

# Quantum Field Theory

a study note based on A. Zee's textbook

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# convention, notation, and units

- 笔记中的度规号差约定为  $(-, +, +, +)$ .
- 使用 Planck units, 此时  $G, \hbar, c, k_B = 1$ , 因此:

name/dimension	expression/value
Planck length ( $L$ )	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} \text{ m}$
Planck time ( $T$ )	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44} \text{ s}$
Planck mass ( $M$ )	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} \text{ kg} \simeq 10^{19} \text{ GeV}$
Planck temperature ( $\Theta$ )	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32} \text{ K}$

- 时空维度用  $d = D + 1$  表示.

**Part I**

**motivation and foundation**

# Chapter 1

## free field theory

### 1.1 partition function

- 考虑标量场

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi), \quad (1.1.1)$$

A. Zee: 在作用量里, 时间的导数项必须是正的, 包括标量场的  $(\partial_0\phi)^2$  和电磁场的  $(\partial_0 A_i)^2$ .

- 含有 source function 的路径积分为

$$Z(J) = \int D\phi e^{i \int d^d x (-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))}. \quad (1.1.2)$$

- 当  $V(\phi) = \frac{1}{2}m^2\phi^2$  时, 称作 free or Gaussian theory.

- 
- 计算 free theory 的 partition function, 得到

$$Z(J) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)}, \quad (1.1.3)$$

另外, 用  $W(J)$  表示指数上的部分 (去除掉虚数  $i$ ).

**proof:**

注意  $\partial^\mu \phi \partial_\mu \phi = \partial^\mu (\phi \partial_\mu \phi) - \phi \partial^2 \phi$ , 忽略全微分项, 那么

$$Z(J) = \int D\phi e^{i \int d^d x (\frac{1}{2} \phi (\partial^2 - m^2) \phi + J(x) \phi(x))}, \quad (1.1.4)$$

代入 (B.1.1), 可知

$$Z(J) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)}, \quad (1.1.5)$$

其中  $D(x-y)$  满足

$$\begin{cases} (\partial^2 - m^2) D(x-y) = \delta^{(d)}(x-y) \\ (-p^2 - m^2) \tilde{D}(p, q) = (2\pi)^d \delta^{(d)}(p-q) \end{cases} \implies D(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{-k^2 - m^2}. \quad (1.1.6)$$

### 1.2 free propagator

- 为了使 (1.1.4) 中的积分在  $\phi$  较大时收敛, 作替换  $m^2 \mapsto m^2 - i\epsilon$ , 这样被积函数中会出现一项  $e^{-\epsilon \int d^d x \phi^2}$ .
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量  $\epsilon$  避免发散,

$$D(x) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot x}}{-k^2 - m^2 + i\epsilon} = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left( \theta(t) e^{i(-\omega_k t + \vec{k} \cdot \vec{x})} + \theta(-t) e^{i(\omega_k t + \vec{k} \cdot \vec{x})} \right). \quad (1.2.1)$$

**calculation:**

对  $k^0$  积分, 注意有两个奇点  $k^0 = \pm(\omega_k - i\epsilon)$ , 当  $t > 0$  时, contour 处于下半平面, ... (另外注意到我们可以任意改变  $\vec{k}$  的符号).

- $D(x)$  的取值与  $x$  的类时, 类空性质关系密切.

– 类时区域,

$$D(t, 0) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left( \theta(t) e^{-i\omega_k t} + \theta(-t) e^{i\omega_k t} \right). \quad (1.2.2)$$

– 类空区域,

$$D(0, \vec{x}) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{i\vec{k} \cdot \vec{x}} \sim e^{-m|\vec{x}|}. \quad (1.2.3)$$

## 1.3 from field to particle to force

### 1.3.1 from field to particle

- 考虑 (1.1.3) 中的  $W(J)$ ,

$$W(J) = -\frac{1}{2} \int d^d x d^d y J(y) D(x-y) J(y) \quad (1.3.1)$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}(k), \quad (1.3.2)$$

其中, 如果  $J(x)$  是实函数, 那么  $\tilde{J}(-k) = \tilde{J}^*(k)$ .

- 考虑  $J(x) = J_1(x) + J_2(x)$ , 那么  $W(J)$  共有 4 项, 其中一个交叉项如下,

$$W_{12}(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_1(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_2(k), \quad (1.3.3)$$

可见  $W(J)$  取值较大的条件是:

1.  $\tilde{J}_1(k), \tilde{J}_2(k)$  有较大重叠,
2. 重叠位置的  $k$  是 on shell (即  $k^2 = -m^2$ ).

- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

### 1.3.2 from particle to force

- 考虑  $J(x) = \delta^{(D)}(\vec{x} - \vec{x}_1) + \delta^{(D)}(\vec{x} - \vec{x}_2) \implies \tilde{J}_a(k) = 2\pi e^{-i\vec{k} \cdot \vec{x}_a} \delta(k^0)$ , 那么

$$\begin{aligned} W_{12}(J) + W_{21}(J) &= \delta(0) \int \frac{d^D k}{(2\pi)^{D-1}} \frac{1}{|\vec{k}|^2 + m^2 - i\epsilon} \cos(\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)) \\ &\stackrel{D=3}{=} 2\pi \delta(0) \frac{1}{4\pi r} e^{-mr}, \end{aligned} \quad (1.3.4)$$

( $-i\epsilon$  显然可以舍去), 注意到  $\langle 0 | e^{-iHT} | 0 \rangle = e^{-iET}$ , 而时间间隔  $T = \int dx^0 = 2\pi \delta(0)$ , 所以

$$E = -\frac{W(J)}{T} \stackrel{D=3}{=} -\frac{1}{4\pi r} e^{-mr}. \quad (1.3.5)$$

**calculation:**

计算 (1.3.4) 中的积分, 令  $\vec{x}_1 - \vec{x}_2 = \vec{r}$ ,

$$I_D = \int \frac{d^D k}{(2\pi)^D} \frac{1}{|\vec{k}|^2 + m^2} \overbrace{\cos(\vec{k} \cdot \vec{r})}^{\mapsto e^{i\vec{k} \cdot \vec{r}}}$$



$$\begin{aligned}
&= \frac{1}{(2\pi)^D} \int (k \sin \theta_1)^{D-2} d\Omega_{D-2} \int k d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1} \\
&= \frac{S_{D-2}}{(2\pi)^D} \int k^{D-1} \sin^{D-2} \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1},
\end{aligned} \tag{1.3.6}$$

取  $D = 3$ , 那么

$$\begin{aligned}
I_{D=3} &= \frac{1}{(2\pi)^2} \int k^2 \sin \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ik \cos \theta_1} \\
&= \frac{1}{2\pi^2 r} \int_0^\infty \sin(kr) \frac{k dk}{k^2 + m^2} = \frac{-i}{4\pi^2 r} \int_{-\infty}^\infty e^{ikr} \frac{k dk}{k^2 + m^2} \\
&= \frac{-i}{4\pi^2 r} 2\pi i \underbrace{\text{Res}(f, im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r} e^{-mr}.
\end{aligned} \tag{1.3.7}$$

## 1.4 vacuum energy

- 注意到

$$Z(J=0) = \langle 0 | e^{-iHT} | 0 \rangle, \tag{1.4.1}$$

所以

$$E_0 = \langle 0 | H | 0 \rangle = V \int \frac{d^D k}{(2\pi)^D} \frac{1}{2} \omega_k + \text{irrelevant terms}. \tag{1.4.2}$$

**calculation:**

代入 (B.1.1) (其中  $N$  是时空格点总数),

$$Z(J=0) = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}}, \tag{1.4.3}$$

其中  $A = -i(\partial^2 - m^2 + i\epsilon)$ .

- 注意到  $\det e^A = e^{\text{tr} A} \implies \det A = e^{\text{tr} \ln A}$ , 代入, 并有

$$(\ln A)\phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \ln(-i(-k^2 - m^2 + i\epsilon)) \tilde{\phi}(k), \tag{1.4.4}$$

对于  $A : v \mapsto u$  以及变换  $P : v \mapsto \tilde{v}$ , 有  $PAP^{-1} : \tilde{v} \mapsto \tilde{u}$ , 且  $\text{tr} A = \text{tr} PAP^{-1}$ , 所以

$$\begin{aligned}
-\frac{1}{2} \text{tr} \ln A &= -\frac{1}{2} \text{tr} \ln(-i(-k^2 - m^2 + i\epsilon)) \\
&= -\frac{1}{2} \sum_k \ln(-i(-k^2 - m^2 + i\epsilon)) \\
&= -\frac{1}{2} \frac{VT}{(2\pi)^d} \int d^d k \ln(-i(-k^2 - m^2 + i\epsilon)),
\end{aligned} \tag{1.4.5}$$

其中, 参考 (A.2.5), 有  $\sum_k = \frac{VT}{(2\pi)^d} \int d^d k$ .

代入 (1.4.1),

$$\begin{aligned}
E_0 &= \frac{i}{T} \left( \frac{N}{2} \ln(2\pi) - \frac{1}{2} \frac{VT}{(2\pi)^d} \int d^d k \ln(-i(-k^2 - m^2 + i\epsilon)) \right) \\
&= \frac{iN}{2T} \ln(2\pi) - \frac{i}{2} V \int \frac{d^d k}{(2\pi)^d} \left( \underbrace{\ln(-k^2 - m^2 + i\epsilon)}_{=(k^0)^2 - \omega_k^2 + i\epsilon} - \frac{\pi}{2} i \right),
\end{aligned} \tag{1.4.6}$$

略去与  $m$  无关的常数项

$$\frac{\Delta E_0}{V} = -\frac{i}{2} \int \frac{d^D k}{(2\pi)^D} \int \frac{dk^0}{2\pi} \ln((k^0)^2 - \omega_k^2 + i\epsilon), \tag{1.4.7}$$

做分部积分,

$$\ln((k^0)^2 - \omega_k^2 + i\epsilon) = \frac{d}{dk^0} (k^0 \ln((k^0)^2 - \omega_k^2 + i\epsilon)) - k^0 \frac{2k^0}{(k^0)^2 - \omega_k^2 + i\epsilon}, \quad (1.4.8)$$

代入,

$$\begin{aligned} \frac{E_0}{V} &= \frac{i}{2} \int \frac{d^D k}{(2\pi)^D} \int \frac{dk^0}{2\pi} \frac{2(k^0)^2}{(k^0)^2 - \omega_k^2 + i\epsilon} \\ &= \frac{i}{2} \int \frac{d^D k}{(2\pi)^D} \left( \frac{1}{2\pi} 2\pi i \frac{2(-\omega_k)^2}{-2\omega_k} \right) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{2} \omega_k. \end{aligned} \quad (1.4.9)$$

另外,  $\ln(z^2 - 1 + i\epsilon)$  的图像如下:

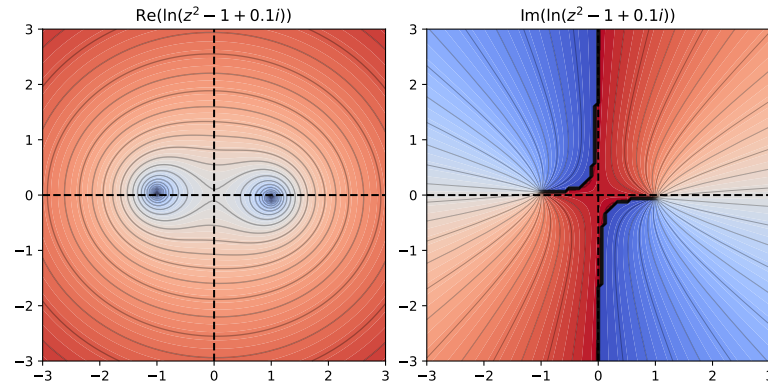


Figure 1.1: graph of  $\ln(z^2 - 1 + i\epsilon)$

## Chapter 2

# Coulomb and Newton: repulsive and attraction

### 2.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu, \quad (2.1.1)$$

其中  $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ .

- 做路径积分,

$$Z(J) = \int DA e^{i \int d^d x (\mathcal{L} + J_\mu A^\mu)} = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J_\mu D^{\mu\nu}(x-y) J_\nu(y)}. \quad (2.1.2)$$

**calculation:**

massive photon 的作用量为

$$\begin{aligned} S(A) &= \int d^d x \frac{1}{2} \left( -(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu \right) \\ &= \int d^d x \frac{1}{2} \left( A_\nu \partial^2 A^\nu - A_\nu \partial^\nu \partial_\mu A^\mu - m^2 A_\mu A^\mu \right) + \text{total differential} \\ &= \int d^d x \frac{1}{2} A_\mu \left( -\partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \right) A_\nu + \text{total differential} \\ &= \int \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(-k) \left( k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2) \right) \tilde{A}_\nu(k) + \text{boundary term}, \end{aligned} \quad (2.1.3)$$

那么, 需要有

$$\begin{aligned} (-\partial^\mu \partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2)) D_{\rho\nu}(x-y) &= \delta_\nu^\mu \delta^{(d)}(x-y) \\ \Rightarrow \tilde{D}_{\mu\nu}(k) &= \frac{k_\mu k_\nu / m^2 + \eta_{\mu\nu}}{-k^2 - m^2}. \end{aligned} \quad (2.1.4)$$

考虑到积分需要收敛, 作替换  $m^2 \mapsto m^2 - i\epsilon$ , (为什么  $A_\mu$  类空, 只知道  $\tilde{A}_\mu$  类空, 见 subsection 2.1.2, 但路径积分中的  $A$  显然不满足 field equation  $\Rightarrow$  路径积分中起主要作用的  $\tilde{A}$  类空, 因此  $-\epsilon|\tilde{A}|^2 < 0$ ).

- 因此

$$W(J) = -\frac{1}{2} \int d^d x d^d y J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) \quad (2.1.5)$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_\mu(-k) \frac{k^\mu k^\nu / m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon} \tilde{J}_\nu(k), \quad (2.1.6)$$

注意到 current conservation, 有  $\partial_\mu J^\mu = 0 \iff k^\mu \tilde{J}_\mu(k) = 0$ , 所以

$$W(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}^\mu(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_\mu(k), \quad (2.1.7)$$

观察电荷分量, 可见同性相斥, 异性相吸.

### 2.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的  $x, y, z$  方向, 在粒子静止系下, 极化矢量  $(\epsilon^i)_\mu = \delta_\mu^i, i = 1, 2, 3$ , 而  $k_\mu = (-m, 0, 0, 0)$ , 所以

$$k^\mu (\epsilon^i)_\mu = 0. \quad (2.1.8)$$

– 注意, 一个粒子的极化方向用  $e^i$  (这不是矢量) 表示, 极化矢量为  $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$ .

- 在粒子静止系下, 考虑

$$\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu}, \quad (2.1.9)$$

可见

$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon}. \quad (2.1.10)$$

### 2.1.2 Maxwell Lagrangian

- 根据 (2.1.1) 中的 Lagrangian density, 得到 field equation,

$$\left( -\partial^\mu \partial^\nu + \eta^{\mu\nu} (\partial^2 - m^2) \right) A_\nu = 0. \quad (2.1.11)$$

– spin-1 particle 有 3 个自旋自由度, 而  $A_\mu$  有 4 个分量, 所以需要一个约束方程 (只在  $m \neq 0$  情况下存在),

$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0. \quad (2.1.12)$$

来源: 在 (2.1.11) 左右两边作用一个  $\partial_\mu$  即可得到这个约束方程.

## 2.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = linearized Einstein Lagrangian.
- 受 subsection 2.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足

$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases}, \quad (2.2.1)$$

其中下指标  $\mu, \nu$  对称,  $a = 1, \dots, 5$ , (可以验证  $(\epsilon^a)_{\mu\nu}$  确实有 5 个独立分量).

- 对  $(\epsilon^a)_{\mu\nu}$  的归一化条件可以定义为  $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$ .
- 与 subsection 2.1.1 中提示一样, 粒子的极化方向用  $e^a$  表示.

- 那么

$$\sum_{a=1}^5 (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma}. \quad (2.2.2)$$

#### calculation:

首先用  $k_\mu$  和  $\eta_{\mu\nu}$  构造最一般的关于  $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$  对称的 4 阶张量, (下式中把  $\frac{k_\mu}{m}$  略写作  $k_\mu$ ),

$$\begin{aligned} & A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ & + D \eta_{\mu\nu} \eta_{\rho\sigma} + E(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}), \end{aligned} \quad (2.2.3)$$

代入 (2.2.1) 得

$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4}, \quad (2.2.4)$$

因此, 这个 4 阶张量最终确定为

$$\frac{3}{4}A\left((G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\right). \quad (2.2.5)$$

- 所以

$$\tilde{D}_{\mu\nu\rho\sigma}(k) = \frac{(G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2 - m^2 + i\epsilon}. \quad (2.2.6)$$

- 计算路径积分中的  $W(T)$ ,

$$W(T) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{\mu\nu}(-k) \frac{(G^{\mu\rho}G^{\nu\sigma} + G^{\mu\sigma}G^{\nu\rho}) - \frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tilde{T}_{\rho\sigma}(k), \quad (2.2.7)$$

注意到  $\partial_\mu T^{\mu\nu}(x) = 0 \iff k_\mu \tilde{T}^{\mu\nu}(k) = 0$ , 并考虑到  $T$  是对称张量, 所以

$$W(T) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{\mu\nu}(-k) \frac{2\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tilde{T}_{\rho\sigma}(k), \quad (2.2.8)$$

考虑能量项, 可见质量互相吸引.

## 2.3 remarks

- 由于 seesaw mechanism (见 subsection C.1.1), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为  $\eta^{00} = -1$ .
- 本 chapter 中的计算都是  $m \neq 0$  的粒子, 与真实世界有差异.

