Quantum Field Theory

a study note based on A. Zee's textbook

Siyang Wan (万思扬)

Email: wansiyang03@gmail.com
https://github.com/siyang03/my-note---A.-Zee-QFT

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convention, notation, and units

- 笔记中的**度规号差**约定为 $(-,+,+,+)$.
- 使用 Planck units, 此时 $G, \hbar, c, k_B = 1$, 因此:

names/dimensions	expressions/values
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

Part I

motivation and foundation

Chapter 1

free field theory

1.1 partition function

- 考虑标量场
$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi), \tag{1.1.1}$$
A. Zee: 在作用量里, 时间的导数项必须是正的, 包括标量场的 $(\partial_0\phi)^2$ 和电磁场的 $(\partial_0 A_i)^2$.
- 含有 source function 的路径积分为
$$Z(J) = \int D\phi\, e^{i\int d^dx\,(-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))}. \tag{1.1.2}$$
- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

- 计算 free theory 的 partition function, 得到
$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\, J(x)D(x-y)J(y)}, \tag{1.1.3}$$
另外, 用 $W(J)$ 表示指数上的部分 (去除掉虚数 i).

proof:

注意 $\partial^\mu\phi\partial_\mu\phi = \partial^\mu(\phi\partial_\mu\phi) - \phi\partial^2\phi$, 忽略全微分项, 那么
$$Z(J) = \int D\phi\, e^{i\int d^dx\,(\frac{1}{2}\phi(\partial^2 - m^2)\phi + J(x)\phi(x))}, \tag{1.1.4}$$
代入 (B.1.1), 可知
$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\, J(x)D(x-y)J(y)}, \tag{1.1.5}$$
其中 $D(x-y)$ 满足
$$\begin{cases}(\partial^2 - m^2)D(x-y) = \delta^{(d)}(x-y)\\ (-p^2 - m^2)\tilde{D}(p,q) = (2\pi)^d\delta^{(d)}(p-q)\end{cases} \Longrightarrow D(x-y) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot(x-y)}}{-k^2 - m^2}. \tag{1.1.6}$$

1.2 free propagator

- 为了使 (1.1.4) 中的积分在 ϕ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon\int d^dx\phi^2}$.
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 ϵ 避免发散,
$$D(x) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot x}}{-k^2 - m^2 + i\epsilon} = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\Big(\theta(t)e^{i(-\omega_k t + \vec{k}\cdot\vec{x})} + \theta(-t)e^{i(\omega_k t + \vec{k}\cdot\vec{x})}\Big). \tag{1.2.1}$$

calculation:

对 k^0 积分, 注意有两个奇点 $k^0=\pm(\omega_k-i\epsilon)$, 当 $t>0$ 时, contour 处于下半平面, ... (另外注意到我们可以任意改变 $\vec{k}$ 的符号).
取 $D=3$, 可以尝试在球坐标下继续计算, 考虑 $\theta(t)$ 项,

$$\begin{aligned} iI(x) &= \frac{1}{(2\pi)^3}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi\int_0^{\infty}k^2dk\,\frac{e^{-i(\omega_k t-kr\cos\theta)}}{2(k^2+m^2)} \\ &= \frac{1}{(2\pi)^2 r}\frac{1}{2i}\int_{-\infty}^{+\infty}\frac{k}{k^2+m^2}e^{-i(\omega_k t-kr)}dk, \end{aligned} \tag{1.2.2}$$

注意到 $k=\pm im$ 既是极点也是 branch cut 的顶点.

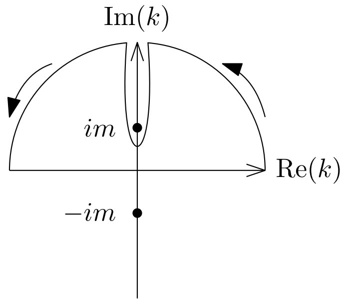


Figure 1.1: contour for evaluating the integral of the free field propagator.

参考 figure C.1, 可知 ω_k 在 branch cut 两侧的取值分别为

$$\omega_k=\begin{cases} i\sqrt{\kappa^2-m^2} & k=\pm(i\kappa+0^+) \\ -i\sqrt{\kappa^2-m^2} & k=\pm(i\kappa+0^-) \end{cases}, \tag{1.2.3}$$

其中 $\kappa>m$. 注意到 $\theta(t)$ 决定了 $t>0$, 那么 contour 的两个上半平面的弧中, 左侧的弧对积分不贡献, 但右侧的弧在 $t>r$ (类时) 情况下会发散, 在 $t<r$ (类空) 的情况下才收敛到零.

- $D(x)$ 的取值与 x 的类时, 类空性质关系密切.
  - 类时区域,
$$D(t,0)=-i\int\frac{d^Dk}{(2\pi)^D2\omega_k}\Big(\theta(t)e^{-i\omega_k t}+\theta(-t)e^{i\omega_k t}\Big). \tag{1.2.4}$$
  - 类空区域,
$$D(0,\vec{x})=-i\int\frac{d^Dk}{(2\pi)^D2\omega_k}e^{i\vec{k}\cdot\vec{x}}\sim e^{-m|\vec{x}|}. \tag{1.2.5}$$

1.3 from field to particle to force

1.3.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,

$$\begin{aligned} W(J) &= -\frac{1}{2}\int d^dxd^dy\,J(y)D(x-y)J(y) &(1.3.1)\\ &= -\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\tilde{J}(-k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}(k), &(1.3.2)\end{aligned}$$

其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k)=\tilde{J}^*(k)$.

- 考虑 $J(x)=J_1(x)+J_2(x)$, 那么 $W(J)$ 共有 4 项, 其中一个交叉项如下,

$$W_{12}(J)=-\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\tilde{J}_1(-k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}_2(k), \tag{1.3.3}$$

可见 $W(J)$ 取值较大的条件是:

1. $\tilde{J}_1(k), \tilde{J}_2(k)$ 有较大重叠,
2. 重叠位置的 k 是 on shell (即 $k^2 = -m^2$).

- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

1.3.2 from particle to force

- 考虑 $J(x) = \delta^{(D)}(\vec{x} - \vec{x}_1) + \delta^{(D)}(\vec{x} - \vec{x}_1) \Longrightarrow \tilde{J}_a(k) = 2\pi e^{-i\vec{k}\cdot\vec{x}_a}\delta(k^0)$, 那么

$$\begin{aligned} W_{12}(J) + W_{21}(J) &= \delta(0)\int \frac{d^Dk}{(2\pi)^{D-1}}\frac{1}{|\vec{k}|^2 + m^2 - i\epsilon}\cos(\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)) \\ &\overset{D=3}{=} 2\pi\delta(0)\frac{1}{4\pi r}e^{-mr}, \end{aligned} \tag{1.3.4}$$

($-i\epsilon$ 显然可以舍去), 注意到 $\langle 0|e^{-iHT}|0\rangle = e^{-iET}$, 而时间间隔 $T = \int dx^0 = 2\pi\delta(0)$, 所以

$$E = -\frac{W(J)}{T} \overset{D=3}{=} -\frac{1}{4\pi r}e^{-mr}. \tag{1.3.5}$$

calculation:

计算 (1.3.4) 中的积分, 令 $\vec{x}_1 - \vec{x}_2 = \vec{r}$,

$$\begin{aligned} I_D &= \int \frac{d^Dk}{(2\pi)^D}\frac{1}{|\vec{k}|^2 + m^2}\overbrace{\cos(\vec{k}\cdot\vec{r})}^{\mapsto e^{i\vec{k}\cdot\vec{r}}} \\ &= \frac{1}{(2\pi)^D}\int (k\sin\theta_1)^{D-2}d\Omega_{D-2}\int kd\theta_1 dk\,\frac{1}{k^2+m^2}e^{ikr\cos\theta_1} \\ &= \frac{S_{D-2}}{(2\pi)^D}\int k^{D-1}\sin^{D-2}\theta_1 d\theta_1 dk\,\frac{1}{k^2+m^2}e^{ikr\cos\theta_1}, \end{aligned} \tag{1.3.6}$$

取 $D = 3$, 那么

$$\begin{aligned} I_{D=3} &= \frac{1}{(2\pi)^2}\int k^2\sin\theta_1 d\theta_1 dk\,\frac{1}{k^2+m^2}e^{ik\cos\theta_1} \\ &= \frac{1}{2\pi^2 r}\int_0^\infty \sin(kr)\frac{kdk}{k^2+m^2} = \frac{-i}{4\pi^2 r}\int_{-\infty}^{\infty} e^{ikr}\frac{kdk}{k^2+m^2} \\ &= \frac{-i}{4\pi^2 r}2\pi i\,\underbrace{\mathrm{Res}(f, im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r}e^{-mr}. \end{aligned} \tag{1.3.7}$$

1.4 vacuum energy

- 注意到

$$Z(J=0) = \langle 0|e^{-iHT}|0\rangle, \tag{1.4.1}$$

所以

$$E_0 = \langle 0|H|0\rangle = V\int \frac{d^Dk}{(2\pi)^D}\frac{1}{2}\omega_k + \text{irrelevant terms}. \tag{1.4.2}$$

calculation:

代入 (B.1.1) (其中 N 是时空格点总数),

$$Z(J=0) = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}}, \tag{1.4.3}$$

其中 $A = -i(\partial^2 - m^2 + i\epsilon)$.

– 注意到 $\det e^A = e^{\operatorname{tr} A} \Longrightarrow \det A = e^{\operatorname{tr}\ln A}$, 代入, 并有

$$(\ln A)\phi(x) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} \ln(-i(-k^2 - m^2 + i\epsilon))\tilde{\phi}(k), \tag{1.4.4}$$

对于 $A : v \mapsto u$ 以及变换 $P : v \mapsto \tilde{v}$, 有 $PAP^{-1} : \tilde{v} \mapsto \tilde{u}$, 且 $\operatorname{tr} A = \operatorname{tr} PAP^{-1}$, 所以

$$\begin{aligned} -\frac{1}{2}\operatorname{tr}\ln A &= -\frac{1}{2}\operatorname{tr}\ln(-i(-k^2 - m^2 + i\epsilon)) \\ &= -\frac{1}{2}\sum_k \ln(-i(-k^2 - m^2 + i\epsilon)) \\ &= -\frac{1}{2}\frac{VT}{(2\pi)^d}\int d^dk\, \ln(-i(-k^2 - m^2 + i\epsilon)), \end{aligned} \tag{1.4.5}$$

其中, 参考 (A.2.5), 有 $\sum_k = \frac{VT}{(2\pi)^d}\int d^dk$.

代入 (1.4.1),

$$\begin{aligned} E_0 &= \frac{i}{T}\left(\frac{N}{2}\ln(2\pi) - \frac{1}{2}\frac{VT}{(2\pi)^d}\int d^dk\, \ln(-i(-k^2 - m^2 + i\epsilon))\right) \\ &= \frac{iN}{2T}\ln(2\pi) - \frac{i}{2}V\int \frac{d^dk}{(2\pi)^d}\left(\ln(\underbrace{-k^2 - m^2 + i\epsilon}_{=(k^0)^2 - \omega_k^2 + i\epsilon}) - \frac{\pi}{2}i\right), \end{aligned} \tag{1.4.6}$$

略去与 m 无关的常数项

$$\frac{\Delta E_0}{V} = -\frac{i}{2}\int \frac{d^Dk}{(2\pi)^D}\int \frac{dk^0}{2\pi}\ln((k^0)^2 - \omega_k^2 + i\epsilon), \tag{1.4.7}$$

做分部积分,

$$\ln((k^0)^2 - \omega_k^2 + i\epsilon) = \frac{d}{dk^0}(k^0\ln((k^0)^2 - \omega_k^2 + i\epsilon)) - k^0\frac{2k^0}{(k^0)^2 - \omega_k^2 + i\epsilon}, \tag{1.4.8}$$

代入,

$$\begin{aligned} \frac{E_0}{V} &= \frac{i}{2}\int \frac{d^Dk}{(2\pi)^D}\int \frac{dk^0}{2\pi}\frac{2(k^0)^2}{(k^0)^2 - \omega_k^2 + i\epsilon} \\ &= \frac{i}{2}\int \frac{d^Dk}{(2\pi)^D}\left(\frac{1}{2\pi}2\pi i\frac{2(-\omega_k)^2}{-2\omega_k}\right) = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{2}\omega_k. \end{aligned} \tag{1.4.9}$$

另外, $\ln(z^2 - 1 + i\epsilon)$ 的图像如下:

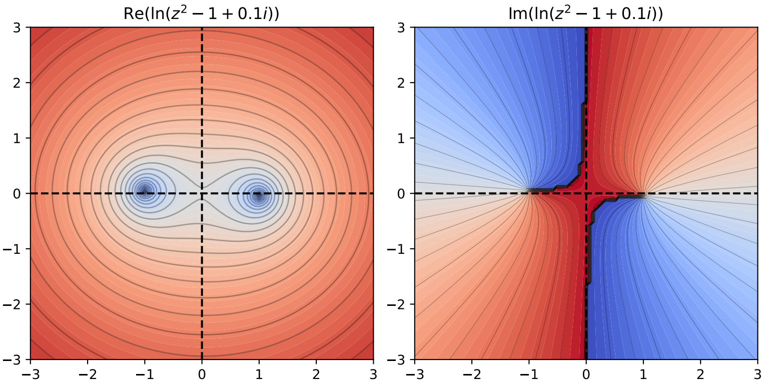


Figure 1.2: graph of $\ln(z^2 - 1 + i\epsilon)$.

Chapter 2

Coulomb and Newton: repulsive and attraction

2.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu, \tag{2.1.1}$$

其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

- 做路径积分,

$$Z(J) = \int DA\, e^{i\int d^dx(\mathcal{L}+J_\mu A^\mu)} = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J_\mu D^{\mu\nu}(x-y)J_\nu(y)}. \tag{2.1.2}$$

calculation:

massive photon 的作用量为

$$\begin{aligned}
S(A) &= \int d^dx\, \frac{1}{2}\Big(-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu \Big) \\
&= \int d^dx\, \frac{1}{2}\Big(A_\nu \partial^2 A^\nu - A_\nu \partial^\nu \partial_\mu A^\mu - m^2 A_\mu A^\mu \Big) + \text{total differential} \\
&= \int d^dx\, \frac{1}{2} A_\mu \Big(-\partial^\mu\partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \Big) A_\nu + \text{total differential} \\
&= \int \frac{d^dk}{(2\pi)^d} \tilde{A}_\mu(-k) \Big(k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2) \Big) \tilde{A}_\nu(k) + \text{boundary term},
\end{aligned} \tag{2.1.3}$$

那么, 需要有

$$\begin{aligned}
&(-\partial^\mu\partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2))D_{\rho\nu}(x-y) = \delta^\mu_\nu \delta^{(d)}(x-y) \\
\Longrightarrow\, &\tilde{D}_{\mu\nu}(k) = \frac{k_\mu k_\nu/m^2 + \eta_{\mu\nu}}{-k^2 - m^2}.
\end{aligned} \tag{2.1.4}$$

考虑到积分需要收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, (为什么 A_μ 类空, 只知道 $\tilde{A}_\mu$ 类空, 见 subsection 2.1.2, 但路径积分中的 A 显然不满足 field equation $\Longrightarrow$ 路径积分中起主要作用的 $\tilde{A}$ 类空, 因此 $-\epsilon|\tilde{A}|^2 < 0$).

- 因此

$$\begin{aligned}
W(J) &= -\frac{1}{2}\int d^dx d^dy\, J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) && (2.1.5) \\
&= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \tilde{J}_\mu(-k) \frac{k^\mu k^\nu/m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon} \tilde{J}_\nu(k), && (2.1.6)
\end{aligned}$$

注意到 current conservation, 有 $\partial_\mu J^\mu = 0 \iff k^\mu \tilde{J}_\mu(k) = 0$, 所以

$$W(J) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \tilde{J}^\mu(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_\mu(k), \tag{2.1.7}$$

观察电荷分量, 可见同性相斥, 异性相吸.

2.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的 x, y, z 方向, 在粒子静止系下, 极化矢量 $(\epsilon^i)_\mu = \delta^i_\mu, i = 1, 2, 3$, 而 $k_\mu = (-m, 0, 0, 0)$, 所以
$$k^\mu (\epsilon^i)_\mu = 0. \tag{2.1.8}$$
  - 注意, 一个粒子的极化方向用 e^i (这不是矢量) 表示, 极化矢量为 $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$.
- 在粒子静止系下, 考虑
$$\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu}, \tag{2.1.9}$$
可见
$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon}. \tag{2.1.10}$$

2.1.2 Maxwell Lagrangian

- 根据 (2.1.1) 中的 Lagrangian density, 得到 field equation,
$$\Big(- \partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \Big) A_\nu = 0. \tag{2.1.11}$$
  - spin-1 particle 有 3 个自旋自由度, 而 A_μ 有 4 个分量, 所以需要一个约束方程 (只在 $m \neq 0$ 情况下存在),
$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0. \tag{2.1.12}$$
  来源: 在 (2.1.11) 左右两边作用一个 ∂_μ 即可得到这个约束方程.

2.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = linearized Einstein Lagrangian.
- 受 subsection 2.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足
$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases}, \tag{2.2.1}$$
其中下指标 μ, ν 对称, $a = 1, \cdots, 5$, (可以验证 $(\epsilon^a)_{\mu\nu}$ 确实有 5 个独立分量).
  - 对 $(\epsilon^a)_{\mu\nu}$ 的归一化条件可以定义为 $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$.
  - 与 subsection 2.1.1 中提示一样, 粒子的极化方向用 e^a 表示.
- 那么
$$\sum_{a=1}^5 (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma}. \tag{2.2.2}$$

calculation:

首先用 k_μ 和 $\eta_{\mu\nu}$ 构造最一般的关于 $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$ 对称的 4 阶张量, (下式中把 $\frac{k_\mu}{m}$ 略写作 k_μ),
$$\begin{aligned} &A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ &+ D \eta_{\mu\nu} \eta_{\rho\sigma} + E(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}), \end{aligned} \tag{2.2.3}$$
代入 (2.2.1) 得
$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4}, \tag{2.2.4}$$

因此, 这个 4 阶张量最终确定为

$$\frac{3}{4}A\Big((G_{\mu\rho}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\rho})-\frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\Big). \tag{2.2.5}$$

- 所以

$$\tilde{D}_{\mu\nu\rho\sigma}(k)=\frac{(G_{\mu\rho}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\rho})-\frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2-m^2+i\epsilon}. \tag{2.2.6}$$

- 计算路径积分中的 $W(T)$,

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{(G^{\mu\rho}G^{\nu\sigma}+G^{\mu\sigma}G^{\nu\rho})-\frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2-m^2+i\epsilon}\tilde{T}_{\rho\sigma}(k), \tag{2.2.7}$$

注意到 $\partial_\mu T^{\mu\nu}(x)=0 \iff k_\mu\tilde{T}^{\mu\nu}(k)=0$, 并考虑到 T 是对称张量, 所以

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{2\eta^{\mu\rho}\eta^{\nu\sigma}-\frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2-m^2+i\epsilon}\tilde{T}_{\rho\sigma}(k), \tag{2.2.8}$$

考虑能量项, 可见质量互相吸引.

2.3 remarks

- 由于 seesaw mechanism (见 subsection C.1.1), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为 $\eta^{00}=-1$.
- 本 chapter 中的计算都是 $m\neq 0$ 的粒子, 与真实世界有差异.

Chapter 3

Feynman diagrams

3.1 a baby problem

- 考虑如下积分,

$$Z(J)=\int_{-\infty}^{+\infty}dq\,e^{-\frac{1}{2}m^2q^2-\frac{\lambda}{4!}q^4+Jq}. \tag{3.1.1}$$

- **Schwinger's way:** 把 integrand 对 λ 展开, 并将 q 用 $\frac{\partial}{\partial J}$ 替代, 得到

$$\begin{aligned}Z(J)&=e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4}\int_{-\infty}^{+\infty}dq\,e^{-\frac{1}{2}m^2q^2+Jq}\\&=\sqrt{\frac{2\pi}{m^2}}e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4}e^{\frac{J^2}{2m^2}},\end{aligned} \tag{3.1.2}$$

后面的计算中忽略 $Z(J=0,\lambda=0)$.

- 每个 vertex 带有 $-\lambda$, 每个 line 带有 $\frac{1}{m^2}$, 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里 q 无法区分, 所以不方便算, 先略了):

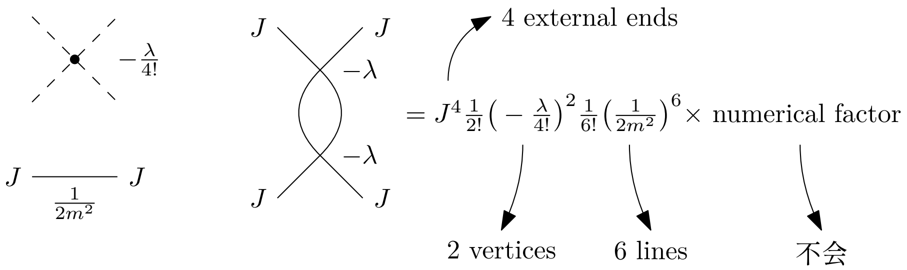
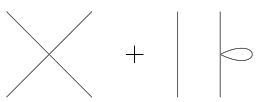


Figure 3.1: baby problem - Feynman diagram.

calculation:

在这里计算 λJ^4 项,



具体暂时不会算, (3.1.3)

但是直接计算 (3.1.2) 的展开项, 得到的结果与 (3.1.5) 一样.

3.1.1 Wick contraction and Green's functions

- 把积分 (3.1.1) 对 J 展开,

$$Z(J)=\sum_{n=0}^{\infty}\frac{1}{n!}J^n\underbrace{\int_{-\infty}^{+\infty}dq\,e^{-\frac{1}{2}m^2q^2-\frac{\lambda}{4!}q^4}q^n}_{=Z(0,0)G^{(n)}}, \tag{3.1.4}$$

其中 Green's function $G^{(n)}$ 对 λ 展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

calculation:

计算 λJ^4 项, 它来自 $G^{(4)}$ 对 λ 展开的一阶项,

$$\begin{aligned}-\frac{\lambda}{4!}\int dq\, e^{-\frac{1}{2}m^2q^2}q^8 &= -\frac{\lambda}{4!}\langle q^8\rangle \\ &= -\frac{\lambda}{4!}\sum_{\text{Wick}}\left(\frac{1}{m^2}\right)^4 \\ &= -\frac{\lambda}{4!}\frac{7\times 5\times 3\times 1}{m^8},\end{aligned} \tag{3.1.5}$$

所以 λJ^4 项等于 $\frac{105}{(4!)^2}\frac{-\lambda J^4}{m^8}$.

3.1.2 connected vs. disconnected

- 考虑
$$Z(J,\lambda) = Z(J=0,\lambda)e^{W(J,\lambda)}, \tag{3.1.6}$$
其中 $Z(J=0,\lambda)$ 由 diagrams with no external source J 组成, 而 $W(J,\lambda)$ 则由 connected diagrams 组成 (?).
- 我们希望计算的是 W, 而不是 Z (?).

3.2 a child problem

- 考虑如下积分
$$Z(J) = \int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q-\frac{\lambda}{4!}q^4+J^T\cdot q}, \tag{3.2.1}$$
其中 $q^4 = \sum_i q_i^4$.
- **Schwinger's way:** 对 λ 展开并把 q 替换为 $\frac{\partial}{\partial J}$, 得到
$$Z(J) = \sqrt{\frac{(2\pi)^N}{\det A}}e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J}, \tag{3.2.2}$$
其中 $\left(\frac{\partial}{\partial J}\right)^4 = \sum_i\left(\frac{\partial}{\partial J_i}\right)^4$.

3.2.1 n-point Green's function

- **Wick's way:** 对 J 展开获得带 Green's function 的表达式,
$$Z(J) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}J_{i_1}\cdots J_{i_n}\underbrace{\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q-\frac{\lambda}{4!}q^4}q_{i_1}\cdots q_{i_n}}_{=Z(0,0)G^{(n)}_{i_1\cdots i_n}}, \tag{3.2.3}$$
其中 $G^{(n)}_{i_1\cdots i_n}$ 称为 n-point Green's function.

Taylor expansion:

多元函数的 Taylor 展开为

$$\begin{aligned}f(x_1,\cdots,x_N) &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_N^{n_N}}{n_N!}\frac{\partial^{n_1}}{\partial x_1^{n_1}}\cdots\frac{\partial^{n_N}}{\partial x_N^{n_N}}f(x=0) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}x_{i_1}\cdots x_{i_n}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_N}}f(x=0),\end{aligned} \tag{3.2.4}$$

这两种求和方法, $x_1^{n_1}\cdots x_N^{n_N}$ 项的 numerical factor 都等于

$$\frac{1}{n!}\times\frac{n!}{n_1!\cdots n_N!} = \frac{1}{n_1!\cdots n_N!}, \tag{3.2.5}$$

其中 $n = n_1 + \cdots + n_N$.

- 在 $\lambda = 0$ 时, 2-point Green's function 就是 propagator,

$$
\begin{aligned}
G_{ij}^{(2)}(\lambda=0) &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} q_i q_j \\
&= \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \Big|_{J=0} = A_{ij}^{-1}.
\end{aligned} \tag{3.2.6}
$$

- 来计算 $2, 3, 4$-point Green's functions,

$$
\begin{cases}
G_{ij}^{(2)} = A_{ij}^{-1} - \dfrac{\lambda}{4!} \displaystyle\sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12 A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \\
G_{ijk}^{(3)} = 0 \\
G_{ijkl}^{(4)} = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \\
\qquad\quad - \dfrac{\lambda}{4!} \displaystyle\sum_m (A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} A_{kl}^{-1} + \cdots + 4! A_{im}^{-1} A_{jm}^{-1} A_{km}^{-1} A_{lm}^{-1}) + O(\lambda^2)
\end{cases}. \tag{3.2.7}
$$

calculation:

2-point Green's function 计算如下,

$$
\begin{aligned}
G_{ij}^{(2)} &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j \\
&= A_{ij}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j \rangle + O(\lambda^2) \\
&= A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12 A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2),
\end{aligned} \tag{3.2.8}
$$

3-point Green's function 计算如下,

$$
G_{ijk}^{(32)} = \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j q_k = 0, \tag{3.2.9}
$$

4-point Green's function 计算如下,

$$
\begin{aligned}
G_{ijkl}^{(4)} &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j q_k q_l \\
&= A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j q_k q_l \rangle + O(\lambda^2).
\end{aligned} \tag{3.2.10}
$$

3.3 perturbative field theory

- 做如下替换即可,

$$
\begin{cases}
A \mapsto -i(\partial^2 - m^2) \\
J \mapsto iJ
\end{cases}. \tag{3.3.1}
$$

- **Schwinger's way:** ϕ^4 theory 的路径积分,

$$
Z(J) = \int D\phi \, e^{i \int d^d x \, (\frac{1}{2}\phi(\partial^2 - m^2)\phi - \frac{\lambda}{4!}\phi^4 + J(x)\phi(x))} \tag{3.3.2}
$$

$$
= Z(0,0) e^{-i\frac{\lambda}{4!} \int d^d z \, (\frac{\delta}{i\delta J(z)})^4} e^{-\frac{i}{2} \int d^d x d^d y \, J(x) D(x-y) J(y)}, \tag{3.3.3}
$$

其中 $D(x-y)$ 是自由场的 propagator, 见 (1.2.1).

- **Wick's way:** 同样, 对 J 展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \cdots d^d x_n \, J(x_1) \cdots J(x_n) G^{(n)}(x_1, \cdots, x_n), \tag{3.3.4}$$

其中

$$G^{(n)}(x_1, \cdots, x_n) = \frac{1}{Z(0,0)} \int D\phi \, e^{i \int d^d x \, (\frac{1}{2}\phi(\partial^2 - m^2)\phi - \frac{\lambda}{4!}\phi^4)} \phi(x_1) \cdots \phi(x_n), \tag{3.3.5}$$

有时 $Z(J)$ 被称为 generating functional, 因为它能生成 Green's functions.

