Quantum Field Theory
a study note based on A. Zee's textbook

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https://github.com/siyang03/my-note---A.-Zee-QFT

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Contents

convention, notation, and units

- 笔记中的**度规号差**约定为 $(-, +, +, +)$.
- 使用 Planck units, 此时 $G, \hbar, c, k_B = 1$, 因此,

name/dimension	expression/value
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

Part I

motivation and foundation

Chapter 1

free field theory

1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{1.1.1}$$

A. Zee 说: 在作用量里, 时间的导数项必须是正的, 包括标量场的 $(\partial_0\phi)^2$ 和电磁场的 $(\partial_0 A_i)^2$.

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi \, e^{i\int d^d x\,(-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \tag{1.1.2}$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^d x d^d y\, J(x)D(x-y)J(y)} \tag{1.1.3}$$

另外, 用 $W(J)$ 表示指数上的部分 (去除掉虚数 i).

> **proof:**
>
> 注意 $\partial^\mu\phi\partial_\mu\phi = \partial^\mu(\phi\partial_\mu\phi) - \phi\partial^2\phi$, 忽略全微分项, 那么,
>
> $$Z(J) = \int D\phi \, e^{i\int d^d x\,(\frac{1}{2}\phi(\partial^2 - m^2)\phi + J(x)\phi(x))} \tag{1.1.4}$$
>
> 代入 (B.1.1), 可知,
>
> $$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^d x d^d y\, J(x)D(x-y)J(y)} \tag{1.1.5}$$
>
> 其中 $D(x-y)$ 满足,
>
> $$\begin{cases} (\partial^2 - m^2)D(x-y) = \delta^{(d)}(x-y) \\ (-p^2 - m^2)\tilde{D}(p,q) = (2\pi)^d\delta^{(d)}(p-q) \end{cases} \implies D(x-y) = \int \frac{d^d k}{(2\pi)^d}\frac{e^{ik\cdot(x-y)}}{-k^2 - m^2} \tag{1.1.6}$$

1.2 free propagator

- 为了使 (1.1.4) 中的积分在 ϕ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon\int d^d x\phi^2}$.

- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 ϵ 避免发散,

$$D(x) = \int \frac{d^d k}{(2\pi)^d}\frac{e^{ik\cdot x}}{-k^2 - m^2 + i\epsilon} = -i\int \frac{d^D k}{(2\pi)^D 2\omega_k}\Big(\theta(t)e^{i(-\omega_k t + \vec{k}\cdot\vec{x})} + \theta(-t)e^{i(\omega_k t + \vec{k}\cdot\vec{x})}\Big) \tag{1.2.1}$$

- $D(x)$ 的取值与 x 的类时, 类空性质关系密切.

  - 类时区域,

$$D(t,0) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left(\theta(t) e^{-i\omega_k t} + \theta(-t) e^{i\omega_k t} \right) \tag{1.2.2}$$

  - 类空区域,

$$D(0,\vec{x}) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \sim e^{-m|\vec{x}|} \tag{1.2.3}$$

1.3 from field to particle to force

1.3.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,

$$W(J) = -\frac{1}{2} \int d^d x d^d y \, J(y) D(x-y) J(y) \tag{1.3.1}$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}(k) \tag{1.3.2}$$

其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k) = \tilde{J}^*(k)$.

- 考虑 $J(x) = J_1(x) + J_2(x)$, 那么 $W(J)$ 共有 4 项, 其中一个交叉项如下,

$$W_{12}(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_1(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_2(k) \tag{1.3.3}$$

可见 $W(J)$ 取值较大的条件是:

  1. $\tilde{J}_1(k), \tilde{J}_2(k)$ 有较大重叠,
  2. 重叠位置的 k 是 on shell (即 $k^2 = -m^2$).

- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

1.3.2 from particle to force

- 考虑 $J(x) = \delta^{(D)}(\vec{x} - \vec{x}_1) + \delta^{(D)}(\vec{x} - \vec{x}_1) \Longrightarrow \tilde{J}_a(k) = 2\pi e^{-i\vec{k}\cdot\vec{x}_a} \delta(k^0)$, 那么,

$$W_{12}(J) + W_{21}(J) = \delta(0) \int \frac{d^D k}{(2\pi)^{D-1}} \frac{1}{|\vec{k}|^2 + m^2 - i\epsilon} \cos(\vec{k} \cdot (\vec{x}_1 - \vec{x}_2))$$

$$\overset{D=3}{=} 2\pi \delta(0) \frac{1}{4\pi r} e^{-mr} \tag{1.3.4}$$

($-i\epsilon$ 显然可以舍去), 注意到 $\langle 0 | e^{-iHT} | 0 \rangle = e^{-iET}$, 而时间间隔 $T = \int dx^0 = 2\pi\delta(0)$, 所以,

$$E = -\frac{W(J)}{T} \overset{D=3}{=} -\frac{1}{4\pi r} e^{-mr} \tag{1.3.5}$$

$$= \frac{1}{(2\pi)^D} \int (k \sin \theta_1)^{D-2} d\Omega_{D-2} \int k d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1}$$

$$= \frac{S_{D-2}}{(2\pi)^D} \int k^{D-1} \sin^{D-2} \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1} \tag{1.3.6}$$

取 $D = 3$, 那么,

$$I_{D=3} = \frac{1}{(2\pi)^2} \int k^2 \sin \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ik \cos \theta_1}$$

$$= \frac{1}{2\pi^2 r} \int_0^\infty \sin(kr) \frac{kdk}{k^2 + m^2} = \frac{-i}{4\pi^2 r} \int_{-\infty}^\infty e^{ikr} \frac{kdk}{k^2 + m^2}$$

$$= \frac{-i}{4\pi^2 r} 2\pi i \underbrace{\operatorname{Res}(f, im)}_{= \frac{1}{2} e^{-mr}} = \frac{1}{4\pi r} e^{-mr} \tag{1.3.7}$$

Chapter 2

Coulomb and Newton: repulsive and attraction

2.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{2.1.1}$$

  其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

- 做路径积分,

$$Z(J) = \int DA\, e^{i\int d^d x(\mathcal{L}+J_\mu A^\mu)} = \mathcal{C}e^{-\frac{i}{2}\int d^d x d^d y\, J_\mu D^{\mu\nu}(x-y)J_\nu(y)} \tag{2.1.2}$$

> **calculation:**
>
> massive photon 的作用量为,
>
> $$S(A) = \int d^d x\, \frac{1}{2}\Big(-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu \Big)$$
>
> $$= \int d^d x\, \frac{1}{2}\Big(A_\nu \partial^2 A^\nu - A_\nu \partial^\nu \partial_\mu A^\mu - m^2 A_\mu A^\mu \Big) + \text{total differential}$$
>
> $$= \int d^d x\, \frac{1}{2} A_\mu \Big(-\partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \Big) A_\nu + \text{total differential}$$
>
> $$= \int \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(-k)\Big(k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2) \Big)\tilde{A}_\nu(k) + \text{boundary term} \tag{2.1.3}$$
>
> 那么, 需要有,
>
> $$(-\partial^\mu \partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2))D_{\rho\nu}(x-y) = \delta^\mu_\nu \delta^{(d)}(x-y)$$
>
> $$\Longrightarrow \tilde{D}_{\mu\nu}(k) = \frac{k_\mu k_\nu/m^2 + \eta_{\mu\nu}}{-k^2 - m^2} \tag{2.1.4}$$

考虑到积分需要收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, (为什么 A_μ 类空, 只知道 $\tilde{A}_\mu$ 类空, 见 subsection 2.1.2, 但路径积分中的 A 显然不满足 field equation $\Longrightarrow$ 路径积分中起主要作用的 $\tilde{A}$ 类空, 因此 $-\epsilon|\tilde{A}|^2 < 0$).

- 因此,

$$W(J) = -\frac{1}{2}\int d^d x d^d y\, J_\mu(x)D^{\mu\nu}(x-y)J_\nu(y) \tag{2.1.5}$$

$$= -\frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\tilde{J}_\mu(-k)\frac{k^\mu k^\nu/m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon}\tilde{J}_\nu(k) \tag{2.1.6}$$

注意到 current conservation, 有 $\partial_\mu J^\mu = 0 \iff k^\mu \tilde{J}_\mu(k) = 0$, 所以,

$$W(J) = -\frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\tilde{J}^\mu(-k)\frac{1}{-k^2 - m^2 + i\epsilon}\tilde{J}_\mu(k) \tag{2.1.7}$$

观察电荷分量, 可见同性相斥, 异性相吸.

2.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的 x, y, z 方向, 在粒子静止系下, 极化矢量 $(\epsilon^i)_\mu = \delta^i_\mu, i = 1, 2, 3$, 而 $k_\mu = (-m, 0, 0, 0)$, 所以,

$$k^\mu (\epsilon^i)_\mu = 0 \tag{2.1.8}$$

  – 注意, 一个粒子的极化方向用 e^i (这不是矢量) 表示, 极化矢量为 $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$.

- 在粒子静止系下, 考虑,

$$\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu} \tag{2.1.9}$$

可见,

$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon} \tag{2.1.10}$$

2.1.2 Maxwell Lagrangian

- 根据 (2.1.1) 中的 Lagrangian density, 得到 field equation 如下,

$$\left(-\partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \right) A_\nu \tag{2.1.11}$$

  – spin-1 particle 有 3 个自旋自由度, 而 A_μ 有 4 个分量, 所以需要一个约束方程,

$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0 \tag{2.1.12}$$

  实际上在 (2.1.11) 左右两边作用一个 ∂_μ 即可得到这个约束方程.

2.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = <mark>linearized</mark> Einstein Lagrangian.

- 受 subsection 2.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足,

$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases} \tag{2.2.1}$$

其中下指标 μ, ν 对称, $a = 1, \cdots, 5$, (可以验证 $(\epsilon^a)_{\mu\nu}$ 确实有 5 个独立分量).

  – 对 $(\epsilon^a)_{\mu\nu}$ 的归一化条件可以定义为 $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$.
  – 与 subsection 2.1.1 中提示一样, 粒子的极化方向用 e^a 表示.

- 那么,

$$\sum_{a=1}^5 (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma} \tag{2.2.2}$$

> **calculation:**
>
> 首先用 k_μ 和 $\eta_{\mu\nu}$ 构造最一般的关于 $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$ 对称的 4 阶张量, (下式中把 $\frac{k_\mu}{m}$ 略写作 k_μ),
>
> $$A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho})$$
> $$+ D\eta_{\mu\nu}\eta_{\rho\sigma} + E(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \tag{2.2.3}$$
>
> 代入 (2.2.1) 得,
>
> $$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4} \tag{2.2.4}$$

因此, 这个 4 阶张量最终确定为,

$$\frac{3}{4}A\Big((G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\Big) \tag{2.2.5}$$

- 所以,

$$\tilde{D}_{\mu\nu\rho\sigma}(k) = \frac{(G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tag{2.2.6}$$

- 计算路径积分中的 $W(T)$,

$$W(T) = -\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{(G^{\mu\rho}G^{\nu\sigma} + G^{\mu\sigma}G^{\nu\rho}) - \frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2 - m^2 + i\epsilon}\tilde{T}_{\rho\sigma}(k) \tag{2.2.7}$$

注意到 $\partial_\mu T^{\mu\nu}(x) = 0 \iff k_\mu\tilde{T}^{\mu\nu}(k) = 0$, 并考虑到 T 是对称张量, 所以,

$$W(T) = -\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{2\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2 - m^2 + i\epsilon}\tilde{T}_{\rho\sigma}(k) \tag{2.2.8}$$

考虑能量项, 可见质量互相吸引.

2.3 remarks

- 由于 seesaw mechanism (见 subsection C.1.1), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为 $\eta^{00} = -1$.

- 本 chapter 中的计算都是 $m \neq 0$ 的粒子, 与真实世界有差异.

Chapter 3

Feynman diagrams

3.1 a baby problem

- 考虑如下积分,

$$Z(J) = \int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!}q^4 + Jq} \tag{3.1.1}$$

- **Schwinger's way:** 把 integrand 对 λ 展开, 并将 q 用 $\frac{\partial}{\partial J}$ 替代, 得到,

$$Z(J) = e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4} \int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2 q^2 + Jq}$$

$$= \sqrt{\frac{2\pi}{m^2}} e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4} e^{\frac{J^2}{2m^2}} \tag{3.1.2}$$

后面的计算中忽略 $Z(J=0, \lambda=0)$.