## Chapter 3

# Feynman diagrams

### 3.1 a baby problem

- 考虑如下积分,

$$Z(J) = \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4 + Jq}. \quad (3.1.1)$$

- Schwinger's way:** 把 integrand 对  $\lambda$  展开, 并将  $q$  用  $\frac{\partial}{\partial J}$  替代, 得到

$$\begin{aligned} Z(J) &= e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4} \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 + Jq} \\ &= \sqrt{\frac{2\pi}{m^2}} e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4} e^{\frac{J^2}{2m^2}}, \end{aligned} \quad (3.1.2)$$

后面的计算中忽略  $Z(J=0, \lambda=0)$ .

- 每个 vertex 带有  $-\lambda$ , 每个 line 带有  $\frac{1}{m^2}$ , 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里  $q$  无法区分, 所以不方便算, 先略了):

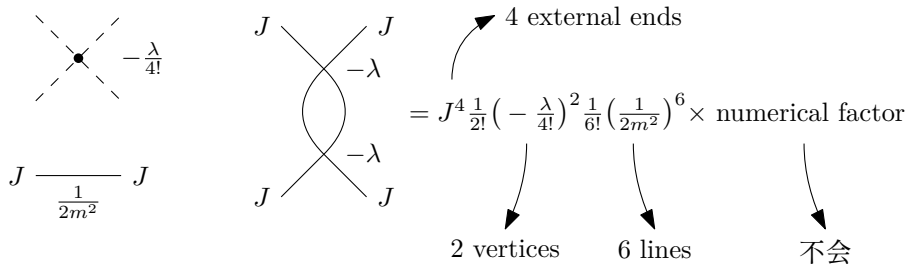


Figure 3.1: baby problem - Feynman diagram

#### calculation:

在这里计算  $\lambda J^4$  项,



(3.1.3)

但是直接计算 (3.1.2) 的展开项, 得到的结果与 (3.1.5) 一样.

#### 3.1.1 Wick contraction and Green's functions

- 把积分 (3.1.1) 对  $J$  展开,

$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} J^n \underbrace{\int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4} q^n}_{=Z(0,0)G^{(n)}}, \quad (3.1.4)$$

其中 Green's function  $G^{(n)}$  对  $\lambda$  展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

#### calculation:

计算  $\lambda J^4$  项, 它来自  $G^{(4)}$  对  $\lambda$  展开的一阶项,

$$\begin{aligned} -\frac{\lambda}{4!} \int dq e^{-\frac{1}{2}m^2 q^2} q^8 &= -\frac{\lambda}{4!} \langle q^8 \rangle \\ &= -\frac{\lambda}{4!} \sum_{\text{Wick}} \left( \frac{1}{m^2} \right)^4 \\ &= -\frac{\lambda}{4!} \frac{7 \times 5 \times 3 \times 1}{m^8}, \end{aligned} \quad (3.1.5)$$

所以  $\lambda J^4$  项等于  $\frac{105}{(4!)^2} \frac{-\lambda J^4}{m^8}$ .

### 3.1.2 connected vs. disconnected

- 考虑

$$Z(J, \lambda) = Z(J=0, \lambda) e^{W(J, \lambda)}, \quad (3.1.6)$$

其中  $Z(J=0, \lambda)$  由 diagrams with no external source  $J$  组成, 而  $W(J, \lambda)$  则由 connected diagrams 组成(?).

- 我们希望计算的是  $W$ , 而不是  $Z$  (?).

### 3.2 a child problem

- 考虑如下积分

$$Z(J) = \int dq_1 \cdots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q - \frac{\lambda}{4!} q^4 + J^T \cdot q}, \quad (3.2.1)$$

其中  $q^4 = \sum_i q_i^4$ .

- Schwinger's way:** 对  $\lambda$  展开并把  $q$  替换为  $\frac{\partial}{\partial J}$ , 得到

$$Z(J) = \sqrt{\frac{(2\pi)^N}{\det A}} e^{-\frac{\lambda}{4!} (\frac{\partial}{\partial J})^4} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J}, \quad (3.2.2)$$

其中  $(\frac{\partial}{\partial J})^4 = \sum_i (\frac{\partial}{\partial J_i})^4$ .

#### 3.2.1 $n$ -point Green's function

- Wick's way:** 对  $J$  展开获得带 Green's function 的表达式,

$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N J_{i_1} \cdots J_{i_n} \underbrace{\int dq_1 \cdots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q - \frac{\lambda}{4!} q^4} q_{i_1} \cdots q_{i_n}}_{=Z(0,0)G_{i_1 \cdots i_n}^{(n)}}, \quad (3.2.3)$$

其中  $G_{i_1 \cdots i_n}^{(n)}$  称为  $n$ -point Green's function.

#### Taylor expansion:

多元函数的 Taylor 展开为

$$\begin{aligned} f(x_1, \cdots, x_N) &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_N^{n_N}}{n_N!} \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_N}}{\partial x_N^{n_N}} f(x=0) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N x_{i_1} \cdots x_{i_n} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}} f(x=0), \end{aligned} \quad (3.2.4)$$

这两种求和方法,  $x_1^{n_1} \cdots x_N^{n_N}$  项的 numerical factor 都等于

$$\frac{1}{n!} \times \frac{n!}{n_1! \cdots n_N!} = \frac{1}{n_1! \cdots n_N!}, \quad (3.2.5)$$

其中  $n = n_1 + \dots + n_N$ .

- 在  $\lambda = 0$  时, 2-point Green's function 就是 propagator,

$$\begin{aligned} G_{ij}^{(2)}(\lambda = 0) &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} q_i q_j \\ &= \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \Big|_{J=0} = A_{ij}^{-1}. \end{aligned} \quad (3.2.6)$$

- 来计算 2, 3, 4-point Green's functions,

$$\begin{cases} G_{ij}^{(2)} = A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \\ G_{ijk}^{(3)} = 0 \\ G_{ijkl}^{(4)} = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \\ \quad - \frac{\lambda}{4!} \sum_m (A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} A_{kl}^{-1} + \dots + 4! A_{im}^{-1} A_{jm}^{-1} A_{km}^{-1} A_{lm}^{-1}) + O(\lambda^2) \end{cases}. \quad (3.2.7)$$

calculation:

2-point Green's function 计算如下,

$$\begin{aligned} G_{ij}^{(2)} &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j \\ &= A_{ij}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j \rangle + O(\lambda^2) \\ &= A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2), \end{aligned} \quad (3.2.8)$$

3-point Green's function 计算如下,

$$G_{ijk}^{(3)} = \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j q_k = 0, \quad (3.2.9)$$

4-point Green's function 计算如下,

$$\begin{aligned} G_{ijkl}^{(4)} &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j q_k q_l \\ &= A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j q_k q_l \rangle + O(\lambda^2). \end{aligned} \quad (3.2.10)$$

### 3.3 perturbative field theory

- 做如下替换即可,

$$\begin{cases} A \mapsto -i(\partial^2 - m^2) \\ J \mapsto iJ \end{cases}. \quad (3.3.1)$$

- Schwinger's way:**  $\phi^4$  theory 的路径积分,

$$Z(J) = \int D\phi e^{i \int d^d x \left( \frac{1}{2} \phi (\partial^2 - m^2) \phi - \frac{\lambda}{4!} \phi^4 + J(x) \phi(x) \right)} \quad (3.3.2)$$

$$= Z(0,0) e^{-i \frac{\lambda}{4!} \int d^d z \left( \frac{\delta}{i \delta J(z)} \right)^4} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)}, \quad (3.3.3)$$

其中  $D(x-y)$  是自由场的 propagator, 见 (1.2.1).



- **Wick's way:** 同样, 对  $J$  展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \cdots d^d x_n J(x_1) \cdots J(x_n) G^{(n)}(x_1, \cdots, x_n), \quad (3.3.4)$$

其中

$$G^{(n)}(x_1, \cdots, x_n) = \frac{1}{Z(0,0)} \int D\phi e^{i \int d^d x (\frac{1}{2} \phi(\partial^2 - m^2)\phi - \frac{\lambda}{4!} \phi^4)} \phi(x_1) \cdots \phi(x_n), \quad (3.3.5)$$

有时  $Z(J)$  被称为 generating functional, 因为它能生成 Green's functions.

### 3.3.1 collision between particles

- 通过 Wick's way, 考虑  $J(x_1)J(x_2)J(x_3)J(x_4)$  项, 实际上就是要计算  $G^{(4)}(x_1, x_2, x_3, x_4)$ , 它的 0 阶项为

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4, \lambda = 0) &= \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta J(x_2)} \frac{\delta}{i\delta J(x_3)} \frac{\delta}{i\delta J(x_4)} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \\ &= -(D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}), \end{aligned} \quad (3.3.6)$$

其中  $D_{ij}$  是  $D(x_i - x_j)$  的简写, 可见, 传播子实际上是  $(-i)^3 D = iD$ .

- $G_{1234}^{(4)}$  的 1 阶项为

$$\begin{aligned} \text{1st order term} &= -\frac{i\lambda}{4!} \int d^d z \langle \phi_1 \cdots \phi_4 \phi^4(z) \rangle \\ &= -\frac{i\lambda}{4!} \int d^d z \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_4} \left( \frac{\delta}{i\delta J(z)} \right)^4 e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \\ &= -\frac{i\lambda}{4!} \int d^d z \left( 4! D_{1z} D_{2z} D_{3z} D_{4z} \right. \\ &\quad \left. + 4 \times 3 D_{12} D_{3z} D_{4z} D_{zz} + \cdots + 3 D_{12} D_{34} D_{zz} D_{zz} + \cdots \right), \end{aligned} \quad (3.3.7)$$

其中各项分别对应如下 Feynman diagrams:

$$\begin{aligned} &-i\lambda \int d^d z D_{1z} D_{2z} D_{3z} D_{4z} \quad \quad \quad -\frac{i\lambda}{2!} \int d^d z D_{13} D_{2z} D_{4z} D_{zz} \quad \quad \quad -\frac{i\lambda}{8} \int d^d z D_{13} D_{24} D_{zz} D_{zz} \end{aligned}$$

Figure 3.2: position space - Feynman diagrams

其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,

$$= (4 \times 3)^2 \times 2 \times \left( \frac{-i\lambda}{4!} \right)^2 \int d^d z_1 d^d z_2 D_{1z_1} D_{2z_1} D_{3z_2} D_{4z_2} D_{z_1 z_2} D_{z_1 z_2}. \quad (3.3.8)$$

### 3.3.2 in momentum space

- 本 subsection 将 (3.3.5) 转换到 momentum space, 注意到  $\tilde{J}(k)$  和  $\tilde{J}(-k)$  并不独立, 所以  $\frac{\partial}{\partial iJ}$  不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\tilde{G}^{(n)}(k_1, \cdots, k_n) = \int d^d x_1 \cdots d^d x_n e^{-i(k_1 \cdot x_1 + \cdots)} G^{(n)}(x_1, \cdots, x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \cdots d^d x_n e^{-i(k_1 \cdot x_1 + \cdots)} \langle \left( -\frac{i\lambda}{4!} \int d^d z \phi_z^4 \right)^n \phi_1 \cdots \phi_n \rangle. \quad (3.3.9)$$

– propagator 的 Fourier transformation 是

$$\tilde{D}_{pq} = \int d^d x d^d y e^{-i(p \cdot x + q \cdot y)} D(x - y) = \frac{(2\pi)^d \delta^{(d)}(p + q)}{-p^2 - m^2 + i\epsilon}, \quad (3.3.10)$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$  的 1 阶项为

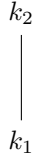
$$\text{1st order term} = -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \langle \phi_z^4 \phi_1 \cdots \phi_4 \rangle, \quad (3.3.11)$$

考虑第 1 项,

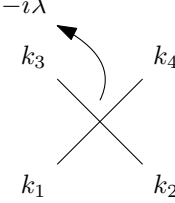
$$\begin{aligned} & -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z 4! D_{1z} \cdots D_{4z} \\ &= -i\lambda \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - z) + \cdots)} \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\ &= -i\lambda \underbrace{\int d^d z e^{-iz \cdot (k_1 + \cdots + k_4)}}_{=(2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4)} \prod_{i=1}^4 \frac{1}{-k_i^2 - m^2 + i\epsilon}. \end{aligned} \quad (3.3.12)$$

– 出射粒子不一定 on-shell (?).

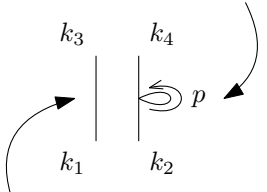
- 得到这些 Feynman diagrams:



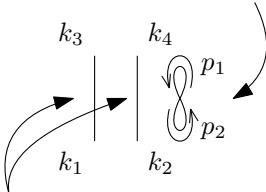
$$(2\pi)^d \delta^{(d)}(k_1 + k_2) \frac{i}{-k_1^2 - m^2 + i\epsilon}$$



$$-i\lambda (2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4) \prod_{i=1}^4 \frac{i}{-k_i^2 - m^2 + i\epsilon}$$



$$(2\pi)^d \delta^{(d)}(k_1 + k_3) \frac{i}{-k_1^2 - m^2 + i\epsilon}$$



$$(2\pi)^d \delta^{(d)}(k_1 + k_3) (2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{i}{-k_i^2 - m^2 + i\epsilon}$$

Figure 3.3: momentum space - Feynman diagrams

**calculation:**

第 3 幅图的计算如下,

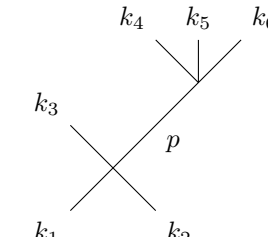
$$-\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z D_{13} D_{2z} D_{4z} D_{zz}$$

$$\begin{aligned}
&= -\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - z) + p_4 \cdot (x_4 - z) + p_4 \cdot 0)} \\
&\quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{2!} \int d^d z e^{-iz \cdot (p_2 + p_4)} \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_4 - k_4) \\
&\quad \prod_{i=1}^4 \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2,4} \frac{1}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{-p^2 - m^2 + i\epsilon}, \tag{3.3.13}
\end{aligned}$$

第 4 幅图的计算如下,

$$\begin{aligned}
&-\frac{i\lambda}{8} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z D_{13} D_{24} D_{zz} D_{zz} \\
&= -\frac{i\lambda}{8} \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - x_4) + p_3 \cdot 0 + p_4 \cdot 0)} \\
&\quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{8} \int d^d z \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_2 + k_4) \\
&\quad \prod_{i=1}^4 \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{1}{-k_i^2 - m^2 + i\epsilon} \\
&\quad \prod_{i=1,2} \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon}. \tag{3.3.14}
\end{aligned}$$

- 再举一个例子 (略去了  $\prod_{i=1}^6 \frac{i}{-k_i^2 - m^2 + i\epsilon}$ ),



$$\begin{aligned}
&= (4!)^2 \times \left(-\frac{i\lambda}{4!}\right)^2 (2\pi)^{2d} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \delta^{(d)}(k_1 + k_2 + k_3 + p) \delta^{(d)}(k_4 + k_5 + k_6 - p) \\
&= (-i\lambda)^2 (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6) \frac{i}{-(k_1 + k_2 + k_3)^2 - m^2 + i\epsilon}. \tag{3.3.15}
\end{aligned}$$

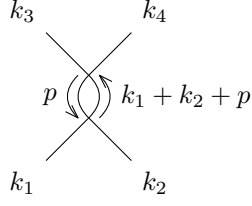
### 3.3.3 loops and a first look at divergence

- subsection 3.3.2 里的 loop diagrams 出现了如下积分,

$$\int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} = \int \frac{d^D p}{(2\pi)^D 2\omega_p} \sim \int \frac{d^D p}{|p|}, \tag{3.3.16}$$

积分发散.

- 再举一个例子 (略去了  $\prod_{i=1}^4 \frac{i}{-k_i^2 - m^2 + i\epsilon}$ ),



$$\begin{aligned}
&= (4 \times 3)^2 \times 2 \times \left( \frac{-i\lambda}{4!} \right)^2 \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{i}{-q^2 - m^2 + i\epsilon} \\
&\quad (2\pi)^d \delta^{(d)}(k_1 + k_2 + p - q) (2\pi)^d \delta^{(d)}(k_3 + k_4 - p + q) \tag{3.3.17}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(-i\lambda)^2}{2} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \frac{i}{-(k_1 + k_2 + p)^2 - m^2 + i\epsilon} \\
&= \frac{(-i\lambda)^2}{2} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int \frac{d^D p}{(2\pi)^D} \left( \frac{1}{2\omega_p} \frac{i}{(k_1^0 + k_2^0 - \omega_p)^2 - \omega_{k_1+k_2+p}^2} \right. \\
&\quad \left. + \frac{i}{(\omega_{k_1+k_2+p} - k_1^0 - k_2^0)^2 - \omega_p^2} \frac{1}{2\omega_{k_1+k_2+p}} \right) \tag{3.3.18}
\end{aligned}$$

$$\sim \int \frac{d^D p}{p^3}, \tag{3.3.19}$$

同样, 积分发散.

## Chapter 4

# canonical quantization

- A. Zee: the canonical and the path integral formalisms often appear complementary, in the sense that results difficult to see in one are clear in the other.
- **nobody is perfect:**
  - **canonical quantization:** 如何定义场算符乘积的顺序.
  - **path integral:** integration measure.

