Quantum Field Theory

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Contents

convention, notation, and units

- 笔记中的**度规号差**约定为 $(+,-,-,-)$.
- 使用 natural units, 此时 $\hbar, c, k_B = 1$, 因此 $1\,\mathrm{m} = \frac{1}{1.97\times10^{-16}\,\mathrm{GeV}}$ 且:

names/dimensions	expressions/values
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616\times10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391\times10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176\times10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417\times10^{32}\,\mathrm{K}$

- 时空维度用 $d = D+1$ 表示.

Part I

Field Theory

Chapter 1

cross sections and decay rates

1.1 cross sections

- cross section 定义为
$$\sigma = \frac{1}{\Phi}\frac{P}{\Delta t}, \tag{1.1.1}$$
其中 $\Phi := nv = \frac{|\vec{v}_1-\vec{v}_2|}{V}$ 是 incoming flux, 是入射粒子数密度乘粒子速度, P 是发生散射的概率.
- 实验上定义 luminosity 为
$$L\Delta t = \frac{dN}{d\sigma}, \tag{1.1.2}$$
其中 dN 是 $d\Omega$ 内发生散射的粒子数.
- 用 S-matrix elements 来表示 cross section, 有
$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}d\Pi, \tag{1.1.3}$$
其中 $d\Pi$ 是末态动量体元
$$d\Pi = \prod_i \delta^{(3)}(\vec{p}=0)d^3p_{f,i} = \prod_i \frac{V}{(2\pi)^3}d^3p_{f,i}, \tag{1.1.4}$$
这保证了无相互作用时 $\int dP = 1$.
- 对于初末态有
$$\begin{cases}\langle i|i\rangle = \langle p_1,p_2|p_1,p_2\rangle = (2\pi)^3 2\omega_{p_1}\delta^{(3)}(0)(2\pi)^3 2\omega_{p_2}\delta^{(3)}(0) = (2\omega_{p_1}V)(2\omega_{p_2}V) \\ \langle f|f\rangle = \prod_i (2\omega_{p_{f,i}}V)\end{cases}. \tag{1.1.5}$$

- 一般将 S-matrix 写为
$$S = I + i\mathcal{T}, \quad \mathcal{T} = (2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\mathcal{M}, \tag{1.1.6}$$
其中 $\mathcal{T}$ 称为 transfer matrix, 而 $\mathcal{M}$ 才是 S-matrix 的 non-trivial part. 有
$$\langle f|S-I|i\rangle = i(2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\langle f|\mathcal{M}|i\rangle. \tag{1.1.7}$$
- 对于 $|f\rangle \neq |i\rangle$ 的情况, 有
$$|\langle f|S|i\rangle|^2 = (2\pi)^4 TV\delta^{(4)}(\textstyle\sum_{i,f} p)|\langle f|\mathcal{M}|i\rangle|^2, \tag{1.1.8}$$
那么
$$dP = \frac{T}{V}\frac{1}{(2\omega_{p_1})(2\omega_{p_2})}|\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}, \tag{1.1.9}$$
其中 LIPS 表示 Lorentz-invariant phase space,
$$d\Pi_{\text{LIPS}} = (2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\displaystyle\prod_i \frac{d^3p_{f,i}}{(2\pi)^3 2\omega_{p_{f,i}}}. \tag{1.1.10}$$
- 最终有 (将 (1.1.1) 中的 Δt 替换为 T)
$$d\sigma = \frac{1}{|\vec{v}_1-\vec{v}_2|(2\omega_{p_1})(2\omega_{p_2})}|\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}. \tag{1.1.11}$$

1.2 decay rates

- decay rate, Γ, 是粒子单位时间发生衰变的概率,

$$d\Gamma = \frac{dP}{T}. \tag{1.2.1}$$

- 因为 $|f\rangle \neq |i\rangle$, 有

$$d\Gamma = \frac{1}{2\omega_p} |\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\mathrm{LIPS}}. \tag{1.2.2}$$