3.3.1 collision between particles

- 通过 Wick's way, 考虑 $J(x_1)J(x_2)J(x_3)J(x_4)$ 项, 实际上就是要计算 $G^{(4)}(x_1, x_2, x_3, x_4)$, 它的 0 阶项为

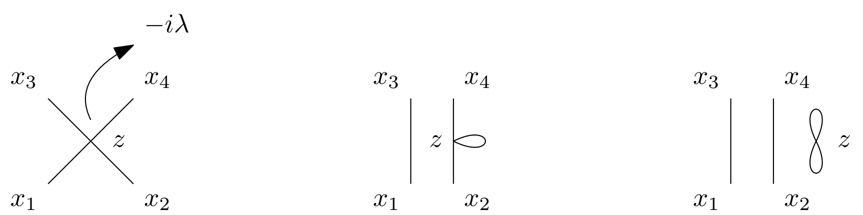
$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4, \lambda = 0) &= \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta J(x_2)} \frac{\delta}{i\delta J(x_3)} \frac{\delta}{i\delta J(x_4)} e^{-\frac{i}{2}\int d^d x d^d y \, J(x) D(x-y) J(y)} \\ &= -(D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}), \end{aligned} \tag{3.3.6}$$

其中 D_{ij} 是 $D(x_i - x_j)$ 的简写, 可见, 传播子实际上是 $(-i)^3 D = iD$.

- $G^{(4)}_{1234}$ 的 1 阶项为

$$\begin{aligned} \text{1st order term} &= -\frac{i\lambda}{4!} \int d^d z \, \langle \phi_1 \cdots \phi_4 \phi^4(z) \rangle \\ &= -\frac{i\lambda}{4!} \int d^d z \, \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_4} \Big(\frac{\delta}{i\delta J(z)} \Big)^4 e^{-\frac{i}{2}\int d^d x d^d y \, J(x) D(x-y) J(y)} \\ &= -\frac{i\lambda}{4!} \int d^d z \, \Big(4! D_{1z} D_{2z} D_{3z} D_{4z} \\ &\quad + 4 \times 3 D_{12} D_{3z} D_{4z} D_{zz} + \cdots + 3 D_{12} D_{34} D_{zz} D_{zz} + \cdots \Big), \end{aligned} \tag{3.3.7}$$

其中各项分别对应如下 Feynman diagrams:

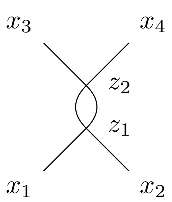


$-i\lambda \int d^d z \, D_{1z} D_{2z} D_{3z} D_{4z}$ $\quad$ $\frac{-i\lambda}{2!} \int d^d z D_{13} D_{2z} D_{4z} D_{zz}$ $\quad$ $\frac{-i\lambda}{8} \int d^d z D_{13} D_{24} D_{zz} D_{zz}$

Figure 3.2: position space - Feynman diagrams.

其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,



$$= (4 \times 3)^2 \times 2 \times \big(\frac{-i\lambda}{4!}\big)^2 \int d^d z_1 d^d z_2 \, D_{1z_1} D_{2z_1} D_{3z_2} D_{4z_2} D_{z_1 z_2} D_{z_1 z_2}. \tag{3.3.8}$$

3.3.2 in momentum space

- 本 subsection 将 (3.3.5) 转换到 momentum space, 注意到 $\tilde{J}(k)$ 和 $\tilde{J}(-k)$ 并不独立, 所以 $\frac{\partial}{\partial i\tilde{J}}$ 不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\begin{aligned} \tilde{G}^{(n)}(k_1, \cdots, k_n) &= \int d^d x_1 \cdots d^d x_n \, e^{-i(k_1 \cdot x_1 + \cdots)} G^{(n)}(x_1, \cdots, x_n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \cdots d^d x_n \, e^{-i(k_1 \cdot x_1 + \cdots)} \langle \Big(-\frac{i\lambda}{4!} \int d^d z \, \phi_z^4 \Big)^n \phi_1 \cdots \phi_n \rangle . \end{aligned} \tag{3.3.9}$$

– propagator 的 Fourier transformation 是

$$\tilde{D}_{pq} = \int d^dx d^dy\, e^{-i(p\cdot x+q\cdot y)} D(x-y) = \frac{(2\pi)^d \delta^{(d)}(p+q)}{-p^2-m^2+i\epsilon}, \tag{3.3.10}$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1,k_2,k_3,k_4)$ 的 1 阶项为

$$\text{1st order term} = -\frac{i\lambda}{4!}\int d^dx_1\cdots d^dx_4\, e^{-i(k_1\cdot x_1+\cdots)}\int d^dz\, \langle \phi_z^4\phi_1\cdots\phi_4\rangle\,, \tag{3.3.11}$$

考虑第 1 项,

$$\begin{aligned}
&-\frac{i\lambda}{4!}\int d^dx_1\cdots d^dx_4\, e^{-i(k_1\cdot x_1+\cdots)}\int d^dz\, 4! D_{1z}\cdots D_{4z}\\
=&-i\lambda\int d^dx_1\cdots d^dx_4 d^dz\, e^{-i(k_1\cdot x_1+\cdots)} e^{i(p_1\cdot(x_1-z)+\cdots)}\prod_{i=1}^4\int\frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon}\\
=&-i\lambda\underbrace{\int d^dz\, e^{-iz\cdot(k_1+\cdots+k_4)}}_{=(2\pi)^d\delta^{(d)}(k_1+\cdots+k_4)}\prod_{i=1}^4\frac{1}{-k_i^2-m^2+i\epsilon}.
\end{aligned} \tag{3.3.12}$$

– 出射粒子不一定 on-shell (?).

- 得到这些 Feynman diagrams:

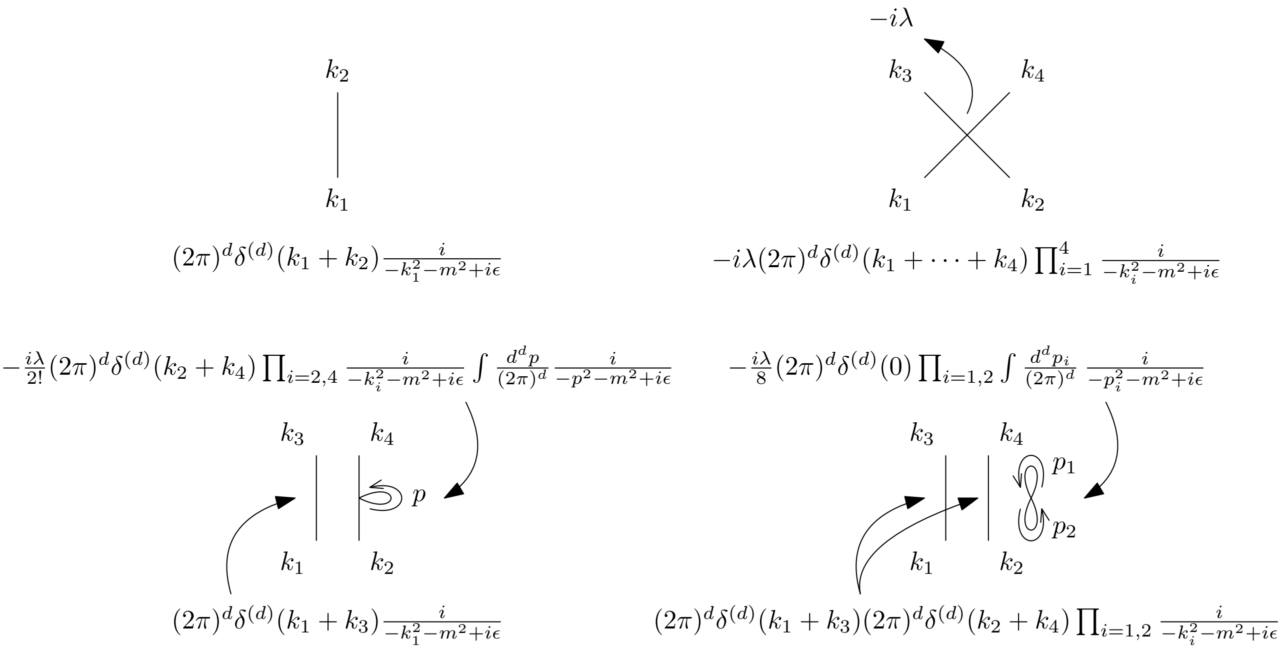


Figure 3.3: momentum space - Feynman diagrams.

calculation:

第 3 幅图的计算如下,

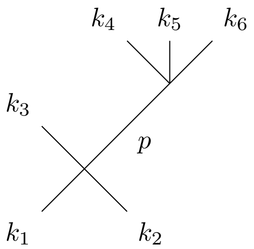
$$\begin{aligned}
&-\frac{i\lambda}{2!}\int d^dx_1\cdots d^dx_4\, e^{-i(k_1\cdot x_1+\cdots)}\int d^dz\, D_{13}D_{2z}D_{4z}D_{zz}\\
=&-\frac{i\lambda}{2!}\int d^dx_1\cdots d^dx_4 d^dz\, e^{-i(k_1\cdot x_1+\cdots)} e^{i(p_1\cdot(x_1-x_3)+p_2\cdot(x_2-z)+p_4\cdot(x_4-z)+p_4\cdot 0)}\\
&\prod_{i=1}^4\int\frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{i\lambda}{2!}\int d^d z\, e^{-iz\cdot(p_2+p_4)}\delta^{(d)}(p_1-k_1)\delta^{(d)}(p_2-k_2)\delta^{(d)}(p_1+k_3)\delta^{(d)}(p_4-k_4) \\
&\quad \prod_{i=1}^{4}\int d^d p_i \frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{2!}(2\pi)^d\delta^{(d)}(k_1+k_3)\delta^{(d)}(k_2+k_4)\prod_{i=1,2,4}\frac{1}{-k_i^2-m^2+i\epsilon}\int\frac{d^dp}{-p^2-m^2+i\epsilon},
\end{aligned}
\tag{3.3.13}
$$

第 4 幅图的计算如下,

$$
\begin{aligned}
&-\frac{i\lambda}{8}\int d^dx_1\cdots d^dx_4\, e^{-i(k_1\cdot x_1+\cdots)}\int d^dz\, D_{13}D_{24}D_{zz}D_{zz} \\
&= -\frac{i\lambda}{8}\int d^dx_1\cdots d^dx_4 d^dz\, e^{-i(k_1\cdot x_1+\cdots)}e^{i(p_1\cdot(x_1-x_3)+p_2\cdot(x_2-x_4)+p_3\cdot 0+p_4\cdot 0)} \\
&\quad \prod_{i=1}^{4}\int\frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{8}\int d^dz\,\delta^{(d)}(p_1-k_1)\delta^{(d)}(p_2-k_2)\delta^{(d)}(p_1+k_3)\delta^{(d)}(p_2+k_4) \\
&\quad \prod_{i=1}^{4}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{8}(2\pi)^d\delta^{(d)}(0)\delta^{(d)}(k_1+k_3)\delta^{(d)}(k_2+k_4)\prod_{i=1,2}\frac{1}{-k_i^2-m^2+i\epsilon} \\
&\quad \prod_{i=1,2}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon}.
\end{aligned}
\tag{3.3.14}
$$

- 再举一个例子 (略去了 $\prod_{i=1}^{6}\frac{i}{-k_i^2-m^2+i\epsilon}$),



$$
\begin{aligned}
&=(4!)^2\times\Big(-\frac{i\lambda}{4!}\Big)^2(2\pi)^{2d}\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\delta^{(d)}(k_1+k_2+k_3+p)\delta^{(d)}(k_4+k_5+k_6-p) \\
&=(-i\lambda)^2(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4+k_5+k_6)\frac{i}{-(k_1+k_2+k_3)^2-m^2+i\epsilon}.
\end{aligned}
\tag{3.3.15}
$$

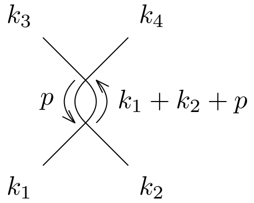
3.3.3 loops and a first look at divergence

- subsection 3.3.2 里的 loop diagrams 出现了如下积分,

$$
\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}=\int\frac{d^Dp}{(2\pi)^D2\omega_p}\sim\int\frac{d^Dp}{|p|},
\tag{3.3.16}
$$

  积分发散.

- 再举一个例子 (略去了 $\prod_{i=1}^{4}\frac{i}{-k_i^2-m^2+i\epsilon}$),



$$
\begin{aligned}
=&(4\times 3)^2\times 2\times\left(\frac{-i\lambda}{4!}\right)^2\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\int\frac{d^dq}{(2\pi)^d}\frac{i}{-q^2-m^2+i\epsilon}\\
&(2\pi)^d\delta^{(d)}(k_1+k_2+p-q)(2\pi)^d\delta^{(d)}(k_3+k_4-p+q) \qquad (3.3.17)\\
=&\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4)\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\frac{i}{-(k_1+k_2+p)^2-m^2+i\epsilon}\\
=&\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4)\int\frac{d^Dp}{(2\pi)^D}\Big(\frac{1}{2\omega_p}\frac{i}{(k_1^0+k_2^0-\omega_p)^2-\omega_{k_1+k_2+p}^2}\\
&+\frac{i}{(\omega_{k_1+k_2+p}-k_1^0-k_2^0)^2-\omega_p^2}\frac{1}{2\omega_{k_1+k_2+p}}\Big) \qquad (3.3.18)\\
\sim&\int\frac{d^Dp}{p^3}, \qquad (3.3.19)
\end{aligned}
$$

同样, 积分发散.

Chapter 4

canonical quantization

- A. Zee: the canonical and the path integral formalisms often appear complementary, in the sense that results difficult to see in one are clear in the other.
- **nobody is perfect:**
  - **canonical quantization:** 如何定义场算符乘积的顺序.
  - **path integral:** integration measure.

4.1 Heisenberg and Dirac

4.1.1 quantum mechanics

- 单粒子的 classical Lagrangian 为

$$L = \frac{1}{2}\dot{q}^2 - V(q) \Longrightarrow \begin{cases} p = \dot{q} \\ H = p\dot{q} - L = \frac{1}{2}p^2 + V(q) \end{cases}. \tag{4.1.1}$$

- canonical commutation relation 如下,

$$[p, q] = -i, \tag{4.1.2}$$

因此, 算符的演化方程为

$$\begin{cases} \frac{dp}{dt} = i[H, p] = -V'(q) \\ \frac{dq}{dt} = i[H, q] = p \end{cases}. \tag{4.1.3}$$

calculation:

$$\begin{cases} [p, q] = -i \\ [p, q^2] = -2iq \\ \qquad \vdots \\ [p, q^n] = -iq^{n-1} + q[p, q^{n-1}] \end{cases} \Longrightarrow [p, q^n] = -inq^{n-1} \Longrightarrow [p, V(q)] = -iV'(q). \tag{4.1.4}$$

- follow Dirac's approach,

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \Longleftrightarrow \begin{cases} q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \\ p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \end{cases} \Longrightarrow [a, a^\dagger] = 1, \tag{4.1.5}$$

算符 a 的演化方程为

$$\frac{da}{dt} = -i\sqrt{\frac{\omega}{2}}\Big(\frac{1}{\omega}V'(q) + ip\Big). \tag{4.1.6}$$

4.1.2 scalar field

- 标量场的 Lagrangian 为

$$L=\int d^Dx\Big(-\frac{1}{2}((\partial\phi)^2+m^2\phi^2)-u(\phi)\Big), \tag{4.1.7}$$

canonical commutation relation 为

$$\pi(\vec{x},t)=\frac{\delta L(t)}{\delta\partial_0\phi(\vec{x},t)}=\partial_0\phi(\vec{x},t)\quad\text{and}\quad[\pi(\vec{x},t),\phi(\vec{y},t)]=-i\delta^{(D)}(\vec{x}-\vec{y}), \tag{4.1.8}$$

标量场的 Hamiltonian 为

$$H=\int d^Dx\,(\pi\phi-\mathcal{L})=\int d^Dx\Big(\frac{1}{2}(\pi^2+|\vec{\nabla}\phi|^2+m^2\phi^2)+u(\phi)\Big). \tag{4.1.9}$$

- 算符的演化方程为

$$\begin{cases}\partial_0\phi=i[H,\phi]=\pi\\ \partial_0\pi=i[H,\pi]=(-\vec{\nabla}^2+m^2)\phi+\dfrac{du}{d\phi}\end{cases}\Longrightarrow(\partial^2-m^2)\phi-\frac{du}{d\phi}=0. \tag{4.1.10}$$

- 当 $u(\phi)=0$ 时, 求解场方程 (4.1.10) 和 canonical commutation relation (4.1.8) 得到

$$\phi(\vec{x},t)=\int\frac{d^Dk}{(2\pi)^D2\omega_k}(\alpha_k(t)e^{i\vec{k}\cdot\vec{x}}+\alpha_k^\dagger(t)e^{-i\vec{k}\cdot\vec{x}}), \tag{4.1.11}$$

其中

$$\alpha_k(t)=\sqrt{(2\pi)^D2\omega_k}\,a_{\vec{k}}e^{-i\omega_kt}\quad\text{and}\quad[a_{\vec{p}},a_{\vec{q}}^\dagger]=\delta^{(D)}(\vec{p}-\vec{q}). \tag{4.1.12}$$

另外, 在后面的笔记中使用简记 $\sqrt{(2\pi)^D2\omega_k}=\rho(k)$.

calculation:

求解场方程 (4.1.10), 得到

$$\phi(\vec{x},t)=\int\frac{d^Dk}{(2\pi)^D}(\alpha_{\vec{k}}e^{i(-\omega_kt+\vec{k}\cdot\vec{x})}+\alpha_{\vec{k}}^\dagger e^{-i(-\omega_kt+\vec{k}\cdot\vec{x})}), \tag{4.1.13}$$

代入 canonical commutation relation (4.1.8), 有 (其中 $x^0=y^0=t,k^0=\omega_k$)

$$\begin{aligned}&\int\frac{d^Dk_2}{(2\pi)^D}\Big(-i\omega_{k_1}[\alpha_{\vec{k}_1},\alpha_{\vec{k}_2}]e^{i(k_1\cdot x+k_2\cdot y)}+i\omega_{k_1}[\alpha_{\vec{k}_1}^\dagger,\alpha_{\vec{k}_2}^\dagger]e^{-i(k_1\cdot x+k_2\cdot y)}\\&-i\omega_{k_1}[\alpha_{\vec{k}_1},\alpha_{\vec{k}_2}^\dagger]e^{i(k_1\cdot x-k_2\cdot y)}+i\omega_{k_1}[\alpha_{\vec{k}_1}^\dagger,\alpha_{\vec{k}_2}]e^{-i(k_1\cdot x-k_2\cdot y)}\Big)=-ie^{i\vec{k}_1\cdot(\vec{x}-\vec{y})}\\&\Longrightarrow\begin{cases}[\alpha_{\vec{k}_1},\alpha_{\vec{k}_2}]=\dfrac{1}{2\omega_{k_1}}\delta^{(D)}(\vec{k}_1+\vec{k}_2)\Longrightarrow[\alpha_{\vec{k}},\alpha_{\vec{k}}]\neq0 & \text{wrong}\\ [\alpha_{\vec{k}_1},\alpha_{\vec{k}_2}^\dagger]=\dfrac{1}{2\omega_{\vec{k}_1}}\delta^{(D)}(\vec{k}_1-\vec{k}_2) & \text{right}\end{cases}.\end{aligned} \tag{4.1.14}$$

- 代入 (4.1.9) 可得 (依然是 $u(\phi)=0$ 的情况下)

$$H=\int d^Dk\,\omega_k\frac{a_{\vec{k}}^\dagger a_{\vec{k}}+a_{\vec{k}}a_{\vec{k}}^\dagger}{2}=\int d^Dk\,\omega_k\Big(a_{\vec{k}}^\dagger a_{\vec{k}}+\frac{1}{2}\delta^{(D)}(0)\Big)\Longrightarrow\langle0|H|0\rangle=V\int\frac{d^Dk}{(2\pi)^D}\frac{1}{2}\omega_k, \tag{4.1.15}$$

其中, $V=\int d^Dx=(2\pi)^D\delta^{(D)}(0)$.

- vacuum state 定义为 $a_{\vec{k}}|0\rangle=0$, 有

$$\langle0|\phi(x)\phi(y)|0\rangle=\int\frac{d^Dk}{(2\pi)^D2\omega_k}e^{ik\cdot(x-y)}, \tag{4.1.16}$$

其中 $k^0=\omega_k$. 因此, 对比 (1.2.1), 有

$$\langle0|T(\phi(x)\phi(y))|0\rangle=iD(x-y). \tag{4.1.17}$$

energy-momentum tensor

- scalar field 的动量算符为
$$P^\mu = \int d^D x\, T^{0\mu} = \int d^D k\, k^\mu a^\dagger_{\vec{k}} a_{\vec{k}}, \tag{4.1.18}$$
其中, energy-momentum tensor 见 subsection D.2.3, 另外 $P^0 = H$ 还有一个 vacuum energy.

4.2 interaction picture

- 注意, 在 $u(\phi) \neq 0$ 的情况下, (即便在 Schrödinger's picture 里, $t = 0$ 时) (4.1.11) 不再成立, 因此无法通过 Schrödinger's picture or Heisenberg's picture 求解存在相互作用的场论.
- 将 Hamiltonian 分成两个部分,
$$H = H_0 + H'. \tag{4.2.1}$$
- operators 以自由场的 Hamiltonian 演化,
$$O_I(t) = U_0^\dagger(t,0) O(0) U_0(t,0) \quad \text{where} \quad U_0(t_2,t_1) = \mathrm{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H_0\Big), \tag{4.2.2}$$
states 以如下方式演化,
$$|\psi(t)\rangle_I = U_0^\dagger(t,0) U(t,0)\, |\psi(0)\rangle \quad \text{where} \quad U(t_2,t_1) = \mathrm{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H\Big), \tag{4.2.3}$$
因此
$$|\psi(t_2)\rangle_I = U_I(t_2,t_1)\, |\psi(t_1)\rangle_I \quad \text{where} \quad U_I(t_2,t_1) = \mathrm{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H_I(t)\Big), \tag{4.2.4}$$
注意, (4.2.2) 和 (4.2.3) 中, Texp 里的 H, H_0 都是 Schrödinger's picture 里的算符.

calculation:

首先有
$$U_I(t_2,t_1) = U_0^\dagger(t_2,0) U(t_2,t_1) U_0(t_1,0), \tag{4.2.5}$$
因此
$$\begin{aligned}\frac{d}{dt} U_I(t,t_0) &= iH_0 U_I(t,t_0) - iU_0^\dagger(t,0) H U(t,t_0) U_0(t_0,0)\\ &= -i\,\underbrace{U_0^\dagger(t,0) H' U_0(t,0)}_{=H_I(t)}\, U_I(t,t_0).\end{aligned} \tag{4.2.6}$$

4.3 scattering amplitude

- 最一般的过程是 $p_1,\cdots,p_m \to q_1,\cdots,q_n$, 其 scattering amplitude 为
$$\langle q_1,\cdots,q_n| U_0^\dagger(-\infty,0) U_I(+\infty,-\infty) U_0(-\infty,0) |p_1,\cdots,p_m\rangle\,, \tag{4.3.1}$$
一般会忽略掉 U_0 产生的相位.

- 考虑 ϕ^4 理论中的 $k_1,k_2 \to k_3,k_4$ 过程,
$$\langle k_3,k_4| e^{-i\int d^d x\, \frac{\lambda}{4!}\phi^4} |k_1,k_2\rangle\,, \tag{4.3.2}$$
对 λ 展开, 0 阶项为
$$\begin{aligned}\text{0th order term} &= \langle k_3,k_4|k_1,k_2\rangle\\ &= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\, \langle 0| a_{\vec{k}_3} a_{\vec{k}_4} a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle\end{aligned}$$

$$
\begin{aligned}
&= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\Big(\underbrace{\langle 0|a_{\vec{k}_3}a_{\vec{k}_4}a^\dagger_{\vec{k}_1}a^\dagger_{\vec{k}_2}|0\rangle}_{=\delta^{(D)}_{31}\delta^{(D)}_{42}} + \underbrace{\langle 0|a_{\vec{k}_3}a_{\vec{k}_4}a^\dagger_{\vec{k}_1}a^\dagger_{\vec{k}_2}|0\rangle}_{=\delta^{(D)}_{32}\delta^{(D)}_{41}}\Big) \\
&= (2\pi)^{2D}4\omega_{k_1}\omega_{k_2}(\delta^{(D)}(\vec{k}_1-\vec{k}_3)\delta^{(D)}(\vec{k}_2-\vec{k}_4)+\delta^{(D)}(\vec{k}_1-\vec{k}_4)\delta^{(D)}(\vec{k}_2-\vec{k}_3)),
\end{aligned}
\tag{4.3.3}
$$

1 阶项为 (其中 $k^0=\omega_k$)

$$
\begin{aligned}
\text{1st order term} =& \frac{-i\lambda}{4!}\int d^dx\ \langle k_3,k_4|\phi^4(x)|k_1,k_2\rangle \\
=& \overbrace{4!\times\frac{-i\lambda}{4!}\int d^dx\, e^{i(k_1+k_2-k_3-k_4)\cdot x}}^{=-i\lambda(2\pi)^d\delta^{(d)}(k_1+k_2-k_3-k_4)} + \rho(k_1)\rho(k_4)\delta^{(D)}_{14}\times 12\times\frac{-i\lambda}{4!}(2\pi)^d\delta^{(d)}_{23}\int\frac{d^Dp}{\rho^2(p)} \\
&+\cdots+\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\delta^{(D)}_{13}\delta^{(D)}_{24}\times 3\times\frac{-i\lambda}{4!}\int d^dx\int\frac{d^Dp_1}{\rho^2(p_1)}\frac{d^Dp_2}{\rho^2(p_2)}+\cdots,
\end{aligned}
\tag{4.3.4}
$$

分别对应如下 Feynman diagrams:

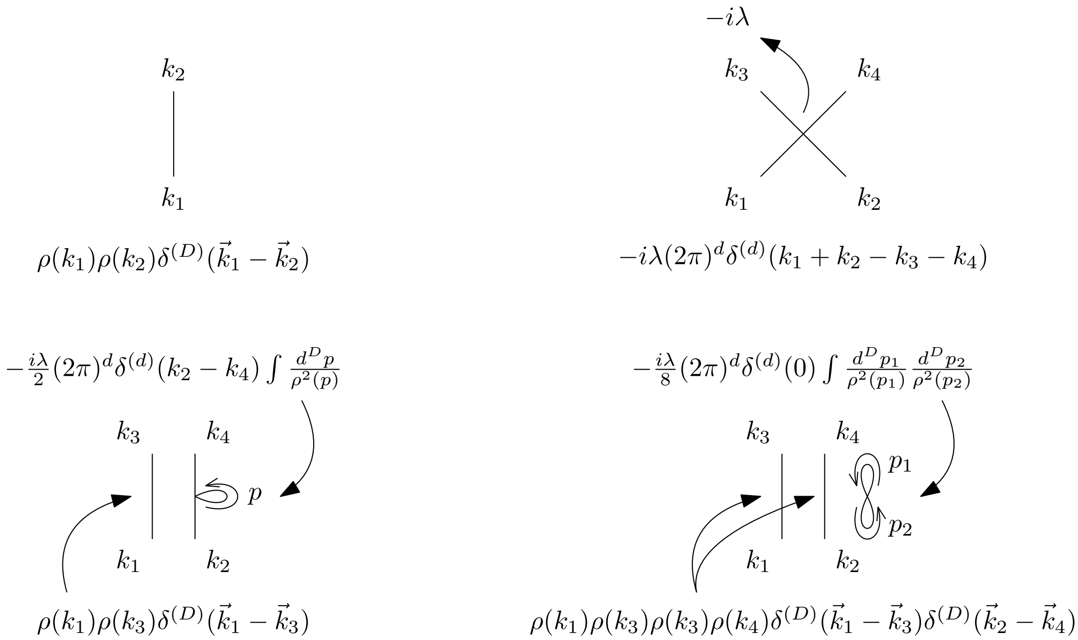
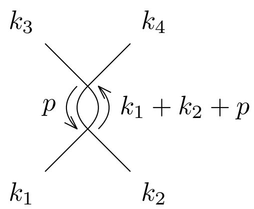


Figure 4.1: canonical quantization - Feynman diagrams.

