

Quantum Field Theory

a study note based on A. Zee's textbook

Siyang Wan (万思扬)

Contents

convention, notation, and units	3
I motivation and foundation	4
1 free field theory	5
1.1 partition function	5
1.2 free propagator	5
1.3 from field to particle to force	6
1.3.1 from field to particle	6
1.3.2 from particle to force	6
2 Coulomb and Newton: repulsive and attraction	8
2.1 massive spin-1 particle & QED	8
2.1.1 spin & polarization vector	9
2.1.2 Maxwell Lagrangian	9
2.2 massive spin-2 particle & gravity	9
2.3 remarks	10
3 Feynman diagrams	11
3.1 a baby problem	11
3.1.1 Wick contraction and Green's functions	11
3.1.2 connected vs. disconnected	12
3.2 a child problem	12
3.2.1 n -point Green's function	12
3.3 perturbative field theory	13
3.3.1 collision between particles	14
3.3.2 in momentum space	14
3.3.3 loops and a first look at divergence	16
4 canonical quantization	18
4.1 Heisenberg and Dirac	18
4.1.1 quantum mechanics	18
4.1.2 scalar field	19
4.2 interaction picture	20
4.3 scattering amplitude	20
4.4 complex scalar field	22
4.4.1 charge	23
5 disturbing the vacuum: Casimir effect	24
II Dirac and spinor	25
6 the Dirac spinor	26
6.1 gamma matrices	26
6.1.1 gamma matrices under Dirac basis	27
6.2 Lorentz transformation and the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation	27
6.2.1 Dirac spinor	28
6.2.2 Dirac bilinears	28

6.2.3	parity and time reversal	29
7	the Dirac equation	30
7.1	Dirac equation	30
7.2	Dirac Lagrangian	30
7.3	chirality or handedness	30
7.3.1	internal vector symmetry	31
7.3.2	axial symmetry	31
7.4	energy-momentum tensor and angular momentum	31
7.5	charge conjugation, parity and time reversal	32
7.5.1	charge conjugation and antimatter	32
7.5.2	parity	32
7.5.3	time reversal	32
7.5.4	CPT theorem	33
7.6	interaction in QED	33
7.7	Majorana neutrino	33
8	quantizing the Dirac field	35
8.1	anticommutation	35
8.2	the Dirac field	35
	Appendices	38
A	Dirac delta function & Fourier transformation	38
A.1	Delta function	38
A.2	Fourier transformation	39
B	Gaussian integrals	41
B.1	N -dim. generalization	41
C	perturbation theory in QM	43
C.1	non-degenerate perturbation theory	43
C.1.1	level repulsion or the seesaw mechanism	45
C.1.2	validity of the perturbation expansion	45
C.2	degenerate perturbation theory	45
D	classical field theory and Noether's theorem	46
D.1	classical field theory	46
D.1.1	Lagrangian density and the action	46
D.1.2	canonical momentum and the Hamiltonian	46
D.2	Noether's theorem	47
D.2.1	in classical particle mechanics	47
D.2.2	in classical field theory	47
D.2.3	spacetime translations and the energy-momentum tensor	48
D.2.4	Lorentz transformations, angular momentum and something else	48
D.3	charge as generators	49
D.4	what the graviton listens to: energy-momentum tensor	49
E	antiunitary operator and time reversal	50
E.1	complex conjugation operator	50
E.2	antiunitary operator	50
E.3	time reversal in QM	51
E.3.1	spin- $\frac{1}{2}$ non-relativistic electron	51

convention, notation, and units

- 笔记中的度规号差约定为 $(-, +, +, +)$.
- 使用 Planck units, 此时 $G, \hbar, c, k_B = 1$, 因此,

name/dimension	expression/value
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} \text{ m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44} \text{ s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} \text{ kg} \simeq 10^{19} \text{ GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32} \text{ K}$

- 时空维度用 $d = D + 1$ 表示.

Part I

motivation and foundation

Chapter 1

free field theory

1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \quad (1.1.1)$$

A. Zee 说: 在作用量里, 时间的导数项必须是正的, 包括标量场的 $(\partial_0\phi)^2$ 和电磁场的 $(\partial_0 A_i)^2$.

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi e^{i \int d^d x (-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \quad (1.1.2)$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

-
- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (1.1.3)$$

另外, 用 $W(J)$ 表示指数上的部分 (去掉虚数 i).

proof:

注意 $\partial^\mu \phi \partial_\mu \phi = \partial^\mu (\phi \partial_\mu \phi) - \phi \partial^2 \phi$, 忽略全微分项, 那么,

$$Z(J) = \int D\phi e^{i \int d^d x (\frac{1}{2} \phi (\partial^2 - m^2) \phi + J(x)\phi(x))} \quad (1.1.4)$$

代入 (B.1.1), 可知,

$$Z(J) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (1.1.5)$$

其中 $D(x-y)$ 满足,

$$\begin{cases} (\partial^2 - m^2) D(x-y) = \delta^{(d)}(x-y) \\ (-p^2 - m^2) \tilde{D}(p, q) = (2\pi)^d \delta^{(d)}(p-q) \end{cases} \implies D(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{-k^2 - m^2} \quad (1.1.6)$$

1.2 free propagator

- 为了使 (1.1.4) 中的积分在 ϕ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon \int d^d x \phi^2}$.
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 ϵ 避免发散,

$$D(x) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot x}}{-k^2 - m^2 + i\epsilon} = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left(\theta(t) e^{i(-\omega_k t + \vec{k} \cdot \vec{x})} + \theta(-t) e^{i(\omega_k t + \vec{k} \cdot \vec{x})} \right) \quad (1.2.1)$$

calculation:

对 k^0 积分, 注意有两个奇点 $k^0 = \pm(\omega_k - i\epsilon)$, 当 $t > 0$ 时, contour 处于下半平面, ... (另外注意到我们可以任意改变 \vec{k} 的符号).

- $D(x)$ 的取值与 x 的类时, 类空性质关系密切.

– 类时区域,

$$D(t, 0) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left(\theta(t) e^{-i\omega_k t} + \theta(-t) e^{i\omega_k t} \right) \quad (1.2.2)$$

– 类空区域,

$$D(0, \vec{x}) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{i\vec{k} \cdot \vec{x}} \sim e^{-m|\vec{x}|} \quad (1.2.3)$$

1.3 from field to particle to force

1.3.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,

$$W(J) = -\frac{1}{2} \int d^d x d^d y J(y) D(x-y) J(y) \quad (1.3.1)$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}(k) \quad (1.3.2)$$

其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k) = \tilde{J}^*(k)$.

- 考虑 $J(x) = J_1(x) + J_2(x)$, 那么 $W(J)$ 共有 4 项, 其中一个交叉项如下,

$$W_{12}(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_1(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_2(k) \quad (1.3.3)$$

可见 $W(J)$ 取值较大的条件是:

1. $\tilde{J}_1(k), \tilde{J}_2(k)$ 有较大重叠,
2. 重叠位置的 k 是 on shell (即 $k^2 = -m^2$).

- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

1.3.2 from particle to force

- 考虑 $J(x) = \delta^{(D)}(\vec{x} - \vec{x}_1) + \delta^{(D)}(\vec{x} - \vec{x}_2) \implies \tilde{J}_a(k) = 2\pi e^{-i\vec{k} \cdot \vec{x}_a} \delta(k^0)$, 那么,

$$W_{12}(J) + W_{21}(J) = \delta(0) \int \frac{d^D k}{(2\pi)^{D-1}} \frac{1}{|\vec{k}|^2 + m^2 - i\epsilon} \cos(\vec{k} \cdot (\vec{x}_1 - \vec{x}_2))$$

$$\stackrel{D=3}{=} 2\pi \delta(0) \frac{1}{4\pi r} e^{-mr} \quad (1.3.4)$$

($-i\epsilon$ 显然可以舍去), 注意到 $\langle 0 | e^{-iHT} | 0 \rangle = e^{-iET}$, 而时间间隔 $T = \int dx^0 = 2\pi \delta(0)$, 所以,

$$E = -\frac{W(J)}{T} \stackrel{D=3}{=} -\frac{1}{4\pi r} e^{-mr} \quad (1.3.5)$$

calculation:

计算 (1.3.4) 中的积分, 令 $\vec{x}_1 - \vec{x}_2 = \vec{r}$,

$$I_D = \int \frac{d^D k}{(2\pi)^D} \frac{1}{|\vec{k}|^2 + m^2} \overbrace{\cos(\vec{k} \cdot \vec{r})}^{\mapsto e^{i\vec{k} \cdot \vec{r}}}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^D} \int (k \sin \theta_1)^{D-2} d\Omega_{D-2} \int k d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1} \\
&= \frac{S_{D-2}}{(2\pi)^D} \int k^{D-1} \sin^{D-2} \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1}
\end{aligned} \tag{1.3.6}$$

取 $D = 3$, 那么,

$$\begin{aligned}
I_{D=3} &= \frac{1}{(2\pi)^2} \int k^2 \sin \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ik \cos \theta_1} \\
&= \frac{1}{2\pi^2 r} \int_0^\infty \sin(kr) \frac{k dk}{k^2 + m^2} = \frac{-i}{4\pi^2 r} \int_{-\infty}^\infty e^{ikr} \frac{k dk}{k^2 + m^2} \\
&= \frac{-i}{4\pi^2 r} 2\pi i \underbrace{\text{Res}(f, im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r} e^{-mr}
\end{aligned} \tag{1.3.7}$$

Chapter 2

Coulomb and Newton: repulsive and attraction

2.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \quad (2.1.1)$$

其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

- 做路径积分,

$$Z(J) = \int DA e^{i \int d^d x (\mathcal{L} + J_\mu A^\mu)} = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J_\mu D^{\mu\nu}(x-y) J_\nu(y)} \quad (2.1.2)$$

calculation:

massive photon 的作用量为,

$$\begin{aligned} S(A) &= \int d^d x \frac{1}{2} \left(-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu \right) \\ &= \int d^d x \frac{1}{2} \left(A_\nu \partial^2 A^\nu - A_\nu \partial^\nu \partial_\mu A^\mu - m^2 A_\mu A^\mu \right) + \text{total differential} \\ &= \int d^d x \frac{1}{2} A_\mu \left(-\partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \right) A_\nu + \text{total differential} \\ &= \int \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(-k) \left(k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2) \right) \tilde{A}_\nu(k) + \text{boundary term} \end{aligned} \quad (2.1.3)$$

那么, 需要有,

$$\begin{aligned} (-\partial^\mu \partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2)) D_{\rho\nu}(x-y) &= \delta_\nu^\mu \delta^{(d)}(x-y) \\ \implies \tilde{D}_{\mu\nu}(k) &= \frac{k_\mu k_\nu / m^2 + \eta_{\mu\nu}}{-k^2 - m^2} \end{aligned} \quad (2.1.4)$$

考虑到积分需要收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, (为什么 A_μ 类空, 只知道 \tilde{A}_μ 类空, 见 subsection 2.1.2, 但路径积分中的 A 显然不满足 field equation \implies 路径积分中起主要作用的 \tilde{A} 类空, 因此 $-\epsilon|\tilde{A}|^2 < 0$).

- 因此,

$$W(J) = -\frac{1}{2} \int d^d x d^d y J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) \quad (2.1.5)$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_\mu(-k) \frac{k^\mu k^\nu / m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon} \tilde{J}_\nu(k) \quad (2.1.6)$$

注意到 current conservation, 有 $\partial_\mu J^\mu = 0 \iff k^\mu \tilde{J}_\mu(k) = 0$, 所以,

$$W(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}^\mu(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_\mu(k) \quad (2.1.7)$$

观察电荷分量, 可见同性相斥, 异性相吸.

2.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的 x, y, z 方向, 在粒子静止系下, 极化矢量 $(\epsilon^i)_\mu = \delta_\mu^i, i = 1, 2, 3$, 而 $k_\mu = (-m, 0, 0, 0)$, 所以,

$$k^\mu (\epsilon^i)_\mu = 0 \quad (2.1.8)$$

– 注意, 一个粒子的极化方向用 e^i (这不是矢量) 表示, 极化矢量为 $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$.

- 在粒子静止系下, 考虑,

$$\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu} \quad (2.1.9)$$

可见,

$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon} \quad (2.1.10)$$

2.1.2 Maxwell Lagrangian

- 根据 (2.1.1) 中的 Lagrangian density, 得到 field equation 如下,

$$\left(-\partial^\mu \partial^\nu + \eta^{\mu\nu} (\partial^2 - m^2) \right) A_\nu \quad (2.1.11)$$

– spin-1 particle 有 3 个自旋自由度, 而 A_μ 有 4 个分量, 所以需要有一个约束方程,

$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0 \quad (2.1.12)$$

实际上在 (2.1.11) 左右两边作用一个 ∂_μ 即可得到这个约束方程.

2.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = **linearized** Einstein Lagrangian.
- 受 subsection 2.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足,

$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases} \quad (2.2.1)$$

其中下指标 μ, ν 对称, $a = 1, \dots, 5$, (可以验证 $(\epsilon^a)_{\mu\nu}$ 确实有 5 个独立分量).

- 对 $(\epsilon^a)_{\mu\nu}$ 的归一化条件可以定义为 $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$.
- 与 subsection 2.1.1 中提示一样, 粒子的极化方向用 e^a 表示.

- 那么,

$$\sum_{a=1}^5 (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma} \quad (2.2.2)$$

calculation:

首先用 k_μ 和 $\eta_{\mu\nu}$ 构造最一般的关于 $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$ 对称的 4 阶张量, (下式中把 $\frac{k_\mu}{m}$ 略写作 k_μ),

$$\begin{aligned} & A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ & + D \eta_{\mu\nu} \eta_{\rho\sigma} + E(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) \end{aligned} \quad (2.2.3)$$

代入 (2.2.1) 得,

$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4} \quad (2.2.4)$$

因此, 这个 4 阶张量最终确定为,

$$\frac{3}{4}A\left((G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\right) \quad (2.2.5)$$

- 所以,

$$\tilde{D}_{\mu\nu\rho\sigma}(k) = \frac{(G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \quad (2.2.6)$$

- 计算路径积分中的 $W(T)$,

$$W(T) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{\mu\nu}(-k) \frac{(G^{\mu\rho}G^{\nu\sigma} + G^{\mu\sigma}G^{\nu\rho}) - \frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tilde{T}_{\rho\sigma}(k) \quad (2.2.7)$$

注意到 $\partial_\mu T^{\mu\nu}(x) = 0 \iff k_\mu \tilde{T}^{\mu\nu}(k) = 0$, 并考虑到 T 是对称张量, 所以,

$$W(T) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{\mu\nu}(-k) \frac{2\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tilde{T}_{\rho\sigma}(k) \quad (2.2.8)$$

考虑能量项, 可见质量互相吸引.

2.3 remarks

- 由于 seesaw mechanism (见 subsection C.1.1), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为 $\eta^{00} = -1$.
- 本 chapter 中的计算都是 $m \neq 0$ 的粒子, 与真实世界有差异.

Chapter 3

Feynman diagrams

3.1 a baby problem

- 考虑如下积分,

$$Z(J) = \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4 + Jq} \quad (3.1.1)$$

- Schwinger's way:** 把 integrand 对 λ 展开, 并将 q 用 $\frac{\partial}{\partial J}$ 替代, 得到,

$$\begin{aligned} Z(J) &= e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4} \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 + Jq} \\ &= \sqrt{\frac{2\pi}{m^2}} e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4} e^{\frac{J^2}{2m^2}} \end{aligned} \quad (3.1.2)$$

后面的计算中忽略 $Z(J=0, \lambda=0)$.

