad (137)$$

and yields two solutions of ψ .

Pre-multiplying (135) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{L^+} :

$$\begin{aligned} & r \sin(\phi_1) (2 \sin(\psi) - 2r^2 \sin(\psi)) + r (4r^4 - 4 (r^2 - 1) r^2 \cos(\psi) - 4r^2 + 1) - 4r (2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right) \cos(\phi_1) \\ &= \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r. \end{aligned} \quad (138)$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$ and defining $\cos \gamma := \frac{4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2})}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$, $\sin \gamma := \frac{r(2 \sin(\psi) - 2r^2 \sin(\psi))}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$. It is obtained that

$$\begin{aligned} \cos(\gamma + \phi_1) &= -\frac{r \left(\alpha_{33} + 4 (r^2 - 1) r^2 \cos(\psi) - (1 - 2r^2)^2 \right) + \alpha_{31} \sqrt{1 - r^2}}{4 \sqrt{-r^2 (r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2 (r^2 - 1) r^2 \cos(\psi) + 2r^2 - 1)}} \\ \Rightarrow \phi_1 &= \cos^{-1} \left(-\frac{r \left(\alpha_{33} + 4 (r^2 - 1) r^2 \cos(\psi) - (1 - 2r^2)^2 \right) + \alpha_{31} \sqrt{1 - r^2}}{4 \sqrt{-r^2 (r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2 (r^2 - 1) r^2 \cos(\psi) + 2r^2 - 1)}} \right) - \tan^{-1} \left(\frac{2r(\sin(\psi) - r^2 \sin(\psi))}{4r(2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right)} \right), \end{aligned} \quad (139)$$

which yields two solutions for each value of ψ .

Pre-multiplying (135) with $\mathbf{u}_{R^-}^T$ and post-multiplying with \mathbf{u}_{G^-} :

$$\begin{aligned} & r \sin(\phi_2) (2r^2 \sin(\psi) - 2 \sin(\psi)) + r (4r^4 - 4 (r^2 - 1) r^2 \cos(\psi) - 4r^2 + 1) - 4r (2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right) \cos(\phi_2) \\ &= \alpha_{13} \sqrt{1 - r^2} + \alpha_{33} r. \end{aligned} \quad (141)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_2 - \gamma) &= -\frac{r \left(\alpha_{33} + 4 (r^2 - 1) r^2 \cos(\psi) - (1 - 2r^2)^2 \right) + \alpha_{13} \sqrt{1 - r^2}}{4 \sqrt{-r^2 (r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2 (r^2 - 1) r^2 \cos(\psi) + 2r^2 - 1)}} \\ \Rightarrow \phi_2 &= \cos^{-1} \left(-\frac{r \left(\alpha_{33} + 4 (r^2 - 1) r^2 \cos(\psi) - (1 - 2r^2)^2 \right) + \alpha_{13} \sqrt{1 - r^2}}{4 \sqrt{-r^2 (r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) (-2r^4 + 2 (r^2 - 1) r^2 \cos(\psi) + 2r^2 - 1)}} \right) + \tan^{-1} \left(\frac{2r(\sin(\psi) - r^2 \sin(\psi))}{4r(2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right)} \right), \end{aligned} \quad (142)$$

which yields two solutions for each value of ψ .

1.8.4 $R^+L_\psi^+|L^-$ Paths

For a $R_{\phi_1}^+L_\psi^+|L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{L^+}(r, \psi)\mathbf{M}_{L^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (144)$$

Pre-multiplying (144) with $\mathbf{u}_{R^+}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$4r^2(r^2 - 1)\cos(\psi) - (1 - 2r^2)^2 = \alpha_{11}(r^2 - 1) + r(\alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} - \alpha_{33}r), \quad (145)$$

which gives

$$\cos(\psi) = \frac{-\alpha_{11} + 4r^4 + \alpha_{11}r^2 - \alpha_{33}r^2 + \alpha_{13}\sqrt{1 - r^2}r + \alpha_{31}\sqrt{1 - r^2}r - 4r^2 + 1}{4r^2(r^2 - 1)}, \quad (146)$$

and yields two solutions of ψ .

Pre-multiplying (144) with $\mathbf{u}_{G^-}^T$ and post-multiplying with \mathbf{u}_{L^-} :

$$\begin{aligned} & r \sin(\phi_1)(2 \sin(\psi) - 2r^2 \sin(\psi)) + r(4r^4 - 4(r^2 - 1)r^2 \cos(\psi) - 4r^2 + 1) - 4r(2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right) \cos(\phi_1) \\ &= \alpha_{33}r - \alpha_{31}\sqrt{1 - r^2}. \end{aligned} \quad (147)$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$ and defining $\cos \gamma := \frac{4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2})}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$, $\sin \gamma := \frac{r(2 \sin(\psi) - 2r^2 \sin(\psi))}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$. It is obtained that

$$\begin{aligned} \cos(\gamma + \phi_1) &= -\frac{r(\alpha_{33} + 4(r^2 - 1)r^2 \cos(\psi) - (1 - 2r^2)^2) - \alpha_{31}\sqrt{1 - r^2}}{4\sqrt{-r^2(r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) - 2r^4 + 2(r^2 - 1)r^2 \cos(\psi) + 2r^2 - 1}} \\ \Rightarrow \phi_1 &= \cos^{-1}\left(\frac{r(\alpha_{33} + 4(r^2 - 1)r^2 \cos(\psi) - (1 - 2r^2)^2) - \alpha_{31}\sqrt{1 - r^2}}{4\sqrt{-r^2(r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) - 2r^4 + 2(r^2 - 1)r^2 \cos(\psi) + 2r^2 - 1}}\right) - \tan^{-1}\left(\frac{2r(\sin(\psi) - r^2 \sin(\psi))}{4r(2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right)}\right), \end{aligned} \quad (148)$$

which yields two solutions for each value of ψ .

Pre-multiplying (144) with $\mathbf{u}_{R^+}^T$ and post-multiplying with \mathbf{u}_{G^+} :

$$\begin{aligned} & r \sin(\phi_2)(2r^2 \sin(\psi) - 2 \sin(\psi)) + r(4r^4 - 4(r^2 - 1)r^2 \cos(\psi) - 4r^2 + 1) - 4r(2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right) \cos(\phi_2) \\ &= \alpha_{33}r - \alpha_{13}\sqrt{1 - r^2}. \end{aligned} \quad (150)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2 \sin(\psi) - 2r^2 \sin(\psi))^2 + (4r(2r^4 - 3r^2 + 1) \sin^2(\frac{\psi}{2}))^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_2 - \gamma) &= -\frac{r(\alpha_{33} + 4(r^2 - 1)r^2 \cos(\psi) - (1 - 2r^2)^2) - \alpha_{13}\sqrt{1 - r^2}}{4\sqrt{-r^2(r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) - 2r^4 + 2(r^2 - 1)r^2 \cos(\psi) + 2r^2 - 1}} \\ \Rightarrow \phi_2 &= \cos^{-1}\left(-\frac{r(\alpha_{33} + 4(r^2 - 1)r^2 \cos(\psi) - (1 - 2r^2)^2) - \alpha_{13}\sqrt{1 - r^2}}{4\sqrt{-r^2(r^2 - 1)^2 \sin^2\left(\frac{\psi}{2}\right) - 2r^4 + 2(r^2 - 1)r^2 \cos(\psi) + 2r^2 - 1}}\right) + \tan^{-1}\left(\frac{2r(\sin(\psi) - r^2 \sin(\psi))}{4r(2r^4 - 3r^2 + 1) \sin^2\left(\frac{\psi}{2}\right)}\right), \end{aligned} \quad (151)$$

which yields two solutions for each value of ψ .

1.9 CGC Paths

1.9.1 $L^+G^+L^+$ Paths

For a $L_{\phi_1}^+ G_{\phi_2}^+ L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{G^+}(\phi_2) \mathbf{M}_{L^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (153)$$

Pre-multiplying (153) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{L^+} :

$$r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2) = r \left(\alpha_{13} \sqrt{1-r^2} + \alpha_{31} \sqrt{1-r^2} + \alpha_{33} r \right) - \alpha_{11} (r^2 - 1), \quad (154)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11} r^2 - \alpha_{33} r^2 - \alpha_{13} \sqrt{1-r^2} r - \alpha_{31} \sqrt{1-r^2} r + r^2}{r^2 - 1}, \quad (155)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (153) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{L^+} :

$$-r^3 + \sin(\phi_1) (r^2 \sin(\phi_2) - \sin(\phi_2)) + 2(r^2 - 1) r \cos(\phi_1) \sin^2\left(\frac{\phi_2}{2}\right) + (r^2 - 1) r \cos(\phi_2) = \alpha_{31} (-\sqrt{1-r^2}) - \alpha_{33} r. \quad (156)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1) r \sin^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{2(r^2 - 1) r \sin^2(\frac{\phi_2}{2})}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1) r \sin^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1) r \sin^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{31} \sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (157)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{31} \sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{2(r^2 - 1) r \sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (158)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (153) with $\mathbf{u}_{L^+}^T$ and post-multiplying with \mathbf{u}_{G^+} :

$$-r^3 + \sin(\phi_3) (r^2 \sin(\phi_2) - \sin(\phi_2)) + 2(r^2 - 1) r \cos(\phi_3) \sin^2\left(\frac{\phi_2}{2}\right) + (r^2 - 1) r \cos(\phi_2) = \alpha_{13} (-\sqrt{1-r^2}) - \alpha_{33} r. \quad (159)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1) r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{13} \sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (160)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{13} \sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{2(r^2 - 1) r \sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (161)$$

which yields two solutions for each value of ϕ_2 .

1.9.2 $R^+G^+R^+$ Paths

For a $R_{\phi_1}^+ G_{\phi_2}^+ R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) \mathbf{M}_{G^+}(\phi_2) \mathbf{M}_{R^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (162)$$

Pre-multiplying (162) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_{R+} :

$$r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2) = r \left(\alpha_{13} \left(-\sqrt{1-r^2} \right) - \alpha_{31} \sqrt{1-r^2} + \alpha_{33} r \right) - \alpha_{11} (r^2 - 1), \quad (163)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11} r^2 - \alpha_{33} r^2 + \alpha_{13} \sqrt{1-r^2} r + \alpha_{31} \sqrt{1-r^2} r + r^2}{r^2 - 1}, \quad (164)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (162) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{R+} :

$$-r^3 + \sin(\phi_1) (r^2 \sin(\phi_2) - \sin(\phi_2)) + 2 (r^2 - 1) r \cos(\phi_1) \sin^2 \left(\frac{\phi_2}{2} \right) + (r^2 - 1) r \cos(\phi_2) = \alpha_{31} \sqrt{1-r^2} - \alpha_{33} r. \quad (165)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1) r \sin^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{2(r^2 - 1) r \sin^2(\frac{\phi_2}{2})}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1) r \sin^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1) r \sin^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r (-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{31} \sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4 \left(\frac{\phi_2}{2} \right) + \sin^2(\phi_2) \right)}} \quad (166)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r (-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{31} \sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4 \left(\frac{\phi_2}{2} \right) + \sin^2(\phi_2) \right)}} \right) + \tan^{-1} \left(\frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{2 (r^2 - 1) r \sin^2 \left(\frac{\phi_2}{2} \right)} \right), \quad (167)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (162) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G-} :

$$-r^3 + \sin(\phi_3) (r^2 \sin(\phi_2) - \sin(\phi_2)) + 2 (r^2 - 1) r \cos(\phi_3) \sin^2 \left(\frac{\phi_2}{2} \right) + (r^2 - 1) r \cos(\phi_2) = \alpha_{13} \sqrt{1-r^2} - \alpha_{33} r. \quad (168)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1) r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r (-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{13} \sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4 \left(\frac{\phi_2}{2} \right) + \sin^2(\phi_2) \right)}} \quad (169)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r (-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{13} \sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4 \left(\frac{\phi_2}{2} \right) + \sin^2(\phi_2) \right)}} \right) + \tan^{-1} \left(\frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{2 (r^2 - 1) r \sin^2 \left(\frac{\phi_2}{2} \right)} \right), \quad (170)$$

which yields two solutions for each value of ϕ_2 .

1.9.3 $R^- G^- R^-$ Paths

For a $R_{\phi_1}^- G_{\phi_2}^- R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R-}(r, \phi_1) \mathbf{M}_{G-}(\phi_2) \mathbf{M}_{R-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (171)$$

Pre-multiplying (171) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_{R-} :

$$r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2) = r \left(\alpha_{13} \sqrt{1-r^2} + \alpha_{31} \sqrt{1-r^2} + \alpha_{33} r \right) - \alpha_{11} (r^2 - 1), \quad (172)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11} r^2 - \alpha_{33} r^2 + \alpha_{13} \sqrt{1-r^2} r + \alpha_{31} \sqrt{1-r^2} r + r^2}{r^2 - 1}, \quad (173)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (171) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{R-} :

$$-r^3 + \sin(\phi_1) (r^2 \sin(\phi_2) - \sin(\phi_2)) + 2(r^2 - 1)r \cos(\phi_1) \sin^2\left(\frac{\phi_2}{2}\right) + (r^2 - 1)r \cos(\phi_2) = \alpha_{31}(-\sqrt{1-r^2}) - \alpha_{33}r. \quad (174)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{2(r^2 - 1)r \sin^2(\frac{\phi_2}{2})}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (175)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{2(r^2 - 1)r \sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (176)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (171) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G+} :

$$-r^3 + \sin(\phi_3) (r^2 \sin(\phi_2) - \sin(\phi_2)) + 2(r^2 - 1)r \cos(\phi_3) \sin^2\left(\frac{\phi_2}{2}\right) + (r^2 - 1)r \cos(\phi_2) = \alpha_{13}(-\sqrt{1-r^2}) - \alpha_{33}r. \quad (177)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (178)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{2(r^2 - 1)r \sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (179)$$

which yields two solutions for each value of ϕ_2 .

1.9.4 $L^-G^-L^-$ Paths

For a $L_{\phi_1}^- G_{\phi_2}^- L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L-}(r, \phi_1) \mathbf{M}_{G-}(\phi_2) \mathbf{M}_{L-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (180)$$

Pre-multiplying (180) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{L-} :

$$r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2) = r \left(\alpha_{13}(-\sqrt{1-r^2}) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r \right) - \alpha_{11}(r^2 - 1), \quad (181)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 - \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2}{r^2 - 1}, \quad (182)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (180) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{L-} :

$$-r^3 + \sin(\phi_1) (r^2 \sin(\phi_2) - \sin(\phi_2)) + 2(r^2 - 1)r \cos(\phi_1) \sin^2\left(\frac{\phi_2}{2}\right) + (r^2 - 1)r \cos(\phi_2) = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \quad (183)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{2(r^2 - 1)r \sin^2(\frac{\phi_2}{2})}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) + \alpha_{31}\sqrt{1 - r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (184)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) + \alpha_{31}\sqrt{1 - r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{2(r^2 - 1)r \sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (185)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (180) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G+} :

$$-r^3 + \sin(\phi_3)(r^2 \sin(\phi_2) - \sin(\phi_2)) + 2(r^2 - 1)r \cos(\phi_3) \sin^2\left(\frac{\phi_2}{2}\right) + (r^2 - 1)r \cos(\phi_2) = \alpha_{13}\sqrt{1 - r^2} - \alpha_{33}r. \quad (186)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) + \alpha_{13}\sqrt{1 - r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (187)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) + \alpha_{13}\sqrt{1 - r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{r^2 \sin(\phi_2) - \sin(\phi_2)}{2(r^2 - 1)r \sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (188)$$

which yields two solutions for each value of ϕ_2 .

1.9.5 $L^+G^+R^+$ Paths

For a $L_{\phi_1}^+ G_{\phi_2}^+ R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L+}(r, \phi_1) \mathbf{M}_{G+}(\phi_2) \mathbf{M}_{R+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (189)$$

Pre-multiplying (189) with \mathbf{u}_{L+}^T and post-multiplying \mathbf{u}_{R+} :

$$(r^2 - 1) \cos(\phi_2) + r^2 = \alpha_{11}(r^2 - 1) + r(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r), \quad (190)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1 - r^2}r - \alpha_{31}\sqrt{1 - r^2}r - r^2}{r^2 - 1}, \quad (191)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (189) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{R+} :

$$(r - r^3) \cos(\phi_2) - r^3 + \sin(\phi_1)(\sin(\phi_2) - r^2 \sin(\phi_2)) + 2(r^2 - 1)r \cos(\phi_1) \cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31}\sqrt{1 - r^2} - \alpha_{33}r. \quad (192)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{2(r^2 - 1)r \cos^2(\frac{\phi_2}{2})}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$. It is obtained

that

$$\cos(\phi_1 - \gamma) = \frac{r^3 \cos(\phi_2) + r^3 + \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (193)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) + r^3 + \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (194)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (189) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_G :

$$-r^3 - (r^2 - 1) \sin(\phi_2) \sin(\phi_3) + 2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right) \cos(\phi_3) - (r^2 - 1)r \cos(\phi_2) = \alpha_{13} \left(-\sqrt{1 - r^2}\right) - \alpha_{33} r. \quad (195)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{13} \sqrt{1 - r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (196)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)) - \alpha_{13} \sqrt{1 - r^2}}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (197)$$

which yields two solutions for each value of ϕ_2 .

1.9.6 $R^+ G^+ L^+$ Paths

For a $R_{\phi_1}^+ G_{\phi_2}^+ L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) \mathbf{M}_{G^+}(\phi_2) \mathbf{M}_{L^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (198)$$

Pre-multiplying (198) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_L :

$$(r^2 - 1) \cos(\phi_2) + r^2 = \alpha_{11} (r^2 - 1) + r \left(\alpha_{13} \left(-\sqrt{1 - r^2}\right) + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right), \quad (199)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11} r^2 + \alpha_{33} r^2 - \alpha_{13} \sqrt{1 - r^2} r + \alpha_{31} \sqrt{1 - r^2} r - r^2}{r^2 - 1}, \quad (200)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (198) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{L+} :

$$(r - r^3) \cos(\phi_2) - r^3 + \sin(\phi_1) (\sin(\phi_2) - r^2 \sin(\phi_2)) + 2(r^2 - 1)r \cos(\phi_1) \cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31} \left(-\sqrt{1 - r^2}\right) - \alpha_{33} r. \quad (201)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$

and defining $\cos \gamma := \frac{2(r^2 - 1)r \cos^2(\frac{\phi_2}{2})}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3 \cos(\phi_2) + r^3 - \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (202)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) + r^3 - \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (203)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (198) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G-} :

$$-r^3 - (r^2 - 1) \sin(\phi_2) \sin(\phi_3) + 2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right) \cos(\phi_3) - (r^2 - 1)r \cos(\phi_2) = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r. \quad (204)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3 \cos(\phi_2) + r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (205)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) + r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (206)$$

which yields two solutions for each value of ϕ_2 .

1.9.7 $R^-G^-L^-$ Paths

For a $R_{\phi_1}^- G_{\phi_2}^- L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R-}(r, \phi_1) \mathbf{M}_{G-}(\phi_2) \mathbf{M}_{L-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (207)$$

Pre-multiplying (207) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_{L-} :

$$(r^2 - 1) \cos(\phi_2) + r^2 = \alpha_{11}(r^2 - 1) + r \left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r \right), \quad (208)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r - r^2}{r^2 - 1}, \quad (209)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (207) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{L-} :

$$(r - r^3) \cos(\phi_2) - r^3 + \sin(\phi_1) (\sin(\phi_2) - r^2 \sin(\phi_2)) + 2(r^2 - 1)r \cos(\phi_1) \cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \quad (210)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$

and defining $\cos \gamma := \frac{2(r^2 - 1)r \cos^2(\frac{\phi_2}{2})}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3 \cos(\phi_2) + r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (211)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) + r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (212)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (207) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G+} :

$$-r^3 - (r^2 - 1) \sin(\phi_2) \sin(\phi_3) + 2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right) \cos(\phi_3) - (r^2 - 1)r \cos(\phi_2) = \alpha_{13}(-\sqrt{1-r^2}) - \alpha_{33}r. \quad (213)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3 \cos(\phi_2) + r^3 - \alpha_{13} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (214)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) + r^3 - \alpha_{13} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (215)$$

which yields two solutions for each value of ϕ_2 .

1.9.8 $L^- G^- R^-$ Paths

For a $L_{\phi_1}^- G_{\phi_2}^- R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1) \mathbf{M}_{G^-}(\phi_2) \mathbf{M}_{R^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (216)$$

Pre-multiplying (216) with $\mathbf{u}_{L^-}^T$ and post-multiplying \mathbf{u}_{R^-} :

$$(r^2 - 1) \cos(\phi_2) + r^2 = \alpha_{11} (r^2 - 1) + r \left(\alpha_{13} (-\sqrt{1 - r^2}) + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right), \quad (217)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11} r^2 + \alpha_{33} r^2 - \alpha_{13} \sqrt{1 - r^2} r + \alpha_{31} \sqrt{1 - r^2} r - r^2}{r^2 - 1}, \quad (218)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (216) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{R^-} :

$$(r - r^3) \cos(\phi_2) - r^3 + \sin(\phi_1) (\sin(\phi_2) - r^2 \sin(\phi_2)) + 2(r^2 - 1)r \cos(\phi_1) \cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31} (-\sqrt{1 - r^2}) - \alpha_{33} r. \quad (219)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{2(r^2 - 1)r \cos^2(\frac{\phi_2}{2})}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \cos^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3 \cos(\phi_2) + r^3 - \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (220)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) + r^3 - \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (221)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (216) with $\mathbf{u}_{L^-}^T$ and post-multiplying with \mathbf{u}_{G^+} :

$$-r^3 - (r^2 - 1) \sin(\phi_2) \sin(\phi_3) + 2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right) \cos(\phi_3) - (r^2 - 1)r \cos(\phi_2) = \alpha_{13} (-\sqrt{1 - r^2}) - \alpha_{33} r. \quad (222)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\phi_2) - \sin(\phi_2))^2 + (2(r^2 - 1)r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3 \cos(\phi_2) + r^3 + \alpha_{13} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \quad (223)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) + r^3 + \alpha_{13} \sqrt{1 - r^2} - \alpha_{33} r - r \cos(\phi_2)}{\sqrt{(r^2 - 1)^2 \left(4r^2 \cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \right) + \tan^{-1} \left(\frac{\sin(\phi_2) - r^2 \sin(\phi_2)}{2(r^2 - 1)r \cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (224)$$

which yields two solutions for each value of ϕ_2 .

1.10 $C|C_\beta G$ Paths

1.10.1 $L^+|L_\beta^- G^-$ Paths

For a $L_{\phi_1}^+|L_\beta^- G_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{L^-}(r, \beta) \mathbf{M}_{G^-}(\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (225)$$

Pre-multiplying (225) with $\mathbf{u}_{G^-}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$2r^4 + \sin(\phi_1) (r^2 \sin(\beta) - \sin(\beta)) - r^2 + \cos(\phi_1) (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)) + (2r^2 - 2r^4) \cos(\beta) = \alpha_{33}, \quad (226)$$

This equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$ and defining $\cos \gamma := \frac{-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$, $\sin \gamma := \frac{r^2 \sin(\beta) - \sin(\beta)}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{33} + r^2 (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)}{\sqrt{(r^2 - 1)^2 (6r^4 + 2(r^2 - 1)r^2 \cos(2\beta) - 2r^2 + (4r^2 - 8r^4) \cos(\beta) + 1)}} \quad (227)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{\alpha_{33} + r^2 (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)}{\sqrt{(r^2 - 1)^2 (6r^4 + 2(r^2 - 1)r^2 \cos(2\beta) - 2r^2 + (4r^2 - 8r^4) \cos(\beta) + 1)}} \right) \\ & + \tan^{-1} \left(\frac{r^2 \sin(\beta) - \sin(\beta)}{-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)} \right), \end{aligned} \quad (228)$$

which yields two solutions.

