Quantum Field Theory

- based on A. Zee's textbook -

万思扬

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Contents

convention, notation, and units

- 笔记中的**度规号差**约定为 $(-,+,+,+)$.
- 使用 Planck units, 此时 $G,\hbar,c,k_B=1$, 因此,

name/dimension	expression/value
Planck length (L)	$l_P=\sqrt{\frac{\hbar G}{c^3}}=1.616\times10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P=\frac{l_P}{c}=5.391\times10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P=\sqrt{\frac{\hbar c}{G}}=2.176\times10^{-8}\,\mathrm{kg}\simeq10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P=\sqrt{\frac{\hbar c^5}{Gk_B^2}}=1.417\times10^{32}\,\mathrm{K}$

- 时空维度用 $d=D+1$ 表示.

Part I

motivation and foundation

Chapter 1

free field theory

1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{1.1.1}$$

A. Zee 说: 在作用量里, 时间的导数项必须是正的, 包括标量场的 $(\partial_0\phi)^2$ 和电磁场的 $(\partial_0 A_i)^2$.

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi\, e^{i\int d^dx\,(-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \tag{1.1.2}$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J(x)D(x-y)J(y)} \tag{1.1.3}$$

另外, 用 $W(J)$ 表示指数上的部分 (去除掉虚数 i).

proof:

注意 $\partial^\mu\phi\partial_\mu\phi = \partial^\mu(\phi\partial_\mu\phi) - \phi\partial^2\phi$, 忽略全微分项, 那么,

$$Z(J) = \int D\phi\, e^{i\int d^dx\,(\frac{1}{2}\phi(\partial^2 - m^2)\phi + J(x)\phi(x))} \tag{1.1.4}$$

代入 (B.1.1), 可知,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J(x)D(x-y)J(y)} \tag{1.1.5}$$

其中 $D(x-y)$ 满足,

$$\begin{cases}(\partial^2 - m^2)D(x-y) = \delta^{(d)}(x-y) \\ (-p^2 - m^2)\tilde{D}(p,q) = (2\pi)^d\delta^{(d)}(p-q)\end{cases} \Longrightarrow D(x-y) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot(x-y)}}{-k^2 - m^2} \tag{1.1.6}$$

1.2 free propagator

- 为了使 (1.1.4) 中的积分在 ϕ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon\int d^dx\phi^2}$.
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 ϵ 避免发散,

$$D(x) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot x}}{-k^2 - m^2 + i\epsilon} = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\Big(e^{i(-\omega_k t + \vec{k}\cdot\vec{x})}\theta(t) + e^{i(\omega_k t + \vec{k}\cdot\vec{x})}\theta(-t)\Big) \tag{1.2.1}$$

calculation:

对 k^0 积分, 注意有两个奇点 $k^0 = \pm(\omega_k - i\epsilon)$, 当 $t > 0$ 时, contour 处于下半平面, ...

- $D(x)$ 的取值与 x 的类时, 类空性质关系密切.
  - 类时区域,
$$D(t,0) = -i\int \frac{d^D k}{(2\pi)^D 2\omega_k}\left(e^{-i\omega_k t}\theta(t) + e^{i\omega_k t}\theta(-t)\right) \tag{1.2.2}$$
  - 类空区域,
$$D(0,\vec{x}) = -i\int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \sim e^{-m|\vec{x}|} \tag{1.2.3}$$

1.3 from field to particle to force

1.3.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,
$$\begin{aligned} W(J) &= -\frac{1}{2}\int d^d x d^d y\, J(y)D(x-y)J(y) &(1.3.1)\\ &= -\frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\tilde{J}(-k)\frac{1}{-k^2 - m^2 + i\epsilon}\tilde{J}(k) &(1.3.2)\end{aligned}$$
  其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k) = \tilde{J}^*(k)$.

- 考虑 $J(x) = J_1(x) + J_2(x)$, 那么 $W(J)$ 共有 4 项, 其中一个交叉项如下,
$$W_{12}(J) = -\frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\tilde{J}_1(-k)\frac{1}{-k^2 - m^2 + i\epsilon}\tilde{J}_2(k) \tag{1.3.3}$$
  可见 $W(J)$ 取值较大的条件是:
  1. $\tilde{J}_1(k), \tilde{J}_2(k)$ 有较大重叠,
  2. 重叠位置的 k 是 on shell (即 $k^2 = -m^2$).
- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

1.3.2 from particle to force

- 考虑 $J(x) = \delta^{(D)}(\vec{x} - \vec{x}_1) + \delta^{(D)}(\vec{x} - \vec{x}_1) \Longrightarrow \tilde{J}_a(k) = 2\pi e^{-i\vec{k}\cdot\vec{x}_a}\delta(k^0)$, 那么,
$$\begin{aligned} W_{12}(J) + W_{21}(J) &= \delta(0)\int \frac{d^D k}{(2\pi)^{D-1}}\frac{1}{|\vec{k}|^2 + m^2 - i\epsilon}\cos(\vec{k}\cdot(\vec{x}_1 - \vec{x}_2))\\ &\overset{D=3}{=} 2\pi\delta(0)\frac{1}{4\pi r}e^{-mr} &(1.3.4)\end{aligned}$$
  ($-i\epsilon$ 显然可以舍去), 注意到 $\langle 0|e^{-iHT}|0\rangle = e^{-iET}$, 而时间间隔 $T = \int dx^0 = 2\pi\delta(0)$, 所以,
$$E = -\frac{W(J)}{T} \overset{D=3}{=} -\frac{1}{4\pi r}e^{-mr} \tag{1.3.5}$$

calculation:

计算 (1.3.4) 中的积分, 令 $\vec{x}_1 - \vec{x}_2 = \vec{r}$,
$$I_D = \int \frac{d^D k}{(2\pi)^D}\frac{1}{|\vec{k}|^2 + m^2}\overbrace{\cos(\vec{k}\cdot\vec{r})}^{\mapsto e^{i\vec{k}\cdot\vec{r}}}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^D}\int (k\sin\theta_1)^{D-2}d\Omega_{D-2}\int kd\theta_1 dk\,\frac{1}{k^2+m^2}e^{ikr\cos\theta_1} \\
&= \frac{S_{D-2}}{(2\pi)^D}\int k^{D-1}\sin^{D-2}\theta_1 d\theta_1 dk\,\frac{1}{k^2+m^2}e^{ikr\cos\theta_1} \qquad (1.3.6)
\end{aligned}
$$

