Quantum Field Theory

- based on A. Zee's textbook -

万思扬

August 16, 2024

Contents

笔记中的**度规号差**约定为 $(-,+,+,+)$.

Part I

Scalar Quantum Field

Chapter 1

free field

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{1.0.1}$$

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi e^{i\int d^dx(-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \tag{1.0.2}$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

Appendices

Appendix A

Dirac delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff 1 = \int dx\, \delta(x) e^{-ikx} \tag{A.0.1}$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \tag{A.0.2}$$

- 第二个常用的公式 (Sokhotski-Plemelj theorem),

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{A.0.3}$$

其中 $\mathcal{P}$ 表示复函数的主值 (principal value).

proof:

考虑,

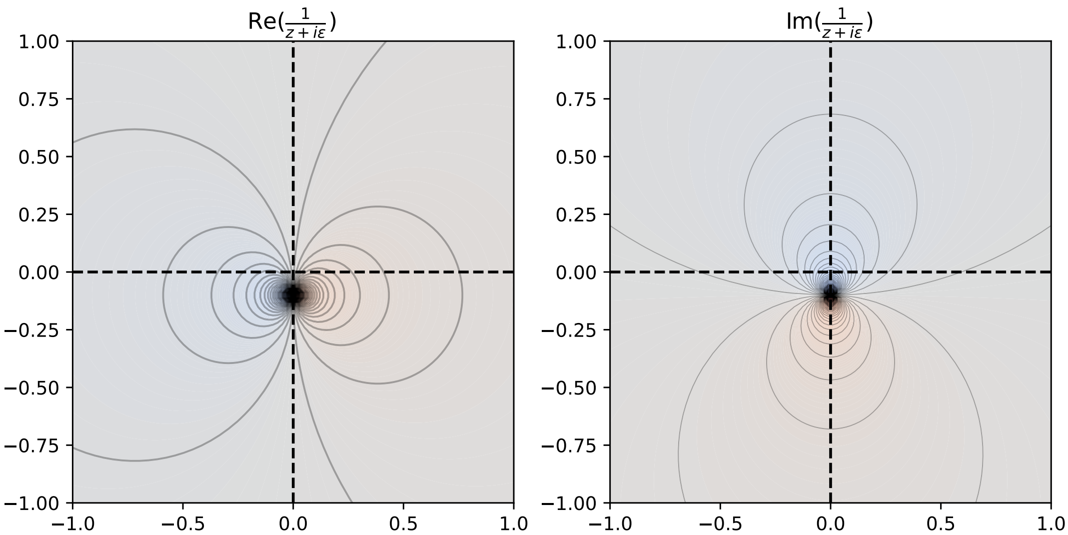
$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \tag{A.0.4}$$

且注意到,

$$\int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \operatorname{Res}(f, i\epsilon) = \pi \tag{A.0.5}$$

所以...

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,



Appendix B

Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{B.0.1}$$

$$\langle x^{2n}\rangle = \frac{\int dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!! \tag{B.0.2}$$

其中 $(2n-1)!! = 1\cdot 3\cdots(2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx\, e^{-\frac{a}{2}x^2+Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \tag{B.0.3}$$

另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 N-dim. generalization

- 考虑如下积分,

$$Z(A,J) = \int dx_1\cdots dx_N\, e^{-\frac{1}{2}x^T\cdot A\cdot x+J^T\cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \tag{B.1.1}$$

其中 x, J 是 N-dim. 列向量, A 是 $N\times N$ 实对称矩阵.

calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1}\cdot D\cdot O \tag{B.1.2}$$

其中 D 是一个 diagonal matrix. 令 $y = O\cdot x$, 那么,

$$\begin{aligned} Z(A,J) &= \int dy_1\cdots dy_N\, e^{-\frac{1}{2}y^T\cdot D\cdot y+(OJ)^T\cdot y} \\ &= \prod_{i=1}^{N}\sqrt{\frac{2\pi}{D_{ii}}} e^{\frac{1}{2D_{ii}}(OJ)_i{}^2} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \end{aligned} \tag{B.1.3}$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O\cdot A^{-1}\cdot O^{-1})_{ii}$ 以及 $\operatorname{tr} D = \det A$.