- 每个 vertex 带有 $-\lambda$, 每个 line 带有 $\frac{1}{m^2}$, 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里 q 无法区分, 所以不方便算, 先略了),
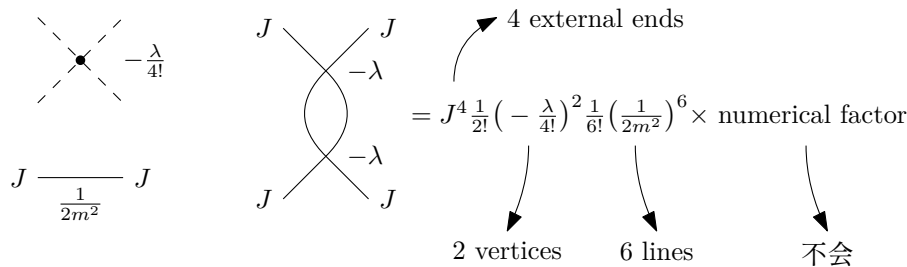
Figure 3.1: baby problem - Feynman diagram

> **calculation:**
>
> 在这里计算 λJ^4 项,
>
>   具体暂时不会算 $\tag{3.1.3}$
>
> 但是直接计算 (3.1.2) 的展开项, 得到的结果与 (3.1.5) 一样.

3.1.1 Wick contraction and Green's functions

- 把积分 (3.1.1) 对 J 展开,

$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} J^n \underbrace{\int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!}q^4} q^n}_{=Z(0,0)G^{(n)}} \tag{3.1.4}$$

其中 Green's function $G^{(n)}$ 对 λ 展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

3.1.2 connected vs. disconnected

- 考虑,
$$Z(J, \lambda) = Z(J=0, \lambda) e^{W(J,\lambda)} \qquad (3.1.6)$$

其中 $Z(J=0,\lambda)$ 由 diagrams with no external source J 组成, 而 $W(J,\lambda)$ 则由 connected diagrams 组成 (?).

- 我们希望计算的是 W, 而不是 Z (?).

3.2 a child problem

- 考虑如下积分,
$$Z(J) = \int dq_1 \cdots dq_N \, e^{-\frac{1}{2}q^T \cdot A \cdot q - \frac{\lambda}{4!}q^4 + J^T \cdot q} \qquad (3.2.1)$$

其中 $q^4 = \sum_i q_i^4$.

- **Schwinger's way:** 对 λ 展开并把 q 替换为 $\frac{\partial}{\partial J}$, 得到,
$$Z(J) = \sqrt{\frac{(2\pi)^N}{\det A}} e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \qquad (3.2.2)$$

其中 $\left(\frac{\partial}{\partial J}\right)^4 = \sum_i \left(\frac{\partial}{\partial J_i}\right)^4$.

3.2.1 n-point Green's function

- **Wick's way:** 对 J 展开获得带 Green's function 的表达式,
$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} J_{i_1} \cdots J_{i_n} \underbrace{\int dq_1 \cdots dq_N \, e^{-\frac{1}{2}q^T \cdot A \cdot q - \frac{\lambda}{4!}q^4} q_{i_1} \cdots q_{i_n}}_{=Z(0,0)G^{(n)}_{i_1 \cdots i_n}} \qquad (3.2.3)$$

其中 $G^{(n)}_{i_1 \cdots i_n}$ 称为 n-point Green's function.

> **Taylor expansion:**
>
> 多元函数的 Taylor 展开如下,
>
> $$f(x_1, \cdots, x_N) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_N^{n_N}}{n_N!} \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_N}}{\partial x_N^{n_N}} f(x=0)$$
> $$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} x_{i_1} \cdots x_{i_n} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}} f(x=0) \qquad (3.2.4)$$
>
> 这两种求和方法, $x_1^{n_1} \cdots x_N^{n_N}$ 项的 numerical factor 都等于,
> $$\frac{1}{n!} \times \frac{n!}{n_1! \cdots n_N!} = \frac{1}{n_1! \cdots n_N!} \qquad (3.2.5)$$

> 其中 $n = n_1 + \cdots + n_N$.

- 在 $\lambda = 0$ 时, 2-point Green's function 就是 propagator,

$$
\begin{aligned}
G_{ij}^{(2)}(\lambda = 0) &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} q_i q_j \\
&= \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \Big|_{J=0} = A_{ij}^{-1}
\end{aligned}
\tag{3.2.6}
$$

- 来计算 $2, 3, 4$-point Green's functions,

$$
\begin{cases}
G_{ij}^{(2)} = A_{ij}^{-1} - \dfrac{\lambda}{4!} \sum_m (3 A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12 A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \\[2mm]
G_{ijk}^{(3)} = 0 \\[2mm]
G_{ijkl}^{(4)} = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \\[2mm]
\qquad - \dfrac{\lambda}{4!} \sum_m (A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} A_{kl}^{-1} + \cdots + 4! A_{im}^{-1} A_{jm}^{-1} A_{km}^{-1} A_{lm}^{-1}) + O(\lambda^2)
\end{cases}
\tag{3.2.7}
$$

---

calculation:

2-point Green's function 计算如下,

$$
\begin{aligned}
G_{ij}^{(2)} &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j \\
&= A_{ij}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j \rangle + O(\lambda^2) \\
&= A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3 A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12 A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2)
\end{aligned}
\tag{3.2.8}
$$

3-point Green's function 计算如下,

$$
G_{ijk}^{(32)} = \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j q_k = 0
\tag{3.2.9}
$$

4-point Green's function 计算如下,

$$
\begin{aligned}
G_{ijkl}^{(4)} &= \frac{1}{Z(0,0)} \int dq_1 \cdots dq_N \, e^{-\frac{1}{2} q^T \cdot A \cdot q} \Big(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\Big) q_i q_j q_k q_l \\
&= A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j q_k q_l \rangle + O(\lambda^2)
\end{aligned}
\tag{3.2.10}
$$

3.3 perturbative field theory

- 做如下替换即可,

$$
\begin{cases}
A \mapsto -i(\partial^2 - m^2) \\
J \mapsto iJ
\end{cases}
\tag{3.3.1}
$$

- **Schwinger's way:** ϕ^4 theory 的路径积分,

$$
Z(J) = \int D\phi \, e^{i \int d^d x \left(\frac{1}{2} \phi(\partial^2 - m^2)\phi - \frac{\lambda}{4!} \phi^4 + J(x)\phi(x)\right)}
\tag{3.3.2}
$$

$$
= Z(0,0) e^{-i \frac{\lambda}{4!} \int d^d z \left(\frac{\delta}{i\delta J(z)}\right)^4} e^{-\frac{i}{2} \int d^d x d^d y \, J(x) D(x-y) J(y)}
\tag{3.3.3}
$$

其中 $D(x-y)$ 是自由场的 propagator, 见 (1.2.1).

- **Wick's way:** 同样, 对 J 展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \cdots d^d x_n \, J(x_1) \cdots J(x_n) G^{(n)}(x_1, \cdots, x_n) \tag{3.3.4}$$

其中,

$$G^{(n)}(x_1, \cdots, x_n) = \frac{1}{Z(0,0)} \int D\phi \, e^{i \int d^d x \left(\frac{1}{2}\phi(\partial^2 - m^2)\phi - \frac{\lambda}{4!}\phi^4\right)} \phi(x_1) \cdots \phi(x_n) \tag{3.3.5}$$

有时 $Z(J)$ 被称为 generating functional, 因为它能生成 Green's functions.

3.3.1 collision between particles

- 通过 Wick's way, 考虑 $J(x_1)J(x_2)J(x_3)J(x_4)$ 项, 实际上就是要计算 $G^{(4)}(x_1, x_2, x_3, x_4)$, 它的 0 阶项为,

$$G^{(4)}(x_1, x_2, x_3, x_4, \lambda = 0) = \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta J(x_2)} \frac{\delta}{i\delta J(x_3)} \frac{\delta}{i\delta J(x_4)} e^{-\frac{i}{2} \int d^d x d^d y \, J(x) D(x-y) J(y)}$$
$$= -(D_{12} D_{34} + D_{13} D_{24} + D_{14} D_{23}) \tag{3.3.6}$$

其中 D_{ij} 是 $D(x_i - x_j)$ 的简写, 可见, 传播子实际上是 $(-i)^3 D = iD$.

- $G^{(4)}_{1234}$ 的 1 阶项为,

$$\begin{aligned} \text{1st order term} = &-\frac{i\lambda}{4!} \int d^d z \, \langle \phi_1 \cdots \phi_4 \phi^4(z) \rangle \\ = &-\frac{i\lambda}{4!} \int d^d z \, \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_4} \left(\frac{\delta}{i\delta J(z)}\right)^4 e^{-\frac{i}{2} \int d^d x d^d y \, J(x) D(x-y) J(y)} \\ = &-\frac{i\lambda}{4!} \int d^d z \, \Big(4! D_{1z} D_{2z} D_{3z} D_{4z} \\ &+ 4 \times 3 D_{12} D_{3z} D_{4z} D_{zz} + \cdots + 3 D_{12} D_{34} D_{zz} D_{zz} + \cdots \Big) \end{aligned} \tag{3.3.7}$$

其中各项分别对应如下 Feynman diagrams,

$$-i\lambda \int d^d z \, D_{1z} D_{2z} D_{3z} D_{4z} \qquad \frac{-i\lambda}{2!} \int d^d z D_{13} D_{2z} D_{4z} D_{zz} \qquad \frac{-i\lambda}{8} \int d^d z D_{13} D_{24} D_{zz} D_{zz}$$

Figure 3.2: position space - Feynman diagrams

其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,

$$= (4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!}\right)^2 \int d^d z_1 d^d z_2 \, D_{1z_1} D_{2z_1} D_{3z_2} D_{4z_2} D_{z_1 z_2} D_{z_1 z_2} \tag{3.3.8}$$

3.3.2 in momentum space

- 本 subsection 将 (3.3.5) 转换到 momentum space, 注意到 $\tilde{J}(k)$ 和 $\tilde{J}(-k)$ 并不独立, 所以 $\frac{\partial}{\partial i\tilde{J}}$ 不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\tilde{G}^{(n)}(k_1, \cdots, k_n) = \int d^d x_1 \cdots d^d x_n \, e^{-i(k_1 \cdot x_1 + \cdots)} G^{(n)}(x_1, \cdots, x_n)$$

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$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \cdots d^d x_n \, e^{-i(k_1 \cdot x_1 + \cdots)} \, \langle \left(-\frac{i\lambda}{4!} \int d^d z \, \phi_z^4 \right)^n \phi_1 \cdots \phi_n \rangle \tag{3.3.9}$$

– propagator 的 Fourier transformation 是,

$$\tilde{D}_{pq} = \int d^d x d^d y \, e^{-i(p \cdot x + q \cdot y)} D(x-y) = \frac{(2\pi)^d \delta^{(d)}(p+q)}{-p^2 - m^2 + i\epsilon} \tag{3.3.10}$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ 的 1 阶项为,

$$\text{1st order term} = -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 \, e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \, \langle \phi_z^4 \phi_1 \cdots \phi_4 \rangle \tag{3.3.11}$$

考虑第 1 项,

$$-\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 \, e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \, 4! D_{1z} \cdots D_{4z}$$

$$= -i\lambda \int d^d x_1 \cdots d^d x_4 d^d z \, e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - z) + \cdots)} \prod_{i=1}^{4} \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon}$$

$$= -i\lambda \underbrace{\int d^d z \, e^{-iz \cdot (k_1 + \cdots + k_4)}}_{=(2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4)} \prod_{i=1}^{4} \frac{1}{-k_i^2 - m^2 + i\epsilon} \tag{3.3.12}$$

– 出射粒子不一定 on-shell (?).