取 $D=3$, 那么,

$$
\begin{aligned}
I_{D=3} &= \frac{1}{(2\pi)^2}\int k^2\sin\theta_1 d\theta_1 dk\,\frac{1}{k^2+m^2}e^{ik\cos\theta_1} \\
&= \frac{1}{2\pi^2 r}\int_0^\infty \sin(kr)\frac{kdk}{k^2+m^2} = \frac{-i}{4\pi^2 r}\int_{-\infty}^\infty e^{ikr}\frac{kdk}{k^2+m^2} \\
&= \frac{-i}{4\pi^2 r}2\pi i\,\underbrace{\mathrm{Res}(f,im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r}e^{-mr} \qquad (1.3.7)
\end{aligned}
$$

Chapter 2

Coulomb and Newton: repulsive and attraction

2.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{2.1.1}$$

  其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

- 做路径积分,

$$Z(J) = \int DA\, e^{i\int d^dx(\mathcal{L}+J_\mu A^\mu)} = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J_\mu D^{\mu\nu}(x-y)J_\nu(y)} \tag{2.1.2}$$

calculation:

massive photon 的作用量为,

$$\begin{aligned} S(A) &= \int d^dx\,\frac{1}{2}\Big(-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu\Big) \\ &= \int d^dx\,\frac{1}{2}\Big(A_\nu\partial^2 A^\nu - A_\nu\partial^\nu\partial_\mu A^\mu - m^2 A_\mu A^\mu\Big) + \text{total differential} \\ &= \int d^dx\,\frac{1}{2}A_\mu\Big(-\partial^\mu\partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2)\Big)A_\nu + \text{total differential} \\ &= \int \frac{d^dk}{(2\pi)^d}\tilde{A}_\mu(-k)\Big(k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2)\Big)\tilde{A}_\nu(k) + \text{boundary term} \end{aligned} \tag{2.1.3}$$

那么, 需要有,

$$\begin{aligned} &(-\partial^\mu\partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2))D_{\rho\nu}(x-y) = \delta^\mu_\nu\delta^{(d)}(x-y) \\ \Longrightarrow &\tilde{D}_{\mu\nu}(k) = \frac{k_\mu k_\nu/m^2 + \eta_{\mu\nu}}{-k^2 - m^2} \end{aligned} \tag{2.1.4}$$

考虑到积分需要收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, (为什么 A_μ 类空, 只知道 $\tilde{A}_\mu$ 类空, 见 subsection 2.1.2, 但路径积分中的 A 显然不满足 field equation $\Longrightarrow$ 路径积分中起主要作用的 $\tilde{A}$ 类空, 因此 $-\epsilon|\tilde{A}|^2 < 0$).

- 因此,

$$\begin{aligned} W(J) &= -\frac{1}{2}\int d^dx d^dy\, J_\mu(x)D^{\mu\nu}(x-y)J_\nu(y) &(2.1.5)\\ &= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}_\mu(-k)\frac{k^\mu k^\nu/m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon}\tilde{J}_\nu(k) &(2.1.6) \end{aligned}$$

  注意到 current conservation, 有 $\partial_\mu J^\mu = 0 \iff k^\mu\tilde{J}_\mu(k) = 0$, 所以,

$$W(J) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}^\mu(-k)\frac{1}{-k^2 - m^2 + i\epsilon}\tilde{J}_\mu(k) \tag{2.1.7}$$

观察电荷分量, 可见同性相斥, 异性相吸.

2.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的 x, y, z 方向, 在粒子静止系下, 极化矢量 $(\epsilon^i)_\mu = \delta^i_\mu, i = 1, 2, 3$, 而 $k_\mu = (-m, 0, 0, 0)$, 所以,
$$k^\mu(\epsilon^i)_\mu = 0 \tag{2.1.8}$$
  - 注意, 一个粒子的极化方向用 e^i (这不是矢量) 表示, 极化矢量为 $\sum_{i=1}^3 e^i(\epsilon^i)_\mu$.
- 在粒子静止系下, 考虑,
$$\sum_{i=1}^{3}(\epsilon^i)_\mu(\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu} \tag{2.1.9}$$
可见,
$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3(\epsilon^i)_\mu(\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon} \tag{2.1.10}$$

2.1.2 Maxwell Lagrangian

- 根据 (2.1.1) 中的 Lagrangian density, 得到 field equation 如下,
$$\Big(-\partial^\mu\partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2)\Big)A_\nu \tag{2.1.11}$$
  - spin-1 particle 有 3 个自旋自由度, 而 A_μ 有 4 个分量, 所以需要一个约束方程,
$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0 \tag{2.1.12}$$
  实际上在 (2.1.11) 左右两边作用一个 ∂_μ 即可得到这个约束方程.

2.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = linearized Einstein Lagrangian.
- 受 subsection 2.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足,
$$\begin{cases} k^\mu(\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu}(\epsilon^a)_{(\mu\nu)} = 0 \end{cases} \tag{2.2.1}$$
其中下指标 μ, ν 对称, $a = 1, \cdots, 5$, (可以验证 $(\epsilon^a)_{\mu\nu}$ 确实有 5 个独立分量).
  - 对 $(\epsilon^a)_{\mu\nu}$ 的归一化条件可以定义为 $\sum_{a=1}^5(\epsilon^a)_{12}(\epsilon^a)_{12} = 1$.
  - 与 subsection 2.1.1 中提示一样, 粒子的极化方向用 e^a 表示.
- 那么,
$$\sum_{a=1}^{5}(\epsilon^a)_{\mu\nu}(\epsilon^a)_{\rho\sigma} = (G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma} \tag{2.2.2}$$

calculation:

首先用 k_μ 和 $\eta_{\mu\nu}$ 构造最一般的关于 $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$ 对称的 4 阶张量, (下式中把 $\frac{k_\mu}{m}$ 略写作 k_μ),
$$\begin{aligned} &Ak_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ &+ D\eta_{\mu\nu}\eta_{\rho\sigma} + E(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \end{aligned} \tag{2.2.3}$$
代入 (2.2.1) 得,
$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4} \tag{2.2.4}$$

因此, 这个 4 阶张量最终确定为,

$$\frac{3}{4}A\Big((G_{\mu\rho}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\rho})-\frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\Big) \tag{2.2.5}$$

- 所以,

$$\tilde{D}_{\mu\nu\rho\sigma}(k)=\frac{(G_{\mu\rho}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\rho})-\frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2-m^2+i\epsilon} \tag{2.2.6}$$

- 计算路径积分中的 $W(T)$,

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{(G^{\mu\rho}G^{\nu\sigma}+G^{\mu\sigma}G^{\nu\rho})-\frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2-m^2+i\epsilon}\tilde{T}_{\rho\sigma}(k) \tag{2.2.7}$$

注意到 $\partial_\mu T^{\mu\nu}(x)=0 \iff k_\mu\tilde{T}^{\mu\nu}(k)=0$, 并考虑到 T 是对称张量, 所以,

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{2\eta^{\mu\rho}\eta^{\nu\sigma}-\frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2-m^2+i\epsilon}\tilde{T}_{\rho\sigma}(k) \tag{2.2.8}$$

考虑能量项, 可见质量互相吸引.

