

# Quantum Field Theory

- based on A. Zee's textbook -

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# convention, notation, and units

- 笔记中的度规号差约定为  $(-, +, +, +)$ .
- 使用 Planck units, 此时  $G, \hbar, c, k_B = 1$ , 因此,

| name/dimension                  | expression/value  |
|---------------------------------|---|
| Planck length ( $L$ )           | $l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} \text{ m}$                          |
| Planck time ( $T$ )             | $t_P = \frac{l_P}{c} = 5.391 \times 10^{-44} \text{ s}$                                       |
| Planck mass ( $M$ )             | $m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} \text{ kg} \simeq 10^{19} \text{ GeV}$ |
| Planck temperature ( $\Theta$ ) | $T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32} \text{ K}$                     |

- 时空维度用  $d = D + 1$  表示.

**Part I**

**motivation and foundation**

# Chapter 1

## free field theory

### 1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \quad (1.1.1)$$

A. Zee 说: 在作用量里, 时间的导数项必须是正的, 包括标量场的  $(\partial_0\phi)^2$  和电磁场的  $(\partial_0 A_i)^2$ .

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi e^{i \int d^d x (-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \quad (1.1.2)$$

- 当  $V(\phi) = \frac{1}{2}m^2\phi^2$  时, 称作 free or Gaussian theory.

- 
- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (1.1.3)$$

另外, 用  $W(J)$  表示指数上的部分 (去掉虚数  $i$ ).

**proof:**

注意  $\partial^\mu \phi \partial_\mu \phi = \partial^\mu (\phi \partial_\mu \phi) - \phi \partial^2 \phi$ , 忽略全微分项, 那么,

$$Z(J) = \int D\phi e^{i \int d^d x (\frac{1}{2} \phi (\partial^2 - m^2) \phi + J(x)\phi(x))} \quad (1.1.4)$$

代入 (B.1.1), 可知,

$$Z(J) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (1.1.5)$$

其中  $D(x-y)$  满足,

$$\begin{cases} (\partial^2 - m^2) D(x-y) = \delta^{(d)}(x-y) \\ (-p^2 - m^2) \tilde{D}(p, q) = (2\pi)^d \delta^{(d)}(p-q) \end{cases} \implies D(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{-k^2 - m^2} \quad (1.1.6)$$

### 1.2 free propagator

- 为了使 (1.1.4) 中的积分在  $\phi$  较大时收敛, 作替换  $m^2 \mapsto m^2 - i\epsilon$ , 这样被积函数中会出现一项  $e^{-\epsilon \int d^d x \phi^2}$ .
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量  $\epsilon$  避免发散,

$$D(x) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot x}}{-k^2 - m^2 + i\epsilon} = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left( e^{i(-\omega_k t + \vec{k} \cdot \vec{x})} \theta(t) + e^{i(\omega_k t + \vec{k} \cdot \vec{x})} \theta(-t) \right) \quad (1.2.1)$$

**calculation:**

对  $k^0$  积分, 注意有两个奇点  $k^0 = \pm(\omega_k - i\epsilon)$ , 当  $t > 0$  时, contour 处于下半平面, ...

- $D(x)$  的取值与  $x$  的类时, 类空性质关系密切.

- 类时区域,

$$D(t, 0) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left( e^{-i\omega_k t} \theta(t) + e^{i\omega_k t} \theta(-t) \right) \quad (1.2.2)$$

- 类空区域,

$$D(0, \vec{x}) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{i\vec{k} \cdot \vec{x}} \sim e^{-m|\vec{x}|} \quad (1.2.3)$$

## Chapter 2

# from field to particle to force

### 2.1 from field to particle

- 考虑 (1.1.3) 中的  $W(J)$ ,

$$W(J) = -\frac{1}{2} \int d^d x d^d y J(y) D(x-y) J(y) \quad (2.1.1)$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}(k) \quad (2.1.2)$$

其中, 如果  $J(x)$  是实函数, 那么  $\tilde{J}(-k) = \tilde{J}^*(k)$ .

- 
- 考虑  $J(x) = J_1(x) + J_2(x)$ , 那么  $W(J)$  共有 4 项, 其中一个交叉项如下,

$$W_{12}(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_1(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_2(k) \quad (2.1.3)$$

可见  $W(J)$  取值较大的条件是:

- $\tilde{J}_1(k), \tilde{J}_2(k)$  有较大重叠,
  - 重叠位置的  $k$  是 on shell (即  $k^2 = -m^2$ ).
- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

### 2.2 from particle to force

- 考虑  $J(x) = \delta^{(D)}(\vec{x} - \vec{x}_1) + \delta^{(D)}(\vec{x} - \vec{x}_2) \implies \tilde{J}_a(k) = 2\pi e^{-i\vec{k} \cdot \vec{x}_a} \delta(k^0)$ , 那么,

$$\begin{aligned} W_{12}(J) + W_{21}(J) &= \delta(0) \int \frac{d^D k}{(2\pi)^{D-1}} \frac{1}{|\vec{k}|^2 + m^2 - i\epsilon} \cos(\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)) \\ &\stackrel{D=3}{=} 2\pi \delta(0) \frac{1}{4\pi r} e^{-mr} \end{aligned} \quad (2.2.1)$$

( $-i\epsilon$  显然可以舍去), 注意到  $\langle 0 | e^{-iHT} | 0 \rangle = e^{-iET}$ , 而时间间隔  $T = \int dx^0 = 2\pi \delta(0)$ , 所以,

$$E = -\frac{W(J)}{T} \stackrel{D=3}{=} -\frac{1}{4\pi r} e^{-mr} \quad (2.2.2)$$

calculation:

计算 (2.2.1) 中的积分, 令  $\vec{x}_1 - \vec{x}_2 = \vec{r}$ ,

$$I_D = \int \frac{d^D k}{(2\pi)^D} \frac{1}{|\vec{k}|^2 + m^2} \overbrace{\cos(\vec{k} \cdot \vec{r})}^{\mapsto e^{i\vec{k} \cdot \vec{r}}}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^D} \int (k \sin \theta_1)^{D-2} d\Omega_{D-2} \int k d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1} \\
&= \frac{S_{D-2}}{(2\pi)^D} \int k^{D-1} \sin^{D-2} \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1}
\end{aligned} \tag{2.2.3}$$

取  $D = 3$ , 那么,

$$\begin{aligned}
I_{D=3} &= \frac{1}{(2\pi)^2} \int k^2 \sin \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ik \cos \theta_1} \\
&= \frac{1}{2\pi^2 r} \int_0^\infty \sin(kr) \frac{k dk}{k^2 + m^2} = \frac{-i}{4\pi^2 r} \int_{-\infty}^\infty e^{ikr} \frac{k dk}{k^2 + m^2} \\
&= \frac{-i}{4\pi^2 r} 2\pi i \underbrace{\text{Res}(f, im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r} e^{-mr}
\end{aligned} \tag{2.2.4}$$



## Chapter 3

# Coulomb and Newton: repulsive and attraction

### 3.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \quad (3.1.1)$$

其中  $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ .

- 做路径积分,

$$Z(J) = \int DA e^{i \int d^d x (\mathcal{L} + J_\mu A^\mu)} = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J_\mu D^{\mu\nu}(x-y) J_\nu(y)} \quad (3.1.2)$$

**calculation:**

massive photon 的作用量为,

$$\begin{aligned} S(A) &= \int d^d x \frac{1}{2} \left( -(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu \right) \\ &= \int d^d x \frac{1}{2} \left( A_\nu \partial^2 A^\nu - A_\nu \partial^\nu \partial_\mu A^\mu - m^2 A_\mu A^\mu \right) + \text{total differential} \\ &= \int d^d x \frac{1}{2} A_\mu \left( -\partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \right) A_\nu + \text{total differential} \\ &= \int \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(-k) \left( k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2) \right) \tilde{A}_\nu(k) + \text{boundary term} \end{aligned} \quad (3.1.3)$$

那么, 需要有,

$$\begin{aligned} (-\partial^\mu \partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2)) D_{\rho\nu}(x-y) &= \delta_\nu^\mu \delta^{(d)}(x-y) \\ \Rightarrow \tilde{D}_{\mu\nu}(k) &= \frac{k_\mu k_\nu / m^2 + \eta_{\mu\nu}}{-k^2 - m^2} \end{aligned} \quad (3.1.4)$$

考虑到积分需要收敛, 作替换  $m^2 \mapsto m^2 - i\epsilon$ , (为什么  $A_\mu$  类空 (?), 只知道  $\tilde{A}_\mu$  类空, 见 subsection 3.1.2, 但路径积分中的  $A$  显然不满足 field equation).

- 因此,

$$W(J) = -\frac{1}{2} \int d^d x d^d y J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) \quad (3.1.5)$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_\mu(-k) \frac{k^\mu k^\nu / m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon} \tilde{J}_\nu(k) \quad (3.1.6)$$

注意到 current conservation, 有  $\partial_\mu J^\mu = 0 \iff k^\mu \tilde{J}_\mu(k) = 0$ , 所以,

$$W(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}^\mu(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_\mu(k) \quad (3.1.7)$$

观察电荷分量, 可见同性相斥, 异性相吸.

### 3.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的  $x, y, z$  方向, 在粒子静止系下, 极化矢量  $(\epsilon^i)_\mu = \delta_\mu^i, i = 1, 2, 3$ , 而  $k_\mu = (-m, 0, 0, 0)$ , 所以,

$$k^\mu (\epsilon^i)_\mu = 0 \quad (3.1.8)$$

– 注意, 一个粒子的极化方向用  $e^i$  (这不是矢量) 表示, 极化矢量为  $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$ .

- 在粒子静止系下, 考虑,

$$\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu} \quad (3.1.9)$$

可见,

$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon} \quad (3.1.10)$$

### 3.1.2 Maxwell Lagrangian

- 根据 (3.1.1) 中的 Lagrangian density, 得到 field equation 如下,

$$\left( -\partial^\mu \partial^\nu + \eta^{\mu\nu} (\partial^2 - m^2) \right) A_\nu \quad (3.1.11)$$

– spin-1 particle 有 3 个自旋自由度, 而  $A_\mu$  有 4 个分量, 所以需要有一个约束方程,

$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0 \quad (3.1.12)$$

实际上在 (3.1.11) 左右两边作用一个  $\partial_\mu$  即可得到这个约束方程.

## 3.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = **linearized** Einstein Lagrangian.
- 受 subsection 3.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足,

$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases} \quad (3.2.1)$$

其中下指标  $\mu, \nu$  对称,  $a = 1, \dots, 5$ , (可以验证  $(\epsilon^a)_{\mu\nu}$  确实有 5 个独立分量).

- 对  $(\epsilon^a)_{\mu\nu}$  的归一化条件可以定义为  $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$ .
- 与 subsection 3.1.1 中提示一样, 粒子的极化方向用  $e^a$  表示.