### 4.1 Heisenberg and Dirac

#### 4.1.1 quantum mechanics

- 单粒子的 classical Lagrangian 为

$$L = \frac{1}{2}\dot{q}^2 - V(q) \implies \begin{cases} p = \dot{q} \\ H = p\dot{q} - L = \frac{1}{2}p^2 + V(q) \end{cases} \quad (4.1.1)$$

- canonical commutation relation 如下,

$$[p, q] = -i, \quad (4.1.2)$$

因此, 算符的演化方程为

$$\begin{cases} \frac{dp}{dt} = i[H, p] = -V'(q) \\ \frac{dq}{dt} = i[H, q] = p \end{cases} \quad (4.1.3)$$

calculation:

$$\begin{cases} [p, q] = -i \\ [p, q^2] = -2iq \\ \vdots \\ [p, q^n] = -iq^{n-1} + q[p, q^{n-1}] \end{cases} \implies [p, q^n] = -inq^{n-1} \implies [p, V(q)] = -iV'(q). \quad (4.1.4)$$

- follow Dirac's approach,

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \iff \begin{cases} q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \\ p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \end{cases} \implies [a, a^\dagger] = 1, \quad (4.1.5)$$

算符  $a$  的演化方程为

$$\frac{da}{dt} = -i\sqrt{\frac{\omega}{2}}\left(\frac{1}{\omega}V'(q) + ip\right). \quad (4.1.6)$$

### 4.1.2 scalar field

- 标量场的 Lagrangian 为

$$L = \int d^D x \left( -\frac{1}{2}((\partial\phi)^2 + m^2\phi^2) - u(\phi) \right), \quad (4.1.7)$$

canonical commutation relation 为

$$\pi(\vec{x}, t) = \frac{\delta L(t)}{\delta \partial_0 \phi(\vec{x}, t)} = \partial_0 \phi(\vec{x}, t) \quad \text{and} \quad [\pi(\vec{x}, t), \phi(\vec{y}, t)] = -i\delta^{(D)}(\vec{x} - \vec{y}), \quad (4.1.8)$$

标量场的 Hamiltonian 为

$$H = \int d^D x (\pi\phi - \mathcal{L}) = \int d^D x \left( \frac{1}{2}(\pi^2 + |\vec{\nabla}\phi|^2 + m^2\phi^2) + u(\phi) \right). \quad (4.1.9)$$

- 算符的演化方程为

$$\begin{cases} \partial_0 \phi = i[H, \phi] = \pi \\ \partial_0 \pi = i[H, \pi] = (-\vec{\nabla}^2 + m^2)\phi + \frac{du}{d\phi} \end{cases} \implies (\partial^2 - m^2)\phi - \frac{du}{d\phi} = 0. \quad (4.1.10)$$

- 当  $u(\phi) = 0$  时, 求解场方程 (4.1.10) 和 canonical commutation relation (4.1.8) 得到

$$\phi(\vec{x}, t) = \int \frac{d^D k}{(2\pi)^D 2\omega_k} (\alpha_k(t) e^{i\vec{k}\cdot\vec{x}} + \alpha_k^\dagger(t) e^{-i\vec{k}\cdot\vec{x}}), \quad (4.1.11)$$

其中

$$\alpha_k(t) = \sqrt{(2\pi)^D 2\omega_k} a_{\vec{k}} e^{-i\omega_k t} \quad \text{and} \quad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = \delta^{(D)}(\vec{p} - \vec{q}). \quad (4.1.12)$$

另外, 在后面的笔记中使用简记  $\sqrt{(2\pi)^D 2\omega_k} = \rho(k)$ .

#### calculation:

求解场方程 (4.1.10), 得到

$$\phi(\vec{x}, t) = \int \frac{d^D k}{(2\pi)^D} (\alpha_{\vec{k}} e^{i(-\omega_k t + \vec{k}\cdot\vec{x})} + \alpha_{\vec{k}}^\dagger e^{-i(-\omega_k t + \vec{k}\cdot\vec{x})}), \quad (4.1.13)$$

代入 canonical commutation relation (4.1.8), 有 (其中  $x^0 = y^0 = t, k^0 = \omega_k$ )

$$\begin{aligned} & \int \frac{d^D k_2}{(2\pi)^D} \left( -i\omega_{k_1} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] e^{i(k_1 \cdot x + k_2 \cdot y)} + i\omega_{k_1} [\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}^\dagger] e^{-i(k_1 \cdot x + k_2 \cdot y)} \right. \\ & \quad \left. - i\omega_{k_1} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] e^{i(k_1 \cdot x - k_2 \cdot y)} + i\omega_{k_1} [\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}] e^{-i(k_1 \cdot x - k_2 \cdot y)} \right) = -i e^{i\vec{k}_1 \cdot (\vec{x} - \vec{y})} \\ \implies & \begin{cases} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] = \frac{1}{2\omega_{k_1}} \delta^{(D)}(\vec{k}_1 + \vec{k}_2) \implies [\alpha_{\vec{k}}, \alpha_{\vec{k}}] \neq 0 & \text{wrong} \\ [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] = \frac{1}{2\omega_{\vec{k}_1}} \delta^{(D)}(\vec{k}_1 - \vec{k}_2) & \text{right} \end{cases}. \end{aligned} \quad (4.1.14)$$

- 代入 (4.1.9) 可得 (依然是  $u(\phi) = 0$  的情况下)

$$H = \int d^D k \omega_k \frac{a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^\dagger}{2} = \int d^D k \omega_k \left( a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2} \delta^{(D)}(0) \right) \implies \langle 0|H|0 \rangle = V \int \frac{d^D k}{(2\pi)^D} \frac{1}{2} \omega_k, \quad (4.1.15)$$

其中,  $V = \int d^D x = (2\pi)^D \delta^{(D)}(0)$ .

- vacuum state 定义为  $a_{\vec{k}} |0\rangle = 0$ , 有

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{ik \cdot (x-y)}, \quad (4.1.16)$$

其中  $k^0 = \omega_k$ . 因此, 对比 (1.2.1), 有

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = iD(x-y). \quad (4.1.17)$$

## energy-momentum tensor

- scalar field 的动量算符为

$$P^\mu = \int d^D x T^{0\mu} = \int d^D k k^\mu a_{\vec{k}}^\dagger a_{\vec{k}}, \quad (4.1.18)$$

其中, energy-momentum tensor 见 subsection D.2.3, 另外  $P^0 = H$  还有一个 vacuum energy.

## 4.2 interaction picture

- 注意, 在  $u(\phi) \neq 0$  的情况下, (即便在 Schrödinger's picture 里,  $t = 0$  时) (4.1.11) 不再成立, 因此无法通过 Schrödinger's picture or Heisenberg's picture 求解存在相互作用的场论.
- 将 Hamiltonian 分成两个部分,

$$H = H_0 + H'. \quad (4.2.1)$$

- operators 以自由场的 Hamiltonian 演化,

$$O_I(t) = U_0^\dagger(t, 0) O(0) U_0(t, 0) \quad \text{where} \quad U_0(t_2, t_1) = \text{Texp}\left(-i \int_{t_1}^{t_2} dt H_0\right), \quad (4.2.2)$$

states 以如下方式演化,

$$|\psi(t)\rangle_I = U_0^\dagger(t, 0) U(t, 0) |\psi(0)\rangle \quad \text{where} \quad U(t_2, t_1) = \text{Texp}\left(-i \int_{t_1}^{t_2} dt H\right), \quad (4.2.3)$$

因此

$$|\psi(t_2)\rangle_I = U_I(t_2, t_1) |\psi(t_1)\rangle_I \quad \text{where} \quad U_I(t_2, t_1) = \text{Texp}\left(-i \int_{t_1}^{t_2} dt H_I(t)\right), \quad (4.2.4)$$

注意, (4.2.2) 和 (4.2.3) 中, Texp 里的  $H, H_0$  都是 Schrödinger's picture 里的算符.

### calculation:

首先有

$$U_I(t_2, t_1) = U_0^\dagger(t_2, 0) U(t_2, t_1) U_0(t_1, 0), \quad (4.2.5)$$

因此

$$\begin{aligned} \frac{d}{dt} U_I(t, t_0) &= i H_0 U_I(t, t_0) - i U_0^\dagger(t, 0) H U(t, t_0) U_0(t_0, 0) \\ &= -i \underbrace{U_0^\dagger(t, 0) H' U_0(t, 0)}_{=H_I(t)} U_I(t, t_0). \end{aligned} \quad (4.2.6)$$

## 4.3 scattering amplitude

- 最一般的过程是  $p_1, \dots, p_m \rightarrow q_1, \dots, q_n$ , 其 scattering amplitude 为

$$\langle q_1, \dots, q_n | U_0^\dagger(-\infty, 0) U_I(+\infty, -\infty) U_0(-\infty, 0) | p_1, \dots, p_m \rangle, \quad (4.3.1)$$

一般会忽略掉  $U_0$  产生的相位.

- 考虑  $\phi^4$  理论中的  $k_1, k_2 \rightarrow k_3, k_4$  过程,

$$\langle k_3, k_4 | e^{-i \int d^d x \frac{\lambda}{4!} \phi^4} | k_1, k_2 \rangle, \quad (4.3.2)$$

对  $\lambda$  展开, 0 阶项为

$$\begin{aligned} \text{0th order term} &= \langle k_3, k_4 | k_1, k_2 \rangle \\ &= \rho(k_1) \rho(k_2) \rho(k_3) \rho(k_4) \langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle \end{aligned}$$

$$\begin{aligned}
&= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4) \left( \underbrace{\langle 0 | \overline{a_{\vec{k}_3}} \overline{a_{\vec{k}_4}} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle}_{=\delta_{31}^{(D)}\delta_{42}^{(D)}} + \underbrace{\langle 0 | \overline{a_{\vec{k}_3}} \overline{a_{\vec{k}_4}} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle}_{=\delta_{32}^{(D)}\delta_{41}^{(D)}} \right) \\
&= (2\pi)^{2D} 4\omega_{k_1}\omega_{k_2} (\delta^{(D)}(\vec{k}_1 - \vec{k}_3)\delta^{(D)}(\vec{k}_2 - \vec{k}_4) + \delta^{(D)}(\vec{k}_1 - \vec{k}_4)\delta^{(D)}(\vec{k}_2 - \vec{k}_3)), \quad (4.3.3)
\end{aligned}$$

1 阶项为 (其中  $k^0 = \omega_k$ )

$$\begin{aligned}
\text{1st order term} &= \frac{-i\lambda}{4!} \int d^d x \langle k_3, k_4 | \phi^4(x) | k_1, k_2 \rangle \\
&= \underbrace{\frac{-i\lambda}{4!} \int d^d x e^{i(k_1+k_2-k_3-k_4)\cdot x}}_{=-i\lambda(2\pi)^d \delta^{(d)}(k_1+k_2-k_3-k_4)} + \rho(k_1)\rho(k_4)\delta_{14}^{(D)} \times 12 \times \frac{-i\lambda}{4!} (2\pi)^d \delta_{23}^{(d)} \int \frac{d^D p}{\rho^2(p)} \\
&\quad + \cdots + \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\delta_{13}^{(D)}\delta_{24}^{(D)} \times 3 \times \frac{-i\lambda}{4!} \int d^d x \int \frac{d^D p_1}{\rho^2(p_1)} \frac{d^D p_2}{\rho^2(p_2)} + \cdots, \quad (4.3.4)
\end{aligned}$$

分别对应如下 Feynman diagrams:

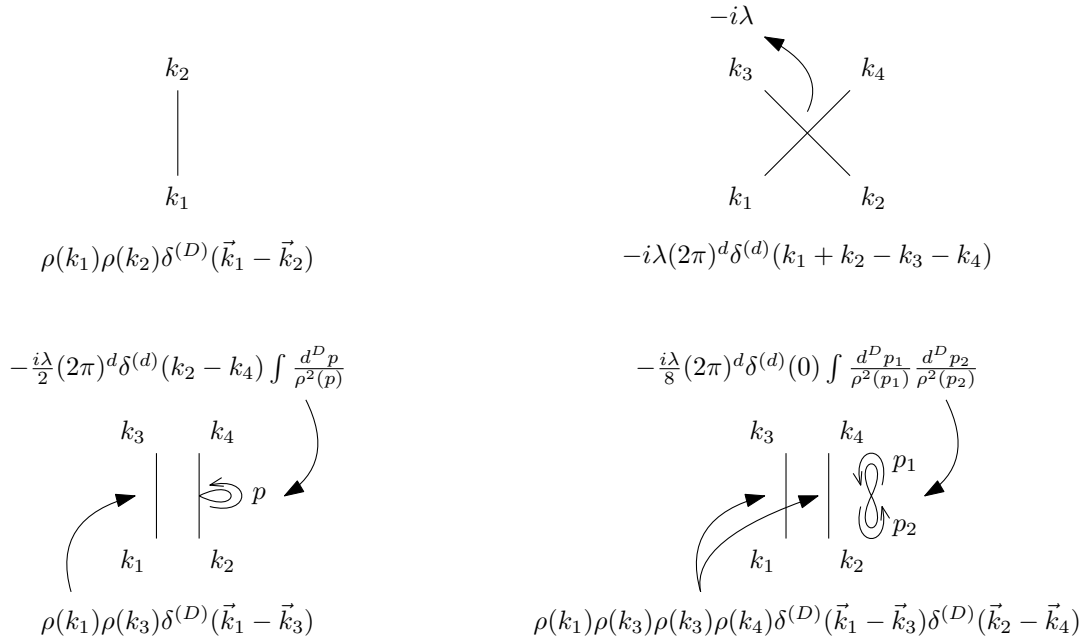


Figure 4.1: canonical quantization - Feynman diagrams

观察可见, 上图和 figure 3.3 有对应关系.

- 再举一个例子,

$$\begin{aligned}
&\text{Diagram: } k_3, k_4 \text{ incoming, } k_1, k_2 \text{ outgoing, loop with momentum } p \text{ and } k_1 + k_2 + p. \\
&= (4 \times 3)^2 \times 2 \times \left( \frac{-i\lambda}{4!} \right)^2 \rho(k_1) \cdots \int d^d x_1 d^d x_2 \int \frac{d^D p_1 \cdots d^D q_1 \cdots}{\rho(p_1) \cdots \rho(q_1) \cdots} e^{i(p_1+p_2-p_3-p_4)\cdot x_1} e^{i(q_1+q_2-q_3-q_4)\cdot x_2} \\
&\quad \left( \theta(t_2 - t_1) \langle 0 | \overline{a_{\vec{k}_3}} \overline{a_{\vec{k}_4}} a_{\vec{q}_1}^\dagger a_{\vec{q}_2}^\dagger a_{\vec{q}_3}^\dagger a_{\vec{q}_4}^\dagger a_{\vec{p}_1}^\dagger a_{\vec{p}_2}^\dagger a_{\vec{p}_3}^\dagger a_{\vec{p}_4}^\dagger a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle + \cdots \right) \\
&= \frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 \int \frac{d^D p_3}{\rho^2(p_3)} \frac{d^D p_4}{\rho^2(p_4)} \left( \theta(t_2 - t_1) e^{i(k_1+k_2-p_3-p_4)\cdot x_1} e^{i(p_3+p_4-k_3-k_4)\cdot x_2} \right. \\
&\quad \left. + \theta(t_1 - t_2) e^{i(k_1+k_2+p_3+p_4)\cdot x_1} e^{i(-p_3-p_4-k_3-k_4)\cdot x_2} \right)
\end{aligned}$$



$$\begin{aligned}
&= \frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 - (k_3+k_4)\cdot x_2)} \int \frac{d^D p_3}{\rho^2(p_3)} \frac{d^D p_4}{\rho^2(p_4)} \left( \theta(t_2 - t_1) e^{i(p_3+p_4)\cdot(x_2-x_1)} \right. \\
&\quad \left. + \theta(t_1 - t_2) e^{i(p_3+p_4)\cdot(x_1-x_2)} \right), \tag{4.3.5}
\end{aligned}$$

同样, 与 (3.3.18) 有对应关系, (注意按时间排序  $\langle k_3 k_4 | T(\phi^4(x_1) \phi^4(x_2)) | k_1 k_2 \rangle$ ).

**calculation:**

从 (3.3.17) 开始 (与 (1.2.1) 类似,  $\vec{p}, \vec{q}$  的符号可以任意改变),

$$\begin{aligned}
&\int d^d x_1 d^d x_2 e^{i(k_1+k_2+p-q)\cdot x_1} e^{i(k_3+k_4-p+q)\cdot x_2} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \frac{i}{-q^2 - m^2 + i\epsilon} \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{i e^{ip\cdot(x_1-x_2)}}{-p^2 - m^2 + i\epsilon} \frac{i e^{iq\cdot(x_2-x_1)}}{-q^2 - m^2 + i\epsilon} \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^D p}{(2\pi)^d} \frac{d^D q}{(2\pi)^d} \left( \theta(t_2 - t_1) \frac{2\pi i^2 e^{-ip\cdot(x_1-x_2)}}{-2\omega_p} \right. \\
&\quad \left. \frac{-2\pi i^2 e^{iq\cdot(x_2-x_1)}}{2\omega_q} + \theta(t_1 - t_2) \frac{-2\pi i^2 e^{ip\cdot(x_1-x_2)}}{2\omega_p} \frac{2\pi i^2 e^{-iq\cdot(x_2-x_1)}}{-2\omega_q} \right) \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^D p}{\rho^2(p)} \frac{d^D q}{\rho^2(q)} \left( \theta(t_2 - t_1) e^{i(p+q)\cdot(x_2-x_1)} \right. \\
&\quad \left. + \theta(t_1 - t_2) e^{i(p+q)\cdot(x_1-x_2)} \right), \tag{4.3.6}
\end{aligned}$$

结果与 (4.3.5) 对应.

## 4.4 complex scalar field

- complex scalar field 的 Lagrangian 为

$$\mathcal{L} = -(\partial\psi^\dagger)(\partial\psi) - m^2\psi^\dagger\psi, \tag{4.4.1}$$

实际上, complex scalar field 可以视为 2 个 real scalar fields 的和

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \implies \left| \frac{\partial\phi_1, \phi_2}{\partial\psi, \psi^\dagger} \right| = i, \tag{4.4.2}$$

因此, 也可以把  $\psi, \psi^\dagger$  视为两个独立的场.