2.3 remarks

- 由于 seesaw mechanism (见 subsection C.1.1), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为 $\eta^{00}=-1$.
- 本 chapter 中的计算都是 $m\neq 0$ 的粒子, 与真实世界有差异.

Chapter 3

Feynman diagrams

3.1 a baby problem

- 考虑如下积分,

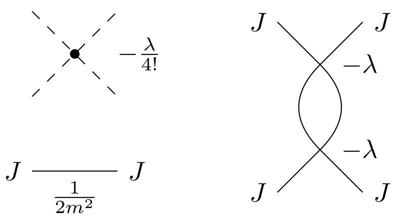
$$Z(J) = \int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq} \tag{3.1.1}$$

- **Schwinger's way:** 把 integrand 对 λ 展开, 并将 q 用 $\frac{\partial}{\partial J}$ 替代, 得到,

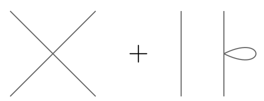
$$\begin{aligned} Z(J) &= e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4} \int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2 + Jq} \\ &= \sqrt{\frac{2\pi}{m^2}} e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4} e^{\frac{J^2}{2m^2}} \end{aligned} \tag{3.1.2}$$

  后面的计算中忽略 $Z(J=0, \lambda=0)$.

- 每个 vertex 带有 $-\lambda$, 每个 line 带有 $\frac{1}{m^2}$, 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里 q 无法区分, 所以不方便算, 先略了),



$$= \overbrace{J^4}^{\text{4 external ends}} \frac{1}{2!}\underbrace{\left(-\frac{\lambda}{4!}\right)^2}_{\text{2 vertices}} \frac{1}{6!}\underbrace{\left(\frac{1}{2m^2}\right)^6}_{\text{6 lines}} \times \underbrace{\text{numerical factor}}_{\text{不会}}$$

> **calculation:**
>
> 在这里计算 λJ^4 项,
>
>  具体暂时不会算 (3.1.3)
>
> 但是直接计算 (3.1.2) 的展开项, 得到的结果与 (3.1.5) 一样.

3.1.1 Wick contraction and Green's functions

- 把积分 (3.1.1) 对 J 展开,

$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} J^n \underbrace{\int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4} q^n}_{=Z(0,0)G^{(n)}} \tag{3.1.4}$$

  其中 Green's function $G^{(n)}$ 对 λ 展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

calculation:

计算 λJ^4 项, 它来自 $G^{(4)}$ 对 λ 展开的一阶项,

$$
\begin{aligned}
-\frac{\lambda}{4!}\int dq\, e^{-\frac{1}{2}m^2q^2}q^8 &= -\frac{\lambda}{4!}\langle q^8\rangle \\
&= -\frac{\lambda}{4!}\sum_{\text{Wick}}\left(\frac{1}{m^2}\right)^4 \\
&= -\frac{\lambda}{4!}\frac{7\times 5\times 3\times 1}{m^8}
\end{aligned}
\tag{3.1.5}
$$

所以 λJ^4 项等于 $\frac{105}{(4!)^2}\frac{-\lambda J^4}{m^8}$.

3.1.2 connected vs. disconnected

- 考虑,
$$
Z(J,\lambda) = Z(J=0,\lambda)e^{W(J,\lambda)} \tag{3.1.6}
$$
其中 $Z(J=0,\lambda)$ 由 diagrams with no external source J 组成, 而 $W(J,\lambda)$ 则由 connected diagrams 组成 (?).
- 我们希望计算的是 W, 而不是 Z (?).

3.2 a child problem

- 考虑如下积分,
$$
Z(J) = \int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q - \frac{\lambda}{4!}q^4 + J^T\cdot q} \tag{3.2.1}
$$
其中 $q^4 = \sum_i q_i^4$.
- **Schwinger's way:** 对 λ 展开并把 q 替换为 $\frac{\partial}{\partial J}$, 得到,
$$
Z(J) = \sqrt{\frac{(2\pi)^N}{\det A}}e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \tag{3.2.2}
$$
其中 $\left(\frac{\partial}{\partial J}\right)^4 = \sum_i\left(\frac{\partial}{\partial J_i}\right)^4$.

3.2.1 n-point Green's function

- **Wick's way:** 对 J 展开获得带 Green's function 的表达式,
$$
Z(J) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}J_{i_1}\cdots J_{i_n}\underbrace{\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q-\frac{\lambda}{4!}q^4}q_{i_1}\cdots q_{i_n}}_{=Z(0,0)G^{(n)}_{i_1\cdots i_n}} \tag{3.2.3}
$$
其中 $G^{(n)}_{i_1\cdots i_n}$ 称为 n-point Green's function.

Taylor expansion:

多元函数的 Taylor 展开如下,

$$
\begin{aligned}
f(x_1,\cdots,x_N) &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_N^{n_N}}{n_N!}\frac{\partial^{n_1}}{\partial x_1^{n_1}}\cdots\frac{\partial^{n_N}}{\partial x_N^{n_N}}f(x=0) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}x_{i_1}\cdots x_{i_n}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_N}}f(x=0)
\end{aligned}
\tag{3.2.4}
$$

这两种求和方法, $x_1^{n_1}\cdots x_N^{n_N}$ 项的 numerical factor 都等于,

$$
\frac{1}{n!}\times\frac{n!}{n_1!\cdots n_N!} = \frac{1}{n_1!\cdots n_N!} \tag{3.2.5}
$$

其中 $n = n_1 + \cdots + n_N$.