- 每个 vertex 带有 $-\lambda$, 每个 line 带有 $\frac{1}{m^2}$, 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里 q 无法区分, 所以不方便算, 先略了),

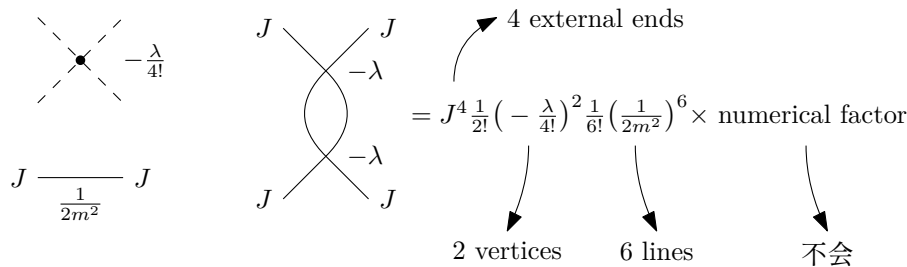


Figure 3.1: baby problem - Feynman diagram

calculation:

在这里计算 λJ^4 项,



具体暂时不会算

(3.1.3)

但是直接计算 (3.1.2) 的展开项, 得到的结果与 (3.1.5) 一样.

3.1.1 Wick contraction and Green's functions

- 把积分 (3.1.1) 对 J 展开,

$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} J^n \underbrace{\int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4} q^n}_{=Z(0,0)G^{(n)}} \quad (3.1.4)$$

其中 Green's function $G^{(n)}$ 对 λ 展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

calculation:

计算 λJ^4 项, 它来自 $G^{(4)}$ 对 λ 展开的一阶项,

$$\begin{aligned}
-\frac{\lambda}{4!} \int dq e^{-\frac{1}{2}m^2 q^2} q^8 &= -\frac{\lambda}{4!} \langle q^8 \rangle \\
&= -\frac{\lambda}{4!} \sum_{\text{Wick}} \left(\frac{1}{m^2} \right)^4 \\
&= -\frac{\lambda}{4!} \frac{7 \times 5 \times 3 \times 1}{m^8}
\end{aligned} \tag{3.1.5}$$

所以 λJ^4 项等于 $\frac{105}{(4!)^2} \frac{-\lambda J^4}{m^8}$.

3.1.2 connected vs. disconnected

- 考虑,

$$Z(J, \lambda) = Z(J=0, \lambda) e^{W(J, \lambda)} \tag{3.1.6}$$

其中 $Z(J=0, \lambda)$ 由 diagrams with no external source J 组成, 而 $W(J, \lambda)$ 则由 connected diagrams 组成(?).

- 我们希望计算的是 W , 而不是 Z (?).

3.2 a child problem

- 考虑如下积分,

$$Z(J) = \int dq_1 \cdots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q - \frac{\lambda}{4!} q^4 + J^T \cdot q} \tag{3.2.1}$$

其中 $q^4 = \sum_i q_i^4$.

- Schwinger's way:** 对 λ 展开并把 q 替换为 $\frac{\partial}{\partial J}$, 得到,

$$Z(J) = \sqrt{\frac{(2\pi)^N}{\det A}} e^{-\frac{\lambda}{4!} \left(\frac{\partial}{\partial J}\right)^4} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \tag{3.2.2}$$

其中 $\left(\frac{\partial}{\partial J}\right)^4 = \sum_i \left(\frac{\partial}{\partial J_i}\right)^4$.

3.2.1 n -point Green's function

- Wick's way:** 对 J 展开获得带 Green's function 的表达式,

$$\begin{aligned}
Z(J) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N J_{i_1} \cdots J_{i_n} \underbrace{\int dq_1 \cdots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q - \frac{\lambda}{4!} q^4} q_{i_1} \cdots q_{i_n}}_{=Z(0,0)G_{i_1 \cdots i_n}^{(n)}}
\end{aligned} \tag{3.2.3}$$

其中 $G_{i_1 \cdots i_n}^{(n)}$ 称为 n -point Green's function.

Taylor expansion:

多元函数的 Taylor 展开如下,

$$\begin{aligned}
f(x_1, \cdots, x_N) &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_N^{n_N}}{n_N!} \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_N}}{\partial x_N^{n_N}} f(x=0) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N x_{i_1} \cdots x_{i_n} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}} f(x=0)
\end{aligned} \tag{3.2.4}$$

这两种求和方法, $x_1^{n_1} \cdots x_N^{n_N}$ 项的 numerical factor 都等于,

$$\frac{1}{n!} \times \frac{n!}{n_1! \cdots n_N!} = \frac{1}{n_1! \cdots n_N!} \tag{3.2.5}$$

其中 $n = n_1 + \dots + n_N$.

- 在 $\lambda = 0$ 时, 2-point Green's function 就是 propagator,

$$\begin{aligned} G_{ij}^{(2)}(\lambda = 0) &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} q_i q_j \\ &= \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \Big|_{J=0} = A_{ij}^{-1} \end{aligned} \quad (3.2.6)$$

- 来计算 2, 3, 4-point Green's functions,

$$\begin{cases} G_{ij}^{(2)} = A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \\ G_{ijk}^{(3)} = 0 \\ G_{ijkl}^{(4)} = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \\ \quad - \frac{\lambda}{4!} \sum_m (A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} A_{kl}^{-1} + \dots + 4! A_{im}^{-1} A_{jm}^{-1} A_{km}^{-1} A_{lm}^{-1}) + O(\lambda^2) \end{cases} \quad (3.2.7)$$

calculation:

2-point Green's function 计算如下,

$$\begin{aligned} G_{ij}^{(2)} &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j \\ &= A_{ij}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j \rangle + O(\lambda^2) \\ &= A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \end{aligned} \quad (3.2.8)$$

3-point Green's function 计算如下,

$$G_{ijk}^{(3)} = \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j q_k = 0 \quad (3.2.9)$$

4-point Green's function 计算如下,

$$\begin{aligned} G_{ijkl}^{(4)} &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j q_k q_l \\ &= A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j q_k q_l \rangle + O(\lambda^2) \end{aligned} \quad (3.2.10)$$

3.3 perturbative field theory

- 做如下替换即可,

$$\begin{cases} A \mapsto -i(\partial^2 - m^2) \\ J \mapsto iJ \end{cases} \quad (3.3.1)$$

- Schwinger's way:** ϕ^4 theory 的路径积分,

$$Z(J) = \int D\phi e^{i \int d^d x (\frac{1}{2} \phi(\partial^2 - m^2) \phi - \frac{\lambda}{4!} \phi^4 + J(x) \phi(x))} \quad (3.3.2)$$

$$= Z(0,0) e^{-i \frac{\lambda}{4!} \int d^d z (\frac{\delta}{i \delta J(z)})^4} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (3.3.3)$$

其中 $D(x-y)$ 是自由场的 propagator, 见 (1.2.1).

- **Wick's way:** 同样, 对 J 展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \cdots d^d x_n J(x_1) \cdots J(x_n) G^{(n)}(x_1, \cdots, x_n) \quad (3.3.4)$$

其中,

$$G^{(n)}(x_1, \cdots, x_n) = \frac{1}{Z(0,0)} \int D\phi e^{i \int d^d x (\frac{1}{2} \phi (\partial^2 - m^2) \phi - \frac{\lambda}{4!} \phi^4)} \phi(x_1) \cdots \phi(x_n) \quad (3.3.5)$$

有时 $Z(J)$ 被称为 generating functional, 因为它能生成 Green's functions.

3.3.1 collision between particles

- 通过 Wick's way, 考虑 $J(x_1)J(x_2)J(x_3)J(x_4)$ 项, 实际上就是要计算 $G^{(4)}(x_1, x_2, x_3, x_4)$, 它的 0 阶项为,

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4, \lambda = 0) &= \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta J(x_2)} \frac{\delta}{i\delta J(x_3)} \frac{\delta}{i\delta J(x_4)} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \\ &= -(D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}) \end{aligned} \quad (3.3.6)$$

其中 D_{ij} 是 $D(x_i - x_j)$ 的简写, 可见, 传播子实际上是 $(-i)^3 D = iD$.

- $G_{1234}^{(4)}$ 的 1 阶项为,

$$\begin{aligned} \text{1st order term} &= -\frac{i\lambda}{4!} \int d^d z \langle \phi_1 \cdots \phi_4 \phi^4(z) \rangle \\ &= -\frac{i\lambda}{4!} \int d^d z \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_4} \left(\frac{\delta}{i\delta J(z)} \right)^4 e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \\ &= -\frac{i\lambda}{4!} \int d^d z \left(4! D_{1z} D_{2z} D_{3z} D_{4z} \right. \\ &\quad \left. + 4 \times 3 D_{12} D_{3z} D_{4z} D_{zz} + \cdots + 3 D_{12} D_{34} D_{zz} D_{zz} + \cdots \right) \end{aligned} \quad (3.3.7)$$

其中各项分别对应如下 Feynman diagrams,

$$-i\lambda \int d^d z D_{1z} D_{2z} D_{3z} D_{4z} \quad \frac{-i\lambda}{2!} \int d^d z D_{13} D_{2z} D_{4z} D_{zz} \quad \frac{-i\lambda}{8} \int d^d z D_{13} D_{24} D_{zz} D_{zz}$$

Figure 3.2: position space - Feynman diagrams

其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,

$$= (4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!} \right)^2 \int d^d z_1 d^d z_2 D_{1z_1} D_{2z_1} D_{3z_2} D_{4z_2} D_{z_1 z_2} D_{z_1 z_2} \quad (3.3.8)$$

3.3.2 in momentum space

- 本 subsection 将 (3.3.5) 转换到 momentum space, 注意到 $\tilde{J}(k)$ 和 $\tilde{J}(-k)$ 并不独立, 所以 $\frac{\partial}{\partial i\tilde{J}}$ 不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\tilde{G}^{(n)}(k_1, \cdots, k_n) = \int d^d x_1 \cdots d^d x_n e^{-i(k_1 \cdot x_1 + \cdots)} G^{(n)}(x_1, \cdots, x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \cdots d^d x_n e^{-i(k_1 \cdot x_1 + \cdots)} \langle \left(-\frac{i\lambda}{4!} \int d^d z \phi_z^4 \right)^n \phi_1 \cdots \phi_n \rangle \quad (3.3.9)$$

– propagator 的 Fourier transformation 是,

$$\tilde{D}_{pq} = \int d^d x d^d y e^{-i(p \cdot x + q \cdot y)} D(x - y) = \frac{(2\pi)^d \delta^{(d)}(p + q)}{-p^2 - m^2 + i\epsilon} \quad (3.3.10)$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ 的 1 阶项为,

$$\text{1st order term} = -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \langle \phi_z^4 \phi_1 \cdots \phi_4 \rangle \quad (3.3.11)$$

考虑第 1 项,

$$\begin{aligned} & -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z 4! D_{1z} \cdots D_{4z} \\ &= -i\lambda \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - z) + \cdots)} \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\ &= -i\lambda \underbrace{\int d^d z e^{-iz \cdot (k_1 + \cdots + k_4)}}_{=(2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4)} \prod_{i=1}^4 \frac{1}{-k_i^2 - m^2 + i\epsilon} \end{aligned} \quad (3.3.12)$$

– 出射粒子不一定 on-shell (?).

- 得到这些 Feynman diagrams,

$$\begin{aligned} & (2\pi)^d \delta^{(d)}(k_1 + k_2) \frac{i}{-k_1^2 - m^2 + i\epsilon} & -i\lambda (2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4) \prod_{i=1}^4 \frac{i}{-k_i^2 - m^2 + i\epsilon} \\ & -\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=2,4} \frac{i}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} & -\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \prod_{i=1,2} \int \frac{d^d p_i}{(2\pi)^d} \frac{i}{-p_i^2 - m^2 + i\epsilon} \\ & (2\pi)^d \delta^{(d)}(k_1 + k_3) \frac{i}{-k_1^2 - m^2 + i\epsilon} & (2\pi)^d \delta^{(d)}(k_1 + k_3) (2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{i}{-k_i^2 - m^2 + i\epsilon} \end{aligned}$$

Figure 3.3: momentum space - Feynman diagrams

calculation:

第 3 幅图的计算如下,

$$-\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z D_{13} D_{2z} D_{4z} D_{zz}$$

$$\begin{aligned}
&= -\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - z) + p_4 \cdot (x_4 - z) + p_4 \cdot 0)} \\
&\quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{2!} \int d^d z e^{-iz \cdot (p_2 + p_4)} \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_4 - k_4) \\
&\quad \prod_{i=1}^4 \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2,4} \frac{1}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{-p^2 - m^2 + i\epsilon} \quad (3.3.13)
\end{aligned}$$

第 4 幅图的计算如下,

$$\begin{aligned}
&-\frac{i\lambda}{8} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z D_{13} D_{24} D_{zz} D_{zz} \\
&= -\frac{i\lambda}{8} \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - x_4) + p_3 \cdot 0 + p_4 \cdot 0)} \\
&\quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{8} \int d^d z \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_2 + k_4) \\
&\quad \prod_{i=1}^4 \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{1}{-k_i^2 - m^2 + i\epsilon} \\
&\quad \prod_{i=1,2} \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \quad (3.3.14)
\end{aligned}$$

- 再举一个例子 (略去了 $\prod_{i=1}^6 \frac{i}{-k_i^2 - m^2 + i\epsilon}$),



$$\begin{aligned}
&= (4!)^2 \times \left(-\frac{i\lambda}{4!}\right)^2 (2\pi)^{2d} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \delta^{(d)}(k_1 + k_2 + k_3 + p) \delta^{(d)}(k_4 + k_5 + k_6 - p) \\
&= (-i\lambda)^2 (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6) \frac{i}{-(k_1 + k_2 + k_3)^2 - m^2 + i\epsilon} \quad (3.3.15)
\end{aligned}$$

3.3.3 loops and a first look at divergence

- subsection 3.3.2 里的 loop diagrams 出现了如下积分,

$$\int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} = \int \frac{d^D p}{(2\pi)^D 2\omega_p} \sim \int \frac{d^D p}{|p|} \quad (3.3.16)$$

积分发散.

- 再举一个例子 (略去了 $\prod_{i=1}^4 \frac{i}{-k_i^2 - m^2 + i\epsilon}$),



$$\begin{aligned}
&= (4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!} \right)^2 \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{i}{-q^2 - m^2 + i\epsilon} \\
&\quad (2\pi)^d \delta^{(d)}(k_1 + k_2 + p - q) (2\pi)^d \delta^{(d)}(k_3 + k_4 - p + q) \quad (3.3.17)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(-i\lambda)^2}{2} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \frac{i}{-(k_1 + k_2 + p)^2 - m^2 + i\epsilon} \\
&= \frac{(-i\lambda)^2}{2} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int \frac{d^D p}{(2\pi)^D} \left(\frac{1}{2\omega_p} \frac{i}{(k_1^0 + k_2^0 - \omega_p)^2 - \omega_{k_1+k_2+p}^2} \right. \\
&\quad \left. + \frac{i}{(\omega_{k_1+k_2+p} - k_1^0 - k_2^0)^2 - \omega_p^2} \frac{1}{2\omega_{k_1+k_2+p}} \right) \quad (3.3.18)
\end{aligned}$$

$$\sim \int \frac{d^D p}{p^3} \quad (3.3.19)$$

同样, 积分发散.

Chapter 4

canonical quantization

- A. Zee: the canonical and the path integral formalisms often appear complementary, in the sense that results difficult to see in one are clear in the other.
- **nobody is perfect:**
 - **canonical quantization:** 如何定义场算符乘积的顺序.
 - **path integral:** integration measure.