- 得到这些 Feynman diagrams,

$$(2\pi)^d \delta^{(d)}(k_1 + k_2) \frac{i}{-k_1^2 - m^2 + i\epsilon}$$

$$-i\lambda (2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4) \prod_{i=1}^{4} \frac{i}{-k_i^2 - m^2 + i\epsilon}$$

$$-\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=2,4} \frac{i}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon}$$

$$-\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \prod_{i=1,2} \int \frac{d^d p_i}{(2\pi)^d} \frac{i}{-p_i^2 - m^2 + i\epsilon}$$

$$(2\pi)^d \delta^{(d)}(k_1 + k_3) \frac{i}{-k_1^2 - m^2 + i\epsilon}$$

$$(2\pi)^d \delta^{(d)}(k_1 + k_3)(2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{i}{-k_i^2 - m^2 + i\epsilon}$$

Figure 3.3: momentum space - Feynman diagrams

calculation:

第 3 幅图的计算如下,

$$-\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 \, e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \, D_{13} D_{2z} D_{4z} D_{zz}$$

$$= -\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 d^d z \, e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - z) + p_4 \cdot (x_4 - z) + p_4 \cdot 0)}$$

$$\prod_{i=1}^{4} \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon}$$

$$= -\frac{i\lambda}{2!} \int d^d z \, e^{-iz \cdot (p_2 + p_4)} \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_4 - k_4)$$

$$\prod_{i=1}^{4} \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon}$$

$$= -\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2,4} \frac{1}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{-p^2 - m^2 + i\epsilon} \quad (3.3.13)$$

第 4 幅图的计算如下,

$$-\frac{i\lambda}{8} \int d^d x_1 \cdots d^d x_4 \, e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \, D_{13} D_{24} D_{zz} D_{zz}$$

$$= -\frac{i\lambda}{8} \int d^d x_1 \cdots d^d x_4 d^d z \, e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - x_4) + p_3 \cdot 0 + p_4 \cdot 0)}$$

$$\prod_{i=1}^{4} \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon}$$

$$= -\frac{i\lambda}{8} \int d^d z \, \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_2 + k_4)$$

$$\prod_{i=1}^{4} \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon}$$

$$= -\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{1}{-k_i^2 - m^2 + i\epsilon}$$

$$\prod_{i=1,2} \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \quad (3.3.14)$$

- 再举一个例子 (略去了 $\prod_{i=1}^{6} \frac{i}{-k_i^2 - m^2 + i\epsilon}$),

$$= (4!)^2 \times \left(-\frac{i\lambda}{4!}\right)^2 (2\pi)^{2d} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \delta^{(d)}(k_1 + k_2 + k_3 + p) \delta^{(d)}(k_4 + k_5 + k_6 - p)$$

$$= (-i\lambda)^2 (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6) \frac{i}{-(k_1 + k_2 + k_3)^2 - m^2 + i\epsilon} \quad (3.3.15)$$

3.3.3 loops and a first look at divergence

- subsection 3.3.2 里的 loop diagrams 出现了如下积分,

$$\int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} = \int \frac{d^D p}{(2\pi)^D 2\omega_p} \sim \int \frac{d^D p}{|p|} \quad (3.3.16)$$

积分发散.

- 再举一个例子 (略去了 $\prod_{i=1}^{4} \frac{i}{-k_i^2 - m^2 + i\epsilon}$),
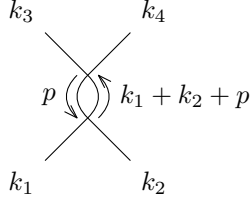
$$
\begin{aligned}
=& (4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!}\right)^2 \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{i}{-q^2 - m^2 + i\epsilon} \\
& (2\pi)^d \delta^{(d)}(k_1 + k_2 + p - q)(2\pi)^d \delta^{(d)}(k_3 + k_4 - p + q) \\
=& \frac{(-i\lambda)^2}{2} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \frac{i}{-(k_1 + k_2 + p)^2 - m^2 + i\epsilon} \\
=& \frac{(-i\lambda)^2}{2} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int \frac{d^D p}{(2\pi)^D} \Bigg(\frac{1}{2\omega_p} \frac{i}{(k_1^0 + k_2^0 - \omega_p)^2 - \omega_{k_1 + k_2 + p}^2} \\
& + \frac{i}{(\omega_{k_1 + k_2 + p} - k_1^0 - k_2^0)^2 - \omega_p^2} \frac{1}{2\omega_{k_1 + k_2 + p}} \Bigg)
\end{aligned}
$$

(3.3.17)

(3.3.18)

$$
\sim \int \frac{d^D p}{p^3}
$$

(3.3.19)

同样, 积分发散.

Chapter 4

canonical quantization

- A. Zee: the canonical and the path integral formalisms often appear complementary, in the sense that results difficult to see in one are clear in the other.

- **nobody is perfect:**

  - **canonical quantization:** 如何定义场算符乘积的顺序.
  - **path integral:** integration measure.

4.1 Heisenberg and Dirac

4.1.1 quantum mechanics

- 单粒子的 classical Lagrangian 为,

$$L = \frac{1}{2}\dot{q}^2 - V(q) \implies \begin{cases} p = \dot{q} \\ H = p\dot{q} - L = \frac{1}{2}p^2 + V(q) \end{cases} \tag{4.1.1}$$

- canonical commutation relation 如下,

$$[p, q] = -i \tag{4.1.2}$$

因此, 算符的演化方程为,

$$\begin{cases} \dfrac{dp}{dt} = i[H, p] = -V'(q) \\ \dfrac{dq}{dt} = i[H, q] = p \end{cases} \tag{4.1.3}$$

---

calculation:

$$\begin{cases} [p, q] = -i \\ [p, q^2] = -2iq \\ \qquad\vdots \\ [p, q^n] = -iq^{n-1} + q[p, q^{n-1}] \end{cases} \implies [p, q^n] = -inq^{n-1} \implies [p, V(q)] = -iV'(q) \tag{4.1.4}$$

- follow Dirac's approach,

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \iff \begin{cases} q = \dfrac{1}{\sqrt{2\omega}}(a + a^\dagger) \\ p = -i\sqrt{\dfrac{\omega}{2}}(a - a^\dagger) \end{cases} \implies [a, a^\dagger] = 1 \tag{4.1.5}$$

算符 a 的演化方程为,

$$\frac{da}{dt} = -i\sqrt{\frac{\omega}{2}}\left(\frac{1}{\omega}V'(q) + ip\right) \tag{4.1.6}$$

4.1.2 scalar field

- 标量场的 Lagrangian 为,

$$L = \int d^D x \left(-\frac{1}{2}((\partial\phi)^2 + m^2\phi^2) - u(\phi) \right) \tag{4.1.7}$$

canonical commutation relation 为,

$$\pi(\vec{x}, t) = \frac{\delta L(t)}{\delta \partial_0 \phi(\vec{x}, t)} = \partial_0 \phi(\vec{x}, t) \quad \text{and} \quad [\pi(\vec{x}, t), \phi(\vec{y}, t)] = -i\delta^{(D)}(\vec{x} - \vec{y}) \tag{4.1.8}$$

标量场的 Hamiltonian 为,

$$H = \int d^D x \, (\pi\phi - \mathcal{L}) = \int d^D x \left(\frac{1}{2}(\pi^2 + |\vec{\nabla}\phi|^2 + m^2\phi^2) + u(\phi) \right) \tag{4.1.9}$$

- -

- 算符的演化方程为,

$$\begin{cases} \partial_0 \phi = i[H, \phi] = \pi \\ \partial_0 \pi = i[H, \pi] = (-\vec{\nabla}^2 + m^2)\phi + \dfrac{du}{d\phi} \end{cases} \implies (\partial^2 - m^2)\phi - \frac{du}{d\phi} = 0 \tag{4.1.10}$$

- 当 $u(\phi) = 0$ 时, 求解场方程 (4.1.10) 和 canonical commutation relation (4.1.8) 得到,

$$\phi(\vec{x}, t) = \int \frac{d^D k}{(2\pi)^D 2\omega_k} (\alpha_k(t) e^{i\vec{k}\cdot\vec{x}} + \alpha_k^\dagger(t) e^{-i\vec{k}\cdot\vec{x}}) \tag{4.1.11}$$

其中,

$$\alpha_k(t) = \sqrt{(2\pi)^D 2\omega_k} \, a_{\vec{k}} e^{-i\omega_k t} \quad \text{and} \quad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = \delta^{(D)}(\vec{p} - \vec{q}) \tag{4.1.12}$$

另外, 在后面的笔记中使用简记 $\sqrt{(2\pi)^D 2\omega_k} = \rho(k)$.

---

calculation:

求解场方程 (4.1.10), 得到,

$$\phi(\vec{x}, t) = \int \frac{d^D k}{(2\pi)^D} (\alpha_{\vec{k}} e^{i(-\omega_k t + \vec{k}\cdot\vec{x})} + \alpha_{\vec{k}}^\dagger e^{-i(-\omega_k t + \vec{k}\cdot\vec{x})}) \tag{4.1.13}$$

代入 canonical commutation relation (4.1.8), 有 (其中 $x^0 = y^0 = t$, $k^0 = \omega_k$),

$$\int \frac{d^D k_2}{(2\pi)^D} \Big(-i\omega_{k_1}[\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] e^{i(k_1 \cdot x + k_2 \cdot y)} + i\omega_{k_1}[\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}^\dagger] e^{-i(k_1 \cdot x + k_2 \cdot y)}$$

$$- i\omega_{k_1}[\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] e^{i(k_1 \cdot x - k_2 \cdot y)} + i\omega_{k_1}[\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}] e^{-i(k_1 \cdot x - k_2 \cdot y)} \Big) = -i e^{i\vec{k}_1 \cdot (\vec{x} - \vec{y})}$$

$$\implies \begin{cases} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] = \dfrac{1}{2\omega_{k_1}} \delta^{(D)}(\vec{k}_1 + \vec{k}_2) \implies [\alpha_{\vec{k}}, \alpha_{\vec{k}}] \neq 0 \quad \text{wrong} \\ [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] = \dfrac{1}{2\omega_{\vec{k}_1}} \delta^{(D)}(\vec{k}_1 - \vec{k}_2) \qquad\qquad\qquad \text{right} \end{cases} \tag{4.1.14}$$

- 代入 (4.1.9) 可得 (依然是 $u(\phi) = 0$ 的情况下),

$$H = \int d^D k \, \omega_k \frac{a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^\dagger}{2} = \int d^D k \, \omega_k \left(a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2}\delta^{(D)}(0) \right) \implies \langle 0|H|0\rangle = V \int \frac{d^D k}{(2\pi)^D} \frac{1}{2}\omega_k \tag{4.1.15}$$

其中, $V = \int d^D x = (2\pi)^D \delta^{(D)}(0)$.

- vacuum state 定义为 $a_{\vec{k}} |0\rangle = 0$, 有,

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{ik\cdot(x-y)} \tag{4.1.16}$$

其中 $k^0 = \omega_k$. 因此, 对比 (1.2.1), 有,

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = iD(x-y) \tag{4.1.17}$$

4.2 interaction picture

- 注意, 在 $u(\phi) \neq 0$ 的情况下, (即便在 Schrödinger's picture 里, $t = 0$ 时) (4.1.11) 不再成立, 因此无法通过 Schrödinger's picture or Heisenberg's picture 求解存在相互作用的场论.

- 将 Hamiltonian 分成两个部分,

$$H = H_0 + H' \tag{4.2.1}$$

- operators 以自由场的 Hamiltonian 演化,

$$O_I(t) = U_0^\dagger(t, 0) O(0) U_0(t, 0) \quad \text{where} \quad U_0(t_2, t_1) = \text{Texp}\Big(-i \int_{t_1}^{t_2} dt\, H_0 \Big) \tag{4.2.2}$$

states 以如下方式演化,

$$|\psi(t)\rangle_I = U_0^\dagger(t, 0) U(t, 0) |\psi(0)\rangle \quad \text{where} \quad U(t_2, t_1) = \text{Texp}\Big(-i \int_{t_1}^{t_2} dt\, H \Big) \tag{4.2.3}$$

因此,

$$|\psi(t_2)\rangle_I = U_I(t_2, t_1) |\psi(t_1)\rangle_I \quad \text{where} \quad U_I(t_2, t_1) = \text{Texp}\Big(-i \int_{t_1}^{t_2} dt\, H_I(t) \Big) \tag{4.2.4}$$

注意, (4.2.2) 和 (4.2.3) 中, Texp 里的 H, H_0 都是 Schrödinger's picture 里的算符.

> **calculation:**
>
> 首先有,
>
> $$U_I(t_2, t_1) = U_0^\dagger(t_2, 0) U(t_2, t_1) U_0(t_1, 0) \tag{4.2.5}$$
>
> 因此,
>
> $$\frac{d}{dt} U_I(t, t_0) = i H_0 U_I(t, t_0) - i U_0^\dagger(t, 0) H U(t, t_0) U_0(t_0, 0)$$
> $$= -i \underbrace{U_0^\dagger(t, 0) H' U_0(t, 0)}_{=H_I(t)} U_I(t, t_0) \tag{4.2.6}$$

4.3 scattering amplitude

- 最一般的过程是 $p_1, \cdots, p_m \to q_1, \cdots, q_n$, 其 scattering amplitude 为,

$$\langle q_1, \cdots, q_n | U_0^\dagger(-\infty, 0) U_I(+\infty, -\infty) U_0(-\infty, 0) | p_1, \cdots, p_m \rangle \tag{4.3.1}$$

一般会忽略掉 U_0 产生的相位.