- 在 $\lambda = 0$ 时, 2-point Green's function 就是 propagator,

$$\begin{aligned} G^{(2)}_{ij}(\lambda = 0) &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} q_i q_j \\ &= \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \Big|_{J=0} = A^{-1}_{ij} \end{aligned} \tag{3.2.6}$$

- 来计算 $2, 3, 4$-point Green's functions,

$$\begin{cases} G^{(2)}_{ij} = A^{-1}_{ij} - \frac{\lambda}{4!} \sum_m (3A^{-1}_{mm} A^{-1}_{mm} A^{-1}_{ij} + 12 A^{-1}_{mm} A^{-1}_{mi} A^{-1}_{mj}) + O(\lambda^2) \\ G^{(3)}_{ijk} = 0 \\ G^{(4)}_{ijkl} = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} \\ \qquad - \frac{\lambda}{4!} \sum_m (A^{-1}_{mm} A^{-1}_{mm} A^{-1}_{ij} A^{-1}_{kl} + \cdots + 4! A^{-1}_{im} A^{-1}_{jm} A^{-1}_{km} A^{-1}_{lm}) + O(\lambda^2) \end{cases} \tag{3.2.7}$$

calculation:

2-point Green's function 计算如下,

$$\begin{aligned} G^{(2)}_{ij} &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j \\ &= A^{-1}_{ij} - \frac{\lambda}{4!} \langle q^4 q_i q_j \rangle + O(\lambda^2) \\ &= A^{-1}_{ij} - \frac{\lambda}{4!} \sum_m (3A^{-1}_{mm} A^{-1}_{mm} A^{-1}_{ij} + 12 A^{-1}_{mm} A^{-1}_{mi} A^{-1}_{mj}) + O(\lambda^2) \end{aligned} \tag{3.2.8}$$

3-point Green's function 计算如下,

$$G^{(32)}_{ijk} = \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j q_k = 0 \tag{3.2.9}$$

4-point Green's function 计算如下,

$$\begin{aligned} G^{(4)}_{ijkl} &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j q_k q_l \\ &= A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} - \frac{\lambda}{4!} \langle q^4 q_i q_j q_k q_l \rangle + O(\lambda^2) \end{aligned} \tag{3.2.10}$$

3.3 perturbative field theory

- 做如下替换即可,

$$\begin{cases} A \mapsto -i(\partial^2 - m^2) \\ J \mapsto iJ \end{cases} \tag{3.3.1}$$

- **Schwinger's way:** ϕ^4 theory 的路径积分,

$$Z(J) = \int D\phi \, e^{i \int d^d x \, (\frac{1}{2} \phi (\partial^2 - m^2) \phi - \frac{\lambda}{4!} \phi^4 + J(x) \phi(x))} \tag{3.3.2}$$

$$= Z(0,0) e^{-i \frac{\lambda}{4!} \int d^d z \, (\frac{\delta}{i \delta J(z)})^4} e^{-\frac{i}{2} \int d^d x d^d y \, J(x) D(x-y) J(y)} \tag{3.3.3}$$

其中 $D(x-y)$ 是自由场的 propagator, 见 (1.2.1).

- **Wick's way:** 同样, 对 J 展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \cdots d^d x_n \, J(x_1) \cdots J(x_n) G^{(n)}(x_1, \cdots, x_n) \tag{3.3.4}$$

其中,

$$G^{(n)}(x_1, \cdots, x_n) = \frac{1}{Z(0,0)} \int D\phi \, e^{i \int d^d x \, (\frac{1}{2}\phi(\partial^2 - m^2)\phi - \frac{\lambda}{4!}\phi^4)} \phi(x_1) \cdots \phi(x_n) \tag{3.3.5}$$

有时 $Z(J)$ 被称为 generating functional, 因为它能生成 Green's functions.

3.3.1 collision between particles

- 通过 Wick's way, 考虑 $J(x_1)J(x_2)J(x_3)J(x_4)$ 项, 实际上就是要计算 $G^{(4)}(x_1, x_2, x_3, x_4)$, 它的 0 阶项为,

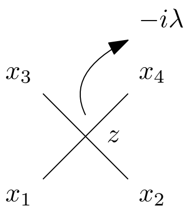
$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4, \lambda = 0) &= \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta J(x_2)} \frac{\delta}{i\delta J(x_3)} \frac{\delta}{i\delta J(x_4)} e^{-\frac{i}{2} \int d^d x d^d y \, J(x) D(x-y) J(y)} \\ &= -(D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}) \end{aligned} \tag{3.3.6}$$

其中 D_{ij} 是 $D(x_i - x_j)$ 的简写, 可见, 传播子实际上是 $(-i)^3 D = iD$.

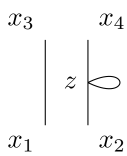
- $G^{(4)}_{1234}$ 的 1 阶项为,

$$\begin{aligned} \text{1st order term} &= -\frac{i\lambda}{4!} \int d^d z \, \langle \phi_1 \cdots \phi_4 \phi^4(z) \rangle \\ &= -\frac{i\lambda}{4!} \int d^d z \, \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_4} \left(\frac{\delta}{i\delta J(z)} \right)^4 e^{-\frac{i}{2} \int d^d x d^d y \, J(x) D(x-y) J(y)} \\ &= -\frac{i\lambda}{4!} \int d^d z \, \Big(4! D_{1z} D_{2z} D_{3z} D_{4z} \\ &\quad + 4 \times 3 D_{12} D_{3z} D_{4z} D_{zz} + \cdots + 3 D_{12} D_{34} D_{zz} D_{zz} + \cdots \Big) \end{aligned} \tag{3.3.7}$$

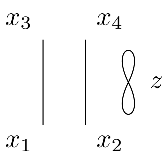
其中各项分别对应如下 Feynman diagrams,



$-i\lambda \int d^d z \, D_{1z} D_{2z} D_{3z} D_{4z}$



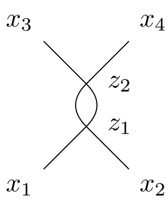
$\frac{-i\lambda}{2!} \int d^d z D_{13} D_{2z} D_{4z} D_{zz}$



$\frac{-i\lambda}{8} \int d^d z D_{13} D_{24} D_{zz} D_{zz}$

其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,



$$(4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!}\right)^2 \int d^d z_1 d^d z_2 \, D_{1z_1} D_{2z_1} D_{3z_2} D_{4z_2} D_{z_1 z_2} D_{z_1 z_2}$$

3.3.2 in momentum space

- 本 subsection 将 (3.3.5) 转换到 momentum space, 注意到 $\tilde{J}(k)$ 和 $\tilde{J}(-k)$ 并不独立, 所以 $\frac{\partial}{\partial i\tilde{J}}$ 不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\tilde{G}^{(n)}(k_1, \cdots, k_n) = \int d^d x_1 \cdots d^d x_n \, e^{-i(k_1 \cdot x_1 + \cdots)} G^{(n)}(x_1, \cdots, x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^dx_1 \cdots d^dx_n \, e^{-i(k_1 \cdot x_1 + \cdots)} \langle \Big(-\frac{i\lambda}{4!} \int d^dz \, \phi_z^4 \Big)^n \phi_1 \cdots \phi_n \rangle \tag{3.3.8}$$