- 一个重要的变体是 $A\mapsto -iA, J\mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j\rangle = \frac{1}{Z(A,0)}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}Z(A,J)\Big|_{J=0} = A^{-1}_{ij} \tag{B.1.4}$$

$$\langle x_i x_j\cdots x_k x_l\rangle = \sum_{\text{Wick}} A^{-1}_{i'j'}\cdots A^{-1}_{k'l'} \tag{B.1.5}$$

其中 (B.1.5) 中有偶数个 x, 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A,J) \Big|_{J=0} = \cdots \tag{B.1.6}$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} \tag{B.1.7}$$

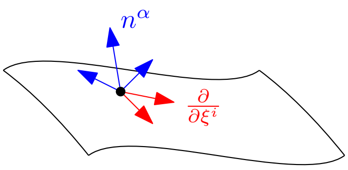
其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ik} A^{-1}_{jl} \tag{B.1.8}$$

Appendix C

the $m_1 + m_2$ decomposition of spacetime

- 将 n-dim. 流形分解为 $m_1 + m_2$ 维 $(m_1 + m_2 = n)$, 其中 m_2 是超曲面的维数.
- 选取与超曲面" 适配" 的坐标, $\{\chi^1, \cdots, \chi^{m_1}, \xi^1, \cdots, \xi^{m_2}\}$, 即超曲面上的点的前 m_1 个坐标值为常数.
  - 用 $\alpha, \beta, \gamma = 1, \cdots, m_1$, 以及 $i, j, k = m_1 + 1, \cdots, n$.



C.1 induced metric

- 对 $d\chi^1, \cdots, d\chi^{m_1}$ 进行 Schmidt 正交化并归一化, 得到 $n^1, \cdots, n^{m_1}$, 有,
$$(n^\alpha)^a (n^\alpha)_a = \epsilon^\alpha = \pm 1 \tag{C.1.1}$$
- 投影张量为,
$$h^a{}_b = \delta^a_b - \sum_\alpha \epsilon^\alpha (n^\alpha)^a (n^\alpha)_b \tag{C.1.2}$$
- 因此, 诱导度规为,
$$h_{ab} = h^c{}_a h^d{}_b g_{cd} = g_{ab} - \sum_\alpha \epsilon^\alpha (n^\alpha)_a (n^\alpha)_b \tag{C.1.3}$$
  - 另外,
$$h^{ab} = g^{ab} - \sum_\alpha \epsilon^\alpha (n^\alpha)^a (n^\alpha)^b \tag{C.1.4}$$
  - 且有 $h^{ac} h_{bc} = h^a{}_b$.

C.2 the decomposition

- **def.:** 定义系数 $N^{\alpha\beta}, M^{\alpha i}$ 如下,
$$\left(\frac{\partial}{\partial \chi^\alpha}\right)^a = \sum_\beta N^{\alpha\beta} (n^\beta)^a + \sum_i M^{\alpha i} \left(\frac{\partial}{\partial \xi^i}\right)^a \tag{C.2.1}$$
  - 有,
$$N^{\alpha\beta} = \epsilon^\beta (n^\beta)_a \left(\frac{\partial}{\partial \chi^\alpha}\right)^a \tag{C.2.2}$$
- 那么,
$$g_{\mu\nu} = \begin{pmatrix} \epsilon^\gamma N^{\alpha\gamma} N^{\beta\gamma} + M^{\alpha i} M^{\beta j} g_{ij} & \{M^{\alpha j} g_{ji}\}^T \\ M^{\alpha j} g_{ji} & g_{ij} \end{pmatrix} \tag{C.2.3}$$
$$\Longrightarrow \det\{g_{\mu\nu}\} = \det\{\epsilon^\gamma N^{\alpha\gamma} N^{\beta\gamma}\} \det\{g_{ij}\} \tag{C.2.4}$$

calculation:

注意到 $(n^\alpha)_a\left(\frac{\partial}{\partial\xi^i}\right)^a=0$, 且 $g_{ab}(n^\alpha)^a(n^\beta)^b=\epsilon^\alpha\delta_{\alpha\beta}$, 所以...

C.3 induced volume form

- **def.:** 令,

$$g=|\det\{g_{\mu\nu}\}|\quad \epsilon N^2=\det\{\epsilon^\gamma N^{\alpha\gamma}N^{\beta\gamma}\}\quad h=|\det\{h_{ij}\}| \tag{C.3.1}$$

其中, 根据 (C.1.3), 有 $h_{ij}=g_{ij}$.