- 考虑 ϕ^4 理论中的 $k_1, k_2 \to k_3, k_4$ 过程,

$$\langle k_3, k_4 | e^{-i \int d^d x\, \frac{\lambda}{4!} \phi^4} | k_1, k_2 \rangle \tag{4.3.2}$$

对 λ 展开, 0 阶项为,

$$\text{0th order term} = \langle k_3, k_4 | k_1, k_2 \rangle$$
$$= \rho(k_1) \rho(k_2) \rho(k_3) \rho(k_4) \langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle$$
$$= \rho(k_1) \rho(k_2) \rho(k_3) \rho(k_4) \Big(\underbrace{\langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle}_{= \delta_{31}^{(D)} \delta_{42}^{(D)}} + \underbrace{\langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle}_{= \delta_{32}^{(D)} \delta_{41}^{(D)}} \Big)$$
$$= (2\pi)^{2D} 4 \omega_{k_1} \omega_{k_2} \big(\delta^{(D)}(\vec{k}_1 - \vec{k}_3) \delta^{(D)}(\vec{k}_2 - \vec{k}_4) + \delta^{(D)}(\vec{k}_1 - \vec{k}_4) \delta^{(D)}(\vec{k}_2 - \vec{k}_3) \big) \tag{4.3.3}$$

1 阶项为 (其中 $k^0 = \omega_k$),

$$\text{1st order term} = \frac{-i\lambda}{4!} \int d^d x\, \langle k_3, k_4 | \phi^4(x) | k_1, k_2 \rangle$$

$$= 4! \times \overbrace{\frac{-i\lambda}{4!} \int d^d x\, e^{i(k_1+k_2-k_3-k_4)\cdot x}}^{=-i\lambda(2\pi)^d \delta^{(d)}(k_1+k_2-k_3-k_4)} + \rho(k_1)\rho(k_4)\delta_{14}^{(D)} \times 12 \times \frac{-i\lambda}{4!}(2\pi)^d \delta_{23}^{(d)} \int \frac{d^D p}{\rho^2(p)}$$

$$+ \cdots + \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\delta_{13}^{(D)}\delta_{24}^{(D)} \times 3 \times \frac{-i\lambda}{4!} \int d^d x \int \frac{d^D p_1}{\rho^2(p_1)}\frac{d^D p_2}{\rho^2(p_2)} + \cdots \quad (4.3.4)$$

分别对应如下 Feynman diagrams,

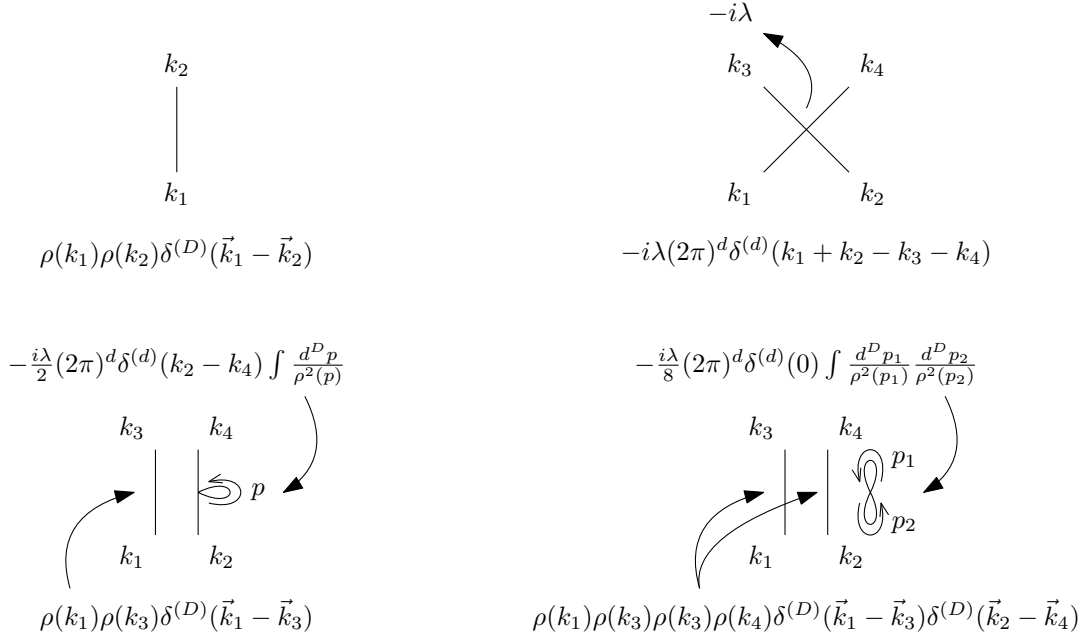$$\rho(k_1)\rho(k_2)\delta^{(D)}(\vec{k}_1 - \vec{k}_2)$$

$$-i\lambda(2\pi)^d\delta^{(d)}(k_1 + k_2 - k_3 - k_4)$$

$$-\frac{i\lambda}{2}(2\pi)^d\delta^{(d)}(k_2 - k_4)\int \frac{d^D p}{\rho^2(p)}$$

$$\rho(k_1)\rho(k_3)\delta^{(D)}(\vec{k}_1 - \vec{k}_3)$$

$$-\frac{i\lambda}{8}(2\pi)^d\delta^{(d)}(0)\int \frac{d^D p_1}{\rho^2(p_1)}\frac{d^D p_2}{\rho^2(p_2)}$$

$$\rho(k_1)\rho(k_3)\rho(k_3)\rho(k_4)\delta^{(D)}(\vec{k}_1 - \vec{k}_3)\delta^{(D)}(\vec{k}_2 - \vec{k}_4)$$

Figure 4.1: canonical quantization - Feynman diagrams

观察可见, 上图和 figure 3.3 有对应关系.

- 再举一个例子,

$$=(4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!}\right)^2 \rho(k_1)\cdots \int d^d x_1 d^d x_2 \int \frac{d^D p_1 \cdots}{\rho(p_1)\cdots}\frac{d^D q_1 \cdots}{\rho(q_1)\cdots} e^{i(p_1+p_2-p_3-p_4)\cdot x_1}e^{i(q_1+q_2-q_3-q_4)\cdot x_2}$$

$$\left(\theta(t_2 - t_1)\langle 0|a_{\vec{k}_3}a_{\vec{k}_4}a_{\vec{q}_1}a_{\vec{q}_2}a_{\vec{q}_3}^\dagger a_{\vec{q}_4}^\dagger a_{\vec{p}_1}a_{\vec{p}_2}a_{\vec{p}_3}^\dagger a_{\vec{p}_4}^\dagger a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger|0\rangle + \cdots\right)$$

$$=\frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 \int \frac{d^D p_3}{\rho^2(p_3)}\frac{d^D p_4}{\rho^2(p_4)}\Big(\theta(t_2 - t_1)e^{i(k_1+k_2-p_3-p_4)\cdot x_1}e^{i(p_3+p_4-k_3-k_4)\cdot x_2}$$

$$+ \theta(t_1 - t_2)e^{i(k_1+k_2+p_3+p_4)\cdot x_1}e^{i(-p_3-p_4-k_3-k_4)\cdot x_2}\Big)$$

$$=\frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2\, e^{i((k_1+k_2)\cdot x_1 - (k_3+k_4)\cdot x_2)} \int \frac{d^D p_3}{\rho^2(p_3)}\frac{d^D p_4}{\rho^2(p_4)}\Big(\theta(t_2 - t_1)e^{i(p_3+p_4)\cdot(x_2-x_1)}$$

$$+ \theta(t_1 - t_2)e^{i(p_3+p_4)\cdot(x_1-x_2)}\Big) \quad (4.3.5)$$

同样, 与 (3.3.18) 有对应关系, (注意按时间排序 $\langle k_3 k_4|T(\phi^4(x_1)\phi^4(x_2))|k_1 k_2\rangle$).

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4.4 complex scalar field

- complex scalar field 的 Lagrangian 为,

$$\mathcal{L} = -(\partial\psi^\dagger)(\partial\psi) - m^2\psi^\dagger\psi \tag{4.4.1}$$

实际上, complex scalar field 可以视为 2 个 real scalar fields 的和,

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \implies \left| \frac{\partial\phi_1, \phi_2}{\partial\psi, \psi^\dagger} \right| = i \tag{4.4.2}$$

因此, 也可以把 $\psi, \psi^\dagger$ 视为两个独立的场.

- 其 canonical momentum 为,

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi} = \partial_0\psi^\dagger \quad \pi^\dagger = \partial_0\psi \tag{4.4.3}$$

其 Hamiltonian 为,

$$\mathcal{H} = \pi^\dagger\pi + (\vec{\nabla}\psi^\dagger)\cdot(\vec{\nabla}\psi) + m^2\psi^\dagger\psi \tag{4.4.4}$$

$$\implies \begin{cases} \partial_0\pi = i[H,\pi] = \vec{\nabla}^2\psi^\dagger - m^2\psi^\dagger \\ \partial_0\psi = i[H,\psi] = \pi^\dagger \end{cases} \implies (-\partial^2 - m^2)\psi = 0 \tag{4.4.5}$$

- 求解得到 (其中 $k^0 = \omega_k$),

$$\psi(x) = \int \frac{d^D k}{\rho(k)} (a_{\vec{k}} e^{ik\cdot x} + b_{\vec{k}}^\dagger e^{-ik\cdot x}) \tag{4.4.6}$$

- 从 path integral 的角度,

$$Z(J, J^\dagger) = \int D\psi D\psi^\dagger \, e^{i\int d^d x \, (\psi^\dagger(\partial^2 - m^2)\psi + J^\dagger\psi + \psi^\dagger J)} \tag{4.4.7}$$

$$= \mathcal{C} e^{-\frac{i}{2}\int d^d x d^d y \, 2J^\dagger(x)D(x-y)J(y)} \tag{4.4.8}$$

$$= \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y \, 2 J^\dagger(x) D(x-y) J(y)} \tag{4.4.9}$$

4.4.1 charge

- 对场算符做如下变换,

$$\psi(x, \lambda) = e^{i\lambda} \psi(x) \implies D_\lambda \mathcal{L} = 0 \tag{4.4.10}$$

- 因此, 得到 conserved current,

$$J^\mu = \pi^\mu D_\lambda \psi + \pi^{\dagger\mu} D_\lambda \psi^\dagger = i(\psi \partial^\mu \psi^\dagger - \psi^\dagger \partial^\mu \psi) \tag{4.4.11}$$

其 0 分量对空间积分就是 charge,

$$Q = \int d^D x \, J^0 = \int d^D x \, i(\psi^\dagger \partial_0 \psi - \psi \partial_0 \psi^\dagger)$$
$$= \int d^D k \, (a_{\vec{k}}^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}}) \tag{4.4.12}$$

calculation:

$$Q = \int d^D x \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} i\Big((a_{\vec{p}}^\dagger e^{-ip\cdot x} + b_{\vec{p}} e^{ip\cdot x})(-i\omega_q)(a_{\vec{q}} e^{iq\cdot x} - b_{\vec{q}}^\dagger e^{-iq\cdot x})$$
$$- (a_{\vec{q}} e^{iq\cdot x} + b_{\vec{q}}^\dagger e^{-iq\cdot x})(i\omega_p)(a_{\vec{p}}^\dagger e^{-ip\cdot x} - b_{\vec{p}} e^{ip\cdot x}) \Big)$$
$$= \int d^D x \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} \Big((\omega_p a_{\vec{q}} a_{\vec{p}}^\dagger + \omega_q a_{\vec{p}}^\dagger a_{\vec{q}}) e^{-i(p-q)\cdot x} - (\omega_p b_{\vec{q}}^\dagger b_{\vec{p}} + \omega_q b_{\vec{p}} b_{\vec{q}}^\dagger) e^{i(p-q)\cdot x}$$
$$+ a_{\vec{p}}^\dagger b_{\vec{q}}^\dagger (\omega_p - \omega_q) e^{-i(p+q)\cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{i(p+q)\cdot x} \Big)$$
$$= \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)}$$
$$\Big(\big((\omega_p a_{\vec{q}} a_{\vec{p}}^\dagger + \omega_q a_{\vec{p}}^\dagger a_{\vec{q}}) e^{i(\omega_p - \omega_q)\cdot t} - (\omega_p b_{\vec{q}}^\dagger b_{\vec{p}} + \omega_q b_{\vec{p}} b_{\vec{q}}^\dagger) e^{-i(\omega_p - \omega_q)\cdot t} \big)(2\pi)^D \delta^{(D)}(\vec{p} - \vec{q})$$
$$+ \big(a_{\vec{p}}^\dagger b_{\vec{q}}^\dagger (\omega_p - \omega_q) e^{i(\omega_p + \omega_q)\cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{-i(\omega_p + \omega_q)\cdot x} \big)(2\pi)^D \delta^{(D)}(\vec{p} + \vec{q}) \Big)$$
$$= \int \frac{d^D k}{2} (a_{\vec{k}} a_{\vec{k}}^\dagger + a_{\vec{k}}^\dagger a_{\vec{k}} - b_{\vec{k}} b_{\vec{k}}^\dagger - b_{\vec{k}}^\dagger b_{\vec{k}}) = \int d^D k \, (a_{\vec{k}}^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}}) \tag{4.4.13}$$

- 代入 (D.3.2), 有 $i[Q, \psi] = -i\psi$, 所以,

$$e^{-i\lambda Q} \psi e^{i\lambda Q} = e^{i\lambda} \psi \tag{4.4.14}$$

Chapter 5

disturbing the vacuum: Casimir effect

- 考虑一个沿 x^1 方向满足 periodic b.c. 的空间, 在垂直于 x^1 方向有两个 plates, s.t. 在 plates 上 $\phi(x) = 0$, 如下图,

Figure 5.1: Casimir effect

- 平板内外, 标量场的波矢的取值为,

$$
\begin{cases}
(n\dfrac{\pi}{d}, k_2, k_3) & \text{平板内} \\[2mm]
(n\dfrac{\pi}{L-d}, k_2, k_3) & \text{平板外}
\end{cases}
\tag{5.0.1}
$$

其中 $n \in \mathbb{Z}^+$.