观察可见, 上图和 figure 3.3 有对应关系.

- 再举一个例子,



$$
\begin{aligned}
=&(4\times 3)^2\times 2\times\left(\frac{-i\lambda}{4!}\right)^2\rho(k_1)\cdots\int d^dx_1d^dx_2\int\frac{d^Dp_1\cdots}{\rho(p_1)\cdots}\frac{d^Dq_1\cdots}{\rho(q_1)\cdots}e^{i(p_1+p_2-p_3-p_4)\cdot x_1}e^{i(q_1+q_2-q_3-q_4)\cdot x_2} \\
&\Big(\theta(t_2-t_1)\langle 0|a_{\vec{k}_3}a_{\vec{k}_4}a_{\vec{q}_1}a_{\vec{q}_2}a^\dagger_{\vec{q}_3}a^\dagger_{\vec{q}_4}a_{\vec{p}_1}a_{\vec{p}_2}a^\dagger_{\vec{p}_3}a^\dagger_{\vec{p}_4}a^\dagger_{\vec{k}_1}a^\dagger_{\vec{k}_2}|0\rangle+\cdots\Big) \\
=&\frac{(-i\lambda)^2}{2}\int d^dx_1d^dx_2\int\frac{d^Dp_3}{\rho^2(p_3)}\frac{d^Dp_4}{\rho^2(p_4)}\Big(\theta(t_2-t_1)e^{i(k_1+k_2-p_3-p_4)\cdot x_1}e^{i(p_3+p_4-k_3-k_4)\cdot x_2} \\
&+\theta(t_1-t_2)e^{i(k_1+k_2+p_3+p_4)\cdot x_1}e^{i(-p_3-p_4-k_3-k_4)\cdot x_2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{(-i\lambda)^2}{2}\int d^dx_1d^dx_2\,e^{i((k_1+k_2)\cdot x_1-(k_3+k_4)\cdot x_2)}\int\frac{d^Dp_3}{\rho^2(p_3)}\frac{d^Dp_4}{\rho^2(p_4)}\Big(\theta(t_2-t_1)e^{i(p_3+p_4)\cdot(x_2-x_1)}\\
&+\theta(t_1-t_2)e^{i(p_3+p_4)\cdot(x_1-x_2)}\Big),
\end{aligned}
\tag{4.3.5}
$$

同样, 与 (3.3.18) 有对应关系, (注意按时间排序 $\langle k_3k_4|T(\phi^4(x_1)\phi^4(x_2))|k_1k_2\rangle$).

calculation:

从 (3.3.17) 开始 (与 (1.2.1) 类似, $\vec{p},\vec{q}$ 的符号可以任意改变),

$$
\begin{aligned}
&\int d^dx_1d^dx_2\,e^{i(k_1+k_2+p-q)\cdot x_1}e^{i(k_3+k_4-p+q)\cdot x_2}\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\frac{i}{-q^2-m^2+i\epsilon}\\
=&\int d^dx_1d^dx_2\,e^{i((k_1+k_2)\cdot x_1+(k_3+k_4)\cdot x_2)}\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{ie^{ip\cdot(x_1-x_2)}}{-p^2-m^2+i\epsilon}\frac{ie^{ie^{iq\cdot(x_2-x_1)}}}{-q^2-m^2+i\epsilon}\\
=&\int d^dx_1d^dx_2\,e^{i((k_1+k_2)\cdot x_1+(k_3+k_4)\cdot x_2)}\int\frac{d^Dp}{(2\pi)^d}\frac{d^Dq}{(2\pi)^d}\Big(\theta(t_2-t_1)\frac{2\pi i^2e^{-ip\cdot(x_1-x_2)}}{-2\omega_p}\\
&\frac{-2\pi i^2e^{iq\cdot(x_2-x_1)}}{2\omega_q}+\theta(t_1-t_2)\frac{-2\pi i^2e^{ip\cdot(x_1-x_2)}}{2\omega_p}\frac{2\pi i^2e^{-iq\cdot(x_2-x_1)}}{-2\omega_q}\Big)\\
=&\int d^dx_1d^dx_2\,e^{i((k_1+k_2)\cdot x_1+(k_3+k_4)\cdot x_2)}\int\frac{d^Dp}{\rho^2(p)}\frac{d^Dq}{\rho^2(q)}\Big(\theta(t_2-t_1)e^{i(p+q)\cdot(x_2-x_1)}\\
&+\theta(t_1-t_2)e^{i(p+q)\cdot(x_1-x_2)}\Big),
\end{aligned}
\tag{4.3.6}
$$

结果与 (4.3.5) 对应.

4.4 complex scalar field

- complex scalar field 的 Lagrangian 为

$$
\mathcal{L}=-(\partial\psi^\dagger)(\partial\psi)-m^2\psi^\dagger\psi,
\tag{4.4.1}
$$

实际上, complex scalar field 可以视为 2 个 real scalar fields 的和

$$
\psi=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)\Longrightarrow\left|\frac{\partial\phi_1,\phi_2}{\partial\psi,\psi^\dagger}\right|=i,
\tag{4.4.2}
$$

因此, 也可以把 $\psi,\psi^\dagger$ 视为两个独立的场.

- 其 canonical momentum 为

$$
\pi(x)=\frac{\delta\mathcal{L}}{\delta\partial_0\psi}=\partial_0\psi^\dagger,\quad\pi^\dagger=\partial_0\psi,
\tag{4.4.3}
$$

其 Hamiltonian 为

$$
\mathcal{H}=\pi^\dagger\pi+(\vec{\nabla}\psi^\dagger)\cdot(\vec{\nabla}\psi)+m^2\psi^\dagger\psi,
\tag{4.4.4}
$$

$$
\Longrightarrow\begin{cases}\partial_0\pi=i[H,\pi]=\vec{\nabla}^2\psi^\dagger-m^2\psi^\dagger\\ \partial_0\psi=i[H,\psi]=\pi^\dagger\end{cases}\Longrightarrow(-\partial^2-m^2)\psi=0.
\tag{4.4.5}
$$

- 求解得到 (其中 $k^0=\omega_k$)

$$
\psi(x)=\int\frac{d^Dk}{\rho(k)}(a_{\vec{k}}e^{ik\cdot x}+b^\dagger_{\vec{k}}e^{-ik\cdot x}).
\tag{4.4.6}
$$

- 从 path integral 的角度,

$$
\begin{aligned}
Z(J,J^\dagger)=&\int D\psi D\psi^\dagger\,e^{i\int d^dx\,(\psi^\dagger(\partial^2-m^2)\psi+J^\dagger\psi+\psi^\dagger J)}
\end{aligned}
\tag{4.4.7}
$$

$$
=\mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\,2J^\dagger(x)D(x-y)J(y)}.
\tag{4.4.8}
$$

calculation:

转换为 ϕ_1, ϕ_2 后计算路径积分,

$$
\begin{aligned}
Z(J, J^\dagger) &= \mathcal{C} e^{-\frac{i}{2}\int d^d x d^d y\,(J_1(x)D(x-y)J_1(y)+J_2(x)D(x-y)J_2(y))} \\
&= \mathcal{C} e^{-\frac{i}{2}\int d^d x d^d y\, 2J^\dagger(x)D(x-y)J(y)}.
\end{aligned} \tag{4.4.9}
$$

4.4.1 charge

- 对场算符做如下变换,

$$
\psi(x,\lambda) = e^{i\lambda}\psi(x) \Longrightarrow D_\lambda \mathcal{L} = 0. \tag{4.4.10}
$$

- 因此, 得到 conserved current,

$$
J^\mu = \pi^\mu D_\lambda \psi + \pi^{\dagger\mu} D_\lambda \psi^\dagger = i(\psi\partial^\mu\psi^\dagger - \psi^\dagger\partial^\mu\psi), \tag{4.4.11}
$$

其 0 分量对空间积分就是 charge,

$$
\begin{aligned}
Q &= \int d^D x\, J^0 = \int d^D x\, i(\psi^\dagger\partial_0\psi - \psi\partial_0\psi^\dagger) \\
&= \int d^D k\,(a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}}).
\end{aligned} \tag{4.4.12}
$$

calculation:

$$
\begin{aligned}
Q = &\int d^D x \int \frac{d^D p}{\rho(p)}\frac{d^D q}{\rho(q)} i\Big((a^\dagger_{\vec{p}} e^{-ip\cdot x} + b_{\vec{p}} e^{ip\cdot x})(-i\omega_q)(a_{\vec{q}} e^{iq\cdot x} - b^\dagger_{\vec{q}} e^{-iq\cdot x}) \\
&- (a_{\vec{q}} e^{iq\cdot x} + b^\dagger_{\vec{q}} e^{-iq\cdot x})(i\omega_p)(a^\dagger_{\vec{p}} e^{-ip\cdot x} - b_{\vec{p}} e^{ip\cdot x})\Big) \\
= &\int d^D x \int \frac{d^D p}{\rho(p)}\frac{d^D q}{\rho(q)} \Big((\omega_p a_{\vec{q}} a^\dagger_{\vec{p}} + \omega_q a^\dagger_{\vec{p}} a_{\vec{q}}) e^{-i(p-q)\cdot x} - (\omega_p b^\dagger_{\vec{q}} b_{\vec{p}} + \omega_q b_{\vec{p}} b^\dagger_{\vec{q}}) e^{i(p-q)\cdot x} \\
&+ a^\dagger_{\vec{p}} b^\dagger_{\vec{q}}(\omega_p - \omega_q) e^{-i(p+q)\cdot x} - a_{\vec{q}} b_{\vec{p}}(\omega_p - \omega_q) e^{i(p+q)\cdot x}\Big) \\
= &\int \frac{d^D p}{\rho(p)}\frac{d^D q}{\rho(q)} \\
&\bigg(\Big((\omega_p a_{\vec{q}} a^\dagger_{\vec{p}} + \omega_q a^\dagger_{\vec{p}} a_{\vec{q}}) e^{i(\omega_p-\omega_q)\cdot t} - (\omega_p b^\dagger_{\vec{q}} b_{\vec{p}} + \omega_q b_{\vec{p}} b^\dagger_{\vec{q}}) e^{-i(\omega_p-\omega_q)\cdot t}\Big)(2\pi)^D \delta^{(D)}(\vec{p}-\vec{q}) \\
&+ \Big(a^\dagger_{\vec{p}} b^\dagger_{\vec{q}}(\omega_p - \omega_q) e^{i(\omega_p+\omega_q)\cdot x} - a_{\vec{q}} b_{\vec{p}}(\omega_p - \omega_q) e^{-i(\omega_p+\omega_q)\cdot x}\Big)(2\pi)^D \delta^{(D)}(\vec{p}+\vec{q})\bigg) \\
= &\int \frac{d^D k}{2}(a_{\vec{k}} a^\dagger_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} - b_{\vec{k}} b^\dagger_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}}) = \int d^D k\,(a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}}).
\end{aligned} \tag{4.4.13}
$$

- 代入 (D.3.2), 有 $i[Q,\psi] = -i\psi$, 所以

$$
e^{-i\lambda Q}\psi e^{i\lambda Q} = e^{i\lambda}\psi. \tag{4.4.14}
$$

Chapter 5

disturbing the vacuum: Casimir effect

- 考虑一个沿 x^1 方向满足 periodic b.c. 的空间, 在垂直于 x^1 方向有两个 plates, s.t. 在 plates 上 $\phi(x)=0$, 如下图:

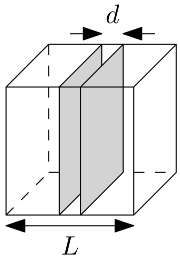


Figure 5.1: Casimir effect.

- 平板内外, 标量场的波矢的取值为

$$\begin{cases} (n\frac{\pi}{d}, k_2, k_3) & \text{平板内} \\ (n\frac{\pi}{L-d}, k_2, k_3) & \text{平板外} \end{cases}, \tag{5.0.1}$$

其中 $n \in \mathbb{Z}^+$.

- 因此, 代入真空能公式 (4.1.15), 平板内的能量为

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2}, \tag{5.0.2}$$

而总能量为 $E = E(d) + E(L-d)$.

- 为解决能量发散的问题, 引入 ultra-violet (UV) cut-off,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2}\, e^{-a\sqrt{(n\frac{\pi}{d})^2 + k_2^2 + k_3^2}}, \tag{5.0.3}$$

for some $a \ll d$.

- 为了简化问题, 考虑 $d = 1+1$ 的情况,

$$E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-\frac{a\pi}{d} n} = \frac{\pi}{2d} \frac{e^{\frac{a\pi}{d}}}{(e^{\frac{a\pi}{d}} - 1)^2} = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O(a^2), \tag{5.0.4}$$

因此

$$E_{1+1} = E_{1+1}(d) + E_{1+1}(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24}\left(\frac{1}{d} + \frac{1}{L-d}\right) + O(a^2), \tag{5.0.5}$$

得到 Casimir force,

$$F_{1+1} = -\frac{\partial E_{1+1}}{\partial d} = -\frac{\pi}{24}\left(\frac{1}{d^2} - \frac{1}{(L-d)^2}\right) + O(a^2) \overset{L\to\infty, a\to 0}{=} -\frac{\pi}{24d^2}. \tag{5.0.6}$$

- 问题中, a 引入了 UV cut-off, L 引入了 infrared cut-off.

Part II

Dirac and spinor

Chapter 6

the Dirac spinor

- 整个 Part II 中, 我们使用 $(+,-,-,-)$ 号差, 因为 $\mathrm{Cl}_{1,3}(\mathbb{R})\not\simeq\mathrm{Cl}_{3,1}(\mathbb{R})$.
- 本笔记中的算符的定义与 A. Zee 的定义不同, 存在如下对应关系:

A. Zee's def.	my def.
$\omega_{\mu\nu}$	$\omega_{\mu\nu}$
$-iJ^{\mu\nu}$	$J^{\mu\nu}$
$-i\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$

- $\Pi(\Lambda)$ 的写法可能不准确, (要考虑 universal cover, $\mathrm{Spin}(1,3)\simeq\mathrm{Spin}(3,1)$), 因为 Lorentz transform 对 spinor 的操作是"path dependent", 因此本 chapter 中的 Λ 都默认沿着以下的 path 做变换,

$$\Lambda(\lambda)=e^{\frac{\lambda}{2}\omega_{\mu\nu}J^{\mu\nu}},\lambda\in[0,1]. \tag{6.0.1}$$

6.1 gamma matrices

- Pauli 矩阵如下,

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{6.1.1}$$

- gamma 矩阵 (also called Dirac matrices) 如下 (其中 $i=1,2,3$),

$$\gamma^0=\begin{pmatrix}&I\\I&\end{pmatrix}=I\otimes\tau_1,\quad\gamma^i=\begin{pmatrix}&\sigma_i\\-\sigma_i&\end{pmatrix}=i\sigma_i\otimes\tau_2,\quad\gamma^5=i\Omega=\begin{pmatrix}-I&\\&I\end{pmatrix}=-I\otimes\tau_3. \tag{6.1.2}$$

其中 $\tau_{2,3}$ 也是 Pauli 矩阵, $\Omega=\gamma^0\gamma^1\gamma^2\gamma^3$, 有时候使用符号 $\sigma^\mu=(I,\vec{\sigma}),\bar{\sigma}^\mu=(I,-\vec{\sigma})$.

  - 另外

$$\begin{cases}\gamma^0\gamma^i=-\sigma_i\otimes\tau_3\\\gamma^i\gamma^j=-(\sigma_i\sigma_j)\otimes I=-i\epsilon_{ijk}\sigma_k\otimes I\end{cases},\quad\begin{cases}\Omega\gamma^0=-I\otimes\tau_2\\\Omega\gamma^i=-\sigma_i\otimes\tau_2\end{cases}, \tag{6.1.3}$$

  其中, 用到了 $\sigma_i\sigma_j=i\epsilon_{ijk}\sigma_k$.

- gamma 矩阵满足

$$\begin{cases}(\gamma^\mu)^2=\eta^{\mu\mu}\\\gamma^\mu\gamma^\nu=-\gamma^\nu\gamma^\mu\quad\mu\neq\nu\end{cases}\Longrightarrow\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}. \tag{6.1.4}$$

- 且存在如下关系,

$$\begin{aligned}&\Omega\gamma^0=-\gamma^1\gamma^2\gamma^3,\quad\Omega\gamma^1=-\gamma^0\gamma^2\gamma^3,\quad\Omega\gamma^2=\gamma^0\gamma^1\gamma^3,\quad\Omega\gamma^3=-\gamma^0\gamma^1\gamma^2,\\\Longleftrightarrow&-\epsilon^{\mu\nu\rho}{}_\sigma\Omega\gamma^\sigma=\gamma^\mu\gamma^\nu\gamma^\rho,\ \text{when }\mu\neq\nu\neq\rho.\end{aligned} \tag{6.1.5}$$

并且有 (注意到 $\Omega^2=-1$),

$$\{\Omega,\gamma^\mu\}=0,\quad\{\Omega,\Omega\gamma^\mu\}=0,\quad[\Omega,\gamma^\mu\gamma^\nu]=0. \tag{6.1.6}$$

- -

- 定义 $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ (注意, 我们的定义中没有虚数 i, 与 A. Zee 的定义不同), 因此

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} + \sigma^{\mu\nu} \Longrightarrow \begin{cases} \sigma^{0i} = \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix} = -\sigma_i \otimes \tau_3 \\ \sigma^{ij} = -i\epsilon^{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk}\sigma_k \otimes I \end{cases}, \tag{6.1.7}$$

与笔记 Lie Groups and Lie Algebras 中 $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ 表示对比, 可见 $\pi_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(J^{\mu\nu}) = \frac{1}{2}\sigma^{\mu\nu}$.

6.1.1 gamma matrices under Dirac basis

- 做如下相似变换 ($B = S^{-1}AS$),

$$S = \frac{\sqrt{2}}{2}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} \Longleftrightarrow S^{-1} = \frac{\sqrt{2}}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}, \tag{6.1.8}$$

得到

$$\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} = I \otimes \tau_3, \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = i\sigma_i \otimes \tau_2, \quad \gamma^5 = \begin{pmatrix} & I \\ I & \end{pmatrix} = I \otimes \tau_1. \tag{6.1.9}$$

- 另外,

$$\begin{cases} \gamma^0\gamma^i = \sigma_i \otimes \tau_1 \\ \gamma^i\gamma^j = -i\epsilon_{ijk}\sigma_k \otimes I \end{cases}, \quad \begin{cases} \Omega\gamma^0 = -I \otimes \tau_2 \\ \Omega\gamma^i = i\sigma_i \otimes \tau_3 \end{cases}, \tag{6.1.10}$$

以及

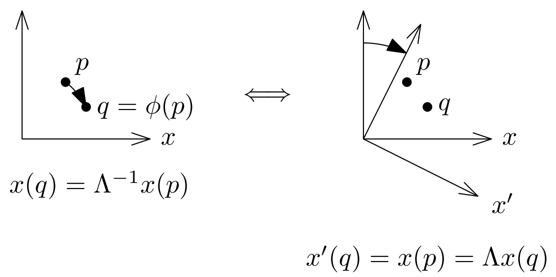
$$\sigma^{0i} = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix} = \sigma_i \otimes \tau_1, \quad \sigma^{ij} = -i\epsilon^{ijk}\begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk}\sigma_k \otimes I. \tag{6.1.11}$$

6.2 Lorentz transformation and the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation

- Lorentz 变换可以写成如下形式,

$$\Lambda = e^{\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}}, \tag{6.2.1}$$

其中 $\omega_{\mu\nu}$ 反对称, J^{0i} generate boosts and J^{ij} generate rotations, (详见笔记 Lie Groups and Lie Algebras).



$$x = \phi^* x' = \Lambda^{-1}x'$$

$$\phi_* \tfrac{\partial}{\partial x^\mu} = \Lambda_\mu{}^\nu \tfrac{\partial}{\partial x^\nu} = \tfrac{\partial}{\partial x'^\mu} \text{ or } \eta\Lambda\eta\tfrac{\partial}{\partial x} = \tfrac{\partial}{\partial x'}$$

$$\phi^* dx'^\mu = \Lambda_\nu{}^\mu dx'^\nu = dx^\mu \text{ or } \Lambda^{-1}dx' = dx$$

e.g. $A'^\mu(\Lambda x(p)) = \Lambda^\mu{}_\nu A^\nu(x(p))$

Figure 6.1: Lorentz transformation.

- Weyl spinor 是 $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ rep. 的 vector space 中的元素,

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \text{with} \quad \Psi_{\text{Dirac}} = S^{-1}\Psi = \frac{\sqrt{2}}{2}\begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix}. \tag{6.2.2}$$

在 Weyl basis 下很容易看出,

$$\Psi_L = \underbrace{\frac{1}{2}(1 - \gamma^5)}_{=P_L}\Psi, \quad \Psi_R = \underbrace{\frac{1}{2}(1 + \gamma^5)}_{=P_R}\Psi. \tag{6.2.3}$$

- 对于 gamma 矩阵, 有

$$\Pi(\Lambda)\gamma^\rho\Pi^{-1}(\Lambda) = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = (\Lambda^{-1})^\rho{}_\sigma\gamma^\sigma. \tag{6.2.4}$$

calculation:

利用 Campbell's identity,

$$e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = e^{\frac{1}{4}\omega_{\mu\nu}\mathrm{ad}_{\sigma^{\mu\nu}}}\gamma^\rho, \tag{6.2.5}$$

其中 (注意 $(J^{\mu\nu})^\rho{}_\sigma = 2\eta^{[\mu|\rho}\delta^{|\nu]}{}_\sigma$, 其中度规号差与笔记 Lie Groups and Lie Algebras 中的不同)

$$\begin{cases} \rho \neq \mu,\nu & \begin{aligned}[\sigma^{\mu\nu},\gamma^\rho] &= \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu) \\ &= -\frac{1}{2}\underbrace{(\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\nu\mu\rho\sigma} - \epsilon^{\rho\mu\nu\sigma} + \epsilon^{\rho\nu\mu\sigma})}_{=0}\Omega\gamma_\sigma = 0\end{aligned} \\ \rho = \mu \text{ or } \nu \text{ and } \mu \neq \nu & [\sigma^{\mu\nu},\gamma^\rho] = 2(\eta^{\mu\rho}\gamma^\mu - \eta^{\nu\rho}\gamma^\nu) \end{cases}$$

$$\Longrightarrow [\sigma^{\mu\nu},\gamma^\rho] = 2(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu) = -2(J^{\mu\nu})^\rho{}_\sigma\gamma^\sigma, \tag{6.2.6}$$

代入, 得到

$$e^{\frac{1}{4}\omega_{\mu\nu}\mathrm{ad}_{\sigma^{\mu\nu}}}\gamma^\rho = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\left(-\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}\right)^n\right)^\rho{}_\sigma\gamma^\sigma = (\Lambda^{-1})^\rho{}_\sigma\gamma^\sigma. \tag{6.2.7}$$

可以用" 无穷小" Lorentz 变换验证以上计算,

$$\begin{aligned}\Pi(1+\delta\omega^\mu{}_\nu)\gamma^\rho\Pi^{-1}(1+\delta\omega^\mu{}_\nu) &= \gamma^\rho + \frac{1}{4}\delta\omega_{\mu\nu}[\sigma^{\mu\nu},\gamma^\rho] \\ &= (1-\delta\omega^\rho{}_\sigma)\gamma^\sigma.\end{aligned} \tag{6.2.8}$$

6.2.1 Dirac spinor

- 对于 Dirac spinor,

$$\Pi(\Lambda)\Psi(x) = \Psi'(\Lambda x), \tag{6.2.9}$$

注意 $\partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu$, 所以

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \iff (i\gamma^\mu\partial'_\mu - m)\Psi'(\Lambda x) = 0. \tag{6.2.10}$$

  - 关键部分在于

$$\gamma^\mu\Psi'(\Lambda x) = \gamma^\mu\Pi(\Lambda)\Psi(x) = \Pi(\Lambda)\Lambda^\mu{}_\nu\gamma^\nu\Psi(x). \tag{6.2.11}$$

calculation:

首先

$$\Lambda^T\eta\Lambda = \eta \Longrightarrow (\Lambda^{-1})^\mu{}_\nu = (\eta\Lambda^T\eta)^\mu{}_\nu = \Lambda_\nu{}^\mu, \tag{6.2.12}$$

考虑

$$\Pi^{-1}(\Lambda)\gamma^\mu\Pi(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu \Longrightarrow \gamma^\mu\Pi(\Lambda) = \Lambda^\mu{}_\nu\Pi(\Lambda)\gamma^\nu, \tag{6.2.13}$$

代入,

$$\begin{aligned}(i\gamma^\mu\partial'_\mu - m)\Psi'(\Lambda x) &= (i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - m)\Pi(\Lambda)\Psi(x) \\ &= \Pi(\Lambda)(i\gamma^\rho\underbrace{\Lambda^\mu{}_\rho\Lambda_\mu{}^\nu}_{=\delta^\nu_\rho}\partial_\nu - m)\Psi(x) \\ &= \Pi(\Lambda)(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0.\end{aligned} \tag{6.2.14}$$

6.2.2 Dirac bilinears

- γ^0 是 Hermitian 矩阵, 而 γ^i 不是, 有

$$\gamma^{i\dagger} = -\gamma^i = \gamma^0\gamma^i\gamma^0, \tag{6.2.15}$$

可以统一写作 $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, 并且有

$$\sigma^{\mu\nu\dagger} = -\gamma^0\sigma^{\mu\nu}\gamma^0, \quad \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0. \tag{6.2.16}$$

calculation:

对于 $\sigma^{\mu\nu}$,

$$\sigma^{\mu\nu\dagger} = \frac{1}{2}(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger}) = \gamma^0\sigma^{\nu\mu}\gamma^0 = -\gamma^0\sigma^{\mu\nu}\gamma^0, \tag{6.2.17}$$

所以

$$((\omega_{\mu\nu}\sigma^{\mu\nu})^\dagger)^n = \gamma^0(-\omega_{\mu\nu}\sigma^{\mu\nu})^n\gamma^0 \Longrightarrow \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0. \tag{6.2.18}$$

- 所以

$$\begin{cases} \bar{\Psi}'(\Lambda x)\Psi'(\Lambda x) = \bar{\Psi}\Psi & \text{scalar field} \\ \bar{\Psi}'\gamma^\mu\Psi' = \Lambda^\mu{}_\nu\bar{\Psi}\gamma^\nu\Psi & \text{vector field} \end{cases}, \tag{6.2.19}$$

其中 $\bar{\Psi} = \Psi^\dagger\gamma^0$.

calculation:

$$\begin{cases} \Psi'^\dagger(\Lambda x)\gamma^0\Psi'(\Lambda x) = \Psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\Pi(\Lambda)\Psi(x) = \Psi^\dagger\gamma^0\Psi \\ \Psi'^\dagger\gamma^0\gamma^\mu\Psi' = \Psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\gamma^\mu\Pi(\Lambda)\Psi(x) = \Lambda^\mu{}_\nu\Psi^\dagger\gamma^0\gamma^\nu\Psi \end{cases}. \tag{6.2.20}$$

此外

$$\begin{cases} \bar{\Psi}'\sigma^{\mu\nu}\Psi' = \Psi^\dagger\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\sigma^{\mu\nu}\Pi(\Lambda)\Psi = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\Psi}\sigma^{\rho\sigma}\Psi & \text{order 2 tensor} \\ \bar{\Psi}'\Omega\gamma^\mu\Psi' = \bar{\Psi}\Pi(\Lambda^{-1})\Omega\gamma^\mu\Pi(\Lambda)\Psi = \det(\Lambda)\Lambda^\mu{}_\nu\bar{\Psi}\Omega\gamma^\nu\Psi & \text{pseudovector} \\ \bar{\Psi}'\Omega\Psi' = \bar{\Psi}\Pi(\Lambda^{-1})\Omega\Pi(\Lambda)\Psi = \det(\Lambda)\bar{\Psi}\Omega\Psi & \text{4-form (pseudoscalar)} \end{cases}, \tag{6.2.21}$$

其中 (注意到下面的计算中, 第二个等号后, 含 η 的项都等于零; 由此可以看出, 对 μ_i 求和的过程中, 任何两个 μ_i, μ_j 相等的项求和之后都等于零)

$$\begin{aligned} \Pi(\Lambda^{-1})\Omega\Pi(\Lambda) &= \prod_{i=0}^{3}\Lambda^i{}_{\mu_i}\gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \\ &= \prod_{i=0}^{3}\Lambda^i{}_{\mu_i}(\eta^{\mu_0\mu_1} + \sigma^{\mu_0\mu_1})(\eta^{\mu_2\mu_3} + \sigma^{\mu_2\mu_3}) \\ &= \prod_{i=0}^{3}\Lambda^i{}_{\mu_i}\gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \quad \text{with} \quad \mu_0 \neq \mu_1 \neq \mu_2 \neq \mu_3 \\ &= \det(\Lambda)\Omega. \end{aligned} \tag{6.2.22}$$

6.2.3 parity and time reversal

- 这里沿用笔记 Lie Groups and Lie Algebras 中的记号, 选择 $O(3,1)$ 而非 $O(1,3)$, 因为他们没有区别.
- $O(3,1)$ 有 4 个联通分支,

$$I \in SO_+(3,1), \quad PT \in SO_-(3,1), \quad P \in O'_+(3,1), \quad T \in O'_-(3,1), \tag{6.2.23}$$

  其中

$$P = \mathrm{diag}(+1,-1,-1,-1), \quad T = \mathrm{diag}(-1,+1,+1,+1), \tag{6.2.24}$$

  另外, $\eta P\eta = P, \eta T\eta = T$.