4.1 Heisenberg and Dirac

4.1.1 quantum mechanics

- 单粒子的 classical Lagrangian 为,

$$L = \frac{1}{2}\dot{q}^2 - V(q) \implies \begin{cases} p = \dot{q} \\ H = p\dot{q} - L = \frac{1}{2}p^2 + V(q) \end{cases} \quad (4.1.1)$$

- canonical commutation relation 如下,

$$[p, q] = -i \quad (4.1.2)$$

因此, 算符的演化方程为,

$$\begin{cases} \frac{dp}{dt} = i[H, p] = -V'(q) \\ \frac{dq}{dt} = i[H, q] = p \end{cases} \quad (4.1.3)$$

calculation:

$$\begin{cases} [p, q] = -i \\ [p, q^2] = -2iq \\ \vdots \\ [p, q^n] = -iq^{n-1} + q[p, q^{n-1}] \end{cases} \implies [p, q^n] = -inq^{n-1} \implies [p, V(q)] = -iV'(q) \quad (4.1.4)$$

- follow Dirac's approach,

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \iff \begin{cases} q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \\ p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \end{cases} \implies [a, a^\dagger] = 1 \quad (4.1.5)$$

算符 a 的演化方程为,

$$\frac{da}{dt} = -i\sqrt{\frac{\omega}{2}}\left(\frac{1}{\omega}V'(q) + ip\right) \quad (4.1.6)$$

4.1.2 scalar field

- 标量场的 Lagrangian 为,

$$L = \int d^D x \left(-\frac{1}{2}((\partial\phi)^2 + m^2\phi^2) - u(\phi) \right) \quad (4.1.7)$$

canonical commutation relation 为,

$$\pi(\vec{x}, t) = \frac{\delta L(t)}{\delta \partial_0 \phi(\vec{x}, t)} = \partial_0 \phi(\vec{x}, t) \quad \text{and} \quad [\pi(\vec{x}, t), \phi(\vec{y}, t)] = -i\delta^{(D)}(\vec{x} - \vec{y}) \quad (4.1.8)$$

标量场的 Hamiltonian 为,

$$H = \int d^D x (\pi\phi - \mathcal{L}) = \int d^D x \left(\frac{1}{2}(\pi^2 + |\vec{\nabla}\phi|^2 + m^2\phi^2) + u(\phi) \right) \quad (4.1.9)$$

- 算符的演化方程为,

$$\begin{cases} \partial_0 \phi = i[H, \phi] = \pi \\ \partial_0 \pi = i[H, \pi] = (-\vec{\nabla}^2 + m^2)\phi + \frac{du}{d\phi} \end{cases} \implies (\partial^2 - m^2)\phi - \frac{du}{d\phi} = 0 \quad (4.1.10)$$

- 当 $u(\phi) = 0$ 时, 求解场方程 (4.1.10) 和 canonical commutation relation (4.1.8) 得到,

$$\phi(\vec{x}, t) = \int \frac{d^D k}{(2\pi)^D 2\omega_k} (\alpha_k(t) e^{i\vec{k}\cdot\vec{x}} + \alpha_k^\dagger(t) e^{-i\vec{k}\cdot\vec{x}}) \quad (4.1.11)$$

其中,

$$\alpha_k(t) = \sqrt{(2\pi)^D 2\omega_k} a_{\vec{k}} e^{-i\omega_k t} \quad \text{and} \quad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = \delta^{(D)}(\vec{p} - \vec{q}) \quad (4.1.12)$$

另外, 在后面的笔记中使用简记 $\sqrt{(2\pi)^D 2\omega_k} = \rho(k)$.

calculation:

求解场方程 (4.1.10), 得到,

$$\phi(\vec{x}, t) = \int \frac{d^D k}{(2\pi)^D} (\alpha_{\vec{k}} e^{i(-\omega_k t + \vec{k}\cdot\vec{x})} + \alpha_{\vec{k}}^\dagger e^{-i(-\omega_k t + \vec{k}\cdot\vec{x})}) \quad (4.1.13)$$

代入 canonical commutation relation (4.1.8), 有 (其中 $x^0 = y^0 = t, k^0 = \omega_k$),

$$\begin{aligned} & \int \frac{d^D k_2}{(2\pi)^D} \left(-i\omega_{k_1} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] e^{i(k_1 \cdot x + k_2 \cdot y)} + i\omega_{k_1} [\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}^\dagger] e^{-i(k_1 \cdot x + k_2 \cdot y)} \right. \\ & \quad \left. - i\omega_{k_1} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] e^{i(k_1 \cdot x - k_2 \cdot y)} + i\omega_{k_1} [\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}] e^{-i(k_1 \cdot x - k_2 \cdot y)} \right) = -ie^{i\vec{k}_1 \cdot (\vec{x} - \vec{y})} \\ \implies & \begin{cases} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] = \frac{1}{2\omega_{k_1}} \delta^{(D)}(\vec{k}_1 + \vec{k}_2) \implies [\alpha_{\vec{k}}, \alpha_{\vec{k}}] \neq 0 & \text{wrong} \\ [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] = \frac{1}{2\omega_{k_1}} \delta^{(D)}(\vec{k}_1 - \vec{k}_2) & \text{right} \end{cases} \end{aligned} \quad (4.1.14)$$

- 代入 (4.1.9) 可得 (依然是 $u(\phi) = 0$ 的情况下),

$$H = \int d^D k \omega_k \frac{a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^\dagger}{2} = \int d^D k \omega_k \left(a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2} \delta^{(D)}(0) \right) \implies \langle 0|H|0 \rangle = V \int \frac{d^D k}{(2\pi)^D} \frac{1}{2} \omega_k \quad (4.1.15)$$

其中, $V = \int d^D x = (2\pi)^D \delta^{(D)}(0)$.

- vacuum state 定义为 $a_{\vec{k}}|0\rangle = 0$, 有,

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{ik \cdot (x-y)} \quad (4.1.16)$$

其中 $k^0 = \omega_k$. 因此, 对比 (1.2.1), 有,

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = iD(x-y) \quad (4.1.17)$$

4.2 interaction picture

- 注意, 在 $u(\phi) \neq 0$ 的情况下, (即便在 Schrödinger's picture 里, $t = 0$ 时) (4.1.11) 不再成立, 因此无法通过 Schrödinger's picture or Heisenberg's picture 求解存在相互作用的场论.
- 将 Hamiltonian 分成两个部分,

$$H = H_0 + H' \quad (4.2.1)$$

- operators 以自由场的 Hamiltonian 演化,

$$O_I(t) = U_0^\dagger(t, 0) O(0) U_0(t, 0) \quad \text{where} \quad U_0(t_2, t_1) = \text{Texp} \left(-i \int_{t_1}^{t_2} dt H_0 \right) \quad (4.2.2)$$

states 以如下方式演化,

$$|\psi(t)\rangle_I = U_0^\dagger(t, 0) U(t, 0) |\psi(0)\rangle \quad \text{where} \quad U(t_2, t_1) = \text{Texp} \left(-i \int_{t_1}^{t_2} dt H \right) \quad (4.2.3)$$

因此,

$$|\psi(t_2)\rangle_I = U_I(t_2, t_1) |\psi(t_1)\rangle_I \quad \text{where} \quad U_I(t_2, t_1) = \text{Texp} \left(-i \int_{t_1}^{t_2} dt H_I(t) \right) \quad (4.2.4)$$

注意, (4.2.2) 和 (4.2.3) 中, Texp 里的 H, H_0 都是 Schrödinger's picture 里的算符.

calculation:

首先有,

$$U_I(t_2, t_1) = U_0^\dagger(t_2, 0) U(t_2, t_1) U_0(t_1, 0) \quad (4.2.5)$$

因此,

$$\begin{aligned} \frac{d}{dt} U_I(t, t_0) &= i H_0 U_I(t, t_0) - i U_0^\dagger(t, 0) H U(t, t_0) U_0(t_0, 0) \\ &= -i \underbrace{U_0^\dagger(t, 0) H' U_0(t, 0)}_{=H_I(t)} U_I(t, t_0) \end{aligned} \quad (4.2.6)$$

4.3 scattering amplitude

- 最一般的过程是 $p_1, \dots, p_m \rightarrow q_1, \dots, q_n$, 其 scattering amplitude 为,

$$\langle q_1, \dots, q_n | U_0^\dagger(-\infty, 0) U_I(+\infty, -\infty) U_0(-\infty, 0) | p_1, \dots, p_m \rangle \quad (4.3.1)$$

一般会忽略掉 U_0 产生的相位.

- 考虑 ϕ^4 理论中的 $k_1, k_2 \rightarrow k_3, k_4$ 过程,

$$\langle k_3, k_4 | e^{-i \int d^d x \frac{\lambda}{4!} \phi^4} | k_1, k_2 \rangle \quad (4.3.2)$$

对 λ 展开, 0 阶项为,

$$\begin{aligned} \text{0th order term} &= \langle k_3, k_4 | k_1, k_2 \rangle \\ &= \rho(k_1) \rho(k_2) \rho(k_3) \rho(k_4) \langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle \\ &= \rho(k_1) \rho(k_2) \rho(k_3) \rho(k_4) \left(\underbrace{\langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle}_{=\delta_{31}^{(D)} \delta_{42}^{(D)}} + \underbrace{\langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle}_{=\delta_{32}^{(D)} \delta_{41}^{(D)}} \right) \\ &= (2\pi)^{2D} 4 \omega_{k_1} \omega_{k_2} (\delta^{(D)}(\vec{k}_1 - \vec{k}_3) \delta^{(D)}(\vec{k}_2 - \vec{k}_4) + \delta^{(D)}(\vec{k}_1 - \vec{k}_4) \delta^{(D)}(\vec{k}_2 - \vec{k}_3)) \end{aligned} \quad (4.3.3)$$

1 阶项为 (其中 $k^0 = \omega_k$),

$$\text{1st order term} = \frac{-i\lambda}{4!} \int d^d x \langle k_3, k_4 | \phi^4(x) | k_1, k_2 \rangle$$

$$\begin{aligned}
& \overbrace{= -i\lambda(2\pi)^d \delta^{(d)}(k_1 + k_2 - k_3 - k_4)} \\
& = 4! \times \frac{-i\lambda}{4!} \int d^d x e^{i(k_1 + k_2 - k_3 - k_4) \cdot x} + \rho(k_1)\rho(k_4)\delta_{14}^{(D)} \times 12 \times \frac{-i\lambda}{4!} (2\pi)^d \delta_{23}^{(d)} \int \frac{d^D p}{\rho^2(p)} \\
& \quad + \cdots + \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\delta_{13}^{(D)}\delta_{24}^{(D)} \times 3 \times \frac{-i\lambda}{4!} \int d^d x \int \frac{d^D p_1}{\rho^2(p_1)} \frac{d^D p_2}{\rho^2(p_2)} + \cdots \quad (4.3.4)
\end{aligned}$$

分别对应如下 Feynman diagrams,

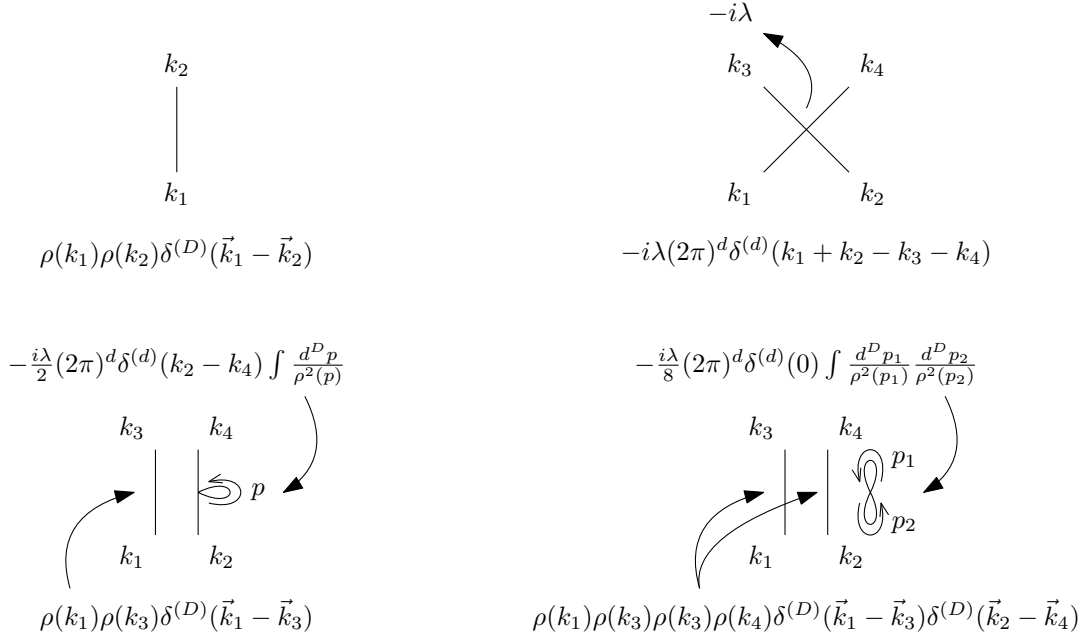


Figure 4.1: canonical quantization - Feynman diagrams

观察可见, 上图和 figure 3.3 有对应关系.

- 再举一个例子,

$$\begin{aligned}
& \text{Diagram: A crossing diagram with external momenta } k_1, k_2, k_3, k_4. \text{ A loop with momentum } p \text{ is attached to the crossing.} \\
& = (4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!} \right)^2 \rho(k_1) \cdots \int d^d x_1 d^d x_2 \int \frac{d^D p_1 \cdots d^D q_1 \cdots}{\rho(p_1) \cdots \rho(q_1) \cdots} e^{i(p_1 + p_2 - p_3 - p_4) \cdot x_1} e^{i(q_1 + q_2 - q_3 - q_4) \cdot x_2} \\
& \quad \left(\theta(t_2 - t_1) \langle 0 | a_{\vec{k}_3}^- a_{\vec{k}_4}^- a_{\vec{q}_1}^- a_{\vec{q}_2}^- a_{\vec{q}_3}^\dagger a_{\vec{q}_4}^\dagger a_{\vec{p}_1}^- a_{\vec{p}_2}^- a_{\vec{p}_3}^\dagger a_{\vec{p}_4}^\dagger a_{\vec{k}_1}^- a_{\vec{k}_2}^- | 0 \rangle + \cdots \right) \\
& = \frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 \int \frac{d^D p_3}{\rho^2(p_3)} \frac{d^D p_4}{\rho^2(p_4)} \left(\theta(t_2 - t_1) e^{i(k_1 + k_2 - p_3 - p_4) \cdot x_1} e^{i(p_3 + p_4 - k_3 - k_4) \cdot x_2} \right. \\
& \quad \left. + \theta(t_1 - t_2) e^{i(k_1 + k_2 + p_3 + p_4) \cdot x_1} e^{i(-p_3 - p_4 - k_3 - k_4) \cdot x_2} \right) \\
& = \frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 e^{i((k_1 + k_2) \cdot x_1 - (k_3 + k_4) \cdot x_2)} \int \frac{d^D p_3}{\rho^2(p_3)} \frac{d^D p_4}{\rho^2(p_4)} \left(\theta(t_2 - t_1) e^{i(p_3 + p_4) \cdot (x_2 - x_1)} \right. \\
& \quad \left. + \theta(t_1 - t_2) e^{i(p_3 + p_4) \cdot (x_1 - x_2)} \right) \quad (4.3.5)
\end{aligned}$$

同样, 与 (3.3.18) 有对应关系, (注意按时间排序 $\langle k_3 k_4 | T(\phi^4(x_1) \phi^4(x_2)) | k_1 k_2 \rangle$).

calculation:

从 (3.3.17) 开始 (与 (1.2.1) 类似, \vec{p}, \vec{q} 的符号可以任意改变),

$$\begin{aligned}
& \int d^d x_1 d^d x_2 e^{i((k_1+k_2+p-q)\cdot x_1 + (k_3+k_4-p+q)\cdot x_2)} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \frac{i}{-q^2 - m^2 + i\epsilon} \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{ie^{ip\cdot(x_1-x_2)}}{-p^2 - m^2 + i\epsilon} \frac{ie^{iq\cdot(x_2-x_1)}}{-q^2 - m^2 + i\epsilon} \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^D p}{(2\pi)^d} \frac{d^D q}{(2\pi)^d} \left(\theta(t_2 - t_1) \frac{2\pi i^2 e^{-ip\cdot(x_1-x_2)}}{-2\omega_p} \right. \\
&\quad \left. \frac{-2\pi i^2 e^{iq\cdot(x_2-x_1)}}{2\omega_q} + \theta(t_1 - t_2) \frac{-2\pi i^2 e^{ip\cdot(x_1-x_2)}}{2\omega_p} \frac{2\pi i^2 e^{-iq\cdot(x_2-x_1)}}{-2\omega_q} \right) \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^D p}{\rho^2(p)} \frac{d^D q}{\rho^2(q)} \left(\theta(t_2 - t_1) e^{i(p+q)\cdot(x_2-x_1)} \right. \\
&\quad \left. + \theta(t_1 - t_2) e^{i(p+q)\cdot(x_1-x_2)} \right) \tag{4.3.6}
\end{aligned}$$

结果与 (4.3.5) 对应.