- 那么,

$$g=N^2h \iff \sqrt{h}=\frac{\sqrt{g}}{N} \tag{C.3.2}$$

- the induced volume form is,

$$\begin{aligned}\tilde{\epsilon}&=\sqrt{h}\underbrace{{h_{a_1}}^{b_1}(d\xi^1)_{b_1}}_{=(d\xi^1)_{a_1}-\sum_\alpha\epsilon^\alpha(n^\alpha)_{a_1}(n^\alpha)^{b_1}(d\xi^1)_{b_1}}\wedge\cdots\wedge {h_{a_{m_2}}}^{b_{m_2}}(d\xi^{m_2})_{b_{m_2}}\\&=\frac{\sqrt{g}}{N}\Big((d\xi^1)_{b_1}\wedge\cdots\wedge(d\xi^{m_2})_{b_{m_2}}-\big(\epsilon^\alpha(n^\alpha)^{b_1}(d\xi^1)_{b_1}\big)(n^\alpha)_{a_1}\wedge\cdots\wedge(d\xi^{m_2})_{b_{m_2}}-\cdots\Big)\end{aligned} \tag{C.3.3}$$

注意到,

$$\epsilon=\sqrt{g}\underbrace{d\chi^1\wedge\cdots d\chi^{m_1}}_{=\frac{1}{N}n^1\wedge\cdots\wedge n^{m_1}}\wedge d\xi^1\wedge\cdots\wedge d\xi^{m_2} \tag{C.3.4}$$

对比 (C.3.3) 和 (C.3.4), 所以,

$$\epsilon=n^1\wedge\cdots\wedge n^{m_1}\wedge\tilde{\epsilon}\Longrightarrow\tilde{\epsilon}=\frac{\prod_\alpha\epsilon^\alpha}{m_1!}(n^1)^{b_1}\wedge\cdots\wedge(n^{m_1})^{b_{m_1}}\epsilon_{b_1\cdots b_{m_1}a_1\cdots a_{m_2}} \tag{C.3.5}$$

  – 其中, 我们还需要证明 $d\chi^1\wedge\cdots d\chi^{m_1}=\frac{1}{N}n^1\wedge\cdots\wedge n^{m_1}$.

proof:

根据 (C.2.2), 可知 Schmidt 正交化并归一化的系数为,

$$n^\alpha=\sum_\beta N^{\alpha\beta}d\chi^\beta \tag{C.3.6}$$

$N^{\alpha\beta}$ 的两条性质 (并不重要),

∗ 考虑到归一化条件, 有,

$$\sum_{\gamma,\delta}g^{\gamma\delta}N^{\alpha\gamma}N^{\beta\delta}=\epsilon^\alpha\delta_{\alpha\beta}\iff N\cdot\{g^{\alpha\beta}\}\cdot N^T=\begin{pmatrix}\epsilon^1&&\\&\ddots&\\&&\epsilon^{m_1}\end{pmatrix} \tag{C.3.7}$$

∗ 另外, 注意到 (C.2.2), 有,

$$\begin{aligned}N^{\alpha\beta}&=\epsilon^\beta\sum_\gamma N^{\beta\gamma}(d\chi^\gamma)_a\left(\frac{\partial}{\partial\chi^\alpha}\right)^a\\\Longrightarrow N^T&=\mathrm{diag}(\epsilon^1,\cdots,\epsilon^{m_1})\cdot N\end{aligned} \tag{C.3.8}$$

所以,

$$\begin{aligned}n^1\wedge\cdots\wedge n^{m_1}&=N^{1\alpha_1}d\chi^{\alpha_1}\wedge\cdots\wedge N^{m_1\alpha_{m_1}}d\chi^{\alpha_{m_1}}\\&=\underbrace{\det N}_{=N}\,d\chi^1\wedge\cdots\wedge d\chi^{m_1}\end{aligned} \tag{C.3.9}$$

Appendix D

classical field theory and Noether's theorem

D.1 classical field theory

D.1.1 Lagrangian density and the action

- Lagrangian density, $\mathcal{L}$, 是 $\phi^a(x), \partial_\mu\phi^a(x), t$ 的函数.
- 对作用量变分得到 Euler-Lagrangian equation of motion,

$$\frac{\partial\mathcal{L}}{\partial\phi^a} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\Big) = 0 \tag{D.1.1}$$

calculation:

对作用量进行变分,

$$\begin{aligned}\delta S &= \int d^4x\Big(\frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\partial_\mu\delta\phi^a\Big)\\ &= \int d^4x\bigg(\Big(\frac{\partial\mathcal{L}}{\partial\phi^a} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\Big)\Big)\delta\phi^a + \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta\phi^a\Big)\bigg)\end{aligned} \tag{D.1.2}$$

由于边界变分为零...