- 那么,

$$\sum_{a=1}^5 (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma} \quad (3.2.2)$$

#### calculation:

首先用  $k_\mu$  和  $\eta_{\mu\nu}$  构造最一般的关于  $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$  对称的 4 阶张量, (下式中把  $\frac{k_\mu}{m}$  略写作  $k_\mu$ ),

$$\begin{aligned} & A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ & + D \eta_{\mu\nu} \eta_{\rho\sigma} + E(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) \end{aligned} \quad (3.2.3)$$

代入 (3.2.1) 得,

$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4} \quad (3.2.4)$$

因此, 这个 4 阶张量最终确定为,

$$\frac{3}{4}A\left((G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\right) \quad (3.2.5)$$

- 所以,

$$\tilde{D}_{\mu\nu\rho\sigma}(k) = \frac{(G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \quad (3.2.6)$$

- 计算路径积分中的  $W(T)$ ,

$$W(T) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{\mu\nu}(-k) \frac{(G^{\mu\rho}G^{\nu\sigma} + G^{\mu\sigma}G^{\nu\rho}) - \frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tilde{T}_{\rho\sigma}(k) \quad (3.2.7)$$

注意到  $\partial_\mu T^{\mu\nu}(x) = 0 \iff k_\mu \tilde{T}^{\mu\nu}(k) = 0$ , 并考虑到  $T$  是对称张量, 所以,

$$W(T) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{\mu\nu}(-k) \frac{2\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tilde{T}_{\rho\sigma}(k) \quad (3.2.8)$$

考虑能量项, 可见质量互相吸引.

### 3.3 remarks

- 由于 seesaw mechanism (见 appendix C), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为  $\eta^{00} = -1$ .
- 本 chapter 中的计算都是  $m \neq 0$  的粒子, 与真实世界有差异.

# Chapter 4

## Feynman diagrams

### 4.1 a baby problem

- 考虑如下积分,

$$Z(J) = \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4 + Jq} \quad (4.1.1)$$

- Schwinger's way:** 把 integrand 对  $\lambda$  展开, 并将  $q$  用  $\frac{\partial}{\partial J}$  替代, 得到,

$$\begin{aligned} Z(J) &= e^{-\frac{\lambda}{4!} (\frac{\partial}{\partial J})^4} \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 + Jq} \\ &= \sqrt{\frac{2\pi}{m^2}} e^{-\frac{\lambda}{4!} (\frac{\partial}{\partial J})^4} e^{\frac{J^2}{2m^2}} \end{aligned} \quad (4.1.2)$$

后面的计算中忽略  $Z(J=0, \lambda=0)$ .

- 每个 vertex 带有  $-\lambda$ , 每个 line 带有  $\frac{1}{m^2}$ , 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里  $q$  无法区分, 所以不方便算, 先略了),

4 external ends

$= J^4 \frac{1}{2!} \left(-\frac{\lambda}{4!}\right)^2 \frac{1}{6!} \left(\frac{1}{2m^2}\right)^6 \times \text{numerical factor}$

2 vertices      6 lines      不会

#### calculation:

在这里计算  $\lambda J^4$  项,

具体暂时不会算

(4.1.3)

但是直接计算 (4.1.2) 的展开项, 得到的结果与 (4.1.5) 一样.

#### 4.1.1 Wick contraction and Green's functions

- 把积分 (4.1.1) 对  $J$  展开,

$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} J^n \underbrace{\int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4} q^n}_{=Z(0,0)G^{(n)}} \quad (4.1.4)$$

其中 Green's function  $G^{(n)}$  对  $\lambda$  展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

### calculation:

计算  $\lambda J^4$  项, 它来自  $G^{(4)}$  对  $\lambda$  展开的一阶项,

$$\begin{aligned}
-\frac{\lambda}{4!} \int dq e^{-\frac{1}{2}m^2 q^2} q^8 &= -\frac{\lambda}{4!} \langle q^8 \rangle \\
&= -\frac{\lambda}{4!} \sum_{\text{Wick}} \left( \frac{1}{m^2} \right)^4 \\
&= -\frac{\lambda}{4!} \frac{7 \times 5 \times 3 \times 1}{m^8}
\end{aligned} \tag{4.1.5}$$

所以  $\lambda J^4$  项等于  $\frac{105}{(4!)^2} \frac{-\lambda J^4}{m^8}$ .

### 4.1.2 connected vs. disconnected

- 考虑,

$$Z(J, \lambda) = Z(J=0, \lambda) e^{W(J, \lambda)} \tag{4.1.6}$$

其中  $Z(J=0, \lambda)$  由 diagrams with no external source  $J$  组成, 而  $W(J, \lambda)$  则由 connected diagrams 组成(?).

- 我们希望计算的是  $W$ , 而不是  $Z$  (?).

## 4.2 a child problem

- 考虑如下积分,

$$Z(J) = \int dq_1 \cdots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q - \frac{\lambda}{4!} q^4 + J^T \cdot q} \tag{4.2.1}$$

其中  $q^4 = \sum_i q_i^4$ .

- Schwinger's way:** 对  $\lambda$  展开并把  $q$  替换为  $\frac{\partial}{\partial J}$ , 得到,

$$Z(J) = \sqrt{\frac{(2\pi)^N}{\det A}} e^{-\frac{\lambda}{4!} \left(\frac{\partial}{\partial J}\right)^4} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \tag{4.2.2}$$

其中  $\left(\frac{\partial}{\partial J}\right)^4 = \sum_i \left(\frac{\partial}{\partial J_i}\right)^4$ .

### 4.2.1 $n$ -point Green's function

- Wick's way:** 对  $J$  展开获得带 Green's function 的表达式,

$$\begin{aligned}
Z(J) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N J_{i_1} \cdots J_{i_n} \underbrace{\int dq_1 \cdots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q - \frac{\lambda}{4!} q^4} q_{i_1} \cdots q_{i_n}}_{=Z(0,0)G_{i_1 \cdots i_n}^{(n)}}
\end{aligned} \tag{4.2.3}$$

其中  $G_{i_1 \cdots i_n}^{(n)}$  称为  $n$ -point Green's function.