Pre-multiplying (225) with $\mathbf{u}_{L^+}^T$ and post-multiplying with \mathbf{u}_{L^+} :

$$\begin{aligned} & 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) - 2(r^2 - 1)r^2 \cos(\beta) + (2r^2 - 1)r^2 + \cos(\phi_2) (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1) \\ & = r (\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r) - \alpha_{11} (r^2 - 1). \end{aligned} \quad (229)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, and defining

$$\cos \theta := \frac{2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}, \quad \sin \theta := \frac{2(r^2 - 1)r \sin(\beta)}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}},$$

it is obtained that

$$\cos(\phi_2 - \theta) = \frac{r (2r^3 \cos(\beta) - 2r^3 + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r - 2r \cos(\beta) + r) - \alpha_{11} (r^2 - 1)}{\sqrt{(r^2 - 1)^2 (4r^4 \cos^2(\beta) + 4r^2 \sin^2(\beta) + (1 - 2r^2)^2 + (4r^2 - 8r^4) \cos(\beta))}} \quad (230)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(\frac{r (2r^3 \cos(\beta) - 2r^3 + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r - 2r \cos(\beta) + r) - \alpha_{11} (r^2 - 1)}{\sqrt{(r^2 - 1)^2 (4r^4 \cos^2(\beta) + 4r^2 \sin^2(\beta) + (1 - 2r^2)^2 + (4r^2 - 8r^4) \cos(\beta))}} \right) \quad (231)$$

$$+ \tan^{-1} \left(\frac{2(r^2 - 1)r \sin(\beta)}{2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1} \right), \quad (232)$$

which yields two solutions.

1.10.2 $R^+|R_\beta^- G^-$ Paths

For a $R_{\phi_1}^+|R_\beta^- G_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) \mathbf{M}_{R^-}(r, \beta) \mathbf{M}_{G^-}(\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (233)$$

Pre-multiplying (233) with $\mathbf{u}_{G^-}^T$ and post-multiplying \mathbf{u}_{G^-} :

$$2r^4 + \sin(\phi_1) (r^2 \sin(\beta) - \sin(\beta)) - r^2 + \cos(\phi_1) (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)) + (2r^2 - 2r^4) \cos(\beta) = \alpha_{33}, \quad (234)$$

This equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$ and defining $\cos \gamma := \frac{-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$, $\sin \gamma := \frac{r^2 \sin(\beta) - \sin(\beta)}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{33} + r^2 (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)}{\sqrt{(r^2 - 1)^2 (6r^4 + 2(r^2 - 1)r^2 \cos(2\beta) - 2r^2 + (4r^2 - 8r^4) \cos(\beta) + 1)}} \quad (235)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{\alpha_{33} + r^2 (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)}{\sqrt{(r^2 - 1)^2 (6r^4 + 2(r^2 - 1)r^2 \cos(2\beta) - 2r^2 + (4r^2 - 8r^4) \cos(\beta) + 1)}} \right) \\ & + \tan^{-1} \left(\frac{r^2 \sin(\beta) - \sin(\beta)}{-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)} \right), \end{aligned} \quad (236)$$

which yields two solutions.

Pre-multiplying (233) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{R+} :

$$\begin{aligned} & 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) - 2(r^2 - 1)r^2 \cos(\beta) + (2r^2 - 1)r^2 + \cos(\phi_2)(2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1) \\ & = r \left(\alpha_{13} \left(-\sqrt{1 - r^2} \right) - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right) - \alpha_{11} (r^2 - 1). \end{aligned} \quad (237)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, and defining $\cos \theta := \frac{2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, $\sin \theta := \frac{2(r^2 - 1)r \sin(\beta)}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, it is obtained that

$$\cos(\phi_2 - \theta) = \frac{r(2r^3 \cos(\beta) - 2r^3 - \alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r - 2r \cos(\beta) + r) - \alpha_{11} (r^2 - 1)}{\sqrt{(r^2 - 1)^2 (4r^4 \cos^2(\beta) + 4r^2 \sin^2(\beta) + (1 - 2r^2)^2 + (4r^2 - 8r^4) \cos(\beta))}} \quad (238)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(\frac{r(2r^3 \cos(\beta) - 2r^3 - \alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r - 2r \cos(\beta) + r) - \alpha_{11} (r^2 - 1)}{\sqrt{(r^2 - 1)^2 (4r^4 \cos^2(\beta) + 4r^2 \sin^2(\beta) + (1 - 2r^2)^2 + (4r^2 - 8r^4) \cos(\beta))}} \right) \quad (239)$$

$$+ \tan^{-1} \left(\frac{2(r^2 - 1)r \sin(\beta)}{2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1} \right), \quad (240)$$

which yields two solutions.

1.10.3 $R^- |R_\beta^+ G^+$ Paths

For a $R_\beta^- |R_\beta^+ G_\phi^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1) \mathbf{M}_{R^+}(r, \beta) \mathbf{M}_{G^+}(\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (241)$$

Pre-multiplying (241) with \mathbf{u}_{G+}^T and post-multiplying \mathbf{u}_{G+} :

$$2r^4 + \sin(\phi_1)(r^2 \sin(\beta) - \sin(\beta)) - r^2 + \cos(\phi_1)(-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)) + (2r^2 - 2r^4) \cos(\beta) = \alpha_{33}, \quad (242)$$

This equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$ and defining $\cos \gamma := \frac{-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$, $\sin \gamma := \frac{r^2 \sin(\beta) - \sin(\beta)}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{33} + r^2 (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)}{\sqrt{(r^2 - 1)^2 (6r^4 + 2(r^2 - 1)r^2 \cos(2\beta) - 2r^2 + (4r^2 - 8r^4) \cos(\beta) + 1)}} \quad (243)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{\alpha_{33} + r^2 (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)}{\sqrt{(r^2 - 1)^2 (6r^4 + 2(r^2 - 1)r^2 \cos(2\beta) - 2r^2 + (4r^2 - 8r^4) \cos(\beta) + 1)}} \right) \\ & + \tan^{-1} \left(\frac{r^2 \sin(\beta) - \sin(\beta)}{-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)} \right), \end{aligned} \quad (244)$$

which yields two solutions.

Pre-multiplying (241) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{R-} :

$$\begin{aligned} & 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) - 2(r^2 - 1)r^2 \cos(\beta) + (2r^2 - 1)r^2 + \cos(\phi_2)(2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1) \\ &= r(\alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r) - \alpha_{11}(r^2 - 1). \end{aligned} \quad (245)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, and defining $\cos \theta := \frac{2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, $\sin \theta := \frac{2(r^2 - 1)r \sin(\beta)}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, it is obtained that

$$\cos(\phi_2 - \theta) = \frac{r(2r^3 \cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r - 2r \cos(\beta) + r) - \alpha_{11}(r^2 - 1)}{\sqrt{(r^2 - 1)^2(4r^4 \cos^2(\beta) + 4r^2 \sin^2(\beta) + (1 - 2r^2)^2 + (4r^2 - 8r^4) \cos(\beta))}} \quad (246)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(\frac{r(2r^3 \cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r - 2r \cos(\beta) + r) - \alpha_{11}(r^2 - 1)}{\sqrt{(r^2 - 1)^2(4r^4 \cos^2(\beta) + 4r^2 \sin^2(\beta) + (1 - 2r^2)^2 + (4r^2 - 8r^4) \cos(\beta))}} \right) \quad (247)$$

$$+ \tan^{-1} \left(\frac{2(r^2 - 1)r \sin(\beta)}{2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1} \right), \quad (248)$$

which yields two solutions.

1.10.4 $L^-|L_\beta^+G^+$ Paths

For a $L_{\phi_1}^-|L_\beta^+G_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L-}(r, \phi_1)\mathbf{M}_{L+}(r, \beta)\mathbf{M}_{G+}(\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (249)$$

Pre-multiplying (249) with \mathbf{u}_{G+}^T and post-multiplying \mathbf{u}_{G+} :

$$2r^4 + \sin(\phi_1)(r^2 \sin(\beta) - \sin(\beta)) - r^2 + \cos(\phi_1)(-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)) + (2r^2 - 2r^4) \cos(\beta) = \alpha_{33}, \quad (250)$$

This equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$ and defining $\cos \gamma := \frac{-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$, $\sin \gamma := \frac{r^2 \sin(\beta) - \sin(\beta)}{\sqrt{(r^2 \sin(\beta) - \sin(\beta))^2 + (-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{33} + r^2(2(r^2 - 1) \cos(\beta) - 2r^2 + 1)}{\sqrt{(r^2 - 1)^2(6r^4 + 2(r^2 - 1)r^2 \cos(2\beta) - 2r^2 + (4r^2 - 8r^4) \cos(\beta) + 1)}} \quad (251)$$

$$\begin{aligned} \Rightarrow \phi_1 &= \cos^{-1} \left(\frac{\alpha_{33} + r^2(2(r^2 - 1) \cos(\beta) - 2r^2 + 1)}{\sqrt{(r^2 - 1)^2(6r^4 + 2(r^2 - 1)r^2 \cos(2\beta) - 2r^2 + (4r^2 - 8r^4) \cos(\beta) + 1)}} \right) \\ &+ \tan^{-1} \left(\frac{r^2 \sin(\beta) - \sin(\beta)}{-2r^4 + 2r^2 + (2r^4 - 3r^2 + 1) \cos(\beta)} \right), \end{aligned} \quad (252)$$

which yields two solutions.

Pre-multiplying (249) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{L-} :

$$\begin{aligned} & 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) - 2(r^2 - 1)r^2 \cos(\beta) + (2r^2 - 1)r^2 + \cos(\phi_2)(2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1) \\ &= r(\alpha_{13}(-\sqrt{1 - r^2}) - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r) - \alpha_{11}(r^2 - 1). \end{aligned} \quad (253)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, and defining $\cos \theta := \frac{2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$, $\sin \theta := \frac{2(r^2 - 1)r \sin(\beta)}{\sqrt{(2(r^2 - 1)r \sin(\beta))^2 + (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1)^2}}$,

it is obtained that

$$\cos(\phi_2 - \theta) = \frac{r(2r^3 \cos(\beta) - 2r^3 - \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r \cos(\beta) + r) - \alpha_{11}(r^2 - 1)}{\sqrt{(r^2 - 1)^2 (4r^4 \cos^2(\beta) + 4r^2 \sin^2(\beta) + (1 - 2r^2)^2 + (4r^2 - 8r^4) \cos(\beta))}} \quad (254)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(\frac{r(2r^3 \cos(\beta) - 2r^3 - \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r \cos(\beta) + r) - \alpha_{11}(r^2 - 1)}{\sqrt{(r^2 - 1)^2 (4r^4 \cos^2(\beta) + 4r^2 \sin^2(\beta) + (1 - 2r^2)^2 + (4r^2 - 8r^4) \cos(\beta))}} \right) \quad (255)$$

$$+ \tan^{-1} \left(\frac{2(r^2 - 1)r \sin(\beta)}{2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1} \right), \quad (256)$$

which yields two solutions.

1.11 CTC Paths

1.11.1 $L^+L^0L^-$ Paths

For a $L_{\phi_1}^+ L_{\phi_2}^0 L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{L^0}(\phi_2) \mathbf{M}_{L^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (257)$$

Pre-multiplying (257) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$r^2(-\cos(\phi_2)) - r^2 + 1 = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \quad (258)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r - r^2 + 1}{r^2}, \quad (259)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (257) with $\mathbf{u}_{G^-}^T$ and post-multiplying with \mathbf{u}_{L^-} :

$$-r\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) + r\cos(\phi_3)(r^2(-\cos(\phi_2)) - r^2 + \cos(\phi_2) + 1) + r(r^2\cos(\phi_2) + r^2 - 1) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r. \quad (260)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$ and

defining $\cos \gamma := \frac{-2(r^2-1)r\cos^2(\frac{\phi_2}{2})}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = -\frac{r(-\alpha_{33} + r^2\cos(\phi_2) + r^2 - 1) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2(r^2 - 1)\cos^2(\frac{\phi_2}{2})(r^2\cos(\phi_2) + r^2 - 2)}} \quad (261)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(-\frac{r(-\alpha_{33} + r^2\cos(\phi_2) + r^2 - 1) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2(r^2 - 1)\cos^2(\frac{\phi_2}{2})(r^2\cos(\phi_2) + r^2 - 2)}} \right) + \tan^{-1} \left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2 - 1)r\cos^2(\frac{\phi_2}{2})} \right), \quad (262)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (257) with $\mathbf{u}_{L^+}^T$ and post-multiplying with \mathbf{u}_{G^+} :

$$r \left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2(r^2 - 1)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2\cos(\phi_2) + r^2 - 1 \right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r. \quad (263)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3(-\cos(\phi_2)) - r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \quad (264)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r^3(-\cos(\phi_2)) - r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \right) + \tan^{-1} \left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (265)$$

which yields two solutions for each value of ϕ_2 .

1.11.2 $R^+R^0R^-$ Paths

For a $R_{\phi_1}^+ R_{\phi_2}^0 R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) \mathbf{M}_{R^0}(\phi_2) \mathbf{M}_{R^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (266)$$

Pre-multiplying (266) with $\mathbf{u}_{R^+}^T$ and post-multiplying \mathbf{u}_{R^-} :

$$r^2(-\cos(\phi_2)) - r^2 + 1 = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}(-\sqrt{1-r^2}) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \quad (267)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r - r^2 + 1}{r^2}, \quad (268)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (266) with $\mathbf{u}_{G^-}^T$ and post-multiplying with \mathbf{u}_{R^-} :

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r(r^2-1)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) + r(r^2\cos(\phi_2) + r^2 - 1) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r. \quad (269)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$ and

defining $\cos \gamma := \frac{-2(r^2-1)r\cos^2(\frac{\phi_2}{2})}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3(-\cos(\phi_2)) - r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \quad (270)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r^3(-\cos(\phi_2)) - r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \right) + \tan^{-1} \left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (271)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (266) with $\mathbf{u}_{R^+}^T$ and post-multiplying with \mathbf{u}_{G^+} :

$$r(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2\cos(\phi_2) + r^2 - 1) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \quad (272)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = -\frac{r(-\alpha_{33} + r^2\cos(\phi_2) + r^2 - 1) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \quad (273)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(-\frac{r(-\alpha_{33} + r^2\cos(\phi_2) + r^2 - 1) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \right) + \tan^{-1} \left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)} \right), \quad (274)$$

which yields two solutions for each value of ϕ_2 .

1.11.3 $R^- R^0 R^+$ Paths

For a $R_{\phi_1}^- R_{\phi_2}^0 R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1) \mathbf{M}_{R^0}(\phi_2) \mathbf{M}_{R^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (275)$$

Pre-multiplying (275) with $\mathbf{u}_{R^-}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$r^2(-\cos(\phi_2)) - r^2 + 1 = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \quad (276)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r - r^2 + 1}{r^2}, \quad (277)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (275) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{R^+} :

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r(r^2 - 1)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) + r(r^2\cos(\phi_2) + r^2 - 1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}. \quad (278)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{-2(r^2-1)r\cos^2(\frac{\phi_2}{2})}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = -\frac{r(-\alpha_{33} + r^2\cos(\phi_2) + r^2 - 1) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2(r^2 - 1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \quad (279)$$

$$\Rightarrow \phi_1 = \cos^{-1}\left(-\frac{r(-\alpha_{33} + r^2\cos(\phi_2) + r^2 - 1) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2(r^2 - 1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}}\right) + \tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2 - 1)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \quad (280)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (275) with $\mathbf{u}_{R^-}^T$ and post-multiplying with \mathbf{u}_{G^-} :

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2(r^2 - 1)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2\cos(\phi_2) + r^2 - 1\right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r. \quad (281)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3(-\cos(\phi_2)) - r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2(r^2 - 1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \quad (282)$$

$$\Rightarrow \phi_3 = \cos^{-1}\left(\frac{r^3(-\cos(\phi_2)) - r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2(r^2 - 1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}}\right) + \tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2 - 1)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \quad (283)$$

which yields two solutions for each value of ϕ_2 .

1.11.4 $L^- L^0 L^+$ Paths

For a $L_{\phi_1}^- L_{\phi_2}^0 L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1) \mathbf{M}_{L^0}(\phi_2) \mathbf{M}_{L^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (284)$$

Pre-multiplying (284) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{L+} :

$$r^2(-\cos(\phi_2)) - r^2 + 1 = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}(-\sqrt{1-r^2}) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \quad (285)$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r - r^2 + 1}{r^2}, \quad (286)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (284) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{L+} :

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r(r^2-1)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) + r(r^2\cos(\phi_2) + r^2 - 1) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r. \quad (287)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$ and

defining $\cos \gamma := \frac{-2(r^2-1)r\cos^2(\frac{\phi_2}{2})}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3(-\cos(\phi|2)) - r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi|2}{2}\right)(r^2\cos(\phi|2) + r^2 - 2)}} \quad (288)$$

$$\Rightarrow \phi_1 = \cos^{-1}\left(\frac{r^3(-\cos(\phi|2)) - r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi|2}{2}\right)(r^2\cos(\phi|2) + r^2 - 2)}}\right) + \tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \quad (289)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (284) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G-} :

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2\cos(\phi_2) + r^2 - 1\right) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \quad (290)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\cos^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = -\frac{r(-\alpha_{33} + r^2\cos(\phi|2) + r^2 - 1) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}} \quad (291)$$

$$\Rightarrow \phi_3 = \cos^{-1}\left(-\frac{r(-\alpha_{33} + r^2\cos(\phi|2) + r^2 - 1) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2(r^2-1)\cos^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) + r^2 - 2)}}\right) + \tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \quad (292)$$

which yields two solutions for each value of ϕ_2 .

1.11.5 $L^+L^0L^+$ Paths

For a $L_{\phi_1}^+L_{\phi_2}^0L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (293)$$

Pre-multiplying (293) with \mathbf{u}_{L+}^T and post-multiplying \mathbf{u}_{L+} :

$$r^2\cos(\phi_2) - r^2 + 1 = r(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r) - \alpha_{11}(r^2 - 1), \quad (294)$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \quad (295)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (293) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{L+} :

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r(r^2-1)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + r(r^2(-\cos(\phi_2)) + r^2 - 1) = \alpha_{31}(-\sqrt{1-r^2}) - \alpha_{33}r. \quad (296)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$ and

defining $\cos \gamma := \frac{-2(r^2-1)r\sin^2(\frac{\phi_2}{2})}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3\cos(\phi_2) - r^3 - \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) - r^2 + 2)}} \quad (297)$$

$$\Rightarrow \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 - \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) - r^2 + 2)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \quad (298)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (293) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G-} :

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2(r^2-1)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2(-\cos(\phi_2)) + r^2 - 1\right) = \alpha_{13}(-\sqrt{1-r^2}) - \alpha_{33}r. \quad (299)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2) + (2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3\cos(\phi_2) - r^3 - \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) - r^2 + 2)}} \quad (300)$$

$$\Rightarrow \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 - \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2\left(\frac{\phi_2}{2}\right)(r^2\cos(\phi_2) - r^2 + 2)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \quad (301)$$

which yields two solutions for each value of ϕ_2 .

1.11.6 $R^+R^0R^+$ Paths

For a $R_{\phi_1}^+R_{\phi_2}^0R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (302)$$

Pre-multiplying (302) with $\mathbf{u}_{R^+}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$r^2\cos(\phi_2) - r^2 + 1 = r\left(\alpha_{13}(-\sqrt{1-r^2}) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}(r^2 - 1), \quad (303)$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \quad (304)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (302) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{R^+} :

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r(r^2-1)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + r(r^2(-\cos(\phi_2)) + r^2 - 1) = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \quad (305)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{-2(r^2-1)r\sin^2(\frac{\phi_2}{2})}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3 \cos(\phi_2) - r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2(\frac{\phi_2}{2})(r^2 \cos(\phi_2) - r^2 + 2)}} \quad (306)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) - r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2(\frac{\phi_2}{2})(r^2 \cos(\phi_2) - r^2 + 2)}} \right) + \tan^{-1} \left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\sin^2(\frac{\phi_2}{2})} \right), \quad (307)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (302) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G-} :

$$r \left(\sqrt{1-r^2} \sin(\phi_2) \sin(\phi_3) - 2(r^2-1)\sin^2\left(\frac{\phi_2}{2}\right) \cos(\phi_3) + r^2(-\cos(\phi_2)) + r^2 - 1 \right) = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r. \quad (308)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3 \cos(\phi_2) - r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2(\frac{\phi_2}{2})(r^2 \cos(\phi_2) - r^2 + 2)}} \quad (309)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) - r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2(\frac{\phi_2}{2})(r^2 \cos(\phi_2) - r^2 + 2)}} \right) + \tan^{-1} \left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\sin^2(\frac{\phi_2}{2})} \right), \quad (310)$$

which yields two solutions for each value of ϕ_2 .

1.11.7 $R^-R^0R^-$ Paths

For a $R_{\phi_1}^- R_{\phi_2}^0 R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R-}(r, \phi_1) \mathbf{M}_{R^0}(\phi_2) \mathbf{M}_{R-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (311)$$

Pre-multiplying (311) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_{R-} :

$$r^2 \cos(\phi_2) - r^2 + 1 = r \left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r \right) - \alpha_{11}(r^2 - 1), \quad (312)$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \quad (313)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (311) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{R-} :

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r(r^2-1)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + r(r^2(-\cos(\phi_2)) + r^2 - 1) = \alpha_{31}(-\sqrt{1-r^2}) - \alpha_{33}r. \quad (314)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$ and defining $\cos \gamma := \frac{-2(r^2-1)r\sin^2(\frac{\phi_2}{2})}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3 \cos(\phi_2) - r^3 - \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2(\frac{\phi_2}{2})(r^2 \cos(\phi_2) - r^2 + 2)}} \quad (315)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) - r^3 - \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2(\frac{\phi_2}{2})(r^2 \cos(\phi_2) - r^2 + 2)}} \right) + \tan^{-1} \left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\sin^2(\frac{\phi_2}{2})} \right), \quad (316)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (311) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G+} :

$$r \left(\sqrt{1-r^2} \sin(\phi_2) \sin(\phi_3) - 2(r^2-1) \sin^2\left(\frac{\phi_2}{2}\right) \cos(\phi_3) + r^2(-\cos(\phi_2)) + r^2 - 1 \right) = \alpha_{13} \left(-\sqrt{1-r^2} \right) - \alpha_{33}r. \quad (317)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2) \sin^2(\phi_2) + (2(r^2-1)r \sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3 \cos(\phi_2) - r^3 - \alpha_{13} \sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2} \sqrt{-r^2(r^2-1) \sin^2\left(\frac{\phi_2}{2}\right) (r^2 \cos(\phi_2) - r^2 + 2)}} \quad (318)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) - r^3 - \alpha_{13} \sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2} \sqrt{-r^2(r^2-1) \sin^2\left(\frac{\phi_2}{2}\right) (r^2 \cos(\phi_2) - r^2 + 2)}} \right) + \tan^{-1} \left(\frac{r \sqrt{1-r^2} \sin(\phi_2)}{-2(r^2-1)r \sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (319)$$

which yields two solutions for each value of ϕ_2 .