- 其 canonical momentum 为

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi} = \partial_0\psi^\dagger, \quad \pi^\dagger = \partial_0\psi, \tag{4.4.3}$$

其 Hamiltonian 为

$$\mathcal{H} = \pi^\dagger\pi + (\vec{\nabla}\psi^\dagger) \cdot (\vec{\nabla}\psi) + m^2\psi^\dagger\psi, \tag{4.4.4}$$

$$\implies \begin{cases} \partial_0\pi = i[H, \pi] = \vec{\nabla}^2\psi^\dagger - m^2\psi^\dagger \\ \partial_0\psi = i[H, \psi] = \pi^\dagger \end{cases} \implies (-\partial^2 - m^2)\psi = 0. \tag{4.4.5}$$

- 求解得到 (其中  $k^0 = \omega_k$ )

$$\psi(x) = \int \frac{d^D k}{\rho(k)} (a_{\vec{k}} e^{ik\cdot x} + b_{\vec{k}}^\dagger e^{-ik\cdot x}). \tag{4.4.6}$$

- 从 path integral 的角度,

$$Z(J, J^\dagger) = \int D\psi D\psi^\dagger e^{i \int d^d x (\psi^\dagger(\partial^2 - m^2)\psi + J^\dagger\psi + \psi^\dagger J)} \tag{4.4.7}$$

$$= \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y 2J^\dagger(x) D(x-y) J(y)}. \tag{4.4.8}$$

calculation:

转换为  $\phi_1, \phi_2$  后计算路径积分,

$$\begin{aligned} Z(J, J^\dagger) &= \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y (J_1(x) D(x-y) J_1(y) + J_2(x) D(x-y) J_2(y))} \\ &= \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y 2J^\dagger(x) D(x-y) J(y)}. \end{aligned} \quad (4.4.9)$$

#### 4.4.1 charge

- 对场算符做如下变换,

$$\psi(x, \lambda) = e^{i\lambda} \psi(x) \implies D_\lambda \mathcal{L} = 0. \quad (4.4.10)$$

- 因此, 得到 conserved current,

$$J^\mu = \pi^\mu D_\lambda \psi + \pi^{\dagger\mu} D_\lambda \psi^\dagger = i(\psi \partial^\mu \psi^\dagger - \psi^\dagger \partial^\mu \psi), \quad (4.4.11)$$

其 0 分量对空间积分就是 charge,

$$\begin{aligned} Q &= \int d^D x J^0 = \int d^D x i(\psi^\dagger \partial_0 \psi - \psi \partial_0 \psi^\dagger) \\ &= \int d^D k (a_k^\dagger a_{\vec{k}} - b_k^\dagger b_{\vec{k}}). \end{aligned} \quad (4.4.12)$$

calculation:

$$\begin{aligned} Q &= \int d^D x \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} i \left( (a_p^\dagger e^{-ip \cdot x} + b_{\vec{p}} e^{ip \cdot x}) (-i\omega_q) (a_{\vec{q}} e^{iq \cdot x} - b_q^\dagger e^{-iq \cdot x}) \right. \\ &\quad \left. - (a_{\vec{q}} e^{iq \cdot x} + b_q^\dagger e^{-iq \cdot x}) (i\omega_p) (a_p^\dagger e^{-ip \cdot x} - b_{\vec{p}} e^{ip \cdot x}) \right) \\ &= \int d^D x \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} \left( (\omega_p a_{\vec{q}} a_p^\dagger + \omega_q a_p^\dagger a_{\vec{q}}) e^{-i(p-q) \cdot x} - (\omega_p b_q^\dagger b_{\vec{p}} + \omega_q b_{\vec{p}} b_q^\dagger) e^{i(p-q) \cdot x} \right. \\ &\quad \left. + a_p^\dagger b_{\vec{q}}^\dagger (\omega_p - \omega_q) e^{-i(p+q) \cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{i(p+q) \cdot x} \right) \\ &= \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} \left( \left( (\omega_p a_{\vec{q}} a_p^\dagger + \omega_q a_p^\dagger a_{\vec{q}}) e^{i(\omega_p - \omega_q) \cdot t} - (\omega_p b_q^\dagger b_{\vec{p}} + \omega_q b_{\vec{p}} b_q^\dagger) e^{-i(\omega_p - \omega_q) \cdot t} \right) (2\pi)^D \delta^{(D)}(\vec{p} - \vec{q}) \right. \\ &\quad \left. + (a_p^\dagger b_{\vec{q}}^\dagger (\omega_p - \omega_q) e^{i(\omega_p + \omega_q) \cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{-i(\omega_p + \omega_q) \cdot x}) (2\pi)^D \delta^{(D)}(\vec{p} + \vec{q}) \right) \\ &= \int \frac{d^D k}{2} (a_k^\dagger a_k^\dagger + a_k^\dagger a_{\vec{k}} - b_k^\dagger b_k^\dagger - b_k^\dagger b_{\vec{k}}) = \int d^D k (a_k^\dagger a_{\vec{k}} - b_k^\dagger b_{\vec{k}}). \end{aligned} \quad (4.4.13)$$

- 代入 (D.3.2), 有  $i[Q, \psi] = -i\psi$ , 所以

$$e^{-i\lambda Q} \psi e^{i\lambda Q} = e^{i\lambda} \psi. \quad (4.4.14)$$

## Chapter 5

# disturbing the vacuum: Casimir effect

- 考虑一个沿  $x^1$  方向满足 periodic b.c. 的空间, 在垂直于  $x^1$  方向有两个 plates, s.t. 在 plates 上  $\phi(x) = 0$ , 如下图:

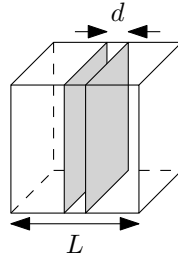


Figure 5.1: Casimir effect

- 平板内外, 标量场的波矢的取值为

$$\begin{cases} (n\frac{\pi}{d}, k_2, k_3) & \text{平板内} \\ (n\frac{\pi}{L-d}, k_2, k_3) & \text{平板外} \end{cases}, \quad (5.0.1)$$

其中  $n \in \mathbb{Z}^+$ .

- 因此, 代入真空能公式 (4.1.15), 平板内的能量为

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2}, \quad (5.0.2)$$

而总能量为  $E = E(d) + E(L-d)$ .

- 为解决能量发散的问题, 引入 ultra-violet (UV) cut-off,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2} e^{-a\sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2}}, \quad (5.0.3)$$

for some  $a \ll d$ .

- 为了简化问题, 考虑  $d = 1 + 1$  的情况,

$$E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-\frac{a\pi}{d}n} = \frac{\pi}{2d} \frac{e^{\frac{a\pi}{d}}}{(e^{\frac{a\pi}{d}} - 1)^2} = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O(a^2), \quad (5.0.4)$$

因此

$$E_{1+1} = E_{1+1}(d) + E_{1+1}(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24} \left( \frac{1}{d} + \frac{1}{L-d} \right) + O(a^2), \quad (5.0.5)$$

得到 Casimir force,

$$F_{1+1} = -\frac{\partial E_{1+1}}{\partial d} = -\frac{\pi}{24} \left( \frac{1}{d^2} - \frac{1}{(L-d)^2} \right) + O(a^2) \stackrel{L \rightarrow \infty, a \rightarrow 0}{\cong} -\frac{\pi}{24d^2}. \quad (5.0.6)$$

- 问题中,  $a$  引入了 UV cut-off,  $L$  引入了 infrared cut-off.

# Part II

## Dirac and spinor

## Chapter 6

# the Dirac spinor

- 整个 Part II 中, 我们使用  $(+, -, -, -)$  号差, 因为  $\text{Cl}_{1,3}(\mathbb{R}) \cong \text{Cl}_{3,1}(\mathbb{R})$ .
- 本笔记中的算符的定义与 A. Zee 的定义不同, 存在如下对应关系:

A. Zee's def.	my def.
$\omega_{\mu\nu}$	$\omega_{\mu\nu}$
$-iJ^{\mu\nu}$	$J^{\mu\nu}$
$-i\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$

- $\Pi(\Lambda)$  的写法可能不准确, (要考虑 universal cover,  $\text{Spin}(1,3) \simeq \text{Spin}(3,1)$ ), 因为 Lorentz transform 对 spinor 的操作是"path dependent", 因此本 chapter 中的  $\Lambda$  都默认沿着以下的 path 做变换,

$$\Lambda(\lambda) = e^{\frac{\lambda}{2}\omega_{\mu\nu}J^{\mu\nu}}, \lambda \in [0, 1]. \quad (6.0.1)$$

### 6.1 gamma matrices

- Pauli 矩阵如下,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.1.1)$$

- gamma 矩阵 (also called Dirac matrices) 如下 (其中  $i = 1, 2, 3$ ),

$$\gamma^0 = \begin{pmatrix} I & \\ & I \end{pmatrix} = I \otimes \tau_1, \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = i\sigma_i \otimes \tau_2, \quad \gamma^5 = i\Omega = \begin{pmatrix} -I & \\ & I \end{pmatrix} = -I \otimes \tau_3. \quad (6.1.2)$$

其中  $\tau_{2,3}$  也是 Pauli 矩阵,  $\Omega = \gamma^0\gamma^1\gamma^2\gamma^3$ , 有时候使用符号  $\sigma^\mu = (I, \vec{\sigma})$ ,  $\bar{\sigma}^\mu = (I, -\vec{\sigma})$ .

– 另外

$$\begin{cases} \gamma^0\gamma^i = -\sigma_i \otimes \tau_3 \\ \gamma^i\gamma^j = -(\sigma_i\sigma_j) \otimes I = -i\epsilon_{ijk}\sigma_k \otimes I \end{cases}, \quad \begin{cases} \Omega\gamma^0 = -I \otimes \tau_2 \\ \Omega\gamma^i = -\sigma_i \otimes \tau_2 \end{cases}, \quad (6.1.3)$$

其中, 用到了  $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$ .

- gamma 矩阵满足

$$\begin{cases} (\gamma^\mu)^2 = \eta^{\mu\mu} \\ \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \mu \neq \nu \end{cases} \implies \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \quad (6.1.4)$$

- 且存在如下关系,

$$\begin{aligned} \Omega\gamma^0 &= -\gamma^1\gamma^2\gamma^3, \quad \Omega\gamma^1 = -\gamma^0\gamma^2\gamma^3, \quad \Omega\gamma^2 = \gamma^0\gamma^1\gamma^3, \quad \Omega\gamma^3 = -\gamma^0\gamma^1\gamma^2, \\ \iff -\epsilon^{\mu\nu\rho}{}_\sigma \Omega\gamma^\sigma &= \gamma^\mu\gamma^\nu\gamma^\rho, \text{ when } \mu \neq \nu \neq \rho. \end{aligned} \quad (6.1.5)$$

并且有 (注意到  $\Omega^2 = -1$ ),

$$\{\Omega, \gamma^\mu\} = 0, \quad \{\Omega, \Omega\gamma^\mu\} = 0, \quad [\Omega, \gamma^\mu\gamma^\nu] = 0. \quad (6.1.6)$$

- 定义  $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$  (注意, 我们的定义中没有虚数  $i$ , 与 A. Zee 的定义不同), 因此

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} + \sigma^{\mu\nu} \implies \begin{cases} \sigma^{0i} = \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix} = -\sigma_i \otimes \tau_3 \\ \sigma^{ij} = -i\epsilon^{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk} \sigma_k \otimes I \end{cases}, \quad (6.1.7)$$

与笔记 [Lie Groups and Lie Algebras](#) 中  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$  表示对比, 可见  $\pi_{(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})}(J^{\mu\nu}) = \frac{1}{2}\sigma^{\mu\nu}$ .

### 6.1.1 gamma matrices under Dirac basis

- 做如下相似变换 ( $B = S^{-1}AS$ ),

$$S = \frac{\sqrt{2}}{2} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \iff S^{-1} = \frac{\sqrt{2}}{2} \begin{pmatrix} I & I \\ -I & I \end{pmatrix}, \quad (6.1.8)$$

得到

$$\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} = I \otimes \tau_3, \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = i\sigma_i \otimes \tau_2, \quad \gamma^5 = \begin{pmatrix} & I \\ I & \end{pmatrix} = I \otimes \tau_1. \quad (6.1.9)$$

- 另外,

$$\begin{cases} \gamma^0 \gamma^i = \sigma_i \otimes \tau_1 \\ \gamma^i \gamma^j = -i\epsilon^{ijk} \sigma_k \otimes I \end{cases}, \quad \begin{cases} \Omega \gamma^0 = -I \otimes \tau_2 \\ \Omega \gamma^i = i\sigma_i \otimes \tau_3 \end{cases}, \quad (6.1.10)$$

以及

$$\sigma^{0i} = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix} = \sigma_i \otimes \tau_1, \quad \sigma^{ij} = -i\epsilon^{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk} \sigma_k \otimes I. \quad (6.1.11)$$

## 6.2 Lorentz transformation and the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation

- Lorentz 变换可以写成如下形式,

$$\Lambda = e^{\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}}, \quad (6.2.1)$$

其中  $\omega_{\mu\nu}$  反对称,  $J^{0i}$  generate boosts and  $J^{ij}$  generate rotations, (详见笔记 [Lie Groups and Lie Algebras](#)).

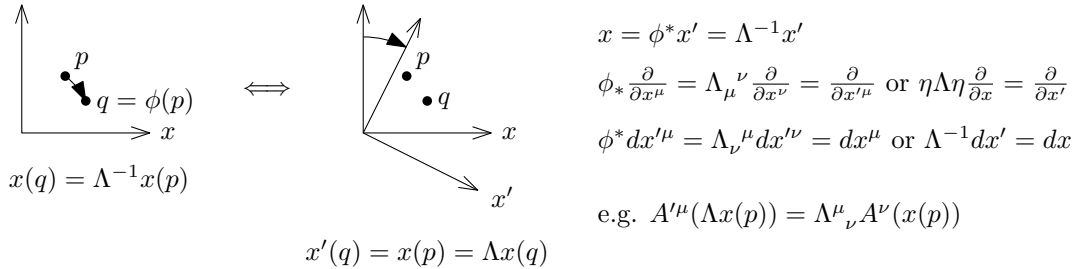


Figure 6.1: Lorentz transformation

- Weyl spinor 是  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$  rep. 的 vector space 中的元素,

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \text{with} \quad \Psi_{\text{Dirac}} = S^{-1}\Psi = \frac{\sqrt{2}}{2} \begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix}. \quad (6.2.2)$$

在 Weyl basis 下很容易看出,

$$\psi_L = \underbrace{\frac{1}{2}(1 - \gamma^5)}_{=P_L} \Psi, \quad \psi_R = \underbrace{\frac{1}{2}(1 + \gamma^5)}_{=P_R} \Psi. \quad (6.2.3)$$

- 对于 gamma 矩阵, 有

$$\Pi(\Lambda)\gamma^\rho\Pi^{-1}(\Lambda) = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = (\Lambda^{-1})^\rho_\sigma \gamma^\sigma. \quad (6.2.4)$$

**calculation:**

利用 Campbell's identity,

$$e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = e^{\frac{1}{4}\omega_{\mu\nu}\text{ad}_{\sigma^{\mu\nu}}}\gamma^\rho, \quad (6.2.5)$$

其中 (注意  $(J^{\mu\nu})^\rho{}_\sigma = 2\eta^{[\mu|\rho}\delta^{|\nu]}{}_\sigma$ , 其中度规号差与笔记 [Lie Groups and Lie Algebras](#) 中的不同)

$$\begin{cases} \rho \neq \mu, \nu & [\sigma^{\mu\nu}, \gamma^\rho] = \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu) \\ & = -\frac{1}{2}(\underbrace{\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\nu\mu\rho\sigma} - \epsilon^{\rho\mu\nu\sigma} + \epsilon^{\rho\nu\mu\sigma}}_{=0})\Omega\gamma_\sigma = 0 \\ \rho = \mu \text{ or } \nu \text{ and } \mu \neq \nu & [\sigma^{\mu\nu}, \gamma^\rho] = 2(\eta^{\mu\rho}\gamma^\mu - \eta^{\nu\rho}\gamma^\nu) \end{cases}$$

$$\Rightarrow [\sigma^{\mu\nu}, \gamma^\rho] = 2(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu) = -2(J^{\mu\nu})^\rho{}_\sigma\gamma^\sigma, \quad (6.2.6)$$

代入, 得到

$$e^{\frac{1}{4}\omega_{\mu\nu}\text{ad}_{\sigma^{\mu\nu}}}\gamma^\rho = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \left( -\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} \right)^n \right)^\rho{}_\sigma \gamma^\sigma = (\Lambda^{-1})^\rho{}_\sigma \gamma^\sigma. \quad (6.2.7)$$

可以用”无穷小” Lorentz 变换验证以上计算,

$$\begin{aligned} \Pi(1 + \delta\omega^\mu{}_\nu)\gamma^\rho\Pi^{-1}(1 + \delta\omega^\mu{}_\nu) &= \gamma^\rho + \frac{1}{4}\delta\omega_{\mu\nu}[\sigma^{\mu\nu}, \gamma^\rho] \\ &= (1 - \delta\omega^\rho{}_\sigma)\gamma^\sigma. \end{aligned} \quad (6.2.8)$$

### 6.2.1 Dirac spinor

- 对于 Dirac spinor,

$$\Pi(\Lambda)\Psi(x) = \Psi'(\Lambda x), \quad (6.2.9)$$

注意  $\partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu$ , 所以

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \iff (i\gamma^\mu\partial'_\mu - m)\Psi'(\Lambda x) = 0. \quad (6.2.10)$$

– 关键部分在于

$$\gamma^\mu\Psi'(\Lambda x) = \gamma^\mu\Pi(\Lambda)\Psi(x) = \Pi(\Lambda)\Lambda^\mu{}_\nu\gamma^\nu\Psi(x). \quad (6.2.11)$$

**calculation:**

首先

$$\Lambda^T\eta\Lambda = \eta \implies (\Lambda^{-1})^\mu{}_\nu = (\eta\Lambda^T\eta)^\mu{}_\nu = \Lambda_\nu{}^\mu, \quad (6.2.12)$$