- 另外, Lorentz algebra 的 representation 不能自然的生成对 P, T 的表示, 因为本质上它只能生成 spin group 的表示, 是 $SO_+(3,1)$ 的 universal cover, 与 Lorentz group 的其它三个连通分支没有直接联系.
- 因此, 对 P, T 的表示要从物理的角度定义, (可能) 无法单纯靠数学的方法给出, 所以这部分放在下一章.

Chapter 7

the Dirac equation

7.1 Dirac equation

- A. Zee: our discussion provides a unified view of the equations of motion in relativistic physics: they just project out the unphysical components.
- the Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \iff (\gamma^\mu p_\mu - m)\tilde{\Psi} = 0 \Longrightarrow \begin{cases} i\sigma^\mu\partial_\mu\psi_R - m\psi_L = 0 \\ i\bar{\sigma}^\mu\partial_\mu\psi_L - m\psi_R = 0 \end{cases}. \tag{7.1.1}$$

首先可以看出 Ψ 满足 Klein-Gordan equation,

$$\begin{aligned}&(i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\Psi = \Big(-\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu - 2im\gamma^\mu\partial_\mu + m^2\Big)\Psi = 0 \\ \Longrightarrow &(-\partial^2 - m^2)\Psi = 0.\end{aligned} \tag{7.1.2}$$

  – 在粒子静止系下 $p_\mu = (m,0,0,0)$, Dirac 方程给出 (这里采用 Dirac basis)

$$(\gamma^0 - 1)\tilde{\Psi}_{\text{Dirac}} = 0 \Longrightarrow \begin{pmatrix} 0 & \\ & I \end{pmatrix}\tilde{\Psi}_{\text{Dirac}} = 0. \tag{7.1.3}$$

  因此, $\tilde{\Psi}$ 的后两个分量为零 $\Longrightarrow$ Ψ 只有两个自由度.

- Dirac 方程的 Lorentz covariance 见 (6.2.10).

7.2 Dirac Lagrangian

- 根据 (6.2.19) 以及之前标量场的计算经验, 可知

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = (-i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi})\Psi + \text{total diff.}, \tag{7.2.1}$$

其中, 与复标量场论中类似, 可以把 $\Psi, \Psi^\dagger$ 或 $\Psi, \bar{\Psi}$ 视为独立变量.

7.3 chirality or handedness

- parity transformation 会把 left spinor 变成 right spinor and vice versa,

$$\gamma^0\Psi_L = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix}, \quad \gamma^0\Psi_R = \begin{pmatrix} \psi_R \\ 0 \end{pmatrix}. \tag{7.3.1}$$

- 把 Lagrangian 中的 Ψ 拆开,

$$\begin{aligned}\mathcal{L} &= \bar{\Psi}_L(i\partial\!\!\!/)\Psi_L + \bar{\Psi}_R(i\partial\!\!\!/)\Psi_R - m(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L) \\ &= \psi_L^\dagger i\bar{\sigma}^\mu\partial_\mu\psi_L + \psi_R^\dagger i\sigma^\mu\partial_\mu\psi_R - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L),\end{aligned} \tag{7.3.2}$$

其中注意到了 $\gamma^0\gamma^\mu$ 的非对角分块为零.

7.3.1 internal vector symmetry

- 做变换 $\Psi \mapsto e^{i\theta}\Psi$, Lagrangian 保持不变, 利用 Noether's theorem 得到守恒流 (见 section D.2),

$$J_V^\mu = \bar{\Psi}\gamma^\mu\Psi, \tag{7.3.3}$$

其中, 按照惯例省略了虚数 i.

calculation:

计算广义动量,

$$\begin{cases} \pi_\Psi^\mu = \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\Psi} = \bar{\Psi}i\gamma^\mu \\ \pi_{\bar{\Psi}}^\mu = 0 \end{cases} \quad \text{or} \quad \begin{cases} \pi_\Psi^\mu = 0 \\ \pi_{\bar{\Psi}}^\mu = \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}} = -i\gamma^\mu\Psi \end{cases}. \tag{7.3.4}$$

这里看起来有点奇怪 (canonical transformation), 需要再说明一下. 对于 (7.2.1) 第一个等号后边,

$$\begin{cases} \pi_\Psi^\mu = \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\Psi} = \bar{\Psi}i\gamma^\mu & \dfrac{\delta\mathcal{L}}{\delta\Psi} = -m\bar{\Psi} \\ \pi_{\bar{\Psi}}^\mu = 0 & \dfrac{\delta\mathcal{L}}{\delta\bar{\Psi}} = (i\gamma^\mu\partial_\mu - m)\Psi \end{cases} \Longrightarrow \begin{cases} -(\partial_\mu\bar{\Psi})i\gamma^\mu - m\bar{\Psi} = 0 \\ (i\gamma^\mu\partial_\mu - m)\Psi = 0 \end{cases}, \tag{7.3.5}$$

对于 (7.2.1) 第二个等号后边, 忽略掉全微分项,

$$\begin{cases} \pi_\Psi^\mu = 0 & \dfrac{\delta\mathcal{L}}{\delta\Psi} = -i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} \\ \pi_{\bar{\Psi}}^\mu = \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}} = -i\gamma^\mu\Psi & \dfrac{\delta\mathcal{L}}{\delta\bar{\Psi}} = -m\Psi \end{cases} \Longrightarrow \begin{cases} -i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \\ (i\gamma^\mu\partial_\mu - m)\Psi = 0 \end{cases}. \tag{7.3.6}$$

7.3.2 axial symmetry

- 做变换

$$\Psi \mapsto e^{i\theta\gamma^5}\Psi = \begin{pmatrix} e^{-i\theta}\Psi_L \\ e^{i\theta}\Psi_R \end{pmatrix}, \tag{7.3.7}$$

在 $m = 0$ 时 Lagrangian 保持不变, 对应的守恒流为

$$J_A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi, \tag{7.3.8}$$

根据 (6.2.21), 是一个 pseudovector.

7.4 energy-momentum tensor and angular momentum

- Dirac 场的 energy-momentum tensor 为

$$T_{\mu\nu} = i\bar{\Psi}\gamma_\mu\partial_\nu\Psi - \eta_{\mu\nu}\mathcal{L}, \tag{7.4.1}$$

其中, 对于满足运动方程的 Dirac 场, $\mathcal{L} = 0$.

- Dirac 场的 angular momentum 为

$$M^{\mu\nu\rho} = \frac{i}{2}\bar{\Psi}\gamma^\mu\sigma^{\nu\rho}\Psi(x) + (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}). \tag{7.4.2}$$

calculation:

做变换 $x \mapsto e^{\frac{1}{2}\lambda\omega_{\mu\nu}J^{\mu\nu}}x$, 那么

$$\begin{aligned} &\Psi(x) \mapsto \Psi'(x') = e^{\frac{1}{4}\lambda\omega_{\mu\nu}\sigma^{\mu\nu}}\Psi(x) \\ \Longrightarrow &D_\lambda\Psi'(x) = \frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\Psi(x) - \frac{1}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma\partial_\rho\Psi(x), \end{aligned} \tag{7.4.3}$$

所以

$$J^\mu = \frac{i}{4}\omega_{\nu\rho}\bar{\Psi}\gamma^\mu\sigma^{\nu\rho}\Psi(x) + \cdots \Longrightarrow M^{\mu\nu\rho} = \frac{i}{2}\bar{\Psi}\gamma^\mu\sigma^{\nu\rho}\Psi(x) + (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}). \tag{7.4.4}$$

7.5 charge conjugation, parity and time reversal

- 沿用 A. Zee 的 notation, 变换映射分别用 $\mathcal{C}, \mathcal{P}, \mathcal{T}$ 表示, 相应的矩阵用 C, P, T 表示.

7.5.1 charge conjugation and antimatter

- 定义矩阵 C,

$$C = -\gamma^0\gamma^2 \Longrightarrow C\gamma^0 = -i\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} = \gamma^2 \Longrightarrow \begin{cases} (\gamma^2)^{-1}\gamma^\mu\gamma^2 = -\gamma^{\mu *} \\ C^{-1}\gamma^\mu C = -(\gamma^\mu)^T \end{cases}, \tag{7.5.1}$$

因此 $-\gamma^{\mu *}$ 同样满足 Clifford algebra.

  – 另外, 有 $(\gamma^2)^{-1} = \gamma^{2*} = -\gamma^2$ 和 $C^{-1} = C$.

calculation:

$$\gamma^0 C^{-1}\gamma^0 C\gamma^0 = -\gamma^{\mu *} \Longrightarrow C^{-1}\gamma^0 C = -\gamma^0\gamma^{\mu *}\gamma^0 = -(\underbrace{\gamma^0\gamma^\mu\gamma^0}_{=\gamma^{\mu\dagger}})^*, \tag{7.5.2}$$

其中用到了 $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$, 见 (6.2.15).

- $\Psi_c = \gamma^2\Psi^*$ 满足如下方程 (对比 (7.6.1)),

$$(-i\gamma^{\mu *}(\partial_\mu - ieA_\mu) - m)\Psi^* = 0 \Longrightarrow (\gamma^2)^{-1}(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\Psi_c = 0, \tag{7.5.3}$$

可见 Ψ_c 满足变换 $-e \mapsto +e$ 后的 Dirac 方程, Ψ_c is the field of positron.

- 对于 Lorentz 变换, $e^{\frac{1}{2}\lambda\omega_{\mu\nu}J^{\mu\nu}}, \lambda \in [0,1]$, 有

$$\begin{cases} \Psi \mapsto \Psi'(x') = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\Psi \\ \Psi_c \mapsto \gamma^2\underbrace{(\gamma^2)^{-1}e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^2\Psi^*}_{=(\Psi'(x'))^*} = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\Psi_c \end{cases}, \tag{7.5.4}$$

可见 Ψ_c 与 Ψ 的变换形式相同.

7.5.2 parity

- 对于 parity, 有 $x \to x' = (x^0, -\vec{x})$, 在 Dirac eq. 中

$$\gamma^0\gamma^\mu = P^\mu{}_\nu\gamma^\nu\gamma^0 \Longrightarrow (i\gamma^\mu\partial'_\mu - m)\gamma^0\Psi(x) = 0, \tag{7.5.5}$$

因此

$$\mathcal{P}: \Psi(x) \mapsto \Psi'(x') = \gamma^0\Psi(x). \tag{7.5.6}$$

7.5.3 time reversal

- 时间反演算符为

$$T = (i\sigma_2 \otimes I)K = \gamma^1\gamma^3 K, \tag{7.5.7}$$

其中 K 是 complex conjugation operator (见 appendix E). 另外, 有 $T^2 = -1$, 符合预期.

proof:

时间反演之后, $\Psi'(t') = T\Psi(t)$ 满足如下方程,

$$i\frac{\partial}{\partial t'}\Psi'(t') = H\Psi'(x'), \tag{7.5.8}$$

其中

$$H = -i\gamma^0\gamma^i\frac{\partial}{\partial x^i} + \gamma^0 m, \tag{7.5.9}$$

且 Hamiltonian 满足时间反演不变, $H'(t') \equiv TH(t)T^\dagger = H(t)$, 即 (其中 $T = UK$)

$$\begin{cases} T(i\gamma^0\gamma^i)T^\dagger = i\gamma^0\gamma^i \\ T\gamma^0 T^\dagger = \gamma^0 \end{cases} \Longrightarrow \begin{cases} U(-i\gamma^0\gamma^{i*})U^\dagger = U(-i\gamma^0\gamma^2\gamma^i\gamma^2)U^\dagger = i\gamma^0\gamma^i \\ [U,\gamma^0] = 0 \end{cases}, \tag{7.5.10}$$

满足以上要求的 U 具有以下形式,

$$U = \begin{pmatrix} a\sigma_2 & b\sigma_2 \\ b\sigma_2 & a\sigma_2 \end{pmatrix}, \quad \text{with} \quad \begin{cases} |a|^2 + |b|^2 = 1 \\ a^*b + b^*a = 0 \end{cases}, \tag{7.5.11}$$

不妨令 $a = i, b = 0$.

7.5.4 CPT theorem

- 在 CPT 变换下

$$\mathcal{CPT} : \Psi(x) \mapsto \gamma^1\gamma^3 K(\gamma^0\gamma^2\Psi^*) = \Omega\Psi = -i\gamma^5\Psi. \tag{7.5.12}$$

- 任何 Lorentz covariant theory 都满足 CPT 不变性.

7.6 interaction in QED

- 注意, 我们采用通常 (比如 Peskin) 的符号 $e = -|e|$, 与 A. Zee 的符号 $e > 0$ 不同.
- QED 的 Lagrangian 为

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu, \tag{7.6.1}$$

其中

$$D_\mu = \partial_\mu + ieA_\mu, \tag{7.6.2}$$

可见电子和电磁场耦合项为 $-eA_\mu J_V^\mu$, 其中 J_V^μ 是 internal vector symmetry 的守恒流, 见 (7.3.3).

- QED 里的 Dirac 方程为

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\Psi = 0 \quad \text{and} \quad -i(\partial_\mu - ieA_\mu)\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0. \tag{7.6.3}$$

7.7 Majorana neutrino

- 因为在 Lorentz 变换下, Ψ, Ψ_c 行为相同, 因此 Majorana 方程同样满足 Lorentz covariance,

$$i\partial\!\!\!/\Psi - m\Psi_c = 0 \quad \text{and} \quad i\partial\!\!\!/\Psi_c - m\Psi = 0, \tag{7.7.1}$$

因此

$$(-\partial^2 - m^2)\Psi = 0, \tag{7.7.2}$$

满足 Klein-Gordon 方程.

calculation:

$$-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\Psi = m(i\partial\!\!\!/)\Psi_c = m^2\Psi. \tag{7.7.3}$$

- Majorana 方程对应的 Lagrangian 为

$$\mathcal{L} = \bar{\Psi} i\partial\!\!\!/ \Psi - \frac{1}{2} m(\Psi^T C \Psi + \bar{\Psi} C \bar{\Psi}^T), \tag{7.7.4}$$

相应的广义动量为

$$\begin{cases} \pi^\mu_\Psi = \bar{\Psi} i\gamma^\mu & \dfrac{\delta \mathcal{L}}{\delta \Psi} = -m\Psi^T C \\ \pi^\mu_{\bar{\Psi}} = 0 & \dfrac{\delta \mathcal{L}}{\delta \bar{\Psi}} = i\partial\!\!\!/ \Psi - mC\bar{\Psi}^T = i\partial\!\!\!/ \Psi - m\Psi_c \end{cases}. \tag{7.7.5}$$

  - 注意, Ψ 应该被当作 Grassmann numbers, 因此, 对于反对称矩阵 C, 有 $\Psi^T C\Psi, \bar{\Psi} C \bar{\Psi}^T \neq 0$.

calculation:

对 Ψ 变分得到

$$\begin{aligned} 0 &= \frac{\delta \mathcal{L}}{\delta \Psi} - \partial_\mu \pi^\mu_\Psi \\ &= -m\Psi^T C - i\partial_\mu \bar{\Psi}\gamma^\mu \\ &= (-m\Psi^T - i\partial_\mu \bar{\Psi}\gamma^\mu C)C \\ &= (-m\Psi + iC(\gamma^\mu)^T \gamma^0 \partial_\mu \Psi^*)^T C, \end{aligned} \tag{7.7.6}$$

其中

$$C(\gamma^\mu)^T\gamma^0 = C(-C^{-1}\gamma^\mu C)\gamma^0 = -\gamma^\mu C\gamma^0 = -\gamma^\mu\gamma^2, \tag{7.7.7}$$

代入, 得到 (?),

$$-i\partial\!\!\!/ \Psi_c - m\Psi = 0. \tag{7.7.8}$$

- Majorana eq. v.s. Dirac eq.:
  - Majorana eq. 只适用于 electrically neutral fields (?).
  - Majorana eq. preserves handedness (?).

Chapter 8

quantizing the Dirac field

8.1 anticommutation

- 用 α, β 表示电子的量子态 (包括动量和自旋), 那么

$$\{b_\alpha, b_\beta\} = 0 \quad \{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}. \tag{8.1.1}$$

comment:

反对称关系 $\{b_\alpha, b_\beta\} = 0$ 由实验发现, 我们希望电子有 number operator,

$$N = \sum_\alpha b_\alpha^\dagger b_\alpha, \quad \text{with} \quad \begin{cases} [N, b_\alpha] = -b_\alpha \\ [N, b_\alpha^\dagger] = b_\alpha^\dagger \end{cases}, \tag{8.1.2}$$

考虑到 $[AB, C] = ABC - CAB = A\{B, C\} - \{A, C\}B$, 所以

$$\begin{cases} [N, b_\alpha] = \sum_\beta (b_\beta^\dagger \{b_\beta, b_\alpha\} - \{b_\beta^\dagger, b_\alpha\} b_\beta) = -\sum_\beta \{b_\beta^\dagger, b_\alpha\} b_\beta \\ [N, b_\alpha^\dagger] = \sum_\beta (b_\beta^\dagger \{b_\beta, b_\alpha^\dagger\} - \{b_\beta^\dagger, b_\alpha^\dagger\} b_\beta) = \sum_\beta b_\beta^\dagger \{b_\beta, b_\alpha^\dagger\} \end{cases}, \tag{8.1.3}$$

可见 $\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}$.

8.2 plane wave solutions

- Dirac 方程的平面波解具有如下形式 (其中 $p^0 = \omega_p$),

$$\Psi = u(\vec{p}) e^{-ip\cdot x} \quad \text{and} \quad \Psi = v(\vec{p}) e^{ip\cdot x}, \tag{8.2.1}$$

代入 Dirac 方程, 得到

$$(\not{p} - m) u(\vec{p}) = 0 \quad \text{and} \quad (-\not{p} - m) v(\vec{p}) = 0, \tag{8.2.2}$$

解为

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix}, \quad v = \begin{pmatrix} -\sqrt{p\cdot\sigma}\chi \\ \sqrt{p\cdot\bar{\sigma}}\chi \end{pmatrix}, \tag{8.2.3}$$

其中 ξ, χ 为任意 $2-\dim$ 列向量, 因此 $u(\vec{p}), v(\vec{p})$ 各有两个独立解, 分别用 $u(\vec{p}, s), v(\vec{p}, s), s = \pm 1$ 表示.

proof:

令 $u^T = (u_1, u_2)$ 代入,

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \Longrightarrow \begin{cases} p\cdot\sigma u_2 = m u_1 \\ p\cdot\bar{\sigma} u_1 = m u_2 \end{cases}, \tag{8.2.4}$$

注意到

$$(p\cdot\sigma)(p\cdot\bar\sigma)=\omega_p^2-p^ip^j\sigma_i\sigma_j=\omega_p^2-|\vec p|^2=m^2, \tag{8.2.5}$$

所以, 令 $u_2=m\xi'$, 那么

$$u=\begin{pmatrix}p\cdot\sigma\xi'\\ m\xi'\end{pmatrix}\Longrightarrow\xi=\sqrt{p\cdot\sigma}\xi'\Longrightarrow\cdots, \tag{8.2.6}$$

其中, ξ 可以任意选取, 并且注意到了 $[(p\cdot\sigma),(p\cdot\bar\sigma)]=0$, 因此

$$\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}=\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}=m. \tag{8.2.7}$$

类似地, 对于 $v^T=(v_1,v_2)$, 代入,

$$\begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar\sigma & m\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=0\Longrightarrow\begin{cases}p\cdot\sigma v_2=-mv_1\\ p\cdot\bar\sigma v_1=-mv_2\end{cases}, \tag{8.2.8}$$

令 $v_1=-\sqrt{p\cdot\sigma}\chi$, 那么...

最后,

$$\begin{cases}\sqrt{p\cdot\sigma}=\sqrt{\dfrac{m+\omega_p}{2}}I+\dfrac{1}{\sqrt{2(m+\omega_p)}}\vec p\cdot\vec\sigma\\ \sqrt{p\cdot\bar\sigma}=\sqrt{\dfrac{m+\omega_p}{2}}I-\dfrac{1}{\sqrt{2(m+\omega_p)}}\vec p\cdot\vec\sigma\end{cases}, \tag{8.2.9}$$

以及一些有用的公式,

$$\begin{cases}\sqrt{p\cdot\sigma}\sigma^\mu\sqrt{p\cdot\sigma}=\begin{cases}\omega_p+\vec p\cdot\vec\sigma & \mu=0\\ \omega_p\sigma^i+p^i+\frac{\vec p\cdot\vec\sigma}{2(m+\omega_p)}2i\epsilon_{ijk}p^j\sigma^k & \mu=i\end{cases}\\ \sqrt{p\cdot\bar\sigma}\sigma^\mu\sqrt{p\cdot\bar\sigma}=\begin{cases}\omega_p-\vec p\cdot\vec\sigma & \mu=0\\ \omega_p\sigma^i-p^i+\frac{\vec p\cdot\vec\sigma}{2(m+\omega_p)}2i\epsilon_{ijk}p^j\sigma^k & \mu=i\end{cases}\\ \sqrt{p\cdot\sigma}\sigma^\mu\sqrt{p\cdot\bar\sigma}=\begin{cases}m & \mu=0\\ m\sigma^i-\frac{\sqrt{p\cdot\sigma}}{\sqrt{2(m+\omega_p)}}2i\epsilon_{ijk}p^j\sigma_k & \mu=i\end{cases}\\ \sqrt{p\cdot\bar\sigma}\sigma^\mu\sqrt{p\cdot\sigma}=\begin{cases}m & \mu=0\\ m\sigma^i+\frac{\sqrt{p\cdot\bar\sigma}}{\sqrt{2(m+\omega_p)}}2i\epsilon_{ijk}p^j\sigma_k & \mu=i\end{cases}\end{cases}, \tag{8.2.10}$$

另外 $(-p)\cdot\sigma=p\cdot\bar\sigma,(-p)\cdot\bar\sigma=p\cdot\sigma$, 其中 $(-p)=(\omega_p,-\vec p)$.

- 选择归一化条件

$$\begin{cases}\bar u(\vec p,s)u(\vec p,s')=2m\delta_{ss'}\\ \bar v(\vec p,s)v(\vec p,s')=-2m\delta_{ss'}\end{cases}\quad\text{and}\quad\bar u(\vec p,s)v(\vec p,s')=0, \tag{8.2.11}$$

其中 $\bar u=u^\dagger\gamma^0,\bar v=v^\dagger\gamma^0$, 那么

$$\begin{cases}\xi^{s\dagger}\xi^{s'}=\delta_{ss'}\\ \chi^{s\dagger}\chi^{s'}=\delta_{ss'}\end{cases}\quad\text{and}\quad\xi^{s\dagger}\chi^{s'}-\chi^{s\dagger}\xi^{s'}=0, \tag{8.2.12}$$

可以选取

$$\xi^{+1}=\chi^{+1}=(1,0)^T,\quad\xi^{-1}=\chi^{-1}=(0,1)^T. \tag{8.2.13}$$

  - 在粒子静止系下, $p_r=(m,0,0,0)$,

$$\frac{u(\vec p_r,+1)}{\sqrt m}=\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\quad\frac{u(\vec p_r,-1)}{\sqrt m}=\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\quad\frac{v(\vec p_r,+1)}{\sqrt m}=\begin{pmatrix}-1\\0\\1\\0\end{pmatrix},\quad\frac{v(\vec p_r,-1)}{\sqrt m}=\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}, \tag{8.2.14}$$

  可见 $s=\pm1$ 分别代表 spin-up 和 spin-down.

– 另外, 我们注意到 (对 v 同样适用)

$$\begin{pmatrix}\omega_p\\ \vec{p}\end{pmatrix} = e^{\lambda J^{01}}\begin{pmatrix}m\\ 0\end{pmatrix} \iff u(\vec{p},s) = e^{\frac{1}{2}\lambda\sigma^{01}}u(\vec{p}_r,s), \quad \text{with} \quad \frac{p_1}{m} = \sinh\lambda, p_2 = p_3 = 0. \tag{8.2.15}$$

- 最后,

$$\begin{cases}\sum_{s=\pm1} u(\vec{p},s)\bar{u}(\vec{p},s) = \not{p} + m\\ \sum_{s=\pm1} v(\vec{p},s)\bar{v}(\vec{p},s) = \not{p} - m\end{cases}. \tag{8.2.16}$$

calculation:

首先,

$$u(\vec{p},s)u^\dagger(\vec{p},s) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}\begin{pmatrix}\xi^{s\dagger}\sqrt{p\cdot\sigma} & \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}, \tag{8.2.17}$$

注意到

$$\sum_{s=\pm1}\xi^s\xi^{s\dagger} = I_{2\times2}, \tag{8.2.18}$$

代入,

$$\sum_{s=\pm1} u(\vec{p},s)u^\dagger(\vec{p},s) = \begin{pmatrix}p\cdot\sigma & m\\ m & p\cdot\bar{\sigma}\end{pmatrix} = (\not{p}+m)\gamma^0. \tag{8.2.19}$$

类似地,

$$\begin{aligned}\sum_{s=\pm1} v(\vec{p},s)v^\dagger(\vec{p},s) &= \sum_{s=\pm1}\begin{pmatrix}\sqrt{p\cdot\sigma}\chi^s\\ -\sqrt{p\cdot\bar{\sigma}}\chi^s\end{pmatrix}\begin{pmatrix}\chi^{s\dagger}\sqrt{p\cdot\sigma} & -\chi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\\ &= \begin{pmatrix}p\cdot\sigma & -m\\ -m & p\cdot\bar{\sigma}\end{pmatrix} = (\not{p}-m)\gamma^0.\end{aligned} \tag{8.2.20}$$

8.3 the Dirac field

- $\Psi(x), \bar{\Psi}$ 有如下形式,

$$\begin{cases}\Psi(x) = \sum_{s=\pm1}\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(b^s_{\vec{p}}u(\vec{p},s)e^{-ip\cdot x} + c^{s\dagger}_{\vec{p}}v(\vec{p},s)e^{ip\cdot x})\\ \bar{\Psi}(x) = \sum_{s=\pm1}\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(b^{s\dagger}_{\vec{p}}\bar{u}(\vec{p},s)e^{ip\cdot x} + c^{s}_{\vec{p}}\bar{v}(\vec{p},s)e^{-ip\cdot x})\end{cases}. \tag{8.3.1}$$

- 回顾 section 4.4 关于 complex scalar field 的内容, 可知 $b^\dagger$ 和 $c^\dagger$ 产生的粒子具有相反的电荷, 不妨令 $b^\dagger$ 产生 electron (带电荷 $-e$), $c^\dagger$ 产生 position (带电荷 e).