D.1.2 canonical momentum and the Hamiltonian

- **def.:** 定义一个叫 π^μ_a 的量,

$$\pi^\mu_a = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)} \tag{D.1.3}$$

其中 $\pi_a \equiv \pi^0_a$ 称作 canonical momentum of the field.

- **def.:** the Hamiltonian density is,

$$\mathcal{H} = \pi_a\partial_0\phi^a - \mathcal{L} \tag{D.1.4}$$

- the Hamilton's equations are,

$$\begin{cases}\partial_0\phi^a = \dfrac{\partial\mathcal{H}}{\partial\pi_a}\\ -\partial_0\pi^a = \dfrac{\partial\mathcal{H}}{\partial\phi^a} - \partial_i\Big(\dfrac{\partial\mathcal{H}}{\partial(\partial_i\phi^a)}\Big)\end{cases} \tag{D.1.5}$$

  - 第二个方程可以写成更紧凑的形式,

$$\partial_\mu\pi^\mu_a = \frac{\partial\mathcal{H}}{\partial\phi^a} \tag{D.1.6}$$

D.2 Noether's theorem

D.2.1 in classical particle mechanics

- 系统的 Lagrangian 为 $L(q^a, \dot{q}^a, t)$.
- 系统通过以下形式变换,

$$q^a(t) \mapsto q^a(\lambda, t) \quad \text{and} \quad q^a(t, 0) = q^a(t) \tag{D.2.1}$$

  并定义,

$$D_\lambda q^a = \frac{\partial q^a}{\partial \lambda}\Big|_{\lambda=0} \tag{D.2.2}$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.,

$$D_\lambda L = \frac{dF(q^a, \dot{q}^a, t)}{dt} \tag{D.2.3}$$

  for some $F(q^a, \dot{q}^a, t)$, and the corresponding **conserved quantity** is,

$$Q = p_a D_\lambda q^a - F(q^a, \dot{q}^a, t) \tag{D.2.4}$$

proof:

$$D_\lambda L = \frac{\partial L}{\partial q^a} D_\lambda q^a + \frac{\partial L}{\partial \dot{q}^a}\frac{dD_\lambda q^a}{dt} = \frac{d}{dt}(p_a D_\lambda q^a) \tag{D.2.5}$$

- 几个例子如下,
  - **空间平移**, $\vec{x}(t) \mapsto \vec{x}(t) + \hat{e}_i\lambda$, 相应地, $D_\lambda \vec{x} = \hat{e}_i$, 且,

$$D_\lambda L = \frac{\partial L}{\partial x^i} \tag{D.2.6}$$

    如果 $\frac{\partial L}{\partial x^i} = 0$, 那么, 有守恒量 p_i.
  - **时间平移**, $q^a(t) \mapsto q^a(t+\lambda)$, 相应地, $D_\lambda q^a = \dot{q}^a$, 且,

$$D_\lambda L = \frac{dL}{dt} - \frac{\partial L}{\partial t} \tag{D.2.7}$$

    如果 $\frac{\partial L}{\partial t} = 0$, 那么, 有守恒量 $H = p_a \dot{q}^a - L$.
  - **转动**, $\vec{x}(t) \mapsto R(\lambda, \hat{e}) \cdot \vec{x}(t)$, 相应地, $D_\lambda \vec{x} = \hat{e} \times \vec{x}$, 且,

$$D_\lambda L = \vec{x} \cdot \left(\frac{\partial L}{\partial \vec{x}} \times \hat{e}\right) + \hat{e}(\dot{\vec{x}} \times \vec{p}) \tag{D.2.8}$$

    如果上式中两个括号内的项都为零, 那么, 有守恒量 $\hat{e} \cdot \vec{J} = \hat{e} \cdot (\vec{x} \times \vec{p})$.