4.4 complex scalar field

- complex scalar field 的 Lagrangian 为,

$$\mathcal{L} = -(\partial\psi^\dagger)(\partial\psi) - m^2\psi^\dagger\psi \tag{4.4.1}$$

实际上, complex scalar field 可以视为 2 个 real scalar fields 的和,

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \implies \left| \frac{\partial\phi_1, \phi_2}{\partial\psi, \psi^\dagger} \right| = i \tag{4.4.2}$$

因此, 也可以把 ψ, ψ^\dagger 视为两个独立的场.

- 其 canonical momentum 为,

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi} = \partial_0\psi^\dagger \quad \pi^\dagger = \partial_0\psi \tag{4.4.3}$$

其 Hamiltonian 为,

$$\mathcal{H} = \pi^\dagger\pi + (\vec{\nabla}\psi^\dagger) \cdot (\vec{\nabla}\psi) + m^2\psi^\dagger\psi \tag{4.4.4}$$

$$\implies \begin{cases} \partial_0\pi = i[H, \pi] = \vec{\nabla}^2\psi^\dagger - m^2\psi^\dagger \\ \partial_0\psi = i[H, \psi] = \pi^\dagger \end{cases} \implies (-\partial^2 - m^2)\psi = 0 \tag{4.4.5}$$

- 求解得到 (其中 $k^0 = \omega_k$),

$$\psi(x) = \int \frac{d^D k}{\rho(k)} (a_{\vec{k}} e^{ik\cdot x} + b_{\vec{k}}^\dagger e^{-ik\cdot x}) \tag{4.4.6}$$

- 从 path integral 的角度,

$$Z(J, J^\dagger) = \int D\psi D\psi^\dagger e^{i \int d^d x (\psi^\dagger(\partial^2 - m^2)\psi + J^\dagger\psi + \psi^\dagger J)} \tag{4.4.7}$$

$$= \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y 2J^\dagger(x) D(x-y) J(y)} \tag{4.4.8}$$

calculation:

转换为 ϕ_1, ϕ_2 后计算路径积分,

$$Z(J, J^\dagger) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y (J_1(x) D(x-y) J_1(y) + J_2(x) D(x-y) J_2(y))}$$

$$= \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y 2J^\dagger(x) D(x-y) J(y)} \quad (4.4.9)$$

4.4.1 charge

- 对场算符做如下变换,

$$\psi(x, \lambda) = e^{i\lambda} \psi(x) \implies D_\lambda \mathcal{L} = 0 \quad (4.4.10)$$

- 因此, 得到 conserved current,

$$J^\mu = \pi^\mu D_\lambda \psi + \pi^{\dagger\mu} D_\lambda \psi^\dagger = i(\psi \partial^\mu \psi^\dagger - \psi^\dagger \partial^\mu \psi) \quad (4.4.11)$$

其 0 分量对空间积分就是 charge,

$$\begin{aligned} Q &= \int d^D x J^0 = \int d^D x i(\psi^\dagger \partial_0 \psi - \psi \partial_0 \psi^\dagger) \\ &= \int d^D k (a_k^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}}) \end{aligned} \quad (4.4.12)$$

calculation:

$$\begin{aligned} Q &= \int d^D x \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} i \left((a_{\vec{p}}^\dagger e^{-ip \cdot x} + b_{\vec{p}} e^{ip \cdot x}) (-i\omega_q) (a_{\vec{q}} e^{iq \cdot x} - b_{\vec{q}}^\dagger e^{-iq \cdot x}) \right. \\ &\quad \left. - (a_{\vec{q}} e^{iq \cdot x} + b_{\vec{q}}^\dagger e^{-iq \cdot x}) (i\omega_p) (a_{\vec{p}}^\dagger e^{-ip \cdot x} - b_{\vec{p}} e^{ip \cdot x}) \right) \\ &= \int d^D x \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} \left((\omega_p a_{\vec{q}} a_{\vec{p}}^\dagger + \omega_q a_{\vec{p}}^\dagger a_{\vec{q}}) e^{-i(p-q) \cdot x} - (\omega_p b_{\vec{q}}^\dagger b_{\vec{p}} + \omega_q b_{\vec{p}} b_{\vec{q}}^\dagger) e^{i(p-q) \cdot x} \right. \\ &\quad \left. + a_{\vec{p}}^\dagger b_{\vec{q}}^\dagger (\omega_p - \omega_q) e^{-i(p+q) \cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{i(p+q) \cdot x} \right) \\ &= \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} \\ &\quad \left(\left((\omega_p a_{\vec{q}} a_{\vec{p}}^\dagger + \omega_q a_{\vec{p}}^\dagger a_{\vec{q}}) e^{i(\omega_p - \omega_q) \cdot t} - (\omega_p b_{\vec{q}}^\dagger b_{\vec{p}} + \omega_q b_{\vec{p}} b_{\vec{q}}^\dagger) e^{-i(\omega_p - \omega_q) \cdot t} \right) (2\pi)^D \delta^{(D)}(\vec{p} - \vec{q}) \right. \\ &\quad \left. + \left(a_{\vec{p}}^\dagger b_{\vec{q}}^\dagger (\omega_p - \omega_q) e^{i(\omega_p + \omega_q) \cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{-i(\omega_p + \omega_q) \cdot x} \right) (2\pi)^D \delta^{(D)}(\vec{p} + \vec{q}) \right) \\ &= \int \frac{d^D k}{2} (a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}}^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}}) = \int d^D k (a_{\vec{k}}^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}}) \end{aligned} \quad (4.4.13)$$

- 代入 (D.3.2), 有 $i[Q, \psi] = -i\psi$, 所以,

$$e^{-i\lambda Q} \psi e^{i\lambda Q} = e^{i\lambda} \psi \quad (4.4.14)$$

Chapter 5

disturbing the vacuum: Casimir effect

- 考虑一个沿 x^1 方向满足 periodic b.c. 的空间, 在垂直于 x^1 方向有两个 plates, s.t. 在 plates 上 $\phi(x) = 0$, 如下图,



Figure 5.1: Casimir effect

- 平板内外, 标量场的波矢的取值为,

$$\begin{cases} (n\frac{\pi}{d}, k_2, k_3) & \text{平板内} \\ (n\frac{\pi}{L-d}, k_2, k_3) & \text{平板外} \end{cases} \quad (5.0.1)$$

其中 $n \in \mathbb{Z}^+$.

- 因此, 代入真空能公式 (4.1.15), 平板内的能量为,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2} \quad (5.0.2)$$

而总能量为 $E = E(d) + E(L-d)$.

- 为解决能量发散的问题, 引入 ultra-violet (UV) cut-off,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2} e^{-a\sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2}} \quad (5.0.3)$$

for some $a \ll d$.

- 为了简化问题, 考虑 $d = 1 + 1$ 的情况,

$$E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-\frac{a\pi}{d}n} = \frac{\pi}{2d} \frac{e^{\frac{a\pi}{d}}}{(e^{\frac{a\pi}{d}} - 1)^2} = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O(a^2) \quad (5.0.4)$$

因此,

$$E_{1+1} = E_{1+1}(d) + E_{1+1}(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24} \left(\frac{1}{d} + \frac{1}{L-d} \right) + O(a^2) \quad (5.0.5)$$

得到 Casimir force,

$$F_{1+1} = -\frac{\partial E_{1+1}}{\partial d} = -\frac{\pi}{24} \left(\frac{1}{d^2} - \frac{1}{(L-d)^2} \right) + O(a^2) \stackrel{L \rightarrow \infty, a \rightarrow 0}{=} -\frac{\pi}{24d^2} \quad (5.0.6)$$

- 问题中, a 引入了 UV cut-off, L 引入了 infrared cut-off.

Part II

Dirac and spinor

Chapter 6

the Dirac spinor

- 整个 Part II 中, 我们使用 $(+, -, -, -)$ 号差, 因为 $\text{Cl}_{1,3}(\mathbb{R}) \cong \text{Cl}_{3,1}(\mathbb{R})$.
- 本笔记中的算符的定义与 A. Zee 的定义不同, 存在如下对应关系,

A. Zee's def.	my def.
$\omega_{\mu\nu}$	$\omega_{\mu\nu}$
$-iJ^{\mu\nu}$	$J^{\mu\nu}$
$-i\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$

- $\Pi(\Lambda)$ 的写法可能不准确, (要考虑 universal cover, $\text{Spin}(1,3) \simeq \text{Spin}(3,1)$), 因为 Lorentz transform 对 spinor 的操作是"path dependent", 因此本 chapter 中的 Λ 都默认沿着以下的 path 做变换,

$$\Lambda(\lambda) = e^{\frac{\lambda}{2}\omega_{\mu\nu}J^{\mu\nu}}, \lambda \in [0, 1] \quad (6.0.1)$$

6.1 gamma matrices

- Pauli 矩阵如下,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (6.1.1)$$

- gamma 矩阵 (also called Dirac matrices) 如下 (其中 $i = 1, 2, 3$),

$$\gamma^0 = \begin{pmatrix} I & \\ & I \end{pmatrix} = I \otimes \tau_1 \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = i\sigma_i \otimes \tau_2 \quad \gamma^5 = i\Omega = \begin{pmatrix} -I & \\ & I \end{pmatrix} = -I \otimes \tau_3 \quad (6.1.2)$$

其中 $\tau_{2,3}$ 也是 Pauli 矩阵, $\Omega = \gamma^0\gamma^1\gamma^2\gamma^3$, 有时候使用符号 $\sigma^\mu = (I, \vec{\sigma})$, $\bar{\sigma}^\mu = (I, -\vec{\sigma})$.

– 另外,

$$\begin{cases} \gamma^0\gamma^i = -\sigma_i \otimes \tau_3 \\ \gamma^i\gamma^j = -(\sigma_i\sigma_j) \otimes I = -i\epsilon_{ijk}\sigma_k \otimes I \end{cases} \quad \begin{cases} \Omega\gamma^0 = -I \otimes \tau_2 \\ \Omega\gamma^i = -\sigma_i \otimes \tau_2 \end{cases} \quad (6.1.3)$$

其中, 用到了 $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$.

- gamma 矩阵满足,

$$\begin{cases} (\gamma^\mu)^2 = \eta^{\mu\mu} \\ \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \mu \neq \nu \end{cases} \implies \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (6.1.4)$$

- 且存在如下关系,

$$\begin{aligned} \Omega\gamma^0 &= -\gamma^1\gamma^2\gamma^3 & \Omega\gamma^1 &= -\gamma^0\gamma^2\gamma^3 & \Omega\gamma^2 &= \gamma^0\gamma^1\gamma^3 & \Omega\gamma^3 &= -\gamma^0\gamma^1\gamma^2 \\ \iff -\epsilon^{\mu\nu\rho}{}_\sigma\Omega\gamma^\sigma &= \gamma^\mu\gamma^\nu\gamma^\rho & \text{when } \mu \neq \nu \neq \rho \end{aligned} \quad (6.1.5)$$

并且有 (注意到 $\Omega^2 = -1$),

$$\{\Omega, \gamma^\mu\} = 0 \quad \{\Omega, \Omega\gamma^\mu\} = 0 \quad [\Omega, \gamma^\mu\gamma^\nu] = 0 \quad (6.1.6)$$

- 定义 $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ (注意, 我们的定义中没有虚数 i , 与 A. Zee 的定义不同),

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} + \sigma^{\mu\nu} \Rightarrow \begin{cases} \sigma^{0i} = \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix} = -\sigma_i \otimes \tau_3 \\ \sigma^{ij} = -i\epsilon^{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk} \sigma_k \otimes I \end{cases} \quad (6.1.7)$$

与笔记 [Lie Groups and Lie Algebras](#) 中 $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ 表示对比, 可见 $\pi_{(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})}(J^{\mu\nu}) = \frac{1}{2}\sigma^{\mu\nu}$.

6.1.1 gamma matrices under Dirac basis

- 做如下相似变换 ($B = S^{-1}AS$),

$$S = \frac{\sqrt{2}}{2} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \iff S^{-1} = \frac{\sqrt{2}}{2} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \quad (6.1.8)$$

得到,

$$\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} = I \otimes \tau_3 \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = i\sigma_i \otimes \tau_2 \quad \gamma^5 = \begin{pmatrix} & I \\ I & \end{pmatrix} = I \otimes \tau_1 \quad (6.1.9)$$

- 另外,

$$\begin{cases} \gamma^0 \gamma^i = \sigma_i \otimes \tau_1 \\ \gamma^i \gamma^j = -i\epsilon_{ijk} \sigma_k \otimes I \end{cases} \quad \begin{cases} \Omega \gamma^0 = -I \otimes \tau_2 \\ \Omega \gamma^i = i\sigma_i \otimes \tau_3 \end{cases} \quad (6.1.10)$$

以及,

$$\sigma^{0i} = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix} = \sigma_i \otimes \tau_1 \quad \sigma^{ij} = -i\epsilon^{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk} \sigma_k \otimes I \quad (6.1.11)$$

6.2 Lorentz transformation and the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation

- Lorentz 变换可以写成如下形式,

$$\Lambda = e^{\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}} \quad (6.2.1)$$

其中 $\omega_{\mu\nu}$ 反对称, J^{0i} generate boosts and J^{ij} generate rotations, (详见笔记 [Lie Groups and Lie Algebras](#)).

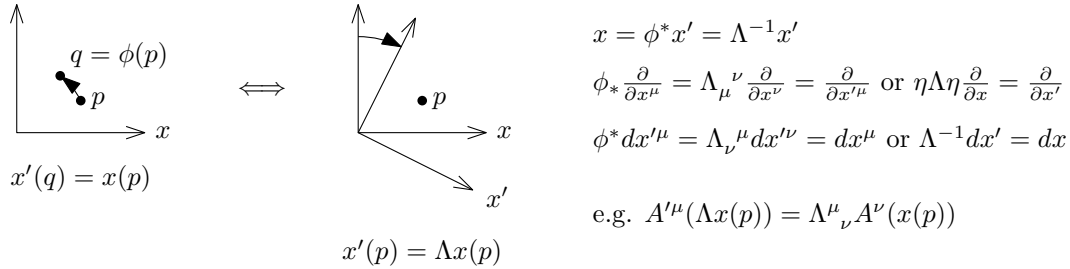


Figure 6.1: Lorentz transformation

- Weyl spinor 是 $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ rep. 的 vector space 中的元素,

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \text{with} \quad \Psi_{\text{Dirac}} = S^{-1}\Psi = \frac{\sqrt{2}}{2} \begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix} \quad (6.2.2)$$

在 Weyl basis 下很容易看出,

$$\Psi_L = \frac{1}{2}(1 - \gamma^5)\Psi \quad \Psi_R = \frac{1}{2}(1 + \gamma^5)\Psi \quad (6.2.3)$$

- 对于 gamma 矩阵, 有,

$$\Pi(\Lambda)\gamma^\rho\Pi^{-1}(\Lambda) = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = (\Lambda^{-1})^\rho_\sigma \gamma^\sigma \quad (6.2.4)$$

calculation:

利用 Campbell's identity,

$$e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = e^{\frac{1}{4}\omega_{\mu\nu}\text{ad}_{\sigma^{\mu\nu}}}\gamma^\rho \quad (6.2.5)$$