– propagator 的 Fourier transformation 是,

$$\tilde{D}_{pq} = \int d^dx d^dy \, e^{-i(p\cdot x + q \cdot y)} D(x-y) = \frac{(2\pi)^d \delta^{(d)}(p+q)}{-p^2 - m^2 + i\epsilon} \tag{3.3.9}$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ 的 1 阶项为,

$$\text{1st order term} = -\frac{i\lambda}{4!} \int d^dx_1 \cdots d^dx_4 \, e^{-i(k_1\cdot x_1 + \cdots)} \int d^dz \, \langle \phi_z^4 \phi_1 \cdots \phi_4 \rangle \tag{3.3.10}$$

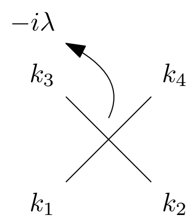
考虑第 1 项,

$$\begin{aligned} & -\frac{i\lambda}{4!} \int d^dx_1 \cdots d^dx_4 \, e^{-i(k_1 \cdot x_1 + \cdots)} \int d^dz \, 4! D_{1z} \cdots D_{4z} \\ = & -i\lambda \int d^dx_1 \cdots d^dx_4 d^dz \, e^{-i(k_1\cdot x_1 + \cdots)} e^{i(p_1\cdot(x_1 - z) + \cdots)} \prod_{i=1}^{4} \int \frac{d^dp_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\ = & -i\lambda \underbrace{\int d^dz \, e^{-iz\cdot(k_1 + \cdots + k_4)}}_{=(2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4)} \prod_{i=1}^{4} \frac{1}{-k_i^2 - m^2 + i\epsilon} \end{aligned} \tag{3.3.11}$$

– 出射粒子不一定 on-shell (?).

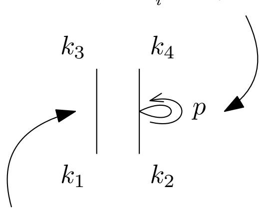
- 得到这些 Feynman diagrams,

$(2\pi)^d \delta^{(d)}(k_1 + k_2) \frac{i}{-k_1^2 - m^2 + i\epsilon}$



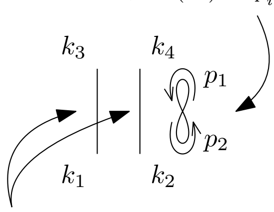
$-i\lambda(2\pi)^4 \delta^{(d)}(k_1 + \cdots + k_4) \prod_{i=1}^{4} \frac{i}{-k_i^2 - m^2 + i\epsilon}$

$-\frac{i\lambda}{2!}(2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=2,4} \frac{i}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^dp}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon}$



$(2\pi)^d \delta^{(d)}(k_1 + k_3) \frac{i}{-k_1^2 - m^2 + i\epsilon}$

$-\frac{i\lambda}{8}(2\pi)^d \delta^{(d)}(0) \prod_{i=1,2} \int \frac{d^dp_i}{(2\pi)^d} \frac{i}{-p_i^2 - m^2 + i\epsilon}$



$(2\pi)^d \delta^{(d)}(k_1 + k_3)(2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{i}{-k_i^2 - m^2 + i\epsilon}$

calculation:

第 3 幅图的计算如下,

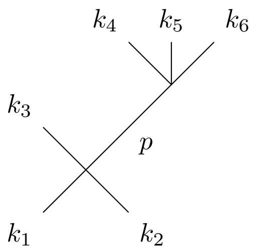
$$\begin{aligned} & -\frac{i\lambda}{2!} \int d^dx_1 \cdots d^dx_4 \, e^{-i(k_1\cdot x_1 + \cdots)} \int d^dz \, D_{13} D_{2z} D_{4z} D_{zz} \\ = & -\frac{i\lambda}{2!} \int d^dx_1 \cdots d^dx_4 d^dz \, e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1\cdot(x_1 - x_3) + p_2 \cdot (x_2 - z) + p_4\cdot(x_4 - z) + p_4 \cdot 0)} \end{aligned}$$

$$\prod_{i=1}^{4}\int \frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon}$$
$$=-\frac{i\lambda}{2!}\int d^dz\, e^{-iz\cdot(p_2+p_4)}\delta^{(d)}(p_1-k_1)\delta^{(d)}(p_2-k_2)\delta^{(d)}(p_1+k_3)\delta^{(d)}(p_4-k_4)$$
$$\prod_{i=1}^{4}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon}$$
$$=-\frac{i\lambda}{2!}(2\pi)^d\delta^{(d)}(k_1+k_3)\delta^{(d)}(k_2+k_4)\prod_{i=1,2,4}\frac{1}{-k_i^2-m^2+i\epsilon}\int\frac{d^dp}{-p^2-m^2+i\epsilon} \tag{3.3.12}$$

第 4 幅图的计算如下,

$$-\frac{i\lambda}{8}\int d^dx_1\cdots d^dx_4\, e^{-i(k_1\cdot x_1+\cdots)}\int d^dz\, D_{13}D_{24}D_{zz}D_{zz}$$
$$=-\frac{i\lambda}{8}\int d^dx_1\cdots d^dx_4d^dz\, e^{-i(k_1\cdot x_1+\cdots)}e^{i(p_1\cdot(x_1-x_3)+p_2\cdot(x_2-x_4)+p_3\cdot 0+p_4\cdot 0)}$$
$$\prod_{i=1}^{4}\int\frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon}$$
$$=-\frac{i\lambda}{8}\int d^dz\,\delta^{(d)}(p_1-k_1)\delta^{(d)}(p_2-k_2)\delta^{(d)}(p_1+k_3)\delta^{(d)}(p_2+k_4)$$
$$\prod_{i=1}^{4}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon}$$
$$=-\frac{i\lambda}{8}(2\pi)^d\delta^{(d)}(0)\delta^{(d)}(k_1+k_3)\delta^{(d)}(k_2+k_4)\prod_{i=1,2}\frac{1}{-k_i^2-m^2+i\epsilon}$$
$$\prod_{i=1,2}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon} \tag{3.3.13}$$

- 再举一个例子 (略去了 $\prod_{i=1}^{6}\frac{i}{-k_i^2-m^2+i\epsilon}$),



$$=\frac{(4!)^2}{2!}\times\Big(-\frac{i\lambda}{4!}\Big)^2(2\pi)^{2d}\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\delta^{(d)}(k_1+k_2+k_3+p)\delta^{(d)}(k_4+k_5+k_6-p)$$
$$=\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4+k_5+k_6)\frac{i}{-(k_1+k_2+k_3)^2-m^2+i\epsilon} \tag{3.3.14}$$

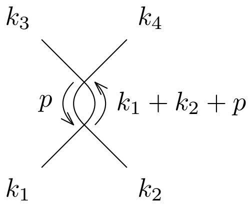
3.3.3 loops and a first look at divergence

- subsection 3.3.2 里的 loop diagrams 出现了如下积分,

$$\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\overset{d=4}{\sim}\int\frac{d^4p}{p^2} \tag{3.3.15}$$

积分发散.