1.11.8 $L^-L^0L^-$ Paths

For a $L_{\phi_1}^- L_{\phi_2}^0 L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1) \mathbf{M}_{L^0}(\phi_2) \mathbf{M}_{L^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (320)$$

Pre-multiplying (320) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{L-} :

$$r^2 \cos(\phi_2) - r^2 + 1 = r \left(\alpha_{13} \left(-\sqrt{1-r^2} \right) - \alpha_{31} \sqrt{1-r^2} + \alpha_{33}r \right) - \alpha_{11} (r^2 - 1), \quad (321)$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \quad (322)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (320) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{L-} :

$$r \sqrt{1-r^2} \sin(\phi_1) \sin(\phi_2) - 2r(r^2-1) \cos(\phi_1) \sin^2\left(\frac{\phi_2}{2}\right) + r(r^2(-\cos(\phi_2)) + r^2 - 1) = \alpha_{31} \sqrt{1-r^2} - \alpha_{33}r. \quad (323)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2) \sin^2(\phi_2) + (2(r^2-1)r \sin^2(\frac{\phi_2}{2}))^2}}$ and

defining $\cos \gamma := \frac{-2(r^2-1)r \sin^2(\frac{\phi_2}{2})}{\sqrt{r^2(1-r^2) \sin^2(\phi_2) + (2(r^2-1)r \sin^2(\frac{\phi_2}{2}))^2}}$, $\sin \gamma := \frac{r \sqrt{1-r^2} \sin(\phi_2)}{\sqrt{r^2(1-r^2) \sin^2(\phi_2) + (2(r^2-1)r \sin^2(\frac{\phi_2}{2}))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r^3 \cos(\phi_2) - r^3 + \alpha_{31} \sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2} \sqrt{-r^2(r^2-1) \sin^2\left(\frac{\phi_2}{2}\right) (r^2 \cos(\phi_2) - r^2 + 2)}} \quad (324)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) - r^3 + \alpha_{31} \sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2} \sqrt{-r^2(r^2-1) \sin^2\left(\frac{\phi_2}{2}\right) (r^2 \cos(\phi_2) - r^2 + 2)}} \right) + \tan^{-1} \left(\frac{r \sqrt{1-r^2} \sin(\phi_2)}{-2(r^2-1)r \sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (325)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (320) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G+} :

$$r \left(\sqrt{1-r^2} \sin(\phi_2) \sin(\phi_3) - 2(r^2-1) \sin^2\left(\frac{\phi_2}{2}\right) \cos(\phi_3) + r^2(-\cos(\phi_2)) + r^2 - 1 \right) = \alpha_{13} \sqrt{1-r^2} - \alpha_{33}r. \quad (326)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+(2(r^2-1)r\sin^2(\frac{\phi_2}{2}))^2}}$, it is obtained that

$$\cos(\phi_3 - \gamma) = \frac{r^3 \cos(\phi_2) - r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2\left(\frac{\phi_2}{2}\right)}(r^2 \cos(\phi_2) - r^2 + 2)} \quad (327)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r^3 \cos(\phi_2) - r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2(r^2-1)\sin^2\left(\frac{\phi_2}{2}\right)}(r^2 \cos(\phi_2) - r^2 + 2)} \right) + \tan^{-1} \left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)} \right), \quad (328)$$

which yields two solutions for each value of ϕ_2 .

1.12 $C|C_\psi C_\psi|C$ Paths

1.12.1 $L^+|L_\psi^- R_\psi^-|R^+$ Paths

For a $L_{\phi_1}^+|L_\psi^- R_\psi^-|R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{L^-}(r, \psi) \mathbf{M}_{R^-}(r, \psi) \mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (329)$$

Pre-multiplying (329) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$\begin{aligned} & 12r^6 - 20r^4 + 10r^2 + 8(r^2 - 1)r^4 \cos^2(\psi) - 4(r^2 - 1)r^4 - 8(2r^6 - 3r^4 + r^2) \cos(\psi) - 1 \\ & = \alpha_{11}(r^2 - 1) + r(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \end{aligned} \quad (330)$$

which gives

$$\cos(\psi) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2}\sqrt{r^4(r^2-1)(\alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + \alpha_{11}(r^2-1) - 1)}}{4r^4(r^2-1)}, \quad (331)$$

and yields four solutions of ψ .

Pre-multiplying (329) with $\mathbf{u}_{G^-}^T$ and post-multiplying with \mathbf{u}_{R^+} :

$$\begin{aligned} & -12r^7 + 20r^5 - 10r^3 + 2(r^2 - 1)r \sin(\psi) \sin(\phi_1) (2r^2 \cos(\psi) - 2r^2 + 1) - 4(r^2 - 1)r^5 \cos(2\psi) \\ & - 2(r^2 - 1)r \cos(\phi_1) ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)) + 8(2r^4 - 3r^2 + 1)r^3 \cos(\psi) + r \\ & = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \end{aligned} \quad (332)$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2(r^2 - 1)r \sin(\psi) (2r^2 \cos(\psi) - 2r^2 + 1) \quad (333)$$

$$B = -2(r^2 - 1)r ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)), \quad (334)$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{31}\sqrt{1-r^2} - r(\alpha_{33} - 4r^6 \cos(2\psi) - 12r^6 + 4r^4 \cos(2\psi) + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2+B^2}} \quad (335)$$

$$\begin{aligned} \Rightarrow \phi_1 &= \cos^{-1} \left(\frac{\alpha_{31}\sqrt{1-r^2} - r(\alpha_{33} - 4r^6 \cos(2\psi) - 12r^6 + 4r^4 \cos(2\psi) + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2+B^2}} \right) \\ &+ \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (336)$$

which yields two solutions for each value of ψ .

Pre-multiplying (329) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & -12r^7 + 20r^5 - 10r^3 + 2(r^2 - 1)r \sin(\psi) \sin(\phi_2) (2r^2 \cos(\psi) - 2r^2 + 1) - 4(r^2 - 1)r^5 \cos(2\psi) \\ & - 2(r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)) + 8(2r^4 - 3r^2 + 1)r^3 \cos(\psi) + r \\ & = \alpha_{13} \left(-\sqrt{1 - r^2} \right) - \alpha_{33}r \end{aligned} \quad (337)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$, it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{\alpha_{13}(-\sqrt{1 - r^2}) - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \quad (338)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{\alpha_{13}(-\sqrt{1 - r^2}) - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (339)$$

which yields two solutions for each value of ψ .

1.12.2 $R^+|R_\psi^- L_\psi^-|L^+$ Paths

For a $R_{\phi_1}^+|R_\psi^- L_\psi^-|L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) \mathbf{M}_{R^-}(r, \psi) \mathbf{M}_{L^-}(r, \psi) \mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (340)$$

Pre-multiplying (340) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{L+} :

$$\begin{aligned} & 12r^6 - 20r^4 + 10r^2 + 8(r^2 - 1)r^4 \cos^2(\psi) - 4(r^2 - 1)r^4 - 8(2r^6 - 3r^4 + r^2) \cos(\psi) - 1 \\ & = \alpha_{11}(r^2 - 1) + r \left(\alpha_{13} \left(-\sqrt{1 - r^2} \right) + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33}r \right), \end{aligned} \quad (341)$$

which gives

$$\cos(\psi) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2} \sqrt{r^4(r^2 - 1)(\alpha_{33}r^2 - \alpha_{13}\sqrt{1 - r^2}r + \alpha_{31}\sqrt{1 - r^2}r + \alpha_{11}(r^2 - 1) - 1)}}{4r^4(r^2 - 1)}, \quad (342)$$

and yields four solutions of ψ .

Pre-multiplying (340) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{L+} :

$$\begin{aligned} & -12r^7 + 20r^5 - 10r^3 + 2(r^2 - 1)r \sin(\psi) \sin(\phi_1) (2r^2 \cos(\psi) - 2r^2 + 1) - 4(r^2 - 1)r^5 \cos(2\psi) \\ & - 2(r^2 - 1)r \cos(\phi_1) ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)) + 8(2r^4 - 3r^2 + 1)r^3 \cos(\psi) + r \\ & = \alpha_{31} \left(-\sqrt{1 - r^2} \right) - \alpha_{33}r. \end{aligned} \quad (343)$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 2(r^2 - 1)r \sin(\psi) (2r^2 \cos(\psi) - 2r^2 + 1) \quad (344)$$

$$B = -2(r^2 - 1)r ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)), \quad (345)$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{31}(-\sqrt{1 - r^2}) - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \quad (346)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{\alpha_{31}(-\sqrt{1 - r^2}) - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (347)$$

which yields two solutions for each value of ψ .

Pre-multiplying (340) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & -12r^7 + 20r^5 - 10r^3 + 2(r^2 - 1)r \sin(\psi) \sin(\phi_2) (2r^2 \cos(\psi) - 2r^2 + 1) - 4(r^2 - 1)r^5 \cos(2\psi) \\ & - 2(r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)) + 8(2r^4 - 3r^2 + 1)r^3 \cos(\psi) + r \\ & = \alpha_{13} \sqrt{1 - r^2} - \alpha_{33} r \end{aligned} \quad (348)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$, it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{\alpha_{13} \sqrt{1 - r^2} - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \quad (349)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{\alpha_{13} \sqrt{1 - r^2} - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (350)$$

which yields two solutions for each value of ψ .

1.12.3 $R^-|R_\psi^+ L_\psi^+|L^-$ Paths

For a $R_{\phi_1}^-|R_\psi^+ L_\psi^+|L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R-}(r, \phi_1) \mathbf{M}_{R+}(r, \psi) \mathbf{M}_{L+}(r, \psi) \mathbf{M}_{L-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (351)$$

Pre-multiplying (351) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{L-} :

$$\begin{aligned} & 12r^6 - 20r^4 + 10r^2 + 8(r^2 - 1)r^4 \cos^2(\psi) - 4(r^2 - 1)r^4 - 8(2r^6 - 3r^4 + r^2) \cos(\psi) - 1 \\ & = \alpha_{11}(r^2 - 1) + r(\alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r), \end{aligned} \quad (352)$$

which gives

$$\cos(\psi) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2} \sqrt{r^4(r^2 - 1)(\alpha_{33} r^2 + \alpha_{13} \sqrt{1 - r^2} r - \alpha_{31} \sqrt{1 - r^2} r + \alpha_{11}(r^2 - 1) - 1)}}{4r^4(r^2 - 1)}, \quad (353)$$

and yields four solutions of ψ .

Pre-multiplying (351) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{L-} :

$$\begin{aligned} & -12r^7 + 20r^5 - 10r^3 + 2(r^2 - 1)r \sin(\psi) \sin(\phi_1) (2r^2 \cos(\psi) - 2r^2 + 1) - 4(r^2 - 1)r^5 \cos(2\psi) \\ & - 2(r^2 - 1)r \cos(\phi_1) ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)) + 8(2r^4 - 3r^2 + 1)r^3 \cos(\psi) + r \\ & = \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r. \end{aligned} \quad (354)$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 2(r^2 - 1)r \sin(\psi) (2r^2 \cos(\psi) - 2r^2 + 1) \quad (355)$$

$$B = -2(r^2 - 1)r ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)), \quad (356)$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{31} \sqrt{1 - r^2} - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \quad (357)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{\alpha_{31} \sqrt{1 - r^2} - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (358)$$

which yields two solutions for each value of ψ .

Pre-multiplying (351) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & -12r^7 + 20r^5 - 10r^3 + 2(r^2 - 1)r \sin(\psi) \sin(\phi_2) (2r^2 \cos(\psi) - 2r^2 + 1) - 4(r^2 - 1)r^5 \cos(2\psi) \\ & - 2(r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)) + 8(2r^4 - 3r^2 + 1)r^3 \cos(\psi) + r \\ & = \alpha_{13} \left(-\sqrt{1 - r^2} \right) - \alpha_{33}r. \end{aligned} \quad (359)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$, it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{\alpha_{13}(-\sqrt{1 - r^2}) - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \quad (360)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{\alpha_{13}(-\sqrt{1 - r^2}) - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (361)$$

which yields two solutions for each value of ψ .

1.12.4 $L^-|L_\psi^+ R_\psi^+|R^-$ Paths

For a $L_{\phi_1}^-|L_\psi^+ R_\psi^+|R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L-}(r, \phi_1) \mathbf{M}_{L+}(r, \psi) \mathbf{M}_{R+}(r, \psi) \mathbf{M}_{R-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (362)$$

Pre-multiplying (362) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{R-} :

$$\begin{aligned} & 12r^6 - 20r^4 + 10r^2 + 8(r^2 - 1)r^4 \cos^2(\psi) - 4(r^2 - 1)r^4 - 8(2r^6 - 3r^4 + r^2) \cos(\psi) - 1 \\ & = \alpha_{11}(r^2 - 1) + r \left(\alpha_{13} \left(-\sqrt{1 - r^2} \right) + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33}r \right), \end{aligned} \quad (363)$$

which gives

$$\cos(\psi) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2} \sqrt{r^4(r^2 - 1)(\alpha_{33}r^2 - \alpha_{13}\sqrt{1 - r^2}r + \alpha_{31}\sqrt{1 - r^2}r + \alpha_{11}(r^2 - 1) - 1)}}{4r^4(r^2 - 1)}, \quad (364)$$

and yields four solutions of ψ .

Pre-multiplying (362) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{R-} :

$$\begin{aligned} & 12r^7 + 20r^5 - 10r^3 + 2(r^2 - 1)r \sin(\psi) \sin(\phi_1) (2r^2 \cos(\psi) - 2r^2 + 1) - 4(r^2 - 1)r^5 \cos(2\psi) \\ & - 2(r^2 - 1)r \cos(\phi_1) ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)) + 8(2r^4 - 3r^2 + 1)r^3 \cos(\psi) + r \\ & = \alpha_{31} \left(-\sqrt{1 - r^2} \right) - \alpha_{33}r. \end{aligned} \quad (365)$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 2(r^2 - 1)r \sin(\psi) (2r^2 \cos(\psi) - 2r^2 + 1) \quad (366)$$

$$B = -2(r^2 - 1)r ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)), \quad (367)$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{31}(-\sqrt{1 - r^2}) - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \quad (368)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{\alpha_{31}(-\sqrt{1 - r^2}) - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (369)$$

which yields two solutions for each value of ψ .

Pre-multiplying (362) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & -12r^7 + 20r^5 - 10r^3 + 2(r^2 - 1)r \sin(\psi) \sin(\phi_2) (2r^2 \cos(\psi) - 2r^2 + 1) - 4(r^2 - 1)r^5 \cos(2\psi) \\ & - 2(r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\psi) - (2r^2 - 1)(r^2 \cos(2\psi) + 3r^2 - 2)) + 8(2r^4 - 3r^2 + 1)r^3 \cos(\psi) + r \\ & = \alpha_{13} \left(-\sqrt{1 - r^2} \right) - \alpha_{33}r. \end{aligned} \quad (370)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$, it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{\alpha_{13}\sqrt{1 - r^2} - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \quad (371)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{\alpha_{13}\sqrt{1 - r^2} - r(\alpha_{33} - (4r^6 - 4r^4) \cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\psi) + 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (372)$$

which yields two solutions for each value of ψ .

1.13 CGC $_{\beta}|C$ Paths

1.13.1 $L^+G^+L_{\beta}^+|L^-$ Paths

For a $L_{\phi_1}^+G_{\phi_2}^+L_{\beta}^+|L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{G^+}(\phi_2) \mathbf{M}_{L^+}(r, \beta) \mathbf{M}_{L^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (373)$$

Pre-multiplying (373) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{L-} :

$$\begin{aligned} & -2r^4 + 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) + 2(r^2 - 1)r^2 \cos(\beta) + r^2 + \cos(\phi_2) (2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1) \\ & = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r), \end{aligned} \quad (374)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 2(r^2 - 1)r \sin(\beta) \quad (375)$$

$$B = 2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1, \quad (376)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{-(\alpha_{11}(r^2 - 1)) - r(2r^3 \cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \quad (377)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{-(\alpha_{11}(r^2 - 1)) - r(2r^3 \cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1 - r^2} - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (378)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (373) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{L-} :

$$\begin{aligned} & 2r^5 - r^3 - (r^2 - 1)r \cos(\phi_1) (2 \cos(\beta) (r^2 \cos(\phi_2) - r^2 + 1) - 2r^2 \cos(\phi_2) + 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) - 1) \\ & + (r^2 - 1) \sin(\phi_1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) - 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) \\ & - (2r^5 - 3r^3 + r) \cos(\phi_2) - 4(r^2 - 1)r^3 \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) = \alpha_{33}r - \alpha_{31}\sqrt{1 - r^2}. \end{aligned} \quad (379)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = (r^2 - 1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) \quad (380)$$

$$D = - (r^2 - 1) r (2 \cos(\beta) (r^2 \cos(\phi_2) - r^2 + 1) - 2r^2 \cos(\phi_2) + 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) - 1), \quad (381)$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{C^2 + D^2}} - \frac{\alpha_{31} \sqrt{1 - r^2}}{\sqrt{C^2 + D^2}} \quad (382)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{C^2 + D^2}} - \frac{\alpha_{31} \sqrt{1 - r^2}}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \quad (383)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (373) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & 2r^5 - r^3 - (r^2 - 1) \sin(\phi_3) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) - 1)) - 2(r^2 - 1) r^2 \sin(\beta) \sin(\phi_2) - (2r^5 - 3r^3 + r) \cos(\phi_2) \\ & - (r^2 - 1) \cos(\phi_3) \left(\sin(\beta) \sin(\phi_2) + 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) - 2(2r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right) \\ & - 4(r^2 - 1) r^3 \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{13} \sqrt{1 - r^2} + \alpha_{33} r. \end{aligned} \quad (384)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3 . Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin \sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos \sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = - (r^2 - 1) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) - 1)) \quad (385)$$

$$F = - (r^2 - 1) \left(\sin(\beta) \sin(\phi_2) + 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) - 2(2r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right), \quad (386)$$

it is obtained that

$$\cos(\phi_3 - \sigma) = \frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{E^2 + F^2}} + \frac{\alpha_{13} \sqrt{1 - r^2}}{\sqrt{E^2 + F^2}} \quad (387)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{E^2 + F^2}} + \frac{\alpha_{13} \sqrt{1 - r^2}}{\sqrt{E^2 + F^2}} \right) + \tan^{-1} \left(\frac{E}{F} \right), \quad (388)$$

which yields two solutions for each value of ϕ_2 .

1.13.2 $R^+ G^+ R_\beta^+ | R^-$ Paths

For a $R_{\phi_1}^+ G_{\phi_2}^+ R_\beta^+ | R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) \mathbf{M}_{G^+}(\phi_2) \mathbf{M}_{R^+}(r, \beta) \mathbf{M}_{R^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (389)$$

Pre-multiplying (389) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_{R-} :

$$\begin{aligned} & -2r^4 + 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) + 2(r^2 - 1)r^2 \cos(\beta) + r^2 + \cos(\phi_2)(2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1) \\ & = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}(-\sqrt{1 - r^2}) + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r), \end{aligned} \quad (390)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 2(r^2 - 1)r \sin(\beta) \quad (391)$$

$$B = 2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1, \quad (392)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{-(\alpha_{11}(r^2 - 1)) - r(2r^3 \cos(\beta) - 2r^3 - \alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \quad (393)$$

$$\begin{aligned} \Rightarrow \phi_2 &= \cos^{-1} \left(\frac{-(\alpha_{11}(r^2 - 1)) - r(2r^3 \cos(\beta) - 2r^3 - \alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \right) \\ &+ \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (394)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (389) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{R-} :

$$\begin{aligned} & 2r^5 - r^3 - (r^2 - 1)r \cos(\phi_1)(2 \cos(\beta)(r^2 \cos(\phi_2) - r^2 + 1) - 2r^2 \cos(\phi_2) + 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) - 1) \\ & + (r^2 - 1) \sin(\phi_1)(\sin(\phi_2)(2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) - 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) \\ & - (2r^5 - 3r^3 + r) \cos(\phi_2) - 4(r^2 - 1)r^3 \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) = \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r. \end{aligned} \quad (395)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$C = (r^2 - 1)(\sin(\phi_2)(2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) \quad (396)$$

$$D = -(r^2 - 1)r(2 \cos(\beta)(r^2 \cos(\phi_2) - r^2 + 1) - 2r^2 \cos(\phi_2) + 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) - 1), \quad (397)$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{r(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1)r \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) + r))}{\sqrt{C^2 + D^2}}$$

$$+ \frac{\alpha_{31}\sqrt{1 - r^2}}{\sqrt{C^2 + D^2}} \quad (398)$$

$$\begin{aligned} \Rightarrow \phi_1 &= \cos^{-1} \left(\frac{r(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1)r \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) + r))}{\sqrt{C^2 + D^2}} \right) \\ &+ \frac{\alpha_{31}\sqrt{1 - r^2}}{\sqrt{C^2 + D^2}} + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (399)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (389) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & 2r^5 - r^3 - (r^2 - 1) \sin(\phi_3)(\cos(\beta) \sin(\phi_2) + r \sin(\beta)(\cos(\phi_2) - 1)) - 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) - (2r^5 - 3r^3 + r) \cos(\phi_2) \\ & - (r^2 - 1) \cos(\phi_3) \left(\sin(\beta) \sin(\phi_2) + 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) - 2(2r^2 - 1)r \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) - 2(r^2 - 1)r \cos(\phi_2) \right) \\ & - 4(r^2 - 1)r^3 \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) = \alpha_{33}r - \alpha_{13}\sqrt{1 - r^2}. \end{aligned} \quad (400)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3 . Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin \sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos \sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = -(r^2 - 1) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) - 1)) \quad (401)$$

$$F = -(r^2 - 1) \left(\sin(\beta) \sin(\phi_2) + 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) - 2(2r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right), \quad (402)$$

it is obtained that

$$\cos(\phi_3 - \sigma) = \frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{E^2 + F^2}} + \frac{-\alpha_{13} \sqrt{1 - r^2}}{\sqrt{E^2 + F^2}} \quad (403)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{E^2 + F^2}} + \frac{-\alpha_{13} \sqrt{1 - r^2}}{\sqrt{E^2 + F^2}} \right) + \tan^{-1} \left(\frac{E}{F} \right), \quad (404)$$

which yields two solutions for each value of ϕ_2 .

1.13.3 $R^- G^- R_\beta^- | R^+$ Paths

For a $R_{\phi_1}^- G_{\phi_2}^- R_\beta^- | R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1) \mathbf{M}_{G^-}(\phi_2) \mathbf{M}_{R^-}(r, \beta) \mathbf{M}_{R^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (405)$$

Pre-multiplying (405) with $\mathbf{u}_{R^-}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$\begin{aligned} & -2r^4 + 2(r^2 - 1) r \sin(\beta) \sin(\phi_2) + 2(r^2 - 1) r^2 \cos(\beta) + r^2 + \cos(\phi_2) (2r^4 - 2(r^2 - 1) r^2 \cos(\beta) - 3r^2 + 1) \\ & = -(\alpha_{11} (r^2 - 1)) - r (\alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r), \end{aligned} \quad (406)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2(r^2 - 1) r \sin(\beta) \quad (407)$$

$$B = 2r^4 - 2(r^2 - 1) r^2 \cos(\beta) - 3r^2 + 1, \quad (408)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{-(\alpha_{11} (r^2 - 1)) - r (2r^3 \cos(\beta) - 2r^3 + \alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \quad (409)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{-(\alpha_{11} (r^2 - 1)) - r (2r^3 \cos(\beta) - 2r^3 + \alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (410)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (405) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{R^+} :

$$\begin{aligned} & 2r^5 - r^3 - (r^2 - 1) r \cos(\phi_1) (2 \cos(\beta) (r^2 \cos(\phi_2) - r^2 + 1) - 2r^2 \cos(\phi_2) + 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) - 1) \\ & + (r^2 - 1) \sin(\phi_1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) - 2(r^2 - 1) r^2 \sin(\beta) \sin(\phi_2) \\ & - (2r^5 - 3r^3 + r) \cos(\phi_2) - 4(r^2 - 1) r^3 \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33} r - \alpha_{31} \sqrt{1 - r^2}. \end{aligned} \quad (411)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = (r^2 - 1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) \quad (412)$$

$$D = - (r^2 - 1) r (2 \cos(\beta) (r^2 \cos(\phi_2) - r^2 + 1) - 2r^2 \cos(\phi_2) + 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) - 1), \quad (413)$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{C^2 + D^2}} + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2 + D^2}} \quad (414)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{C^2 + D^2}} + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \quad (415)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (405) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & 2r^5 - r^3 - (r^2 - 1) \sin(\phi_3) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) - 1)) - 2(r^2 - 1) r^2 \sin(\beta) \sin(\phi_2) - (2r^5 - 3r^3 + r) \cos(\phi_2) \\ & - (r^2 - 1) \cos(\phi_3) \left(\sin(\beta) \sin(\phi_2) + 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) - 2(2r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right) \\ & - 4(r^2 - 1) r^3 \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \end{aligned} \quad (416)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3 . Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin \sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos \sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = - (r^2 - 1) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) - 1)) \quad (417)$$

$$F = - (r^2 - 1) \left(\sin(\beta) \sin(\phi_2) + 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) - 2(2r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right), \quad (418)$$

it is obtained that

$$\cos(\phi_3 - \sigma) = \frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{E^2 + F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \quad (419)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{E^2 + F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \right) + \tan^{-1} \left(\frac{E}{F} \right), \quad (420)$$

which yields two solutions for each value of ϕ_2 .