### Taylor expansion:

多元函数的 Taylor 展开如下,

$$\begin{aligned}
f(x_1, \cdots, x_N) &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_N^{n_N}}{n_N!} \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_N}}{\partial x_N^{n_N}} f(x=0) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N x_{i_1} \cdots x_{i_n} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}} f(x=0)
\end{aligned} \tag{4.2.4}$$

这两种求和方法,  $x_1^{n_1} \cdots x_N^{n_N}$  项的 numerical factor 都等于,

$$\frac{1}{n!} \times \frac{n!}{n_1! \cdots n_N!} = \frac{1}{n_1! \cdots n_N!} \tag{4.2.5}$$

其中  $n = n_1 + \dots + n_N$ .

- 在  $\lambda = 0$  时, 2-point Green's function 就是 propagator,

$$\begin{aligned} G_{ij}^{(2)}(\lambda = 0) &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} q_i q_j \\ &= \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \Big|_{J=0} = A_{ij}^{-1} \end{aligned} \quad (4.2.6)$$

- 来计算 2, 3, 4-point Green's functions,

$$\begin{cases} G_{ij}^{(2)} = A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \\ G_{ijk}^{(3)} = 0 \\ G_{ijkl}^{(4)} = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \\ \quad - \frac{\lambda}{4!} \sum_m (A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} A_{kl}^{-1} + \dots + 4! A_{im}^{-1} A_{jm}^{-1} A_{km}^{-1} A_{lm}^{-1}) + O(\lambda^2) \end{cases} \quad (4.2.7)$$

calculation:

2-point Green's function 计算如下,

$$\begin{aligned} G_{ij}^{(2)} &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j \\ &= A_{ij}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j \rangle + O(\lambda^2) \\ &= A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \end{aligned} \quad (4.2.8)$$

3-point Green's function 计算如下,

$$G_{ijk}^{(3)} = \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j q_k = 0 \quad (4.2.9)$$

4-point Green's function 计算如下,

$$\begin{aligned} G_{ijkl}^{(4)} &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j q_k q_l \\ &= A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j q_k q_l \rangle + O(\lambda^2) \end{aligned} \quad (4.2.10)$$

### 4.3 perturbative field theory

- 做如下替换即可,

$$\begin{cases} A \mapsto -i(\partial^2 - m^2) \\ J \mapsto iJ \end{cases} \quad (4.3.1)$$

- Schwinger's way:**  $\phi^4$  theory 的路径积分,

$$Z(J) = \int D\phi e^{i \int d^d x \left( \frac{1}{2} \phi(\partial^2 - m^2) \phi - \frac{\lambda}{4!} \phi^4 + J(x) \phi(x) \right)} \quad (4.3.2)$$

$$= Z(0,0) e^{-i \frac{\lambda}{4!} \int d^d z \left( \frac{\delta}{i \delta J(z)} \right)^4} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (4.3.3)$$

其中  $D(x-y)$  是自由场的 propagator, 见 (1.2.1).

- **Wick's way:** 同样, 对  $J$  展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \cdots d^d x_n J(x_1) \cdots J(x_n) G^{(n)}(x_1, \cdots, x_n) \quad (4.3.4)$$

其中,

$$G^{(n)}(x_1, \cdots, x_n) = \frac{1}{Z(0,0)} \int D\phi e^{i \int d^d x (\frac{1}{2} \phi (\partial^2 - m^2) \phi - \frac{\lambda}{4!} \phi^4)} \phi(x_1) \cdots \phi(x_n) \quad (4.3.5)$$

有时  $Z(J)$  被称为 generating functional, 因为它能生成 Green's functions.

#### 4.3.1 collision between particles

- 通过 Wick's way, 考虑  $J(x_1)J(x_2)J(x_3)J(x_4)$  项, 实际上就是要计算  $G^{(4)}(x_1, x_2, x_3, x_4)$ , 它的 0 阶项为,

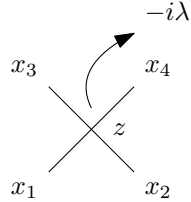
$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4, \lambda = 0) &= \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta J(x_2)} \frac{\delta}{i\delta J(x_3)} \frac{\delta}{i\delta J(x_4)} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \\ &= -(D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}) \end{aligned} \quad (4.3.6)$$

其中  $D_{ij}$  是  $D(x_i - x_j)$  的简写, 可见, 传播子实际上是  $(-i)^3 D = iD$ .

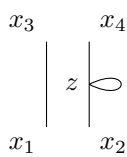
- $G_{1234}^{(4)}$  的 1 阶项为,

$$\begin{aligned} \text{1st order term} &= -\frac{i\lambda}{4!} \int d^d z \langle \phi_1 \cdots \phi_4 \phi^4(z) \rangle \\ &= -\frac{i\lambda}{4!} \int d^d z \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_4} \left( \frac{\delta}{i\delta J(z)} \right)^4 e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \\ &= -\frac{i\lambda}{4!} \int d^d z \left( 4! D_{1z} D_{2z} D_{3z} D_{4z} \right. \\ &\quad \left. + 4 \times 3 D_{12} D_{3z} D_{4z} D_{zz} + \cdots + 3 D_{12} D_{34} D_{zz} D_{zz} + \cdots \right) \end{aligned} \quad (4.3.7)$$

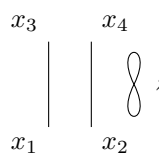
其中各项分别对应如下 Feynman diagrams,



$$-i\lambda \int d^d z D_{1z} D_{2z} D_{3z} D_{4z}$$



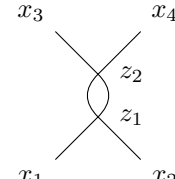
$$\frac{-i\lambda}{2!} \int d^d z D_{13} D_{2z} D_{4z} D_{zz}$$



$$\frac{-i\lambda}{8} \int d^d z D_{13} D_{24} D_{zz} D_{zz}$$

其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,



$$(4 \times 3)^2 \times 2 \times \left( \frac{-i\lambda}{4!} \right)^2 \int d^d z_1 d^d z_2 D_{1z_1} D_{2z_1} D_{3z_2} D_{4z_2} D_{z_1 z_2} D_{z_1 z_2}$$