Chapter 2

the S-matrix and time-ordered products

2.1 the LSZ reduction formula

- S-matrix element 为

$$\begin{cases} |i\rangle = \sqrt{(2\pi)^3 2\omega_{p_1}}\sqrt{(2\pi)^3 2\omega_{p_2}} a^\dagger_{\vec{p}_1}(-\infty) a^\dagger_{\vec{p}_2}(-\infty)\,|\Omega\rangle \\ |f\rangle = \sqrt{(2\pi)^3 2\omega_{p_3}}\cdots\sqrt{(2\pi)^3 2\omega_{p_n}} a^\dagger_{\vec{p}_3}(+\infty)\cdots a^\dagger_{\vec{p}_n}(+\infty)\,|\Omega\rangle \\ \langle f|S|i\rangle = (2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega| a_{\vec{p}_3}(+\infty)\cdots a_{\vec{p}_n}(+\infty) a^\dagger_{\vec{p}_1}(-\infty) a^\dagger_{\vec{p}_2}(-\infty)|\Omega\rangle \end{cases}. \tag{2.1.1}$$

remark:

注意到

$$\begin{cases} a_{\vec{p}}(t) = U^\dagger(t) a_{\vec{p}}(0) U(t) \\ |\vec{p}(t)\rangle = U(t)\,|\vec{p}(0)\rangle \end{cases} \Longrightarrow |\vec{p}(t)\rangle = a^\dagger_{\vec{p}}(-t)\,|\Omega\rangle\,, \tag{2.1.2}$$

那么, 对于初末态, 有

$$\begin{aligned} &|i(-\infty)\rangle = \sqrt{(2\pi)^3 2\omega_{p_1}}\sqrt{(2\pi)^3 2\omega_{p_2}} a^\dagger_{\vec{p}_1}(0) a^\dagger_{\vec{p}_2}(0)\,|\Omega\rangle \\ \Longrightarrow &|i(0)\rangle = \cdots\,, \end{aligned} \tag{2.1.3}$$

可见 (2.1.1) 中的 $|i\rangle\,, |f\rangle$ 都是 $t=0$ 即 Heisenberg picture 中的形式.

- LSZ 需要用到以下公式,

$$i\int d^4x\, e^{ip\cdot x}(\partial^2 + m^2)\phi(x) = \sqrt{(2\pi)^3 2\omega_p}(e^{i\omega_p t} a_{\vec{p}}(t))\Big|_{-\infty}^{\infty}. \tag{2.1.4}$$

proof:

注意到

$$\begin{aligned} &\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}(t)e^{-i\vec{p}\cdot\vec{x}}) \\ \Longrightarrow &(\partial^2 + m^2)\phi(x) = (\partial_t^2 + \omega_p^2)\phi(x), \end{aligned} \tag{2.1.5}$$

代入, 得

$$\begin{aligned} \text{LHS} &= i\int d^4x\, e^{ip\cdot x}(\partial_t^2 + \omega_p^2)\phi(x) \\ &= i\int dt\, e^{i\omega_p t}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}(\partial_t^2 + \omega_p^2)\int \frac{d^3q}{(2\pi)^{3/2}\sqrt{2\omega_q}}(a_{\vec{q}}(t)e^{i\vec{q}\cdot\vec{x}} + a^\dagger_{\vec{q}}(t)e^{-i\vec{q}\cdot\vec{x}}) \end{aligned}$$

$$= i\frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}}\int dt\, e^{i\omega_p t}(\partial_t^2+\omega_p^2)(a_{\vec{p}}(t)+a^\dagger_{-\vec{p}}(t)), \tag{2.1.6}$$

注意到

$$e^{i\omega_p t}(\partial_t^2+\omega_p^2)O(t) = \partial_t(e^{i\omega_p t}(\partial_t - i\omega_p)O(t)), \tag{2.1.7}$$

因此

$$\text{LHS} = i\frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}}(e^{i\omega_p t}(\partial_t - i\omega_p)(a_{\vec{p}}(t)+a^\dagger_{-\vec{p}}(t)))\Big|_{-\infty}^{\infty}, \tag{2.1.8}$$