1.13.4 $L^-G^-L_\beta^-|L^+$ Paths

For a $L_{\phi_1}^-G_{\phi_2}^-L_\beta^-|L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L-}(r, \phi_1) \mathbf{M}_{G-}(\phi_2) \mathbf{M}_{L-}(r, \beta) \mathbf{M}_{L+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (421)$$

Pre-multiplying (421) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{L+} :

$$\begin{aligned} & -2r^4 + 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) + 2(r^2 - 1)r^2 \cos(\beta) + r^2 + \cos(\phi_2)(2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1) \\ & = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}(-\sqrt{1 - r^2}) + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r), \end{aligned} \quad (422)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 2(r^2 - 1)r \sin(\beta) \quad (423)$$

$$B = 2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1, \quad (424)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{-(\alpha_{11}(r^2 - 1)) - r(2r^3 \cos(\beta) - 2r^3 - \alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \quad (425)$$

$$\begin{aligned} \Rightarrow \phi_2 &= \cos^{-1} \left(\frac{-(\alpha_{11}(r^2 - 1)) - r(2r^3 \cos(\beta) - 2r^3 - \alpha_{13}\sqrt{1 - r^2} + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \right) \\ &+ \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (426)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (421) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{L+} :

$$\begin{aligned} & 2r^5 - r^3 - (r^2 - 1)r \cos(\phi_1)(2 \cos(\beta)(r^2 \cos(\phi_2) - r^2 + 1) - 2r^2 \cos(\phi_2) + 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) - 1) \\ & + (r^2 - 1) \sin(\phi_1)(\sin(\phi_2)(2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) - 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) \\ & - (2r^5 - 3r^3 + r) \cos(\phi_2) - 4(r^2 - 1)r^3 \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) = \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r. \end{aligned} \quad (427)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$C = (r^2 - 1)(\sin(\phi_2)(2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) \quad (428)$$

$$D = -(r^2 - 1)r(2 \cos(\beta)(r^2 \cos(\phi_2) - r^2 + 1) - 2r^2 \cos(\phi_2) + 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) - 1), \quad (429)$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{r(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1)r \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) + r))}{\sqrt{C^2 + D^2}}$$

$$+ \frac{\alpha_{31}\sqrt{1 - r^2}}{\sqrt{C^2 + D^2}} \quad (430)$$

$$\begin{aligned} \Rightarrow \phi_1 &= \cos^{-1} \left(\frac{r(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1)r \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) + r))}{\sqrt{C^2 + D^2}} \right) \\ &+ \frac{\alpha_{31}\sqrt{1 - r^2}}{\sqrt{C^2 + D^2}} + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (431)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (421) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & 2r^5 - r^3 - (r^2 - 1) \sin(\phi_3)(\cos(\beta) \sin(\phi_2) + r \sin(\beta)(\cos(\phi_2) - 1)) - 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) - (2r^5 - 3r^3 + r) \cos(\phi_2) \\ & - (r^2 - 1) \cos(\phi_3) \left(\sin(\beta) \sin(\phi_2) + 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) - 2(2r^2 - 1)r \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) - 2(r^2 - 1)r \cos(\phi_2) \right) \\ & - 4(r^2 - 1)r^3 \cos(\beta) \sin^2 \left(\frac{\phi_2}{2} \right) = \alpha_{33}r - \alpha_{13}\sqrt{1 - r^2}. \end{aligned} \quad (432)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3 . Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin \sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos \sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = -(r^2 - 1) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) - 1)) \quad (433)$$

$$F = -(r^2 - 1) \left(\sin(\beta) \sin(\phi_2) + 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) - 2(2r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right), \quad (434)$$

it is obtained that

$$\cos(\phi_3 - \sigma) = \frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{E^2 + F^2}} + \frac{-\alpha_{13} \sqrt{1 - r^2}}{\sqrt{E^2 + F^2}} \quad (435)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{r \left(\alpha_{33} + (2r^4 - 3r^2 + 1) \cos(\phi_2) + r \left(-2r^3 + 2(r^2 - 1) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r \cos(\beta) \sin^2\left(\frac{\phi_2}{2}\right) + r \right) \right)}{\sqrt{E^2 + F^2}} + \frac{-\alpha_{13} \sqrt{1 - r^2}}{\sqrt{E^2 + F^2}} \right) + \tan^{-1} \left(\frac{E}{F} \right), \quad (436)$$

which yields two solutions for each value of ϕ_2 .

1.13.5 $L^+ G^+ R_\beta^+ | R^-$ Paths

For a $L_{\phi_1}^+ G_{\phi_2}^+ R_\beta^+ | R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{G^+}(\phi_2) \mathbf{M}_{R^+}(r, \beta) \mathbf{M}_{R^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (437)$$

Pre-multiplying (437) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{R^-} :

$$-2r^4 - 2(r^2 - 1) r \sin(\beta) \sin(\phi_2) + 2(r^2 - 1) r^2 \cos(\beta) + r^2 + \cos(\phi_2) (-2r^4 + 2(r^2 - 1) r^2 \cos(\beta) + 3r^2 - 1) = \alpha_{11} (r^2 - 1) - r (\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r), \quad (438)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = -2(r^2 - 1) r \sin(\beta) \quad (439)$$

$$B = -(2r^4 - 2(r^2 - 1) r^2 \cos(\beta) - 3r^2 + 1), \quad (440)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{\alpha_{11} (r^2 - 1) - r (2r^3 \cos(\beta) - 2r^3 + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \quad (441)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(\frac{\alpha_{11} (r^2 - 1) - r (2r^3 \cos(\beta) - 2r^3 + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right), \quad (442)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (437) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{R^-} :

$$\begin{aligned} & 2r^5 - r^3 + (r^2 - 1) r \cos(\phi_1) (2 \cos(\beta) (r^2 \cos(\phi_2) + r^2 - 1) - 2r^2 \cos(\phi_2) - 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) + 1) \\ & - (r^2 - 1) \sin(\phi_1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) + 2(r^2 - 1) r^2 \sin(\beta) \sin(\phi_2) \\ & + (2r^5 - 3r^3 + r) \cos(\phi_2) - 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r. \end{aligned} \quad (443)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -(r^2 - 1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) \quad (444)$$

$$D = (r^2 - 1) r (2 \cos(\beta) (r^2 \cos(\phi_2) + r^2 - 1) - 2r^2 \cos(\phi_2) - 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) + 1), \quad (445)$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{C^2 + D^2}} + \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2 + D^2}} \quad (446)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{C^2 + D^2}} + \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \quad (447)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (437) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & 2r^5 - r^3 + (r^2 - 1) \sin(\phi_3)(\cos(\beta) \sin(\phi_2) + r \sin(\beta)(\cos(\phi_2) + 1)) + 2(r^2 - 1) r^2 \sin(\beta) \sin(\phi_2) + (2r^5 - 3r^3 + r) \cos(\phi_2) \\ & + (r^2 - 1) \cos(\phi_3) \left(\sin(\beta) \sin(\phi_2) - 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) + 2(2r^2 - 1) r \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right) \\ & - 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r. \end{aligned} \quad (448)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3 . Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin \sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos \sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = (r^2 - 1) (\cos(\beta) \sin(\phi_2) + r \sin(\beta)(\cos(\phi_2) + 1)) \quad (449)$$

$$F = (r^2 - 1) \left(\sin(\beta) \sin(\phi_2) - 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) + 2(2r^2 - 1) r \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right), \quad (450)$$

it is obtained that

$$\cos(\phi_3 - \sigma) = \frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{E^2 + F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \quad (451)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{E^2 + F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \right) + \tan^{-1} \left(\frac{E}{F} \right), \quad (452)$$

which yields two solutions for each value of ϕ_2 .

1.13.6 $R^+G^+L_\beta^+|L^-$ Paths

For a $R_{\phi_1}^+ G_{\phi_2}^+ L_\beta^+ | L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) \mathbf{M}_{G^+}(\phi_2) \mathbf{M}_{L^+}(r, \beta) \mathbf{M}_{L^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (453)$$

Pre-multiplying (453) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_L :-

$$\begin{aligned} & -2r^4 - 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) + 2(r^2 - 1)r^2 \cos(\beta) + r^2 + \cos(\phi_2) (-2r^4 + 2(r^2 - 1)r^2 \cos(\beta) + 3r^2 - 1) \\ & = \alpha_{11}(r^2 - 1) + r(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r), \end{aligned} \quad (454)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = -2(r^2 - 1)r \sin(\beta) \quad (455)$$

$$B = -(2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1), \quad (456)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{r(-r(\alpha_{33} + 2(r^2 - 1)\cos(\beta) - 2r^2 + 1) + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2}) + \alpha_{11}(r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (457)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{r(-r(\alpha_{33} + 2(r^2 - 1)\cos(\beta) - 2r^2 + 1) + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2}) + \alpha_{11}(r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (458)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (453) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_L :-

$$\begin{aligned} & 2r^5 - r^3 + (r^2 - 1)r \cos(\phi_1) (2\cos(\beta)(r^2 \cos(\phi_2) + r^2 - 1) - 2r^2 \cos(\phi_2) - 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) + 1) \\ & - (r^2 - 1) \sin(\phi_1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) + 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) \\ & + (2r^5 - 3r^3 + r) \cos(\phi_2) - 4(r^2 - 1)r^3 \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}. \end{aligned} \quad (459)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -(r^2 - 1)(\sin(\phi_2)(2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) \quad (460)$$

$$D = (r^2 - 1)r(2\cos(\beta)(r^2 \cos(\phi_2) + r^2 - 1) - 2r^2 \cos(\phi_2) - 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) + 1), \quad (461)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1)r^3 \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) + \alpha_{33}r}{\sqrt{C^2 + D^2}} \\ & + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2 + D^2}} \end{aligned} \quad (462)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1)r^3 \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) + \alpha_{33}r}{\sqrt{C^2 + D^2}} \right) \\ & + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (463)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (453) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G+} :-

$$\begin{aligned} & 2r^5 - r^3 + (r^2 - 1) \sin(\phi_3) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) + 1)) + 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) + (2r^5 - 3r^3 + r) \cos(\phi_2) \\ & + (r^2 - 1) \cos(\phi_3) \left(\sin(\beta) \sin(\phi_2) - 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) + 2(2r^2 - 1)r \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) - 2(r^2 - 1)r \cos(\phi_2) \right) \\ & - 4(r^2 - 1)r^3 \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \end{aligned} \quad (464)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3 . Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin \sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos \sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = (r^2 - 1) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) + 1)) \quad (465)$$

$$F = (r^2 - 1) \left(\sin(\beta) \sin(\phi_2) - 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) + 2(2r^2 - 1) r \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right), \quad (466)$$

it is obtained that

$$\cos(\phi_3 - \sigma) = \frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{E^2 + F^2}} + \frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \quad (467)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{E^2 + F^2}} + \frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \right) + \tan^{-1} \left(\frac{E}{F} \right), \quad (468)$$

which yields two solutions for each value of ϕ_2 .

1.13.7 $R^-G^-L_\beta^-|L^+$ Paths

For a $R_{\phi_1}^-G_{\phi_2}^-L_{\beta}^-|L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1) \mathbf{M}_{G^-}(\phi_2) \mathbf{M}_{L^-}(r, \beta) \mathbf{M}_{L^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (469)$$

Pre-multiplying (469) with $\mathbf{u}_{R^-}^T$ and post-multiplying \mathbf{u}_{L^+} :

$$\begin{aligned} & -2r^4 - 2(r^2 - 1) r \sin(\beta) \sin(\phi_2) + 2(r^2 - 1) r^2 \cos(\beta) + r^2 + \cos(\phi_2) (-2r^4 + 2(r^2 - 1) r^2 \cos(\beta) + 3r^2 - 1) \\ & = \alpha_{11} (r^2 - 1) - r (\alpha_{13} \sqrt{1-r^2} + \alpha_{31} \sqrt{1-r^2} + \alpha_{33}r), \end{aligned} \quad (470)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = -2(r^2 - 1) r \sin(\beta) \quad (471)$$

$$B = -(2r^4 - 2(r^2 - 1) r^2 \cos(\beta) - 3r^2 + 1), \quad (472)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{\alpha_{11} (r^2 - 1) - r (2r^3 \cos(\beta) - 2r^3 + \alpha_{13} \sqrt{1-r^2} + \alpha_{31} \sqrt{1-r^2} + \alpha_{33}r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \quad (473)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{\alpha_{11} (r^2 - 1) - r (2r^3 \cos(\beta) - 2r^3 + \alpha_{13} \sqrt{1-r^2} + \alpha_{31} \sqrt{1-r^2} + \alpha_{33}r - 2r \cos(\beta) + r)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (474)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (469) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{L^+} :

$$\begin{aligned} & 2r^5 - r^3 + (r^2 - 1) r \cos(\phi_1) (2 \cos(\beta) (r^2 \cos(\phi_2) + r^2 - 1) - 2r^2 \cos(\phi_2) - 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) + 1) \\ & - (r^2 - 1) \sin(\phi_1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) + 2(r^2 - 1) r^2 \sin(\beta) \sin(\phi_2) \\ & + (2r^5 - 3r^3 + r) \cos(\phi_2) - 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{31} \sqrt{1-r^2}. \end{aligned} \quad (475)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -(r^2 - 1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) \quad (476)$$

$$D = (r^2 - 1) r (2 \cos(\beta) (r^2 \cos(\phi_2) + r^2 - 1) - 2r^2 \cos(\phi_2) - 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) + 1), \quad (477)$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{C^2 + D^2}} + \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2 + D^2}} \quad (478)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{C^2 + D^2}} + \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \quad (479)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (469) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & 2r^5 - r^3 + (r^2 - 1) \sin(\phi_3)(\cos(\beta) \sin(\phi_2) + r \sin(\beta)(\cos(\phi_2) + 1)) + 2(r^2 - 1) r^2 \sin(\beta) \sin(\phi_2) + (2r^5 - 3r^3 + r) \cos(\phi_2) \\ & + (r^2 - 1) \cos(\phi_3) \left(\sin(\beta) \sin(\phi_2) - 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) + 2(2r^2 - 1) r \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right) \\ & - 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r. \end{aligned} \quad (480)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3 . Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin \sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos \sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = (r^2 - 1) (\cos(\beta) \sin(\phi_2) + r \sin(\beta)(\cos(\phi_2) + 1)) \quad (481)$$

$$F = (r^2 - 1) \left(\sin(\beta) \sin(\phi_2) - 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) + 2(2r^2 - 1) r \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right), \quad (482)$$

it is obtained that

$$\cos(\phi_3 - \sigma) = \frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{E^2 + F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \quad (483)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{E^2 + F^2}} + \frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \right) + \tan^{-1} \left(\frac{E}{F} \right), \quad (484)$$

which yields two solutions for each value of ϕ_2 .

1.13.8 $L^-G^-R_\beta^-|R^+$ Paths

For a $L_{\phi_1}^-G_{\phi_2}^-R_\beta^-|R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L-}(r, \phi_1) \mathbf{M}_{G-}(\phi_2) \mathbf{M}_{R-}(r, \beta) \mathbf{M}_{R+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (485)$$

Pre-multiplying (485) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{R+} :

$$\begin{aligned} & -2r^4 - 2(r^2 - 1)r \sin(\beta) \sin(\phi_2) + 2(r^2 - 1)r^2 \cos(\beta) + r^2 + \cos(\phi_2) (-2r^4 + 2(r^2 - 1)r^2 \cos(\beta) + 3r^2 - 1) \\ & = \alpha_{11}(r^2 - 1) + r(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r), \end{aligned} \quad (486)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = -2(r^2 - 1)r \sin(\beta) \quad (487)$$

$$B = -(2r^4 - 2(r^2 - 1)r^2 \cos(\beta) - 3r^2 + 1), \quad (488)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{r(-r(\alpha_{33} + 2(r^2 - 1)\cos(\beta) - 2r^2 + 1) + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2}) + \alpha_{11}(r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (489)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{r(-r(\alpha_{33} + 2(r^2 - 1)\cos(\beta) - 2r^2 + 1) + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2}) + \alpha_{11}(r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (490)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (485) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{R+} :

$$\begin{aligned} & 2r^5 - r^3 + (r^2 - 1)r \cos(\phi_1) (2\cos(\beta)(r^2 \cos(\phi_2) + r^2 - 1) - 2r^2 \cos(\phi_2) - 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) + 1) \\ & - (r^2 - 1) \sin(\phi_1) (\sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) + 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) \\ & + (2r^5 - 3r^3 + r) \cos(\phi_2) - 4(r^2 - 1)r^3 \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}. \end{aligned} \quad (491)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -(r^2 - 1)(\sin(\phi_2)(2r^2 \cos(\beta) - 2r^2 + 1) + 2r \sin(\beta) \cos(\phi_2)) \quad (492)$$

$$D = (r^2 - 1)r(2\cos(\beta)(r^2 \cos(\phi_2) + r^2 - 1) - 2r^2 \cos(\phi_2) - 2r^2 - 2r \sin(\beta) \sin(\phi_2) + \cos(\phi_2) + 1), \quad (493)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1)r^3 \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) + \alpha_{33}r}{\sqrt{C^2 + D^2}} \\ & + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2 + D^2}} \end{aligned} \quad (494)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1)r^3 \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) + \alpha_{33}r}{\sqrt{C^2 + D^2}} \right) \\ & + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (495)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (485) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & 2r^5 - r^3 + (r^2 - 1) \sin(\phi_3) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) + 1)) + 2(r^2 - 1)r^2 \sin(\beta) \sin(\phi_2) + (2r^5 - 3r^3 + r) \cos(\phi_2) \\ & + (r^2 - 1) \cos(\phi_3) \left(\sin(\beta) \sin(\phi_2) - 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) + 2(2r^2 - 1)r \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) - 2(r^2 - 1)r \cos(\phi_2) \right) \\ & - 4(r^2 - 1)r^3 \cos(\beta) \cos^2 \left(\frac{\phi_2}{2} \right) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \end{aligned} \quad (496)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3 . Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin \sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos \sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = (r^2 - 1) (\cos(\beta) \sin(\phi_2) + r \sin(\beta) (\cos(\phi_2) + 1)) \quad (497)$$

$$F = (r^2 - 1) \left(\sin(\beta) \sin(\phi_2) - 2r^3 - 2r^2 \sin(\beta) \sin(\phi_2) + 2(2r^2 - 1) r \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) - 2(r^2 - 1) r \cos(\phi_2) \right), \quad (498)$$

it is obtained that

$$\cos(\phi_3 - \sigma) = \frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{E^2 + F^2}} + \frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \quad (499)$$

$$\Rightarrow \phi_3 = \cos^{-1} \left(\frac{-2r^5 + r^3 + (-2r^5 + 3r^3 - r) \cos(\phi_2) + (2r^2 - 2r^4) \sin(\beta) \sin(\phi_2) + 4(r^2 - 1) r^3 \cos(\beta) \cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33}r}{\sqrt{E^2 + F^2}} + \frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2 + F^2}} \right) + \tan^{-1} \left(\frac{E}{F} \right), \quad (500)$$

which yields two solutions for each value of ϕ_2 .

1.14 $CC_\mu|C_\mu C$ Paths

1.14.1 $L^+R_\mu^+|R_\mu^-L^-$ Paths

For a $L_{\phi_1}^+R_\mu^+|R_\mu^-L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{R^+}(r, \mu) \mathbf{M}_{R^-}(r, \mu) \mathbf{M}_{L^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (501)$$

Pre-multiplying (501) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$\begin{aligned} & -12r^6 + 16r^4 - 8(r^2 - 1)^2 r^2 \cos^2(\mu) + 4(r^2 - 1)^2 r^2 - 6r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\mu) + 1 \\ & = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \end{aligned} \quad (502)$$

which gives

$$\cos(\mu) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2}\sqrt{r^2(r^2 - 1)^2(\alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + \alpha_{11}(r^2 - 1) + 1)}}{4r^2(r^2 - 1)^2}, \quad (503)$$

and yields four solutions of μ .

Pre-multiplying (501) with $\mathbf{u}_{G^-}^T$ and post-multiplying with \mathbf{u}_{L^-} :

$$\begin{aligned} & r \left(12r^6 - 16r^4 + 2(r^2 - 1) \sin(\mu) \sin(\phi_1) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos(2\mu) + 6r^2 \right. \\ & \quad \left. - 2(r^2 - 1) \cos(\phi_1) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)) - 8(2r^4 - 3r^2 + 1) r^2 \cos(\mu) - 1 \right) \\ & = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}. \end{aligned} \quad (504)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2r(r^2 - 1) \sin(\mu) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) \quad (505)$$

$$B = -2r(r^2 - 1) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)), \quad (506)$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \quad (507)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(\frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \right) \\ + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (508)$$

which yields two solutions for each value of μ .

Pre-multiplying (501) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} r \left(12r^6 - 16r^4 + 2(r^2-1) \sin(\mu) \sin(\phi_2) (2(r^2-1) \cos(\mu) - 2r^2 + 1) + 4(r^2-1)^2 r^2 \cos(2\mu) + 6r^2 \right. \\ \left. - 2(r^2-1) \cos(\phi_2) ((2r^2-1) ((r^2-1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)) - 8(2r^6 - 3r^4 + r^2) \cos(\mu) - 1 \right) \\ = \alpha_{13} \sqrt{1-r^2} + \alpha_{33}r. \end{aligned} \quad (509)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_2 - \gamma) = \frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \quad (510) \\ \Rightarrow \phi_2 = \cos^{-1} \left(\frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \right) \\ + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (511)$$

which yields two solutions for each value of μ .

1.14.2 $R^+L_\mu^+|L_\mu^-R^-$ Paths

For a $R_{\phi_1}^+L_\mu^+|L_\mu^-R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) \mathbf{M}_{L^+}(r, \mu) \mathbf{M}_{L^-}(r, \mu) \mathbf{M}_{R^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (512)$$

Pre-multiplying (512) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_{R-} :

$$\begin{aligned} -12r^6 + 16r^4 - 8(r^2-1)^2 r^2 \cos^2(\mu) + 4(r^2-1)^2 r^2 - 6r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\mu) + 1 \\ = -(\alpha_{11}(r^2-1)) - r \left(\alpha_{13}(-\sqrt{1-r^2}) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r \right), \end{aligned} \quad (513)$$

which gives

$$\cos(\mu) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2} \sqrt{r^2(r^2-1)^2 (\alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + \alpha_{11}(r^2-1) + 1)}}{4r^2(r^2-1)^2}, \quad (514)$$

and yields four solutions of μ .