#### 4.3.2 in momentum space

- 本 subsection 将 (4.3.5) 转换到 momentum space, 注意到  $\tilde{J}(k)$  和  $\tilde{J}(-k)$  并不独立, 所以  $\frac{\partial}{\partial i\tilde{J}}$  不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\tilde{G}^{(n)}(k_1, \cdots, k_n) = \int d^d x_1 \cdots d^d x_n e^{-i(k_1 \cdot x_1 + \cdots)} G^{(n)}(x_1, \cdots, x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \cdots d^d x_n e^{-i(k_1 \cdot x_1 + \cdots)} \langle \left( -\frac{i\lambda}{4!} \int d^d z \phi_z^4 \right)^n \phi_1 \cdots \phi_n \rangle \quad (4.3.8)$$

– propagator 的 Fourier transformation 是,

$$\tilde{D}_{pq} = \int d^d x d^d y e^{-i(p \cdot x + q \cdot y)} D(x - y) = \frac{(2\pi)^d \delta^{(d)}(p + q)}{-p^2 - m^2 + i\epsilon} \quad (4.3.9)$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$  的 1 阶项为,

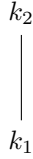
$$\text{1st order term} = -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \langle \phi_z^4 \phi_1 \cdots \phi_4 \rangle \quad (4.3.10)$$

考虑第 1 项,

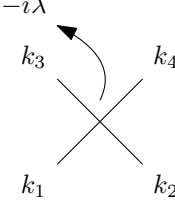
$$\begin{aligned} & -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z 4! D_{1z} \cdots D_{4z} \\ &= -i\lambda \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - z) + \cdots)} \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\ &= -i\lambda \underbrace{\int d^d z e^{-iz \cdot (k_1 + \cdots + k_4)}}_{=(2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4)} \prod_{i=1}^4 \frac{1}{-k_i^2 - m^2 + i\epsilon} \end{aligned} \quad (4.3.11)$$

– 出射粒子不一定 on-shell (?).

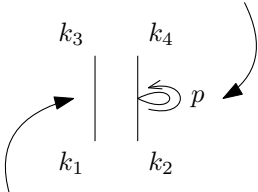
- 得到这些 Feynman diagrams,



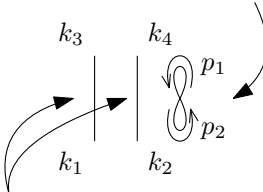
$$(2\pi)^d \delta^{(d)}(k_1 + k_2) \frac{i}{-k_1^2 - m^2 + i\epsilon}$$



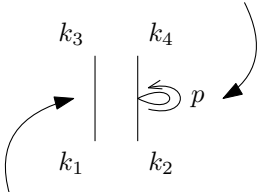
$$-i\lambda (2\pi)^4 \delta^{(d)}(k_1 + \cdots + k_4) \prod_{i=1}^4 \frac{i}{-k_i^2 - m^2 + i\epsilon}$$



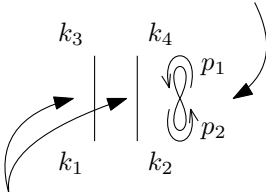
$$-\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=2,4} \frac{i}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon}$$



$$-\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \prod_{i=1,2} \int \frac{d^d p_i}{(2\pi)^d} \frac{i}{-p_i^2 - m^2 + i\epsilon}$$



$$(2\pi)^d \delta^{(d)}(k_1 + k_3) \frac{i}{-k_1^2 - m^2 + i\epsilon}$$



$$(2\pi)^d \delta^{(d)}(k_1 + k_3) (2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{i}{-k_i^2 - m^2 + i\epsilon}$$

**calculation:**

第 3 幅图的计算如下,

$$\begin{aligned} & -\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z D_{13} D_{2z} D_{4z} D_{zz} \\ &= -\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - z) + p_4 \cdot (x_4 - z) + p_4 \cdot 0)} \end{aligned}$$

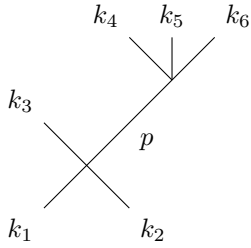


$$\begin{aligned}
& \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{2!} \int d^d z e^{-iz \cdot (p_2 + p_4)} \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_4 - k_4) \\
& \quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2,4} \frac{1}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{-p^2 - m^2 + i\epsilon} \quad (4.3.12)
\end{aligned}$$

第 4 幅图的计算如下,

$$\begin{aligned}
& -\frac{i\lambda}{8} \int d^d x_1 \dots d^d x_4 e^{-i(k_1 \cdot x_1 + \dots)} \int d^d z D_{13} D_{24} D_{zz} D_{zz} \\
&= -\frac{i\lambda}{8} \int d^d x_1 \dots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \dots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - x_4) + p_3 \cdot 0 + p_4 \cdot 0)} \\
& \quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{8} \int d^d z \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_2 + k_4) \\
& \quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{1}{-k_i^2 - m^2 + i\epsilon} \\
& \quad \prod_{i=1,2} \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \quad (4.3.13)
\end{aligned}$$

- 再举一个例子 (略去了  $\prod_{i=1}^6 \frac{i}{-k_i^2 - m^2 + i\epsilon}$ ),



$$\begin{aligned}
&= (4!)^2 \times \left(-\frac{i\lambda}{4!}\right)^2 (2\pi)^{2d} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \delta^{(d)}(k_1 + k_2 + k_3 + p) \delta^{(d)}(k_4 + k_5 + k_6 - p) \\
&= (-i\lambda)^2 (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6) \frac{i}{-(k_1 + k_2 + k_3)^2 - m^2 + i\epsilon}
\end{aligned}$$

## 4.4 loops and a first look at divergence

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# Appendices

## Appendix A

# Dirac delta function & Fourier transformation

### A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx \delta(x) e^{-ikx} \quad (\text{A.1.1})$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \quad (\text{A.1.2})$$

- 第二个常用的公式 ([Sokhotski-Plemelj theorem](#)),

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{x + i\epsilon} = \mathcal{P} \frac{1}{x} - i\pi \delta(x) \quad (\text{A.1.3})$$

其中  $\mathcal{P}$  表示复函数的主值 (principal value).