- 因此, 代入真空能公式 (4.1.15), 平板内的能量为,

$$
\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2}
\tag{5.0.2}
$$

而总能量为 $E = E(d) + E(L-d)$.

- 为解决能量发散的问题, 引入 ultra-violet (UV) cut-off,

$$
\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2} \, e^{-a\sqrt{(n\frac{\pi}{d})^2 + k_2^2 + k_3^2}}
\tag{5.0.3}
$$

for some $a \ll d$.

- 为了简化问题, 考虑 $d = 1+1$ 的情况,

$$
E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-\frac{a\pi}{d}n} = \frac{\pi}{2d} \frac{e^{\frac{a\pi}{d}}}{(e^{\frac{a\pi}{d}} - 1)^2} = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O(a^2)
\tag{5.0.4}
$$

因此,

$$
E_{1+1} = E_{1+1}(d) + E_{1+1}(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24}\left(\frac{1}{d} + \frac{1}{L-d}\right) + O(a^2)
\tag{5.0.5}
$$

得到 Casimir force,

$$
F_{1+1} = -\frac{\partial E_{1+1}}{\partial d} = -\frac{\pi}{24}\left(\frac{1}{d^2} - \frac{1}{(L-d)^2}\right) + O(a^2) \overset{L\to\infty, a\to 0}{=} -\frac{\pi}{24d^2}
\tag{5.0.6}
$$

- 问题中, a 引入了 UV cut-off, L 引入了 infrared cut-off.

Part II

Dirac and spinor

Chapter 6

the Dirac equation

- 整个 Part II 中, 我们使用 $(+, -, -, -)$ 号差, 因为 $\mathrm{Cl}_{1,3}(\mathbb{R}) \not\simeq \mathrm{Cl}_{3,1}(\mathbb{R})$.

- 本笔记中的算符的定义与 A. Zee 的定义不同, 存在如下对应关系,

A. Zee's def.	my def.
$\omega_{\mu\nu}$	$\omega_{\mu\nu}$
$-iJ^{\mu\nu}$	$J^{\mu\nu}$
$-i\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$

6.1 gamma matrices

- Pauli 矩阵如下,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.1.1}$$

- gamma 矩阵 (also called Dirac matrices) 如下 (其中 $i = 1, 2, 3$),

$$\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} = I \otimes \tau_3 \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = i\sigma_i \otimes \tau_2 \quad \Omega = \gamma^0\gamma^1\gamma^2\gamma^3 = -i\begin{pmatrix} & I \\ I & \end{pmatrix} = -iI \otimes \tau_1 \tag{6.1.2}$$

其中 $\tau_{2,3}$ 也是 Pauli 矩阵, 最后, 按照惯例, 定义 $\gamma^5 = i\Omega = I \otimes \tau_1$.

  – 另外,

$$\begin{cases} \gamma^0\gamma^i = \sigma_i \otimes \tau_1 \\ \gamma^i\gamma^j = -(\sigma_i\sigma_j) \otimes I \end{cases} \qquad \begin{cases} \Omega\gamma^0 = -I \otimes \sigma_2 \\ \Omega\gamma^i = i\sigma_i \otimes \tau_3 \end{cases} \tag{6.1.3}$$

- gamma 矩阵满足,

$$\begin{cases} (\gamma^\mu)^2 = \eta^{\mu\mu} \\ \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \mu \neq \nu \end{cases} \implies \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{6.1.4}$$

- 且存在如下关系,

$$\Omega\gamma^0 = -\gamma^1\gamma^2\gamma^3 \quad \Omega\gamma^1 = -\gamma^0\gamma^2\gamma^3 \quad \Omega\gamma^2 = \gamma^0\gamma^1\gamma^3 \quad \Omega\gamma^3 = -\gamma^0\gamma^1\gamma^2$$
$$\iff -\epsilon^{\mu\nu\rho}{}_\sigma \Omega\gamma^\sigma = \gamma^\mu\gamma^\nu\gamma^\rho \quad \text{when} \quad \mu \neq \nu \neq \rho \tag{6.1.5}$$

并且有 (注意到 $\Omega^2 = -1$),

$$\{\Omega, \gamma^\mu\} = 0 \quad \{\Omega, \Omega\gamma^\mu\} = 0 \quad [\Omega, \gamma^\mu\gamma^\nu] = 0 \tag{6.1.6}$$

- -

- 定义 $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ (注意, 我们的定义中没有虚数 i, 与 A. Zee 的定义不同),

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} + \sigma^{\mu\nu} \implies \begin{cases} \sigma^{0i} = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix} = \sigma_i \otimes \tau_1 \\ \sigma^{ij} = -i\epsilon^{ijk}\begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk}\sigma_k \otimes I \end{cases} \tag{6.1.7}$$

6.2 Lorentz transformation and the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation

- Lorentz 变换可以写成如下形式,

$$\Lambda = e^{\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{6.2.1}$$

其中 $\omega_{\mu\nu}$ 反对称, J^{0i} generate boosts and J^{ij} generate rotations, (详见笔记 Lie Groups and Lie Algebras).

$$x = \phi^* x' = \Lambda^{-1} x'$$
$$\phi_* \frac{\partial}{\partial x^\mu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x'^\mu} \text{ or } \eta\Lambda\eta\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$$
$$\phi^* dx'^\mu = \Lambda_\nu{}^\mu dx'^\nu = dx^\mu \text{ or } \Lambda^{-1}dx' = dx$$

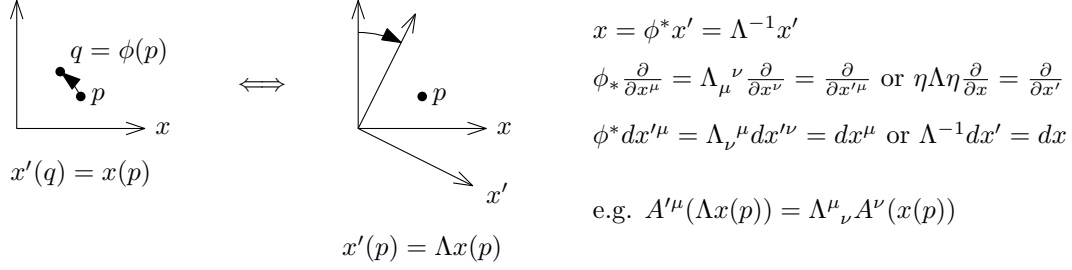$$\text{e.g. } A'^\mu(\Lambda x(p)) = \Lambda^\mu{}_\nu A^\nu(x(p))$$

Figure 6.1: Lorentz transformation

- 有 $\pi_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(J^{\mu\nu}) = \frac{1}{2}\sigma^{\mu\nu}$ (up to a similarity transformation).

> **calculation:**
>
> 做如下相似变换,
>
> $$S = \frac{\sqrt{2}}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix} \iff S^{-1} = \frac{\sqrt{2}}{2}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} \tag{6.2.2}$$
>
> 得到,
>
> $$S^{-1}\sigma^{0i}S = \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix} \quad S^{-1}\sigma^{ij}S = -i\epsilon^{ijk}\begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} \tag{6.2.3}$$
>
> 得到的结果和笔记 Lie Groups and Lie Algebras 中 $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ 表示是完全一样的.

- Dirac spinor 是 $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ rep. 的 vector space 中的元素,

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad \text{with} \quad S^{-1}\psi = \frac{\sqrt{2}}{2}\begin{pmatrix} \phi - \chi \\ \phi + \chi \end{pmatrix} \tag{6.2.4}$$

其中,

$$\frac{\sqrt{2}}{2}\begin{pmatrix} \phi \\ -\phi \end{pmatrix} \in (\frac{1}{2},0) \quad \text{and} \quad \frac{\sqrt{2}}{2}\begin{pmatrix} \chi \\ \chi \end{pmatrix} \in (0,\frac{1}{2}) \tag{6.2.5}$$

- 对于 gamma 矩阵, 有,

$$\Pi(\Lambda)\gamma^\rho\Pi^{-1}(\Lambda) = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = (\Lambda^{-1})^\rho{}_\sigma\gamma^\sigma \tag{6.2.6}$$

> **calculation:**
>
> 利用 Campbell's identity,
>
> $$e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = e^{\frac{1}{4}\omega_{\mu\nu}\text{ad}_{\sigma^{\mu\nu}}}\gamma^\rho \tag{6.2.7}$$
>
> 其中 (注意 $(J^{\mu\nu})^\rho{}_\sigma = 2\eta^{[\mu|\rho}\delta^{|\nu]}{}_\sigma$, 其中度规号差与笔记 Lie Groups and Lie Algebras 中的不同),
>
> $$\begin{cases} \rho \neq \mu, \nu & [\sigma^{\mu\nu}, \gamma^\rho] = \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu) \\ & \quad\quad\quad = -\frac{1}{2}\underbrace{(\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\nu\mu\rho\sigma} - \epsilon^{\rho\mu\nu\sigma} + \epsilon^{\rho\nu\mu\sigma})}_{=0}\Omega\gamma_\sigma = 0 \\ \rho = \mu \text{ or } \nu \text{ and } \mu \neq \nu & [\sigma^{\mu\nu}, \gamma^\rho] = 2(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\nu) \end{cases}$$
>
> $$\implies [\sigma^{\mu\nu}, \gamma^\rho] = 2(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu) = -2(J^{\mu\nu})^\rho{}_\sigma\gamma^\sigma \tag{6.2.8}$$

代入, 得到,

$$e^{\frac{1}{4}\omega_{\mu\nu}\mathrm{ad}_{\sigma^{\mu\nu}}}\gamma^\rho = \sum_{n=0}^\infty \frac{1}{n!}\left(\left(-\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}\right)^n\right)^\rho_{\ \sigma}\gamma^\sigma = (\Lambda^{-1})^\rho_{\ \sigma}\gamma^\sigma \tag{6.2.9}$$

可以用" 无穷小" Lorentz 变换验证以上计算,

$$\Pi(1+\delta\omega^\mu_{\ \nu})\gamma^\rho\Pi^{-1}(1+\delta\omega^\mu_{\ \nu}) = \gamma^\rho + \frac{1}{4}\delta\omega_{\mu\nu}[\sigma^{\mu\nu},\gamma^\rho]$$
$$= (1-\delta\omega^\rho_{\ \sigma})\gamma^\sigma \tag{6.2.10}$$

6.2.1 Dirac spinor

- 对于 Dirac spinor,

$$\Pi(\Lambda)\psi(x) = \psi'(\Lambda x) \tag{6.2.11}$$

注意 $\partial'_\mu = \Lambda_\mu^{\ \nu}\partial_\nu$, 所以,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \iff (i\gamma^\mu\partial'_\mu - m)\psi'(\Lambda x) = 0 \tag{6.2.12}$$

calculation:

首先,
$$\Lambda^T\eta\Lambda = \eta \implies (\Lambda^{-1})^\mu_{\ \nu} = (\eta\Lambda^T\eta)^\mu_{\ \nu} = \Lambda_\nu^{\ \mu} \tag{6.2.13}$$

考虑,
$$\Pi^{-1}(\Lambda)\gamma^\mu\Pi(\Lambda) = \Lambda^\mu_{\ \nu}\gamma^\nu \implies \gamma^\mu\Pi(\Lambda) = \Lambda^\mu_{\ \nu}\Pi(\Lambda)\gamma^\nu \tag{6.2.14}$$

代入,

$$(i\gamma^\mu\partial'_\mu - m)\psi'(\Lambda x) = (i\gamma^\mu\Lambda_\mu^{\ \nu}\partial_\nu - m)\Pi(\Lambda)\psi(x)$$
$$= \Pi(\Lambda)(i\gamma^\rho\underbrace{\Lambda^\mu_{\ \rho}\Lambda_\mu^{\ \nu}}_{=\delta^\nu_\rho}\partial_\nu - m)\psi(x)$$
$$= \Pi(\Lambda)(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{6.2.15}$$

– 关键部分在于,

$$\gamma^\mu\psi'(\Lambda x) = \gamma^\mu\Pi(\Lambda)\psi(x) = \Pi(\Lambda)\Lambda^\mu_{\ \nu}\gamma^\nu\psi(x) \tag{6.2.16}$$

6.2.2 Dirac bilinears

- γ^0 是 Hermitian 矩阵, 而 γ^i 不是, 有,

$$\gamma^{i\dagger} = -\gamma^i = \gamma^0\gamma^i\gamma^0 \tag{6.2.17}$$

可以统一写作 $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, 并且有,

$$\sigma^{\mu\nu\dagger} = -\gamma^0\sigma^{\mu\nu}\gamma^0 \quad \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0 \tag{6.2.18}$$

calculation:

对于 $\sigma^{\mu\nu}$,
$$\sigma^{\mu\nu\dagger} = \frac{1}{2}(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger}) = \gamma^0\sigma^{\nu\mu}\gamma^0 = -\gamma^0\sigma^{\mu\nu}\gamma^0 \tag{6.2.19}$$

所以,
$$((\omega_{\mu\nu}\sigma^{\mu\nu})^\dagger)^n = \gamma^0(-\omega_{\mu\nu}\sigma^{\mu\nu})^n\gamma^0 \implies \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0 \tag{6.2.20}$$

- 所以,

$$\begin{cases} \bar{\psi}'(\Lambda x)\psi'(\Lambda x) = \bar{\psi}\psi & \text{scalar field} \\ \bar{\psi}'\gamma^\mu\psi' = \Lambda^\mu_{\ \nu}\bar{\psi}\gamma^\nu\psi & \text{vector field} \end{cases} \tag{6.2.21}$$

其中 $\bar{\psi} = \psi^\dagger\gamma^0$.

> **calculation:**
>
> $$\begin{cases} \psi'^{\dagger}(\Lambda x)\gamma^0\psi'(\Lambda x) = \psi^{\dagger}(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\Pi(\Lambda)\psi(x) = \psi^{\dagger}\gamma^0\psi \\ \psi'^{\dagger}\gamma^0\gamma^{\mu}\psi' = \psi^{\dagger}(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\gamma^{\mu}\Pi(\Lambda)\psi(x) = \Lambda^{\mu}{}_{\nu}\psi^{\dagger}\gamma^0\gamma^{\nu}\psi \end{cases} \tag{6.2.22}$$
>
> ---
>
> 此外,
>
> $$\begin{cases} \bar{\psi}'\sigma^{\mu\nu}\psi' = \psi^{\dagger}\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\sigma^{\mu\nu}\Pi(\Lambda)\psi = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\bar{\psi}\sigma^{\rho\sigma}\psi & \text{order 2 tensor} \\ \bar{\psi}'\Omega\gamma^{\mu}\psi' = \bar{\psi}\Pi(\Lambda^{-1})\Omega\gamma^{\mu}\Pi(\Lambda)\psi = \det(\Lambda)\Lambda^{\mu}{}_{\nu}\bar{\psi}\Omega\gamma^{\nu}\psi & \text{pseudovector} \\ \bar{\psi}'\Omega\psi' = \bar{\psi}\Pi(\Lambda^{-1})\Omega\Pi(\Lambda)\psi = \det(\Lambda)\bar{\psi}\Omega\psi & \text{4-form, aka. pseudoscalar} \end{cases} \tag{6.2.23}$$
>
> 其中 (注意到下面的计算中, 第二个等号后, 含 η 的项都等于零; 由此可以看出, 对 μ_i 求和的过程中, 任何两个 μ_i, μ_j 相等的项求和之后都等于零),
>
> $$\begin{aligned} \Pi(\Lambda^{-1})\Omega\Pi(\Lambda) &= \prod_{i=0}^{3}\Lambda^{i}{}_{\mu_i}\gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \\ &= \prod_{i=0}^{3}\Lambda^{i}{}_{\mu_i}(\eta^{\mu_0\mu_1}+\sigma^{\mu_0\mu_1})(\eta^{\mu_2\mu_3}+\sigma^{\mu_2\mu_3}) \\ &= \prod_{i=0}^{3}\Lambda^{i}{}_{\mu_i}\gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \quad \text{with} \quad \mu_0 \neq \mu_1 \neq \mu_2 \neq \mu_3 \\ &= \det(\Lambda)\Omega \end{aligned} \tag{6.2.24}$$

6.2.3 parity and time reversal

- 这里沿用笔记 Lie Groups and Lie Algebras 中的记号, 选择 $\mathrm{O}(3,1)$ 而非 $\mathrm{O}(1,3)$, 因为他们没有区别.
- $\mathrm{O}(3,1)$ 有 4 个联通分支,

$$I \in \mathrm{SO}_+(3,1) \quad PT \in \mathrm{SO}_-(3,1) \quad P \in \mathrm{O}'_+(3,1) \quad T \in \mathrm{O}'_-(3,1) \tag{6.2.25}$$

其中,

$$P = \mathrm{diag}(+1,-1,-1,-1) \quad T = \mathrm{diag}(-1,+1,+1,+1) \tag{6.2.26}$$

另外, $\eta P\eta = P, \eta T\eta = T$.

parity

- 对于 parity, 有,

$$\Pi(P) = \gamma^0 \quad \gamma^0\gamma^{\mu} = P^{\mu}{}_{\nu}\gamma^{\nu}\gamma^0 \tag{6.2.27}$$

6.3 Dirac equation

- A. Zee: our discussion provides a unified view of the equations of motion in relativistic physics: they just project out the unphysical components.
- the Dirac equation is,

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \iff (\gamma^{\mu}p_{\mu} - m)\tilde{\psi} = 0 \tag{6.3.1}$$

首先可以看出 ψ 满足 Klein-Gordan equation,

$$(i\gamma^{\mu}\partial_{\mu} - m)(i\gamma^{\nu}\partial_{\nu} - m)\psi = \Big(-\frac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\}\partial_{\mu}\partial_{\nu} - 2im\gamma^{\mu}\partial_{\mu} + m^2\Big)\psi = 0$$
$$\Longrightarrow (-\partial^2 - m^2)\psi = 0 \tag{6.3.2}$$

– 在粒子静止系下 $p_{\mu} = (m,0,0,0)$, Dirac 方程给出,

$$(\gamma^0 - 1)\tilde{\psi} = 0 \Longrightarrow \begin{pmatrix} 0 & \\ & I \end{pmatrix}\tilde{\psi} = 0 \tag{6.3.3}$$

因此, $\tilde{\psi}$ 的后两个分量为零 $\Longrightarrow \psi$ 只有两个自由度.

Appendices

Appendix A

Dirac delta function & Fourier transformation

A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx \, \delta(x) e^{-ikx} \tag{A.1.1}$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x))g(x)dx = \sum_{\{i,f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \tag{A.1.2}$$

- 第二个常用的公式 (Sokhotski-Plemelj theorem),

$$\lim_{\epsilon \to 0^+} \frac{1}{x+i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{A.1.3}$$

其中 $\mathcal{P}$ 表示复函数的主值 (principal value).

> **proof:**
>
> 考虑,
>
> $$\frac{1}{x+i\epsilon} = \frac{x-i\epsilon}{x^2+\epsilon^2} \quad \text{and} \quad \int \frac{\epsilon}{x^2+\epsilon^2} dx = 2\pi i \operatorname{Res}(f, i\epsilon) = \pi \tag{A.1.4}$$
>
> 所以...
>
> 取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,
>
>
>
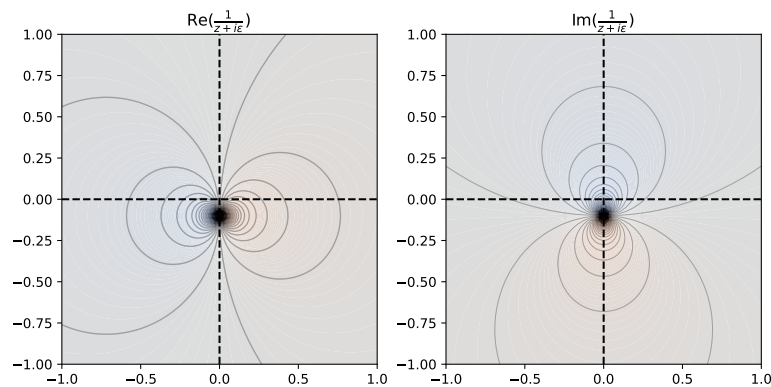> Figure A.1: graph of $\frac{1}{z+i\epsilon}$

- 另外, $\delta(x-a)\delta(x-b) = \delta(b-a)\delta(x-a)$.

A.2 Fourier transformation

- d-dim. Fourier transformation 如下,

$$
\begin{cases}
\phi(x) = \displaystyle\int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \tilde{\phi}(k) \\
\tilde{\phi}(k) = \displaystyle\int d^d x \, e^{-ik\cdot x} \phi(x)
\end{cases}
\tag{A.2.1}
$$

- 因此,

$$
\partial_\mu \phi(x) \mapsto ik_\mu \tilde{\phi}(k) \tag{A.2.2}
$$

- 对于**实函数**, Fourier transformation 是正交变换, 其 Jacobi determinant 为,

$$
\left| \frac{\partial \phi(x)\cdots}{\partial \mathrm{Re}\tilde{\phi}(k)\cdots \partial \mathrm{Im}\tilde{\phi}(k)\cdots} \right| = \left(\frac{2}{V}\right)^{(2N+1)^d} \det A = \left(\frac{2(2N)^d}{V^2}\right)^{\frac{(2N+1)^d}{2}} \tag{A.2.3}
$$

---

proof:

position space 和 momentum space 的格点分别为,

$$
\begin{cases}
x_i^\mu = i^\mu \epsilon \in \{0, \pm\epsilon, \cdots, \dfrac{L}{2}\} \\
k_n^\mu = n^\mu \dfrac{2\pi}{L} \in \{0, \pm\dfrac{2\pi}{L}, \cdots, \dfrac{\pi}{\epsilon}\}
\end{cases}
\iff i^\mu, n^\mu \in \{0, \pm 1, \cdots, N\} \tag{A.2.4}
$$

x^μ, k^μ 分别有 $2N+1$ 个取值, 其中 $N\epsilon = \frac{L}{2}$, 时空总体积为 $V = L^d$, momentum space 的总体积为 $\tilde{V} = \frac{(4\pi N)^d}{V}$.

将 (A.2.1) 写成格点求和的形式,

$$
\begin{cases}
\phi(x_i) = \dfrac{1}{(2\pi)^d}\left(\dfrac{2\pi}{L}\right)^d \displaystyle\sum_n e^{ik_n\cdot x_i} \tilde{\phi}(k_n) \\
\qquad = \dfrac{2}{V}\displaystyle\sum_{n^0>0}\left(\cos(k_n\cdot x_i)\mathrm{Re}\tilde{\phi}(k_n) - \sin(k_n\cdot x_i)\mathrm{Im}\tilde{\phi}(k_n)\right) \\
\tilde{\phi}(k_n) = \epsilon^d \displaystyle\sum_i e^{-ik_n\cdot x_i}\phi(x_i) \\
\qquad = \dfrac{V}{(2N)^d}\displaystyle\sum_i\left(\cos(k_n\cdot x_i) - i\sin(k_n\cdot x_i)\right)\phi(x_i)
\end{cases}
\tag{A.2.5}
$$

- -

proof:
$\phi(x_i)$ 的变换需要做一些说明. 注意到 $\tilde{\phi}$ 的分量的数量是 ϕ 的两倍 (考虑到实部与虚部), 但在 $\phi \in \mathbb{R}^{(2N+1)^d}$ 时,

$$
\tilde{\phi}^*(k) = \tilde{\phi}(-k) \tag{A.2.6}
$$

可见 $\tilde{\phi}$ 的分量并不独立, 取 $k^0 > 0$ 的部分为独立分量, 那么...