其中 (注意 $(J^{\mu\nu})^\rho{}_\sigma = 2\eta^{[\mu|\rho}\delta^{|\nu]}\sigma$, 其中度规号差与笔记 [Lie Groups and Lie Algebras](#) 中的不同),

$$\begin{cases} \rho \neq \mu, \nu & [\sigma^{\mu\nu}, \gamma^\rho] = \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu) \\ & = -\frac{1}{2}(\underbrace{\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\nu\mu\rho\sigma} - \epsilon^{\rho\mu\nu\sigma} + \epsilon^{\rho\nu\mu\sigma}}_{=0})\Omega\gamma_\sigma = 0 \\ \rho = \mu \text{ or } \nu \text{ and } \mu \neq \nu & [\sigma^{\mu\nu}, \gamma^\rho] = 2(\eta^{\mu\rho}\gamma^\mu - \eta^{\nu\rho}\gamma^\nu) \end{cases}$$

$$\Rightarrow [\sigma^{\mu\nu}, \gamma^\rho] = 2(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu) = -2(J^{\mu\nu})^\rho{}_\sigma\gamma^\sigma \quad (6.2.6)$$

代入, 得到,

$$e^{\frac{1}{4}\omega_{\mu\nu}\text{ad}_{\sigma^{\mu\nu}}}\gamma^\rho = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left(-\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} \right)^n \right)^\rho{}_\sigma \gamma^\sigma = (\Lambda^{-1})^\rho{}_\sigma \gamma^\sigma \quad (6.2.7)$$

可以用”无穷小” Lorentz 变换验证以上计算,

$$\begin{aligned} \Pi(1 + \delta\omega^\mu{}_\nu)\gamma^\rho\Pi^{-1}(1 + \delta\omega^\mu{}_\nu) &= \gamma^\rho + \frac{1}{4}\delta\omega_{\mu\nu}[\sigma^{\mu\nu}, \gamma^\rho] \\ &= (1 - \delta\omega^\rho{}_\sigma)\gamma^\sigma \end{aligned} \quad (6.2.8)$$

6.2.1 Dirac spinor

- 对于 Dirac spinor,

$$\Pi(\Lambda)\Psi(x) = \Psi'(\Lambda x) \quad (6.2.9)$$

注意 $\partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu$, 所以,

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \iff (i\gamma^\mu\partial'_\mu - m)\Psi'(\Lambda x) = 0 \quad (6.2.10)$$

– 关键部分在于,

$$\gamma^\mu\Psi'(\Lambda x) = \gamma^\mu\Pi(\Lambda)\Psi(x) = \Pi(\Lambda)\Lambda^\mu{}_\nu\gamma^\nu\Psi(x) \quad (6.2.11)$$

calculation:

首先,

$$\Lambda^T\eta\Lambda = \eta \implies (\Lambda^{-1})^\mu{}_\nu = (\eta\Lambda^T\eta)^\mu{}_\nu = \Lambda_\nu{}^\mu \quad (6.2.12)$$

考虑,

$$\Pi^{-1}(\Lambda)\gamma^\mu\Pi(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu \implies \gamma^\mu\Pi(\Lambda) = \Lambda^\mu{}_\nu\Pi(\Lambda)\gamma^\nu \quad (6.2.13)$$

代入,

$$\begin{aligned} (i\gamma^\mu\partial'_\mu - m)\Psi'(\Lambda x) &= (i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - m)\Pi(\Lambda)\Psi(x) \\ &= \Pi(\Lambda)(i\underbrace{\gamma^\rho\Lambda^\mu{}_\rho\Lambda_\mu{}^\nu}_{=\delta^\nu_\rho}\partial_\nu - m)\Psi(x) \\ &= \Pi(\Lambda)(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \end{aligned} \quad (6.2.14)$$

6.2.2 Dirac bilinears

- γ^0 是 Hermitian 矩阵, 而 γ^i 不是, 有,

$$\gamma^{i\dagger} = -\gamma^i = \gamma^0\gamma^i\gamma^0 \quad (6.2.15)$$

可以统一写作 $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, 并且有,

$$\sigma^{\mu\nu\dagger} = -\gamma^0\sigma^{\mu\nu}\gamma^0 \quad \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0 \quad (6.2.16)$$

calculation:

对于 $\sigma^{\mu\nu}$,

$$\sigma^{\mu\nu\dagger} = \frac{1}{2}(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger}) = \gamma^0\sigma^{\nu\mu}\gamma^0 = -\gamma^0\sigma^{\mu\nu}\gamma^0 \quad (6.2.17)$$

所以,

$$((\omega_{\mu\nu}\sigma^{\mu\nu})^\dagger)^n = \gamma^0(-\omega_{\mu\nu}\sigma^{\mu\nu})^n\gamma^0 \implies \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0 \quad (6.2.18)$$

- 所以,

$$\begin{cases} \bar{\Psi}'(\Lambda x)\Psi'(\Lambda x) = \bar{\Psi}\Psi & \text{scalar field} \\ \bar{\Psi}'\gamma^\mu\Psi' = \Lambda^\mu{}_\nu\bar{\Psi}\gamma^\nu\Psi & \text{vector field} \end{cases} \quad (6.2.19)$$

其中 $\bar{\Psi} = \Psi^\dagger\gamma^0$.

calculation:

$$\begin{cases} \Psi'^\dagger(\Lambda x)\gamma^0\Psi'(\Lambda x) = \Psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\Pi(\Lambda)\Psi(x) = \Psi^\dagger\gamma^0\Psi \\ \Psi'^\dagger\gamma^0\gamma^\mu\Psi' = \Psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\gamma^\mu\Pi(\Lambda)\Psi(x) = \Lambda^\mu{}_\nu\Psi^\dagger\gamma^0\gamma^\nu\Psi \end{cases} \quad (6.2.20)$$

此外,

$$\begin{cases} \bar{\Psi}'\sigma^{\mu\nu}\Psi' = \Psi^\dagger\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\sigma^{\mu\nu}\Pi(\Lambda)\Psi = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\Psi}\sigma^{\rho\sigma}\Psi & \text{order 2 tensor} \\ \bar{\Psi}'\Omega\gamma^\mu\Psi' = \bar{\Psi}\Pi(\Lambda^{-1})\Omega\gamma^\mu\Pi(\Lambda)\Psi = \det(\Lambda)\Lambda^\mu{}_\nu\bar{\Psi}\Omega\gamma^\nu\Psi & \text{pseudovector} \\ \bar{\Psi}'\Omega\Psi' = \bar{\Psi}\Pi(\Lambda^{-1})\Omega\Pi(\Lambda)\Psi = \det(\Lambda)\bar{\Psi}\Omega\Psi & \text{4-form (pseudoscalar)} \end{cases} \quad (6.2.21)$$

其中 (注意到下面的计算中, 第二个等号后, 含 η 的项都等于零; 由此可以看出, 对 μ_i 求和的过程中, 任何两个 μ_i, μ_j 相等的项求和之后都等于零),

$$\begin{aligned} \Pi(\Lambda^{-1})\Omega\Pi(\Lambda) &= \prod_{i=0}^3 \Lambda^i{}_{\mu_i} \gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \\ &= \prod_{i=0}^3 \Lambda^i{}_{\mu_i} (\eta^{\mu_0\mu_1} + \sigma^{\mu_0\mu_1})(\eta^{\mu_2\mu_3} + \sigma^{\mu_2\mu_3}) \\ &= \prod_{i=0}^3 \Lambda^i{}_{\mu_i} \gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \quad \text{with } \mu_0 \neq \mu_1 \neq \mu_2 \neq \mu_3 \\ &= \det(\Lambda)\Omega \end{aligned} \quad (6.2.22)$$

6.2.3 parity and time reversal

- 这里沿用笔记 [Lie Groups and Lie Algebras](#) 中的记号, 选择 $O(3,1)$ 而非 $O(1,3)$, 因为他们没有区别.
- $O(3,1)$ 有 4 个联通分支,

$$I \in \text{SO}_+(3,1) \quad PT \in \text{SO}_-(3,1) \quad P \in O'_+(3,1) \quad T \in O'_-(3,1) \quad (6.2.23)$$

其中,

$$P = \text{diag}(+1, -1, -1, -1) \quad T = \text{diag}(-1, +1, +1, +1) \quad (6.2.24)$$

另外, $\eta P \eta = P, \eta T \eta = T$.

- 另外, Lorentz algebra 的 representation 不能自然的生成对 P, T 的表示, 因为本质上它只能生成 spin group 的表示, 是 $\text{SO}_+(3,1)$ 的 universal cover, 与 Lorentz group 的其它三个连通分支没有直接联系.
- 因此, 对 P, T 的表示要从物理的角度定义, (可能) 无法单纯靠数学的方法给出, 所以这部分放在下一章.

Chapter 7

the Dirac equation

7.1 Dirac equation

- A. Zee: our discussion provides a unified view of the equations of motion in relativistic physics: they just project out the unphysical components.
- the Dirac equation is,

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 \iff (\gamma^\mu p_\mu - m)\tilde{\Psi} = 0 \implies \begin{cases} i\sigma^\mu \partial_\mu \psi_R - m\psi_L = 0 \\ i\bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_R = 0 \end{cases} \quad (7.1.1)$$

首先可以看出 Ψ 满足 Klein-Gordon equation,

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\Psi &= \left(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu - 2im\gamma^\mu \partial_\mu + m^2\right)\Psi = 0 \\ \implies (-\partial^2 - m^2)\Psi &= 0 \end{aligned} \quad (7.1.2)$$

– 在粒子静止系下 $p_\mu = (m, 0, 0, 0)$, Dirac 方程给出 (这里采用 Dirac basis),

$$(\gamma^0 - 1)\tilde{\Psi}_{\text{Dirac}} = 0 \implies \begin{pmatrix} 0 & \\ & I \end{pmatrix} \tilde{\Psi}_{\text{Dirac}} = 0 \quad (7.1.3)$$

因此, $\tilde{\Psi}$ 的后两个分量为零 $\implies \Psi$ 只有两个自由度.

- Dirac 方程的 Lorentz covariance 见 (6.2.10).

7.2 Dirac Lagrangian

- 根据 (6.2.19) 以及之前标量场的计算经验, 可知,

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = (-i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi})\Psi + \text{total diff.} \quad (7.2.1)$$

其中, 与复标量场论中类似, 可以把 Ψ, Ψ^\dagger 或 $\Psi, \bar{\Psi}$ 视为独立变量.

7.3 chirality or handedness

- parity transformation 会把 left spinor 变成 right spinor and vice versa,

$$\gamma^0 \Psi_L = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix} \quad \gamma^0 \Psi_R = \begin{pmatrix} \psi_R \\ 0 \end{pmatrix} \quad (7.3.1)$$

- 把 Lagrangian 中的 Ψ 拆开,

$$\begin{aligned} \mathcal{L} &= \bar{\Psi}_L(i\not{\partial})\Psi_L + \bar{\Psi}_R(i\not{\partial})\Psi_R - m(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L) \\ &= \psi_L^\dagger i\bar{\sigma}^\mu \partial_\mu \psi_L + \psi_R^\dagger i\sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \end{aligned} \quad (7.3.2)$$

其中注意到了 $\gamma^0 \gamma^\mu$ 的非对角分块为零.

7.3.1 internal vector symmetry

- 做变换 $\Psi \mapsto e^{i\theta}\Psi$, Lagrangian 保持不变, 利用 Noether's theorem 得到守恒流 (见 section D.2),

$$J_V^\mu = \bar{\Psi}\gamma^\mu\Psi \quad (7.3.3)$$

其中, 按照惯例省略了虚数 i .

calculation:

计算广义动量,

$$\begin{cases} \pi_\Psi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\Psi} = \bar{\Psi}i\gamma^\mu \\ \pi_{\bar{\Psi}}^\mu = 0 \end{cases} \quad \text{or} \quad \begin{cases} \pi_\Psi^\mu = 0 \\ \pi_{\bar{\Psi}}^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}} = -i\gamma^\mu\Psi \end{cases} \quad (7.3.4)$$

这里看起来有点奇怪, 需要再说明一下. 对于 (7.2.1) 第一个等号后边,

$$\begin{cases} \pi_\Psi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\Psi} = \bar{\Psi}i\gamma^\mu & \frac{\delta\mathcal{L}}{\delta\Psi} = -m\bar{\Psi} \\ \pi_{\bar{\Psi}}^\mu = 0 & \frac{\delta\mathcal{L}}{\delta\bar{\Psi}} = (i\gamma^\mu\partial_\mu - m)\Psi \end{cases} \implies \begin{cases} -(\partial_\mu\bar{\Psi})i\gamma^\mu - m\bar{\Psi} = 0 \\ (i\gamma^\mu\partial_\mu - m)\Psi = 0 \end{cases} \quad (7.3.5)$$

对于 (7.2.1) 第二个等号后边, 忽略掉全微分项,

$$\begin{cases} \pi_\Psi^\mu = 0 & \frac{\delta\mathcal{L}}{\delta\Psi} = -i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} \\ \pi_{\bar{\Psi}}^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}} = -i\gamma^\mu\Psi & \frac{\delta\mathcal{L}}{\delta\bar{\Psi}} = -m\Psi \end{cases} \implies \begin{cases} -i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \\ (i\gamma^\mu\partial_\mu - m)\Psi = 0 \end{cases} \quad (7.3.6)$$

7.3.2 axial symmetry

- 做变换,

$$\Psi \mapsto e^{i\theta\gamma^5}\Psi = \begin{pmatrix} e^{-i\theta}\Psi_L \\ e^{i\theta}\Psi_R \end{pmatrix} \quad (7.3.7)$$

在 $m = 0$ 时 Lagrangian 保持不变, 对应的守恒流为,

$$J_A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi \quad (7.3.8)$$

根据 (6.2.21), 是一个 pseudovector.

7.4 energy-momentum tensor and angular momentum

- Dirac 场的 energy-momentum tensor 为,

$$T_{\mu\nu} = i\bar{\Psi}\gamma_\mu\partial_\nu\Psi - \eta_{\mu\nu}\mathcal{L} \quad (7.4.1)$$

其中, 对于满足运动方程的 Dirac 场, $\mathcal{L} = 0$.

- Dirac 场的 angular momentum 为,

$$M^{\mu\nu\rho} = \frac{i}{2}\bar{\Psi}\gamma^\mu\sigma^{\nu\rho}\Psi(x) + (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}) \quad (7.4.2)$$

calculation:

做变换 $x \mapsto e^{\frac{1}{2}\lambda\omega_{\mu\nu}J^{\mu\nu}}x$, 那么,

$$\begin{aligned} \Psi(x) &\mapsto \Psi'(x') = e^{\frac{1}{4}\lambda\omega_{\mu\nu}\sigma^{\mu\nu}}\Psi(x) \\ \implies D_\lambda\Psi'(\mathbf{x}) &= \frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\Psi(x) - \frac{1}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma\partial_\rho\Psi(x) \end{aligned} \quad (7.4.3)$$

所以,

$$J^\mu = \frac{i}{4} \omega_{\nu\rho} \bar{\Psi} \gamma^\mu \sigma^{\nu\rho} \Psi(x) + \dots \implies M^{\mu\nu\rho} = \frac{i}{2} \bar{\Psi} \gamma^\mu \sigma^{\nu\rho} \Psi(x) + (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}) \quad (7.4.4)$$

7.5 charge conjugation, parity and time reversal

- 沿用 A. Zee 的 notation, 变换映射分别用 $\mathcal{C}, \mathcal{P}, \mathcal{T}$ 表示, 相应的矩阵用 C, P, T 表示.

7.5.1 charge conjugation and antimatter

- 定义矩阵 C ,

$$C = -\gamma^0 \gamma^2 \implies C \gamma^0 = -i \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} = \gamma^2 \implies \begin{cases} (\gamma^2)^{-1} \gamma^\mu \gamma^2 = -\gamma^{\mu*} \\ C^{-1} \gamma^\mu C = -(\gamma^\mu)^T \end{cases} \quad (7.5.1)$$

因此 $-\gamma^{\mu*}$ 同样满足 Clifford algebra.

– 另外, 有 $(\gamma^2)^{-1} = \gamma^{2*} = -\gamma^2$ 和 $C^{-1} = C$.

calculation:

$$\gamma^0 C^{-1} \gamma^0 C \gamma^0 = -\gamma^{\mu*} \implies C^{-1} \gamma^0 C = -\gamma^0 \gamma^{\mu*} \gamma^0 = -\underbrace{(\gamma^0 \gamma^\mu \gamma^0)^*}_{=\gamma^{\mu\dagger}} \quad (7.5.2)$$

其中用到了 $\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger}$, 见 (6.2.15).