**proof:**

考虑,

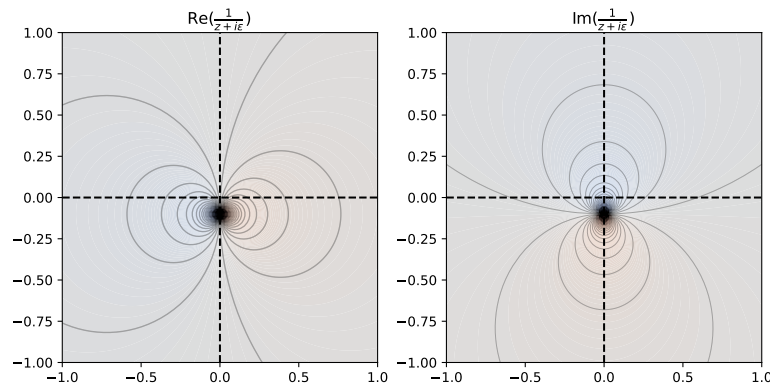
$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \quad (\text{A.1.4})$$

且注意到,

$$\int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \text{Res}(f, i\epsilon) = \pi \quad (\text{A.1.5})$$

所以...

取  $\epsilon = 0.1$  时, 复变函数的实部, 虚部分别如下,



- 另外,  $\delta(x-a)\delta(x-b) = \delta(b-a)\delta(x-a)$ .

## A.2 Fourier transformation

- $d$ -dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^d x e^{-ik \cdot x} \phi(x) \end{cases} \quad (\text{A.2.1})$$

- 因此,

$$\partial_\mu \phi(x) \mapsto ik_\mu \tilde{\phi}(k) \quad (\text{A.2.2})$$

- 对于实函数, Fourier transformation 是正交变换, 其 Jacobi determinant 为,

$$\left| \frac{\partial \phi(x) \cdots}{\partial \text{Re} \tilde{\phi}(k) \cdots \partial \text{Im} \tilde{\phi}(k) \cdots} \right| = \left( \frac{2}{V} \right)^{(2N+1)^d} \det A = \left( \frac{2(2N)^d}{V^2} \right)^{\frac{(2N+1)^d}{2}} \quad (\text{A.2.3})$$

**proof:**

position space 和 momentum space 的格点分别为,

$$\begin{cases} x_i^\mu = i^\mu \epsilon \in \{0, \pm\epsilon, \dots, \frac{L}{2}\} \\ k_n^\mu = n^\mu \frac{2\pi}{L} \in \{0, \pm \frac{2\pi}{L}, \dots, \frac{\pi}{\epsilon}\} \end{cases} \iff i^\mu, n^\mu \in \{0, \pm 1, \dots, N\} \quad (\text{A.2.4})$$

$x^\mu, k^\mu$  分别有  $2N+1$  个取值, 其中  $N\epsilon = \frac{L}{2}$ , 时空总体积为  $V = L^d$ , momentum space 的总体积为  $\tilde{V} = \frac{(4\pi N)^d}{V}$ .

将 (A.2.1) 写成格点求和的形式,

$$\begin{cases} \phi(x_i) = \frac{1}{(2\pi)^d} \left( \frac{2\pi}{L} \right)^d \sum_n e^{ik_n \cdot x_i} \tilde{\phi}(k_n) \\ \quad = \frac{2}{V} \sum_{n^0 > 0} \left( \cos(k_n \cdot x_i) \text{Re} \tilde{\phi}(k_n) - \sin(k_n \cdot x_i) \text{Im} \tilde{\phi}(k_n) \right) \\ \tilde{\phi}(k_n) = \epsilon^d \sum_i e^{-ik_n \cdot x_i} \phi(x_i) \\ \quad = \frac{V}{(2N)^d} \sum_i \left( \cos(k_n \cdot x_i) - i \sin(k_n \cdot x_i) \right) \phi(x_i) \end{cases} \quad (\text{A.2.5})$$

**proof:**

$\phi(x_i)$  的变换需要做一些说明. 注意到  $\tilde{\phi}$  的分量的数量是  $\phi$  的两倍 (考虑到实部与虚部), 但在  $\phi \in \mathbb{R}^{(2N+1)^d}$  时,

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k) \quad (\text{A.2.6})$$

可见  $\tilde{\phi}$  的分量并不独立, 取  $k^0 > 0$  的部分为独立分量, 那么...

将 (A.2.5) 写成矩阵的形式,

$$\begin{cases} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} = \frac{2}{V} \overbrace{\begin{pmatrix} \cos k_0 \cdot x_0 & \cdots & \cos k_{\max} \cdot x_0 & -\sin k_0 \cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A} \begin{pmatrix} \text{Re} \tilde{\phi}(k_0) \\ \vdots \\ \text{Im} \tilde{\phi}(k_0) \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \text{Re} \tilde{\phi}(k_0) \\ \vdots \\ \text{Im} \tilde{\phi}(k_0) \\ \vdots \end{pmatrix} = \frac{V}{(2N)^d} \begin{pmatrix} \cos k_0 \cdot x_0 & \cdots & \cos k_0 \cdot x_{\max} \\ \vdots & \ddots & \vdots \\ -\sin k_0 \cdot x_0 & \cdots & -\sin k_0 \cdot x_{\max} \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} \end{cases} \quad (\text{A.2.7})$$

观察可见  $\tilde{\phi}$  的变换中的矩阵是  $A^T$ , 所以,

$$\frac{2}{V} \frac{V}{(2N)^d} A A^T = I \implies \det A = \left( \frac{(2N)^d}{2} \right)^{\frac{(2N+1)^d}{2}} \quad (\text{A.2.8})$$

因此...