- section 8.1 中的讨论说明

$$\begin{cases}\{b^s_{\vec{p}}, b^{s'}_{\vec{p}'}\} = 0\\ \{b^s_{\vec{p}}, b^{s'\dagger}_{\vec{p}'}\} = \delta^{(3)}(\vec{p}-\vec{p}')\delta_{ss'}\end{cases}. \tag{8.3.2}$$

- Ψ 的 momentum conjecture 为 (π^μ_Ψ 见 (7.3.4))

$$\pi_\Psi = \frac{\delta\mathcal{L}}{\delta\partial_0\Psi} = \pi^0_\Psi = \bar{\Psi}i\gamma^0 = i\Psi^\dagger, \tag{8.3.3}$$

存在如下 anticommutation relation,

$$\{\Psi_\alpha(t,\vec{x}), i\Psi^\dagger_\beta(t,\vec{y})\} = i\delta^{(3)}(\vec{x}-\vec{y})\delta_{\alpha\beta}. \tag{8.3.4}$$

calculation:

代入 (8.3.2), (下式中 $x=(t,\vec{x}), y=(t,\vec{y})$, 另外注意到 $u\bar{u}=uu^\dagger\gamma^0$),

$$\begin{aligned}
\{\Psi_\alpha(t,\vec{x}),\Psi^\dagger_\beta(t,\vec{y})\} &= \sum_{s=\pm}\int\frac{d^3p_1d^3p_2}{(2\pi)^3\sqrt{4\omega_{p_1}\omega_{p_2}}}\Big(\{b^s_{\vec{p}_1},b^{s\dagger}_{\vec{p}_2}\}u(\vec{p}_1,s)u^\dagger(\vec{p}_2,s)e^{i(-p_1\cdot x+p_2\cdot y)}\\
&\quad+\{c^{s\dagger}_{\vec{p}_1},c^s_{\vec{p}_2}\}v(\vec{p}_1,s)v^\dagger(\vec{p}_2,s)e^{i(p_1\cdot x-p_2\cdot y)}\Big)\\
&=\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2\omega_p}\Big(u(\vec{p},s)u^\dagger(\vec{p},s)e^{ip\cdot(-x+y)}+v(\vec{p},s)v^\dagger(\vec{p},s)e^{ip\cdot(x-y)}\Big)\\
&=\int\frac{d^3p}{(2\pi)^3 2\omega_p}\Big((\not{p}+m)\gamma^0e^{i\vec{p}\cdot(\vec{x}-\vec{y})}+(\not{p}-m)\gamma^0e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big)\\
&=\int\frac{d^3p}{(2\pi)^3 2\omega_p}\Big(2\omega_pI\cos(\vec{p}\cdot(\vec{x}-\vec{y}))-2p^i\gamma^i\gamma^0\cos(\vec{p}\cdot(\vec{x}-\vec{y}))\\
&\quad+2im\gamma^0\sin(\vec{p}\cdot(\vec{x}-\vec{y}))\Big),
\end{aligned}\tag{8.3.5}$$

注意, 只有第一项是偶函数, 积分后不为零,

$$\begin{aligned}
\{\Psi_\alpha(t,\vec{x}),\Psi^\dagger_\beta(t,\vec{y})\} &= \int\frac{d^3p}{(2\pi)^3}I\cos(\vec{p}\cdot(\vec{x}-\vec{y}))\\
&=\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}=\delta^{(3)}(\vec{x}-\vec{y})I.
\end{aligned}\tag{8.3.6}$$

- 另外, 显然有
$$\{\Psi(x),\Psi(y)\}=\{\Psi^\dagger(x),\Psi^\dagger(y)\}=0.\tag{8.3.7}$$

8.4 Hamiltonian, energy-momentum tensor and angular momentum

8.4.1 Hamiltonian

- 计算 Hamiltonian,
$$H=\sum_{s=\pm1}\int d^3p\,\omega_p(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}-c^s_{\vec{p}}c^{s\dagger}_{\vec{p}})=\sum_{s=\pm1}\int d^3p\,\omega_p(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}+c^{s\dagger}_{\vec{p}}c^s_{\vec{p}})+E_0,\tag{8.4.1}$$
其中 vacuum energy,
$$E_0=-2\delta^{(3)}(0)\int d^3p\,\omega_p=-2V\int\frac{d^3p}{(2\pi)^3}\omega_p,\tag{8.4.2}$$
的符号与标量场的正好相反.

calculation:

the Hamiltonian density is

$$\begin{aligned}
\mathcal{H}&=i\Psi^\dagger\partial_0\Psi-\mathcal{L}=-\bar{\Psi}(i\gamma^i\partial_i-m)\Psi\\
&=\sum_{s_1,s_2=\pm1}\int\frac{d^3p_1d^3p_2}{(2\pi)^3\sqrt{4\omega_{p_1}\omega_{p_2}}}(b^{s_1\dagger}_{\vec{p}_1}\bar{u}(\vec{p}_1,s_1)e^{ip_1\cdot x}+c^{s_1}_{\vec{p}_1}\bar{v}(\vec{p}_1,s_1)e^{-ip_1\cdot x})\\
&\quad\big(\underbrace{(\gamma^ip^i_2+m)}_{\mapsto\omega_{p_2}\gamma^0}b^{s_2}_{\vec{p}_2}u(\vec{p}_2,s_2)e^{-ip_2\cdot x}+\underbrace{(-\gamma^ip^i_2+m)}_{\mapsto-\omega_{p_2}\gamma^0}c^{s_2\dagger}_{\vec{p}_2}v(\vec{p}_2,s_2)e^{ip_2\cdot x}\big),
\end{aligned}\tag{8.4.3}$$

代入,

$$\begin{aligned}
H=\int d^3x\,\mathcal{H}=\sum_{s_1,s_2=\pm1}\int\frac{d^3p}{2\omega_p}\Big(&b^{s_1\dagger}_{\vec{p}}\bar{u}(\vec{p},s_1)\omega_p\gamma^0b^{s_2}_{\vec{p}}u(\vec{p},s_2)\\
&-b^{s_1\dagger}_{\vec{p}}\bar{u}(\vec{p},s_1)\omega_p\gamma^0c^{s_2\dagger}_{-\vec{p}}v(-\vec{p},s_2)e^{2i\omega_pt}\\
&+c^{s_1}_{\vec{p}}\bar{v}(\vec{p},s_1)\omega_p\gamma^0b^{s_2}_{-\vec{p}}u(-\vec{p},s_2)e^{-2i\omega_pt}
\end{aligned}$$

$$-c^{s_1}_{\vec{p}}\bar{v}(\vec{p},s_1)\omega_p\gamma^0 c^{s_2\dagger}_{\vec{p}}v(\vec{p},s_2)\Big), \tag{8.4.4}$$

注意到

$$\begin{cases} u^\dagger(\vec{p},s_1)u(\vec{p},s_2)=2\omega_p\delta_{s_1s_2} \\ u^\dagger(\vec{p},s_1)v(-\vec{p},s_2)=0 \\ v^\dagger(\vec{p},s_1)u(-\vec{p},s_2)=0 \\ v^\dagger(\vec{p},s_1)v(\vec{p},s_2)=2\omega_p\delta_{s_1s_2} \end{cases}, \tag{8.4.5}$$

代入,

$$H=\sum_{s_1,s_2=\pm1}\int\frac{d^3p}{2\omega_p}\Big(b^{s_1\dagger}_{\vec{p}}b^{s_2}_{\vec{p}}(2\omega_p^2)\delta_{s_1s_2}+c^{s_1}_{\vec{p}}c^{s_2\dagger}_{\vec{p}}(-2\omega_p^2)\delta_{s_1s_2}\Big)=\cdots \tag{8.4.6}$$

8.4.2 energy-momentum tensor

- Dirac field 的动量算符为

$$P^\mu=\int d^3x\,T^{0\mu}=\int d^3p\,p^\mu(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}+c^{s\dagger}_{\vec{p}}c^s_{\vec{p}}), \tag{8.4.7}$$

另外 $P^0=H$ 还有一个 vacuum energy.

calculation:

energy-momentum tensor 的 $0,\mu$ 分量为 (见 (7.4.1))

$$\begin{aligned} T^{0\mu}=&i\bar{\Psi}\gamma^0\partial^\mu\Psi=i\Psi^\dagger\partial^\mu\Psi \\ =&\sum_{s_1,s_2=\pm1}\int\frac{d^3p_1d^3p_2}{(2\pi)^3\sqrt{4\omega_{p_1}\omega_{p_2}}}(b^{s_1\dagger}_{\vec{p}_1}u^\dagger(\vec{p}_1,s_1)e^{ip_1\cdot x}+c^{s_1}_{\vec{p}_1}v^\dagger(\vec{p}_1,s_1)e^{-ip_1\cdot x}) \\ &p_2^\mu(b^{s_2}_{\vec{p}_2}u(\vec{p}_2,s_2)e^{-ip_2\cdot x}-c^{s_2\dagger}_{\vec{p}_2}v(\vec{p}_2,s_2)e^{ip_2\cdot x}), \end{aligned} \tag{8.4.8}$$

代入,

$$\begin{aligned} P^\mu=&\sum_{s_1,s_2=\pm1}\int\frac{d^3p}{2\omega_p}\Big(p^\mu b^{s_1\dagger}_{\vec{p}}u^\dagger(\vec{p},s_1)b^{s_2}_{\vec{p}}u(\vec{p},s_2)-(-p^\mu)b^{s_1\dagger}_{\vec{p}}u^\dagger(\vec{p},s_1)c^{s_2\dagger}_{-\vec{p}}v(-\vec{p},s_2)e^{2i\omega_pt} \\ &+(-p^\mu)c^{s_1}_{\vec{p}}v^\dagger(\vec{p},s_1)b^{s_2}_{-\vec{p}}u(-\vec{p},s_2)-p^\mu c^{s_1}_{\vec{p}}v^\dagger(\vec{p},s_1)c^{s_2\dagger}_{\vec{p}}v(\vec{p},s_2)\Big) \\ =&\sum_{s_1,s_2=\pm1}\int\frac{d^3p}{2\omega_p}\Big(p^\mu b^{s_1\dagger}_{\vec{p}}b^{s_2}_{\vec{p}}(2\omega_p\delta_{s_1s_2})-p^\mu c^{s_1}_{\vec{p}}c^{s_2\dagger}_{\vec{p}}(2\omega_p\delta_{s_1s_2})\Big) \\ =&\int d^3p\,p^\mu(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}-c^s_{\vec{p}}c^{s\dagger}_{\vec{p}}). \end{aligned} \tag{8.4.9}$$

8.4.3 angular momentum

- Dirac field 的角动量算符为 (这部分在 Peskin 上有)

$$\begin{aligned} J^{ij}=&\int d^3\,M^{0ij} \\ =&\epsilon^{ijk}\sum_{s_1,s_2=\pm1}\int d^3p\,\frac{m}{2\omega_p}(b^{s_1\dagger}_{\vec{p}}b^{s_2}_{\vec{p}}+c^{s_1}_{\vec{p}}c^{s_2\dagger}_{\vec{p}})\xi^{s_1\dagger}\sigma_k\xi^{s_2}+\int d^3x\,(x^iT^{0j}-x^jT^{0i}), \end{aligned} \tag{8.4.10}$$

其中, $M^{\mu\nu\rho}$ 见 (7.4.2).

  - 把角动量算符中 spin 的部分表示为 S^{ij}, 那么

$$\begin{cases} S^{12}b^{s\dagger}_{\vec{p}}\left|0\right\rangle=s\dfrac{m}{2\omega_p}b^{s\dagger}_{\vec{p}}\left|0\right\rangle \\ S^{12}c^{s\dagger}_{\vec{p}}\left|0\right\rangle=-s\dfrac{m}{2\omega_p}c^{s\dagger}_{\vec{p}}\left|0\right\rangle \end{cases}. \tag{8.4.11}$$

calculation:

角动量张量为

$$
\begin{aligned}
M^{0\mu\nu} =& \frac{i}{2}\underbrace{\bar{\Psi}\gamma^0}_{\Psi^\dagger}\sigma^{\mu\nu}\Psi + (x^\mu T^{0\nu} - x^\nu T^{0\mu}) \\
=& \frac{i}{2}\sum_{s_1,s_2=\pm1}\int \frac{d^3p_1 d^3p_2}{(2\pi)^3\sqrt{4\omega_{p_1}\omega_{p_2}}}(b^{s_1\dagger}_{\vec{p}_1}u^\dagger(\vec{p}_1,s_1)e^{ip_1\cdot x} + c^{s_1}_{\vec{p}_1}v^\dagger(\vec{p}_1,s_1)e^{-ip_1\cdot x}) \\
& \sigma^{\mu\nu}(b^{s_2}_{\vec{p}_2}u(\vec{p}_2,s_2)e^{-ip_2\cdot x} + c^{s_2\dagger}_{\vec{p}_2}v(\vec{p}_2,s_2)e^{ip_2\cdot x}) + (x^\mu T^{0\nu} - x^\nu T^{0\mu}),
\end{aligned}
\tag{8.4.12}
$$

代入,

$$
\begin{aligned}
J^{\mu\nu} - \int d^3x\,(x^\mu T^{0\nu} - x^\nu T^{0\mu}) =& \frac{i}{2}\sum_{s_1,s_2=\pm1}\int\frac{d^3p}{2\omega_p}\Big(b^{s_1\dagger}_{\vec{p}}u^\dagger(\vec{p},s_1)\sigma^{\mu\nu}b^{s_2}_{\vec{p}}u(\vec{p},s_2) \\
& + b^{s_1\dagger}_{\vec{p}}u^\dagger(\vec{p},s_1)\sigma^{\mu\nu}c^{s_2\dagger}_{-\vec{p}}v(-\vec{p},s_2)e^{2i\omega_p t} \\
& + c^{s_1}_{\vec{p}}v^\dagger(\vec{p},s_1)\sigma^{\mu\nu}b^{s_2}_{-\vec{p}}u(-\vec{p},s_2)e^{-2i\omega_p t} \\
& + c^{s_1}_{\vec{p}}v^\dagger(\vec{p},s_1)\sigma^{\mu\nu}c^{s_2\dagger}_{\vec{p}}v(\vec{p},s_2)\Big),
\end{aligned}
\tag{8.4.13}
$$

其中

$$
\begin{cases}
u^\dagger(\vec{p},s_1)\sigma^{ij}u(\vec{p},s_2) = -2i\epsilon^{ijk}m\xi^{s_1\dagger}\sigma_k\xi^{s_2} \\
u^\dagger(\vec{p},s_1)\sigma^{ij}v(-\vec{p},s_2) = 0 \\
v^\dagger(\vec{p},s_1)\sigma^{ij}u(-\vec{p},s_2) = 0 \\
v^\dagger(\vec{p},s_1)\sigma^{ij}v(\vec{p},s_2) = -2i\epsilon^{ijk}m\chi^{s_1\dagger}\sigma_k\chi^{s_2}
\end{cases},
\tag{8.4.14}
$$

代入 (注意到 $\xi^s = \chi^s$),

$$
J^{ij} - \int d^3x\,(x^i T^{0j} - x^j T^{0i}) = \epsilon^{ijk}\sum_{s_1,s_2=\pm1}\int\frac{d^3p}{2\omega_p}m(b^{s_1\dagger}_{\vec{p}}b^{s_2}_{\vec{p}} + c^{s_1}_{\vec{p}}c^{s_2\dagger}_{\vec{p}})\xi^{s_1\dagger}\sigma_k\xi^{s_2}.
\tag{8.4.15}
$$

8.5 electric current

- internal vector symmetry 对应的守恒流就是电流, 见 subsection 7.3.1, 有

$$
Q = \int d^3x\,J^0_V = \sum_{s=\pm1}\int d^3p\,(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} - c^{s\dagger}_{\vec{p}}c^s_{\vec{p}}) - 2\delta^{(3)}(0)\int d^3p.
\tag{8.5.1}
$$

calculation:

首先,

$$
\begin{aligned}
\int d^3x\,J^\mu_V = \int d^3x\,\bar{\Psi}\gamma^\mu\Psi = \sum_{s_1,s_2=\pm1}\int\frac{d^3p}{2\omega_p}\Big(& b^{s_1\dagger}_{\vec{p}}b^{s_2}_{\vec{p}}\bar{u}(\vec{p},s_1)\gamma^\mu u(\vec{p},s_2) \\
& + b^{s_1\dagger}_{\vec{p}}c^{s_2\dagger}_{-\vec{p}}\bar{u}(\vec{p},s_1)\gamma^\mu v(-\vec{p},s_2)e^{2i\omega_p t} \\
& + c^{s_1}_{\vec{p}}b^{s_2}_{-\vec{p}}\bar{v}(\vec{p},s_1)\gamma^\mu u(-\vec{p},s_2)e^{-2i\omega_p t} \\
& + c^{s_1}_{\vec{p}}c^{s_2\dagger}_{\vec{p}}\bar{v}(\vec{p},s_1)\gamma^\mu v(\vec{p},s_2)\Big),
\end{aligned}
\tag{8.5.2}
$$

其中

$$
\begin{cases}
\bar{u}(\vec{p},s_1)\gamma^\mu u(\vec{p},s_2) = 2p_\mu\delta_{s_1s_2} \quad (?) \\
\bar{u}(\vec{p},s_1)\gamma^0 v(-\vec{p},s_2) = 0 \\
\bar{u}(\vec{p},s_1)\gamma^i v(-\vec{p},s_2) = \xi^{s_1\dagger}(2m\sigma^i)\xi^{s_2} \\
\bar{v}(\vec{p},s_1)\gamma^\mu u(-\vec{p},s_2) = \bar{u}(\vec{p},s_1)\gamma^\mu v(-\vec{p},s_2) \\
\bar{v}(\vec{p},s_1)\gamma^\mu v(\vec{p},s_2) = \bar{u}(\vec{p},s_1)\gamma^\mu u(\vec{p},s_2)
\end{cases},
\tag{8.5.3}
$$

代入,

$$Q = \sum_{s=\pm1} \int d^3p \, (b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} + c^s_{\vec{p}} c^{s\dagger}_{\vec{p}}), \tag{8.5.4}$$

$$J^i = \int d^3x \, J^i_V \overset{(?)}{=} \int d^3p \Big(\sum_{s=\pm1} -\frac{p^i}{\omega_p} (b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} + c^s_{\vec{p}} c^{s\dagger}_{\vec{p}}) \\ + \sum_{s_1,s_2=\pm1} (b^{s_1\dagger}_{\vec{p}} c^{s_2\dagger}_{-\vec{p}} e^{2i\omega_p t} + c^{s_1}_{\vec{p}} b^{s_2}_{-\vec{p}} e^{-2i\omega_p t}) \xi^{s_1\dagger} (2m\sigma^i) \xi^{s_2} \Big). \tag{8.5.5}$$

8.6 free propagator

- 参考 scalar field 中的 propagator (见 (4.1.17)), the propagator of the Dirac field is

$$\begin{aligned} iS(x-y) &= \langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \Big(\theta(x^0 - y^0)(\not{p} + m) e^{-ip\cdot(x-y)} - \theta(y^0 - x^0)(\not{p} - m) e^{-ip\cdot(y-x)} \Big) \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p} - m + i\epsilon} e^{-ip\cdot(x-y)}, \end{aligned} \tag{8.6.1}$$

其中

$$(T\Psi(x)\bar{\Psi}(y))_{\alpha\beta} = \theta(x^0 - y^0)\Psi_\alpha(x)\bar{\Psi}_\beta(y) - \theta(y^0 - x^0)\bar{\Psi}_\beta(y)\Psi_\alpha(x), \tag{8.6.2}$$

注意到这里交换 $\Psi, \bar{\Psi}$ 是产生湮灭算符层面上的, 不是 spinor 层面上的.

calculation:

分别计算 $\langle 0|\Psi_\alpha(x)\bar{\Psi}_\beta(y)|0\rangle$ 和 $\langle 0|\bar{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle$,

$$\begin{aligned} \langle 0|\Psi_\alpha(x)\bar{\Psi}_\beta(y)|0\rangle &= \sum_{s=\pm1} \int \frac{d^3p}{(2\pi)^3 2\omega_p} u_\alpha(\vec{p},s)\bar{u}_\beta(\vec{p},s) e^{-ip\cdot(x-y)} \\ &= \int \frac{d^3p}{(2\pi)^3 2\omega_p} (\not{p}+m)_{\alpha\beta} e^{-ip\cdot(x-y)}, \end{aligned} \tag{8.6.3}$$

$$\langle 0|\bar{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle = \sum_{s=\pm1} \int \frac{d^3p}{(2\pi)^3 2\omega_p} v_\alpha(\vec{p},s)\bar{v}_\beta(\vec{p},s) e^{-ip\cdot(y-x)}, \tag{8.6.4}$$

代入, 得到...

把 $iS(x)$ 的第二项中的 $\vec{p}$ 变成 $-\vec{p}$,

$$\begin{aligned} iS(x) &= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \Big(\theta(t)(\omega_p\gamma^0 - p^i\gamma^i + m) e^{-ip\cdot x} - \theta(-t)(\omega_p\gamma^0 + p^i\gamma^i - m) e^{i(\omega_p t + \vec{p}\cdot\vec{x})} \Big) \\ &= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \Big(\theta(t)(\omega_p\gamma^0 - p^i\gamma^i + m) e^{-ip\cdot x} + \theta(-t)(\omega_p \mapsto -\omega_p) \Big) \\ &= \int \frac{dp^0}{-2\pi i} \frac{1}{(p^0 - (\omega_p - i\epsilon))(p^0 + (\omega_p - i\epsilon))} \int \frac{d^3p}{(2\pi)^3} (\not{p}+m) e^{-ip\cdot x} = \cdots, \end{aligned} \tag{8.6.5}$$

最后,

$$\frac{\not{p}+m}{p^2 - m^2 + i\epsilon}(\not{p} - m + i\epsilon) = I. \tag{8.6.6}$$

Chapter 9

spin-statistics connection

- **spin-statistics theorem:** 在 3 维空间中, 具有整数自旋的粒子遵守 Bose-Einstein statistics, 具有半整数自旋的粒子遵守 Fermi-Dirac statistics.
- 本 chapter 不对此做出证明, 只是举例说明不能满足 spin-statistics theorem 会导致什么样的后果.

9.1 the price of perversity

9.1.1 scalar field

- 如果 scalar field 满足 anticommutation relation, 那么

$$\{\phi(\vec{x},t),\phi(\vec{y},t)\} = \int \frac{d^Dk}{(2\pi)^D\omega_k}\cos(\vec{k}\cdot(\vec{x}-\vec{y})) \neq 0, \tag{9.1.1}$$

违反狭义相对论.

calculation:

代入 (4.1.11),

$$\{\phi(\vec{x},t),\phi(\vec{y},t)\} = \int \frac{d^Dk}{(2\pi)^D 2\omega_k}(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}) = \cdots \tag{9.1.2}$$

9.1.2 Dirac field

- 如果 Dirac field 满足 commutation relation, 那么

$$[\Psi(\vec{x},t),\Psi^\dagger(\vec{y},t)] = \int \frac{d^3p}{(2\pi)^3\omega_p}(i\not{p}\gamma^0\sin(\vec{p}\cdot(\vec{x}-\vec{y})) + m\gamma^0\cos(\vec{p}\cdot(\vec{x}-\vec{y}))), \tag{9.1.3}$$

考虑可观测量 $J_V^0 = \Psi^\dagger\Psi$ (其中 $x=(\vec{x},t), y=(\vec{y},t)$),

$$[J_V^0(x),J_V^0(y)] = \Psi_\alpha^\dagger(x)[\Psi_\alpha(x),\Psi_\beta^\dagger(y)]\Psi_\beta(y) - \Psi_\beta^\dagger(y)[\Psi_\beta(y),\Psi_\alpha^\dagger(x)]\Psi_\alpha(x). \tag{9.1.4}$$

calculation:

代入 (8.3.1),

$$\begin{aligned}
[\Psi(\vec{x},t),\Psi^\dagger(\vec{y},t)] &= \sum_{s=\pm1}\int \frac{d^3p}{(2\pi)^3 2\omega_p}(u(\vec{p},s)u^\dagger(\vec{p},s)e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - v(\vec{p},s)v^\dagger(\vec{p},s)e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}) \\
&= \int \frac{d^3p}{(2\pi)^3 2\omega_p}((\not{p}+m)\gamma^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - (\not{p}-m)\gamma^0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}) \\
&= \int \frac{d^3p}{(2\pi)^3 2\omega_p}(2i\not{p}\gamma^0\sin(\vec{p}\cdot(\vec{x}-\vec{y})) + 2m\gamma^0\cos(\vec{p}\cdot(\vec{x}-\vec{y}))).
\end{aligned} \tag{9.1.5}$$

然后,

$$
\begin{aligned}
[J_V^0(x), J_V^0(y)] =& \Psi_\alpha^\dagger(x)[\Psi_\alpha(x), \Psi_\beta^\dagger(y)]\Psi_\beta(y) - \Psi_\beta^\dagger(y)[\Psi_\beta(y), \Psi_\alpha^\dagger(x)]\Psi_\alpha(x) \\
=& \sum_{s_1,s_2=\pm1}\int\frac{d^3p_1d^3p_2d^3q}{(2\pi)^6\sqrt{4\omega_{p_1}\omega_{p_2}}\omega_q}(b_{\vec{p}_1}^{s_1\dagger}u^\dagger(\vec{p}_1,s_1)e^{ip_1\cdot x} + c_{\vec{p}_1}^{s_1}v^\dagger(\vec{p}_1,s_1)e^{-ip_1\cdot x}) \\
& (i\not{q}\gamma^0\sin(\vec{q}\cdot(\vec{x}-\vec{y})) + m\gamma^0\cos(\vec{q}\cdot(\vec{x}-\vec{y}))) \\
& (b_{\vec{p}_2}^{s_2}u(\vec{p}_2,s_2)e^{-ip_2\cdot y} + c_{\vec{p}_2}^{s_2}v(\vec{p}_2,s_2)e^{ip_2\cdot y}) - (x\leftrightarrow y),
\end{aligned}
\tag{9.1.6}
$$

注意到 $p_1 \neq p_2$, 这种情况怎么算 (?).

Chapter 10

Grassmann path integrals and Feynman diagrams for Fermions

- Grassmann number 和 Gaussian-Berezin integrals 见 section B.2.

10.1 Grassmann path integral

- Dirac field 的 partition function 为

$$
\begin{aligned}
Z(\eta,\bar\eta) &= \int D\Psi D\bar\Psi\, e^{i\int d^4x\,(\bar\Psi(i\not\partial - m + i\epsilon)\Psi + \bar\eta\Psi + \bar\Psi\eta)} \\
&= e^{iE_0T} e^{-i\int \frac{d^4p}{(2\pi)^4}\tilde{\bar\eta}(-p)\frac{1}{\not p - m + i\epsilon}\tilde\eta(p)},
\end{aligned}
\tag{10.1.1}
$$

其中 vacuum energy 为

$$
E_0 = -4V\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\omega_p + \text{irrelevant terms.} \tag{10.1.2}
$$

calculation:

代入 (B.2.13),

$$
Z(\eta,\bar\eta) = \det(\underbrace{i(i\not\partial - m + i\epsilon)}_{=iA})e^{-i^2(-i)\bar\eta A^{-1}\eta}, \tag{10.1.3}
$$

其中

$$
\begin{aligned}
&\begin{cases}
\det(i(i\not\partial - m + i\epsilon)) = \det(i\,\underbrace{\gamma^5(i\not\partial - m + i\epsilon)\gamma^5}_{=(-i\not\partial - m + i\epsilon)}) \\
(i\not\partial - m + i\epsilon)(-i\not\partial - m + i\epsilon) = (\partial^2 + m^2 - i\epsilon)I_{4\times 4}
\end{cases} \\
\Longrightarrow &\det(i(i\not\partial - m + i\epsilon)) = \sqrt{\det((-\partial^2 - m^2 + i\epsilon)I_{4\times 4})} = e^{iE_0T},
\end{aligned}
\tag{10.1.4}
$$

注意到 $I_{4\times 4}$ 会带来一个 4 次方的系数.