D.2.2 in classical field theory

- 类似地, 系统通过以下形式变换,

$$\phi^a(x) \mapsto \phi^a(x, \lambda) \quad \text{and} \quad \phi^a(x, 0) = \phi^a(x) \tag{D.2.9}$$

  并定义,

$$D_\lambda \phi^a = \frac{\partial \phi^a}{\partial \lambda}\Big|_{\lambda=0} \tag{D.2.10}$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.,

$$D_\lambda \mathcal{L} = \partial_\mu F^\mu(\phi^a, \partial_\mu \phi^a, t) \tag{D.2.11}$$

  for some $F^\mu(\phi^a, \partial_\mu \phi^a, t)$, and the **conserved current** is,

$$J^\mu = \pi^\mu_a D_\lambda \phi^a - F^\mu \tag{D.2.12}$$

proof:

$$
\begin{aligned}
D_\lambda \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi^a} D_\lambda \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\mu D_\lambda \phi^a \\
&= \Big(\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \Big) \Big) D_\lambda \phi^a + \partial_\mu \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}}_{=\pi_a^\mu} D_\lambda \phi^a \Big)
\end{aligned}
\tag{D.2.13}
$$

代入 (D.1.1), 得...

- 注意, conserved current 并不是唯一确定的, 考虑如下变换,

$$
F^\mu \mapsto F'^\mu = F^\mu + \partial_\nu A^{\mu\nu} \quad \text{with} \quad A^{\mu\nu} = A^{[\mu\nu]} \tag{D.2.14}
$$

新 F'^μ 依然能满足 (D.2.11).

- 但是, 守恒荷是唯一确定的.

proof:

$$
Q' = \int d^3x J^0 = \int d^3x (\pi_a D_\lambda \phi^a - F^0) - \int d^3x \, \partial_\mu A^{0\mu} \tag{D.2.15}
$$

考虑到边界值为零, 且 $A^{00} = 0$, 所以 $Q' = Q$.

D.2.3 spacetime translations and the energy-momentum tensor

- 时空平移变换为,

$$
\phi^a(x) \mapsto \phi^a(x + \lambda e) \tag{D.2.16}
$$

- 所以,

$$
D_\lambda \phi^a = e^\mu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = e^\mu \partial_\mu \mathcal{L} \tag{D.2.17}
$$

代入 (D.2.12),

$$
J^\mu = e^\nu \underbrace{(\pi_a^\mu \partial_\nu \phi^a - \delta_\nu^\mu \mathcal{L})}_{=T^\mu{}_\nu} \tag{D.2.18}
$$

- 并且有,

$$
\partial_\mu T^{\mu\nu} = 0 \Longrightarrow P^\mu = \int d^3x \, T^{0\mu} = \text{Const.} \tag{D.2.19}
$$

来自守恒流散度为零.

D.2.4 Lorentz transformations, angular momentum and something else

- Lorentz transformation 下坐标做变换 $x'^\mu = \Lambda^\mu_\nu x^\nu$, 其中 Λ 满足,

$$
\eta = \Lambda^T \eta \Lambda \tag{D.2.20}
$$

- infinitesimal Lorentz transformation 是,

$$
\Lambda = I + \epsilon \tag{D.2.21}
$$

其中 $\{\epsilon^{\mu\nu}\} = \epsilon\eta$ 是反对称矩阵.

proof:

考虑,

$$
\eta = (\Lambda\eta)^T \eta (\Lambda\eta) = (\eta + \epsilon\eta)^T \eta (\eta + \epsilon\eta)
$$

$$= \eta + \eta\epsilon^T + \epsilon\eta + O(\epsilon^2) \tag{D.2.22}$$

- 标量场在 Lorentz transform 下的变换为,

$$\Lambda : \phi^a(x) \mapsto \phi^a(\Lambda^{-1}x') \tag{D.2.23}$$

- 有,

$$D_\lambda \phi^a = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\epsilon_{\mu\nu}\partial^\mu(x^\nu \mathcal{L}) \tag{D.2.24}$$

代入 (D.2.12),

$$J^\mu = \frac{1}{2}\epsilon_{\nu\rho} M^{\mu\nu\rho} \quad \text{where} \quad M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{D.2.25}$$

且有,

$$\partial_\mu M^{\mu\nu\rho} = 0 \tag{D.2.26}$$

- 对全空间积分, 得到 6 个守恒量,

$$J^{\mu\nu} = \int d^3x\, M^{0\mu\nu} = \text{Const.} \tag{D.2.27}$$

不难发现 J^{ij} 对应角动量, 现在来讨论 J^{0i} 的物理意义,

$$0 = \frac{d}{dt}J^{0i} = \frac{d}{dt}\int d^3x (tT^{0i} - x^i T^{00}) = P^i - \frac{d}{dt}\int d^3x\, x^i T^{00} \tag{D.2.28}$$

其中, 用到了 $\frac{dP^i}{dt} = 0$ (见 (D.2.19)), 可以将上式的第二项理解为质心运动的动量.