# Appendix B

## Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \quad (\text{B.0.1})$$

$$\langle x^{2n} \rangle = \frac{\int dx e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n} (2n-1)!! \quad (\text{B.0.2})$$

其中  $(2n-1)!! = 1 \cdot 3 \cdots (2n-3)(2n-1)$ .

- 一个重要的变体如下,

$$\int dx e^{-\frac{a}{2}x^2 + Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \quad (\text{B.0.3})$$

另外, 将  $a, J$  分别替换为  $-ia, iJ$  也是重要的变体.

### B.1 $N$ -dim. generalization

- 考虑如下积分,

$$Z(A, J) = \int dx_1 \cdots dx_N e^{-\frac{1}{2}x^T \cdot A \cdot x + J^T \cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \quad (\text{B.1.1})$$

其中  $x, J$  是  $N$ -dim. 列向量,  $A$  是  $N \times N$  实对称矩阵.

#### calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1} \cdot D \cdot O \quad (\text{B.1.2})$$

其中  $D$  是一个 diagonal matrix. 令  $y = O \cdot x$ , 那么,

$$\begin{aligned} Z(A, J) &= \int dy_1 \cdots dy_N e^{-\frac{1}{2}y^T \cdot D \cdot y + (OJ)^T \cdot y} \\ &= \prod_{i=1}^N \sqrt{\frac{2\pi}{D_{ii}}} e^{\frac{1}{2D_{ii}}(OJ)_i^2} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \end{aligned} \quad (\text{B.1.3})$$

其中, 注意到了  $\frac{1}{D_{ii}} = (O \cdot A^{-1} \cdot O^{-1})_{ii}$  以及  $\text{tr } D = \det A$ .

- 一个重要的变体是  $A \mapsto -iA, J \mapsto iJ$ .
- 考虑 (B.0.2) 的变体, (注意  $A$  是对称的),

$$\langle x_i x_j \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} Z(A, J) \Big|_{J=0} = A_{ij}^{-1} \quad (\text{B.1.4})$$

$$\langle x_i x_j \cdots x_k x_l \rangle = \sum_{\text{Wick}} A_{i'j'}^{-1} \cdots A_{k'l'}^{-1} \quad (\text{B.1.5})$$

其中 (B.1.5) 中有偶数个  $x$ , 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A, J) \Big|_{J=0} = \cdots \quad (\text{B.1.6})$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \quad (\text{B.1.7})$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle \overbrace{x_i x_j x_k x_l} \rangle = A_{ik}^{-1} A_{jl}^{-1} \quad (\text{B.1.8})$$

# Appendix C

## perturbation theory in QM

- this chapter is based on MIT OpenCourseWare [Quantum Physics III Chapter 1: Perturbation Theory](#).

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda \delta H \quad (\text{C.0.1})$$

其中  $\lambda \in [0, 1]$ .

- 考虑  $H^{(0)}$  的本征态为,

$$H^{(0)} |k^{(0)}\rangle = E_k^{(0)} |k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)} | l^{(0)} \rangle = \delta_{kl} \\ E_0^{(0)} \leq E_1^{(0)} \leq E_2^{(0)} \leq \dots \end{cases} \quad (\text{C.0.2})$$

### C.1 non-degenerate perturbation theory

- 考虑 non-degenerate 能级  $k$ , 有  $\dots \leq E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \leq \dots$ , 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \dots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \dots \end{cases} \quad (\text{C.1.1})$$

– 注意, 我们可以选取修正项满足,

$$\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \dots \quad (\text{C.1.2})$$

**proof:**

假设我们求解得到的修正项不满足  $\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \dots$ , 考虑,

$$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n |k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)} | k^{(n)} \rangle' = 0 \quad (\text{C.1.3})$$

那么, (注意到态矢量可以乘一个常数,  $\frac{1}{1-a_1\lambda-a_2\lambda^2-\dots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \dots$ ),

$$\begin{aligned} |k\rangle_\lambda &= (1 - a_1\lambda - a_2\lambda^2 - \dots) |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \dots \\ |k\rangle_\lambda' &= |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \dots} (\lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \dots) \\ &= |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 (a_1 |k^{(1)}\rangle' + |k^{(2)}\rangle') + \dots \end{aligned} \quad (\text{C.1.4})$$

可见修正项都与  $|k^{(0)}\rangle$  正交.

- 注意, 不能要求  ${}_\lambda \langle k | k \rangle_\lambda = 1$ , 否则  $|k^{(n)}\rangle$  将与  $\lambda$  相关 (包括  $|k^{(0)}\rangle$ ),

$$\begin{aligned} {}_\lambda \langle k | k \rangle_\lambda &= \langle k^{(0)} | k^{(0)} \rangle \\ &\quad + \lambda (\langle k^{(1)} | k^{(0)} \rangle + \langle k^{(0)} | k^{(1)} \rangle) \\ &\quad + \lambda^2 (\langle k^{(2)} | k^{(0)} \rangle + \langle k^{(1)} | k^{(1)} \rangle + \langle k^{(0)} | k^{(2)} \rangle) \end{aligned}$$



$$\begin{aligned} & \vdots \\ & + \lambda^n (\langle k^{(n)} | k^{(0)} \rangle + \langle k^{(n-1)} | k^{(1)} \rangle + \dots + \langle k^{(0)} | k^{(n)} \rangle) \end{aligned} \quad (\text{C.1.5})$$