Pre-multiplying (512) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{R-} :

$$\begin{aligned} r \left(12r^6 - 16r^4 + 2(r^2-1) \sin(\mu) \sin(\phi_1) (2(r^2-1) \cos(\mu) - 2r^2 + 1) + 4(r^2-1)^2 r^2 \cos(2\mu) + 6r^2 \right. \\ \left. - 2(r^2-1) \cos(\phi_1) ((2r^2-1) ((r^2-1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)) - 8(2r^4 - 3r^2 + 1) r^2 \cos(\mu) - 1 \right) \\ = \alpha_{31} \sqrt{1-r^2} + \alpha_{33}r. \end{aligned} \quad (515)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2r(r^2 - 1) \sin(\mu) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) \quad (516)$$

$$B = -2r(r^2 - 1) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)), \quad (517)$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{31}\sqrt{1-r^2} - 4(r^2 - 1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \quad (518)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} & \left(\frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{31}\sqrt{1-r^2} - 4(r^2 - 1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (519)$$

which yields two solutions for each value of μ .

Pre-multiplying (512) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & r \left(12r^6 - 16r^4 + 2(r^2 - 1) \sin(\mu) \sin(\phi_2) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos(2\mu) + 6r^2 \right. \\ & \quad \left. - 2(r^2 - 1) \cos(\phi_2) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)) - 8(2r^6 - 3r^4 + r^2) \cos(\mu) - 1 \right) \\ & = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \end{aligned} \quad (520)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2 - 1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \quad (521)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2 - 1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (522)$$

which yields two solutions for each value of μ .

1.14.3 $R^-L_\mu^-|L_\mu^+R^+$ Paths

For a $R_{\phi_1}^-L_\mu^-|L_\mu^+R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R-}(r, \phi_1) \mathbf{M}_{L-}(r, \mu) \mathbf{M}_{L+}(r, \mu) \mathbf{M}_{R+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (523)$$

Pre-multiplying (523) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_{R+} :

$$\begin{aligned} & -12r^6 + 16r^4 - 8(r^2 - 1)^2 r^2 \cos^2(\mu) + 4(r^2 - 1)^2 r^2 - 6r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\mu) + 1 \\ & = -(\alpha_{11}(r^2 - 1)) - r(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \end{aligned} \quad (524)$$

which gives

$$\cos(\mu) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2} \sqrt{r^2(r^2 - 1)^2 (\alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + \alpha_{11}(r^2 - 1) + 1)}}{4r^2(r^2 - 1)^2}, \quad (525)$$

and yields four solutions of μ .

Pre-multiplying (523) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{R+} :

$$r \left(12r^6 - 16r^4 + 2(r^2 - 1) \sin(\mu) \sin(\phi_1) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos(2\mu) + 6r^2 \right. \\ \left. - 2(r^2 - 1) \cos(\phi_1) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)) - 8(2r^4 - 3r^2 + 1) r^2 \cos(\mu) - 1 \right) \\ = \alpha_{33}r - \alpha_{31} \sqrt{1 - r^2}. \quad (526)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2r(r^2 - 1) \sin(\mu) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) \quad (527)$$

$$B = -2r(r^2 - 1) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)), \quad (528)$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}} \quad (529)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}} \right) \\ + \tan^{-1} \left(\frac{A}{B} \right), \quad (530)$$

which yields two solutions for each value of μ .

Pre-multiplying (523) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G-} :

$$r \left(12r^6 - 16r^4 + 2(r^2 - 1) \sin(\mu) \sin(\phi_2) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos(2\mu) + 6r^2 \right. \\ \left. - 2(r^2 - 1) \cos(\phi_2) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)) - 8(2r^6 - 3r^4 + r^2) \cos(\mu) - 1 \right) \\ = \alpha_{13} \sqrt{1 - r^2} + \alpha_{33}r. \quad (531)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}} \quad (532)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(\frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33}r + r}{\sqrt{A^2+B^2}} \right) \\ + \tan^{-1} \left(\frac{A}{B} \right), \quad (533)$$

which yields two solutions for each value of μ .

1.14.4 $L^- R_\mu^- | R_\mu^+ L^+$ Paths

For a $L_{\phi_1}^- R_\mu^- | R_\mu^+ L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L-}(r, \phi_1) \mathbf{M}_{R-}(r, \mu) \mathbf{M}_{R+}(r, \mu) \mathbf{M}_{L+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (534)$$

Pre-multiplying (534) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{L+} :

$$-12r^6 + 16r^4 - 8(r^2 - 1)^2 r^2 \cos^2(\mu) + 4(r^2 - 1)^2 r^2 - 6r^2 + 8(2r^6 - 3r^4 + r^2) \cos(\mu) + 1 \\ = -(\alpha_{11}(r^2 - 1)) - r \left(\alpha_{13} \left(-\sqrt{1 - r^2} \right) + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33}r \right), \quad (535)$$

which gives

$$\cos(\mu) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2} \sqrt{r^2 (r^2 - 1)^2 (\alpha_{33} r^2 - \alpha_{13} \sqrt{1 - r^2} r + \alpha_{31} \sqrt{1 - r^2} r + \alpha_{11} (r^2 - 1) + 1)}}{4r^2 (r^2 - 1)^2}, \quad (536)$$

and yields four solutions of μ .

Pre-multiplying (534) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{L+} :

$$\begin{aligned} & r \left(12r^6 - 16r^4 + 2(r^2 - 1) \sin(\mu) \sin(\phi_1) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos(2\mu) + 6r^2 \right. \\ & \quad \left. - 2(r^2 - 1) \cos(\phi_1) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)) - 8(2r^4 - 3r^2 + 1) r^2 \cos(\mu) - 1 \right) \\ & = \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r. \end{aligned} \quad (537)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 2r(r^2 - 1) \sin(\mu) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) \quad (538)$$

$$B = -2r(r^2 - 1) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)), \quad (539)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \gamma) &= \frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{31} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33} r + r}{\sqrt{A^2 + B^2}} \quad (540) \\ \Rightarrow \phi_1 &= \cos^{-1} \left(\frac{-12r^7 + 16r^5 - 6r^3 + \alpha_{31} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33} r + r}{\sqrt{A^2 + B^2}} \right) \\ & \quad + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (541)$$

which yields two solutions for each value of μ .

Pre-multiplying (534) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & r \left(12r^6 - 16r^4 + 2(r^2 - 1) \sin(\mu) \sin(\phi_2) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos(2\mu) + 6r^2 \right. \\ & \quad \left. - 2(r^2 - 1) \cos(\phi_2) ((2r^2 - 1) ((r^2 - 1) \cos(2\mu) + 3r^2 - 1) + (-8r^4 + 8r^2 - 1) \cos(\mu)) - 8(2r^6 - 3r^4 + r^2) \cos(\mu) - 1 \right) \\ & = \alpha_{33} r - \alpha_{13} \sqrt{1 - r^2}. \end{aligned} \quad (542)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_2 - \gamma) &= \frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{13} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33} r + r}{\sqrt{A^2 + B^2}} \quad (543) \\ \Rightarrow \phi_2 &= \cos^{-1} \left(\frac{-12r^7 + 16r^5 - 6r^3 - \alpha_{13} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^3 \cos(2\mu) + 8(2r^7 - 3r^5 + r^3) \cos(\mu) + \alpha_{33} r + r}{\sqrt{A^2 + B^2}} \right) \\ & \quad + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (544)$$

which yields two solutions for each value of μ .

1.15 $C|C_\beta G C_\beta|C$ Paths

1.15.1 $L^+|L_\beta^- G^- L_\beta^-|L^+$ Paths

For a $L_{\phi_1}^+|L_\beta^- G_{\phi_2}^- L_\beta^-|L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{L^-}(r, \beta) \mathbf{M}_{G^-}(\phi_2) \mathbf{M}_{L^-}(r, \beta) \mathbf{M}_{L^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (545)$$

Pre-multiplying (545) with \mathbf{u}_{L+}^T and post-multiplying \mathbf{u}_{L+} :

$$\begin{aligned} & 4(r^2 - 1)r \sin(\beta) \sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos^2(\beta) + 4(1 - 2r^2)(r^2 - 1)r^2 \cos(\beta) + (1 - 2r^2)^2 r^2 \\ & - (r^2 - 1) \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1) \\ & = r(\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r) - \alpha_{11}(r^2 - 1), \end{aligned} \quad (546)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 4(r^2 - 1)r \sin(\beta) (2r^2 \cos(\beta) - 2r^2 + 1) \quad (547)$$

$$B = -(r^2 - 1)(6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1), \quad (548)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{r \left(r \left(\alpha_{33} - (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 \right) + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} \right) - \alpha_{11}(r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (549)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{r \left(r \left(\alpha_{33} - (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 \right) + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} \right) - \alpha_{11}(r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (550)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (545) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{L+} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 - 4(r^2 - 1)r^4 \sin(2\beta) \sin(\phi_2) + 4(2r^4 - 3r^2 + 1)r^2 \sin(\beta) \sin(\phi_2) \\ & + (r^2 - 1)r \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1) - 4(r^2 - 1)^2 r^3 \cos^2(\beta) \\ & + 4(2r^7 - 3r^5 + r^3) \cos(\beta) + \sin(\phi_1) \left((r^5 + r) \sin(2\beta) - (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \right. \\ & \left. + r \sin(\beta) ((2r^4 - 3r^2 + 1)(\cos(\phi_2) - 1) - 2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) + 2r^2)) \right) + \cos(\phi_1) (4r^7 - 6r^5 - 6r^4 \sin(2\beta) \sin(\phi_2) \\ & + 2r^3 + 2(r^2 - 1)^2 (2r^2 - 1)r \cos^2(\beta) + (r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\beta) - (2r^2 - 1)((r^2 + 1) \cos(2\beta) + 3(r^2 - 1))) \\ & \left. + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2) \cos(\beta) + 1) \right) = -\alpha_{33}r - \alpha_{31} \sqrt{1 - r^2}. \end{aligned} \quad (551)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$\begin{aligned} C = & (r^5 + r) \sin(2\beta) - (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \\ & + r \sin(\beta) ((2r^4 - 3r^2 + 1)(\cos(\phi_2) - 1) - 2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) + 2r^2)) \end{aligned} \quad (552)$$

$$\begin{aligned} D = & 4r^7 - 6r^5 - 6r^4 \sin(2\beta) \sin(\phi_2) + 2r^3 + 2(r^2 - 1)^2 (2r^2 - 1)r \cos^2(\beta) \\ & + (r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\beta) - (2r^2 - 1)((r^2 + 1) \cos(2\beta) + 3(r^2 - 1))) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2) \cos(\beta) + 1), \end{aligned} \quad (553)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \end{aligned} \quad (554)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \right. \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & \left. + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (555)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (545) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 - 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1) \\ & + (r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) - 4(r^2-1)^2r^3\cos^2(\beta) \\ & + 4(2r^7 - 3r^5 + r^3)\cos(\beta) + \sin(\phi_3) \left((r^5 + r)\sin(2\beta) - (r^2-1)\sin(\phi_2)(\cos(\beta) - 2r^2\cos(\beta) + 2r^2\cos(2\beta)) \right) \\ & + r\sin(\beta) \left((2r^4 - 3r^2 + 1)(\cos(\phi_2) - 1) - 2\cos(\beta)((r^4 - 1)\cos(\phi_2) + 2r^2) \right) + \cos(\phi_3) \left(4r^7 - 6r^5 - 6r^4\sin(2\beta)\sin(\phi_2) \right. \\ & + 2r^3 + 2(r^2-1)^2(2r^2-1)r\cos^2(\beta) + (r^2-1)r\cos(\phi_2)((8r^4 - 8r^2 + 1)\cos(\beta) - (2r^2-1)((r^2+1)\cos(2\beta) + 3(r^2-1))) \\ & \left. + (-8r^7 + 16r^5 - 9r^3 + r)\cos(\beta) + \sin(\beta)\sin(\phi_2)(-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2)\cos(\beta) + 1) \right) = \alpha_{13}(-\sqrt{1-r^2}) - \alpha_{33}r. \end{aligned} \quad (556)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_3 - \theta) = & \frac{4r^7 - 4r^5 + r^3 - \alpha_{13}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \end{aligned} \quad (557)$$

$$\begin{aligned} \Rightarrow \phi_3 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 - \alpha_{13}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \right. \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & \left. + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (558)$$

which yields two solutions for each value of ϕ_2 .

1.15.2 $R^+|R_\beta^-G^-R_\beta^-|R^+$ Paths

For a $R_{\phi_1}^+|R_\beta^-G_{\phi_2}^-R_\beta^-|R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{R^-}(r, \beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r, \beta)\mathbf{M}_{R^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (559)$$

Pre-multiplying (559) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_{R+} :

$$\begin{aligned} & 4(r^2 - 1)r \sin(\beta) \sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos^2(\beta) + 4(1 - 2r^2)(r^2 - 1)r^2 \cos(\beta) + (1 - 2r^2)^2 r^2 \\ & - (r^2 - 1) \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1) \\ & = r \left(\alpha_{13} \left(-\sqrt{1 - r^2} \right) - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right) - \alpha_{11} (r^2 - 1), \end{aligned} \quad (560)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 4(r^2 - 1)r \sin(\beta) (2r^2 \cos(\beta) - 2r^2 + 1) \quad (561)$$

$$B = -(r^2 - 1) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1), \quad (562)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = -\frac{r \left(r \left((2(r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 - \alpha_{33} \right) + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} \right) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (563)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(-\frac{r \left(r \left((2(r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 - \alpha_{33} \right) + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} \right) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (564)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (559) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{R+} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 - 4(r^2 - 1)r^4 \sin(2\beta) \sin(\phi_2) + 4(2r^4 - 3r^2 + 1)r^2 \sin(\beta) \sin(\phi_2) \\ & + (r^2 - 1)r \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1) - 4(r^2 - 1)^2 r^3 \cos^2(\beta) \\ & + 4(2r^7 - 3r^5 + r^3) \cos(\beta) + \sin(\phi_1) \left((r^5 + r) \sin(2\beta) - (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \right. \\ & \left. + r \sin(\beta) ((2r^4 - 3r^2 + 1) (\cos(\phi_2) - 1) - 2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) + 2r^2)) \right) + \cos(\phi_1) (4r^7 - 6r^5 - 6r^4 \sin(2\beta) \sin(\phi_2) \\ & + 2r^3 + 2(r^2 - 1)^2 (2r^2 - 1)r \cos^2(\beta) + (r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\beta) - (2r^2 - 1)((r^2 + 1) \cos(2\beta) + 3(r^2 - 1))) \\ & \left. + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2) \cos(\beta) + 1) \right) = \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r. \end{aligned} \quad (565)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$\begin{aligned} C = & (r^5 + r) \sin(2\beta) - (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \\ & + r \sin(\beta) ((2r^4 - 3r^2 + 1) (\cos(\phi_2) - 1) - 2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) + 2r^2)) \end{aligned} \quad (566)$$

$$\begin{aligned} D = & 4r^7 - 6r^5 - 6r^4 \sin(2\beta) \sin(\phi_2) + 2r^3 + 2(r^2 - 1)^2 (2r^2 - 1)r \cos^2(\beta) \\ & + (r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\beta) - (2r^2 - 1)((r^2 + 1) \cos(2\beta) + 3(r^2 - 1))) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2) \cos(\beta) + 1), \end{aligned} \quad (567)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{4r^7 - 4r^5 + r^3 + \alpha_{31}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \end{aligned} \quad (568)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 + \alpha_{31}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \right. \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & \left. + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (569)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (559) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 - 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1) \\ & + (r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) - 4(r^2-1)^2r^3\cos^2(\beta) \\ & + 4(2r^7 - 3r^5 + r^3)\cos(\beta) + \sin(\phi_3) \left((r^5 + r)\sin(2\beta) - (r^2-1)\sin(\phi_2)(\cos(\beta) - 2r^2\cos(\beta) + 2r^2\cos(2\beta)) \right) \\ & + r\sin(\beta) \left((2r^4 - 3r^2 + 1)(\cos(\phi_2) - 1) - 2\cos(\beta)((r^4 - 1)\cos(\phi_2) + 2r^2) \right) + \cos(\phi_3) \left(4r^7 - 6r^5 - 6r^4\sin(2\beta)\sin(\phi_2) \right. \\ & + 2r^3 + 2(r^2-1)^2(2r^2-1)r\cos^2(\beta) + (r^2-1)r\cos(\phi_2)((8r^4 - 8r^2 + 1)\cos(\beta) - (2r^2-1)((r^2+1)\cos(2\beta) + 3(r^2-1))) \\ & \left. + (-8r^7 + 16r^5 - 9r^3 + r)\cos(\beta) + \sin(\beta)\sin(\phi_2)(-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2)\cos(\beta) + 1) \right) = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r. \end{aligned} \quad (570)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_3 - \theta) = & \frac{4r^7 - 4r^5 + r^3 + \alpha_{13}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \end{aligned} \quad (571)$$

$$\begin{aligned} \Rightarrow \phi_3 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 + \alpha_{13}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \right. \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & \left. + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (572)$$

which yields two solutions for each value of ϕ_2 .

1.15.3 $R^-|R_\beta^+G^+R_\beta^+|R^-$ Paths

For a $R_{\phi_1}^-|R_\beta^+G_{\phi_2}^+R_\beta^+|R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1)\mathbf{M}_{R^+}(r, \beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r, \beta)\mathbf{M}_{R^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (573)$$

Pre-multiplying (573) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_{R-} :

$$\begin{aligned} & 4(r^2 - 1)r \sin(\beta) \sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos^2(\beta) + 4(1 - 2r^2)(r^2 - 1)r^2 \cos(\beta) + (1 - 2r^2)^2 r^2 \\ & - (r^2 - 1) \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1) \\ & = r(\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r) - \alpha_{11}(r^2 - 1), \end{aligned} \quad (574)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 4(r^2 - 1)r \sin(\beta) (2r^2 \cos(\beta) - 2r^2 + 1) \quad (575)$$

$$B = -(r^2 - 1)(6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1), \quad (576)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{r \left(r \left(\alpha_{33} - (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 \right) + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} \right) - \alpha_{11}(r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (577)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(\frac{r \left(r \left(\alpha_{33} - (2(r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 \right) + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} \right) - \alpha_{11}(r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (578)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (573) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{R-} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 - 4(r^2 - 1)r^4 \sin(2\beta) \sin(\phi_2) + 4(2r^4 - 3r^2 + 1)r^2 \sin(\beta) \sin(\phi_2) \\ & + (r^2 - 1)r \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1) - 4(r^2 - 1)^2 r^3 \cos^2(\beta) \\ & + 4(2r^7 - 3r^5 + r^3) \cos(\beta) + \sin(\phi_1) \left((r^5 + r) \sin(2\beta) - (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \right) \\ & + r \sin(\beta) \left((2r^4 - 3r^2 + 1) (\cos(\phi_2) - 1) - 2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) + 2r^2) \right) + \cos(\phi_1) (4r^7 - 6r^5 - 6r^4 \sin(2\beta) \sin(\phi_2) \\ & + 2r^3 + 2(r^2 - 1)^2 (2r^2 - 1)r \cos^2(\beta) + (r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\beta) - (2r^2 - 1)((r^2 + 1) \cos(2\beta) + 3(r^2 - 1))) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2) \cos(\beta) + 1) = \alpha_{31}(-\sqrt{1 - r^2}) - \alpha_{33}r. \end{aligned} \quad (579)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$\begin{aligned} C = & (r^5 + r) \sin(2\beta) - (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \\ & + r \sin(\beta) ((2r^4 - 3r^2 + 1) (\cos(\phi_2) - 1) - 2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) + 2r^2)) \end{aligned} \quad (580)$$

$$\begin{aligned} D = & 4r^7 - 6r^5 - 6r^4 \sin(2\beta) \sin(\phi_2) + 2r^3 + 2(r^2 - 1)^2 (2r^2 - 1)r \cos^2(\beta) \\ & + (r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\beta) - (2r^2 - 1)((r^2 + 1) \cos(2\beta) + 3(r^2 - 1))) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2) \cos(\beta) + 1), \end{aligned} \quad (581)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \end{aligned} \quad (582)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(\frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \right. \\ \left. + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \right. \\ \left. + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (583)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (573) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 - 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1) \\ & + (r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) - 4(r^2-1)^2r^3\cos^2(\beta) \\ & + 4(2r^7 - 3r^5 + r^3)\cos(\beta) + \sin(\phi_3) \left((r^5 + r)\sin(2\beta) - (r^2-1)\sin(\phi_2)(\cos(\beta) - 2r^2\cos(\beta) + 2r^2\cos(2\beta)) \right) \\ & + r\sin(\beta) \left((2r^4 - 3r^2 + 1)(\cos(\phi_2) - 1) - 2\cos(\beta)((r^4 - 1)\cos(\phi_2) + 2r^2) \right) + \cos(\phi_3) \left(4r^7 - 6r^5 - 6r^4\sin(2\beta)\sin(\phi_2) \right. \\ & \left. + 2r^3 + 2(r^2-1)^2(2r^2-1)r\cos^2(\beta) + (r^2-1)r\cos(\phi_2)((8r^4 - 8r^2 + 1)\cos(\beta) - (2r^2-1)((r^2+1)\cos(2\beta) + 3(r^2-1))) \right. \\ & \left. + (-8r^7 + 16r^5 - 9r^3 + r)\cos(\beta) + \sin(\beta)\sin(\phi_2)(-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2)\cos(\beta) + 1) \right) = \alpha_{13}(-\sqrt{1-r^2}) - \alpha_{33}r. \end{aligned} \quad (584)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_3 - \theta) = & \frac{4r^7 - 4r^5 + r^3 - \alpha_{13}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \\ & + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \end{aligned} \quad (585)$$

$$\begin{aligned} \Rightarrow \phi_3 = \cos^{-1} \left(\frac{4r^7 - 4r^5 + r^3 - \alpha_{13}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta) - 2r^2 + 1)}{\sqrt{C^2 + D^2}} \right. \\ \left. + \frac{-(r^2-1)r\cos(\phi_2)(6r^4 - 6r^2 + (4r^2 - 8r^4)\cos(\beta) + 2(r^4 + r^2)\cos(2\beta) + 1) + 4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2 + D^2}} \right. \\ \left. + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta) - \alpha_{33}r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (586)$$

which yields two solutions for each value of ϕ_2 .