- 再举一个例子 (略去了 $\prod_{i=1}^4 \frac{i}{-k_i^2-m^2+i\epsilon}$),



$$
\begin{aligned}
&=(4\times 3)^2\times 2\times\frac{1}{2!}\times\left(\frac{-i\lambda}{4!}\right)^2\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\int\frac{d^dq}{(2\pi)^d}\frac{i}{-q^2-m^2+i\epsilon}\\
&\quad(2\pi)^d\delta^{(d)}(k_1+k_2+p-q)(2\pi)^d\delta^{(d)}(k_3+k_4-p+q)\\
&=\frac{(-i\lambda)^2}{4}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4)\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\frac{i}{-(k_1+k_2+p)^2-m^2+i\epsilon}\\
&\overset{d=4}{\sim}\int\frac{d^4p}{p^4}
\end{aligned}
\tag{3.3.16}
$$

同样, 积分发散.

Chapter 4

canonical quantization

- A. Zee: the canonical and the path integral formalisms often appear complementary, in the sense that results difficult to see in one are clear in the other.

Appendices

Appendix A

Dirac delta function & Fourier transformation

A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx\, \delta(x) e^{-ikx} \tag{A.1.1}$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \tag{A.1.2}$$

- 第二个常用的公式 (Sokhotski-Plemelj theorem),

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{A.1.3}$$

其中 $\mathcal{P}$ 表示复函数的主値 (principal value).

proof:

考虑,

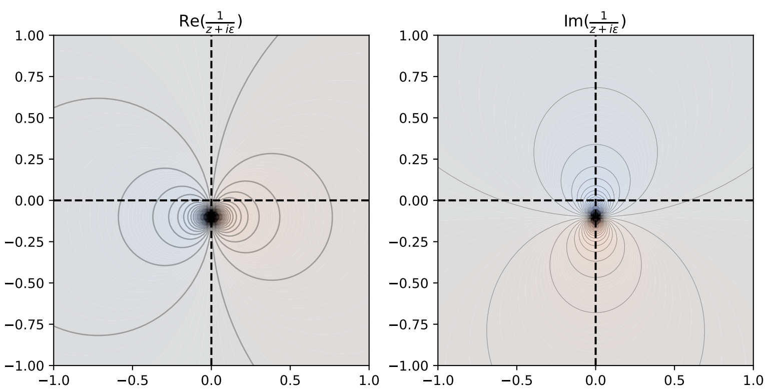
$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \tag{A.1.4}$$

且注意到,

$$\int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \operatorname{Res}(f, i\epsilon) = \pi \tag{A.1.5}$$

所以...

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,



- 另外, $\delta(x-a)\delta(x-b) = \delta(b-a)\delta(x-a)$.

A.2 Fourier transformation

- d-dim. Fourier transformation 如下,

$$\begin{cases}\phi(x) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x}\tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^dx\, e^{-ik\cdot x}\phi(x)\end{cases} \tag{A.2.1}$$

- 因此,

$$\partial_\mu\phi(x) \mapsto ik_\mu\tilde{\phi}(k) \tag{A.2.2}$$

- 对于**实函数**, Fourier transformation 是正交变换, 其 Jacobi determinant 为,

$$\left|\frac{\partial\phi(x)\cdots}{\partial\mathrm{Re}\tilde{\phi}(k)\cdots\partial\mathrm{Im}\tilde{\phi}(k)\cdots}\right| = \left(\frac{2}{V}\right)^{(2N+1)^d}\det A = \left(\frac{2(2N)^d}{V^2}\right)^{\frac{(2N+1)^d}{2}} \tag{A.2.3}$$

proof:

position space 和 momentum space 的格点分别为,

$$\begin{cases}x_i^\mu = i^\mu\epsilon \in \{0, \pm\epsilon, \cdots, \frac{L}{2}\} \\ k_n^\mu = n^\mu\frac{2\pi}{L} \in \{0, \pm\frac{2\pi}{L}, \cdots, \frac{\pi}{\epsilon}\}\end{cases} \iff i^\mu, n^\mu \in \{0, \pm1, \cdots, N\} \tag{A.2.4}$$

x^μ, k^μ 分别有 $2N+1$ 个取值, 其中 $N\epsilon = \frac{L}{2}$, 时空总体积为 $V = L^d$, momentum space 的总体积为 $\tilde{V} = \frac{(4\pi N)^d}{V}$.

将 (A.2.1) 写成格点求和的形式,

$$\begin{cases}\phi(x_i) = \frac{1}{(2\pi)^d}\left(\frac{2\pi}{L}\right)^d\sum_n e^{ik_n\cdot x_i}\tilde{\phi}(k_n) \\ \qquad = \frac{2}{V}\sum_{n^0>0}\Big(\cos(k_n\cdot x_i)\mathrm{Re}\tilde{\phi}(k_n) - \sin(k_n\cdot x_i)\mathrm{Im}\tilde{\phi}(k_n)\Big) \\ \tilde{\phi}(k_n) = \epsilon^d\sum_i e^{-ik_n\cdot x_i}\phi(x_i) \\ \qquad = \frac{V}{(2N)^d}\sum_i\Big(\cos(k_n\cdot x_i) - i\sin(k_n\cdot x_i)\Big)\phi(x_i)\end{cases} \tag{A.2.5}$$

- -

proof:
$\phi(x_i)$ 的变换需要做一些说明. 注意到 $\tilde{\phi}$ 的分量的数量是 ϕ 的两倍 (考虑到实部与虚部), 但在 $\phi \in \mathbb{R}^{(2N+1)^d}$ 时,

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k) \tag{A.2.6}$$

可见 $\tilde{\phi}$ 的分量并不独立, 取 $k^0 > 0$ 的部分为独立分量, 那么...