考虑

$$\Pi^{-1}(\Lambda)\gamma^\mu\Pi(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu \implies \gamma^\mu\Pi(\Lambda) = \Lambda^\mu{}_\nu\Pi(\Lambda)\gamma^\nu, \quad (6.2.13)$$

代入,

$$\begin{aligned} (i\gamma^\mu\partial'_\mu - m)\Psi'(\Lambda x) &= (i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - m)\Pi(\Lambda)\Psi(x) \\ &= \Pi(\Lambda)(i\underbrace{\gamma^\rho\Lambda^\mu{}_\rho\Lambda_\mu{}^\nu}_{=\delta^\nu{}_\rho}\partial_\nu - m)\Psi(x) \\ &= \Pi(\Lambda)(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0. \end{aligned} \quad (6.2.14)$$

### 6.2.2 Dirac bilinears

- $\gamma^0$  是 Hermitian 矩阵, 而  $\gamma^i$  不是, 有

$$\gamma^{i\dagger} = -\gamma^i = \gamma^0\gamma^i\gamma^0, \quad (6.2.15)$$

可以统一写作  $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ , 并且有

$$\sigma^{\mu\nu\dagger} = -\gamma^0\sigma^{\mu\nu}\gamma^0, \quad \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0. \quad (6.2.16)$$

calculation:

对于  $\sigma^{\mu\nu}$ ,

$$\sigma^{\mu\nu\dagger} = \frac{1}{2}(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger}) = \gamma^0 \sigma^{\nu\mu} \gamma^0 = -\gamma^0 \sigma^{\mu\nu} \gamma^0, \quad (6.2.17)$$

所以

$$((\omega_{\mu\nu}\sigma^{\mu\nu})^\dagger)^n = \gamma^0(-\omega_{\mu\nu}\sigma^{\mu\nu})^n\gamma^0 \implies \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0. \quad (6.2.18)$$

• 所以

$$\begin{cases} \bar{\Psi}'(\Lambda x)\Psi'(\Lambda x) = \bar{\Psi}\Psi & \text{scalar field} \\ \bar{\Psi}'\gamma^\mu\Psi' = \Lambda^\mu_\nu\bar{\Psi}\gamma^\nu\Psi & \text{vector field} \end{cases}, \quad (6.2.19)$$

其中  $\bar{\Psi} = \Psi^\dagger\gamma^0$ .

calculation:

$$\begin{cases} \Psi'^\dagger(\Lambda x)\gamma^0\Psi'(\Lambda x) = \Psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\Pi(\Lambda)\Psi(x) = \Psi^\dagger\gamma^0\Psi \\ \Psi'^\dagger\gamma^0\gamma^\mu\Psi' = \Psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\gamma^\mu\Pi(\Lambda)\Psi(x) = \Lambda^\mu_\nu\Psi^\dagger\gamma^0\gamma^\nu\Psi \end{cases}. \quad (6.2.20)$$

此外

$$\begin{cases} \bar{\Psi}'\sigma^{\mu\nu}\Psi' = \Psi^\dagger\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\sigma^{\mu\nu}\Pi(\Lambda)\Psi = \Lambda^\mu_\rho\Lambda^\nu_\sigma\bar{\Psi}\sigma^{\rho\sigma}\Psi & \text{order 2 tensor} \\ \bar{\Psi}'\Omega\gamma^\mu\Psi' = \bar{\Psi}\Pi(\Lambda^{-1})\Omega\gamma^\mu\Pi(\Lambda)\Psi = \det(\Lambda)\Lambda^\mu_\nu\bar{\Psi}\Omega\gamma^\nu\Psi & \text{pseudovector} \\ \bar{\Psi}'\Omega\Psi' = \bar{\Psi}\Pi(\Lambda^{-1})\Omega\Pi(\Lambda)\Psi = \det(\Lambda)\bar{\Psi}\Omega\Psi & \text{4-form (pseudoscalar)} \end{cases}, \quad (6.2.21)$$

其中 (注意到下面的计算中, 第二个等号后, 含  $\eta$  的项都等于零; 由此可以看出, 对  $\mu_i$  求和的过程中, 任何两个  $\mu_i, \mu_j$  相等的项求和之后都等于零)

$$\begin{aligned} \Pi(\Lambda^{-1})\Omega\Pi(\Lambda) &= \prod_{i=0}^3 \Lambda^i_{\mu_i} \gamma^{\mu_0} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \\ &= \prod_{i=0}^3 \Lambda^i_{\mu_i} (\eta^{\mu_0\mu_1} + \sigma^{\mu_0\mu_1})(\eta^{\mu_2\mu_3} + \sigma^{\mu_2\mu_3}) \\ &= \prod_{i=0}^3 \Lambda^i_{\mu_i} \gamma^{\mu_0} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \quad \text{with } \mu_0 \neq \mu_1 \neq \mu_2 \neq \mu_3 \\ &= \det(\Lambda)\Omega. \end{aligned} \quad (6.2.22)$$

### 6.2.3 parity and time reversal

- 这里沿用笔记 [Lie Groups and Lie Algebras](#) 中的记号, 选择  $O(3,1)$  而非  $O(1,3)$ , 因为他们没有区别.
- $O(3,1)$  有 4 个联通分支,

$$I \in SO_+(3,1), \quad PT \in SO_-(3,1), \quad P \in O'_+(3,1), \quad T \in O'_-(3,1), \quad (6.2.23)$$

其中

$$P = \text{diag}(+1, -1, -1, -1), \quad T = \text{diag}(-1, +1, +1, +1), \quad (6.2.24)$$

另外,  $\eta P \eta = P, \eta T \eta = T$ .

- 另外, Lorentz algebra 的 representation 不能自然的生成对  $P, T$  的表示, 因为本质上它只能生成 spin group 的表示, 是  $SO_+(3,1)$  的 universal cover, 与 Lorentz group 的其它三个连通分支没有直接联系.
- 因此, 对  $P, T$  的表示要从物理的角度定义, (可能) 无法单纯靠数学的方法给出, 所以这部分放在下一章.



# Chapter 7

## the Dirac equation

### 7.1 Dirac equation

- A. Zee: our discussion provides a unified view of the equations of motion in relativistic physics: they just project out the unphysical components.
- the Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 \iff (\gamma^\mu p_\mu - m)\tilde{\Psi} = 0 \implies \begin{cases} i\sigma^\mu \partial_\mu \psi_R - m\psi_L = 0 \\ i\bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_R = 0 \end{cases}. \quad (7.1.1)$$

首先可以看出  $\Psi$  满足 Klein-Gordon equation,

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\Psi &= \left(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu - 2im\gamma^\mu \partial_\mu + m^2\right)\Psi = 0 \\ \implies (-\partial^2 - m^2)\Psi &= 0. \end{aligned} \quad (7.1.2)$$

– 在粒子静止系下  $p_\mu = (m, 0, 0, 0)$ , Dirac 方程给出 (这里采用 Dirac basis)

$$(\gamma^0 - 1)\tilde{\Psi}_{\text{Dirac}} = 0 \implies \begin{pmatrix} 0 & \\ & I \end{pmatrix} \tilde{\Psi}_{\text{Dirac}} = 0. \quad (7.1.3)$$

因此,  $\tilde{\Psi}$  的后两个分量为零  $\implies \Psi$  只有两个自由度.

- Dirac 方程的 Lorentz covariance 见 (6.2.10).

### 7.2 Dirac Lagrangian

- 根据 (6.2.19) 以及之前标量场的计算经验, 可知

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = (-i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi})\Psi + \text{total diff.}, \quad (7.2.1)$$

其中, 与复标量场论中类似, 可以把  $\Psi, \Psi^\dagger$  或  $\Psi, \bar{\Psi}$  视为独立变量.

### 7.3 chirality or handedness

- parity transformation 会把 left spinor 变成 right spinor and vice versa,

$$\gamma^0 \Psi_L = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix}, \quad \gamma^0 \Psi_R = \begin{pmatrix} \psi_R \\ 0 \end{pmatrix}. \quad (7.3.1)$$

- 把 Lagrangian 中的  $\Psi$  拆开,

$$\begin{aligned} \mathcal{L} &= \bar{\Psi}_L(i\cancel{\partial})\Psi_L + \bar{\Psi}_R(i\cancel{\partial})\Psi_R - m(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L) \\ &= \psi_L^\dagger i\bar{\sigma}^\mu \partial_\mu \psi_L + \psi_R^\dagger i\sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L), \end{aligned} \quad (7.3.2)$$

其中注意到了  $\gamma^0 \gamma^\mu$  的非对角分块为零.

### 7.3.1 internal vector symmetry

- 做变换  $\Psi \mapsto e^{i\theta}\Psi$ , Lagrangian 保持不变, 利用 Noether's theorem 得到守恒流 (见 section D.2),

$$J_V^\mu = \bar{\Psi}\gamma^\mu\Psi, \quad (7.3.3)$$

其中, 按照惯例省略了虚数  $i$ .

calculation:

计算广义动量,

$$\begin{cases} \pi_\Psi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\Psi} = \bar{\Psi}i\gamma^\mu \\ \pi_{\bar{\Psi}}^\mu = 0 \end{cases} \quad \text{or} \quad \begin{cases} \pi_\Psi^\mu = 0 \\ \pi_{\bar{\Psi}}^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}} = -i\gamma^\mu\Psi \end{cases}. \quad (7.3.4)$$

这里看起来有点奇怪 (canonical transformation), 需要再说明一下. 对于 (7.2.1) 第一个等号后边,

$$\begin{cases} \pi_\Psi^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\Psi} = \bar{\Psi}i\gamma^\mu & \frac{\delta\mathcal{L}}{\delta\Psi} = -m\bar{\Psi} \\ \pi_{\bar{\Psi}}^\mu = 0 & \frac{\delta\mathcal{L}}{\delta\bar{\Psi}} = (i\gamma^\mu\partial_\mu - m)\Psi \end{cases} \implies \begin{cases} -(\partial_\mu\bar{\Psi})i\gamma^\mu - m\bar{\Psi} = 0 \\ (i\gamma^\mu\partial_\mu - m)\Psi = 0 \end{cases}, \quad (7.3.5)$$

对于 (7.2.1) 第二个等号后边, 忽略掉全微分项,

$$\begin{cases} \pi_\Psi^\mu = 0 & \frac{\delta\mathcal{L}}{\delta\Psi} = -i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} \\ \pi_{\bar{\Psi}}^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}} = -i\gamma^\mu\Psi & \frac{\delta\mathcal{L}}{\delta\bar{\Psi}} = -m\Psi \end{cases} \implies \begin{cases} -i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \\ (i\gamma^\mu\partial_\mu - m)\Psi = 0 \end{cases}. \quad (7.3.6)$$

### 7.3.2 axial symmetry

- 做变换

$$\Psi \mapsto e^{i\theta\gamma^5}\Psi = \begin{pmatrix} e^{-i\theta}\Psi_L \\ e^{i\theta}\Psi_R \end{pmatrix}, \quad (7.3.7)$$

在  $m = 0$  时 Lagrangian 保持不变, 对应的守恒流为

$$J_A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi, \quad (7.3.8)$$

根据 (6.2.21), 是一个 pseudovector.

## 7.4 energy-momentum tensor and angular momentum

- Dirac 场的 energy-momentum tensor 为

$$T_{\mu\nu} = i\bar{\Psi}\gamma_\mu\partial_\nu\Psi - \eta_{\mu\nu}\mathcal{L}, \quad (7.4.1)$$

其中, 对于满足运动方程的 Dirac 场,  $\mathcal{L} = 0$ .

- Dirac 场的 angular momentum 为

$$M^{\mu\nu\rho} = \frac{i}{2}\bar{\Psi}\gamma^\mu\sigma^{\nu\rho}\Psi(x) + (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}). \quad (7.4.2)$$

calculation:

做变换  $x \mapsto e^{\frac{1}{2}\lambda\omega_{\mu\nu}J^{\mu\nu}}x$ , 那么

$$\begin{aligned} \Psi(x) &\mapsto \Psi'(x') = e^{\frac{1}{4}\lambda\omega_{\mu\nu}\sigma^{\mu\nu}}\Psi(x) \\ \implies D_\lambda\Psi'(\mathbf{x}) &= \frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\Psi(x) - \frac{1}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma\partial_\rho\Psi(x), \end{aligned} \quad (7.4.3)$$

所以

$$J^\mu = \frac{i}{4} \omega_{\nu\rho} \bar{\Psi} \gamma^\mu \sigma^{\nu\rho} \Psi(x) + \dots \implies M^{\mu\nu\rho} = \frac{i}{2} \bar{\Psi} \gamma^\mu \sigma^{\nu\rho} \Psi(x) + (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}). \quad (7.4.4)$$

## 7.5 charge conjugation, parity and time reversal

- 沿用 A. Zee 的 notation, 变换映射分别用  $\mathcal{C}, \mathcal{P}, \mathcal{T}$  表示, 相应的矩阵用  $C, P, T$  表示.

### 7.5.1 charge conjugation and antimatter

- 定义矩阵  $C$ ,

$$C = -\gamma^0 \gamma^2 \implies C \gamma^0 = -i \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} = \gamma^2 \implies \begin{cases} (\gamma^2)^{-1} \gamma^\mu \gamma^2 = -\gamma^{\mu*} \\ C^{-1} \gamma^\mu C = -(\gamma^\mu)^T \end{cases}, \quad (7.5.1)$$

因此  $-\gamma^{\mu*}$  同样满足 Clifford algebra.

– 另外, 有  $(\gamma^2)^{-1} = \gamma^{2*} = -\gamma^2$  和  $C^{-1} = C$ .

calculation:

$$\gamma^0 C^{-1} \gamma^0 C \gamma^0 = -\gamma^{\mu*} \implies C^{-1} \gamma^0 C = -\gamma^0 \gamma^{\mu*} \gamma^0 = -\underbrace{(\gamma^0 \gamma^\mu \gamma^0)^*}_{=\gamma^{\mu\dagger}}, \quad (7.5.2)$$

其中用到了  $\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger}$ , 见 (6.2.15).

- $\Psi_c = \gamma^2 \Psi^*$  满足如下方程,

$$(-i\gamma^{\mu*}(\partial_\mu - ieA_\mu) - m)\Psi^* = 0 \implies (\gamma^2)^{-1}(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\Psi_c = 0, \quad (7.5.3)$$

可见  $\Psi_c$  满足变换  $-e \mapsto +e$  后的 Dirac 方程,  $\Psi_c$  is the field of positron.

- 对于 Lorentz 变换,  $e^{\frac{1}{2}\lambda\omega_{\mu\nu}J^{\mu\nu}}, \lambda \in [0, 1]$ , 有

$$\begin{cases} \Psi \mapsto \Psi'(x') = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \Psi \\ \Psi_c \mapsto \gamma^2 \underbrace{(\gamma^2)^{-1} e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \gamma^2 \Psi^*}_{=(\Psi'(x'))^*} = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \Psi_c \end{cases}, \quad (7.5.4)$$

可见  $\Psi_c$  与  $\Psi$  的变换形式相同.

### 7.5.2 parity

- 对于 parity, 有  $x \rightarrow x' = (x^0, -\vec{x})$ , 在 Dirac eq. 中

$$\gamma^0 \gamma^\mu = P^\mu_\nu \gamma^\nu \gamma^0 \implies (i\gamma^\mu \partial'_\mu - m)\gamma^0 \Psi(x) = 0, \quad (7.5.5)$$

因此

$$\mathcal{P} : \Psi(x) \mapsto \Psi'(x') = \gamma^0 \Psi(x). \quad (7.5.6)$$

### 7.5.3 time reversal

- 时间反演算符为

$$T = (i\sigma_2 \otimes I)K = \gamma^1 \gamma^3 K, \quad (7.5.7)$$

其中  $K$  是 complex conjugation operator (见 appendix E). 另外, 有  $T^2 = -1$ , 符合预期.

**proof:**

时间反演之后,  $\Psi'(t') = T\Psi(t)$  满足如下方程,

$$i\frac{\partial}{\partial t'}\Psi'(t') = H\Psi'(x'), \quad (7.5.8)$$

其中

$$H = -i\gamma^0\gamma^i\frac{\partial}{\partial x^i} + \gamma^0m, \quad (7.5.9)$$

且 Hamiltonian 满足时间反演不变,  $H'(t') \equiv TH(t)T^\dagger = H(t)$ , 即 (其中  $T = UK$ )

$$\begin{cases} T(i\gamma^0\gamma^i)T^\dagger = i\gamma^0\gamma^i \\ T\gamma^0T^\dagger = \gamma^0 \end{cases} \implies \begin{cases} U(-i\gamma^0\gamma^{i*})U^\dagger = U(-i\gamma^0\gamma^2\gamma^i\gamma^2)U^\dagger = i\gamma^0\gamma^i \\ [U, \gamma^0] = 0 \end{cases}, \quad (7.5.10)$$

满足以上要求的  $U$  具有以下形式,

$$U = \begin{pmatrix} a\sigma_2 & b\sigma_2 \\ b\sigma_2 & a\sigma_2 \end{pmatrix}, \quad \text{with} \quad \begin{cases} |a|^2 + |b|^2 = 1 \\ a^*b + b^*a = 0 \end{cases}, \quad (7.5.11)$$

不妨令  $a = i, b = 0$ .

#### 7.5.4 CPT theorem

- 在 CPT 变换下

$$\mathcal{CPT} : \Psi(x) \mapsto \gamma^1\gamma^3K(\gamma^0\gamma^2\Psi^*) = \Omega\Psi = -i\gamma^5\Psi. \quad (7.5.12)$$

- 任何 Lorentz covariant theory 都满足 CPT 不变性.