对于指数项, 考虑

$$
(i\not\partial - m + i\epsilon)\Psi(x) = \int d^4y\, A(x-y)\Psi(y), \tag{10.1.5}
$$

其中

$$
\begin{aligned}
&A(x-y) = \int \frac{d^4p}{(2\pi)^4}(\not p - m + i\epsilon)e^{-ip\cdot(x-y)} \\
\Longrightarrow &A^{-1}(x-y) = S(x-y),
\end{aligned}
\tag{10.1.6}
$$

其中 $S(x-y)$ 是传播子, 见 (8.6.1), 所以指数项为

$$
e^{-i\bar\eta A^{-1}\eta} = e^{-i\int d^4x d^4y\, \bar\eta(x)S(x-y)\eta(y)} = \cdots \tag{10.1.7}
$$

10.2 Feynman rules for Yukawa interaction

- 考虑如下 Lagrangian,

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi + \frac{1}{2}((\partial\phi)^2 - \mu^2\phi^2) - \frac{\lambda}{4!}\phi^4 + g\bar{\Psi}\phi\Psi, \tag{10.2.1}$$

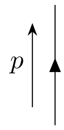
对应如下 partition function,

$$\begin{aligned}\frac{Z(\bar{\eta},\eta,J;\lambda,g)}{Z(0;0)} =& e^{i\int d^4x\,(-\frac{\lambda}{4!}(\frac{\delta}{\delta iJ(x)})^4 + g\frac{\delta}{\delta i\eta_\alpha(x)}\frac{\delta}{\delta iJ(x)}\frac{\delta}{\delta i\bar{\eta}_\alpha(x)})} e^{-\frac{i}{2}JDJ - i\bar{\eta}_\alpha S_{\alpha\beta}\eta_\beta}, \quad \text{Schwinger's way,}\\ =& \sum_{l,m,n=0}^{\infty} \frac{i^{l+m+n}}{l!m!n!}(-1)^{\frac{m(m-1)+n(n-1)}{2}} \int d^4x_1\cdots d^4x_l d^4y_1\cdots d^4y_m d^4z_1\cdots d^4z_n\\ & J(x_1)\cdots\bar{\eta}_{\alpha_1}(y_1)\cdots G^{(l,m,n)}_{\alpha_1\cdots\beta_1\cdots}(x_1,\cdots,z_n)\eta_{\beta_1}(z_1)\cdots, \quad \text{Weyl's way,}\end{aligned} \tag{10.2.2}$$

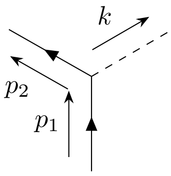
其中

$$G^{(l,m,n)}_{\alpha_1\cdots\beta_1\cdots}(x_1,\cdots,z_n) = e^{i\int d^4x\,\mathcal{L}(\lambda,g)}\phi(x_1)\cdots\Psi_{\alpha_1}(y_1)\cdots\bar{\Psi}_{\beta_1}(z_1)\cdots \tag{10.2.3}$$

- 下面给出一些 Feynman diagrams 作为例子, 先用正则量子化方法计算, 首先,

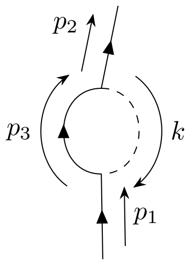


$$= \rho^2(p_1)\delta^{(3)}(\vec{p}_1 - \vec{p}_2)\delta_{s_1s_2}, \tag{10.2.4}$$



$$\begin{aligned}&= \rho(p_1)\rho(p_2)\rho(k)(-ig)\int d^4x\,\langle 0|b^{s_2}_{\vec{p}_2}a_{\vec{k}}(\bar{\Psi}(x)\phi(x)\Psi(x))b^{s_1\dagger}_{\vec{p}_1}|0\rangle\\ &= (-ig)\int d^4x\,e^{-i(p_1-p_2-k)\cdot x}\bar{u}(\vec{p}_2,s_2)u(\vec{p}_1,s_1),\end{aligned} \tag{10.2.5}$$

再算一个复杂一点的例子,



$$\begin{aligned}&=(-ig)^2(2\pi)^4\delta^{(4)}(p_1-p_2)\\ &\bar{u}_\alpha(\vec{p}_2,s_2)\Big(\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{\not{p}_3 - m + i\epsilon}\frac{i}{(p_2-p_3)^2-\mu^2+i\epsilon}\Big)_{\alpha\beta}u_\beta(\vec{p}_1,s_1).\end{aligned} \tag{10.2.6}$$

calculation:

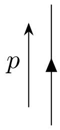
注意不要忘了算符按时间排序,

$$\cdots$$

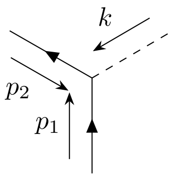
$$\begin{aligned}=&\rho(p_1)\rho(p_2)\frac{(-ig)^2}{2!}\int d^4x_1d^4x_2\,\Big(\theta(t_1-t_2)\Big(\langle 0|b^{s_2}_{\vec{p}_2}(\bar{\Psi}(x_1)\phi(x_1)\Psi(x_1)\bar{\Psi}(x_2)\phi(x_2)\Psi(x_2))b^{s_1\dagger}_{\vec{p}_1}|0\rangle\\ &+\langle 0|b^{s_2}_{\vec{p}_2}(\bar{\Psi}(x_1)\phi(x_1)\Psi(x_1)\bar{\Psi}(x_2)\phi(x_2)\Psi(x_2))b^{s_1\dagger}_{\vec{p}_1}|0\rangle\Big) - \theta(t_2-t_1)\cdots\Big)\\ =&2\times\frac{(-ig)^2}{2!}\int d^4x_1d^4x_2\end{aligned}$$

$$
\begin{aligned}
&e^{i(p_2\cdot x_1-p_1\cdot x_2)}\bar{u}_\alpha(\vec{p}_2,s_2)\Big(\int\frac{d^4p_3}{(2\pi)^4}\frac{ie^{-ip_3\cdot(x_1-x_2)}}{\not{p}_3-m+i\epsilon}\Big)_{\alpha\beta}u_\beta(\vec{p}_1,s_1)\int\frac{d^4k}{(2\pi)^4}\frac{ie^{-ik\cdot(x_1-x_2)}}{k^2-\mu^2+i\epsilon}\\
=&(-ig)^2(2\pi)^4\delta^{(4)}(p_1-p_2)\\
&\bar{u}_\alpha(\vec{p}_2,s_2)\Big(\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{\not{p}_3-m+i\epsilon}\frac{i}{(p_2-p_3)^2-\mu^2+i\epsilon}\Big)_{\alpha\beta}u_\beta(\vec{p}_1,s_1).
\end{aligned}
\tag{10.2.7}
$$

- 对于 Weyl's way, 首先,

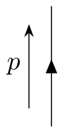


$$
=(2\pi)^4\delta^{(4)}(p_1+p_2)\Big(\frac{i}{\not{p}_1-m+i\epsilon}\Big)_{\alpha\beta},
\tag{10.2.8}
$$

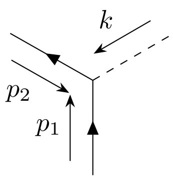


$$
=ig(2\pi)^4\delta^{(4)}(p_1+p_2+k)\Big(\frac{i}{\not{p}_1-m+i\epsilon}\frac{i}{-\not{p}_2-m+i\epsilon}\Big)_{\alpha\beta}\frac{i}{k^2-\mu^2+i\epsilon},
\tag{10.2.9}
$$

calculation:

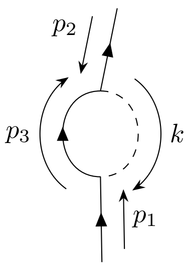


$$
\begin{aligned}
&=\int d^4xd^4y\,e^{ip_1\cdot x+ip_2\cdot y}\,\langle\Psi_\alpha(x)\bar{\Psi}_\beta(y)\rangle\\
&=\int d^4xd^4y\,e^{ip_1\cdot x+ip_2\cdot y}(-i)^2(-iS_{\alpha\beta}(x-y))=\cdots,
\end{aligned}
\tag{10.2.10}
$$



$$
\begin{aligned}
&=\int d^4x_1d^4x_2d^4y\,e^{ip_1\cdot x_1+ip_2\cdot x_2+ik\cdot y}\\
&\int d^4z\,\langle(ig\bar{\Psi}_\gamma(z)\phi(z)\Psi_\gamma(z))\phi(y)\Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\rangle\\
=&ig\int d^4x_1d^4x_2d^4y\,e^{ip_1\cdot x_1+ip_2\cdot x_2+ik\cdot y}\\
&\int d^4z(-(-i)^2iS_{\alpha\gamma}(x_1-z))(-(-i)^2iS_{\gamma\beta}(z-x_2))(-(-i)^2iD(z-y))\\
=&ig\int d^4z\Big(\frac{ie^{ip_1\cdot z}}{\not{p}_1-m+i\epsilon}\frac{ie^{ip_2\cdot z}}{-\not{p}_2-m+i\epsilon}\Big)_{\alpha\beta}\frac{ie^{ik\cdot z}}{k^2-\mu^2+i\epsilon}=\cdots
\end{aligned}
\tag{10.2.11}
$$

再算 (10.2.6),



$$
=(ig)^2(2\pi)^4\delta^{(4)}(p_1+p_2)
$$

$$
\frac{i}{\not{p}_1-m+i\epsilon}\int\frac{d^4p_3}{(2\pi)^4}\Big(\frac{i}{\not{p}_3-m+i\epsilon}\frac{i}{(p_1-p_3)^2-\mu^2+i\epsilon}\Big)\frac{i}{\not{p}_2-m+i\epsilon}.
\tag{10.2.12}
$$

calculation:

$$
\begin{aligned}
&\ldots \\
=&\frac{(ig)^2}{2!}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2\, e^{ip_1\cdot x_1+ip_2\cdot x_2}\Big(\langle(\bar\Psi(y_1)\phi(y_1)\Psi(y_1)\bar\Psi(y_2)\phi(y_2)\Psi(y_2))\Psi_1\bar\Psi_2\rangle \\
&+(y_1\leftrightarrow y_2)\Big) \\
=&(ig)^2\int d^4x_1 d^4x_2 d^4y_1 d^4y_2\, e^{ip_1\cdot x_1+ip_2\cdot x_2} iS(x_1-y_1)iS(y_1-y_2)iS(y_2-x_2)iD(y_1-y_2) \\
=&(ig)^2(2\pi)^4\delta^{(4)}(p_1+p_2) \\
&\frac{i}{\not{p}_1-m+i\epsilon}\int\frac{d^4p_3}{(2\pi)^4}\Big(\frac{i}{\not{p}_3-m+i\epsilon}\frac{i}{(p_1-p_3)^2-\mu^2+i\epsilon}\Big)\frac{i}{\not{p}_2-m+i\epsilon}.
\end{aligned}
\tag{10.2.13}
$$

Part III

quantum electrodynamics

Chapter 11

Maxwell's equations

11.1 the $(1,0)\oplus(0,1)$ representation of the Lorentz algebra

- 反对称张量 $F_{[\mu\nu]}$ 是 $(1,0)\oplus(0,1)$ rep. 中的向量, 详见笔记 Lie Groups and Lie Algebras.

11.2 Maxwell's equations

- 电磁场的 Lagrangian 为 (现实中 $\mu=0$)

$$\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}\mu^2A^\mu A_\mu,\quad \text{with}\quad F_{\mu\nu}=2\partial_{[\mu}A_{\nu]},\tag{11.2.1}$$

其中 $A_\mu=(\phi,-\vec{A})$, 对作用量变分得到运动方程 (Proca equation),

$$\partial^\nu F_{\nu\mu}+\mu^2A_\mu=0.\tag{11.2.2}$$

calculation:

场方程还可以写成

$$\Big(-\partial^\mu\partial^\nu+\eta^{\mu\nu}(\partial^2+\mu^2)\Big)A_\nu=0\iff(k^2-\mu^2)\tilde{A}_\mu(k)=k_\mu k^\nu\tilde{A}_\nu(k)\tag{11.2.3}$$

和

$$\begin{cases}-\nabla^2A_0-\nabla\cdot\dfrac{\partial\vec{A}}{\partial t}+\mu^2A_0=0\\ \dfrac{\partial}{\partial t}\Big(-\nabla A_0-\dfrac{\partial\vec{A}}{\partial t}\Big)+\Big(\underbrace{\nabla^2\vec{A}-\nabla(\nabla\cdot\vec{A})}_{=-\nabla\times(\nabla\times\vec{A})}\Big)-\mu^2\vec{A}=0\end{cases}.\tag{11.2.4}$$

  – 如果引入 Lorentz gauge condition (如 (2.1.12) 所示, 在 $\mu\neq0$ 时必然成立),

$$\text{field eq. (11.2.2)}\overset{\mu\neq0}{\iff}\begin{cases}(\partial^2+\mu^2)A_\mu=0\\ \partial^\mu A_\mu=0\end{cases}.\tag{11.2.5}$$

- 此外, $F_{\mu\nu}$ 满足 Bianchi identity,

$$\nabla_\rho F_{\mu\nu}+\nabla_\nu F_{\rho\mu}+\nabla_\mu F_{\nu\rho}=0.\tag{11.2.6}$$

calculation:

代入定义式,

$$\begin{aligned}\nabla_\rho\nabla_{[\mu}A_{\nu]}+\cdots=&+\rho\mu\nu-\rho\nu\mu\\&+\nu\rho\mu-\nu\mu\rho\\&+\mu\nu\rho-\mu\rho\nu\end{aligned}$$

$$=\underbrace{(R_{\rho\mu\nu}{}^{\sigma}+R_{\nu\rho\mu}{}^{\sigma}+R_{\mu\nu\rho}{}^{\sigma})}_{=0}A_{\sigma}. \tag{11.2.7}$$

- 令

$$F_{\mu\nu}=\begin{pmatrix}0 & E_1 & E_2 & E_3\\ -E_1 & 0 & -B_3 & B_2\\ -E_2 & B_3 & 0 & -B_1\\ -E_3 & -B_2 & B_1 & 0\end{pmatrix} \Longleftrightarrow \begin{cases}F_{0i}=\vec{E}=-\nabla\phi-\dfrac{\partial\vec{A}}{\partial t}\\ -\dfrac{1}{2}\epsilon^{ijk}F_{jk}=\vec{B}=\nabla\times\vec{A}\end{cases}, \tag{11.2.8}$$

代入场方程 (11.2.2),

$$\begin{cases}\nabla\cdot\vec{E}+\mu^2\phi=0\\ -\dfrac{\partial\vec{E}}{\partial t}+\nabla\times\vec{B}+\mu^2\vec{A}=0\end{cases}, \tag{11.2.9}$$

代入 Bianchi identity (11.2.6),

$$\begin{cases}\nabla\cdot\vec{B}=0 & \rho,\mu,\nu=1,2,3\\ \nabla\times\vec{E}-\dfrac{\partial\vec{B}}{\partial t}=0 & \rho,\mu,\nu=0,i,j\end{cases}. \tag{11.2.10}$$

- 最后, 电磁场的能动量张量见 subsection D.4.1.

11.2.1 gauge symmetry (gauge redundancy)

- A_μ 有 4 个分量, 但光子只有 2 个自由度 (偏振态).
- 首先, 考虑 $\tilde{A}_\mu$ 的场方程, 对于指标 $\mu=0$ 有 (计算过程见 subsection A.2.1)

$$\tilde{A}_0=\frac{k^0}{\omega_k^2}\vec{k}\cdot\vec{\tilde{A}}\Longrightarrow A_0(t,\vec{x})=\int d^3y\,\frac{e^{-\mu|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}\nabla_y\cdot\frac{\partial\vec{A}(t,\vec{y})}{\partial t}, \tag{11.2.11}$$

这是一个约束条件, 将此式代入剩余的场方程, 得到

$$(k^2-\mu^2)\vec{\tilde{A}}=(k^2-\mu^2)\frac{\vec{k}(\vec{k}\cdot\vec{\tilde{A}})}{\omega_k^2}, \tag{11.2.12}$$

因此:

- 当 $\mu=0$ 时,

$$\begin{cases}\text{on shell:} & \vec{\tilde{A}}\text{ 取值任意}\\ \text{off shell:} & \tilde{A}_0=\dfrac{k^0}{|\vec{k}|}|\vec{\tilde{A}}|,\vec{\tilde{A}}=|\vec{\tilde{A}}|\hat{e}_k\end{cases}\Longrightarrow \tilde{A}_\mu(k)=\tilde{\mathcal{A}}_\mu(k)2\pi\delta(k^2)-ik_\mu\tilde{\lambda}(k), \tag{11.2.13}$$

其中 $\vec{\tilde{\mathcal{A}}}(k),\tilde{\lambda}(k)=\frac{|\vec{\tilde{A}}|}{|\vec{k}|}$ 是任意函数, 且 $\text{sign}(k^0)|\vec{k}|\tilde{\mathcal{A}}_0-\vec{k}\cdot\vec{\tilde{\mathcal{A}}}=0$. 因此

$$A_\mu(x)=\partial_\mu\lambda(x)+\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}\Big(\tilde{\mathcal{A}}_\mu(|\vec{k}|,\vec{k})e^{-i(|\vec{k}|x^0-\vec{k}\cdot\vec{x})}+\tilde{\mathcal{A}}^*_\mu(|\vec{k}|,\vec{k})e^{i(|\vec{k}|x^0-\vec{k}\cdot\vec{x})}\Big), \tag{11.2.14}$$

第二项是 on shell 平面波解的叠加, 并且振幅满足 $k^\mu\tilde{\mathcal{A}}_\mu(|\vec{k}|,\vec{k})=0$.

- 当 $\mu\neq0$ 时,

$$\begin{cases}\text{on shell:} & \vec{\tilde{A}}\text{ 取值任意}\\ \text{off shell:} & \vec{k}\cdot\vec{\tilde{A}}=0\Longrightarrow\vec{\tilde{A}}=\tilde{A}_0=0\end{cases}\Longrightarrow \tilde{A}_\mu(k)=\tilde{\mathcal{A}}_\mu(k)2\pi\delta(k^2-\mu^2). \tag{11.2.15}$$

其中 $\vec{\tilde{\mathcal{A}}}(k)$ 是任意函数, 且 $\text{sign}(k^0)\omega_k\tilde{\mathcal{A}}_0-\vec{k}\cdot\vec{\tilde{\mathcal{A}}}=0$. 因此

$$A_\mu(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(\tilde{\mathcal{A}}_\mu(\omega_k,\vec{k})e^{-i(\omega_k x^0-\vec{k}\cdot\vec{x})}+\tilde{\mathcal{A}}^*_\mu(\omega_k,\vec{k})e^{i(\omega_k x^0-\vec{k}\cdot\vec{x})}\Big). \tag{11.2.16}$$

calculation:

对 (11.2.12) 两边同时内积 $\vec{k}$, 有

$$(k^2-\mu^2)(\vec{k}\cdot\vec{\tilde{A}})=0 \quad \text{or} \quad \omega_k=|\vec{k}|, \tag{11.2.17}$$

因此 $\mu=0$ 时 (11.2.12) 自然成立, 而 massive 情况下 $\tilde{A}_0=\vec{k}\cdot\vec{\tilde{A}}=0$ 除非 on shell.

- 除此之外还需要一个约束条件 (接下来默认 $\mu=0$).
- 考虑 (11.2.14), 注意到给定初始条件 $A_\mu(t_0,\vec{x}),\partial_\nu A_\mu(t_0,\vec{x})$ 无法唯一确定参数 $\lambda(x),\tilde{A}_\mu(|\vec{k}|,\vec{k})$.

calculation:

对 A_μ 做平面波分解,

$$\begin{cases}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}A_0(x)=\dfrac{1}{2|\vec{p}|}\left(e^{-i|\vec{p}|x^0}\tilde{A}_0(|\vec{p}|,\vec{p})+e^{i|\vec{p}|x^0}\tilde{A}_0(-|\vec{p}|,\vec{p})\right)+\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\dfrac{\partial\lambda}{\partial t}\\ \int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\vec{A}(x)=\dfrac{1}{2|\vec{p}|}\left(e^{-i|\vec{p}|x^0}\vec{\tilde{A}}(|\vec{p}|,\vec{p})+e^{i|\vec{p}|x^0}\vec{\tilde{A}}(-|\vec{p}|,\vec{p})\right)+i\vec{p}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\lambda\end{cases}, \tag{11.2.18}$$

对 $\partial_\nu A_\mu$ 做平面波分解,

$$\begin{cases}\dfrac{\partial A_0(x)}{\partial t}\mapsto\dfrac{-i}{2}\left(e^{-i|\vec{p}|x^0}\tilde{A}_0(|\vec{p}|,\vec{p})-e^{i|\vec{p}|x^0}\tilde{A}_0(-|\vec{p}|,\vec{p})\right)+\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\dfrac{\partial^2\lambda}{\partial t^2}\\ \dfrac{\partial\vec{A}(x)}{\partial t}\mapsto\dfrac{-i}{2}\left(e^{-i|\vec{p}|x^0}\vec{\tilde{A}}(|\vec{p}|,\vec{p})-e^{i|\vec{p}|x^0}\vec{\tilde{A}}(-|\vec{p}|,\vec{p})\right)-i\vec{p}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\dfrac{\partial\lambda}{\partial t}\\ \nabla A_\mu\mapsto i\vec{p}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}A_\mu(x)\quad(\text{这是冗余的})\end{cases}, \tag{11.2.19}$$

可见, 由于 λ 的存在, 我们无法唯一确定参数 $\tilde{A}_\mu$.

也就是说, 给定初始条件, 我们可以求解 $A_\mu(x)$ up to a function $\partial_\mu\lambda$.

- gauge redundancy: 将 A_μ 和 $A_\mu+\partial_\mu\lambda$ 认为是同一个物理态.
- Lorentz gauge 是

$$\partial_\mu A^\mu=0, \tag{11.2.20}$$

注意到方程 $\partial^2\lambda=-f$ 总是有解, 它的 Green's function 是

$$G^{(\pm)}(x)=\frac{1}{4\pi|\vec{x}|}\delta(x^0\mp|\vec{x}|). \tag{11.2.21}$$

  - Lorentz gauge 下, A_μ 的通解是

$$A_\mu(x)=\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}\left(\tilde{A}_\mu(|\vec{k}|,\vec{k})e^{-i(|\vec{k}|x^0-\vec{k}\cdot\vec{x})}+\tilde{A}^*_\mu(|\vec{k}|,\vec{k})e^{i(|\vec{k}|x^0-\vec{k}\cdot\vec{x})}\right). \tag{11.2.22}$$

- Coulomb gauge 是

$$\nabla\cdot\vec{A}=0, \tag{11.2.23}$$

注意到方程 $\nabla^2\lambda=-f$ 总是有解. 且根据 (11.2.11) 可知 $A_0(x)=0$.

  - Coulomb gauge 下, A_μ 的通解是

$$\vec{A}(x)=\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}\left(\vec{\tilde{A}}_\perp(|\vec{k}|,\vec{k})e^{-i(|\vec{k}|x^0-\vec{k}\cdot\vec{x})}+\vec{\tilde{A}}^*_\perp(|\vec{k}|,\vec{k})e^{i(|\vec{k}|x^0-\vec{k}\cdot\vec{x})}\right), \tag{11.2.24}$$

  其中 $\vec{\tilde{A}}_\perp=\vec{\tilde{A}}-\hat{e}_k\hat{e}_k\cdot\vec{\tilde{A}}$.

Chapter 12

the quantization of the electromagnetic field

12.1 massive

- 场的广义动量为

$$\pi^0 = 0, \quad \vec{\pi} = \vec{E} = i\int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k\Big(1 - \frac{\vec{k}\vec{k}}{\omega_k^2}\Big)\cdot\Big(\vec{\mathcal{A}}(k)e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} - \vec{\mathcal{A}}^*(k)e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\Big). \tag{12.1.1}$$

- 场算符是

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\sum_{i=1}^{3}\Big(\epsilon_\mu^{(i)}(\vec{k})a_{\vec{k}}^{(i)}e^{-i(\omega_k x^0 - \vec{k}\cdot\vec{x})} + \epsilon_\mu^{(i)*}(\vec{k})a_{\vec{k}}^{(i)\dagger}e^{i(\omega_k x^0 - \vec{k}\cdot\vec{x})}\Big), \tag{12.1.2}$$

  其中 $k^\mu\epsilon_\mu^{(i)}(\vec{k}) = 0$, 并满足归一化条件,

$$\begin{cases} \epsilon_\mu^{(i)*}(\vec{k})\epsilon^{(j)\mu}(\vec{k}) = -\delta_{ij} \\ \displaystyle\sum_{i=1}^{3}\epsilon_\mu^{(i)*}(\vec{k})\epsilon_\nu^{(i)}(\vec{k}) = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2} \end{cases}. \tag{12.1.3}$$

  – 对于 $k^\mu = (\omega_k, 0, 0, k)^T$, $\epsilon_\mu^{(i)}(\vec{k})$ 的具体形式为

$$\epsilon_\mu^{(1)}(\vec{k}) = (0,1,0,0), \quad \epsilon_\mu^{(2)}(\vec{k}) = (0,0,1,0), \quad \epsilon_\mu^{(3)}(\vec{k}) = (-\frac{k}{\mu}, 0, 0, \frac{\omega_k}{\mu}). \tag{12.1.4}$$

  – 产生湮灭算符满足

$$[a_{\vec{k}_1}^{(i)}, a_{\vec{k}_2}^{(j)\dagger}] = \delta_{ij}\delta^{(3)}(\vec{k}_1 - \vec{k}_2). \tag{12.1.5}$$

  正则对易关系为

$$\begin{cases} [\vec{\pi}(t,\vec{x}), A_0(t,\vec{y})] = 0 \\ [\pi_i(t,\vec{x}), A_j(t,\vec{y})] = i\displaystyle\int\frac{d^3k}{(2\pi)^3}\Big(-\eta_{ij} + \frac{k_ik_j}{\mu^2}\Big)e^{i\vec{k}(\vec{x}-\vec{y})} = i\Big(\delta_{ij} - \frac{\partial_i\partial_j}{\mu^2}\Big)\delta^{(3)}(\vec{x}-\vec{y}) \end{cases}. \tag{12.1.6}$$

calculation:

对易关系 (下式中所有动量都 on shell)

$$\begin{aligned}
&[\pi^i(t,\vec{x}), A_\mu(t,\vec{y})] \\
=& i\int\frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}}\omega_{k_1}\Big(\eta^{ij} + \frac{k_1^ik_1^j}{\omega_{k_1}^2}\Big)\sum_{k=1}^{3}\Big(\epsilon_j^{(k)}(\vec{k}_1)\epsilon_\mu^{(k)*}(\vec{k}_1)e^{-ik_1\cdot(x-y)} + \epsilon_j^{(k)*}(\vec{k}_1)\epsilon_\mu^{(k)}(\vec{k}_1)e^{ik_1\cdot(x-y)}\Big) \\
=& i\int\frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}}\omega_{k_1}\Big(-\delta_\mu^i + \frac{k_1^ik_{1\mu}}{\mu^2}\Big)(e^{-ik_1\cdot(x-y)} + e^{ik_1\cdot(x-y)}),
\end{aligned} \tag{12.1.7}$$

如果指标 $\mu=0$, 被积函数是奇函数, 结果为零, 所以...