- -

将 (A.2.5) 写成矩阵的形式,

$$\begin{cases}\begin{pmatrix}\phi(x_0) \\ \vdots \\ \phi(x_{\max})\end{pmatrix} = \frac{2}{V}\overbrace{\begin{pmatrix}\cos k_0\cdot x_0 & \cdots & \cos k_{\max}\cdot x_0 & -\sin k_0\cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A}\begin{pmatrix}\mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots\end{pmatrix} \\ \begin{pmatrix}\mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots\end{pmatrix} = \frac{V}{(2N)^d}\begin{pmatrix}\cos k_0\cdot x_0 & \cdots & \cos k_0\cdot x_{\max} \\ \vdots & \ddots & \\ -\sin k_0\cdot x_0 & \cdots & -\sin k_0\cdot x_{\max} \\ \vdots & & \ddots\end{pmatrix}\begin{pmatrix}\phi(x_0) \\ \vdots \\ \phi(x_{\max})\end{pmatrix}\end{cases} \tag{A.2.7}$$

观察可见 $\tilde{\phi}$ 的变换中的矩阵是 A^T, 所以,

$$\frac{2}{V}\frac{V}{(2N)^d}AA^T = I \Longrightarrow \det A = \left(\frac{(2N)^d}{2}\right)^{\frac{(2N+1)^d}{2}} \tag{A.2.8}$$

因此...

– 顺便,

$$\int d^d x\, f(x)g(x) = \int \frac{d^d k}{(2\pi)^d}\tilde{f}(-k)\tilde{g}(k) \tag{A.2.9}$$

Appendix B

Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{B.0.1}$$

$$\langle x^{2n}\rangle = \frac{\int dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!! \tag{B.0.2}$$

其中 $(2n-1)!! = 1\cdot 3\cdots(2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx\, e^{-\frac{a}{2}x^2+Jx} = \sqrt{\frac{2\pi}{a}}e^{\frac{J^2}{2a}} \tag{B.0.3}$$

另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 N-dim. generalization

- 考虑如下积分,

$$Z(A,J) = \int dx_1\cdots dx_N\, e^{-\frac{1}{2}x^T\cdot A\cdot x+J^T\cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \tag{B.1.1}$$

其中 x, J 是 N-dim. 列向量, A 是 $N\times N$ 实对称矩阵.

calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1}\cdot D\cdot O \tag{B.1.2}$$

其中 D 是一个 diagonal matrix. 令 $y = O\cdot x$, 那么,

$$\begin{aligned} Z(A,J) &= \int dy_1\cdots dy_N\, e^{-\frac{1}{2}y^T\cdot D\cdot y+(OJ)^T\cdot y} \\ &= \prod_{i=1}^N \sqrt{\frac{2\pi}{D_{ii}}}e^{\frac{1}{2D_{ii}}(OJ)_i{}^2} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \end{aligned} \tag{B.1.3}$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O\cdot A^{-1}\cdot O^{-1})_{ii}$ 以及 $\mathrm{tr}\, D = \det A$.

- 一个重要的变体是 $A\mapsto -iA, J\mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j\rangle = \frac{1}{Z(A,0)}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}Z(A,J)\Big|_{J=0} = A^{-1}_{ij} \tag{B.1.4}$$

$$\langle x_i x_j\cdots x_k x_l\rangle = \sum_{\text{Wick}} A^{-1}_{i'j'}\cdots A^{-1}_{k'l'} \tag{B.1.5}$$

其中 (B.1.5) 中有偶数个 x, 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A,J) \Big|_{J=0} = \cdots \tag{B.1.6}$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} \tag{B.1.7}$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ik} A^{-1}_{jl} \tag{B.1.8}$$

Appendix C

perturbation theory in QM

- this chapter is based on MIT OpenCourseWare Quantum Physics III Chapter 1: Perturbation Theory.

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda\delta H \tag{C.0.1}$$

其中 $\lambda \in [0,1]$.

- 考虑 $H^{(0)}$ 的本征态为,

$$H^{(0)}|k^{(0)}\rangle = E_k^{(0)}|k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)}|l^{(0)}\rangle = \delta_{kl} \\ E_0^{(0)} \leq E_1^{(0)} \leq E_2^{(0)} \leq \cdots \end{cases} \tag{C.0.2}$$

C.1 non-degenerate perturbation theory

- 考虑 non-degenerate 能级 k, 有 $\cdots \leq E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \leq \cdots$, 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda|k^{(1)}\rangle + \lambda^2|k^{(2)}\rangle + \cdots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \cdots \end{cases} \tag{C.1.1}$$

  - 注意, 我们可以选取修正项满足,

$$\langle k^{(0)}|k^{(n)}\rangle = 0, n = 1, 2, \cdots \tag{C.1.2}$$

proof:

假设我们求解得到的修正项不满足 $\langle k^{(0)}|k^{(n)}\rangle = 0, n = 1, 2, \cdots$, 考虑,

$$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n|k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)}|k^{(n)}\rangle' = 0 \tag{C.1.3}$$

那么, (注意到态矢量可以乘一个常数, $\frac{1}{1-a_1\lambda-a_2\lambda^2-\cdots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \cdots$),

$$\begin{aligned} |k\rangle_\lambda &= (1 - a_1\lambda - a_2\lambda^2 - \cdots)|k^{(0)}\rangle + \lambda|k^{(1)}\rangle' + \lambda^2|k^{(2)}\rangle' + \cdots \\ |k\rangle'_\lambda &= |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \cdots}\left(\lambda|k^{(1)}\rangle' + \lambda^2|k^{(2)}\rangle' + \cdots\right) \\ &= |k^{(0)}\rangle + \lambda|k^{(1)}\rangle' + \lambda^2(a_1|k^{(1)}\rangle' + |k^{(2)}\rangle') + \cdots \end{aligned} \tag{C.1.4}$$

可见修正项都与 $|k^{(0)}\rangle$ 正交.

