Quantum Field Theory

- based on A. Zee's textbook -

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Contents

convention, notation, and units

- 笔记中的**度规号差**约定为 $(-,+,+,+)$.
- 使用 Planck units, 此时 $G, \hbar, c, k_B = 1$, 因此,

name/dimension	expression/value
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

Part I

Scalar Quantum Field

Chapter 1

free field theory

1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{1.1.1}$$

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi\, e^{i\int d^dx(-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \tag{1.1.2}$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J(x)D(x-y)J(y)} \tag{1.1.3}$$

另外, 用 $W(J)$ 表示指数上的部分 (去除掉虚数 i).

proof:

注意 $\partial^\mu\phi\partial_\mu\phi = \partial^\mu(\phi\partial_\mu\phi) - \phi\partial^2\phi$, 忽略全微分项, 那么,

$$Z(J) = \int D\phi\, e^{i\int d^dx \frac{1}{2}(\phi(\partial^2 - m^2)\phi + J(x)\phi(x))} \tag{1.1.4}$$

代入 (B.1.1), 可知,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J(x)D(x-y)J(y)} \tag{1.1.5}$$

其中 $D(x-y)$ 满足,

$$\begin{cases} (\partial^2 - m^2)D(x-y) = \delta^{(d)}(x-y) \\ (-p^2 - m^2)\tilde{D}(p,q) = (2\pi)^d\delta^{(d)}(p-q) \end{cases} \Longrightarrow D(x-y) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot(x-y)}}{-k^2 - m^2} \tag{1.1.6}$$

1.2 free propagator

- 为了使 (1.1.4) 中的积分在 ϕ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon\int d^dx\phi^2}$.
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 ϵ 避免发散,

$$D(x) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot x}}{-k^2 - m^2 + i\epsilon} = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\left(e^{i(-\omega_k t + \vec{k}\cdot\vec{x})}\theta(t) + e^{i(\omega_k t + \vec{k}\cdot\vec{x})}\theta(-t)\right) \tag{1.2.1}$$

calculation:

对 k^0 积分, 注意有两个奇点 $k^0 = \pm(\omega_k - i\epsilon)$, 当 $t > 0$ 时, contour 处于下半平面, ...

- $D(x)$ 的取值与 x 的类时, 类空性质关系密切.
  - 类时区域,
$$D(t,0) = -i\int \frac{d^D k}{(2\pi)^D 2\omega_k}\left(e^{-i\omega_k t}\theta(t) + e^{i\omega_k t}\theta(-t)\right) \tag{1.2.2}$$
  - 类空区域,
$$D(0,\vec{x}) = -i\int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \sim e^{-m|\vec{x}|} \tag{1.2.3}$$

Chapter 2

from field to particle to force

2.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,

$$W(J) = -\frac{1}{2}\int d^dxd^dy\, J(y)D(x-y)J(y) \tag{2.1.1}$$

$$= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}(k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}(-k) \tag{2.1.2}$$

其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k) = \tilde{J}^*(k)$.

- 考虑 $J(x) = J_1(x) + J_2(x)$, 那么 $W(J)$ 共有 4 项,

Appendices

Appendix A

Dirac delta function & Fourier transformation

A.1 Delta function

- 可以认为以下是定义式,
$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff 1 = \int dx\, \delta(x) e^{-ikx} \tag{A.1.1}$$

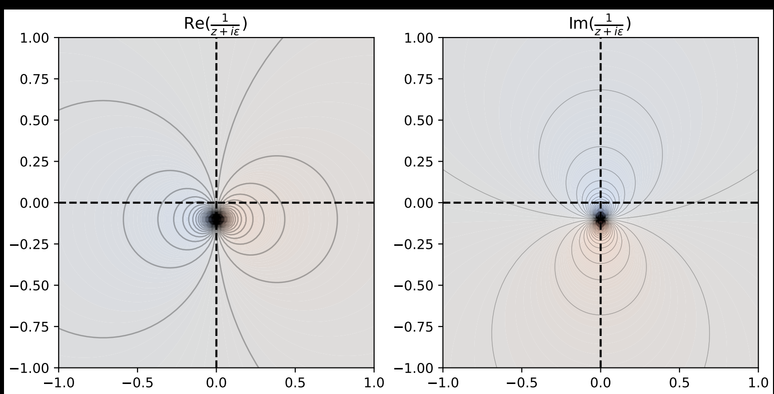
- 第一个常用的公式,
$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \tag{A.1.2}$$

- 第二个常用的公式 (Sokhotski-Plemelj theorem),
$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{A.1.3}$$
其中 $\mathcal{P}$ 表示复函数的主值 (principal value).

proof:

考虑,
$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \tag{A.1.4}$$
且注意到,
$$\int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \operatorname{Res}(f, i\epsilon) = \pi \tag{A.1.5}$$
所以...

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,



A.2 Fourier transformation

- d-dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^d x \, e^{-ik\cdot x} \phi(x) \end{cases} \tag{A.2.1}$$

- 因此,

$$\partial_\mu \phi(x) \mapsto ik_\mu \tilde{\phi}(k) \tag{A.2.2}$$

Appendix B

Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{B.0.1}$$

$$\langle x^{2n}\rangle = \frac{\int dx\, e^{-\frac{1}{2}ax^2}x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!! \tag{B.0.2}$$

其中 $(2n-1)!! = 1\cdot 3\cdots(2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx\, e^{-\frac{a}{2}x^2+Jx} = \sqrt{\frac{2\pi}{a}}e^{\frac{J^2}{2a}} \tag{B.0.3}$$

另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 N-dim. generalization

- 考虑如下积分,

$$Z(A,J) = \int dx_1\cdots dx_N\, e^{-\frac{1}{2}x^T\cdot A\cdot x + J^T\cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \tag{B.1.1}$$

其中 x, J 是 N-dim. 列向量, A 是 $N\times N$ 实对称矩阵.

calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1}\cdot D\cdot O \tag{B.1.2}$$

其中 D 是一个 diagonal matrix. 令 $y = O\cdot x$, 那么,

$$\begin{aligned} Z(A,J) &= \int dy_1\cdots dy_N\, e^{-\frac{1}{2}y^T\cdot D\cdot y + (OJ)^T\cdot y} \\ &= \prod_{i=1}^{N}\sqrt{\frac{2\pi}{D_{ii}}}e^{\frac{1}{2D_{ii}}(OJ)_i{}^2} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \end{aligned} \tag{B.1.3}$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O\cdot A^{-1}\cdot O^{-1})_{ii}$ 以及 $\operatorname{tr} D = \det A$.

- 一个重要的变体是 $A\mapsto -iA, J\mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j\rangle = \frac{1}{Z(A,0)}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}Z(A,J)\Big|_{J=0} = A^{-1}_{ij} \tag{B.1.4}$$

$$\langle x_i x_j\cdots x_k x_l\rangle = \sum_{\text{Wick}} A^{-1}_{i'j'}\cdots A^{-1}_{k'l'} \tag{B.1.5}$$

其中 (B.1.5) 中有偶数个 x, 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A,J)\Big|_{J=0} = \cdots \tag{B.1.6}$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} \tag{B.1.7}$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ik} A^{-1}_{jl} \tag{B.1.8}$$