- $\Psi_c = \gamma^2 \Psi^*$ 满足如下方程,

$$(-i\gamma^{\mu*}(\partial_\mu - ieA_\mu) - m)\Psi^* = 0 \implies (\gamma^2)^{-1}(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\Psi_c = 0 \quad (7.5.3)$$

可见 Ψ_c 满足 $-e \mapsto +e$ 后的 Dirac 方程, Ψ_c is the field of positron.

- 对于 Lorentz 变换, $e^{\frac{1}{2}\lambda\omega_{\mu\nu}J^{\mu\nu}}, \lambda \in [0, 1]$, 有,

$$\begin{cases} \Psi \mapsto \Psi'(x') = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \Psi \\ \Psi_c \mapsto \gamma^2 \underbrace{(\gamma^2)^{-1} e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \gamma^2 \Psi^*}_{=(\Psi'(x'))^*} = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \Psi_c \end{cases} \quad (7.5.4)$$

可见 Ψ_c 与 Ψ 的变换形式相同.

7.5.2 parity

- 对于 parity, 有 $x \rightarrow x' = (x^0, -\vec{x})$, 在 Dirac eq. 中,

$$\gamma^0 \gamma^\mu = P^\mu_\nu \gamma^\nu \gamma^0 \implies (i\gamma^\mu \partial'_\mu - m)\gamma^0 \Psi(x) = 0 \quad (7.5.5)$$

因此,

$$\mathcal{P} : \Psi(x) \mapsto \Psi'(x') = \gamma^0 \Psi(x) \quad (7.5.6)$$

7.5.3 time reversal

- 时间反演算符为,

$$T = (i\sigma_2 \otimes I)K = \gamma^1 \gamma^3 K \quad (7.5.7)$$

其中 K 是 complex conjugation operator (见 appendix E). 另外, 有 $T^2 = -1$, 符合预期.

proof:

时间反演之后, $\Psi'(t') = T\Psi(t)$ 满足如下方程,

$$i\frac{\partial}{\partial t'}\Psi'(t') = H\Psi'(x') \quad (7.5.8)$$

其中,

$$H = -i\gamma^0\gamma^i\frac{\partial}{\partial x^i} + \gamma^0m \quad (7.5.9)$$

且 Hamiltonian 满足时间反演不变, $H'(t') \equiv TH(t)T^\dagger = H(t)$, 即 (其中 $T = UK$),

$$\begin{cases} T(i\gamma^0\gamma^i)T^\dagger = i\gamma^0\gamma^i \\ T\gamma^0T^\dagger = \gamma^0 \end{cases} \implies \begin{cases} U(-i\gamma^0\gamma^{i*})U^\dagger = U(-i\gamma^0\gamma^2\gamma^i\gamma^2)U^\dagger = i\gamma^0\gamma^i \\ [U, \gamma^0] = 0 \end{cases} \quad (7.5.10)$$

满足以上要求的 U 具有一下形式,

$$U = \begin{pmatrix} a\sigma_2 & b\sigma_2 \\ b\sigma_2 & a\sigma_2 \end{pmatrix} \quad \text{with} \quad \begin{cases} |a|^2 + |b|^2 = 1 \\ a^*b + b^*a = 0 \end{cases} \quad (7.5.11)$$

不妨令 $a = i, b = 0$.

7.5.4 CPT theorem

- 在 CPT 变换下,

$$CPT : \Psi(x) \mapsto \gamma^1\gamma^3K(\gamma^0\gamma^2\Psi^*) = \Omega\Psi = -i\gamma^5\Psi \quad (7.5.12)$$

- 任何 Lorentz covariant theory 都满足 CPT 不变性.

7.6 interaction in QED

- 注意, 我们采用通常的符号 $e > 0$, 与 A. Zee 的符号 $e = -|e|$ 不同.
- QED 的 Lagrangian 为,

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\mu^2 A^\mu A_\mu \quad (7.6.1)$$

其中,

$$D_\mu = \partial_\mu + ieA_\mu \quad (7.6.2)$$

可见电子和电磁场耦合项为 $-eA_\mu J_V^\mu$, 其中 J_V^μ 是 internal vector symmetry 的守恒流, 见 (7.3.3).

- QED 里的 Dirac 方程为,

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\Psi = 0 \quad \text{and} \quad -i(\partial_\mu - ieA_\mu)\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \quad (7.6.3)$$

7.7 Majorana neutrino

- 因为在 Lorentz 变换下, Ψ, Ψ_c 行为相同, 因此 Majorana 方程同样满足 Lorentz covariance,

$$i\not{\partial}\Psi - m\Psi_c = 0 \quad \text{and} \quad i\not{\partial}\Psi_c - m\Psi = 0 \quad (7.7.1)$$

因此,

$$(-\partial^2 - m^2)\Psi = 0 \quad (7.7.2)$$

满足 Klein-Gordon 方程.

calculation:

$$-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\Psi = m(i\not{\partial})\Psi_c = m^2\Psi \quad (7.7.3)$$

- Majorana 方程对应的 Lagrangian 为,

$$\mathcal{L} = \bar{\Psi} i \not{\partial} \Psi - \frac{1}{2} m (\Psi^T C \Psi + \bar{\Psi} C \bar{\Psi}^T) \quad (7.7.4)$$

相应的广义动量为,

$$\begin{cases} \pi_{\Psi}^{\mu} = \bar{\Psi} i \gamma^{\mu} & \frac{\delta \mathcal{L}}{\delta \Psi} = -m \Psi^T C \\ \pi_{\bar{\Psi}}^{\mu} = 0 & \frac{\delta \mathcal{L}}{\delta \bar{\Psi}} = i \not{\partial} \Psi - m C \bar{\Psi}^T = i \not{\partial} \Psi - m \Psi_c \end{cases} \quad (7.7.5)$$

- 注意, Ψ 应该被当作 Grassmann numbers, 因此, 对于反对称矩阵 C , 有 $\Psi^T C \Psi, \bar{\Psi} C \bar{\Psi}^T \neq 0$.

calculation:

对 Ψ 变分得到,

$$\begin{aligned} 0 &= \frac{\delta \mathcal{L}}{\delta \Psi} - \partial_{\mu} \pi_{\Psi}^{\mu} \\ &= -m \Psi^T C - i \partial_{\mu} \bar{\Psi} \gamma^{\mu} \\ &= (-m \Psi^T - i \partial_{\mu} \bar{\Psi} \gamma^{\mu} C) C \\ &= (-m \Psi + i C (\gamma^{\mu})^T \gamma^0 \partial_{\mu} \Psi^*)^T C \end{aligned} \quad (7.7.6)$$

其中,

$$C (\gamma^{\mu})^T \gamma^0 = C (-C^{-1} \gamma^{\mu} C) \gamma^0 = -\gamma^{\mu} C \gamma^0 = -\gamma^{\mu} \gamma^2 \quad (7.7.7)$$

代入, 得到 (?),

$$-i \not{\partial} \Psi_c - m \Psi = 0 \quad (7.7.8)$$

- Majorana eq. v.s. Dirac eq.:

- Majorana eq. 只适用于 electrically neutral fields (?).
- Majorana eq. preserves handedness (?).

Chapter 8

quantizing the Dirac field

8.1 anticommutation

- 用 α, β 表示电子的量子态 (包括动量和自旋), 那么,

$$\begin{cases} \{b_\alpha, b_\beta\} = 0 \\ \{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta} \end{cases} \quad (8.1.1)$$

comment:

反对称关系 $\{b_\alpha, b_\beta\} = 0$ 由实验发现, 我们希望电子有 number operator,

$$N = \sum_{\alpha} b_{\alpha}^{\dagger} b_{\alpha} \quad \text{with} \quad \begin{cases} [N, b_{\alpha}] = -b_{\alpha} \\ [N, b_{\alpha}^{\dagger}] = b_{\alpha}^{\dagger} \end{cases} \quad (8.1.2)$$

考虑到 $[AB, C] = ABC - CAB = A\{B, C\} - \{A, C\}B$, 所以,

$$\begin{cases} [N, b_{\alpha}] = \sum_{\beta} (b_{\beta}^{\dagger} \{b_{\beta}, b_{\alpha}\} - \{b_{\beta}^{\dagger}, b_{\alpha}\} b_{\beta}) = -\sum_{\beta} \{b_{\beta}^{\dagger}, b_{\alpha}\} b_{\beta} \\ [N, b_{\alpha}^{\dagger}] = \sum_{\beta} (b_{\beta}^{\dagger} \{b_{\beta}, b_{\alpha}^{\dagger}\} - \{b_{\beta}^{\dagger}, b_{\alpha}^{\dagger}\} b_{\beta}) = \sum_{\beta} b_{\beta}^{\dagger} \{b_{\beta}, b_{\alpha}^{\dagger}\} \end{cases} \quad (8.1.3)$$

可见 $\{b_{\alpha}, b_{\beta}^{\dagger}\} = \delta_{\alpha\beta}$.

8.2 the Dirac field

- $\Psi(x), \bar{\Psi}$ 有如下形式,

$$\begin{cases} \Psi(x) = \sum_{s=\pm 1} \int \frac{d^D p}{(2\pi)^{D/2} \sqrt{\omega_p/m}} (b_{\vec{p}}^s u(\vec{p}, s) e^{-ip \cdot x} + c_{\vec{p}}^{s\dagger} v(\vec{p}, s) e^{ip \cdot x}) \\ \bar{\Psi}(x) = \sum_{s=\pm 1} \int \frac{d^D p}{(2\pi)^{D/2} \sqrt{\omega_p/m}} (b_{\vec{p}}^{s\dagger} \bar{u}(\vec{p}, s) e^{ip \cdot x} + c_{\vec{p}}^s \bar{v}(\vec{p}, s) e^{-ip \cdot x}) \end{cases} \quad (8.2.1)$$

其中 $\bar{u} = u^\dagger \gamma^0, \bar{v} = v^\dagger \gamma^0$, 且 $p^0 = \omega_p$, 参数 $u(\vec{p}, s), v(\vec{p}, s)$ 分别满足,

$$\begin{cases} (\not{p} - m)u(\vec{p}, s) = 0 \\ (-\not{p} - m)v(\vec{p}, s) = 0 \end{cases} \quad (8.2.2)$$

并选择归一化条件,

$$\begin{cases} \bar{u}(\vec{p}, s)u(\vec{p}, s') = \delta_{ss'} \\ \bar{v}(\vec{p}, s)v(\vec{p}, s') = -\delta_{ss'} \end{cases} \quad \text{and} \quad \bar{u}(\vec{p}, s)v(\vec{p}, s') = 0 \quad (8.2.3)$$

- 在粒子静止系下, 方程化为 $(\gamma^0 - 1)u(\vec{p}_r, s) = (\gamma^0 + 1)v(\vec{p}_r, s) = 0$, (其中 $p_r = (m, 0, 0, 0)$), 解为,

$$\begin{cases} u(\vec{p}_r, +1) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & u(\vec{p}_r, -1) = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ v(\vec{p}_r, +1) = \frac{\sqrt{2}}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & v(\vec{p}_r, -1) = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \end{cases} \quad (8.2.4)$$

可见 $s = \pm 1$ 分别代表 spin-up 和 spin-down.

- 一般情况下, 解为,

$$\begin{cases} u(\vec{p}, +1) = \sqrt{\frac{\omega_p + m}{2m}} S \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{m + \omega_p} \\ \frac{p_1 + ip_2}{m + \omega_p} \end{pmatrix} & u(\vec{p}, -1) = \sqrt{\frac{\omega_p + m}{2m}} S \begin{pmatrix} 0 \\ 1 \\ \frac{p_1 - ip_2}{m + \omega_p} \\ -\frac{p_3}{m + \omega_p} \end{pmatrix} \\ v(\vec{p}, +1) = \sqrt{\frac{\omega_p + m}{2m}} S \begin{pmatrix} p_3 \frac{\omega_p - m}{|p|^2} \\ (p_1 + ip_2) \frac{\omega_p - m}{|p|^2} \\ 1 \\ 0 \end{pmatrix} & v(\vec{p}, -1) = \sqrt{\frac{\omega_p + m}{2m}} S \begin{pmatrix} (p_1 - ip_2) \frac{\omega_p - m}{|p|^2} \\ -p_3 \frac{\omega_p - m}{|p|^2} \\ 0 \\ 1 \end{pmatrix} \end{cases} \quad (8.2.5)$$

其中 S 是相似变换矩阵, 见 (6.1.8), 可以验证解满足归一化条件.

- 另外,

$$\begin{cases} \sum_{s=\pm 1} u(\vec{p}, s) \bar{u}(\vec{p}, s) = \frac{\not{p} + m}{2m} \\ \sum_{s=\pm 1} v(\vec{p}, s) \bar{v}(\vec{p}, s) = \frac{\not{p} - m}{2m} \end{cases} \quad (8.2.6)$$

- 回顾 section 4.4 关于 complex scalar field 的内容, 可知 b^\dagger 和 c^\dagger 产生的粒子具有相反的电荷, 不妨令 b^\dagger 产生 electron (带电荷 $-e$), c^\dagger 产生 positron (带电荷 e).
- section 8.1 中的讨论说明,

$$\begin{cases} \{b_{\vec{p}}^s, b_{\vec{p}'}^{s'}\} = 0 \\ \{b_{\vec{p}}^s, b_{\vec{p}'}^{s'\dagger}\} = \delta^{(D)}(\vec{p} - \vec{p}') \delta_{ss'} \end{cases} \quad (8.2.7)$$

- Ψ 的 momentum conjecture 为 (π_Ψ^μ 见 (7.3.4)),

$$\pi = \frac{\delta \mathcal{L}}{\delta \partial_0 \Psi} = \pi_\Psi^0 = \bar{\Psi} i \gamma^0 = i \Psi^\dagger \quad (8.2.8)$$

存在如下 anticommutation relation,

$$\{\Psi_\alpha(t, \vec{x}), i\Psi_\beta^\dagger(t, \vec{y})\} = i\delta^{(D)}(\vec{x} - \vec{y})\delta_{\alpha\beta} \quad (8.2.9)$$

Appendices

Appendix A

Dirac delta function & Fourier transformation

A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx \delta(x) e^{-ikx} \quad (\text{A.1.1})$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x))g(x)dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \quad (\text{A.1.2})$$

- 第二个常用的公式 ([Sokhotski-Plemelj theorem](#)),

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{x + i\epsilon} = \mathcal{P} \frac{1}{x} - i\pi\delta(x) \quad (\text{A.1.3})$$

其中 \mathcal{P} 表示复函数的主值 (principal value).

proof:

考虑,

$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \quad \text{and} \quad \int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \text{Res}(f, i\epsilon) = \pi \quad (\text{A.1.4})$$

所以...

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,

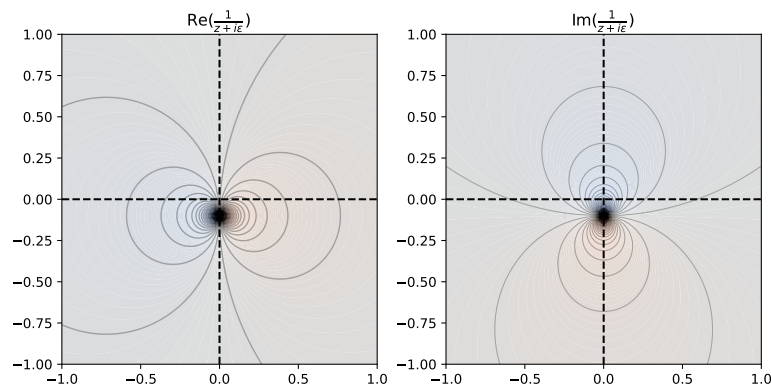


Figure A.1: graph of $\frac{1}{z + i\epsilon}$

- 另外, $\delta(x - a)\delta(x - b) = \delta(b - a)\delta(x - a)$.