1.15.4 $L^-|L_\beta^+G^+L_\beta^+|L^-$ Paths

For a $L_{\phi_1}^-|L_\beta^+G^+L_{\phi_2}^+|L_\beta^-|L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1)\mathbf{M}_{L^+}(r, \beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r, \beta)\mathbf{M}_{L^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (587)$$

Pre-multiplying (587) with \mathbf{u}_L^T and post-multiplying \mathbf{u}_L :-

$$\begin{aligned} & 4(r^2 - 1)r \sin(\beta) \sin(\phi_2) (2r^2 \cos(\beta) - 2r^2 + 1) + 4(r^2 - 1)^2 r^2 \cos^2(\beta) + 4(1 - 2r^2)(r^2 - 1)r^2 \cos(\beta) + (1 - 2r^2)^2 r^2 \\ & - (r^2 - 1) \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1) \\ & = r \left(\alpha_{13} \left(-\sqrt{1 - r^2} \right) - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right) - \alpha_{11} (r^2 - 1), \end{aligned} \quad (588)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 4(r^2 - 1)r \sin(\beta) (2r^2 \cos(\beta) - 2r^2 + 1) \quad (589)$$

$$B = -(r^2 - 1) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1), \quad (590)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = -\frac{r \left(r \left((2(r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 - \alpha_{33} \right) + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} \right) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (591)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} & \left(-\frac{r \left(r \left((2(r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 - \alpha_{33} \right) + \alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} \right) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ & + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (592)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (587) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_L :-

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 - 4(r^2 - 1)r^4 \sin(2\beta) \sin(\phi_2) + 4(2r^4 - 3r^2 + 1)r^2 \sin(\beta) \sin(\phi_2) \\ & + (r^2 - 1)r \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2(r^4 + r^2) \cos(2\beta) + 1) - 4(r^2 - 1)^2 r^3 \cos^2(\beta) \\ & + 4(2r^7 - 3r^5 + r^3) \cos(\beta) + \sin(\phi_1) \left((r^5 + r) \sin(2\beta) - (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \right. \\ & \left. + r \sin(\beta) ((2r^4 - 3r^2 + 1) (\cos(\phi_2) - 1) - 2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) + 2r^2)) \right) + \cos(\phi_1) (4r^7 - 6r^5 - 6r^4 \sin(2\beta) \sin(\phi_2) \\ & + 2r^3 + 2(r^2 - 1)^2 (2r^2 - 1)r \cos^2(\beta) + (r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\beta) - (2r^2 - 1)((r^2 + 1) \cos(2\beta) + 3(r^2 - 1))) \\ & \left. + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2) \cos(\beta) + 1) \right) = \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r. \end{aligned} \quad (593)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$\begin{aligned} C = & (r^5 + r) \sin(2\beta) - (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \\ & + r \sin(\beta) ((2r^4 - 3r^2 + 1) (\cos(\phi_2) - 1) - 2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) + 2r^2)) \end{aligned} \quad (594)$$

$$\begin{aligned} D = & 4r^7 - 6r^5 - 6r^4 \sin(2\beta) \sin(\phi_2) + 2r^3 + 2(r^2 - 1)^2 (2r^2 - 1)r \cos^2(\beta) \\ & + (r^2 - 1)r \cos(\phi_2) ((8r^4 - 8r^2 + 1) \cos(\beta) - (2r^2 - 1)((r^2 + 1) \cos(2\beta) + 3(r^2 - 1))) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (-8r^6 + 16r^4 - 9r^2 + 4(2r^6 + r^2) \cos(\beta) + 1), \end{aligned} \quad (595)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{4r^7 - 4r^5 + r^3 + \alpha_{31}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1)}{\sqrt{C^2+D^2}} \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (596)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(\frac{4r^7 - 4r^5 + r^3 + \alpha_{31}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1)}{\sqrt{C^2+D^2}} \right. \\ \left. + \frac{-(r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \right. \\ \left. + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (597)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (587) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 - 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1) \\ & + (r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1) - 4(r^2-1)^2r^3\cos^2(\beta) \\ & + 4(2r^7-3r^5+r^3)\cos(\beta) + \sin(\phi_3) \left((r^5+r)\sin(2\beta) - (r^2-1)\sin(\phi_2)(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)) \right) \\ & + r\sin(\beta) \left((2r^4-3r^2+1)(\cos(\phi_2)-1) - 2\cos(\beta)((r^4-1)\cos(\phi_2)+2r^2) \right) + \cos(\phi_3) \left(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2) \right. \\ & \left. + 2r^3+2(r^2-1)^2(2r^2-1)r\cos^2(\beta) + (r^2-1)r\cos(\phi_2)((8r^4-8r^2+1)\cos(\beta)-(2r^2-1)((r^2+1)\cos(2\beta)+3(r^2-1))) \right. \\ & \left. + (-8r^7+16r^5-9r^3+r)\cos(\beta) + \sin(\beta)\sin(\phi_2)(-8r^6+16r^4-9r^2+4(2r^6+r^2)\cos(\beta)+1) \right) = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r. \end{aligned} \quad (598)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_3 - \theta) = & \frac{4r^7 - 4r^5 + r^3 + \alpha_{13}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1)}{\sqrt{C^2+D^2}} \\ & + \frac{-(r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (599)$$

$$\begin{aligned} \Rightarrow \phi_3 = \cos^{-1} \left(\frac{4r^7 - 4r^5 + r^3 + \alpha_{13}\sqrt{1-r^2} + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1)}{\sqrt{C^2+D^2}} \right. \\ \left. + \frac{-(r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \right. \\ \left. + \frac{4(1-2r^2)(r^2-1)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (600)$$

which yields two solutions for each value of ϕ_2 .

1.15.5 $L^+|L_\beta^-G^-R_\beta^-|R^+$ Paths

For a $L_{\phi_1}^+|L_\beta^-G_{\phi_2}^-R_\beta^-|R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1)\mathbf{M}_{L^-}(r, \beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r, \beta)\mathbf{M}_{R^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (601)$$

Pre-multiplying (601) with \mathbf{u}_{L+}^T and post-multiplying \mathbf{u}_{R+} :

$$\begin{aligned} & r^2 (2r^2 - 1)^2 + \cos(\phi_2) (-4 (r^2 - 1) (2r^2 - 1) r^2 \cos(\beta) + 4 (r^2 - 1) r^4 \cos^2(\beta) + (r^2 - 1) (4r^4 + 2r^2 \cos(2\beta) - 6r^2 + 1)) \\ & + (4r^4 (r^2 - 1) - 4r^2 (r^2 - 1)) \cos^2(\beta) + (4r^2 (r^2 - 1) - 8r^4 (r^2 - 1)) \cos(\beta) \\ & + \sin(\phi_2) (4r (r^2 - 1) (2r^2 - 1) \sin(\beta) - 8r^3 (r^2 - 1) \sin(\beta) \cos(\beta)) \\ & = \alpha_{11} (r^2 - 1) + r (\alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r), \end{aligned} \quad (602)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 4r (r^2 - 1) (2r^2 - 1) \sin(\beta) - 8r^3 (r^2 - 1) \sin(\beta) \cos(\beta) \quad (603)$$

$$B = -4 (r^2 - 1) (2r^2 - 1) r^2 \cos(\beta) + 4 (r^2 - 1) r^4 \cos^2(\beta) + (r^2 - 1) (4r^4 + 2r^2 \cos(2\beta) - 6r^2 + 1), \quad (604)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{r (r (\alpha_{33} - (2 (r^2 - 1) \cos(\beta) - 2r^2 + 1)^2) + \alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2}) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (605)$$

$$\begin{aligned} \Rightarrow \phi_2 &= \cos^{-1} \left(\frac{r (r (\alpha_{33} - (2 (r^2 - 1) \cos(\beta) - 2r^2 + 1)^2) + \alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2}) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ &+ \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (606)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (601) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{R+} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4 (r^2 - 1) r^4 \sin(2\beta) \sin(\phi_2) - 4 (2r^4 - 3r^2 + 1) r^2 \sin(\beta) \sin(\phi_2) \\ & - (r^2 - 1) r \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2 (r^4 + r^2) \cos(2\beta) + 1) - 4 (r^2 - 1)^2 r^3 \cos^2(\beta) \\ & + 4 (2r^7 - 3r^5 + r^3) \cos(\beta) + \sin(\phi_1) ((r^5 + r) \sin(2\beta) + (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta))) \\ & + r \sin(\beta) (2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) - 2r^2) - (2r^4 - 3r^2 + 1) (\cos(\phi_2) + 1)) + \cos(\phi_1) (4r^7 - 6r^5 + 6r^4 \sin(2\beta) \sin(\phi_2) \\ & + 2r^3 + 2 (r^2 - 1)^2 (2r^2 - 1) r \cos^2(\beta) + (r^2 - 1) r \cos(\phi_2) ((2r^2 - 1) ((r^2 + 1) \cos(2\beta) + 3 (r^2 - 1)) + (-8r^4 + 8r^2 - 1) \cos(\beta)) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (8r^6 - 16r^4 + 9r^2 - 4 (2r^6 + r^2) \cos(\beta) - 1)) = \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r. \end{aligned} \quad (607)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C &= (r^5 + r) \sin(2\beta) + (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \\ &+ r \sin(\beta) (2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) - 2r^2) - (2r^4 - 3r^2 + 1) (\cos(\phi_2) + 1)) \end{aligned} \quad (608)$$

$$\begin{aligned} D &= 4r^7 - 6r^5 + 6r^4 \sin(2\beta) \sin(\phi_2) + 2r^3 + 2 (r^2 - 1)^2 (2r^2 - 1) r \cos^2(\beta) \\ &+ (r^2 - 1) r \cos(\phi_2) ((2r^2 - 1) ((r^2 + 1) \cos(2\beta) + 3 (r^2 - 1)) + (-8r^4 + 8r^2 - 1) \cos(\beta)) \\ &+ (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (8r^6 - 16r^4 + 9r^2 - 4 (2r^6 + r^2) \cos(\beta) - 1), \end{aligned} \quad (609)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{4r^7 - 4r^5 + r^3 + \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (610)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 + \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right. \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & \left. + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \quad (611)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (601) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1) \\ & - (r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1)-4(r^2-1)^2r^3\cos^2(\beta) \\ & + 4(2r^7-3r^5+r^3)\cos(\beta)+\sin(\phi_3)\left((r^5+r)\sin(2\beta)+(r^2-1)\sin(\phi_2)(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta))\right. \\ & \left.+ r\sin(\beta)(2\cos(\beta)((r^4-1)\cos(\phi_2)-2r^2)-(2r^4-3r^2+1)(\cos(\phi_2)+1))\right) + \cos(\phi_3)\left(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\right. \\ & \left.+ 2r^3+2(r^2-1)^2(2r^2-1)r\cos^2(\beta)+(r^2-1)r\cos(\phi_2)((2r^2-1)((r^2+1)\cos(2\beta)+3(r^2-1))+(-8r^4+8r^2-1)\cos(\beta))\right. \\ & \left.+ (-8r^7+16r^5-9r^3+r)\cos(\beta)+\sin(\beta)\sin(\phi_2)(8r^6-16r^4+9r^2-4(2r^6+r^2)\cos(\beta)-1)\right) = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \end{aligned} \quad (612)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_3 - \theta) = & \frac{4r^7 - 4r^5 + r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (613)$$

$$\begin{aligned} \Rightarrow \phi_3 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right. \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & \left. + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \quad (614)$$

which yields two solutions for each value of ϕ_2 .

1.15.6 $R^+|R_\beta^-G^-L_\beta^-|L^+$ Paths

For a $R_{\phi_1}^+|R_\beta^-G_{\phi_2}^-L_\beta^-|L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{R^-}(r, \beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r, \beta)\mathbf{M}_{L^+}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (615)$$

Pre-multiplying (615) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_{L+} :

$$\begin{aligned} & r^2 (2r^2 - 1)^2 + \cos(\phi_2) (-4 (r^2 - 1) (2r^2 - 1) r^2 \cos(\beta) + 4 (r^2 - 1) r^4 \cos^2(\beta) + (r^2 - 1) (4r^4 + 2r^2 \cos(2\beta) - 6r^2 + 1)) \\ & + (4r^4 (r^2 - 1) - 4r^2 (r^2 - 1)) \cos^2(\beta) + (4r^2 (r^2 - 1) - 8r^4 (r^2 - 1)) \cos(\beta) \\ & + \sin(\phi_2) (4r (r^2 - 1) (2r^2 - 1) \sin(\beta) - 8r^3 (r^2 - 1) \sin(\beta) \cos(\beta)) \\ & = \alpha_{11} (r^2 - 1) + r \left(\alpha_{13} (-\sqrt{1 - r^2}) + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right), \end{aligned} \quad (616)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 4r (r^2 - 1) (2r^2 - 1) \sin(\beta) - 8r^3 (r^2 - 1) \sin(\beta) \cos(\beta) \quad (617)$$

$$B = -4 (r^2 - 1) (2r^2 - 1) r^2 \cos(\beta) + 4 (r^2 - 1) r^4 \cos^2(\beta) + (r^2 - 1) (4r^4 + 2r^2 \cos(2\beta) - 6r^2 + 1), \quad (618)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{r \left(r \left(\alpha_{33} - (2 (r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 \right) + \alpha_{13} (-\sqrt{1 - r^2}) + \alpha_{31} \sqrt{1 - r^2} \right) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (619)$$

$$\begin{aligned} \Rightarrow \phi_2 &= \cos^{-1} \left(\frac{r \left(r \left(\alpha_{33} - (2 (r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 \right) + \alpha_{13} (-\sqrt{1 - r^2}) + \alpha_{31} \sqrt{1 - r^2} \right) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ &+ \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (620)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (615) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{L+} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4 (r^2 - 1) r^4 \sin(2\beta) \sin(\phi_2) - 4 (2r^4 - 3r^2 + 1) r^2 \sin(\beta) \sin(\phi_2) \\ & - (r^2 - 1) r \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2 (r^4 + r^2) \cos(2\beta) + 1) - 4 (r^2 - 1)^2 r^3 \cos^2(\beta) \\ & + 4 (2r^7 - 3r^5 + r^3) \cos(\beta) + \sin(\phi_1) \left((r^5 + r) \sin(2\beta) + (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \right. \\ & \left. + r \sin(\beta) (2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) - 2r^2) - (2r^4 - 3r^2 + 1) (\cos(\phi_2) + 1)) \right) + \cos(\phi_1) (4r^7 - 6r^5 + 6r^4 \sin(2\beta) \sin(\phi_2) \\ & + 2r^3 + 2 (r^2 - 1)^2 (2r^2 - 1) r \cos^2(\beta) + (r^2 - 1) r \cos(\phi_2) ((2r^2 - 1) ((r^2 + 1) \cos(2\beta) + 3 (r^2 - 1)) + (-8r^4 + 8r^2 - 1) \cos(\beta)) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (8r^6 - 16r^4 + 9r^2 - 4 (2r^6 + r^2) \cos(\beta) - 1) \Big) = \alpha_{31} (-\sqrt{1 - r^2}) - \alpha_{33} r. \end{aligned} \quad (621)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C &= (r^5 + r) \sin(2\beta) + (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \\ &+ r \sin(\beta) (2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) - 2r^2) - (2r^4 - 3r^2 + 1) (\cos(\phi_2) + 1)) \end{aligned} \quad (622)$$

$$\begin{aligned} D &= 4r^7 - 6r^5 + 6r^4 \sin(2\beta) \sin(\phi_2) + 2r^3 + 2 (r^2 - 1)^2 (2r^2 - 1) r \cos^2(\beta) \\ &+ (r^2 - 1) r \cos(\phi_2) ((2r^2 - 1) ((r^2 + 1) \cos(2\beta) + 3 (r^2 - 1)) + (-8r^4 + 8r^2 - 1) \cos(\beta)) \\ &+ (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (8r^6 - 16r^4 + 9r^2 - 4 (2r^6 + r^2) \cos(\beta) - 1), \end{aligned} \quad (623)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (624)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right. \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & \left. + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \quad (625)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (615) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1) \\ & - (r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1)-4(r^2-1)^2r^3\cos^2(\beta) \\ & + 4(2r^7-3r^5+r^3)\cos(\beta)+\sin(\phi_3)\left((r^5+r)\sin(2\beta)+(r^2-1)\sin(\phi_2)(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta))\right. \\ & \left.+ r\sin(\beta)(2\cos(\beta)((r^4-1)\cos(\phi_2)-2r^2)-(2r^4-3r^2+1)(\cos(\phi_2)+1))\right) + \cos(\phi_3)\left(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\right. \\ & \left.+ 2r^3+2(r^2-1)^2(2r^2-1)r\cos^2(\beta)+(r^2-1)r\cos(\phi_2)((2r^2-1)((r^2+1)\cos(2\beta)+3(r^2-1))+(-8r^4+8r^2-1)\cos(\beta))\right. \\ & \left.+ (-8r^7+16r^5-9r^3+r)\cos(\beta)+\sin(\beta)\sin(\phi_2)(8r^6-16r^4+9r^2-4(2r^6+r^2)\cos(\beta)-1)\right) = \alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \end{aligned} \quad (626)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_3 - \theta) = & \frac{4r^7 - 4r^5 + r^3 + \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (627)$$

$$\begin{aligned} \Rightarrow \phi_3 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 + \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right. \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & \left. + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \quad (628)$$

which yields two solutions for each value of ϕ_2 .

1.15.7 $R^-|R_\beta^+G^+L_\beta^+|L^-$ Paths

For a $R_{\phi_1}^-|R_\beta^+G_{\phi_2}^+L_\beta^+|L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1)\mathbf{M}_{R^+}(r, \beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r, \beta)\mathbf{M}_{L^-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (629)$$

Pre-multiplying (629) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_L- :

$$\begin{aligned} & r^2 (2r^2 - 1)^2 + \cos(\phi_2) (-4 (r^2 - 1) (2r^2 - 1) r^2 \cos(\beta) + 4 (r^2 - 1) r^4 \cos^2(\beta) + (r^2 - 1) (4r^4 + 2r^2 \cos(2\beta) - 6r^2 + 1)) \\ & + (4r^4 (r^2 - 1) - 4r^2 (r^2 - 1)) \cos^2(\beta) + (4r^2 (r^2 - 1) - 8r^4 (r^2 - 1)) \cos(\beta) \\ & + \sin(\phi_2) (4r (r^2 - 1) (2r^2 - 1) \sin(\beta) - 8r^3 (r^2 - 1) \sin(\beta) \cos(\beta)) \\ & = \alpha_{11} (r^2 - 1) + r (\alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r), \end{aligned} \quad (630)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 4r (r^2 - 1) (2r^2 - 1) \sin(\beta) - 8r^3 (r^2 - 1) \sin(\beta) \cos(\beta) \quad (631)$$

$$B = -4 (r^2 - 1) (2r^2 - 1) r^2 \cos(\beta) + 4 (r^2 - 1) r^4 \cos^2(\beta) + (r^2 - 1) (4r^4 + 2r^2 \cos(2\beta) - 6r^2 + 1), \quad (632)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{r (r (\alpha_{33} - (2 (r^2 - 1) \cos(\beta) - 2r^2 + 1)^2) + \alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2}) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (633)$$

$$\begin{aligned} \Rightarrow \phi_2 &= \cos^{-1} \left(\frac{r (r (\alpha_{33} - (2 (r^2 - 1) \cos(\beta) - 2r^2 + 1)^2) + \alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2}) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ &+ \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (634)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (629) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_L- :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4 (r^2 - 1) r^4 \sin(2\beta) \sin(\phi_2) - 4 (2r^4 - 3r^2 + 1) r^2 \sin(\beta) \sin(\phi_2) \\ & - (r^2 - 1) r \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2 (r^4 + r^2) \cos(2\beta) + 1) - 4 (r^2 - 1)^2 r^3 \cos^2(\beta) \\ & + 4 (2r^7 - 3r^5 + r^3) \cos(\beta) + \sin(\phi_1) ((r^5 + r) \sin(2\beta) + (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta))) \\ & + r \sin(\beta) (2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) - 2r^2) - (2r^4 - 3r^2 + 1) (\cos(\phi_2) + 1)) + \cos(\phi_1) (4r^7 - 6r^5 + 6r^4 \sin(2\beta) \sin(\phi_2) \\ & + 2r^3 + 2 (r^2 - 1)^2 (2r^2 - 1) r \cos^2(\beta) + (r^2 - 1) r \cos(\phi_2) ((2r^2 - 1) ((r^2 + 1) \cos(2\beta) + 3 (r^2 - 1)) + (-8r^4 + 8r^2 - 1) \cos(\beta)) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (8r^6 - 16r^4 + 9r^2 - 4 (2r^6 + r^2) \cos(\beta) - 1)) = \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r. \end{aligned} \quad (635)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C &= (r^5 + r) \sin(2\beta) + (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \\ &+ r \sin(\beta) (2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) - 2r^2) - (2r^4 - 3r^2 + 1) (\cos(\phi_2) + 1)) \end{aligned} \quad (636)$$

$$\begin{aligned} D &= 4r^7 - 6r^5 + 6r^4 \sin(2\beta) \sin(\phi_2) + 2r^3 + 2 (r^2 - 1)^2 (2r^2 - 1) r \cos^2(\beta) \\ &+ (r^2 - 1) r \cos(\phi_2) ((2r^2 - 1) ((r^2 + 1) \cos(2\beta) + 3 (r^2 - 1)) + (-8r^4 + 8r^2 - 1) \cos(\beta)) \\ &+ (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (8r^6 - 16r^4 + 9r^2 - 4 (2r^6 + r^2) \cos(\beta) - 1), \end{aligned} \quad (637)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{4r^7 - 4r^5 + r^3 + \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (638)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 + \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right. \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & \left. + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \quad (639)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (629) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1) \\ & - (r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1)-4(r^2-1)^2r^3\cos^2(\beta) \\ & + 4(2r^7-3r^5+r^3)\cos(\beta)+\sin(\phi_3)\left((r^5+r)\sin(2\beta)+(r^2-1)\sin(\phi_2)(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta))\right. \\ & \left.+ r\sin(\beta)(2\cos(\beta)((r^4-1)\cos(\phi_2)-2r^2)-(2r^4-3r^2+1)(\cos(\phi_2)+1))\right)+\cos(\phi_3)\left(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\right. \\ & \left.+ 2r^3+2(r^2-1)^2(2r^2-1)r\cos^2(\beta)+(r^2-1)r\cos(\phi_2)((2r^2-1)((r^2+1)\cos(2\beta)+3(r^2-1))+(-8r^4+8r^2-1)\cos(\beta))\right. \\ & \left.+ (-8r^7+16r^5-9r^3+r)\cos(\beta)+\sin(\beta)\sin(\phi_2)(8r^6-16r^4+9r^2-4(2r^6+r^2)\cos(\beta)-1)\right) = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \end{aligned} \quad (640)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_3 - \theta) = & \frac{4r^7 - 4r^5 + r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (641)$$

$$\begin{aligned} \Rightarrow \phi_3 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right. \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & \left. + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \quad (642)$$

which yields two solutions for each value of ϕ_2 .

1.15.8 $L^-|L_\beta^+G^+R_\beta^+|R^-$ Paths

For a $L_{\phi_1}^-|L_\beta^+G_{\phi_2}^+R_\beta^+|R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L-}(r, \phi_1)\mathbf{M}_{L+}(r, \beta)\mathbf{M}_{G+}(\phi_2)\mathbf{M}_{R+}(r, \beta)\mathbf{M}_{R-}(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (643)$$

Pre-multiplying (643) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{R-} :

$$\begin{aligned} & r^2 (2r^2 - 1)^2 + \cos(\phi_2) (-4 (r^2 - 1) (2r^2 - 1) r^2 \cos(\beta) + 4 (r^2 - 1) r^4 \cos^2(\beta) + (r^2 - 1) (4r^4 + 2r^2 \cos(2\beta) - 6r^2 + 1)) \\ & + (4r^4 (r^2 - 1) - 4r^2 (r^2 - 1)) \cos^2(\beta) + (4r^2 (r^2 - 1) - 8r^4 (r^2 - 1)) \cos(\beta) \\ & + \sin(\phi_2) (4r (r^2 - 1) (2r^2 - 1) \sin(\beta) - 8r^3 (r^2 - 1) \sin(\beta) \cos(\beta)) \\ & = \alpha_{11} (r^2 - 1) + r \left(\alpha_{13} (-\sqrt{1 - r^2}) + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right), \end{aligned} \quad (644)$$

Since β is known, this equation can be utilized to calculate ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \gamma := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \gamma := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$A = 4r (r^2 - 1) (2r^2 - 1) \sin(\beta) - 8r^3 (r^2 - 1) \sin(\beta) \cos(\beta) \quad (645)$$

$$B = -4 (r^2 - 1) (2r^2 - 1) r^2 \cos(\beta) + 4 (r^2 - 1) r^4 \cos^2(\beta) + (r^2 - 1) (4r^4 + 2r^2 \cos(2\beta) - 6r^2 + 1), \quad (646)$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{r \left(r \left(\alpha_{33} - (2 (r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 \right) + \alpha_{13} (-\sqrt{1 - r^2}) + \alpha_{31} \sqrt{1 - r^2} \right) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \quad (647)$$

$$\begin{aligned} \Rightarrow \phi_2 &= \cos^{-1} \left(\frac{r \left(r \left(\alpha_{33} - (2 (r^2 - 1) \cos(\beta) - 2r^2 + 1)^2 \right) + \alpha_{13} (-\sqrt{1 - r^2}) + \alpha_{31} \sqrt{1 - r^2} \right) + \alpha_{11} (r^2 - 1)}{\sqrt{A^2 + B^2}} \right) \\ &+ \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (648)$$

and yields two solutions of ϕ_2 .