注意到 $t=\pm\infty$ 时, fields are free, 所以

$$\begin{cases} \lim\limits_{t\to\pm\infty} a_{\vec{p}}(t) = e^{-i\omega_p t}a_{\vec{p}} \\ \lim\limits_{t\to\pm\infty} a^\dagger_{-\vec{p}}(t) = e^{i\omega_p t}a^\dagger_{-\vec{p}} \end{cases} \Longrightarrow \begin{cases} \lim\limits_{t\to\pm\infty} (\partial_t - i\omega_p)a_{\vec{p}}(t) = -2i\omega_p a_{\vec{p}}(t) \\ \lim\limits_{t\to\pm\infty} (\partial_t - i\omega_p)a^\dagger_{-\vec{p}}(t) = 0 \end{cases}, \tag{2.1.9}$$

代入得到

$$\text{LHS} = \sqrt{(2\pi)^3 2\omega_p}(e^{i\omega_p t}a_{\vec{p}}(t))\Big|_{-\infty}^{\infty}. \tag{2.1.10}$$

- 那么 (up to an infinite phase)

$$\begin{aligned} &\langle f|S|i\rangle \\ =&(2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega|a_{\vec{p}_3}(+\infty)\cdots a_{\vec{p}_n}(+\infty)a^\dagger_{\vec{p}_1}(-\infty)a^\dagger_{\vec{p}_2}(-\infty)|\Omega\rangle \\ =&(2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega|T((a_{\vec{p}_3}(+\infty)-a_{\vec{p}_3}(-\infty))\cdots(a^\dagger_{\vec{p}_2}(-\infty)-a^\dagger_{\vec{p}_2}(+\infty)))|\Omega\rangle \\ =&\left(i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)\right)\cdots\left(i\int d^4x_n\, e^{ip_n\cdot x_n}(\partial_n^2+m^2)\right)\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle\,, \end{aligned} \tag{2.1.11}$$

得到 LSZ reduction formula.

2.2 LSZ for operators

- 考虑

$$\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}, \tag{2.2.1}$$

且 $P|\Omega\rangle = 0$, 因此

$$\begin{cases} \langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle \\ \langle p|\phi(x)|\Omega\rangle = e^{ip\cdot x}\langle p|\phi(0)|\Omega\rangle \end{cases}. \tag{2.2.2}$$

- LSZ reduction formula 的前提是场算符满足

$$\begin{cases} \langle\Omega|\phi(x)|\Omega\rangle = 0 \\ \langle p|\phi(x)|\Omega\rangle = e^{ip\cdot x} \end{cases}. \tag{2.2.3}$$

remark:

前提是场算符具有以下形式,

$$\phi(x) = \int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}(t)e^{-i\vec{p}\cdot\vec{x}}), \tag{2.2.4}$$

且 $a^\dagger_{\vec{p}}(\pm\infty)|\Omega\rangle \in \text{span}(|p\rangle)$ 是一个单粒子态.

Chapter 3

Feynman rules

3.1 Lagrangian derivation

- 假设存在相互作用时, 场算符依然满足

$$\begin{cases}[\phi(t,\vec{x}),\phi(t,\vec{y})]=0\\ [\phi(t,\vec{x}),\partial_t\phi(t,\vec{y})]=i\delta^{(3)}(\vec{x}-\vec{y})\end{cases}, \tag{3.1.1}$$

  即 causality and canonical commutation relation.

- 一个重要的中间公式为

$$(\partial_x^2+m^2)\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle=\langle\Omega|T((\partial_x^2+m^2)\phi(x)\phi(y))|\Omega\rangle-i\delta^{(4)}(x-y). \tag{3.1.2}$$

proof:

考虑

$$\begin{aligned}\partial_t\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle=&\partial_t(\langle\Omega|\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(y)\phi(x)|\Omega\rangle\theta(t'-t))\\ =&\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(x)\phi(y)|\Omega\rangle\delta(t-t')\\ &+\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\theta(t'-t)-\langle\Omega|\phi(y)\phi(x)|\Omega\rangle\delta(t'-t)\\ =&\langle\Omega|T(\partial_t\phi(x)\phi(y))|\Omega\rangle+\langle\Omega|[\phi(t,\vec{x}),\phi(t,\vec{y})]|\Omega\rangle\delta(t-t')\\ =&\langle\Omega|T(\partial_t\phi(x)\phi(y))|\Omega\rangle,\end{aligned} \tag{3.1.3}$$