#### 7.6 interaction in QED

- 注意, 我们采用通常的符号  $e > 0$ , 与 A. Zee 的符号  $e = -|e|$  不同.
- QED 的 Lagrangian 为

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu, \quad (7.6.1)$$

其中

$$D_\mu = \partial_\mu + ieA_\mu, \quad (7.6.2)$$

可见电子和电磁场耦合项为  $-eA_\mu J_V^\mu$ , 其中  $J_V^\mu$  是 internal vector symmetry 的守恒流, 见 (7.3.3).

- QED 里的 Dirac 方程为

$$(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\Psi = 0 \quad \text{and} \quad -i(\partial_\mu - ieA_\mu)\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0. \quad (7.6.3)$$

#### 7.7 Majorana neutrino

- 因为在 Lorentz 变换下,  $\Psi, \Psi_c$  行为相同, 因此 Majorana 方程同样满足 Lorentz covariance,

$$i\not{\partial}\Psi - m\Psi_c = 0 \quad \text{and} \quad i\not{\partial}\Psi_c - m\Psi = 0, \quad (7.7.1)$$

因此

$$(-\partial^2 - m^2)\Psi = 0, \quad (7.7.2)$$

满足 Klein-Gordon 方程.

**calculation:**

$$-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\Psi = m(i\not{\partial})\Psi_c = m^2\Psi. \quad (7.7.3)$$

- Majorana 方程对应的 Lagrangian 为

$$\mathcal{L} = \bar{\Psi} i \not{\partial} \Psi - \frac{1}{2} m (\Psi^T C \Psi + \bar{\Psi} C \bar{\Psi}^T), \quad (7.7.4)$$

相应的广义动量为

$$\begin{cases} \pi_{\Psi}^{\mu} = \bar{\Psi} i \gamma^{\mu} & \frac{\delta \mathcal{L}}{\delta \bar{\Psi}} = -m \Psi^T C \\ \pi_{\bar{\Psi}}^{\mu} = 0 & \frac{\delta \mathcal{L}}{\delta \Psi} = i \not{\partial} \Psi - m C \bar{\Psi}^T = i \not{\partial} \Psi - m \Psi_c \end{cases}. \quad (7.7.5)$$

– 注意,  $\Psi$  应该被当作 Grassmann numbers, 因此, 对于反对称矩阵  $C$ , 有  $\Psi^T C \Psi, \bar{\Psi} C \bar{\Psi}^T \neq 0$ .

**calculation:**

对  $\Psi$  变分得到

$$\begin{aligned} 0 &= \frac{\delta \mathcal{L}}{\delta \bar{\Psi}} - \partial_{\mu} \pi_{\Psi}^{\mu} \\ &= -m \Psi^T C - i \partial_{\mu} \bar{\Psi} \gamma^{\mu} \\ &= (-m \Psi^T - i \partial_{\mu} \bar{\Psi} \gamma^{\mu} C) C \\ &= (-m \Psi + i C (\gamma^{\mu})^T \gamma^0 \partial_{\mu} \Psi^*)^T C, \end{aligned} \quad (7.7.6)$$

其中

$$C(\gamma^{\mu})^T \gamma^0 = C(-C^{-1} \gamma^{\mu} C) \gamma^0 = -\gamma^{\mu} C \gamma^0 = -\gamma^{\mu} \gamma^2, \quad (7.7.7)$$

代入, 得到 (?),

$$-i \not{\partial} \Psi_c - m \Psi = 0. \quad (7.7.8)$$

- Majorana eq. v.s. Dirac eq.:

- Majorana eq. 只适用于 electrically neutral fields (?).
- Majorana eq. preserves handedness (?).

## Chapter 8

# quantizing the Dirac field

### 8.1 anticommutation

- 用  $\alpha, \beta$  表示电子的量子态 (包括动量和自旋), 那么

$$\{b_\alpha, b_\beta\} = 0 \quad \{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}. \quad (8.1.1)$$

**comment:**

反对称关系  $\{b_\alpha, b_\beta\} = 0$  由实验发现, 我们希望电子有 number operator,

$$N = \sum_\alpha b_\alpha^\dagger b_\alpha, \quad \text{with} \quad \begin{cases} [N, b_\alpha] = -b_\alpha \\ [N, b_\alpha^\dagger] = b_\alpha^\dagger \end{cases}, \quad (8.1.2)$$

考虑到  $[AB, C] = ABC - CAB = A\{B, C\} - \{A, C\}B$ , 所以

$$\begin{cases} [N, b_\alpha] = \sum_\beta (b_\beta^\dagger \{b_\beta, b_\alpha\} - \{b_\beta^\dagger, b_\alpha\} b_\beta) = -\sum_\beta \{b_\beta^\dagger, b_\alpha\} b_\beta \\ [N, b_\alpha^\dagger] = \sum_\beta (b_\beta^\dagger \{b_\beta, b_\alpha^\dagger\} - \{b_\beta^\dagger, b_\alpha^\dagger\} b_\beta) = \sum_\beta b_\beta^\dagger \{b_\beta, b_\alpha^\dagger\} \end{cases}, \quad (8.1.3)$$

可见  $\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}$ .

### 8.2 plane wave solutions

- Dirac 方程的平面波解具有如下形式 (其中  $p^0 = \omega_p$ ),

$$\Psi = u(\vec{p})e^{-ip \cdot x} \quad \text{and} \quad \Psi = v(\vec{p})e^{ip \cdot x}, \quad (8.2.1)$$

代入 Dirac 方程, 得到

$$(\not{p} - m)u(\vec{p}) = 0 \quad \text{and} \quad (-\not{p} - m)v(\vec{p}) = 0, \quad (8.2.2)$$

解为

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix}, \quad v = \begin{pmatrix} -\sqrt{p \cdot \sigma} \chi \\ \sqrt{p \cdot \bar{\sigma}} \chi \end{pmatrix}, \quad (8.2.3)$$

其中  $\xi, \chi$  为任意 2-dim 列向量, 因此  $u(\vec{p}), v(\vec{p})$  各有两个独立解, 分别用  $u(\vec{p}, s), v(\vec{p}, s), s = \pm 1$  表示.

**proof:**

令  $u^T = (u_1, u_2)$  代入,

$$\begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \implies \begin{cases} p \cdot \sigma u_2 = m u_1 \\ p \cdot \bar{\sigma} u_1 = m u_2 \end{cases}, \quad (8.2.4)$$

注意到

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = \omega_p^2 - p^i p^j \sigma_i \sigma_j = \omega_p^2 - |\vec{p}|^2 = m^2, \quad (8.2.5)$$

所以, 令  $u_2 = m\xi'$ , 那么

$$u = \begin{pmatrix} p \cdot \sigma \xi' \\ m \xi' \end{pmatrix} \Rightarrow \xi = \sqrt{p \cdot \sigma} \xi' \Rightarrow \dots, \quad (8.2.6)$$

其中,  $\xi$  可以任意选取, 并且注意到了  $[(p \cdot \sigma), (p \cdot \bar{\sigma})] = 0$ , 因此

$$\sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} = \sqrt{(p \cdot \sigma)(p \cdot \bar{\sigma})} = m. \quad (8.2.7)$$

类似地, 对于  $v^T = (v_1, v_2)$ , 代入,

$$\begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} p \cdot \sigma v_2 = -m v_1 \\ p \cdot \bar{\sigma} v_1 = -m v_2 \end{cases}, \quad (8.2.8)$$

令  $v_1 = -\sqrt{p \cdot \sigma} \chi$ , 那么...

最后,

$$\begin{cases} \sqrt{p \cdot \sigma} = \sqrt{\frac{m + \omega_p}{2}} I + \frac{1}{\sqrt{2(m + \omega_p)}} \vec{p} \cdot \vec{\sigma} \\ \sqrt{p \cdot \bar{\sigma}} = \sqrt{\frac{m + \omega_p}{2}} I - \frac{1}{\sqrt{2(m + \omega_p)}} \vec{p} \cdot \vec{\sigma} \end{cases}, \quad (8.2.9)$$

以及一些有用的公式,

$$\begin{cases} \sqrt{p \cdot \sigma} \sigma^\mu \sqrt{p \cdot \sigma} = \begin{cases} \omega_p + \vec{p} \cdot \vec{\sigma} & \mu = 0 \\ \omega_p \sigma^i + p^i + \frac{\vec{p} \cdot \vec{\sigma}}{2(m + \omega_p)} 2i\epsilon_{ijk} p^j \sigma^k & \mu = i \end{cases} \\ \sqrt{p \cdot \bar{\sigma}} \sigma^\mu \sqrt{p \cdot \bar{\sigma}} = \begin{cases} \omega_p - \vec{p} \cdot \vec{\sigma} & \mu = 0 \\ \omega_p \sigma^i - p^i + \frac{\vec{p} \cdot \vec{\sigma}}{2(m + \omega_p)} 2i\epsilon_{ijk} p^j \sigma^k & \mu = i \end{cases} \\ \sqrt{p \cdot \sigma} \sigma^\mu \sqrt{p \cdot \bar{\sigma}} = \begin{cases} m & \mu = 0 \\ m \sigma^i - \frac{\sqrt{p \cdot \sigma}}{\sqrt{2(m + \omega_p)}} 2i\epsilon_{ijk} p^j \sigma^k & \mu = i \end{cases} \\ \sqrt{p \cdot \bar{\sigma}} \sigma^\mu \sqrt{p \cdot \sigma} = \begin{cases} m & \mu = 0 \\ m \sigma^i + \frac{\sqrt{p \cdot \bar{\sigma}}}{\sqrt{2(m + \omega_p)}} 2i\epsilon_{ijk} p^j \sigma^k & \mu = i \end{cases} \end{cases}, \quad (8.2.10)$$

另外  $(-p) \cdot \sigma = p \cdot \bar{\sigma}, (-p) \cdot \bar{\sigma} = p \cdot \sigma$ , 其中  $(-p) = (\omega_p, -\vec{p})$ .

- 选择归一化条件

$$\begin{cases} \bar{u}(\vec{p}, s) u(\vec{p}, s') = 2m \delta_{ss'} \\ \bar{v}(\vec{p}, s) v(\vec{p}, s') = -2m \delta_{ss'} \end{cases} \quad \text{and} \quad \bar{u}(\vec{p}, s) v(\vec{p}, s') = 0, \quad (8.2.11)$$

其中  $\bar{u} = u^\dagger \gamma^0, \bar{v} = v^\dagger \gamma^0$ , 那么

$$\begin{cases} \xi^{s\dagger} \xi^{s'} = \delta_{ss'} \\ \chi^{s\dagger} \chi^{s'} = \delta_{ss'} \end{cases} \quad \text{and} \quad \xi^{s\dagger} \chi^{s'} - \chi^{s\dagger} \xi^{s'} = 0, \quad (8.2.12)$$

可以选取

$$\xi^{+1} = \chi^{+1} = (1, 0)^T, \quad \xi^{-1} = \chi^{-1} = (0, 1)^T. \quad (8.2.13)$$

– 在粒子静止系下,  $p_r = (m, 0, 0, 0)$ ,

$$\frac{u(\vec{p}_r, +1)}{\sqrt{m}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{u(\vec{p}_r, -1)}{\sqrt{m}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{v(\vec{p}_r, +1)}{\sqrt{m}} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{v(\vec{p}_r, -1)}{\sqrt{m}} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad (8.2.14)$$

可见  $s = \pm 1$  分别代表 spin-up 和 spin-down.

– 另外, 我们注意到 (对  $v$  同样适用)

$$\begin{pmatrix} \omega_p \\ \vec{p} \end{pmatrix} = e^{\lambda J^{01}} \begin{pmatrix} m \\ 0 \end{pmatrix} \iff u(\vec{p}, s) = e^{\frac{1}{2}\lambda\sigma^{01}} u(\vec{p}_r, s), \quad \text{with } \frac{p_1}{m} = \sinh \lambda, p_2 = p_3 = 0. \quad (8.2.15)$$

• 最后,

$$\begin{cases} \sum_{s=\pm 1} u(\vec{p}, s) \bar{u}(\vec{p}, s) = \not{p} + m \\ \sum_{s=\pm 1} v(\vec{p}, s) \bar{v}(\vec{p}, s) = \not{p} - m \end{cases}. \quad (8.2.16)$$

**calculation:**

首先,

$$u(\vec{p}, s) u^\dagger(\vec{p}, s) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} (\xi^{s\dagger} \sqrt{p \cdot \sigma} \quad \xi^{s\dagger} \sqrt{p \cdot \bar{\sigma}}), \quad (8.2.17)$$

注意到

$$\sum_{s=\pm 1} \xi^s \xi^{s\dagger} = I_{2 \times 2}, \quad (8.2.18)$$

代入,

$$\sum_{s=\pm 1} u(\vec{p}, s) u^\dagger(\vec{p}, s) = \begin{pmatrix} p \cdot \sigma & m \\ m & p \cdot \bar{\sigma} \end{pmatrix} = (\not{p} + m) \gamma^0. \quad (8.2.19)$$

类似地,

$$\begin{aligned} \sum_{s=\pm 1} v(\vec{p}, s) v^\dagger(\vec{p}, s) &= \sum_{s=\pm 1} \begin{pmatrix} \sqrt{p \cdot \sigma} \chi^s \\ -\sqrt{p \cdot \bar{\sigma}} \chi^s \end{pmatrix} (\chi^{s\dagger} \sqrt{p \cdot \sigma} \quad -\chi^{s\dagger} \sqrt{p \cdot \bar{\sigma}}) \\ &= \begin{pmatrix} p \cdot \sigma & -m \\ -m & p \cdot \bar{\sigma} \end{pmatrix} = (\not{p} - m) \gamma^0. \end{aligned} \quad (8.2.20)$$

### 8.3 the Dirac field

•  $\Psi(x), \bar{\Psi}$  有如下形式,

$$\begin{cases} \Psi(x) = \sum_{s=\pm 1} \int \frac{d^3 p}{(2\pi)^{3/2} \sqrt{2\omega_p}} (b_p^s u(\vec{p}, s) e^{-ip \cdot x} + c_p^{s\dagger} v(\vec{p}, s) e^{ip \cdot x}) \\ \bar{\Psi}(x) = \sum_{s=\pm 1} \int \frac{d^3 p}{(2\pi)^{3/2} \sqrt{2\omega_p}} (b_p^{s\dagger} \bar{u}(\vec{p}, s) e^{ip \cdot x} + c_p^s \bar{v}(\vec{p}, s) e^{-ip \cdot x}) \end{cases}. \quad (8.3.1)$$

• 回顾 section 4.4 关于 complex scalar field 的内容, 可知  $b^\dagger$  和  $c^\dagger$  产生的粒子具有相反的电荷, 不妨令  $b^\dagger$  产生 electron (带电荷  $-e$ ),  $c^\dagger$  产生 positron (带电荷  $e$ ).

• section 8.1 中的讨论说明

$$\begin{cases} \{b_p^s, b_{p'}^{s'}\} = 0 \\ \{b_p^s, b_{p'}^{s'\dagger}\} = \delta^{(3)}(\vec{p} - \vec{p}') \delta_{ss'} \end{cases}. \quad (8.3.2)$$

•  $\Psi$  的 momentum conjecture 为 ( $\pi_\Psi^\mu$  见 (7.3.4))

$$\pi_\Psi = \frac{\delta \mathcal{L}}{\delta \partial_0 \Psi} = \pi_\Psi^0 = \bar{\Psi} i \gamma^0 = i \Psi^\dagger, \quad (8.3.3)$$

存在如下 anticommutation relation,

$$\{\Psi_\alpha(t, \vec{x}), i \Psi_\beta^\dagger(t, \vec{y})\} = i \delta^{(3)}(\vec{x} - \vec{y}) \delta_{\alpha\beta}. \quad (8.3.4)$$



calculation:

代入 (8.3.2), (下式中  $x = (t, \vec{x}), y = (t, \vec{y})$ , 另外注意到  $u\bar{u} = uu^\dagger\gamma^0$ ),

$$\begin{aligned}
\{\Psi_\alpha(t, \vec{x}), \Psi_\beta^\dagger(t, \vec{y})\} &= \sum_{s=\pm} \int \frac{d^3p_1 d^3p_2}{(2\pi)^3 \sqrt{4\omega_{p_1}\omega_{p_2}}} \left( \{b_{\vec{p}_1}^s, b_{\vec{p}_2}^{s\dagger}\} u(\vec{p}_1, s) u^\dagger(\vec{p}_2, s) e^{i(-p_1 \cdot x + p_2 \cdot y)} \right. \\
&\quad \left. + \{c_{\vec{p}_1}^{s\dagger}, c_{\vec{p}_2}^s\} v(\vec{p}_1, s) v^\dagger(\vec{p}_2, s) e^{i(p_1 \cdot x - p_2 \cdot y)} \right) \\
&= \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left( u(\vec{p}, s) u^\dagger(\vec{p}, s) e^{ip \cdot (-x+y)} + v(\vec{p}, s) v^\dagger(\vec{p}, s) e^{ip \cdot (x-y)} \right) \\
&= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left( (\not{p} + m) \gamma^0 e^{i\vec{p} \cdot (\vec{x} - \vec{y})} + (\not{p} - m) \gamma^0 e^{-i\vec{p} \cdot (\vec{x} - \vec{y})} \right) \\
&= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left( 2\omega_p I \cos(\vec{p} \cdot (\vec{x} - \vec{y})) - 2p^i \gamma^i \gamma^0 \cos(\vec{p} \cdot (\vec{x} - \vec{y})) \right. \\
&\quad \left. + 2im\gamma^0 \sin(\vec{p} \cdot (\vec{x} - \vec{y})) \right), \tag{8.3.5}
\end{aligned}$$

注意, 只有第一项是偶函数, 积分后不为零,

$$\begin{aligned}
\{\Psi_\alpha(t, \vec{x}), \Psi_\beta^\dagger(t, \vec{y})\} &= \int \frac{d^3p}{(2\pi)^3} I \cos(\vec{p} \cdot (\vec{x} - \vec{y})) \\
&= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} = \delta^{(3)}(\vec{x} - \vec{y}) I. \tag{8.3.6}
\end{aligned}$$