- 传播子为

$$iD_{\mu\nu}(x-y)\equiv\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle=\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}\Big(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2}\Big)e^{-ik\cdot(x-y)}. \tag{12.1.8}$$

calculation:

考虑

$$\begin{aligned}\langle 0|A_\mu(x)A_\nu(y)|0\rangle&=\int\frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}}\sum_{i=1}^{3}\epsilon_\mu^{(i)}(\vec{k}_1)\epsilon_\nu^{(i)*}(\vec{k}_1)e^{-ik_1\cdot(x-y)}\\&=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2}\Big)e^{-ik\cdot(x-y)},\end{aligned} \tag{12.1.9}$$

因此,

$$iD_{\mu\nu}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2}\Big)\Big(\theta(t)e^{-i(\omega_k t-\vec{k}\cdot\vec{x})}+\theta(-t)e^{i(\omega_k t-\vec{k}\cdot\vec{x})}\Big), \tag{12.1.10}$$

对于第二项,

$$\begin{aligned}&I_{\mu\nu}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2}\Big)\theta(-t)e^{i(\omega_k t-\vec{k}\cdot\vec{x})}\\&\Longrightarrow\begin{cases}I_{00}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(-\eta_{00}+\frac{\omega_k\omega_k}{\mu^2}\Big)\theta(-t)e^{i(\omega_k t+\vec{k}\cdot\vec{x})}\\I_{0i}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(-\eta_{0i}+\frac{\omega_k(-k_i)}{\mu^2}\Big)\theta(-t)e^{i(\omega_k t+\vec{k}\cdot\vec{x})}\\I_{ij}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(-\eta_{ij}+\frac{(-k_i)(-k_j)}{\mu^2}\Big)\theta(-t)e^{i(\omega_k t+\vec{k}\cdot\vec{x})}\end{cases}\\&\Longrightarrow I_{\mu\nu}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{\mu^2}\Big)\theta(-t)e^{-ik\cdot x}\quad\text{with}\quad k_0=-\omega_k,\end{aligned} \tag{12.1.11}$$

因此...

12.2 massless

- 本节分别在 Lorentz gauge 和 Coulomb gauge 下对电磁场量子化.
- 电磁场的广义动量为

$$\pi^{\mu,\nu}=\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}=-\partial^\mu A^\nu+\partial^\nu A^\mu=F^{\nu\mu}, \tag{12.2.1}$$

简写 $\pi^\mu\equiv\pi^{0\mu}$, 有

$$\pi^0=0,\quad \vec{\pi}=\vec{E}. \tag{12.2.2}$$

- $\vec{E}$ 是可观测量, 因此其形式不受 gauges 影响,

$$\vec{\pi}=i\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}|\vec{k}|\Big(\vec{\mathcal{A}}_\perp(k)e^{-i(|\vec{k}|x^0-\vec{k}\cdot\vec{x})}-\vec{\mathcal{A}}_\perp^*(k)e^{i(|\vec{k}|x^0-\vec{k}\cdot\vec{x})}\Big). \tag{12.2.3}$$

12.2.1 in Coulomb gauge

- Coulomb gauge 似乎更常见.
- 场算符是

$$\vec{A}(x)=\int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2|\vec{k}|}}\sum_{i=1}^{2}\Big(\vec{\epsilon}^{(i)}(\vec{k})a_{\vec{k}}^{(i)}e^{-i(\omega_k x^0-\vec{k}\cdot\vec{x})}+\vec{\epsilon}^{(i)*}(\vec{k})a_{\vec{k}}^{(i)\dagger}e^{i(\omega_k x^0-\vec{k}\cdot\vec{x})}\Big), \tag{12.2.4}$$

其中 $\vec{k}\cdot\vec{\epsilon}^{(i)}(\vec{k})=0$, 并满足归一化条件

$$\begin{cases}\vec{\epsilon}^{(i)*}(\vec{k})\cdot\vec{\epsilon}^{(j)}(\vec{k})=\delta_{ij}\\ \sum_{i=1}^{2}\vec{\epsilon}^{(i)*}(\vec{k})\vec{\epsilon}^{(i)}(\vec{k})=1-\frac{\vec{k}\vec{k}}{|\vec{k}|^2}\end{cases}. \tag{12.2.5}$$

– 对于 $\vec{k}=(0,0,k)^T$, $\vec{\epsilon}^{(i)}(\vec{k})$ 的具体形式为 (分别对应线偏振和圆偏振)

$$\begin{cases}\vec{\epsilon}^{(1)}(\vec{k})=(1,0,0)^T\\ \vec{\epsilon}^{(2)}(\vec{k})=(0,1,0)^T\end{cases}\quad\text{or}\quad\begin{cases}\vec{\epsilon}^{(1)}(\vec{k})=\frac{1}{\sqrt{2}}(1,-i,0)^T\\ \vec{\epsilon}^{(2)}(\vec{k})=\frac{1}{\sqrt{2}}(1,+i,0)^T\end{cases}. \tag{12.2.6}$$

– 产生湮灭算符满足 $[a^{(i)}_{\vec{k}_1},a^{(j)\dagger}_{\vec{k}_2}]=\delta_{ij}\delta^{(3)}(\vec{k}_1-\vec{k}_2)$. 正则对易关系为 (差一个负号 (?) → 来自度规)

$$[\pi_i(t,\vec{x}),A_j(t,\vec{y})]=i\Big(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2}\Big)\delta^{(3)}(\vec{x}-\vec{y}). \tag{12.2.7}$$

calculation:

$$\begin{aligned}&[\pi_i(t,\vec{x}),A_j(t,\vec{y})]\\ =&i\int\frac{d^3k_1}{(2\pi)^3 2}\sum_{k=1}^{2}\Big(\epsilon_i^{(k)}(\vec{k}_1)\epsilon_j^{(k)*}(\vec{k}_1)e^{-ik_1\cdot(x-y)}+\epsilon_i^{(k)*}(\vec{k}_1)\epsilon_j^{(k)}(\vec{k}_1)e^{ik_1\cdot(x-y)}\Big)\\ =&i\int\frac{d^3k}{(2\pi)^3}\Big(\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\Big)e^{i\vec{k}\cdot(\vec{x}-\vec{y})}.\end{aligned} \tag{12.2.8}$$

- 传播子为

$$iD_{ij}(x-y)=\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\Big(\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\Big)e^{-ik\cdot(x-y)}. \tag{12.2.9}$$

calculation:

考虑

$$\begin{aligned}\langle 0|A_i(x)A_j(y)|0\rangle&=\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}\sum_{k=1}^{2}\epsilon_i^{(k)}(\vec{k})\epsilon_j^{(k)*}(\vec{k})e^{-ik\cdot(x-y)}\\ &=\int\frac{d^3k}{(2\pi)^3 2|\vec{k}|}\Big(\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\Big)e^{-ik\cdot(x-y)},\end{aligned} \tag{12.2.10}$$

因此

$$iD_{ij}(x-y)=\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\Big(\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\Big)e^{-ik\cdot(x-y)}. \tag{12.2.11}$$

12.2.2 in Lorentz gauge

- 完全没懂.
- 在 Lorentz gauge 下, 算符 $\partial_\mu A^\mu\neq 0$, 并修改 Lagrangian 为

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2, \tag{12.2.12}$$

那么

$$\begin{cases}(1-\frac{1}{\alpha})\partial_\mu\partial_\nu A^\nu-\partial_\nu\partial^\nu A_\mu=0\\ \pi^{\mu,\nu}=\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}=F^{\nu\mu}-\frac{1}{\alpha}(\partial_\rho A^\rho)\eta^{\mu\nu}\end{cases}. \tag{12.2.13}$$

因此 $\pi^0=-\frac{1}{\alpha}\partial_\mu A^\mu,\vec{\pi}=\vec{E}$.

  - $\alpha = 1$ 称作 Feynman gauge, $\alpha = 0$ 称作 Landau gauge, $\alpha \to \infty$ 称作 unitary gauge.

- 场方程 (12.2.13) 的 Green's function 为

$$\tilde{G}^{(\pm)\mu\nu}(k) = -\frac{1}{k^2 \pm i\epsilon}\Big(\eta^{\mu\nu} - (1-\alpha)\frac{k^\mu k^\nu}{k^2}\Big). \tag{12.2.14}$$

- 取 $\alpha = 1$, 那么场算符为

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2|\vec{k}|}} \sum_{\nu=0}^{3} \Big(\epsilon_\mu^{(\nu)}(\vec{k}) a_{\vec{k}}^{(\nu)} e^{-ik\cdot x} + \epsilon_\mu^{(\nu)*}(\vec{k}) a_{\vec{k}}^{(\nu)\dagger} e^{ik\cdot x}\Big), \tag{12.2.15}$$

其中 $\epsilon_\mu^{(\nu)}(\vec{k})$ 满足

$$\begin{cases} \epsilon^{(\mu)}(\vec{k}) \cdot \epsilon^{(\nu)}(\vec{k}) = \eta^{\mu\nu} \\ \displaystyle\sum_{\rho,\sigma=0}^{3} \epsilon_\mu^{(\rho)}(\vec{k}) \epsilon_\nu^{(\sigma)}(\vec{k}) \eta_{\rho\sigma} = \eta_{\mu\nu} \\ k \cdot \epsilon^{(\mu=1,2)}(\vec{k}) = 0, \quad k \cdot \epsilon^{(\mu=0,3)}(\vec{k}) = |\vec{k}| \end{cases}. \tag{12.2.16}$$

  - 对于 $k^\mu = (k, 0, 0, k)^T$, $\epsilon_\mu^{(\nu)}(\vec{k})$ 的具体形式为

$$\epsilon_\mu^{(0)}(\vec{k}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_\mu^{(1)}(\vec{k}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_\mu^{(2)}(\vec{k}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon_\mu^{(3)}(\vec{k}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{12.2.17}$$

  - 正则对易关系为

$$[\pi_\mu(t, \vec{x}), A_\nu(t, \vec{y})] = \tag{12.2.18}$$

calculation:

首先计算 $\pi^\mu(x)$,

$$\begin{aligned} \pi^0(x) = -\partial_\mu A^\mu &= i \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2|\vec{k}|}} \sum_{\mu=0,3} \Big(k \cdot \epsilon^{(\mu)}(\vec{k}) a_{\vec{k}}^{(\mu)} e^{-ik\cdot x} - k \cdot \epsilon^{(\mu)*}(\vec{k}) a_{\vec{k}}^{(\mu)\dagger} e^{ik\cdot x}\Big) \\ &= i \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2|\vec{k}|}} |\vec{k}| \sum_{\mu=0,3} \Big(a_{\vec{k}}^{(\mu)} e^{-ik\cdot x} - a_{\vec{k}}^{(\mu)\dagger} e^{ik\cdot x}\Big) \end{aligned} \tag{12.2.19}$$

注意到 $\sum_{\mu=0}^{3} \epsilon^{(\mu)0}(\vec{k}) a_{\vec{k}}^{(\mu)} = \sum_{\mu=0,3} a_{\vec{k}}^{(\mu)}$ (?).
David Tong 的 (6.40) 式不满足 $\pi^0 = -\partial_\mu A^\mu$ (?).

Appendices

Appendix A

Dirac delta function & Fourier transformation

A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx\, \delta(x) e^{-ikx}. \tag{A.1.1}$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|}. \tag{A.1.2}$$

- 第二个常用的公式 (Sokhotski-Plemelj theorem),

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x), \tag{A.1.3}$$

其中 $\mathcal{P}$ 表示复函数的主值 (principal value).

proof:

考虑

$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \quad \text{and} \quad \int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \,\mathrm{Res}(f, i\epsilon) = \pi, \tag{A.1.4}$$

所以...

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下:

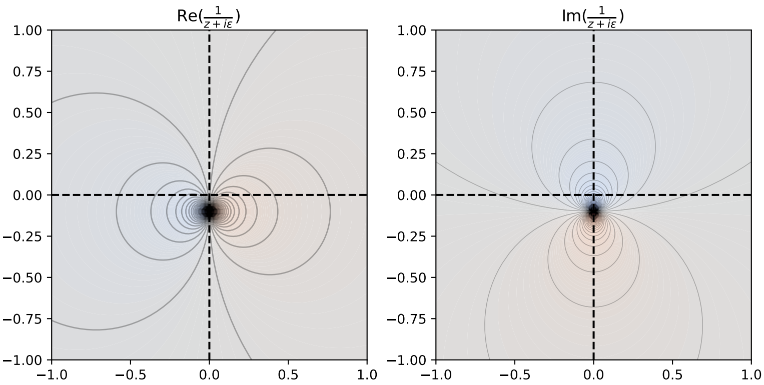


Figure A.1: graph of $\frac{1}{z+i\epsilon}$.

- 另外, $\delta(x-a)\delta(x-b) = \delta(b-a)\delta(x-a)$.

A.2 Fourier transformation

- d-dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x}\tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^d x\, e^{-ik\cdot x}\phi(x) \end{cases}, \tag{A.2.1}$$

- 因此

$$\partial_\mu \phi(x) \mapsto i k_\mu \tilde{\phi}(k). \tag{A.2.2}$$

- 对于**实函数**, Fourier transformation 是正交变换, 其 Jacobi determinant 为

$$\left|\frac{\partial\phi(x)\cdots}{\partial\mathrm{Re}\tilde{\phi}(k)\cdots\partial\mathrm{Im}\tilde{\phi}(k)\cdots}\right| = \left(\frac{2}{V}\right)^{(2N+1)^d}\det A = \left(\frac{2(2N)^d}{V^2}\right)^{\frac{(2N+1)^d}{2}}. \tag{A.2.3}$$

proof:

position space 和 momentum space 的格点分别为

$$\begin{cases} x_i^\mu = i^\mu\epsilon \in \{0, \pm\epsilon, \cdots, \frac{L}{2}\} \\ k_n^\mu = n^\mu\frac{2\pi}{L} \in \{0, \pm\frac{2\pi}{L}, \cdots, \frac{\pi}{\epsilon}\} \end{cases} \iff i^\mu, n^\mu \in \{0, \pm 1, \cdots, \pm N\}, \tag{A.2.4}$$

x^μ, k^μ 分别有 $2N+1$ 个取值, 其中 $N\epsilon = \frac{L}{2}$, 时空总体积为 $V = L^d$, momentum space 的总体积为 $\tilde{V} = \frac{(4\pi N)^d}{V}$.

将 (A.2.1) 写成格点求和的形式,

$$\begin{cases} \phi(x_i) = \frac{1}{(2\pi)^d}\left(\frac{2\pi}{L}\right)^d \sum_n e^{ik_n\cdot x_i}\tilde{\phi}(k_n) \\ \qquad = \frac{2}{V}\sum_{n^0>0}\Big(\cos(k_n\cdot x_i)\mathrm{Re}\tilde{\phi}(k_n) - \sin(k_n\cdot x_i)\mathrm{Im}\tilde{\phi}(k_n)\Big) \\ \tilde{\phi}(k_n) = \epsilon^d \sum_i e^{-ik_n\cdot x_i}\phi(x_i) \\ \qquad = \frac{V}{(2N)^d}\sum_i \Big(\cos(k_n\cdot x_i) - i\sin(k_n\cdot x_i)\Big)\phi(x_i) \end{cases}. \tag{A.2.5}$$

- -

proof:
$\phi(x_i)$ 的变换需要做一些说明. 注意到 $\tilde{\phi}$ 的分量的数量是 ϕ 的两倍 (考虑到实部与虚部), 但在 $\phi \in \mathbb{R}^{(2N+1)^d}$ 时,

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k), \tag{A.2.6}$$

可见 $\tilde{\phi}$ 的分量并不独立, 取 $k^0 > 0$ 的部分为独立分量, 那么...

- -

将 (A.2.5) 写成矩阵的形式,

$$\begin{cases} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} = \frac{2}{V}\overbrace{\begin{pmatrix} \cos k_0\cdot x_0 & \cdots & \cos k_{\max}\cdot x_0 & -\sin k_0\cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A}\begin{pmatrix} \mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots \end{pmatrix} = \frac{V}{(2N)^d}\begin{pmatrix} \cos k_0\cdot x_0 & \cdots & \cos k_0\cdot x_{\max} \\ \vdots & \ddots & \\ -\sin k_0\cdot x_0 & \cdots & -\sin k_0\cdot x_{\max} \\ \vdots & & \ddots \end{pmatrix}\begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} \end{cases}, \tag{A.2.7}$$

观察可见 $\tilde{\phi}$ 的变换中的矩阵是 A^T, 所以

$$\frac{2}{V}\frac{V}{(2N)^d}AA^T = I \Longrightarrow \det A = \left(\frac{(2N)^d}{2}\right)^{\frac{(2N+1)^d}{2}}, \tag{A.2.8}$$

因此...

– 顺便,

$$\int d^dx\, f(x)g(x) = \int \frac{d^dk}{(2\pi)^d}\tilde{f}(-k)\tilde{g}(k). \tag{A.2.9}$$

A.2.1 an important example

- 考虑如下 PDE,

$$(\nabla^2 - \mu^2)\phi(\vec{x}) = f(\vec{x}), \tag{A.2.10}$$

其 Green's function 为

$$G(\vec{x}) = -\frac{1}{4\pi}\frac{e^{-\mu r}}{r}, \tag{A.2.11}$$

其中 $r = |\vec{x}|$.

calculation:

Green's function 满足

$$(\nabla^2 - \mu^2)G(\vec{x}) = \delta^{(3)}(\vec{x}) \Longrightarrow \tilde{G}(\vec{k}) = -\frac{1}{|\vec{k}|^2 + \mu^2}, \tag{A.2.12}$$

因此

$$\begin{aligned} G(\vec{x}) &= -\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2 + \mu^2} \\ &= -\int \frac{k^2 \sin\theta d\theta d\phi dk}{(2\pi)^3}\frac{e^{i\cos\theta kr}}{k^2 + \mu^2} = -\frac{1}{(2\pi)^3}\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^\infty dk\, \frac{k^2 e^{i\cos\theta kr}}{k^2 + \mu^2} \\ &= -\frac{1}{(2\pi)^2}\int_0^\infty \frac{k^2}{k^2 + \mu^2}\frac{2\sin kr}{kr}dk, \end{aligned} \tag{A.2.13}$$

注意到 (在复平面上考虑以下积分并使用 residue theorem)

$$\begin{aligned} \int_0^\infty \frac{k\sin kr}{k^2 + \mu^2}dk &= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{k\sin kr}{k^2 + \mu^2}dk = \frac{1}{2i}\int_{-\infty}^{+\infty} \frac{ke^{ikr}}{k^2 + \mu^2}dk \\ &= \frac{1}{2i}\int_{-\infty}^{+\infty} \frac{ke^{ikr}}{(k + i\mu)(k - i\mu)}dk = \frac{1}{2i}2\pi i \mathrm{Res}(f, k = i\mu) = \frac{\pi}{2}e^{-\mu r}. \end{aligned} \tag{A.2.14}$$

Appendix B

Gaussian integrals and Gaussian-Berezin integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{B.0.1}$$

$$\langle x^{2n}\rangle = \frac{\int dx\, e^{-\frac{1}{2}ax^2}x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!!, \tag{B.0.2}$$

  其中 $(2n-1)!! = 1\cdot 3\cdots(2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx\, e^{-\frac{a}{2}x^2+Jx} = \sqrt{\frac{2\pi}{a}}e^{\frac{J^2}{2a}}, \tag{B.0.3}$$

  另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 generalize to N-dim.

- 考虑如下积分,

$$Z(A,J) = \int dx_1\cdots dx_N\, e^{-\frac{1}{2}x^T\cdot A\cdot x+J^T\cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J}, \tag{B.1.1}$$

  其中 x, J 是 N-dim. 列向量, A 是 $N\times N$ 实对称矩阵.

calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得

$$A = O^{-1}\cdot D\cdot O, \tag{B.1.2}$$

其中 D 是一个 diagonal matrix. 令 $y = O\cdot x$, 那么

$$\begin{aligned} Z(A,J) &= \int dy_1\cdots dy_N\, e^{-\frac{1}{2}y^T\cdot D\cdot y+(OJ)^T\cdot y} \\ &= \prod_{i=1}^{N}\sqrt{\frac{2\pi}{D_{ii}}}e^{\frac{1}{2D_{ii}}(OJ)_i{}^2} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J}, \end{aligned} \tag{B.1.3}$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O\cdot A^{-1}\cdot O^{-1})_{ii}$ 以及 $\operatorname{tr} D = \det A$.

- 一个重要的变体是 $A\mapsto -iA, J\mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j\rangle = \frac{1}{Z(A,0)}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}Z(A,J)\Big|_{J=0} = A^{-1}_{ij}, \tag{B.1.4}$$

$$\langle x_i x_j \cdots x_k x_l \rangle = \sum_{\text{Wick}} A^{-1}_{i'j'} \cdots A^{-1}_{k'l'}, \tag{B.1.5}$$

其中 (B.1.5) 中有偶数个 x, 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A,J)\Big|_{J=0} = \cdots . \tag{B.1.6}$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk}, \tag{B.1.7}$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ik} A^{-1}_{jl}. \tag{B.1.8}$$

B.2 Grassmann number and Grassmann integrals

- 对于 Grassmann number θ_1, θ_2, 有反对易关系,

$$\theta_1 \theta_2 = -\theta_2 \theta_1, \tag{B.2.1}$$

因此 $\theta^2 = 0$, 且关于 Grassmann number 最一般的函数为

$$f(\theta) = a\theta + b, \tag{B.2.2}$$

其中 $a, b \in \mathbb{C}$.

- 注意到 $(\theta_1\theta_2)\theta_3 = \theta_3(\theta_1\theta_2)$, (但是 $(\theta_1\theta_2)^2 = 0$, 所以 $\theta_1\theta_2 \notin \mathbb{C}$), 且有

$$(\theta_1\theta_2)(\theta_3\theta_4) = \theta_3(\theta_1\theta_2)\theta_4 = (\theta_3\theta_4)(\theta_1\theta_2). \tag{B.2.3}$$

- 定义 Grassmann integral (也称作 Berezin integral),

$$\int d\theta\, \theta = 1, \quad \int d\theta = 0, \tag{B.2.4}$$

并且具有 linearity.

comment:

我们希望积分在 integration variable been shifted 之后 $(\theta \mapsto \theta + \eta)$ 保持不变,

$$\int d\theta\, (a\theta + b) = \int d\theta\, (a\theta + a\eta + b), \tag{B.2.5}$$

因此, 积分结果应该与常数无关, 只与斜率有关, 所以直接定义

$$\int d\theta\, (a\theta + b) = a. \tag{B.2.6}$$

  - 另外, 对于 $f(\theta) = \eta\theta + b$, 有

$$\int d\theta\, (\eta\theta + b) = \int d\theta\, (-\theta\eta + b) = -\eta. \tag{B.2.7}$$

B.2.1 Gaussian-Berezin integrals

- 回顾 section 1.4 和 (8.4.2), 我们希望 Gauss 积分中出现正号而不是符号, 即

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{2\pi} e^{-\frac{1}{2}\ln a} \mapsto \propto e^{+\frac{1}{2}\ln a}. \tag{B.2.8}$$

- 对于两个独立的 Grassmann number $\theta, \bar{\theta}$, 有 Gauss 积分,

$$\int d\theta \int d\bar{\theta}\, e^{\bar{\theta} a \theta} = \int d\theta \int d\bar{\theta}\,(1 + \bar{\theta} a \theta) = a = e^{+\ln a}. \tag{B.2.9}$$

- 推广以上积分, 对于 $\theta = (\theta_1, \cdots, \theta_N) \in V, \bar{\theta} = (\bar{\theta}_1, \cdots, \bar{\theta}_N) \in V^*$, 有

$$\int d\theta \int d\bar{\theta}\, e^{\bar{\theta} A \theta} = \det A, \tag{B.2.10}$$

其中 A 是 $N \times N$ normal matrix.

calculation:

对向量做幺正变换, $\eta = U\theta, \bar{\eta} = \bar{\theta} U^\dagger$, 使得 A 对角化 $D = UAU^\dagger$, (注意对**积分顺序**的定义),

$$I = \int d\eta \int d\bar{\eta}\, e^{\bar{\eta} D \eta} = \sum_{n=0}^{\infty} \int d\eta_N \cdots d\eta_1 \int d\bar{\eta}_1 \cdots d\bar{\eta}_N \frac{\left(\sum_{i=1}^N \bar{\eta}_i D_i \eta_i\right)^n}{n!}, \tag{B.2.11}$$

其中, 唯一不为零的项是 $\propto \prod_{i=1}^N (\bar{\eta}_i D_i \eta_i)$, 并且注意到 $(\bar{\eta}_i D_i \eta_i)$ 互相对易, 所以

$$\begin{aligned} I &= \int d\eta_N \cdots d\eta_1 \int d\bar{\eta}_1 \cdots d\bar{\eta}_N \frac{n! \prod_{i=1}^N (\bar{\eta}_i D_i \eta_i)}{n!} \\ &= \int d\eta_N \cdots d\eta_1 \int d\bar{\eta}_1 \cdots d\bar{\eta}_N\, (\bar{\eta}_N D_N \eta_N) \cdots (\bar{\eta}_1 D_1 \eta_1) \\ &= \int d\eta_N \cdots d\eta_1 \int d\bar{\eta}_1 \cdots d\bar{\eta}_{N-1}\, \overbrace{(\bar{\eta}_{N-1} D_{N-1} \eta_{N-1}) \cdots (\bar{\eta}_1 D_1 \eta_1)}^{\text{commutes with } \eta_N} D_N \eta_N \\ &= \cdots = \int d\eta_N \cdots d\eta_1\, D_1 \eta_1 \cdots D_N \eta_N = \prod_{i=1}^N D_i = \det A, \end{aligned} \tag{B.2.12}$$

注意到, 由于 $(\bar{\eta}_i D_i \eta_i)$ 互相对易, 所以 $\eta, \bar{\eta}$ 的积分顺序并不重要, 唯一的要求是 η 和 $\bar{\eta}$ 的积分顺序互相对应 (顺序正好**相反**), 即 $d\eta_j d\eta_i \leftrightarrow d\bar{\eta}_i d\bar{\eta}_j$, (Coleman 对积分顺序的定义是 $d\eta d\bar{\eta} = d\eta_1 d\bar{\eta}_1 \cdots d\eta_N d\bar{\eta}_N$, 这与我们的定义是等效的).

- 进一步推广,

$$Z(A, \eta, \bar{\eta}) = \int d\theta \int d\bar{\theta}\, e^{\bar{\theta} A \theta + \bar{\eta}\theta + \bar{\theta}\eta} = \det A\, e^{-\bar{\eta} A^{-1} \eta}, \tag{B.2.13}$$

只需要注意到 $(\bar{\theta} + \bar{\eta} A^{-1}) A (\theta + A^{-1}\eta) = \bar{\theta} A \theta + \bar{\eta}\theta + \bar{\theta}\eta + \bar{\eta} A^{-1} \eta$, 其中 $\eta \in V, \bar{\eta} \in V^*$ 都是 Grassmann number 组成的向量.

- 最后, 考虑 (B.1.5) 的变体,

$$\langle \theta_i \rangle = \langle \bar{\theta}_j \rangle = 0, \tag{B.2.14}$$

$$\langle \cdots \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \cdots \rangle = \underbrace{\langle \cdots \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \cdots \rangle}_{= \cdots (-A^{-1}_{jk})(-A^{-1}_{il}) \cdots} + \underbrace{\langle \cdots \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \cdots \rangle}_{= -\cdots (-A^{-1}_{ik})(-A^{-1}_{jl}) \cdots} + \cdots, \tag{B.2.15}$$

技巧在于先把 $\cdots \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \cdots$ 的顺序调整到 $\theta, \bar{\theta}$ 互相对应 (像 (B.2.15) 等号右边第一项), 然后再做 contraction.

calculation:

考虑

$$\begin{aligned} \langle \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \rangle &= \frac{\partial}{\partial \bar{\eta}_i} \frac{\partial}{\partial \bar{\eta}_j} \left(-\frac{\partial}{\partial \eta_k} \right) \left(-\frac{\partial}{\partial \eta_l} \right) e^{-\bar{\eta} A^{-1} \eta} = \frac{\partial}{\partial \bar{\eta}_i} \frac{\partial}{\partial \bar{\eta}_j} (-\eta_{j'} A^{-1}_{j'k})(-\eta_{i'} A^{-1}_{i'l}) \\ &= \underbrace{\langle \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \rangle}_{= (-A^{-1}_{jk})(-A^{-1}_{il})} + \underbrace{\langle \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \rangle}_{= -(-A^{-1}_{ik})(-A^{-1}_{jl})}. \end{aligned} \tag{B.2.16}$$

Appendix C

perturbation theory in QM

- this chapter is based on MIT OpenCourseWare Quantum Physics III Chapter 1: Perturbation Theory.