Pre-multiplying (643) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{R-} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4 (r^2 - 1) r^4 \sin(2\beta) \sin(\phi_2) - 4 (2r^4 - 3r^2 + 1) r^2 \sin(\beta) \sin(\phi_2) \\ & - (r^2 - 1) r \cos(\phi_2) (6r^4 - 6r^2 + (4r^2 - 8r^4) \cos(\beta) + 2 (r^4 + r^2) \cos(2\beta) + 1) - 4 (r^2 - 1)^2 r^3 \cos^2(\beta) \\ & + 4 (2r^7 - 3r^5 + r^3) \cos(\beta) + \sin(\phi_1) \left((r^5 + r) \sin(2\beta) + (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \right. \\ & \left. + r \sin(\beta) (2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) - 2r^2) - (2r^4 - 3r^2 + 1) (\cos(\phi_2) + 1)) \right) + \cos(\phi_1) (4r^7 - 6r^5 + 6r^4 \sin(2\beta) \sin(\phi_2) \\ & + 2r^3 + 2 (r^2 - 1)^2 (2r^2 - 1) r \cos^2(\beta) + (r^2 - 1) r \cos(\phi_2) ((2r^2 - 1) ((r^2 + 1) \cos(2\beta) + 3 (r^2 - 1)) + (-8r^4 + 8r^2 - 1) \cos(\beta)) \\ & + (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (8r^6 - 16r^4 + 9r^2 - 4 (2r^6 + r^2) \cos(\beta) - 1) \Big) = \alpha_{31} (-\sqrt{1 - r^2}) - \alpha_{33} r. \end{aligned} \quad (649)$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \theta := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \theta := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$\begin{aligned} C &= (r^5 + r) \sin(2\beta) + (r^2 - 1) \sin(\phi_2) (\cos(\beta) - 2r^2 \cos(\beta) + 2r^2 \cos(2\beta)) \\ &+ r \sin(\beta) (2 \cos(\beta) ((r^4 - 1) \cos(\phi_2) - 2r^2) - (2r^4 - 3r^2 + 1) (\cos(\phi_2) + 1)) \end{aligned} \quad (650)$$

$$\begin{aligned} D &= 4r^7 - 6r^5 + 6r^4 \sin(2\beta) \sin(\phi_2) + 2r^3 + 2 (r^2 - 1)^2 (2r^2 - 1) r \cos^2(\beta) \\ &+ (r^2 - 1) r \cos(\phi_2) ((2r^2 - 1) ((r^2 + 1) \cos(2\beta) + 3 (r^2 - 1)) + (-8r^4 + 8r^2 - 1) \cos(\beta)) \\ &+ (-8r^7 + 16r^5 - 9r^3 + r) \cos(\beta) + \sin(\beta) \sin(\phi_2) (8r^6 - 16r^4 + 9r^2 - 4 (2r^6 + r^2) \cos(\beta) - 1), \end{aligned} \quad (651)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (652)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right. \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & \left. + \frac{4r^7 - 4r^5 + r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (653)$$

which yields two solutions for each value of ϕ_2 .

Pre-multiplying (643) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4(r^2-1)r^2\sin(\beta)\sin(\phi_2)(2r^2\cos(\beta)-2r^2+1) \\ & - (r^2-1)r\cos(\phi_2)(6r^4-6r^2+(4r^2-8r^4)\cos(\beta)+2(r^4+r^2)\cos(2\beta)+1)-4(r^2-1)^2r^3\cos^2(\beta) \\ & + 4(2r^7-3r^5+r^3)\cos(\beta)+\sin(\phi_3)\left((r^5+r)\sin(2\beta)+(r^2-1)\sin(\phi_2)(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta))\right) \\ & + r\sin(\beta)(2\cos(\beta)((r^4-1)\cos(\phi_2)-2r^2)-(2r^4-3r^2+1)(\cos(\phi_2)+1)) + \cos(\phi_3)(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2) \\ & + 2r^3+2(r^2-1)^2(2r^2-1)r\cos^2(\beta)+(r^2-1)r\cos(\phi_2)((2r^2-1)((r^2+1)\cos(2\beta)+3(r^2-1))+(-8r^4+8r^2-1)\cos(\beta)) \\ & + (-8r^7+16r^5-9r^3+r)\cos(\beta)+\sin(\beta)\sin(\phi_2)(8r^6-16r^4+9r^2-4(2r^6+r^2)\cos(\beta)-1)) = \alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \end{aligned} \quad (654)$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned} \cos(\phi_3 - \theta) = & \frac{4r^7 - 4r^5 + r^3 + \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \quad (655)$$

$$\begin{aligned} \Rightarrow \phi_3 = \cos^{-1} \left(& \frac{4r^7 - 4r^5 + r^3 + \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \right. \\ & + \frac{(r^2-1)r\cos(\phi_2)(6r^4-6r^2+2(r^4+r^2)\cos(2\beta)+1)+4(r^2-1)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & \left. + \frac{4(2r^6-3r^4+r^2)\sin(\beta)\sin(\phi_2)-8(2r^4-3r^2+1)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (656)$$

which yields two solutions for each value of ϕ_2 .

1.16 $C|C_\mu C_\mu|C_\mu C$ Paths

1.16.1 $L^+|L_\mu^- R_\mu^-|R_\mu^+ L^+$ Paths

For a $L_{\phi_1}^+|L_\mu^- R_\mu^-|R_\mu^+ L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1)\mathbf{M}_{L^-}(r, \mu)\mathbf{M}_{R^-}(r, \mu)\mathbf{M}_{R^+}(r, \mu)\mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (657)$$

Pre-multiplying (657) with \mathbf{u}_{L+}^T and post-multiplying \mathbf{u}_{L+} :

$$\begin{aligned} & 16r^8 - 32r^6 + 24r^4 - 8r^2 + 1 + (-16r^8 + 32r^6 - 16r^4) \cos^3(\mu) + (48r^8 - 96r^6 + 56r^4 - 8r^2) \cos^2(\mu) \\ & + (-48r^8 + 96r^6 - 64r^4 + 16r^2) \cos(\mu) \\ & = r \left(\alpha_{13} \sqrt{1-r^2} + \alpha_{31} \sqrt{1-r^2} + \alpha_{33} r \right) - \alpha_{11} (r^2 - 1), \end{aligned} \quad (658)$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ .

Pre-multiplying (657) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{L+} :

$$\begin{aligned} & r \left(-40r^8 + 80r^6 - 52r^4 + 8(r^2 - 1)r^2 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) \sin(\phi_1) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 12r^2 \right. \\ & + 8(r^2 - 1) \sin^2\left(\frac{\mu}{2}\right) \cos(\phi_1) ((r^2 - 1)(6r^4 + (2r^2 - 1)r^2 \cos(2\mu) - 3r^2 + 1) - r^2(8r^4 - 12r^2 + 5) \cos(\mu)) \\ & + 4(r^2 - 1)^2 r^4 \cos(3\mu) + 4(15r^6 - 30r^4 + 19r^2 - 4)r^2 \cos(\mu) - 4(6r^6 - 12r^4 + 7r^2 - 1)r^2 \cos(2\mu) - 1 \Big) \\ & = \alpha_{31} \left(-\sqrt{1-r^2} \right) - \alpha_{33} r. \end{aligned} \quad (659)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A &= 8(r^2 - 1)r^3 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) \\ B &= 8(r^2 - 1)r \sin^2\left(\frac{\mu}{2}\right) ((r^2 - 1)(6r^4 + (2r^2 - 1)r^2 \cos(2\mu) - 3r^2 + 1) - r^2(8r^4 - 12r^2 + 5) \cos(\mu)), \end{aligned} \quad (660)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) &= \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{31} \sqrt{1-r^2} - 4(r^2 - 1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{A^2+B^2}} \\ &+ \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33} r + r}{\sqrt{A^2+B^2}} \\ \implies \phi_1 &= \cos^{-1} \left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{31} \sqrt{1-r^2} - 4(r^2 - 1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{A^2+B^2}} \right. \\ &\left. + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33} r + r}{\sqrt{A^2+B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (661)$$

which yields two solutions for each value of μ .

Pre-multiplying (657) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & r \left(-40r^8 + 80r^6 - 52r^4 - 8(r^2 - 1)^2 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) \sin(\phi_2) (2r^2 \cos(\mu) - 2r^2 + 1) + 12r^2 + 4(r^2 - 1)^2 r^4 \cos(3\mu) \right. \\ & + 8(r^2 - 1) \sin^2\left(\frac{\mu}{2}\right) \cos(\phi_2) (r^2(6r^4 - 9r^2 + (2r^4 - 3r^2 + 1) \cos(2\mu) + 4) + (-8r^6 + 12r^4 - 5r^2 + 1) \cos(\mu)) \\ & + 4(15r^6 - 30r^4 + 19r^2 - 4)r^2 \cos(\mu) - 4(6r^6 - 12r^4 + 7r^2 - 1)r^2 \cos(2\mu) - 1 \Big) = \alpha_{13} \left(-\sqrt{1-r^2} \right) - \alpha_{33} r. \end{aligned} \quad (662)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -8(r^2 - 1)^2 r \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) (2r^2 \cos(\mu) - 2r^2 + 1) \quad (664)$$

$$D = 8(r^2 - 1)r \sin^2\left(\frac{\mu}{2}\right) (r^2(6r^4 - 9r^2 + (2r^4 - 3r^2 + 1) \cos(2\mu) + 4) + (-8r^6 + 12r^4 - 5r^2 + 1) \cos(\mu)), \quad (665)$$

it is obtained that

$$\begin{aligned} \cos(\phi_2 - \sigma) = & \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \\ & + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{C^2 + D^2}} \end{aligned} \quad (666)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} \left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \right. \\ \left. + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (667)$$

which yields two solutions for each value of μ .

1.16.2 $R^+|R_\mu^-L_\mu^-|L_\mu^+R^+$ Paths

For a $R_{\phi_1}^+|R_\mu^-L_\mu^-|L_\mu^+R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{R^-}(r, \mu)\mathbf{M}_{L^-}(r, \mu)\mathbf{M}_{L^+}(r, \mu)\mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (668)$$

Pre-multiplying (668) with $\mathbf{u}_{R^+}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$\begin{aligned} & 16r^8 - 32r^6 + 24r^4 - 8r^2 + 1 + (-16r^8 + 32r^6 - 16r^4) \cos^3(\mu) + (48r^8 - 96r^6 + 56r^4 - 8r^2) \cos^2(\mu) \\ & + (-48r^8 + 96r^6 - 64r^4 + 16r^2) \cos(\mu) \\ = & r \left(\alpha_{13} \left(-\sqrt{1-r^2} \right) - \alpha_{31} \sqrt{1-r^2} + \alpha_{33}r \right) - \alpha_{11} (r^2 - 1), \end{aligned} \quad (669)$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ .

Pre-multiplying (668) with $\mathbf{u}_{G^-}^T$ and post-multiplying with \mathbf{u}_{R^+} :

$$\begin{aligned} & r \left(-40r^8 + 80r^6 - 52r^4 + 8(r^2-1)r^2 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) \sin(\phi_1) (2(r^2-1) \cos(\mu) - 2r^2 + 1) + 12r^2 \right. \\ & + 8(r^2-1) \sin^2\left(\frac{\mu}{2}\right) \cos(\phi_1) ((r^2-1)(6r^4 + (2r^2-1)r^2 \cos(2\mu) - 3r^2 + 1) - r^2(8r^4 - 12r^2 + 5) \cos(\mu)) \\ & + 4(r^2-1)^2 r^4 \cos(3\mu) + 4(15r^6 - 30r^4 + 19r^2 - 4)r^2 \cos(\mu) - 4(6r^6 - 12r^4 + 7r^2 - 1)r^2 \cos(2\mu) - 1 \Big) \\ = & \alpha_{31} \sqrt{1-r^2} - \alpha_{33}r. \end{aligned} \quad (670)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A = & 8(r^2-1)r^3 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) (2(r^2-1) \cos(\mu) - 2r^2 + 1) \\ B = & 8(r^2-1)r \sin^2\left(\frac{\mu}{2}\right) ((r^2-1)(6r^4 + (2r^2-1)r^2 \cos(2\mu) - 3r^2 + 1) - r^2(8r^4 - 12r^2 + 5) \cos(\mu)), \end{aligned} \quad (671)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} \\ & + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \end{aligned} \quad (672)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} \right. \\ \left. + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (673)$$

which yields two solutions for each value of μ .

Pre-multiplying (668) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned} & r \left(-40r^8 + 80r^6 - 52r^4 - 8(r^2 - 1)^2 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) \sin(\phi_2) (2r^2 \cos(\mu) - 2r^2 + 1) + 12r^2 + 4(r^2 - 1)^2 r^4 \cos(3\mu) \right. \\ & + 8(r^2 - 1) \sin^2\left(\frac{\mu}{2}\right) \cos(\phi_2) (r^2 (6r^4 - 9r^2 + (2r^4 - 3r^2 + 1) \cos(2\mu) + 4) + (-8r^6 + 12r^4 - 5r^2 + 1) \cos(\mu)) \\ & \left. + 4(15r^6 - 30r^4 + 19r^2 - 4) r^2 \cos(\mu) - 4(6r^6 - 12r^4 + 7r^2 - 1) r^2 \cos(2\mu) - 1 \right) = \alpha_{13} \sqrt{1 - r^2} - \alpha_{33} r. \end{aligned} \quad (674)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \sigma := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \sigma := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$C = -8(r^2 - 1)^2 r \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) (2r^2 \cos(\mu) - 2r^2 + 1) \quad (675)$$

$$D = 8(r^2 - 1) r \sin^2\left(\frac{\mu}{2}\right) (r^2 (6r^4 - 9r^2 + (2r^4 - 3r^2 + 1) \cos(2\mu) + 4) + (-8r^6 + 12r^4 - 5r^2 + 1) \cos(\mu)), \quad (676)$$

it is obtained that

$$\begin{aligned} \cos(\phi_2 - \sigma) &= \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{13} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4) r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \\ &+ \frac{4(6r^6 - 12r^4 + 7r^2 - 1) r^3 \cos(2\mu) - \alpha_{33} r + r}{\sqrt{C^2 + D^2}} \end{aligned} \quad (677)$$

$$\begin{aligned} \Rightarrow \phi_2 &= \cos^{-1} \left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{13} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4) r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \right. \\ &\left. + \frac{4(6r^6 - 12r^4 + 7r^2 - 1) r^3 \cos(2\mu) - \alpha_{33} r + r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (678)$$

which yields two solutions for each value of μ .

1.16.3 $R^-|R_\mu^+ L_\mu^+|L_\mu^- R^-$ Paths

For a $R_{\phi_1}^-|R_\mu^+ L_\mu^+|L_\mu^- R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1) \mathbf{M}_{R^+}(r, \mu) \mathbf{M}_{L^+}(r, \mu) \mathbf{M}_{L^-}(r, \mu) \mathbf{M}_{R^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (679)$$

Pre-multiplying (679) with \mathbf{u}_{R-}^T and post-multiplying \mathbf{u}_{R-} :

$$\begin{aligned} & 16r^8 - 32r^6 + 24r^4 - 8r^2 + 1 + (-16r^8 + 32r^6 - 16r^4) \cos^3(\mu) + (48r^8 - 96r^6 + 56r^4 - 8r^2) \cos^2(\mu) \\ & + (-48r^8 + 96r^6 - 64r^4 + 16r^2) \cos(\mu) \\ & = r \left(\alpha_{13} \sqrt{1 - r^2} + \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right) - \alpha_{11} (r^2 - 1), \end{aligned} \quad (680)$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ .

Pre-multiplying (679) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_{R-} :

$$\begin{aligned} & r \left(-40r^8 + 80r^6 - 52r^4 + 8(r^2 - 1) r^2 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) \sin(\phi_1) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 12r^2 \right. \\ & + 8(r^2 - 1) \sin^2\left(\frac{\mu}{2}\right) \cos(\phi_1) ((r^2 - 1) (6r^4 + (2r^2 - 1) r^2 \cos(2\mu) - 3r^2 + 1) - r^2 (8r^4 - 12r^2 + 5) \cos(\mu)) \\ & + 4(r^2 - 1)^2 r^4 \cos(3\mu) + 4(15r^6 - 30r^4 + 19r^2 - 4) r^2 \cos(\mu) - 4(6r^6 - 12r^4 + 7r^2 - 1) r^2 \cos(2\mu) - 1 \left. \right) \\ & = \alpha_{31} (-\sqrt{1 - r^2}) - \alpha_{33} r. \end{aligned} \quad (681)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \theta := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \theta := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$\begin{aligned} A &= 8(r^2 - 1) r^3 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) \\ B &= 8(r^2 - 1) r \sin^2\left(\frac{\mu}{2}\right) ((r^2 - 1) (6r^4 + (2r^2 - 1) r^2 \cos(2\mu) - 3r^2 + 1) - r^2 (8r^4 - 12r^2 + 5) \cos(\mu)), \end{aligned} \quad (682)$$

it is obtained that

$$\cos(\phi_1 - \theta) = \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \quad (683)$$

$$\Rightarrow \phi_1 = \cos^{-1} \left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{31}\sqrt{1-r^2} - 4(r^2-1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right), \quad (684)$$

which yields two solutions for each value of μ .

Pre-multiplying (679) with \mathbf{u}_{R-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & r \left(-40r^8 + 80r^6 - 52r^4 - 8(r^2-1)^2 \sin^2 \left(\frac{\mu}{2} \right) \sin(\mu) \sin(\phi_2) (2r^2 \cos(\mu) - 2r^2 + 1) + 12r^2 + 4(r^2-1)^2 r^4 \cos(3\mu) \right. \\ & + 8(r^2-1) \sin^2 \left(\frac{\mu}{2} \right) \cos(\phi_2) (r^2 (6r^4 - 9r^2 + (2r^4 - 3r^2 + 1) \cos(2\mu) + 4) + (-8r^6 + 12r^4 - 5r^2 + 1) \cos(\mu)) \\ & \left. + 4(15r^6 - 30r^4 + 19r^2 - 4)r^2 \cos(\mu) - 4(6r^6 - 12r^4 + 7r^2 - 1)r^2 \cos(2\mu) - 1 \right) = \alpha_{13} \left(-\sqrt{1-r^2} \right) - \alpha_{33}r. \end{aligned} \quad (685)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -8(r^2-1)^2 r \sin^2 \left(\frac{\mu}{2} \right) \sin(\mu) (2r^2 \cos(\mu) - 2r^2 + 1) \quad (686)$$

$$D = 8(r^2-1)r \sin^2 \left(\frac{\mu}{2} \right) (r^2 (6r^4 - 9r^2 + (2r^4 - 3r^2 + 1) \cos(2\mu) + 4) + (-8r^6 + 12r^4 - 5r^2 + 1) \cos(\mu)), \quad (687)$$

it is obtained that

$$\cos(\phi_2 - \sigma) = \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{C^2 + D^2}} \quad (688)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 - \alpha_{13}\sqrt{1-r^2} - 4(r^2-1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \quad (689)$$

which yields two solutions for each value of μ .

1.16.4 $L^-|L_\mu^+ R_\mu^+|R_\mu^- L^-$ Paths

For a $L_{\phi_1}^-|L_\mu^+ R_\mu^+|R_\mu^- L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L-}(r, \phi_1) \mathbf{M}_{L+}(r, \mu) \mathbf{M}_{R+}(r, \mu) \mathbf{M}_{R-}(r, \mu) \mathbf{M}_{L-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (690)$$

Pre-multiplying (690) with \mathbf{u}_{L-}^T and post-multiplying \mathbf{u}_{L-} :

$$\begin{aligned} & 16r^8 - 32r^6 + 24r^4 - 8r^2 + 1 + (-16r^8 + 32r^6 - 16r^4) \cos^3(\mu) + (48r^8 - 96r^6 + 56r^4 - 8r^2) \cos^2(\mu) \\ & + (-48r^8 + 96r^6 - 64r^4 + 16r^2) \cos(\mu) \\ & = r \left(\alpha_{13} \left(-\sqrt{1-r^2} \right) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r \right) - \alpha_{11}(r^2-1), \end{aligned} \quad (691)$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ .

Pre-multiplying (690) with \mathbf{u}_{G+}^T and post-multiplying with \mathbf{u}_L -:

$$\begin{aligned}
& r \left(-40r^8 + 80r^6 - 52r^4 + 8(r^2 - 1)r^2 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) \sin(\phi_1) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) + 12r^2 \right. \\
& + 8(r^2 - 1) \sin^2\left(\frac{\mu}{2}\right) \cos(\phi_1) ((r^2 - 1)(6r^4 + (2r^2 - 1)r^2 \cos(2\mu) - 3r^2 + 1) - r^2(8r^4 - 12r^2 + 5) \cos(\mu)) \\
& \left. + 4(r^2 - 1)^2 r^4 \cos(3\mu) + 4(15r^6 - 30r^4 + 19r^2 - 4)r^2 \cos(\mu) - 4(6r^6 - 12r^4 + 7r^2 - 1)r^2 \cos(2\mu) - 1 \right) \\
& = \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r.
\end{aligned} \tag{692}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \theta := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \theta := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$\begin{aligned}
A &= 8(r^2 - 1)r^3 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) (2(r^2 - 1) \cos(\mu) - 2r^2 + 1) \\
B &= 8(r^2 - 1)r \sin^2\left(\frac{\mu}{2}\right) ((r^2 - 1)(6r^4 + (2r^2 - 1)r^2 \cos(2\mu) - 3r^2 + 1) - r^2(8r^4 - 12r^2 + 5) \cos(\mu)),
\end{aligned} \tag{693}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1 - \theta) &= \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{31} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} \\
&+ \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \\
\Rightarrow \phi_1 &= \cos^{-1} \left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{31} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} \right. \\
&\left. + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{A^2 + B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right),
\end{aligned} \tag{694}$$

which yields two solutions for each value of μ .

Pre-multiplying (690) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned}
& r \left(-40r^8 + 80r^6 - 52r^4 - 8(r^2 - 1)^2 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) \sin(\phi_2) (2r^2 \cos(\mu) - 2r^2 + 1) + 12r^2 + 4(r^2 - 1)^2 r^4 \cos(3\mu) \right. \\
& + 8(r^2 - 1) \sin^2\left(\frac{\mu}{2}\right) \cos(\phi_2) (r^2(6r^4 - 9r^2 + (2r^4 - 3r^2 + 1) \cos(2\mu) + 4) + (-8r^6 + 12r^4 - 5r^2 + 1) \cos(\mu)) \\
& \left. + 4(15r^6 - 30r^4 + 19r^2 - 4)r^2 \cos(\mu) - 4(6r^6 - 12r^4 + 7r^2 - 1)r^2 \cos(2\mu) - 1 \right) = \alpha_{13} \left(-\sqrt{1 - r^2} \right) - \alpha_{33} r.
\end{aligned} \tag{696}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \sigma := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \sigma := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$C = -8(r^2 - 1)^2 r \sin^2\left(\frac{\mu}{2}\right) \sin(\mu) (2r^2 \cos(\mu) - 2r^2 + 1) \tag{697}$$

$$D = 8(r^2 - 1)r \sin^2\left(\frac{\mu}{2}\right) (r^2(6r^4 - 9r^2 + (2r^4 - 3r^2 + 1) \cos(2\mu) + 4) + (-8r^6 + 12r^4 - 5r^2 + 1) \cos(\mu)), \tag{698}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_2 - \sigma) &= \frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{13} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \\
&+ \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{C^2 + D^2}} \\
\Rightarrow \phi_2 &= \cos^{-1} \left(\frac{40r^9 - 80r^7 + 52r^5 - 12r^3 + \alpha_{13} \sqrt{1 - r^2} - 4(r^2 - 1)^2 r^5 \cos(3\mu) - 4(15r^6 - 30r^4 + 19r^2 - 4)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \right. \\
&\left. + \frac{4(6r^6 - 12r^4 + 7r^2 - 1)r^3 \cos(2\mu) - \alpha_{33}r + r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right),
\end{aligned} \tag{699}$$

which yields two solutions for each value of μ .