- 另外, 显然有

$$\{\Psi(x), \Psi(y)\} = \{\Psi^\dagger(x), \Psi^\dagger(y)\} = 0. \tag{8.3.7}$$

## 8.4 Hamiltonian, energy-momentum tensor and angular momentum

### 8.4.1 Hamiltonian

- 计算 Hamiltonian,

$$H = \sum_{s=\pm 1} \int d^3p \omega_p (b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s - c_{\vec{p}}^s c_{\vec{p}}^{s\dagger}) = \sum_{s=\pm 1} \int d^3p \omega_p (b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s + c_{\vec{p}}^{s\dagger} c_{\vec{p}}^s) + E_0, \tag{8.4.1}$$

其中 vacuum energy,

$$E_0 = -2\delta^{(3)}(0) \int d^3p \omega_p = -2V \int \frac{d^3p}{(2\pi)^3} \omega_p, \tag{8.4.2}$$

的符号与标量场的正好相反.

calculation:

the Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= i\Psi^\dagger \partial_0 \Psi - \mathcal{L} = -\bar{\Psi} (i\gamma^i \partial_i - m) \Psi \\
&= \sum_{s_1, s_2 = \pm 1} \int \frac{d^3p_1 d^3p_2}{(2\pi)^3 \sqrt{4\omega_{p_1}\omega_{p_2}}} (b_{\vec{p}_1}^{s_1\dagger} \bar{u}(\vec{p}_1, s_1) e^{ip_1 \cdot x} + c_{\vec{p}_1}^{s_1\dagger} \bar{v}(\vec{p}_1, s_1) e^{-ip_1 \cdot x}) \\
&\quad \left( \underbrace{((\gamma^i p_2^i + m) b_{\vec{p}_2}^{s_2} u(\vec{p}_2, s_2) e^{-ip_2 \cdot x})}_{\mapsto \omega_{p_2} \gamma^0} + \underbrace{(-\gamma^i p_2^i + m) c_{\vec{p}_2}^{s_2\dagger} v(\vec{p}_2, s_2) e^{ip_2 \cdot x}}_{\mapsto -\omega_{p_2} \gamma^0} \right), \tag{8.4.3}
\end{aligned}$$

代入,

$$\begin{aligned}
H &= \int d^3x \mathcal{H} = \sum_{s_1, s_2 = \pm 1} \int \frac{d^3p}{2\omega_p} \left( b_{\vec{p}}^{s_1\dagger} \bar{u}(\vec{p}, s_1) \omega_p \gamma^0 b_{\vec{p}}^{s_2} u(\vec{p}, s_2) \right. \\
&\quad \left. - b_{\vec{p}}^{s_1\dagger} \bar{u}(\vec{p}, s_1) \omega_p \gamma^0 c_{-\vec{p}}^{s_2\dagger} v(-\vec{p}, s_2) e^{2i\omega_p t} \right. \\
&\quad \left. + c_{\vec{p}}^{s_1} \bar{v}(\vec{p}, s_1) \omega_p \gamma^0 b_{-\vec{p}}^{s_2} u(-\vec{p}, s_2) e^{-2i\omega_p t} \right)
\end{aligned}$$

$$-c_{\vec{p}}^{s_1} \bar{v}(\vec{p}, s_1) \omega_p \gamma^0 c_{\vec{p}}^{s_2 \dagger} v(\vec{p}, s_2) \Big), \quad (8.4.4)$$

注意到

$$\begin{cases} u^\dagger(\vec{p}, s_1) u(\vec{p}, s_2) = 2\omega_p \delta_{s_1 s_2} \\ u^\dagger(\vec{p}, s_1) v(-\vec{p}, s_2) = 0 \\ v^\dagger(\vec{p}, s_1) u(-\vec{p}, s_2) = 0 \\ v^\dagger(\vec{p}, s_1) v(\vec{p}, s_2) = 2\omega_p \delta_{s_1 s_2} \end{cases}, \quad (8.4.5)$$

代入,

$$H = \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p}{2\omega_p} \left( b_{\vec{p}}^{s_1 \dagger} b_{\vec{p}}^{s_2} (2\omega_p^2) \delta_{s_1 s_2} + c_{\vec{p}}^{s_1} c_{\vec{p}}^{s_2 \dagger} (-2\omega_p^2) \delta_{s_1 s_2} \right) = \dots \quad (8.4.6)$$

### 8.4.2 energy-momentum tensor

- Dirac field 的动量算符为

$$P^\mu = \int d^3 x T^{0\mu} = \int d^3 p p^\mu (b_{\vec{p}}^{s \dagger} b_{\vec{p}}^s + c_{\vec{p}}^{s \dagger} c_{\vec{p}}^s), \quad (8.4.7)$$

另外  $P^0 = H$  还有一个 vacuum energy.

calculation:

energy-momentum tensor 的  $0, \mu$  分量为 (见 (7.4.1))

$$\begin{aligned} T^{0\mu} &= i \bar{\Psi} \gamma^0 \partial^\mu \Psi = i \Psi^\dagger \partial^\mu \Psi \\ &= \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^3 \sqrt{4\omega_{p_1} \omega_{p_2}}} (b_{\vec{p}_1}^{s_1 \dagger} u^\dagger(\vec{p}_1, s_1) e^{ip_1 \cdot x} + c_{\vec{p}_1}^{s_1} v^\dagger(\vec{p}_1, s_1) e^{-ip_1 \cdot x}) \\ &\quad p_2^\mu (b_{\vec{p}_2}^{s_2} u(\vec{p}_2, s_2) e^{-ip_2 \cdot x} - c_{\vec{p}_2}^{s_2 \dagger} v(\vec{p}_2, s_2) e^{ip_2 \cdot x}), \end{aligned} \quad (8.4.8)$$

代入,

$$\begin{aligned} P^\mu &= \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p}{2\omega_p} \left( p^\mu b_{\vec{p}}^{s_1 \dagger} u^\dagger(\vec{p}, s_1) b_{\vec{p}}^{s_2} u(\vec{p}, s_2) - (-p^\mu) b_{\vec{p}}^{s_1 \dagger} u^\dagger(\vec{p}, s_1) c_{-\vec{p}}^{s_2 \dagger} v(-\vec{p}, s_2) e^{2i\omega_p t} \right. \\ &\quad \left. + (-p^\mu) c_{\vec{p}}^{s_1} v^\dagger(\vec{p}, s_1) b_{-\vec{p}}^{s_2} u(-\vec{p}, s_2) - p^\mu c_{\vec{p}}^{s_1} v^\dagger(\vec{p}, s_1) c_{\vec{p}}^{s_2 \dagger} v(\vec{p}, s_2) \right) \\ &= \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p}{2\omega_p} \left( p^\mu b_{\vec{p}}^{s_1 \dagger} b_{\vec{p}}^{s_2} (2\omega_p \delta_{s_1 s_2}) - p^\mu c_{\vec{p}}^{s_1} c_{\vec{p}}^{s_2 \dagger} (2\omega_p \delta_{s_1 s_2}) \right) \\ &= \int d^3 p p^\mu (b_{\vec{p}}^{s \dagger} b_{\vec{p}}^s - c_{\vec{p}}^s c_{\vec{p}}^{s \dagger}). \end{aligned} \quad (8.4.9)$$

### 8.4.3 angular momentum

- Dirac field 的角动量算符为 (这部分在 Peskin 上有)

$$\begin{aligned} J^{ij} &= \int d^3 x M^{0ij} \\ &= \epsilon^{ijk} \sum_{s_1, s_2 = \pm 1} \int d^3 p \frac{m}{2\omega_p} (b_{\vec{p}}^{s_1 \dagger} b_{\vec{p}}^{s_2} + c_{\vec{p}}^{s_1} c_{\vec{p}}^{s_2 \dagger}) \xi^{s_1 \dagger} \sigma_k \xi^{s_2} + \int d^3 x (x^i T^{0j} - x^j T^{0i}), \end{aligned} \quad (8.4.10)$$

其中,  $M^{\mu\nu\rho}$  见 (7.4.2).

– 把角动量算符中 spin 的部分表示为  $S^{ij}$ , 那么

$$\begin{cases} S^{12} b_{\vec{p}}^{s \dagger} |0\rangle = s \frac{m}{2\omega_p} b_{\vec{p}}^{s \dagger} |0\rangle \\ S^{12} c_{\vec{p}}^{s \dagger} |0\rangle = -s \frac{m}{2\omega_p} c_{\vec{p}}^{s \dagger} |0\rangle \end{cases}. \quad (8.4.11)$$

calculation:

角动量张量为

$$\begin{aligned}
 M^{0\mu\nu} &= \frac{i}{2} \underbrace{\bar{\Psi} \gamma^0}_{\Psi^\dagger} \sigma^{\mu\nu} \Psi + (x^\mu T^{0\nu} - x^\nu T^{0\mu}) \\
 &= \frac{i}{2} \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^3 \sqrt{4\omega_{p_1} \omega_{p_2}}} (b_{\vec{p}_1}^{s_1 \dagger} u^\dagger(\vec{p}_1, s_1) e^{ip_1 \cdot x} + c_{\vec{p}_1}^{s_1} v^\dagger(\vec{p}_1, s_1) e^{-ip_1 \cdot x}) \\
 &\quad \sigma^{\mu\nu} (b_{\vec{p}_2}^{s_2} u(\vec{p}_2, s_2) e^{-ip_2 \cdot x} + c_{\vec{p}_2}^{s_2 \dagger} v(\vec{p}_2, s_2) e^{ip_2 \cdot x}) + (x^\mu T^{0\nu} - x^\nu T^{0\mu}), \tag{8.4.12}
 \end{aligned}$$

代入,

$$\begin{aligned}
 J^{\mu\nu} - \int d^3 x (x^\mu T^{0\nu} - x^\nu T^{0\mu}) &= \frac{i}{2} \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p}{2\omega_p} \left( b_{\vec{p}}^{s_1 \dagger} u^\dagger(\vec{p}, s_1) \sigma^{\mu\nu} b_{\vec{p}}^{s_2} u(\vec{p}, s_2) \right. \\
 &\quad + b_{\vec{p}}^{s_1 \dagger} u^\dagger(\vec{p}, s_1) \sigma^{\mu\nu} c_{-\vec{p}}^{s_2 \dagger} v(-\vec{p}, s_2) e^{2i\omega_p t} \\
 &\quad + c_{\vec{p}}^{s_1} v^\dagger(\vec{p}, s_1) \sigma^{\mu\nu} b_{-\vec{p}}^{s_2} u(-\vec{p}, s_2) e^{-2i\omega_p t} \\
 &\quad \left. + c_{\vec{p}}^{s_1} v^\dagger(\vec{p}, s_1) \sigma^{\mu\nu} c_{\vec{p}}^{s_2 \dagger} v(\vec{p}, s_2) \right), \tag{8.4.13}
 \end{aligned}$$

其中

$$\begin{cases} u^\dagger(\vec{p}, s_1) \sigma^{ij} u(\vec{p}, s_2) = -2i\epsilon^{ijk} m \xi^{s_1 \dagger} \sigma_k \xi^{s_2} \\ u^\dagger(\vec{p}, s_1) \sigma^{ij} v(-\vec{p}, s_2) = 0 \\ v^\dagger(\vec{p}, s_1) \sigma^{ij} u(-\vec{p}, s_2) = 0 \\ v^\dagger(\vec{p}, s_1) \sigma^{ij} v(\vec{p}, s_2) = -2i\epsilon^{ijk} m \chi^{s_1 \dagger} \sigma_k \chi^{s_2} \end{cases}, \tag{8.4.14}$$

代入 (注意到  $\xi^s = \chi^s$ ),

$$J^{ij} - \int d^3 x (x^i T^{0j} - x^j T^{0i}) = \epsilon^{ijk} \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p}{2\omega_p} m (b_{\vec{p}}^{s_1 \dagger} b_{\vec{p}}^{s_2} + c_{\vec{p}}^{s_1} c_{\vec{p}}^{s_2 \dagger}) \xi^{s_1 \dagger} \sigma_k \xi^{s_2}. \tag{8.4.15}$$

## 8.5 electric current

- internal vector symmetry 对应的守恒流就是电流, 见 subsection 7.3.1, 有

$$Q = \int d^3 x J_V^0 = \sum_{s=\pm 1} \int d^3 p (b_{\vec{p}}^{s \dagger} b_{\vec{p}}^s - c_{\vec{p}}^{s \dagger} c_{\vec{p}}^s) - 2\delta^{(3)}(0) \int d^3 p. \tag{8.5.1}$$

calculation:

首先,

$$\begin{aligned}
 \int d^3 x J_V^\mu &= \int d^3 x \bar{\Psi} \gamma^\mu \Psi = \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p}{2\omega_p} \left( b_{\vec{p}}^{s_1 \dagger} b_{\vec{p}}^{s_2} \bar{u}(\vec{p}, s_1) \gamma^\mu u(\vec{p}, s_2) \right. \\
 &\quad + b_{\vec{p}}^{s_1 \dagger} c_{-\vec{p}}^{s_2 \dagger} \bar{u}(\vec{p}, s_1) \gamma^\mu v(-\vec{p}, s_2) e^{2i\omega_p t} \\
 &\quad + c_{\vec{p}}^{s_1} b_{-\vec{p}}^{s_2} \bar{v}(\vec{p}, s_1) \gamma^\mu u(-\vec{p}, s_2) e^{-2i\omega_p t} \\
 &\quad \left. + c_{\vec{p}}^{s_1} c_{\vec{p}}^{s_2 \dagger} \bar{v}(\vec{p}, s_1) \gamma^\mu v(\vec{p}, s_2) \right), \tag{8.5.2}
 \end{aligned}$$

其中

$$\begin{cases} \bar{u}(\vec{p}, s_1) \gamma^\mu u(\vec{p}, s_2) = 2p_\mu \delta_{s_1 s_2} \quad (?) \\ \bar{u}(\vec{p}, s_1) \gamma^0 v(-\vec{p}, s_2) = 0 \\ \bar{u}(\vec{p}, s_1) \gamma^i v(-\vec{p}, s_2) = \xi^{s_1 \dagger} (2m \sigma^i) \xi^{s_2} \\ \bar{v}(\vec{p}, s_1) \gamma^\mu u(-\vec{p}, s_2) = \bar{u}(\vec{p}, s_1) \gamma^\mu v(-\vec{p}, s_2) \\ \bar{v}(\vec{p}, s_1) \gamma^\mu v(\vec{p}, s_2) = \bar{u}(\vec{p}, s_1) \gamma^\mu u(\vec{p}, s_2) \end{cases}, \tag{8.5.3}$$

代入,

$$Q = \sum_{s=\pm 1} \int d^3 p (b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s + c_{\vec{p}}^s c_{\vec{p}}^{s\dagger}), \quad (8.5.4)$$

$$\begin{aligned} J^i &= \int d^3 x J_V^i \stackrel{(?)}{=} \int d^3 p \left( \sum_{s=\pm 1} -\frac{p^i}{\omega_p} (b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s + c_{\vec{p}}^s c_{\vec{p}}^{s\dagger}) \right. \\ &\quad \left. + \sum_{s_1, s_2=\pm 1} (b_{\vec{p}}^{s_1\dagger} c_{-\vec{p}}^{s_2\dagger} e^{2i\omega_p t} + c_{\vec{p}}^{s_1} b_{-\vec{p}}^{s_2} e^{-2i\omega_p t}) \xi^{s_1\dagger} (2m\sigma^i) \xi^{s_2} \right). \end{aligned} \quad (8.5.5)$$

## 8.6 free propagator

- 参考 scalar field 中的 propagator (见 (4.1.17)), the propagator of the Dirac field is

$$\begin{aligned} iS(x-y) &= \langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle \\ &= \int \frac{d^3 p}{(2\pi)^3 2\omega_p} \left( \theta(x^0 - y^0) (\not{p} + m) e^{-ip \cdot (x-y)} - \theta(y^0 - x^0) (\not{p} - m) e^{-ip \cdot (y-x)} \right) \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{\not{p} - m + i\epsilon} e^{-ip \cdot (x-y)}, \end{aligned} \quad (8.6.1)$$

其中

$$(T\Psi(x)\bar{\Psi}(y))_{\alpha\beta} = \theta(x^0 - y^0) \Psi_{\alpha}(x) \bar{\Psi}_{\beta}(y) - \theta(y^0 - x^0) \bar{\Psi}_{\beta}(y) \Psi_{\alpha}(x), \quad (8.6.2)$$

注意到这里交换  $\Psi, \bar{\Psi}$  是产生湮灭算符层面上的, 不是 spinor 层面上的.

calculation:

分别计算  $\langle 0 | \Psi_{\alpha}(x) \bar{\Psi}_{\beta}(y) | 0 \rangle$  和  $\langle 0 | \bar{\Psi}_{\beta}(y) \Psi_{\alpha}(x) | 0 \rangle$ ,

$$\begin{aligned} \langle 0 | \Psi_{\alpha}(x) \bar{\Psi}_{\beta}(y) | 0 \rangle &= \sum_{s=\pm 1} \int \frac{d^3 p}{(2\pi)^3 2\omega_p} u_{\alpha}(\vec{p}, s) \bar{u}_{\beta}(\vec{p}, s) e^{-ip \cdot (x-y)} \\ &= \int \frac{d^3 p}{(2\pi)^3 2\omega_p} (\not{p} + m)_{\alpha\beta} e^{-ip \cdot (x-y)}, \end{aligned} \quad (8.6.3)$$

$$\langle 0 | \bar{\Psi}_{\beta}(y) \Psi_{\alpha}(x) | 0 \rangle = \sum_{s=\pm 1} \int \frac{d^3 p}{(2\pi)^3 2\omega_p} v_{\alpha}(\vec{p}, s) \bar{v}_{\beta}(\vec{p}, s) e^{-ip \cdot (y-x)}, \quad (8.6.4)$$

代入, 得到...