A.2 Fourier transformation

- d -dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^d x e^{-ik \cdot x} \phi(x) \end{cases} \quad (\text{A.2.1})$$

- 因此,

$$\partial_\mu \phi(x) \mapsto ik_\mu \tilde{\phi}(k) \quad (\text{A.2.2})$$

- 对于**实函数**, Fourier transformation 是正交变换, 其 Jacobi determinant 为,

$$\left| \frac{\partial \phi(x) \cdots}{\partial \text{Re} \tilde{\phi}(k) \cdots \partial \text{Im} \tilde{\phi}(k) \cdots} \right| = \left(\frac{2}{V} \right)^{(2N+1)^d} \det A = \left(\frac{2(2N)^d}{V^2} \right)^{\frac{(2N+1)^d}{2}} \quad (\text{A.2.3})$$

proof:

position space 和 momentum space 的格点分别为,

$$\begin{cases} x_i^\mu = i^\mu \epsilon \in \{0, \pm\epsilon, \dots, \frac{L}{2}\} \\ k_n^\mu = n^\mu \frac{2\pi}{L} \in \{0, \pm\frac{2\pi}{L}, \dots, \frac{\pi}{\epsilon}\} \end{cases} \iff i^\mu, n^\mu \in \{0, \pm 1, \dots, N\} \quad (\text{A.2.4})$$

x^μ, k^μ 分别有 $2N+1$ 个取值, 其中 $N\epsilon = \frac{L}{2}$, 时空总体积为 $V = L^d$, momentum space 的总体积为 $\tilde{V} = \frac{(4\pi N)^d}{V}$.

将 (A.2.1) 写成格点求和的形式,

$$\begin{cases} \phi(x_i) = \frac{1}{(2\pi)^d} \left(\frac{2\pi}{L} \right)^d \sum_n e^{ik_n \cdot x_i} \tilde{\phi}(k_n) \\ = \frac{2}{V} \sum_{n^0 > 0} \left(\cos(k_n \cdot x_i) \text{Re} \tilde{\phi}(k_n) - \sin(k_n \cdot x_i) \text{Im} \tilde{\phi}(k_n) \right) \\ \tilde{\phi}(k_n) = \epsilon^d \sum_i e^{-ik_n \cdot x_i} \phi(x_i) \\ = \frac{V}{(2N)^d} \sum_i \left(\cos(k_n \cdot x_i) - i \sin(k_n \cdot x_i) \right) \phi(x_i) \end{cases} \quad (\text{A.2.5})$$

proof:

$\phi(x_i)$ 的变换需要做一些说明. 注意到 $\tilde{\phi}$ 的分量的数量是 ϕ 的两倍 (考虑到实部与虚部), 但在 $\phi \in \mathbb{R}^{(2N+1)^d}$ 时,

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k) \quad (\text{A.2.6})$$

可见 $\tilde{\phi}$ 的分量并不独立, 取 $k^0 > 0$ 的部分为独立分量, 那么...

将 (A.2.5) 写成矩阵的形式,

$$\begin{cases} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} = \frac{2}{V} \overbrace{\begin{pmatrix} \cos k_0 \cdot x_0 & \cdots & \cos k_{\max} \cdot x_0 & -\sin k_0 \cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A} \begin{pmatrix} \text{Re} \tilde{\phi}(k_0) \\ \vdots \\ \text{Im} \tilde{\phi}(k_0) \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \text{Re} \tilde{\phi}(k_0) \\ \vdots \\ \text{Im} \tilde{\phi}(k_0) \\ \vdots \end{pmatrix} = \frac{V}{(2N)^d} \begin{pmatrix} \cos k_0 \cdot x_0 & \cdots & \cos k_0 \cdot x_{\max} \\ \vdots & \ddots & \vdots \\ -\sin k_0 \cdot x_0 & \cdots & -\sin k_0 \cdot x_{\max} \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} \end{cases} \quad (\text{A.2.7})$$

观察可见 $\tilde{\phi}$ 的变换中的矩阵是 A^T , 所以,

$$\frac{2}{V} \frac{V}{(2N)^d} A A^T = I \implies \det A = \left(\frac{(2N)^d}{2} \right)^{\frac{(2N+1)^d}{2}} \quad (\text{A.2.8})$$

因此...

– 顺便,

$$\int d^d x f(x) g(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{f}(-k) \tilde{g}(k) \quad (\text{A.2.9})$$

Appendix B

Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \quad (\text{B.0.1})$$

$$\langle x^{2n} \rangle = \frac{\int dx e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n} (2n-1)!! \quad (\text{B.0.2})$$

其中 $(2n-1)!! = 1 \cdot 3 \cdots (2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx e^{-\frac{a}{2}x^2 + Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \quad (\text{B.0.3})$$

另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 N -dim. generalization

- 考虑如下积分,

$$Z(A, J) = \int dx_1 \cdots dx_N e^{-\frac{1}{2}x^T \cdot A \cdot x + J^T \cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \quad (\text{B.1.1})$$

其中 x, J 是 N -dim. 列向量, A 是 $N \times N$ 实对称矩阵.

calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1} \cdot D \cdot O \quad (\text{B.1.2})$$

其中 D 是一个 diagonal matrix. 令 $y = O \cdot x$, 那么,

$$\begin{aligned} Z(A, J) &= \int dy_1 \cdots dy_N e^{-\frac{1}{2}y^T \cdot D \cdot y + (OJ)^T \cdot y} \\ &= \prod_{i=1}^N \sqrt{\frac{2\pi}{D_{ii}}} e^{\frac{1}{2D_{ii}}(OJ)_i^2} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \end{aligned} \quad (\text{B.1.3})$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O \cdot A^{-1} \cdot O^{-1})_{ii}$ 以及 $\text{tr } D = \det A$.

- 一个重要的变体是 $A \mapsto -iA, J \mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} Z(A, J) \Big|_{J=0} = A_{ij}^{-1} \quad (\text{B.1.4})$$

$$\langle x_i x_j \cdots x_k x_l \rangle = \sum_{\text{Wick}} A_{i'j'}^{-1} \cdots A_{k'l'}^{-1} \quad (\text{B.1.5})$$

其中 (B.1.5) 中有偶数个 x , 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A, J) \Big|_{J=0} = \cdots \quad (\text{B.1.6})$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \quad (\text{B.1.7})$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle \overbrace{x_i x_j x_k x_l} \rangle = A_{ik}^{-1} A_{jl}^{-1} \quad (\text{B.1.8})$$

Appendix C

perturbation theory in QM

- this chapter is based on MIT OpenCourseWare [Quantum Physics III Chapter 1: Perturbation Theory](#).

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda \delta H \quad (\text{C.0.1})$$

其中 $\lambda \in [0, 1]$.

- 考虑 $H^{(0)}$ 的本征态为,

$$H^{(0)} |k^{(0)}\rangle = E_k^{(0)} |k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)} | l^{(0)} \rangle = \delta_{kl} \\ E_0^{(0)} \leq E_1^{(0)} \leq E_2^{(0)} \leq \dots \end{cases} \quad (\text{C.0.2})$$

C.1 non-degenerate perturbation theory

- 考虑 non-degenerate 能级 k , 有 $\dots \leq E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \leq \dots$, 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \dots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \dots \end{cases} \quad (\text{C.1.1})$$

– 注意, 我们可以选取修正项满足,

$$\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \dots \quad (\text{C.1.2})$$

proof:

假设我们求解得到的修正项不满足 $\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \dots$, 考虑,

$$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n |k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)} | k^{(n)} \rangle' = 0 \quad (\text{C.1.3})$$

那么, (注意到态矢量可以乘一个常数, $\frac{1}{1-a_1\lambda-a_2\lambda^2-\dots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \dots$),

$$\begin{aligned} |k\rangle_\lambda &= (1 - a_1\lambda - a_2\lambda^2 - \dots) |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \dots \\ |k\rangle_\lambda' &= |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \dots} (\lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \dots) \\ &= |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 (a_1 |k^{(1)}\rangle' + |k^{(2)}\rangle') + \dots \end{aligned} \quad (\text{C.1.4})$$

可见修正项都与 $|k^{(0)}\rangle$ 正交.

- 注意, 不能要求 ${}_\lambda \langle k | k \rangle_\lambda = 1$, 否则 $|k^{(n)}\rangle$ 将与 λ 相关 (包括 $|k^{(0)}\rangle$),

$$\begin{aligned} {}_\lambda \langle k | k \rangle_\lambda &= \langle k^{(0)} | k^{(0)} \rangle \\ &\quad + \lambda (\langle k^{(1)} | k^{(0)} \rangle + \langle k^{(0)} | k^{(1)} \rangle) \\ &\quad + \lambda^2 (\langle k^{(2)} | k^{(0)} \rangle + \langle k^{(1)} | k^{(1)} \rangle + \langle k^{(0)} | k^{(2)} \rangle) \end{aligned}$$

$$\begin{aligned} & \vdots \\ & + \lambda^n (\langle k^{(n)} | k^{(0)} \rangle + \langle k^{(n-1)} | k^{(1)} \rangle + \dots + \langle k^{(0)} | k^{(n)} \rangle) \end{aligned} \quad (\text{C.1.5})$$

- 将 (C.1.1) 代入 Schrödinger's eq., 得到,

$$\begin{array}{ll} \lambda^0 & (H^{(0)} - E_k^{(0)}) |k^{(0)}\rangle = 0 \\ \lambda^1 & (H^{(0)} - E_k^{(0)}) |k^{(1)}\rangle = (E_k^{(1)} - \delta H) |k^{(0)}\rangle \\ \lambda^2 & (H^{(0)} - E_k^{(0)}) |k^{(2)}\rangle = (E_k^{(1)} - \delta H) |k^{(1)}\rangle + E_k^{(2)} |k^{(0)}\rangle \\ \vdots & \vdots \\ \lambda^n & (H^{(0)} - E_k^{(0)}) |k^{(n)}\rangle = (E_k^{(1)} - \delta H) |k^{(n-1)}\rangle + E_k^{(2)} |k^{(n-2)}\rangle + \dots + E_k^{(n)} |k^{(0)}\rangle \end{array}$$

calculation:

Schrödinger's eq. 为,

$$(H^{(0)} + \lambda \delta H - E_k(\lambda)) |k\rangle_\lambda = 0 \quad (\text{C.1.6})$$

展开为,

$$\left((H^{(0)} - E_k^{(0)}) + \lambda(\delta H - E_k^{(1)}) - \lambda^2 E_k^{(2)} - \dots \right) (|k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \dots) = 0 \quad (\text{C.1.7})$$

- 现在来计算 $\langle l^{(0)} | k^{(n)} \rangle$, 有,

$$\left\{ \begin{array}{l} (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(1)} \rangle = E_k^{(1)} \delta_{lk} - \delta H_{lk} \\ (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(2)} \rangle = E_k^{(1)} \langle l^{(0)} | k^{(1)} \rangle - \langle l^{(0)} | \delta H | k^{(1)} \rangle + E_k^{(2)} \delta_{lk} \\ \vdots \\ (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(n)} \rangle = E_k^{(1)} \langle l^{(0)} | k^{(n-1)} \rangle - \langle l^{(0)} | \delta H | k^{(n-1)} \rangle \\ \quad + E_k^{(2)} \langle l^{(0)} | k^{(n-2)} \rangle + \dots + E_k^{(n)} \delta_{lk} \end{array} \right. \quad (\text{C.1.8})$$

其中 $\delta H_{lk} = \langle l^{(0)} | \delta H | k^{(0)} \rangle$, 对于满足 (C.1.2) 的解, 有,

$$E_k^{(n)} = \langle k^{(0)} | \delta H | k^{(n-1)} \rangle, n = 1, 2, \dots \quad (\text{C.1.9})$$

并且,

$$|k^{(1)}\rangle = - \sum_{l \neq k} \frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} |l^{(0)}\rangle \implies E_k^{(2)} = - \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} \quad (\text{C.1.10})$$

calculation:

将 (C.1.10) 代入 (C.1.8), 得到 ($l \neq k$),

$$(E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(2)} \rangle = -E_k^{(1)} \frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} + \sum_{m \neq k} \frac{\delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \quad (\text{C.1.11})$$

所以,

$$\left\{ \begin{array}{l} |k^{(2)}\rangle = \sum_{l \neq k} \left(- \frac{\delta H_{00} \delta H_{lk}}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m \neq k} \frac{\delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \right) |l^{(0)}\rangle \\ E_k^{(3)} = \sum_{l \neq k} \left(- \frac{\delta H_{00} |\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m \neq k} \frac{\delta H_{kl} \delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \right) \end{array} \right. \quad (\text{C.1.12})$$

计算归一化系数,

$${}_l \langle k | k \rangle_\lambda = 1 + \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + O(\lambda^3) \quad (\text{C.1.13})$$

C.1.1 level repulsion or the seesaw mechanism

- 能量的展开式为,

$$E_k(\lambda) = E_k^{(0)} + \lambda \delta H_{kk} - \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3) \quad (\text{C.1.14})$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

C.1.2 validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即,

$$|\lambda V| < \frac{1}{2} \Delta E^{(0)} \quad (\text{C.1.15})$$

因此, 对于能级简并的情况, $\Delta E^{(0)} = 0$, 情况会更复杂.

calculation:

对于两能级系统,

$$H(\lambda) = H^{(0)} + \lambda \hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix} \quad (\text{C.1.16})$$

$H(\lambda)$ 的本征值可以直接计算,

$$E_{\pm}(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)}) \sqrt{1 + \left(\frac{\lambda |V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})} \right)^2} \quad (\text{C.1.17})$$

考虑 $\sqrt{1+z^2}$ 的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1} \frac{(2n-3)!!}{2^n n!} z^{2n} + \cdots \quad (\text{C.1.18})$$

注意到 $\sqrt{1+z^2}$ 在 $z = \pm i$ 有 branch cut, 因此 $z = 0$ 附近的 Taylor expansion 只有在 $|z| < 1$ 内才收敛.

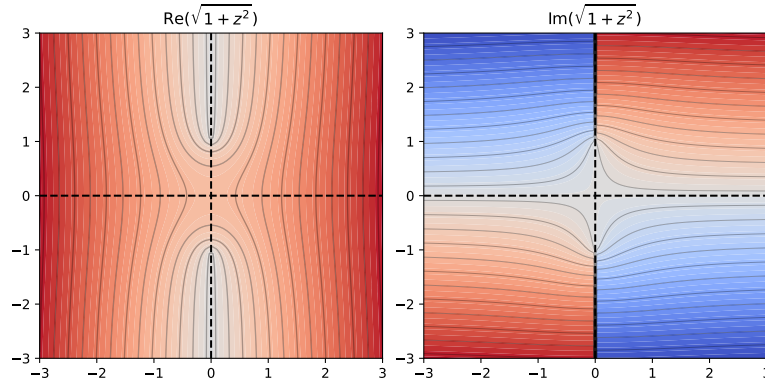


Figure C.1: graph of $\sqrt{1+z^2}$

C.2 degenerate perturbation theory

- 暂时先跳过.

Appendix D

classical field theory and Noether's theorem

D.1 classical field theory

D.1.1 Lagrangian density and the action

- Lagrangian density, \mathcal{L} , 是 $\phi^a(x), \partial_\mu \phi^a(x), t$ 的函数.
- 对作用量变分得到 Euler-Lagrangian equation of motion,

$$\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) = 0 \quad (\text{D.1.1})$$

calculation:

对作用量进行变分,

$$\begin{aligned} \delta S &= \int d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^a} \delta \phi^a + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \delta \partial_\mu \phi^a \right) \\ &= \int d^4x \left(\left(\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) \right) \delta \phi^a + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \delta \phi^a \right) \right) \end{aligned} \quad (\text{D.1.2})$$

由于边界变分为零...

D.1.2 canonical momentum and the Hamiltonian

- **def.:** 定义一个叫 π_a^μ 的量,

$$\pi_a^\mu = \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \quad (\text{D.1.3})$$

其中 $\pi_a \equiv \pi_a^0$ 称作 canonical momentum of the field.