- -

将 (A.2.5) 写成矩阵的形式,

$$
\begin{cases}
\begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} = \dfrac{2}{V} \overbrace{\begin{pmatrix} \cos k_0\cdot x_0 & \cdots & \cos k_{\max}\cdot x_0 & -\sin k_0\cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A} \begin{pmatrix} \mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots \end{pmatrix} \\[3em]
\begin{pmatrix} \mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots \end{pmatrix} = \dfrac{V}{(2N)^d} \begin{pmatrix} \cos k_0\cdot x_0 & \cdots & \cos k_0\cdot x_{\max} \\ \vdots & \ddots & \\ -\sin k_0\cdot x_0 & \cdots & -\sin k_0\cdot x_{\max} \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix}
\end{cases}
\tag{A.2.7}
$$

观察可见 $\tilde{\phi}$ 的变换中的矩阵是 A^T, 所以,

$$\frac{2}{V}\frac{V}{(2N)^d}AA^T = I \implies \det A = \left(\frac{(2N)^d}{2}\right)^{\frac{(2N+1)^d}{2}} \tag{A.2.8}$$

因此...

– 顺便,

$$\int d^d x\, f(x)g(x) = \int \frac{d^d k}{(2\pi)^d}\,\tilde{f}(-k)\tilde{g}(k) \tag{A.2.9}$$

Appendix B

Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{B.0.1}$$

$$\langle x^{2n} \rangle = \frac{\int dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!! \tag{B.0.2}$$

其中 $(2n-1)!! = 1 \cdot 3 \cdots (2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx\, e^{-\frac{a}{2}x^2 + Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \tag{B.0.3}$$

另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 N-dim. generalization

- 考虑如下积分,

$$Z(A, J) = \int dx_1 \cdots dx_N\, e^{-\frac{1}{2}x^T \cdot A \cdot x + J^T \cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \tag{B.1.1}$$

其中 x, J 是 N-dim. 列向量, A 是 $N \times N$ 实对称矩阵.

> **calculation:**
>
> 根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,
>
> $$A = O^{-1} \cdot D \cdot O \tag{B.1.2}$$
>
> 其中 D 是一个 diagonal matrix. 令 $y = O \cdot x$, 那么,
>
> $$Z(A, J) = \int dy_1 \cdots dy_N\, e^{-\frac{1}{2}y^T \cdot D \cdot y + (OJ)^T \cdot y}$$
>
> $$= \prod_{i=1}^{N} \sqrt{\frac{2\pi}{D_{ii}}} e^{\frac{1}{2D_{ii}}(OJ)_i^2} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \tag{B.1.3}$$
>
> 其中, 注意到了 $\frac{1}{D_{ii}} = (O \cdot A^{-1} \cdot O^{-1})_{ii}$ 以及 $\operatorname{tr} D = \det A$.

- 一个重要的变体是 $A \mapsto -iA, J \mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} Z(A, J)\Big|_{J=0} = A_{ij}^{-1} \tag{B.1.4}$$

$$\langle x_i x_j \cdots x_k x_l \rangle = \sum_{\text{Wick}} A_{i'j'}^{-1} \cdots A_{k'l'}^{-1} \tag{B.1.5}$$

其中 (B.1.5) 中有偶数个 x, 否则等于零.

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A,J)\Big|_{J=0} = \cdots \tag{B.1.6}$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \tag{B.1.7}$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle \overbrace{x_i x_j x_k x_l} \rangle = A_{ik}^{-1} A_{jl}^{-1} \tag{B.1.8}$$

Appendix C

perturbation theory in QM

- this chapter is based on MIT OpenCourseWare Quantum Physics III Chapter 1: Perturbation Theory.

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda \delta H \tag{C.0.1}$$

其中 $\lambda \in [0,1]$.

- 考虑 $H^{(0)}$ 的本征态为,

$$H^{(0)} |k^{(0)}\rangle = E_k^{(0)} |k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)}|l^{(0)}\rangle = \delta_{kl} \\ E_0^{(0)} \leq E_1^{(0)} \leq E_2^{(0)} \leq \cdots \end{cases} \tag{C.0.2}$$

C.1   non-degenerate perturbation theory

- 考虑 non-degenerate 能级 k, 有 $\cdots \leq E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \leq \cdots$, 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \cdots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \cdots \end{cases} \tag{C.1.1}$$

  – 注意, 我们可以选取修正项满足,
$$\langle k^{(0)}|k^{(n)}\rangle = 0, n = 1, 2, \cdots \tag{C.1.2}$$

> **proof:**
>
> 假设我们求解得到的修正项不满足 $\langle k^{(0)}|k^{(n)}\rangle = 0, n = 1, 2, \cdots$, 考虑,
>
> $$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n |k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)}|k^{(n)}\rangle' = 0 \tag{C.1.3}$$
>
> 那么, (注意到态矢量可以乘一个常数, $\frac{1}{1 - a_1\lambda - a_2\lambda^2 - \cdots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \cdots$),
>
> $$|k\rangle_\lambda = (1 - a_1\lambda - a_2\lambda^2 - \cdots) |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \cdots$$
> $$|k\rangle_\lambda' = |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \cdots} \left(\lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \cdots \right)$$
> $$= |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 (a_1 |k^{(1)}\rangle' + |k^{(2)}\rangle') + \cdots \tag{C.1.4}$$
>
> 可见修正项都与 $|k^{(0)}\rangle$ 正交.

  – 注意, 不能要求 $_\lambda\langle k|k\rangle_\lambda = 1$, 否则 $|k^{(n)}\rangle$ 将与 λ 相关 (包括 $|k^{(0)}\rangle$),

$$\begin{aligned} _\lambda\langle k|k\rangle_\lambda = & \langle k^{(0)}|k^{(0)}\rangle \\ & + \lambda(\langle k^{(1)}|k^{(0)}\rangle + \langle k^{(0)}|k^{(1)}\rangle) \\ & + \lambda^2(\langle k^{(2)}|k^{(0)}\rangle + \langle k^{(1)}|k^{(1)}\rangle + \langle k^{(0)}|k^{(2)}\rangle) \end{aligned}$$

$$\vdots$$
$$+ \lambda^n (\langle k^{(n)}|k^{(0)}\rangle + \langle k^{(n-1)}|k^{(1)}\rangle + \cdots + \langle k^{(0)}|k^{(n)}\rangle) \tag{C.1.5}$$

- 将 (C.1.1) 代入 Schrödinger's eq., 得到,

$$
\begin{array}{ll}
\lambda^0 & (H^{(0)} - E_k^{(0)})|k^{(0)}\rangle = 0 \\
\lambda^1 & (H^{(0)} - E_k^{(0)})|k^{(1)}\rangle = (E_k^{(1)} - \delta H)|k^{(0)}\rangle \\
\lambda^2 & (H^{(0)} - E_k^{(0)})|k^{(2)}\rangle = (E_k^{(1)} - \delta H)|k^{(1)}\rangle + E_k^{(2)}|k^{(0)}\rangle \\
\vdots & \qquad\qquad\qquad\qquad\vdots \\
\lambda^n & (H^{(0)} - E_k^{(0)})|k^{(n)}\rangle = (E_k^{(1)} - \delta H)|k^{(n-1)}\rangle + E_k^{(2)}|k^{(n-2)}\rangle + \cdots + E_k^{(n)}|k^{(0)}\rangle
\end{array}
$$

> **calculation:**
>
> Schrödinger's eq. 为,
> $$(H^{(0)} + \lambda\delta H - E_k(\lambda))|k\rangle_\lambda = 0 \tag{C.1.6}$$
> 展开为,
> $$\Big((H^{(0)} - E_k^{(0)}) + \lambda(\delta H - E_k^{(1)}) - \lambda^2 E_k^{(2)} - \cdots\Big)(|k^{(0)}\rangle + \lambda|k^{(1)}\rangle + \lambda^2|k^{(2)}\rangle + \cdots) = 0 \tag{C.1.7}$$

- 现在来计算 $\langle l^{(0)}|k^{(n)}\rangle$, 有,

$$
\begin{cases}
(E_l^{(0)} - E_k^{(0)})\langle l^{(0)}|k^{(1)}\rangle = E_k^{(1)}\delta_{lk} - \delta H_{lk} \\
(E_l^{(0)} - E_k^{(0)})\langle l^{(0)}|k^{(2)}\rangle = E_k^{(1)}\langle l^{(0)}|k^{(1)}\rangle - \langle l^{(0)}|\delta H|k^{(1)}\rangle + E_k^{(2)}\delta_{lk} \\
\qquad\qquad\vdots \qquad\qquad\qquad\qquad\qquad\vdots \\
(E_l^{(0)} - E_k^{(0)})\langle l^{(0)}|k^{(n)}\rangle = E_k^{(1)}\langle l^{(0)}|k^{(n-1)}\rangle - \langle l^{(0)}|\delta H|k^{(n-1)}\rangle \\
\qquad\qquad\qquad\qquad\qquad + E_k^{(2)}\langle l^{(0)}|k^{(n-2)}\rangle + \cdots + E_k^{(n)}\delta_{lk}
\end{cases} \tag{C.1.8}
$$

其中 $\delta H_{lk} = \langle l^{(0)}|\delta H|k^{(0)}\rangle$, 对于满足 (C.1.2) 的解, 有,

$$E_k^{(n)} = \langle k^{(0)}|\delta H|k^{(n-1)}\rangle, n = 1, 2, \cdots \tag{C.1.9}$$

并且,

$$|k^{(1)}\rangle = -\sum_{l\neq k}\frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}}|l^{(0)}\rangle \Longrightarrow E_k^{(2)} = -\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} \tag{C.1.10}$$

> **calculation:**
>
> 将 (C.1.10) 代入 (C.1.8), 得到 $(l \neq k)$,
> $$(E_l^{(0)} - E_k^{(0)})\langle l^{(0)}|k^{(2)}\rangle = -E_k^{(1)}\frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} + \sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \tag{C.1.11}$$
>
> 所以,
> $$
> \begin{cases}
> |k^{(2)}\rangle = \sum_{l\neq k}\Big(-\frac{\delta H_{00}\delta H_{lk}}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}\Big)|l^{(0)}\rangle \\
> E_k^{(3)} = \sum_{l\neq k}\Big(-\frac{\delta H_{00}|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m\neq k}\frac{\delta H_{kl}\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}\Big)
> \end{cases} \tag{C.1.12}
> $$
> ───────────────────────────
> 计算归一化系数,
> $$_\lambda\langle k|k\rangle_\lambda = 1 + \lambda^2\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + O(\lambda^3) \tag{C.1.13}$$

C.1.1   level repulsion or the seesaw mechanism

- 能量的展开式为,

$$E_k(\lambda) = E_k^{(0)} + \lambda \delta H_{kk} - \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3) \tag{C.1.14}$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

C.1.2   validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即,

$$|\lambda V| < \frac{1}{2} \Delta E^{(0)} \tag{C.1.15}$$

因此, 对于能级简并的情况, $\Delta E^{(0)} = 0$, 情况会更复杂.

---

calculation:

对于两能级系统,

$$H(\lambda) = H^{(0)} + \lambda \hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix} \tag{C.1.16}$$

$H(\lambda)$ 的本征值可以直接计算,

$$E_\pm(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)}) \sqrt{1 + \left(\frac{\lambda|V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})} \right)^2} \tag{C.1.17}$$

考虑 $\sqrt{1+z^2}$ 的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1} \frac{(2n-3)!!}{2^n n!} z^{2n} + \cdots \tag{C.1.18}$$

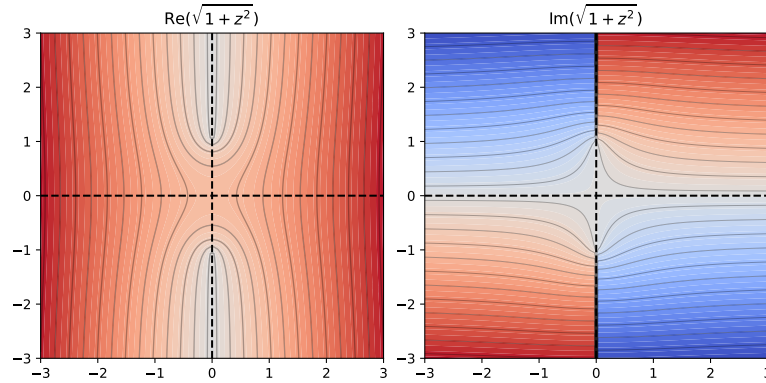注意到 $\sqrt{1+z^2}$ 在 $z = \pm i$ 有 branch cut, 因此 $z = 0$ 附件的 Taylor expansion 只有在 $|z| < 1$ 内才收敛.