  - 注意, 不能要求 ${}_\lambda\langle k|k\rangle_\lambda = 1$, 否则 $|k^{(n)}\rangle$ 将与 λ 相关 (包括 $|k^{(0)}\rangle$),

$$\begin{aligned} {}_\lambda\langle k|k\rangle_\lambda = &\langle k^{(0)}|k^{(0)}\rangle \\ &+ \lambda(\langle k^{(1)}|k^{(0)}\rangle + \langle k^{(0)}|k^{(1)}\rangle) \\ &+ \lambda^2(\langle k^{(2)}|k^{(0)}\rangle + \langle k^{(1)}|k^{(1)}\rangle + \langle k^{(0)}|k^{(2)}\rangle) \end{aligned}$$

$$
\vdots
$$
$$
+\lambda^n(\langle k^{(n)}|k^{(0)}\rangle + \langle k^{(n-1)}|k^{(1)}\rangle + \cdots + \langle k^{(0)}|k^{(n)}\rangle) \tag{C.1.5}
$$

- 将 (C.1.1) 代入 Schrodinger's eq., 得到,

λ^0	$(H^{(0)} - E_k^{(0)})\,\vert k^{(0)}\rangle = 0$
λ^1	$(H^{(0)} - E_k^{(0)})\,\vert k^{(1)}\rangle = (E_k^{(1)} - \delta H)\,\vert k^{(0)}\rangle$
λ^2	$(H^{(0)} - E_k^{(0)})\,\vert k^{(2)}\rangle = (E_k^{(1)} - \delta H)\,\vert k^{(1)}\rangle + E_k^{(2)}\,\vert k^{(0)}\rangle$
$\vdots$	$\vdots$
λ^n	$(H^{(0)} - E_k^{(0)})\,\vert k^{(n)}\rangle = (E_k^{(1)} - \delta H)\,\vert k^{(n-1)}\rangle + E_k^{(2)}\,\vert k^{(n-2)}\rangle + \cdots + E_k^{(n)}\,\vert k^{(0)}\rangle$

calculation:

Schrodinger's eq. 为,

$$
(H^{(0)} + \lambda\delta H - E_k(\lambda))\,|k\rangle_\lambda = 0 \tag{C.1.6}
$$

展开为,

$$
\Big((H^{(0)} - E_k^{(0)}) + \lambda(\delta H - E_k^{(1)}) - \lambda^2 E_k^{(2)} - \cdots\Big)(|k^{(0)}\rangle + \lambda\,|k^{(1)}\rangle + \lambda^2\,|k^{(2)}\rangle + \cdots) = 0 \tag{C.1.7}
$$

- 现在来计算 $\langle l^{(0)}|k^{(n)}\rangle$, 有,

$$
\begin{cases}
(E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(1)}\rangle = E_k^{(1)}\delta_{lk} - \delta H_{lk} \\
(E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(2)}\rangle = E_k^{(1)}\,\langle l^{(0)}|k^{(1)}\rangle - \langle l^{(0)}|\delta H|k^{(1)}\rangle + E_k^{(2)}\delta_{lk} \\
\qquad\vdots \qquad\qquad\qquad\qquad \vdots \\
(E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(n)}\rangle = E_k^{(1)}\,\langle l^{(0)}|k^{(n-1)}\rangle - \langle l^{(0)}|\delta H|k^{(n-1)}\rangle \\
\qquad\qquad + E_k^{(2)}\,\langle l^{(0)}|k^{(n-2)}\rangle + \cdots + E_k^{(n)}\delta_{lk}
\end{cases} \tag{C.1.8}
$$

其中 $\delta H_{lk} = \langle l^{(0)}|\delta H|k^{(0)}\rangle$, 对于满足 (C.1.2) 的解, 有,

$$
E_k^{(n)} = \langle k^{(0)}|\delta H|k^{(n-1)}\rangle\,, n = 1, 2, \cdots \tag{C.1.9}
$$

并且,

$$
|k^{(1)}\rangle = -\sum_{l\neq k}\frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}}\,|l^{(0)}\rangle \Longrightarrow E_k^{(2)} = -\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} \tag{C.1.10}
$$

calculation:

将 (C.1.10) 代入 (C.1.8), 得到 $(l \neq k)$,

$$
(E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(2)}\rangle = -E_k^{(1)}\frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} + \sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \tag{C.1.11}
$$

所以,

$$
\begin{cases}
|k^{(2)}\rangle = \sum_{l\neq k}\Big(-\frac{\delta H_{00}\delta H_{lk}}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}\Big)\,|l^{(0)}\rangle \\
E_k^{(3)} = \sum_{l\neq k}\Big(-\frac{\delta H_{00}|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m\neq k}\frac{\delta H_{kl}\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}\Big)
\end{cases} \tag{C.1.12}
$$

计算归一化系数,

$$
{}_\lambda\langle k|k\rangle_\lambda = 1 + \lambda^2\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + O(\lambda^3) \tag{C.1.13}
$$

C.1.1 level repulsion or the seesaw mechanism

- 能量的展开式为,

$$E_k(\lambda) = E_k^{(0)} + \lambda \delta H_{kk} - \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3) \tag{C.1.14}$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

C.1.2 validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即,

$$|\lambda V| < \frac{1}{2}\Delta E^{(0)} \tag{C.1.15}$$

因此, 对于能级简并的情况, $\Delta E^{(0)} = 0$, 情况会更复杂.

calculation:

对于两能级系统,

$$H(\lambda) = H^{(0)} + \lambda \hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix} \tag{C.1.16}$$

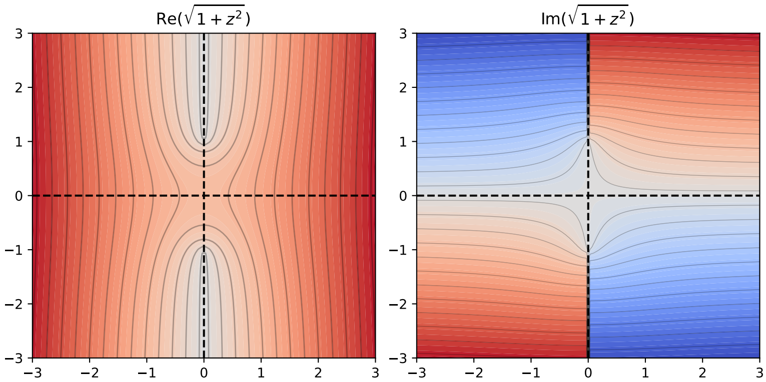
$H(\lambda)$ 的本征值可以直接计算,

$$E_\pm(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)})\sqrt{1 + \left(\frac{\lambda|V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})}\right)^2} \tag{C.1.17}$$

考虑 $\sqrt{1+z^2}$ 的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1}\frac{(2n-3)!!}{2^n n!} z^{2n} + \cdots \tag{C.1.18}$$

注意到 $\sqrt{1+z^2}$ 在 $z = \pm i$ 有 branch cut, 因此 $z = 0$ 附件的 Taylor expansion 只有在 $|z| < 1$ 内才收敛.



C.2 degenerate perturbation theory

- 暂时先跳过.