- **def.:** the Hamiltonian density is,

$$\mathcal{H} = \pi_a \partial_0 \phi^a - \mathcal{L} \quad (\text{D.1.4})$$

- the Hamilton's equations are,

$$\begin{cases} \partial_0 \phi^a = \frac{\delta \mathcal{H}}{\delta \pi_a} \\ -\partial_0 \pi^a = \frac{\delta \mathcal{H}}{\delta \phi^a} - \partial_i \left(\frac{\delta \mathcal{H}}{\delta (\partial_i \phi^a)} \right) \end{cases} \quad (\text{D.1.5})$$

- 第二个方程可以写成更紧凑的形式,

$$\partial_\mu \pi_a^\mu = \frac{\delta \mathcal{H}}{\delta \phi^a} \quad (\text{D.1.6})$$

D.2 Noether's theorem

D.2.1 in classical particle mechanics

- 系统的 Lagrangian 为 $L(q^a, \dot{q}^a, t)$.
- 系统通过以下形式变换,

$$q^a(t) \mapsto q^a(\lambda, t) \quad \text{and} \quad q^a(t, 0) = q^a(t) \quad (\text{D.2.1})$$

并定义,

$$D_\lambda q^a = \left. \frac{\partial q^a}{\partial \lambda} \right|_{\lambda=0} \quad (\text{D.2.2})$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.,

$$D_\lambda L = \frac{dF(q^a, \dot{q}^a, t)}{dt} \quad (\text{D.2.3})$$

for some $F(q^a, \dot{q}^a, t)$, and the corresponding **conserved quantity** is,

$$Q = p_a D_\lambda q^a - F(q^a, \dot{q}^a, t) \quad (\text{D.2.4})$$

proof:

$$D_\lambda L = \frac{\partial L}{\partial q^a} D_\lambda q^a + \frac{\partial L}{\partial \dot{q}^a} \frac{dD_\lambda q^a}{dt} = \frac{d}{dt} (p_a D_\lambda q^a) \quad (\text{D.2.5})$$

- 几个例子如下,

- **空间平移**, $\vec{x}(t) \mapsto \vec{x}(t) + \hat{e}_i \lambda$, 相应地, $D_\lambda \vec{x} = \hat{e}_i$, 且,

$$D_\lambda L = \frac{\partial L}{\partial x^i} \quad (\text{D.2.6})$$

如果 $\frac{\partial L}{\partial x^i} = 0$, 那么, 有守恒量 p_i .

- **时间平移**, $q^a(t) \mapsto q^a(t + \lambda)$, 相应地, $D_\lambda q^a = \dot{q}^a$, 且,

$$D_\lambda L = \frac{dL}{dt} - \frac{\partial L}{\partial t} \quad (\text{D.2.7})$$

如果 $\frac{\partial L}{\partial t} = 0$, 那么, 有守恒量 $H = p_a \dot{q}^a - L$.

- **转动**, $\vec{x}(t) \mapsto R(\lambda, \hat{e}) \cdot \vec{x}(t)$, 相应地, $D_\lambda \vec{x} = \hat{e} \times \vec{x}$, 且,

$$D_\lambda L = \vec{x} \cdot \left(\frac{\partial L}{\partial \vec{x}} \times \hat{e} \right) + \hat{e} \cdot (\dot{\vec{x}} \times \vec{p}) \quad (\text{D.2.8})$$

如果上式中两个括号内的项都为零, 那么, 有守恒量 $\hat{e} \cdot \vec{J} = \hat{e} \cdot (\vec{x} \times \vec{p})$.

D.2.2 in classical field theory

- 类似地, 系统通过以下形式变换,

$$\phi^a(x) \mapsto \phi^a(x, \lambda) \quad \text{and} \quad \phi^a(x, 0) = \phi^a(x) \quad (\text{D.2.9})$$

并定义,

$$D_\lambda \phi^a = \left. \frac{\partial \phi^a}{\partial \lambda} \right|_{\lambda=0} \quad (\text{D.2.10})$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.,

$$D_\lambda \mathcal{L} = \partial_\mu F^\mu(\phi^a, \partial_\mu \phi^a, t) \quad (\text{D.2.11})$$

for some $F^\mu(\phi^a, \partial_\mu \phi^a, t)$, and the **conserved current** is,

$$J^\mu = \pi_a^\mu D_\lambda \phi^a - F^\mu \quad (\text{D.2.12})$$

proof:

$$\begin{aligned} D_\lambda \mathcal{L} &= \frac{\delta \mathcal{L}}{\delta \phi^a} D_\lambda \phi^a + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \partial_\mu D_\lambda \phi^a \\ &= \left(\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) \right) D_\lambda \phi^a + \partial_\mu \underbrace{\left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} D_\lambda \phi^a \right)}_{=\pi_a^\mu} \end{aligned} \quad (\text{D.2.13})$$

代入 (D.1.1), 得...

- 注意, conserved current 并不是唯一确定的, 考虑如下变换,

$$F^\mu \mapsto F'^\mu = F^\mu + \partial_\nu A^{\mu\nu} \quad \text{with} \quad A^{\mu\nu} = A^{[\mu\nu]} \quad (\text{D.2.14})$$

新 F'^μ 依然能满足 (D.2.11).

- 但是, 守恒荷是唯一确定的.

proof:

$$Q' = \int d^3x J^0 = \int d^3x (\pi_a D_\lambda \phi^a - F^0) - \int d^3x \partial_\mu A^{0\mu} \quad (\text{D.2.15})$$

考虑到边界值为零, 且 $A^{00} = 0$, 所以 $Q' = Q$.

D.2.3 spacetime translations and the energy-momentum tensor

- 时空平移变换为,

$$\phi^a(x) \mapsto \phi^a(x + \lambda e) \quad (\text{D.2.16})$$

- 所以,

$$D_\lambda \phi^a = e^\mu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = e^\mu \partial_\mu \mathcal{L} \quad (\text{D.2.17})$$

代入 (D.2.12),

$$J^\mu = e^\nu \underbrace{(\pi_a^\mu \partial_\nu \phi^a - \delta_\nu^\mu \mathcal{L})}_{=T^\mu_\nu} \quad (\text{D.2.18})$$

- 并且有,

$$\partial_\mu T^{\mu\nu} = 0 \implies P^\mu = \int d^3x T^{0\mu} = \text{Const.} \quad (\text{D.2.19})$$

来自守恒流散度为零.

D.2.4 Lorentz transformations, angular momentum and something else

- Lorentz transformation 下坐标做变换 $x'^\mu = \Lambda^\mu_\nu x^\nu$, 其中 Λ 满足,

$$\eta = \Lambda^T \eta \Lambda \quad (\text{D.2.20})$$

- infinitesimal Lorentz transformation 是,

$$\Lambda = I + \epsilon \quad (\text{D.2.21})$$

其中 $\{\epsilon^{\mu\nu}\} = \epsilon \eta$ 是反对称矩阵.

proof:

考虑,

$$\eta = (\Lambda \eta)^T \eta (\Lambda \eta) = (\eta + \epsilon \eta)^T \eta (\eta + \epsilon \eta)$$

$$= \eta + \eta \epsilon^T + \epsilon \eta + O(\epsilon^2) \quad (\text{D.2.22})$$

- 标量场在 Lorentz transform 下的变换为,

$$\Lambda : \phi^a(x) \mapsto \phi^a(\Lambda^{-1}x') \quad (\text{D.2.23})$$

- 有,

$$D_\lambda \phi^a = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\epsilon_{\mu\nu} \partial^\mu (x^\nu \mathcal{L}) \quad (\text{D.2.24})$$

代入 (D.2.12),

$$J^\mu = \frac{1}{2} \epsilon_{\nu\rho} M^{\mu\nu\rho} \quad \text{where} \quad M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \quad (\text{D.2.25})$$

且有,

$$\partial_\mu M^{\mu\nu\rho} = 0 \quad (\text{D.2.26})$$

- 对全空间积分, 得到 6 个守恒量,

$$J^{\mu\nu} = \int d^3x M^{0\mu\nu} = \text{Const.} \quad (\text{D.2.27})$$

不难发现 J^{ij} 对应角动量, 现在来讨论 J^{0i} 的物理意义,

$$0 = \frac{d}{dt} J^{0i} = \frac{d}{dt} \int d^3x (x^i T^{00} - t T^{0i}) = P^i - \frac{d}{dt} \int d^3x x^i T^{00} \quad (\text{D.2.28})$$

其中, 用到了 $\frac{dP^i}{dt} = 0$ (见 (D.2.19)), 可以将上式的第二项理解为质心运动的动量.

D.3 charge as generators

- the charge associated with the conserved current is,

$$Q = \int d^D x J^0 = \int d^D x (\pi_a D_\lambda \phi^a - F^0) \quad (\text{D.3.1})$$

在 $F^\mu = 0$ 且 $[D_\lambda \phi^a, \phi^a] = 0$ 的情况下,

$$i[Q, \phi^a] = D_\lambda \phi^a \quad (\text{D.3.2})$$

D.4 what the graviton listens to: energy-momentum tensor

- the energy-momentum tensor is defined as (其中 $g = |\det\{g_{\mu\nu}\}|$),

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2 \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M \quad (\text{D.4.1})$$

- 如果将 \mathcal{L}_M 对 $g^{\mu\nu}$ 做展开 $\mathcal{L}_M = A + g^{\mu\nu} B_{\mu\nu} + g^{\mu\nu} g^{\rho\sigma} C_{\mu\nu\rho\sigma} + \dots$, 那么,

$$T_{\mu\nu} = -2(B_{\mu\nu} + 2g^{\rho\sigma} C_{\mu\nu\rho\sigma} + 3\dots) + g_{\mu\nu} \mathcal{L}_M \quad (\text{D.4.2})$$

另外, the trace of the energy-momentum tensor is,

$$T = g^{\mu\nu} T_{\mu\nu} = d \times A + (d-2)g^{\mu\nu} B_{\mu\nu} + (d-4)g^{\mu\nu} g^{\rho\sigma} C_{\mu\nu\rho\sigma} \quad (\text{D.4.3})$$

可见 $d=4$ 时, T 与 $C_{\mu\nu\rho\sigma}$ 无关.

- 以 electromagnetic field 为例, $d=4$,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^\mu A_\mu \implies \begin{cases} T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho + m^2 A_{\mu\nu} + g_{\mu\nu} \mathcal{L}_M \\ T = -m^2 A^\mu A_\mu \end{cases} \quad (\text{D.4.4})$$

可见 the energy-momentum tensor of electromagnetic field (when $m=0$) is traceless.

- $\mathcal{L} = -\frac{1}{2}((\partial\phi)^2 - m^2\phi^2)$ 和 $\mathcal{L} = \frac{1}{2}\phi(\partial^2 - m^2)\phi$ 对应的 energy-momentum tensor 一样吗 (?).

Appendix E

antiunitary operator and time reversal

E.1 complex conjugation operator

- complex conjugation operator, K , is an antiunitary operator on the complex plane,

$$\begin{cases} Kz = z^* \\ zK^* = z^* \end{cases} \implies K^2 = K^{*2} = 1 \quad (\text{E.1.1})$$

- $K^*I : V^* \rightarrow V^*$ 是 dual space 上的算符.
- 对于一组 orthonormal basis, 有,

$$\langle i | K^* I K | j \rangle = \delta_{ij} \quad (\text{E.1.2})$$

并且可以证明在基矢变换后这个等式依然成立.

proof:

- 对基矢做 unitary transformation,

$$|i'\rangle = U |i\rangle = \sum_j |j\rangle U_{ji} \quad \text{where} \quad U_{ji} = \langle j | U | i \rangle \quad (\text{E.1.3})$$

那么,

$$\langle i' | K^* I K | j' \rangle = \sum_{kl} \langle k | U_{ki}^* K^* I K U_{lj} | l \rangle = \sum_{kl} U_{ki} U_{lj}^* \delta_{kl} = \delta_{ij} \quad (\text{E.1.4})$$

- 对基矢做 antiunitary transformation, 只需要证明 $|i'\rangle = K |i\rangle$ 的情况, 此时,

$$\langle i' | K^* I K | j' \rangle = \langle i | j \rangle = \delta_{ij} \quad (\text{E.1.5})$$

E.2 antiunitary operator

- 对于一个 unitary operator, U , $\Omega = UK$ 是一个 antiunitary operator.
- 定义其 Hermitian conjugate,

$$\Omega^\dagger = K^* U^\dagger \iff \langle i | \Omega j \rangle = \langle j | \Omega^\dagger i \rangle^* \quad (\text{E.2.1})$$

那么,

$$\begin{cases} \langle \phi | \Omega \psi \rangle = \langle \psi | \Omega^\dagger \phi \rangle^* \\ \langle \Omega \phi | \Omega \psi \rangle = \langle \psi | \phi \rangle \end{cases} \quad (\text{E.2.2})$$

proof:

首先,

$$\langle \phi | \Omega \psi \rangle = \sum_{ij} \langle i | \phi_i^* U K \psi_j | j \rangle$$

$$\begin{aligned}
&= \sum_{ij} \phi_i^* \psi_j^* \langle i | UK | j \rangle \\
&= \left(\sum_{ij} \langle j | K^* U^\dagger | i \rangle \phi_i \psi_j \right)^* \\
&= \left(\sum_{ij} \langle j | \psi_j^* K^* U^\dagger \phi_i | i \rangle \right)^* = \langle \psi | K^* U^\dagger | \phi \rangle^*
\end{aligned} \tag{E.2.3}$$

其次,

$$\begin{aligned}
\langle \Omega \phi | \Omega \psi \rangle &= \langle \phi | \Omega^\dagger \Omega \psi \rangle = \langle \phi | K^* IK | \psi \rangle \\
&= \sum_{ij} \langle i | \phi_i^* K^* IK \psi_j | j \rangle \\
&= \sum_{ij} \phi_i \psi_j^* \langle i | K^* IK | j \rangle = \langle \psi | \phi \rangle
\end{aligned} \tag{E.2.4}$$

E.3 time reversal in QM

- 在量子力学中,

$$\mathcal{T} : |\psi\rangle \mapsto |\psi'(t')\rangle = \int d^D x |x\rangle K \langle x | \psi(t) \rangle \quad \text{where } t' = -t \tag{E.3.1}$$

- 因此, 对于动量本征态,

$$T |p\rangle = \int d^D x |x\rangle K e^{i\vec{p}\cdot\vec{x}} = |-p\rangle \tag{E.3.2}$$

- 对于动量算符,

$$TPT^\dagger = \int d^D p |-p\rangle p \langle -p| = -P \tag{E.3.3}$$

- 对于角动量算符,

$$TLT^\dagger = T(X \times P)T^{-1} = -L \tag{E.3.4}$$

- 对于平面波,

$$\psi(t) = e^{i(\vec{k}\cdot\vec{x}-Et)} \mapsto \psi'(t') = \langle x | K^* IK | \psi(t) \rangle = e^{-i(\vec{k}\cdot\vec{x}-Et)} \tag{E.3.5}$$

注意到 $t' = -t$, 代入,

$$\psi'(t) = e^{i(-\vec{k}\cdot\vec{x}-Et)} \tag{E.3.6}$$

E.3.1 spin- $\frac{1}{2}$ non-relativistic electron

- 时间反演算符作用到 spin-up state 应该得到 spin-down state, 所以,

$$T = \sigma_2 K \tag{E.3.7}$$

- 因此,

$$T^2 = \sigma_2 K \sigma_2 K = \sigma_2^* \sigma_2 = -1 \tag{E.3.8}$$

- 具体地,

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} \tag{E.3.9}$$

- Kramer's degeneracy:** 含有奇数个电子的时间反演不变系统, 其能级是 twofold degenerate.

proof:

因为系统时间反演不变, 所以 ψ 和 $T\psi$ 有相同的能级, 且 $T\psi \neq e^{i\alpha}\psi, \forall \alpha$.

考虑 $T\psi = e^{i\alpha}\psi$, 那么,

$$T^2\psi = T e^{i\alpha}\psi = e^{-i\alpha} e^{i\alpha}\psi = \psi \tag{E.3.10}$$

与 $T^2 = -1$ 矛盾.