那么

$$\begin{aligned}\partial_t^2\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle=&\partial_t\langle\Omega|T(\partial_t\phi(x)\phi(y))|\Omega\rangle\\ =&\partial_t(\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\theta(t'-t))\\ =&\langle\Omega|\partial_t^2\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\delta(t-t')\\ &+\langle\Omega|\phi(y)\partial_t^2\phi(x)|\Omega\rangle\theta(t'-t)-\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\delta(t'-t)\\ =&\langle\Omega|T(\partial_t^2\phi(x)\phi(y))|\Omega\rangle+\langle\Omega|[\partial_t\phi(t,\vec{x}),\phi(t,\vec{y})]|\Omega\rangle\delta(t-t')\\ =&\langle\Omega|T(\partial_t^2\phi(x)\phi(y))|\Omega\rangle-i\delta^{(4)}(x-y).\end{aligned} \tag{3.1.4}$$

- (3.1.2) 可以推广为 (其中 ϕ_i 是 $\phi(x_i)$ 的简写)

$$\begin{aligned}&(\partial_1^2+m^2)\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle\\ =&\langle\Omega|T((\partial_1^2+m^2)\phi(x_1)\cdots\phi(x_n))|\Omega\rangle-i\sum_{i=2}^{n}\delta^{(4)}(x_i-x_1)\langle\Omega|\phi_2\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_n|\Omega\rangle.\end{aligned} \tag{3.1.5}$$

proof:

首先

$$\partial_{t_1}\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle=\langle\Omega|T(\partial_{t_1}\phi(x_1)\cdots\phi(x_n))|\Omega\rangle, \tag{3.1.6}$$

那么

$$\partial_{t_1}^2 \langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \partial_{t_1} \langle\Omega|T(\partial_{t_1}\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \cdots \tag{3.1.7}$$

- canonical commutation relation 保证了 quantum field 满足 Euler–Lagrange equation, 因此

$$(\partial^2 + m^2)\phi = -\frac{\delta}{\delta\phi}V(\phi). \tag{3.1.8}$$

- 结合 (3.1.5) 和 (3.1.8) 得到 Schwinger–Dyson equations,

$$\begin{aligned}&(\partial_1^2 + m^2)\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \\ &= \langle\Omega|T(-\frac{\delta}{\delta\phi_1}V(\phi_1)\cdots\phi(x_n))|\Omega\rangle - i\sum_{i=2}^{n}\delta^{(4)}(x_i - x_1)\langle\Omega|\phi_2\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_n|\Omega\rangle .\end{aligned} \tag{3.1.9}$$

3.1.1 position-space Feynman rules

- Feynman propagator 为 ($\phi_0(x)$ 是 free field, 本 chapter 不做特别说明都是存在相互作用的 Heisenberg picture)

$$D_F(x-y) := \langle 0|T(\phi_0(x)\phi_0(y))|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}e^{ik\cdot(x-y)}, \tag{3.1.10}$$

满足

$$(\partial^2 + m^2)D_F(x-y) = -i\delta^{(4)}(x-y). \tag{3.1.11}$$

- 2-point correlation function 可以重写作 (注意到 $\lim_{x\to\infty} D_{xy} = \lim_{x\to\infty}\partial_x D_{xy} = 0$)

$$\begin{aligned}\langle\Omega|T(\phi_1\phi_2)|\Omega\rangle &= i\int d^4x\,((\partial_x^2 + m^2)D_{x1})\langle\Omega|T(\phi_x\phi_2)|\Omega\rangle \\ &= i\int d^4x\,D_{x1}(\partial_x^2 + m^2)\langle\Omega|T(\phi_x\phi_2)|\Omega\rangle \\ &= D_{12} + i\int d^4x\,D_{x1}\langle\Omega|T(-\frac{\delta}{\delta\phi_x}V(\phi_x)\phi_2)|\Omega\rangle .\end{aligned} \tag{3.1.12}$$

Appendices