- 将 (C.1.1) 代入 Schrodinger's eq., 得到,

---


$$\begin{array}{ll} \lambda^0 & (H^{(0)} - E_k^{(0)}) |k^{(0)}\rangle = 0 \\ \lambda^1 & (H^{(0)} - E_k^{(0)}) |k^{(1)}\rangle = (E_k^{(1)} - \delta H) |k^{(0)}\rangle \\ \lambda^2 & (H^{(0)} - E_k^{(0)}) |k^{(2)}\rangle = (E_k^{(1)} - \delta H) |k^{(1)}\rangle + E_k^{(2)} |k^{(0)}\rangle \\ \vdots & \vdots \\ \lambda^n & (H^{(0)} - E_k^{(0)}) |k^{(n)}\rangle = (E_k^{(1)} - \delta H) |k^{(n-1)}\rangle + E_k^{(2)} |k^{(n-2)}\rangle + \dots + E_k^{(n)} |k^{(0)}\rangle \end{array}$$


---

**calculation:**

Schrodinger's eq. 为,

$$(H^{(0)} + \lambda \delta H - E_k(\lambda)) |k\rangle_\lambda = 0 \quad (\text{C.1.6})$$

展开为,

$$\left( (H^{(0)} - E_k^{(0)}) + \lambda(\delta H - E_k^{(1)}) - \lambda^2 E_k^{(2)} - \dots \right) (|k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \dots) = 0 \quad (\text{C.1.7})$$

- 现在来计算  $\langle l^{(0)} | k^{(n)} \rangle$ , 有,

$$\left\{ \begin{array}{l} (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(1)} \rangle = E_k^{(1)} \delta_{lk} - \delta H_{lk} \\ (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(2)} \rangle = E_k^{(1)} \langle l^{(0)} | k^{(1)} \rangle - \langle l^{(0)} | \delta H | k^{(1)} \rangle + E_k^{(2)} \delta_{lk} \\ \vdots \\ (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(n)} \rangle = E_k^{(1)} \langle l^{(0)} | k^{(n-1)} \rangle - \langle l^{(0)} | \delta H | k^{(n-1)} \rangle \\ \quad + E_k^{(2)} \langle l^{(0)} | k^{(n-2)} \rangle + \dots + E_k^{(n)} \delta_{lk} \end{array} \right. \quad (\text{C.1.8})$$

其中  $\delta H_{lk} = \langle l^{(0)} | \delta H | k^{(0)} \rangle$ , 对于满足 (C.1.2) 的解, 有,

$$E_k^{(n)} = \langle k^{(0)} | \delta H | k^{(n-1)} \rangle, n = 1, 2, \dots \quad (\text{C.1.9})$$

并且,

$$|k^{(1)}\rangle = - \sum_{l \neq k} \frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} |l^{(0)}\rangle \implies E_k^{(2)} = - \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} \quad (\text{C.1.10})$$

**calculation:**

将 (C.1.10) 代入 (C.1.8), 得到 ( $l \neq k$ ),

$$(E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(2)} \rangle = -E_k^{(1)} \frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} + \sum_{m \neq k} \frac{\delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \quad (\text{C.1.11})$$

所以,

$$\left\{ \begin{array}{l} |k^{(2)}\rangle = \sum_{l \neq k} \left( - \frac{\delta H_{00} \delta H_{lk}}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m \neq k} \frac{\delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \right) |l^{(0)}\rangle \\ E_k^{(3)} = \sum_{l \neq k} \left( - \frac{\delta H_{00} |\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m \neq k} \frac{\delta H_{kl} \delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \right) \end{array} \right. \quad (\text{C.1.12})$$

计算归一化系数,

$${}_l \langle k | k \rangle_\lambda = 1 + \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + O(\lambda^3) \quad (\text{C.1.13})$$

### C.1.1 level repulsion or the seesaw mechanism

- 能量的展开式为,

$$E_k(\lambda) = E_k^{(0)} + \lambda \delta H_{kk} - \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3) \quad (\text{C.1.14})$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

### C.1.2 validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即,

$$|\lambda V| < \frac{1}{2} \Delta E^{(0)} \quad (\text{C.1.15})$$

因此, 对于能级简并的情况,  $\Delta E^{(0)} = 0$ , 情况会更复杂.

#### calculation:

对于两能级系统,

$$H(\lambda) = H^{(0)} + \lambda \hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix} \quad (\text{C.1.16})$$

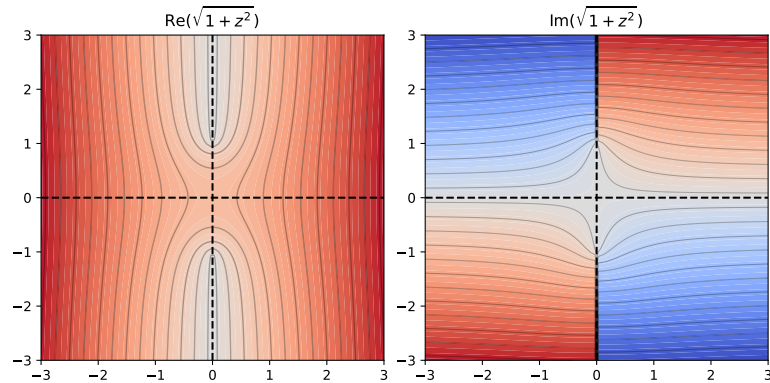
$H(\lambda)$  的本征值可以直接计算,

$$E_{\pm}(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)}) \sqrt{1 + \left( \frac{\lambda |V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})} \right)^2} \quad (\text{C.1.17})$$

考虑  $\sqrt{1+z^2}$  的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1} \frac{(2n-3)!!}{2^n n!} z^{2n} + \cdots \quad (\text{C.1.18})$$

注意到  $\sqrt{1+z^2}$  在  $z = \pm i$  有 branch cut, 因此  $z = 0$  附近的 Taylor expansion 只有在  $|z| < 1$  内才收敛.



## C.2 degenerate perturbation theory

- 暂时先跳过.