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda \delta H, \tag{C.0.1}$$

  其中 $\lambda \in [0,1]$.

- 考虑 $H^{(0)}$ 的本征态为

$$H^{(0)} |k^{(0)}\rangle = E_k^{(0)} |k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)}|l^{(0)}\rangle = \delta_{kl} \\ E_0^{(0)} \le E_1^{(0)} \le E_2^{(0)} \le \cdots \end{cases}. \tag{C.0.2}$$

C.1 non-degenerate perturbation theory

- 考虑 non-degenerate 能级 k, 有 $\cdots \le E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \le \cdots$, 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \cdots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \cdots \end{cases}. \tag{C.1.1}$$

  - 注意, 我们可以选取修正项满足

$$\langle k^{(0)}|k^{(n)}\rangle = 0, n = 1, 2, \cdots \tag{C.1.2}$$

proof:

假设我们求解得到的修正项不满足 $\langle k^{(0)}|k^{(n)}\rangle = 0, n = 1, 2, \cdots$, 考虑

$$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n |k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)}|k^{(n)}\rangle' = 0, \tag{C.1.3}$$

那么 (注意到态矢量可以乘一个常数, $\frac{1}{1-a_1\lambda-a_2\lambda^2-\cdots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \cdots$)

$$\begin{aligned} |k\rangle_\lambda &= (1 - a_1\lambda - a_2\lambda^2 - \cdots) |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \cdots \\ |k\rangle'_\lambda &= |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \cdots}\left(\lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \cdots\right) \\ &= |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2(a_1 |k^{(1)}\rangle' + |k^{(2)}\rangle') + \cdots, \end{aligned} \tag{C.1.4}$$

可见修正项都与 $|k^{(0)}\rangle$ 正交.

  - 注意, 不能要求 ${}_\lambda\langle k|k\rangle_\lambda = 1$, 否则 $|k^{(n)}\rangle$ 将与 λ 相关 (包括 $|k^{(0)}\rangle$),

$$\begin{aligned} {}_\lambda\langle k|k\rangle_\lambda = \langle k^{(0)}|k^{(0)}\rangle \\ &+ \lambda(\langle k^{(1)}|k^{(0)}\rangle + \langle k^{(0)}|k^{(1)}\rangle) \\ &+ \lambda^2(\langle k^{(2)}|k^{(0)}\rangle + \langle k^{(1)}|k^{(1)}\rangle + \langle k^{(0)}|k^{(2)}\rangle) \end{aligned}$$

$$\vdots$$
$$+\lambda^n(\langle k^{(n)}|k^{(0)}\rangle + \langle k^{(n-1)}|k^{(1)}\rangle + \cdots + \langle k^{(0)}|k^{(n)}\rangle). \tag{C.1.5}$$

- 将 (C.1.1) 代入 Schrödinger's eq., 得到:

λ^0	$(H^{(0)} - E_k^{(0)})\,|k^{(0)}\rangle = 0$
λ^1	$(H^{(0)} - E_k^{(0)})\,|k^{(1)}\rangle = (E_k^{(1)} - \delta H)\,|k^{(0)}\rangle$
λ^2	$(H^{(0)} - E_k^{(0)})\,|k^{(2)}\rangle = (E_k^{(1)} - \delta H)\,|k^{(1)}\rangle + E_k^{(2)}\,|k^{(0)}\rangle$
$\vdots$	$\vdots$
λ^n	$(H^{(0)} - E_k^{(0)})\,|k^{(n)}\rangle = (E_k^{(1)} - \delta H)\,|k^{(n-1)}\rangle + E_k^{(2)}\,|k^{(n-2)}\rangle + \cdots + E_k^{(n)}\,|k^{(0)}\rangle$

calculation:

Schrödinger's eq. 为

$$(H^{(0)} + \lambda\delta H - E_k(\lambda))\,|k\rangle_\lambda = 0, \tag{C.1.6}$$

展开为

$$\Big((H^{(0)} - E_k^{(0)}) + \lambda(\delta H - E_k^{(1)}) - \lambda^2 E_k^{(2)} - \cdots\Big)(|k^{(0)}\rangle + \lambda\,|k^{(1)}\rangle + \lambda^2\,|k^{(2)}\rangle + \cdots) = 0. \tag{C.1.7}$$

- 现在来计算 $\langle l^{(0)}|k^{(n)}\rangle$, 有

$$\begin{cases} (E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(1)}\rangle = E_k^{(1)}\delta_{lk} - \delta H_{lk} \\ (E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(2)}\rangle = E_k^{(1)}\,\langle l^{(0)}|k^{(1)}\rangle - \langle l^{(0)}|\delta H|k^{(1)}\rangle + E_k^{(2)}\delta_{lk} \\ \qquad\vdots \qquad\qquad\qquad\qquad \vdots \\ (E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(n)}\rangle = E_k^{(1)}\,\langle l^{(0)}|k^{(n-1)}\rangle - \langle l^{(0)}|\delta H|k^{(n-1)}\rangle \\ \qquad\qquad\qquad + E_k^{(2)}\,\langle l^{(0)}|k^{(n-2)}\rangle + \cdots + E_k^{(n)}\delta_{lk} \end{cases}, \tag{C.1.8}$$

其中 $\delta H_{lk} = \langle l^{(0)}|\delta H|k^{(0)}\rangle$, 对于满足 (C.1.2) 的解, 有

$$E_k^{(n)} = \langle k^{(0)}|\delta H|k^{(n-1)}\rangle\,, n = 1, 2, \cdots, \tag{C.1.9}$$

并且

$$|k^{(1)}\rangle = -\sum_{l\neq k}\frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}}\,|l^{(0)}\rangle \Longrightarrow E_k^{(2)} = -\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}}. \tag{C.1.10}$$

calculation:

将 (C.1.10) 代入 (C.1.8), 得到 $(l\neq k)$

$$(E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(2)}\rangle = -E_k^{(1)}\frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} + \sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}, \tag{C.1.11}$$

所以

$$\begin{cases} |k^{(2)}\rangle = \sum_{l\neq k}\Big(-\frac{\delta H_{00}\delta H_{lk}}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}\Big)\,|l^{(0)}\rangle \\ E_k^{(3)} = \sum_{l\neq k}\Big(-\frac{\delta H_{00}|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m\neq k}\frac{\delta H_{kl}\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}\Big) \end{cases}. \tag{C.1.12}$$

计算归一化系数

$${}_\lambda\langle k|k\rangle_\lambda = 1 + \lambda^2\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + O(\lambda^3). \tag{C.1.13}$$

C.1.1 level repulsion or the seesaw mechanism

- 能量的展开式为

$$E_k(\lambda) = E_k^{(0)} + \lambda\delta H_{kk} - \lambda^2 \sum_{l\neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3), \tag{C.1.14}$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

C.1.2 validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即

$$|\lambda V| < \frac{1}{2}\Delta E^{(0)}, \tag{C.1.15}$$

因此, 对于能级简并的情况, $\Delta E^{(0)} = 0$, 情况会更复杂.

calculation:

对于两能级系统

$$H(\lambda) = H^{(0)} + \lambda\hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix}, \tag{C.1.16}$$

$H(\lambda)$ 的本征值可以直接计算,

$$E_\pm(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)})\sqrt{1 + \left(\frac{\lambda|V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})}\right)^2}, \tag{C.1.17}$$

考虑 $\sqrt{1+z^2}$ 的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1}\frac{(2n-3)!!}{2^n n!}z^{2n} + \cdots, \tag{C.1.18}$$

注意到 $\sqrt{1+z^2}$ 在 $z = \pm i$ 有 branch cut, 因此 $z = 0$ 附近的 Taylor expansion 只有在 $|z| < 1$ 内才收敛.

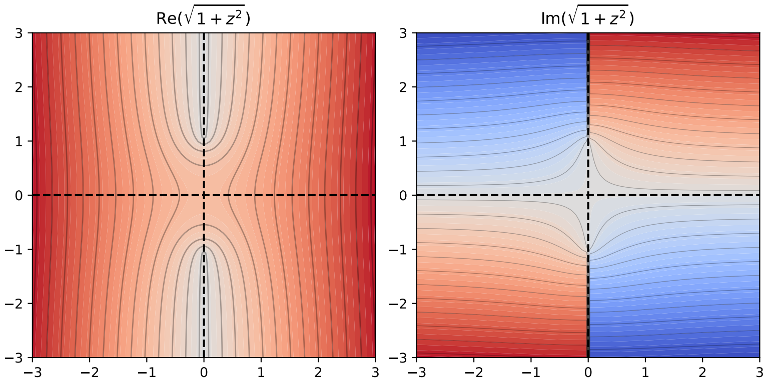


Figure C.1: graph of $\sqrt{1+z^2}$.

C.2 degenerate perturbation theory

- 暂时先跳过.

Appendix D

classical field theory and Noether's theorem

D.1 classical field theory

D.1.1 Lagrangian density and the action

- Lagrangian density, $\mathcal{L}$, 是 $\phi^a(x), \partial_\mu\phi^a(x), t$ 的函数.
- 对作用量变分得到 Euler-Lagrangian equation of motion,

$$\frac{\delta\mathcal{L}}{\delta\phi^a} - \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\Big) = 0. \tag{D.1.1}$$

calculation:

对作用量进行变分,

$$\begin{aligned}\delta S &= \int d^4x\Big(\frac{\delta\mathcal{L}}{\delta\phi^a}\delta\phi^a + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\delta\partial_\mu\phi^a\Big) \\ &= \int d^4x\bigg(\Big(\frac{\delta\mathcal{L}}{\delta\phi^a} - \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\Big)\Big)\delta\phi^a + \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\delta\phi^a\Big)\bigg),\end{aligned} \tag{D.1.2}$$

由于边界变分为零...

D.1.2 canonical momentum and the Hamiltonian

- **def.:** 定义一个叫 π^μ_a 的量,

$$\pi^\mu_a = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}, \tag{D.1.3}$$

  其中 $\pi_a \equiv \pi^0_a$ 称作 canonical momentum of the field.

- **def.:** the Hamiltonian density is

$$\mathcal{H} = \pi_a\partial_0\phi^a - \mathcal{L}. \tag{D.1.4}$$

- the Hamilton's equations are

$$\begin{cases}\partial_0\phi^a = \dfrac{\delta\mathcal{H}}{\delta\pi_a} \\ -\partial_0\pi^a = \dfrac{\delta\mathcal{H}}{\delta\phi^a} - \partial_i\Big(\dfrac{\delta\mathcal{H}}{\delta(\partial_i\phi^a)}\Big)\end{cases}. \tag{D.1.5}$$

  - 第二个方程可以写成更紧凑的形式

$$\partial_\mu\pi^\mu_a = \frac{\delta\mathcal{H}}{\delta\phi^a}. \tag{D.1.6}$$

D.2 Noether's theorem

- Noether 定理中的对称性是 global symmetry, 不是 gauge symmetry.
- strictly speaking, gauge symmetry is not a symmetry but a redundancy.

D.2.1 in classical particle mechanics

- 系统的 Lagrangian 为 $L(q^a, \dot{q}^a, t)$.
- 系统通过以下形式变换,

$$q^a(t) \mapsto q^a(\lambda, t) \quad \text{and} \quad q^a(t,0) = q^a(t), \tag{D.2.1}$$

并定义

$$D_\lambda q^a = \frac{\partial q^a}{\partial \lambda}\Big|_{\lambda=0}. \tag{D.2.2}$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.

$$D_\lambda L = \frac{dF(q^a, \dot{q}^a, t)}{dt}, \tag{D.2.3}$$

for some $F(q^a, \dot{q}^a, t)$, and the corresponding **conserved quantity** is

$$Q = p_a D_\lambda q^a - F(q^a, \dot{q}^a, t). \tag{D.2.4}$$

proof:

$$D_\lambda L = \frac{\partial L}{\partial q^a} D_\lambda q^a + \frac{\partial L}{\partial \dot{q}^a}\frac{dD_\lambda q^a}{dt} = \frac{d}{dt}(p_a D_\lambda q^a). \tag{D.2.5}$$

- 几个例子如下,
  - **空间平移**, $\vec{x}(t) \mapsto \vec{x}(t) + \hat{e}_i\lambda$, 相应地, $D_\lambda \vec{x} = \hat{e}_i$, 且

$$D_\lambda L = \frac{\partial L}{\partial x^i}, \tag{D.2.6}$$

如果 $\frac{\partial L}{\partial x^i} = 0$, 那么, 有守恒量 p_i.

  - **时间平移**, $q^a(t) \mapsto q^a(t+\lambda)$, 相应地, $D_\lambda q^a = \dot{q}^a$, 且

$$D_\lambda L = \frac{dL}{dt} - \frac{\partial L}{\partial t}, \tag{D.2.7}$$

如果 $\frac{\partial L}{\partial t} = 0$, 那么, 有守恒量 $H = p_a\dot{q}^a - L$.

  - **转动**, $\vec{x}(t) \mapsto R(\lambda, \hat{e}) \cdot \vec{x}(t)$, 相应地, $D_\lambda \vec{x} = \hat{e} \times \vec{x}$, 且

$$D_\lambda L = \vec{x} \cdot \left(\frac{\partial L}{\partial \vec{x}} \times \hat{e}\right) + \hat{e}(\dot{\vec{x}} \times \vec{p}), \tag{D.2.8}$$

如果上式中两个括号内的项都为零, 那么, 有守恒量 $\hat{e} \cdot \vec{J} = \hat{e} \cdot (\vec{x} \times \vec{p})$.

D.2.2 in classical field theory

- 类似地, 系统通过以下形式变换,

$$\phi^a(x) \mapsto \phi^a(x, \lambda) \quad \text{and} \quad \phi^a(x,0) = \phi^a(x), \tag{D.2.9}$$

并定义

$$D_\lambda \phi^a = \frac{\partial \phi^a}{\partial \lambda}\Big|_{\lambda=0}. \tag{D.2.10}$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.

$$D_\lambda \mathcal{L} = \partial_\mu F^\mu(\phi^a, \partial_\mu\phi^a, t), \tag{D.2.11}$$

for some $F^\mu(\phi^a, \partial_\mu\phi^a, t)$, and the **conserved current** is

$$J^\mu = \pi^\mu_a D_\lambda \phi^a - F^\mu. \tag{D.2.12}$$

proof:

$$\begin{aligned} D_\lambda \mathcal{L} &= \frac{\delta \mathcal{L}}{\delta \phi^a} D_\lambda \phi^a + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \partial_\mu D_\lambda \phi^a \\ &= \Big(\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \Big(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \Big) \Big) D_\lambda \phi^a + \partial_\mu \Big(\underbrace{\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)}}_{=\pi_a^\mu} D_\lambda \phi^a \Big), \end{aligned} \tag{D.2.13}$$

代入 (D.1.1), 得...

- 注意, conserved current 并不是唯一确定的, 考虑如下变换,

$$F^\mu \mapsto F'^\mu = F^\mu + \partial_\nu A^{\mu\nu} \quad \text{with} \quad A^{\mu\nu} = A^{[\mu\nu]}, \tag{D.2.14}$$

新 F'^μ 依然能满足 (D.2.11).

- 但是, 守恒荷是唯一确定的.

proof:

$$Q' = \int d^3x J^0 = \int d^3x (\pi_a D_\lambda \phi^a - F^0) - \int d^3x \, \partial_\mu A^{0\mu}, \tag{D.2.15}$$

考虑到边界值为零, 且 $A^{00} = 0$, 所以 $Q' = Q$.

D.2.3 spacetime translations and the energy-momentum tensor

- 时空平移变换为

$$\phi^a(x) \mapsto \phi^a(x + \lambda e). \tag{D.2.16}$$

- 所以

$$D_\lambda \phi^a = e^\mu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = e^\mu \partial_\mu \mathcal{L}, \tag{D.2.17}$$

代入 (D.2.12),

$$J^\mu = e^\nu \underbrace{(\pi_a^\mu \partial_\nu \phi^a - \delta_\nu^\mu \mathcal{L})}_{=T^\mu{}_\nu}. \tag{D.2.18}$$

- 并且有

$$\partial_\mu T^{\mu\nu} = 0 \Longrightarrow P^\mu = \int d^3x \, T^{0\mu} = \text{Const.}, \tag{D.2.19}$$

来自守恒流散度为零.

D.2.4 Lorentz transformations, angular momentum and something else

- Lorentz transformation 下坐标做变换 $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, 其中 Λ 满足

$$\eta = \Lambda^T \eta \Lambda. \tag{D.2.20}$$

- infinitesimal Lorentz transformation 是

$$\Lambda = I + \epsilon, \tag{D.2.21}$$

其中 $\{\epsilon^{\mu\nu}\} = \epsilon\eta$ 是反对称矩阵.

proof:

考虑

$$\eta = (\Lambda\eta)^T \eta (\Lambda\eta) = (\eta + \epsilon\eta)^T \eta (\eta + \epsilon\eta)$$

$$= \eta + \eta\epsilon^T + \epsilon\eta + O(\epsilon^2). \tag{D.2.22}$$

- 标量场在 Lorentz transform 下的变换为

$$\Lambda : \phi^a(x) \mapsto \phi^a(\Lambda^{-1}x'). \tag{D.2.23}$$

- 有

$$D_\lambda \phi^a = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\epsilon_{\mu\nu} \partial^\mu (x^\nu \mathcal{L}), \tag{D.2.24}$$

代入 (D.2.12),

$$J^\mu = \frac{1}{2} \epsilon_{\nu\rho} M^{\mu\nu\rho} \quad \text{where} \quad M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}, \tag{D.2.25}$$

且有

$$\partial_\mu M^{\mu\nu\rho} = 0. \tag{D.2.26}$$

- 对全空间积分, 得到 6 个守恒量,

$$J^{\mu\nu} = \int d^3x \, M^{0\mu\nu} = \text{Const.}, \tag{D.2.27}$$

不难发现 J^{ij} 对应角动量. 现在来讨论 J^{0i} 的物理意义,

$$0 = \frac{d}{dt} J^{0i} = \frac{d}{dt} \int d^3x (tT^{0i} - x^i T^{00}) = P^i - \frac{d}{dt} \int d^3x \, x^i T^{00}, \tag{D.2.28}$$

其中, 用到了 $\frac{dP^i}{dt} = 0$ (见 (D.2.19)), 可以将上式的第二项理解为质心运动的动量.

D.3 charge as generators

- the charge associated with the conserved current is

$$Q = \int d^D \, J^0 = \int d^D x \, (\pi_a D_\lambda \phi^a - F^0), \tag{D.3.1}$$

在 $F^\mu = 0$ 且 $[D_\lambda \phi^a, \phi^a] = 0$ 的情况下,

$$i[Q, \phi^a] = D_\lambda \phi^a. \tag{D.3.2}$$

D.4 what the graviton listens to: energy-momentum tensor

- the energy-momentum tensor is defined as (其中 $g = |\det\{g_{\mu\nu}\}|$)

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M. \tag{D.4.1}$$

- 如果将 $\mathcal{L}_M$ 对 $g^{\mu\nu}$ 做展开 $\mathcal{L}_M = A + g^{\mu\nu} B_{\mu\nu} + g^{\mu\nu} g^{\rho\sigma} C_{\mu\nu\rho\sigma} + \cdots$, 那么

$$T_{\mu\nu} = -2(B_{\mu\nu} + 2g^{\rho\sigma} C_{\mu\nu\rho\sigma} + 3\cdots) + g_{\mu\nu}\mathcal{L}_M, \tag{D.4.2}$$

另外, the trace of the energy-momentum tensor is

$$T = g^{\mu\nu} T_{\mu\nu} = d \times A + (d-2) g^{\mu\nu} B_{\mu\nu} + (d-4) g^{\mu\nu} g^{\rho\sigma} C_{\mu\nu\rho\sigma}, \tag{D.4.3}$$

可见 $d = 4$ 时, T 与 $C_{\mu\nu\rho\sigma}$ 无关.

- $\mathcal{L} = -\frac{1}{2}((\partial\phi)^2 - m^2\phi^2)$ 和 $\mathcal{L} = \frac{1}{2}\phi(\partial^2 - m^2)\phi$ 对应的 energy-momentum tensor 一样吗 (?).

D.4.1 example: energy-momentum tensor of the electromagnetic field

- 在 $(+,-,-,-)$ 号差下，定义 $A_\mu = (\phi, -\vec{A})$，这很容易让人误以为 4-potential 是 vector，而事实上它是 covector，按照这种定义

$$\frac{\delta F_{\rho\sigma}}{\delta g^{\mu\nu}} = 0. \tag{D.4.4}$$

  – Wikipedia: Maxwell's equations in curved spacetime, Electromagnetic potential.

- 代入电磁场的 Lagrangian，见 (11.2.1)，所以

$$T_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \mu^2 A_\mu A_\nu + g_{\mu\nu}\mathcal{L} \tag{D.4.5}$$

  – 在 Minkowski 时空中，

$$\begin{aligned}
&T_{\mu\nu}\\
=&\begin{pmatrix} -\mathcal{E} - \frac{1}{2}\mu^2(\phi^2 + |\vec{A}|^2) & (\vec{E}\times\vec{B})_i - \mu^2\phi A_i \\ \vdots & -\delta_{ij}(\mathcal{E} + \frac{1}{2}\mu^2(\phi^2 - |\vec{A}|^2)) - \mu^2 A_i A_j + E_i E_j + B_i B_j \end{pmatrix}\\
=&\begin{pmatrix} -\mathcal{E} & (\vec{E}\times\vec{B})_i \\ \vdots & -\delta_{ij}\mathcal{E} + E_i E_j + B_i B_j \end{pmatrix} - \mu^2 \begin{pmatrix} \frac{1}{2}(\phi^2 + |\vec{A}|^2) & \phi A_i \\ \vdots & \frac{\delta_{ij}}{2}(\phi^2 - |\vec{A}|^2) + A_i A_j \end{pmatrix},
\end{aligned} \tag{D.4.6}$$

  其中 $\mathcal{E} = \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2)$.

  calculate:

  在 Minkowski 时空中

$$\begin{cases} \mathcal{L} = \dfrac{1}{2}(|\vec{E}|^2 - |\vec{B}|^2) + \dfrac{1}{2}\mu^2(\phi^2 - |\vec{A}|^2) \\ F_0{}^\mu F_{0\mu} = -|\vec{E}|^2 \\ F_i{}^\mu F_{0\mu} = (\vec{E}\times\vec{B})_i \\ F_i{}^\mu F_{j\mu} = E_i E_j - \delta^i_j|\vec{B}|^2 + B_i B_j \end{cases}. \tag{D.4.7}$$

  – 另外注意到

$$T = -\mu^2 A^\mu A_\mu, \tag{D.4.8}$$

  可见 the energy-momentum tensor of electromagnetic field (when $m = 0$) is traceless.

Appendix E

antiunitary operator and time reversal

E.1 complex conjugation operator

- complex conjugation operator, K, is an antiunitary operator on the complex plane,

$$\begin{cases} Kz = z^* \\ zK^* = z^* \end{cases} \implies K^2 = K^{*2} = 1. \tag{E.1.1}$$

- $K^* I : V^* \to V^*$ 是 dual space 上的算符.
- 对于一组 orthonormal basis, 有

$$\langle i|K^* IK|j\rangle = \delta_{ij}, \tag{E.1.2}$$

并且可以证明在基矢变换后这个等式依然成立.

proof:

- 对基矢做 unitary transformation,

$$|i'\rangle = U|i\rangle = \sum_j |j\rangle U_{ji} \quad \text{where} \quad U_{ji} = \langle j|U|i\rangle, \tag{E.1.3}$$

那么

$$\langle i'|K^* IK|j'\rangle = \sum_{kl} \langle k|U^*_{ki} K^* IKU_{lj}|l\rangle = \sum_{kl} U_{ki} U^*_{lj} \delta_{kl} = \delta_{ij}. \tag{E.1.4}$$

- 对基矢做 antiunitary transformation, 只需要证明 $|i'\rangle = K|i\rangle$ 的情况, 此时

$$\langle i'|K^* IK|j'\rangle = \langle i|j\rangle = \delta_{ij}. \tag{E.1.5}$$

E.2 antiunitary operator

- 对于一个 unitary operator, U, $\Omega = UK$ 是一个 antiunitary operator.
- 定义其 Hermitian conjugate

$$\Omega^\dagger = K^* U^\dagger \iff \langle i|\Omega j\rangle = \langle j|\Omega^\dagger i\rangle^*, \tag{E.2.1}$$

那么

$$\begin{cases} \langle \phi|\Omega\psi\rangle = \langle \psi|\Omega^\dagger \phi\rangle^* \\ \langle \Omega\phi|\Omega\psi\rangle = \langle \psi|\phi\rangle \end{cases}. \tag{E.2.2}$$

proof:

首先,

$$\langle \phi|\Omega\psi\rangle = \sum_{ij} \langle i|\phi_i^* UK\psi_j|j\rangle$$

$$
\begin{aligned}
&= \sum_{ij} \phi_i^* \psi_j^* \langle i|UK|j\rangle \\
&= \Big(\sum_{ij} \langle j|K^* U^\dagger|i\rangle \, \phi_i \psi_j \Big)^* \\
&= \Big(\sum_{ij} \langle j|\psi_j^* K^* U^\dagger \phi_i|i\rangle \Big)^* = \langle \psi|K^* U^\dagger|\phi\rangle^* ,
\end{aligned}
\tag{E.2.3}
$$

其次,

$$
\begin{aligned}
\langle \Omega\phi|\Omega\psi\rangle = \langle \phi|\Omega^\dagger \Omega\psi\rangle = \langle \phi|K^* I K|\psi\rangle \\
&= \sum_{ij} \langle i|\phi_i^* K^* I K \psi_j|j\rangle \\
&= \sum_{ij} \phi_i \psi_j^* \langle i|K^* I K|j\rangle = \langle \psi|\phi\rangle .
\end{aligned}
\tag{E.2.4}
$$

E.3 time reversal in QM

- 在量子力学中,

$$
\mathcal{T} : |\psi\rangle \mapsto |\psi'(t')\rangle = \int d^D x \, |x\rangle \, K \, \langle x|\psi(t)\rangle , \quad \text{where} \quad t' = -t. \tag{E.3.1}
$$

  - 因此, 对于动量本征态,

$$
T\,|p\rangle = \int d^D x \, |x\rangle \, K e^{i\vec{p}\cdot\vec{x}} = |-p\rangle . \tag{E.3.2}
$$

  - 对于动量算符,

$$
TPT^\dagger = \int d^D p \, |-p\rangle \, p \, \langle -p| = -P. \tag{E.3.3}
$$

  - 对于角动量算符,

$$
TLT^\dagger = T(X \times P)T^{-1} = -L. \tag{E.3.4}
$$

- 对于平面波,

$$
\psi(t) = e^{i(\vec{k}\cdot\vec{x} - Et)} \mapsto \psi'(t') = \langle x|K^* I K|\psi(t)\rangle = e^{-i(\vec{k}\cdot\vec{x} - Et)}, \tag{E.3.5}
$$

  注意到 $t' = -t$, 代入,

$$
\psi'(t) = e^{i(-\vec{k}\cdot\vec{x} - Et)}. \tag{E.3.6}
$$

E.3.1 spin-$\frac{1}{2}$ non-relativistic electron

- 时间反演算符作用到 spin-up state 应该得到 spin-down state, 所以

$$
T = \sigma_2 K. \tag{E.3.7}
$$

  - 因此

$$
T^2 = \sigma_2 K \sigma_2 K = \sigma_2^* \sigma_2 = -1. \tag{E.3.8}
$$

  - 具体地,

$$
T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix}. \tag{E.3.9}
$$

- **Kramer's degeneracy:** 含有奇数个电子的时间反演不变系统, 其能级是 twofold degenerate.

proof:

因为系统时间反演不变, 所以 ψ 和 $T\psi$ 有相同的能级, 且 $T\psi \neq e^{i\alpha}\psi, \forall\alpha$.

考虑 $T\psi = e^{i\alpha}\psi$, 那么

$$
T^2\psi = T e^{i\alpha}\psi = e^{-i\alpha} e^{i\alpha}\psi = \psi, \tag{E.3.10}
$$

与 $T^2 = -1$ 矛盾.