1.17 $CC_\mu|C_\mu C_\mu|C_\mu C$ Paths

1.17.1 $L^+ R_\mu^+ |R_\mu^- L_\mu^- |L_\mu^+ R^+$ Paths

For a $L_{\phi_1}^+ R_\mu^+ |R_\mu^- L_\mu^- |L_\mu^+ R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) \mathbf{M}_{R^+}(r, \mu) \mathbf{M}_{R^-}(r, \mu) \mathbf{M}_{L^-}(r, \mu) \mathbf{M}_{L^+}(r, \mu) \mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (701)$$

Pre-multiplying (701) with $\mathbf{u}_{L^+}^T$ and post-multiplying \mathbf{u}_{R^+} :

$$\begin{aligned} & 32r^{10} - 64r^8 + 56r^6 - 32r^4 + 10r^2 + (32r^{10} - 96r^8 + 96r^6 - 32r^4) \cos^4(\mu) + (-128r^{10} + 352r^8 - 320r^6 + 96r^4) \cos^3(\mu) \\ & + (192r^{10} - 480r^8 + 408r^6 - 136r^4 + 16r^2) \cos^2(\mu) + (-128r^{10} + 288r^8 - 240r^6 + 104r^4 - 24r^2) \cos(\mu) - 1 \\ & = \alpha_{11} (r^2 - 1) + r \left(\alpha_{13} \sqrt{1 - r^2} - \alpha_{31} \sqrt{1 - r^2} + \alpha_{33} r \right), \end{aligned} \quad (702)$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ .

Pre-multiplying (701) with $\mathbf{u}_{G^-}^T$ and post-multiplying with \mathbf{u}_{R^+} :

$$\begin{aligned} & -140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8(r^2 - 1)^2 (4r^2 - 3) r^5 \cos(3\mu) \\ & - 4(r^2 - 1)^3 r^5 \cos(4\mu) + 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \sin(\phi_1) \left(6r^4 + 2(r^2 - 1) r^2 \cos(2\mu) - 4r^2 \right. \\ & + (6r^2 - 8r^4) \cos(\mu) + 1) + 2(r^2 - 1) r \cos(\phi_1) \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) \right. \\ & - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 \\ & + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2) \\ & + 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3) r^3 \cos(\mu) - 4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2) r^3 \cos(2\mu) + r \\ & = \alpha_{31} \sqrt{1 - r^2} - \alpha_{33} r. \end{aligned} \quad (703)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2 + B^2}}$ and defining $\sin \theta := \frac{A}{\sqrt{A^2 + B^2}}$, $\cos \theta := \frac{B}{\sqrt{A^2 + B^2}}$, where

$$\begin{aligned} A &= 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \left(6r^4 + 2(r^2 - 1) r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1\right) \\ B &= 2(r^2 - 1) r \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\ & + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \\ & + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2), \end{aligned} \quad (704)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) &= \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{31} \sqrt{1 - r^2} - 8(r^2 - 1)^2 (4r^2 - 3) r^5 \cos(3\mu)}{\sqrt{A^2 + B^2}} \\ &+ \frac{4(r^2 - 1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3) r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} \\ &+ \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2) r^3 \cos(2\mu) - \alpha_{33} r - r}{\sqrt{A^2 + B^2}} \end{aligned} \quad (705)$$

$$\begin{aligned} \Rightarrow \phi_1 &= \cos^{-1} \left(\frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{31} \sqrt{1 - r^2} - 8(r^2 - 1)^2 (4r^2 - 3) r^5 \cos(3\mu)}{\sqrt{A^2 + B^2}} \right. \\ &+ \frac{4(r^2 - 1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3) r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} \\ &+ \left. \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2) r^3 \cos(2\mu) - \alpha_{33} r - r}{\sqrt{A^2 + B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (706)$$

which yields two solutions for each value of μ .

Pre-multiplying (701) with \mathbf{u}_{L+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned}
& -140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu) - 4(r^2 - 1)^3 r^5 \cos(4\mu) \\
& + 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \sin(\phi_2) \left(6r^4 + 2(r^2 - 1)r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1\right) \\
& + 2(r^2 - 1)r \cos(\phi_2) \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\
& + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \\
& + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \left. \right) + 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu) \\
& - 4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) + r = \alpha_{13} \left(-\sqrt{1 - r^2}\right) - \alpha_{33}r.
\end{aligned} \tag{707}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{C^2 + D^2}}$ and defining $\sin \sigma := \frac{C}{\sqrt{C^2 + D^2}}$, $\cos \sigma := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$C = 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \left(6r^4 + 2(r^2 - 1)r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1\right) \tag{708}$$

$$\begin{aligned}
D = & 2(r^2 - 1)r \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\
& + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \\
& + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \left. \right),
\end{aligned} \tag{709}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_2 - \sigma) = & \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 - \alpha_{13}\sqrt{1 - r^2} - 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu)}{\sqrt{C^2 + D^2}} \\
& + \frac{4(r^2 - 1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \\
& + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) - \alpha_{33}r - r}{\sqrt{C^2 + D^2}}
\end{aligned} \tag{710}$$

$$\begin{aligned}
\Rightarrow \phi_2 = \cos^{-1} \left(& \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 - \alpha_{13}\sqrt{1 - r^2} - 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu)}{\sqrt{C^2 + D^2}} \right. \\
& + \frac{4(r^2 - 1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \\
& \left. + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) - \alpha_{33}r - r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1}\left(\frac{C}{D}\right),
\end{aligned} \tag{711}$$

which yields two solutions for each value of μ .

1.17.2 $R^+L_\mu^+|L_\mu^-R_\mu^-|R_\mu^+L^+$ Paths

For a $R_{\phi_1}^+L_\mu^+|L_\mu^-R_\mu^-|R_\mu^+L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{L^+}(r, \mu)\mathbf{M}_{L^-}(r, \mu)\mathbf{M}_{R^-}(r, \mu)\mathbf{M}_{R^+}(r, \mu)\mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{712}$$

Pre-multiplying (712) with \mathbf{u}_{R+}^T and post-multiplying \mathbf{u}_{L+} :

$$\begin{aligned}
& 32r^{10} - 64r^8 + 56r^6 - 32r^4 + 10r^2 + (32r^{10} - 96r^8 + 96r^6 - 32r^4) \cos^4(\mu) + (-128r^{10} + 352r^8 - 320r^6 + 96r^4) \cos^3(\mu) \\
& + (192r^{10} - 480r^8 + 408r^6 - 136r^4 + 16r^2) \cos^2(\mu) + (-128r^{10} + 288r^8 - 240r^6 + 104r^4 - 24r^2) \cos(\mu) - 1 \\
& = \alpha_{11}(r^2 - 1) + r \left(\alpha_{13} \left(-\sqrt{1 - r^2}\right) + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33}r \right),
\end{aligned} \tag{713}$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ .

Pre-multiplying (712) with \mathbf{u}_{G-}^T and post-multiplying with \mathbf{u}_{L+} :

$$\begin{aligned}
& -140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu) - 4(r^2 - 1)^3 r^5 \cos(4\mu) \\
& + 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \sin(\phi_1) \left(6r^4 + 2(r^2 - 1)r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1\right) \\
& + 2(r^2 - 1)r \cos(\phi_1) \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\
& + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \\
& + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \left. \right) + 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu) \\
& - 4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) + r = \alpha_{31} \left(-\sqrt{1 - r^2}\right) - \alpha_{33}r.
\end{aligned} \tag{714}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}
A &= 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \left(6r^4 + 2(r^2 - 1)r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1\right) \\
B &= 2(r^2 - 1)r \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\
& + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \\
& + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \left. \right),
\end{aligned} \tag{715}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1 - \theta) &= \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 - \alpha_{31}\sqrt{1 - r^2} - 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu)}{\sqrt{A^2 + B^2}} \\
& + \frac{4(r^2 - 1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} \\
& + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) - \alpha_{33}r - r}{\sqrt{A^2 + B^2}} \\
\Rightarrow \phi_1 &= \cos^{-1} \left(\frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 - \alpha_{31}\sqrt{1 - r^2} - 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu)}{\sqrt{A^2 + B^2}} \right. \\
& + \frac{4(r^2 - 1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu)}{\sqrt{A^2 + B^2}} \\
& + \left. \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) - \alpha_{33}r - r}{\sqrt{A^2 + B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right),
\end{aligned} \tag{716}$$

which yields two solutions for each value of μ .

Pre-multiplying (712) with \mathbf{u}_{R+}^T and post-multiplying with \mathbf{u}_{G-} :

$$\begin{aligned}
& -140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu) - 4(r^2 - 1)^3 r^5 \cos(4\mu) \\
& + 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \sin(\phi_2) \left(6r^4 + 2(r^2 - 1)r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1\right) \\
& + 2(r^2 - 1)r \cos(\phi_2) \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\
& + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \\
& + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \left. \right) + 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu) \\
& - 4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) + r = \alpha_{13}\sqrt{1 - r^2} - \alpha_{33}r.
\end{aligned} \tag{718}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \sigma := \frac{C}{\sqrt{C^2+D^2}}$,

$\cos \sigma := \frac{D}{\sqrt{C^2 + D^2}}$, where

$$C = 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \left(6r^4 + 2(r^2 - 1)r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1\right) \quad (719)$$

$$D = 2(r^2 - 1)r \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\ \left. + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \right. \\ \left. + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \right), \quad (720)$$

it is obtained that

$$\cos(\phi_2 - \sigma) = \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{13}\sqrt{1-r^2} - 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu)}{\sqrt{C^2 + D^2}} \\ + \frac{4(r^2 - 1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \\ + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) - \alpha_{33}r - r}{\sqrt{C^2 + D^2}} \quad (721)$$

$$\Rightarrow \phi_2 = \cos^{-1} \left(\frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{13}\sqrt{1-r^2} - 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu)}{\sqrt{C^2 + D^2}} \right. \\ \left. + \frac{4(r^2 - 1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu)}{\sqrt{C^2 + D^2}} \right. \\ \left. + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) - \alpha_{33}r - r}{\sqrt{C^2 + D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \quad (722)$$

which yields two solutions for each value of μ .

1.17.3 $R^-L_\mu^-|L_\mu^+R_\mu^+|R_\mu^-L^-$ Paths

For a $R_{\phi_1}^-L_\mu^-|L_\mu^+R_\mu^+|R_\mu^-L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1)\mathbf{M}_{L^-}(r, \mu)\mathbf{M}_{L^+}(r, \mu)\mathbf{M}_{R^+}(r, \mu)\mathbf{M}_{R^-}(r, \mu)\mathbf{M}_{L^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (723)$$

Pre-multiplying (723) with $\mathbf{u}_{R^-}^T$ and post-multiplying \mathbf{u}_{L^-} :

$$32r^{10} - 64r^8 + 56r^6 - 32r^4 + 10r^2 + (32r^{10} - 96r^8 + 96r^6 - 32r^4) \cos^4(\mu) + (-128r^{10} + 352r^8 - 320r^6 + 96r^4) \cos^3(\mu) \\ + (192r^{10} - 480r^8 + 408r^6 - 136r^4 + 16r^2) \cos^2(\mu) + (-128r^{10} + 288r^8 - 240r^6 + 104r^4 - 24r^2) \cos(\mu) - 1 \\ = \alpha_{11}(r^2 - 1) + r(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r), \quad (724)$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ .

Pre-multiplying (723) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{L^-} :

$$-140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8(r^2 - 1)^2(4r^2 - 3)r^5 \cos(3\mu) \\ - 4(r^2 - 1)^3 r^5 \cos(4\mu) + 16(r^2 - 1)^2 r \sin^3\left(\frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) \sin(\phi_1) \left(6r^4 + 2(r^2 - 1)r^2 \cos(2\mu) - 4r^2 \right. \\ \left. + (6r^2 - 8r^4) \cos(\mu) + 1\right) + 2(r^2 - 1)r \cos(\phi_1) \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) \right. \\ \left. - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 \right. \\ \left. + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \right) \\ + 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu) - 4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) + r \\ = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \quad (725)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A &= 16 (r^2 - 1)^2 r \sin^3 \left(\frac{\mu}{2} \right) \cos \left(\frac{\mu}{2} \right) \left(6r^4 + 2 (r^2 - 1) r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1 \right) \\ B &= 2 (r^2 - 1) r \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\ &\quad \left. + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \right. \\ &\quad \left. + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \right), \end{aligned} \quad (726)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) &= \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{31}\sqrt{1-r^2} - 8(r^2-1)^2(4r^2-3)r^5 \cos(3\mu)}{\sqrt{A^2+B^2}} \\ &\quad + \frac{4(r^2-1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu)}{\sqrt{A^2+B^2}} \\ &\quad + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) - \alpha_{33}r - r}{\sqrt{A^2+B^2}} \end{aligned} \quad (727)$$

$$\begin{aligned} \Rightarrow \phi_1 &= \cos^{-1} \left(\frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{31}\sqrt{1-r^2} - 8(r^2-1)^2(4r^2-3)r^5 \cos(3\mu)}{\sqrt{A^2+B^2}} \right. \\ &\quad \left. + \frac{4(r^2-1)^3 r^5 \cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu)}{\sqrt{A^2+B^2}} \right. \\ &\quad \left. + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) - \alpha_{33}r - r}{\sqrt{A^2+B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (728)$$

which yields two solutions for each value of μ .

Pre-multiplying (723) with \mathbf{u}_R^T and post-multiplying with \mathbf{u}_G :

$$\begin{aligned} &-140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8(r^2-1)^2(4r^2-3)r^5 \cos(3\mu) - 4(r^2-1)^3 r^5 \cos(4\mu) \\ &+ 16(r^2-1)^2 r \sin^3 \left(\frac{\mu}{2} \right) \cos \left(\frac{\mu}{2} \right) \sin(\phi_2) \left(6r^4 + 2(r^2-1)r^2 \cos(2\mu) - 4r^2 + (6r^2-8r^4) \cos(\mu) + 1 \right) \\ &+ 2(r^2-1)r \cos(\phi_2) \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\ &+ 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \\ &+ (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \Big) + 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3 \cos(\mu) \\ &- 4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3 \cos(2\mu) + r = \alpha_{13} \left(-\sqrt{1-r^2} \right) - \alpha_{33}r. \end{aligned} \quad (729)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = 16 (r^2 - 1)^2 r \sin^3 \left(\frac{\mu}{2} \right) \cos \left(\frac{\mu}{2} \right) \left(6r^4 + 2 (r^2 - 1) r^2 \cos(2\mu) - 4r^2 + (6r^2 - 8r^4) \cos(\mu) + 1 \right) \quad (730)$$

$$\begin{aligned} D &= 2 (r^2 - 1) r \left(-16r^8 \cos(3\mu) + 2r^8 \cos(4\mu) + 70r^8 + 36r^6 \cos(3\mu) - 5r^6 \cos(4\mu) - 135r^6 - 25r^4 \cos(3\mu) \right. \\ &\quad \left. + 4r^4 \cos(4\mu) + 88r^4 + 5r^2 \cos(3\mu) - r^2 \cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2) \cos(\mu) \right. \\ &\quad \left. + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1) \cos(2\mu) + 2 \right), \end{aligned} \quad (731)$$

it is obtained that

$$\begin{aligned} \cos(\phi_2 - \sigma) = & \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 - \alpha_{13}\sqrt{1-r^2} - 8(r^2-1)^2(4r^2-3)r^5\cos(3\mu)}{\sqrt{C^2+D^2}} \\ & + \frac{4(r^2-1)^3r^5\cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \\ & + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3\cos(2\mu) - \alpha_{33}r - r}{\sqrt{C^2+D^2}} \end{aligned} \quad (732)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} \left(\frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 - \alpha_{13}\sqrt{1-r^2} - 8(r^2-1)^2(4r^2-3)r^5\cos(3\mu)}{\sqrt{C^2+D^2}} \right. \\ \left. + \frac{4(r^2-1)^3r^5\cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \right. \\ \left. + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3\cos(2\mu) - \alpha_{33}r - r}{\sqrt{C^2+D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (733)$$

which yields two solutions for each value of μ .

1.17.4 $L^-R_\mu^-|R_\mu^+L_\mu^+|L_\mu^-R^-$ Paths

For a $L_{\phi_1}^-R_\mu^-|R_\mu^+L_\mu^+|L_\mu^-R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r, \phi_1)\mathbf{M}_{R^-}(r, \mu)\mathbf{M}_{R^+}(r, \mu)\mathbf{M}_{L^+}(r, \mu)\mathbf{M}_{L^-}(r, \mu)\mathbf{M}_{R^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (734)$$

Pre-multiplying (734) with $\mathbf{u}_{L^-}^T$ and post-multiplying \mathbf{u}_{R^-} :

$$\begin{aligned} & 32r^{10} - 64r^8 + 56r^6 - 32r^4 + 10r^2 + (32r^{10} - 96r^8 + 96r^6 - 32r^4)\cos^4(\mu) + (-128r^{10} + 352r^8 - 320r^6 + 96r^4)\cos^3(\mu) \\ & + (192r^{10} - 480r^8 + 408r^6 - 136r^4 + 16r^2)\cos^2(\mu) + (-128r^{10} + 288r^8 - 240r^6 + 104r^4 - 24r^2)\cos(\mu) - 1 \\ = & \alpha_{11}(r^2-1) + r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \end{aligned} \quad (735)$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ .

Pre-multiplying (734) with $\mathbf{u}_{G^+}^T$ and post-multiplying with \mathbf{u}_{R^-} :

$$\begin{aligned} & -140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8(r^2-1)^2(4r^2-3)r^5\cos(3\mu) - 4(r^2-1)^3r^5\cos(4\mu) \\ & + 16(r^2-1)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\left(6r^4 + 2(r^2-1)r^2\cos(2\mu) - 4r^2 + (6r^2-8r^4)\cos(\mu) + 1\right) \\ & + 2(r^2-1)r\cos(\phi_1)\left(-16r^8\cos(3\mu) + 2r^8\cos(4\mu) + 70r^8 + 36r^6\cos(3\mu) - 5r^6\cos(4\mu) - 135r^6 - 25r^4\cos(3\mu) \right. \\ & + 4r^4\cos(4\mu) + 88r^4 + 5r^2\cos(3\mu) - r^2\cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2)\cos(\mu) \\ & + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1)\cos(2\mu) + 2\left.) + 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3\cos(\mu) \right. \\ & \left. - 4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3\cos(2\mu) + r = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \right. \end{aligned} \quad (736)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1 . Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin \theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos \theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A = & 16(r^2-1)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4 + 2(r^2-1)r^2\cos(2\mu) - 4r^2 + (6r^2-8r^4)\cos(\mu) + 1\right) \\ B = & 2(r^2-1)r\left(-16r^8\cos(3\mu) + 2r^8\cos(4\mu) + 70r^8 + 36r^6\cos(3\mu) - 5r^6\cos(4\mu) - 135r^6 - 25r^4\cos(3\mu) \right. \\ & + 4r^4\cos(4\mu) + 88r^4 + 5r^2\cos(3\mu) - r^2\cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2)\cos(\mu) \\ & \left. + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1)\cos(2\mu) + 2\right), \end{aligned} \quad (737)$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) = & \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 - \alpha_{31}\sqrt{1-r^2} - 8(r^2-1)^2(4r^2-3)r^5\cos(3\mu)}{\sqrt{A^2+B^2}} \\ & + \frac{4(r^2-1)^3r^5\cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\ & + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3\cos(2\mu) - \alpha_{33}r - r}{\sqrt{A^2+B^2}} \end{aligned} \quad (738)$$

$$\begin{aligned} \Rightarrow \phi_1 = \cos^{-1} \left(\frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 - \alpha_{31}\sqrt{1-r^2} - 8(r^2-1)^2(4r^2-3)r^5\cos(3\mu)}{\sqrt{A^2+B^2}} \right. \\ \left. + \frac{4(r^2-1)^3r^5\cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \right. \\ \left. + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3\cos(2\mu) - \alpha_{33}r - r}{\sqrt{A^2+B^2}} \right) + \tan^{-1} \left(\frac{A}{B} \right), \end{aligned} \quad (739)$$

which yields two solutions for each value of μ .

Pre-multiplying (734) with \mathbf{u}_{L-}^T and post-multiplying with \mathbf{u}_{G+} :

$$\begin{aligned} & -140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8(r^2-1)^2(4r^2-3)r^5\cos(3\mu) - 4(r^2-1)^3r^5\cos(4\mu) \\ & + 16(r^2-1)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\left(6r^4 + 2(r^2-1)r^2\cos(2\mu) - 4r^2 + (6r^2 - 8r^4)\cos(\mu) + 1\right) \\ & + 2(r^2-1)r\cos(\phi_2)\left(-16r^8\cos(3\mu) + 2r^8\cos(4\mu) + 70r^8 + 36r^6\cos(3\mu) - 5r^6\cos(4\mu) - 135r^6 - 25r^4\cos(3\mu) \right. \\ & + 4r^4\cos(4\mu) + 88r^4 + 5r^2\cos(3\mu) - r^2\cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2)\cos(\mu) \\ & + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1)\cos(2\mu) + 2\left.) + 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3\cos(\mu) \right. \\ & \left. - 4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3\cos(2\mu) + r = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r. \right. \end{aligned} \quad (740)$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2 . Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin \sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos \sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = 16(r^2-1)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4 + 2(r^2-1)r^2\cos(2\mu) - 4r^2 + (6r^2 - 8r^4)\cos(\mu) + 1\right) \quad (741)$$

$$\begin{aligned} D = & 2(r^2-1)r\left(-16r^8\cos(3\mu) + 2r^8\cos(4\mu) + 70r^8 + 36r^6\cos(3\mu) - 5r^6\cos(4\mu) - 135r^6 - 25r^4\cos(3\mu) \right. \\ & + 4r^4\cos(4\mu) + 88r^4 + 5r^2\cos(3\mu) - r^2\cos(4\mu) - 22r^2 + (-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2)\cos(\mu) \\ & \left. + (56r^8 - 116r^6 + 76r^4 - 17r^2 + 1)\cos(2\mu) + 2\right), \end{aligned} \quad (742)$$

it is obtained that

$$\begin{aligned} \cos(\phi_2 - \sigma) = & \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{13}\sqrt{1-r^2} - 8(r^2-1)^2(4r^2-3)r^5\cos(3\mu)}{\sqrt{C^2+D^2}} \\ & + \frac{4(r^2-1)^3r^5\cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \\ & + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3\cos(2\mu) - \alpha_{33}r - r}{\sqrt{C^2+D^2}} \end{aligned} \quad (743)$$

$$\begin{aligned} \Rightarrow \phi_2 = \cos^{-1} \left(\frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{13}\sqrt{1-r^2} - 8(r^2-1)^2(4r^2-3)r^5\cos(3\mu)}{\sqrt{C^2+D^2}} \right. \\ \left. + \frac{4(r^2-1)^3r^5\cos(4\mu) - 8(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \right. \\ \left. + \frac{4(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2)r^3\cos(2\mu) - \alpha_{33}r - r}{\sqrt{C^2+D^2}} \right) + \tan^{-1} \left(\frac{C}{D} \right), \end{aligned} \quad (744)$$

which yields two solutions for each value of μ .

References