Figure C.1: graph of $\sqrt{1+z^2}$

C.2   degenerate perturbation theory

- 暂时先跳过.

Appendix D

classical field theory and Noether's theorem

D.1 classical field theory

D.1.1 Lagrangian density and the action

- Lagrangian density, $\mathcal{L}$, 是 $\phi^a(x), \partial_\mu \phi^a(x), t$ 的函数.

- 对作用量变分得到 Euler-Lagrangian equation of motion,

$$\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \Big(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \Big) = 0 \tag{D.1.1}$$

> **calculation:**
>
> 对作用量进行变分,
>
> $$\begin{aligned} \delta S &= \int d^4 x \Big(\frac{\delta \mathcal{L}}{\delta \phi^a} \delta \phi^a + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \delta \partial_\mu \phi^a \Big) \\ &= \int d^4 x \Big(\Big(\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \Big(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \Big) \Big) \delta \phi^a + \partial_\mu \Big(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \delta \phi^a \Big) \Big) \end{aligned} \tag{D.1.2}$$
>
> 由于边界变分为零...

D.1.2 canonical momentum and the Hamiltonian

- **def.:** 定义一个叫 π_a^μ 的量,

$$\pi_a^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \tag{D.1.3}$$

其中 $\pi_a \equiv \pi_a^0$ 称作 canonical momentum of the field.

- **def.:** the Hamiltonian density is,

$$\mathcal{H} = \pi_a \partial_0 \phi^a - \mathcal{L} \tag{D.1.4}$$

- the Hamilton's equations are,

$$\begin{cases} \partial_0 \phi^a = \dfrac{\delta \mathcal{H}}{\delta \pi_a} \\ -\partial_0 \pi^a = \dfrac{\delta \mathcal{H}}{\delta \phi^a} - \partial_i \Big(\dfrac{\delta \mathcal{H}}{\delta(\partial_i \phi^a)} \Big) \end{cases} \tag{D.1.5}$$

  - 第二个方程可以写成更紧凑的形式,

$$\partial_\mu \pi_a^\mu = \frac{\delta \mathcal{H}}{\delta \phi^a} \tag{D.1.6}$$

D.2 Noether's theorem

D.2.1 in classical particle mechanics

- 系统的 Lagrangian 为 $L(q^a, \dot{q}^a, t)$.
- 系统通过以下形式变换,

$$q^a(t) \mapsto q^a(\lambda, t) \quad \text{and} \quad q^a(t, 0) = q^a(t) \tag{D.2.1}$$

  并定义,

$$D_\lambda q^a = \frac{\partial q^a}{\partial \lambda}\Big|_{\lambda=0} \tag{D.2.2}$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.,

$$D_\lambda L = \frac{dF(q^a, \dot{q}^a, t)}{dt} \tag{D.2.3}$$

  for some $F(q^a, \dot{q}^a, t)$, and the corresponding **conserved quantity** is,

$$Q = p_a D_\lambda q^a - F(q^a, \dot{q}^a, t) \tag{D.2.4}$$

> **proof:**
>
> $$D_\lambda L = \frac{\partial L}{\partial q^a} D_\lambda q^a + \frac{\partial L}{\partial \dot{q}^a} \frac{dD_\lambda q^a}{dt} = \frac{d}{dt}(p_a D_\lambda q^a) \tag{D.2.5}$$

- 几个例子如下,
  - **空间平移**, $\vec{x}(t) \mapsto \vec{x}(t) + \hat{e}_i \lambda$, 相应地, $D_\lambda \vec{x} = \hat{e}_i$, 且,

$$D_\lambda L = \frac{\partial L}{\partial x^i} \tag{D.2.6}$$

    如果 $\frac{\partial L}{\partial x^i} = 0$, 那么, 有守恒量 p_i.
  - **时间平移**, $q^a(t) \mapsto q^a(t + \lambda)$, 相应地, $D_\lambda q^a = \dot{q}^a$, 且,

$$D_\lambda L = \frac{dL}{dt} - \frac{\partial L}{\partial t} \tag{D.2.7}$$

    如果 $\frac{\partial L}{\partial t} = 0$, 那么, 有守恒量 $H = p_a \dot{q}^a - L$.
  - **转动**, $\vec{x}(t) \mapsto R(\lambda, \hat{e}) \cdot \vec{x}(t)$, 相应地, $D_\lambda \vec{x} = \hat{e} \times \vec{x}$, 且,

$$D_\lambda L = \vec{x} \cdot \left(\frac{\partial L}{\partial \vec{x}} \times \hat{e}\right) + \hat{e}(\dot{\vec{x}} \times \vec{p}) \tag{D.2.8}$$

    如果上式中两个括号内的项都为零, 那么, 有守恒量 $\hat{e} \cdot \vec{J} = \hat{e} \cdot (\vec{x} \times \vec{p})$.

D.2.2 in classical field theory

- 类似地, 系统通过以下形式变换,

$$\phi^a(x) \mapsto \phi^a(x, \lambda) \quad \text{and} \quad \phi^a(x, 0) = \phi^a(x) \tag{D.2.9}$$

  并定义,

$$D_\lambda \phi^a = \frac{\partial \phi^a}{\partial \lambda}\Big|_{\lambda=0} \tag{D.2.10}$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.,

$$D_\lambda \mathcal{L} = \partial_\mu F^\mu(\phi^a, \partial_\mu \phi^a, t) \tag{D.2.11}$$

  for some $F^\mu(\phi^a, \partial_\mu \phi^a, t)$, and the **conserved current** is,

$$J^\mu = \pi_a^\mu D_\lambda \phi^a - F^\mu \tag{D.2.12}$$

> **proof:**
>
> $$D_\lambda \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi^a} D_\lambda \phi^a + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \partial_\mu D_\lambda \phi^a$$
> $$= \Big(\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \Big(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \Big) \Big) D_\lambda \phi^a + \partial_\mu \Big(\underbrace{\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)}}_{=\pi_a^\mu} D_\lambda \phi^a \Big) \qquad (\text{D.2.13})$$
>
> 代入 (D.1.1), 得...

- 注意, conserved current 并不是唯一确定的, 考虑如下变换,

$$F^\mu \mapsto F'^\mu = F^\mu + \partial_\nu A^{\mu\nu} \quad \text{with} \quad A^{\mu\nu} = A^{[\mu\nu]} \qquad (\text{D.2.14})$$

新 F'^μ 依然能满足 (D.2.11).

- 但是, 守恒荷是唯一确定的.

> **proof:**
>
> $$Q' = \int d^3x J^0 = \int d^3x (\pi_a D_\lambda \phi^a - F^0) - \int d^3x \, \partial_\mu A^{0\mu} \qquad (\text{D.2.15})$$
>
> 考虑到边界值为零, 且 $A^{00} = 0$, 所以 $Q' = Q$.

D.2.3 spacetime translations and the energy-momentum tensor

- 时空平移变换为,

$$\phi^a(x) \mapsto \phi^a(x + \lambda e) \qquad (\text{D.2.16})$$

- 所以,

$$D_\lambda \phi^a = e^\mu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = e^\mu \partial_\mu \mathcal{L} \qquad (\text{D.2.17})$$

代入 (D.2.12),

$$J^\mu = e^\nu \underbrace{(\pi_a^\mu \partial_\nu \phi^a - \delta_\nu^\mu \mathcal{L})}_{=T^\mu{}_\nu} \qquad (\text{D.2.18})$$

- 并且有,

$$\partial_\mu T^{\mu\nu} = 0 \implies P^\mu = \int d^3x \, T^{0\mu} = \text{Const.} \qquad (\text{D.2.19})$$

来自守恒流散度为零.

D.2.4 Lorentz transformations, angular momentum and something else

- Lorentz transformation 下坐标做变换 $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, 其中 Λ 满足,

$$\eta = \Lambda^T \eta \Lambda \qquad (\text{D.2.20})$$

- infinitesimal Lorentz transformation 是,

$$\Lambda = I + \epsilon \qquad (\text{D.2.21})$$

其中 $\{\epsilon^{\mu\nu}\} = \epsilon\eta$ 是反对称矩阵.

> **proof:**
>
> 考虑,
>
> $$\eta = (\Lambda\eta)^T \eta (\Lambda\eta) = (\eta + \epsilon\eta)^T \eta (\eta + \epsilon\eta)$$

$$= \eta + \eta\epsilon^T + \epsilon\eta + O(\epsilon^2) \tag{D.2.22}$$

- -

- 标量场在 Lorentz transform 下的变换为,

$$\Lambda : \phi^a(x) \mapsto \phi^a(\Lambda^{-1}x') \tag{D.2.23}$$

___

- 有,

$$D_\lambda \phi^a = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\epsilon_{\mu\nu} \partial^\mu (x^\nu \mathcal{L}) \tag{D.2.24}$$

代入 (D.2.12),

$$J^\mu = \frac{1}{2}\epsilon_{\nu\rho} M^{\mu\nu\rho} \quad \text{where} \quad M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{D.2.25}$$

且有,

$$\partial_\mu M^{\mu\nu\rho} = 0 \tag{D.2.26}$$

- -

- 对全空间积分, 得到 6 个守恒量,

$$J^{\mu\nu} = \int d^3x \, M^{0\mu\nu} = \text{Const.} \tag{D.2.27}$$

不难发现 J^{ij} 对应角动量, 现在来讨论 J^{0i} 的物理意义,

$$0 = \frac{d}{dt}J^{0i} = \frac{d}{dt}\int d^3x(tT^{0i} - x^i T^{00}) = P^i - \frac{d}{dt}\int d^3x \, x^i T^{00} \tag{D.2.28}$$

其中, 用到了 $\frac{dP^i}{dt} = 0$ (见 (D.2.19)), 可以将上式的第二项理解为质心运动的动量.

D.3 charge as generators

- the charge associated with the conserved current is,

$$Q = \int d^D J^0 = \int d^D x \left(\pi_a D_\lambda \phi^a - F^0\right) \tag{D.3.1}$$

在 $F^\mu = 0$ 且 $[D_\lambda \phi^a, \phi^a] = 0$ 的情况下,

$$i[Q, \phi^a] = D_\lambda \phi^a \tag{D.3.2}$$

D.4 what the graviton listens to: energy-momentum tensor

- the energy-momentum tensor is defined as (其中 $g = |\det\{g_{\mu\nu}\}|$),

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M \tag{D.4.1}$$

- 如果将 $\mathcal{L}_M$ 对 $g^{\mu\nu}$ 做展开 $\mathcal{L}_M = A + g^{\mu\nu}B_{\mu\nu} + g^{\mu\nu}g^{\rho\sigma}C_{\mu\nu\rho\sigma} + \cdots$, 那么,

$$T_{\mu\nu} = -2(B_{\mu\nu} + 2g^{\rho\sigma}C_{\mu\nu\rho\sigma} + 3\cdots) + g_{\mu\nu}\mathcal{L}_M \tag{D.4.2}$$

另外, the trace of the energy-momentum tensor is,

$$T = g^{\mu\nu}T_{\mu\nu} = d \times A + (d-2)g^{\mu\nu}B_{\mu\nu} + (d-4)g^{\mu\nu}g^{\rho\sigma}C_{\mu\nu\rho\sigma} \tag{D.4.3}$$

可见 $d = 4$ 时, T 与 $C_{\mu\nu\rho\sigma}$ 无关.

___

- 以 electromagnetic field 为例, $d = 4$,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A^\mu A_\mu \implies \begin{cases} T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + m^2 A_{\mu\nu} + g_{\mu\nu}\mathcal{L}_M \\ T = -m^2 A^\mu A_\mu \end{cases} \tag{D.4.4}$$

可见 the energy-momentum tensor of electromagnetic field (when $m = 0$) is traceless.

- $\mathcal{L} = -\frac{1}{2}((\partial\phi)^2 - m^2\phi^2)$ 和 $\mathcal{L} = \frac{1}{2}\phi(\partial^2 - m^2)\phi$ 对应的 energy-momentum tensor 一样吗 (?).