把  $iS(x)$  的第二项中的  $\vec{p}$  变成  $-\vec{p}$ ,

$$\begin{aligned} iS(x) &= \int \frac{d^3 p}{(2\pi)^3 2\omega_p} \left( \theta(t) (\omega_p \gamma^0 - p^i \gamma^i + m) e^{-ip \cdot x} - \theta(-t) (\omega_p \gamma^0 + p^i \gamma^i - m) e^{i(\omega_p t + \vec{p} \cdot \vec{x})} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3 2\omega_p} \left( \theta(t) (\omega_p \gamma^0 - p^i \gamma^i + m) e^{-ip \cdot x} + \theta(-t) (\omega_p \mapsto -\omega_p) \right) \\ &= \int \frac{dp^0}{-2\pi i} \frac{1}{(p^0 - (\omega_p - i\epsilon))(p^0 + (\omega_p - i\epsilon))} \int \frac{d^3 p}{(2\pi)^3} (\not{p} + m) e^{-ip \cdot x} = \dots, \end{aligned} \quad (8.6.5)$$

最后,

$$\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} (\not{p} - m + i\epsilon) = I. \quad (8.6.6)$$

## Chapter 9

# spin-statistics connection

- **spin-statistics theorem:** 在 3 维空间中, 具有整数自旋的粒子遵守 Bose-Einstein statistics, 具有半整数自旋的粒子遵守 Fermi-Dirac statistics.
- 本 chapter 不对此做出证明, 只是举例说明不能满足 spin-statistics theorem 会导致什么样的后果.

### 9.1 the price of perversity

#### 9.1.1 scalar field

- 如果 scalar field 满足 anticommutation relation, 那么

$$\{\phi(\vec{x}, t), \phi(\vec{y}, t)\} = \int \frac{d^D k}{(2\pi)^D \omega_k} \cos(\vec{k} \cdot (\vec{x} - \vec{y})) \neq 0, \quad (9.1.1)$$

违反狭义相对论.

calculation:

代入 (4.1.11),

$$\{\phi(\vec{x}, t), \phi(\vec{y}, t)\} = \int \frac{d^D k}{(2\pi)^D 2\omega_k} (e^{i\vec{k} \cdot (\vec{x} - \vec{y})} + e^{-i\vec{k} \cdot (\vec{x} - \vec{y})}) = \dots \quad (9.1.2)$$

#### 9.1.2 Dirac field

- 如果 Dirac field 满足 commutation relation, 那么

$$[\Psi(\vec{x}, t), \Psi^\dagger(\vec{y}, t)] = \int \frac{d^3 p}{(2\pi)^3 \omega_p} (i\vec{p} \gamma^0 \sin(\vec{p} \cdot (\vec{x} - \vec{y})) + m \gamma^0 \cos(\vec{p} \cdot (\vec{x} - \vec{y}))), \quad (9.1.3)$$

考虑可观测量  $J_V^0 = \Psi^\dagger \Psi$  (其中  $x = (\vec{x}, t), y = (\vec{y}, t)$ ),

$$[J_V^0(x), J_V^0(y)] = \Psi_\alpha^\dagger(x) [\Psi_\alpha(x), \Psi_\beta^\dagger(y)] \Psi_\beta(y) - \Psi_\beta^\dagger(y) [\Psi_\beta(y), \Psi_\alpha^\dagger(x)] \Psi_\alpha(x). \quad (9.1.4)$$

calculation:

代入 (8.3.1),

$$\begin{aligned} [\Psi(\vec{x}, t), \Psi^\dagger(\vec{y}, t)] &= \sum_{s=\pm 1} \int \frac{d^3 p}{(2\pi)^3 2\omega_p} (u(\vec{p}, s) u^\dagger(\vec{p}, s) e^{i\vec{p} \cdot (\vec{x} - \vec{y})} - v(\vec{p}, s) v^\dagger(\vec{p}, s) e^{-i\vec{p} \cdot (\vec{x} - \vec{y})}) \\ &= \int \frac{d^3 p}{(2\pi)^3 2\omega_p} ((\not{p} + m) \gamma^0 e^{i\vec{p} \cdot (\vec{x} - \vec{y})} - (\not{p} - m) \gamma^0 e^{-i\vec{p} \cdot (\vec{x} - \vec{y})}) \\ &= \int \frac{d^3 p}{(2\pi)^3 2\omega_p} (2i\not{p} \gamma^0 \sin(\vec{p} \cdot (\vec{x} - \vec{y})) + 2m \gamma^0 \cos(\vec{p} \cdot (\vec{x} - \vec{y}))). \end{aligned} \quad (9.1.5)$$

然后,

$$\begin{aligned}
[J_V^0(x), J_V^0(y)] &= \Psi_\alpha^\dagger(x) [\Psi_\alpha(x), \Psi_\beta^\dagger(y)] \Psi_\beta(y) - \Psi_\beta^\dagger(y) [\Psi_\beta(y), \Psi_\alpha^\dagger(x)] \Psi_\alpha(x) \\
&= \sum_{s_1, s_2 = \pm 1} \int \frac{d^3 p_1 d^3 p_2 d^3 q}{(2\pi)^6 \sqrt{4\omega_{p_1} \omega_{p_2} \omega_q}} (b_{\vec{p}_1}^{s_1 \dagger} u^\dagger(\vec{p}_1, s_1) e^{i p_1 \cdot x} + c_{\vec{p}_1}^{s_1} v^\dagger(\vec{p}_1, s_1) e^{-i p_1 \cdot x}) \\
&\quad (i \not{q} \gamma^0 \sin(\vec{q} \cdot (\vec{x} - \vec{y})) + m \gamma^0 \cos(\vec{q} \cdot (\vec{x} - \vec{y}))) \\
&\quad (b_{\vec{p}_2}^{s_2} u(\vec{p}_2, s_2) e^{-i p_2 \cdot y} + c_{\vec{p}_2}^{s_2} v(\vec{p}_2, s_2) e^{i p_2 \cdot y}) - (x \leftrightarrow y), \tag{9.1.6}
\end{aligned}$$

注意到  $p_1 \neq p_2$ , 这种情况怎么算 (?).

## Chapter 10

# Grassmann path integrals and Feynman diagrams for Fermions

- Grassmann number 和 Gaussian-Berezin integrals 见 section B.2.

### 10.1 Grassmann path integral

- Dirac field 的 partition function 为

$$\begin{aligned} Z(\eta, \bar{\eta}) &= \int D\Psi D\bar{\Psi} e^{i \int d^4x (\bar{\Psi}(i\not{\partial} - m + i\epsilon)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta)} \\ &= e^{iE_0 T} e^{-i \int \frac{d^4p}{(2\pi)^4} \bar{\eta}(-p) \frac{1}{\not{p} - m + i\epsilon} \tilde{\eta}(p)}, \end{aligned} \quad (10.1.1)$$

其中 vacuum energy 为

$$E_0 = -4V \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \omega_p + \text{irrelevant terms}. \quad (10.1.2)$$

calculation:

代入 (B.2.13),

$$Z(\eta, \bar{\eta}) = \det(\underbrace{i(\not{\partial} - m + i\epsilon)}_{=iA}) e^{-i^2(-i)\bar{\eta}A^{-1}\eta}, \quad (10.1.3)$$

其中

$$\begin{cases} \det(i(\not{\partial} - m + i\epsilon)) = \det(\underbrace{i\gamma^5(\not{\partial} - m + i\epsilon)\gamma^5}_{=(-i\not{\partial} - m + i\epsilon)}) \\ (i\not{\partial} - m + i\epsilon)(-i\not{\partial} - m + i\epsilon) = (\partial^2 + m^2 - i\epsilon)I_{4 \times 4} \\ \implies \det(i(\not{\partial} - m + i\epsilon)) = \sqrt{\det((-\partial^2 - m^2 + i\epsilon)I_{4 \times 4})} = e^{iE_0 T}, \end{cases} \quad (10.1.4)$$

注意到  $I_{4 \times 4}$  会带来一个 4 次方的系数.

对于指数项, 考虑

$$(i\not{\partial} - m + i\epsilon)\Psi(x) = \int d^4y A(x-y)\Psi(y), \quad (10.1.5)$$

其中

$$\begin{aligned} A(x-y) &= \int \frac{d^4p}{(2\pi)^4} (\not{p} - m + i\epsilon) e^{-ip \cdot (x-y)} \\ \implies A^{-1}(x-y) &= S(x-y), \end{aligned} \quad (10.1.6)$$

其中  $S(x-y)$  是传播子, 见 (8.6.1), 所以指数项为

$$e^{-i\bar{\eta}A^{-1}\eta} = e^{-i \int d^4x d^4y \bar{\eta}(x) S(x-y) \eta(y)} = \dots \quad (10.1.7)$$

## 10.2 Feynman rules for Yukawa interaction

- 考虑如下 Lagrangian,

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi + \frac{1}{2}((\partial\phi)^2 - \mu^2\phi^2) - \frac{\lambda}{4!}\phi^4 + g\bar{\Psi}\phi\Psi, \quad (10.2.1)$$

对应如下 partition function,

$$\begin{aligned} \frac{Z(\bar{\eta}, \eta, J; \lambda, g)}{Z(0; 0)} &= e^{i \int d^4x \left( -\frac{\lambda}{4!} \left( \frac{\delta}{\delta i J(x)} \right)^4 + g \frac{\delta}{\delta i \bar{\eta}_\alpha(x)} \frac{\delta}{\delta i J(x)} \frac{\delta}{\delta i \eta_\alpha(x)} \right)} e^{-\frac{i}{2} J D J - i \bar{\eta}_\alpha S_{\alpha\beta} \eta_\beta}, \quad \text{Schwinger's way,} \\ &= \sum_{l, m, n=0}^{\infty} \frac{i^{l+m+n}}{l!m!n!} (-1)^{\frac{m(m-1)+n(n-1)}{2}} \int d^4x_1 \cdots d^4x_l d^4y_1 \cdots d^4y_m d^4z_1 \cdots d^4z_n \\ &\quad J(x_1) \cdots \bar{\eta}_{\alpha_1}(y_1) \cdots G_{\alpha_1 \cdots \beta_1}^{(l, m, n)}(x_1, \cdots, z_n) \eta_{\beta_1}(z_1) \cdots, \quad \text{Weyl's way,} \end{aligned} \quad (10.2.2)$$

其中

$$G_{\alpha_1 \cdots \beta_1}^{(l, m, n)}(x_1, \cdots, z_n) = e^{i \int d^4x \mathcal{L}(\lambda, g)} \phi(x_1) \cdots \Psi_{\alpha_1}(y_1) \cdots \bar{\Psi}_{\beta_1}(z_1) \cdots. \quad (10.2.3)$$

- 下面给出一些 Feynman diagrams 作为例子, 先用正则量子化方法计算, 首先,

$$p \uparrow \quad \uparrow = \rho^2(p_1) \delta^{(3)}(\vec{p}_1 - \vec{p}_2) \delta_{s_1 s_2}, \quad (10.2.4)$$

$$\begin{aligned} \begin{array}{c} k \\ \swarrow \\ p_2 \swarrow \quad \uparrow \\ \quad \quad \quad \uparrow \\ \quad \quad \quad p_1 \end{array} &= \rho(p_1) \rho(p_2) \rho(k) (-ig) \int d^4x \langle 0 | \overbrace{b_{p_2}^{s_2} a_{\vec{k}}^{s_1}} (\bar{\Psi}(x) \phi(x) \Psi(x)) \overbrace{b_{p_1}^{s_1 \dagger}} | 0 \rangle \\ &= (-ig) \int d^4x e^{-i(p_1 - p_2 - k) \cdot x} \bar{u}(\vec{p}_2, s_2) u(\vec{p}_1, s_1), \end{aligned} \quad (10.2.5)$$

再算一个复杂一点的例子,

$$\begin{array}{c} p_2 \uparrow \\ \swarrow \\ p_3 \uparrow \quad \uparrow \\ \quad \quad \quad \uparrow \\ \quad \quad \quad p_1 \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \quad k = (-ig)^2 (2\pi)^4 \delta^{(4)}(p_1 - p_2) \quad (10.2.6)$$

$$\bar{u}_\alpha(\vec{p}_2, s_2) \left( \int \frac{d^4p_3}{(2\pi)^4} \frac{i}{\not{p}_3 - m + i\epsilon} \frac{i}{(p_2 - p_3)^2 - \mu^2 + i\epsilon} \right)_{\alpha\beta} u_\beta(\vec{p}_1, s_1).$$

### calculation:

注意不要忘了算符按时间排序,

$$\begin{aligned} &\cdots \\ &= \rho(p_1) \rho(p_2) \frac{(-ig)^2}{2!} \int d^4x_1 d^4x_2 \left( \theta(t_1 - t_2) \left( \langle 0 | \overbrace{b_{p_2}^{s_2} (\bar{\Psi}(x_1) \phi(x_1) \Psi(x_1) \bar{\Psi}(x_2) \phi(x_2) \Psi(x_2))} b_{p_1}^{s_1 \dagger} | 0 \rangle \right. \right. \\ &\quad \left. \left. + \langle 0 | \overbrace{b_{p_2}^{s_2} (\bar{\Psi}(x_1) \phi(x_1) \Psi(x_1) \bar{\Psi}(x_2) \phi(x_2) \Psi(x_2))} b_{p_1}^{s_1 \dagger} | 0 \rangle \right) - \theta(t_2 - t_1) \cdots \right) \\ &= 2 \times \frac{(-ig)^2}{2!} \int d^4x_1 d^4x_2 \end{aligned}$$



$$\begin{aligned}
& e^{i(p_2 \cdot x_1 - p_1 \cdot x_2)} \bar{u}_\alpha(\vec{p}_2, s_2) \left( \int \frac{d^4 p_3}{(2\pi)^4} \frac{i e^{-i p_3 \cdot (x_1 - x_2)}}{\not{p}_3 - m + i\epsilon} \right)_{\alpha\beta} u_\beta(\vec{p}_1, s_1) \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-i k \cdot (x_1 - x_2)}}{k^2 - \mu^2 + i\epsilon} \\
& = (-ig)^2 (2\pi)^4 \delta^{(4)}(p_1 - p_2) \\
& \bar{u}_\alpha(\vec{p}_2, s_2) \left( \int \frac{d^4 p_3}{(2\pi)^4} \frac{i}{\not{p}_3 - m + i\epsilon} \frac{i}{(p_2 - p_3)^2 - \mu^2 + i\epsilon} \right)_{\alpha\beta} u_\beta(\vec{p}_1, s_1). \tag{10.2.7}
\end{aligned}$$

• 对于 Weyl's way, 首先,

$$\begin{array}{c} \uparrow \\ | \\ p \end{array} = (2\pi)^4 \delta^{(4)}(p_1 + p_2) \left( \frac{i}{\not{p}_1 - m + i\epsilon} \right)_{\alpha\beta}, \tag{10.2.8}$$

$$\begin{array}{c} \nearrow p_2 \\ \nwarrow k \\ \uparrow p_1 \end{array} = ig(2\pi)^4 \delta^{(4)}(p_1 + p_2 + k) \left( \frac{i}{\not{p}_1 - m + i\epsilon} \frac{i}{-\not{p}_2 - m + i\epsilon} \right)_{\alpha\beta} \frac{i}{k^2 - \mu^2 + i\epsilon}, \tag{10.2.9}$$

calculation:

$$\begin{aligned}
\begin{array}{c} \uparrow \\ | \\ p \end{array} &= \int d^4 x d^4 y e^{i p_1 \cdot x + i p_2 \cdot y} \langle \overline{\Psi}_\alpha(x) \bar{\Psi}_\beta(y) \rangle \\
&= \int d^4 x d^4 y e^{i p_1 \cdot x + i p_2 \cdot y} (-i)^2 (-i S_{\alpha\beta}(x - y)) = \dots, \tag{10.2.10}
\end{aligned}$$

$$\begin{aligned}
\begin{array}{c} \nearrow p_2 \\ \nwarrow k \\ \uparrow p_1 \end{array} &= \int d^4 x_1 d^4 x_2 d^4 y e^{i p_1 \cdot x_1 + i p_2 \cdot x_2 + i k \cdot y} \\
&\int d^4 z \langle (ig \overline{\Psi}_\gamma(z) \phi(z) \Psi_\gamma(z)) \phi(y) \Psi_\alpha(x_1) \bar{\Psi}_\beta(x_2) \rangle \\
&= ig \int d^4 x_1 d^4 x_2 d^4 y e^{i p_1 \cdot x_1 + i p_2 \cdot x_2 + i k \cdot y} \\
&\int d^4 z (-(-i)^2 i S_{\alpha\gamma}(x_1 - z)) (-(-i)^2 i S_{\gamma\beta}(z - x_2)) (-(-i)^2 i D(z - y)) \\
&= ig \int d^4 z \left( \frac{i e^{i p_1 \cdot z}}{\not{p}_1 - m + i\epsilon} \frac{i e^{i p_2 \cdot z}}{-\not{p}_2 - m + i\epsilon} \right)_{\alpha\beta} \frac{i e^{i k \cdot z}}{k^2 - \mu^2 + i\epsilon} = \dots. \tag{10.2.11}
\end{aligned}$$

再算 (10.2.6),

$$\begin{array}{c} \nearrow p_2 \\ \circlearrowleft p_3 \\ \circlearrowright k \\ \nwarrow p_1 \end{array} = (ig)^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2)$$

$$\frac{i}{\not{p}_1 - m + i\epsilon} \int \frac{d^4 p_3}{(2\pi)^4} \left( \frac{i}{\not{p}_3 - m + i\epsilon} \frac{i}{(p_1 - p_3)^2 - \mu^2 + i\epsilon} \right) \frac{i}{\not{p}_2 - m + i\epsilon}. \tag{10.2.12}$$

calculation:

$$\begin{aligned}
& \dots \\
&= \frac{(ig)^2}{2!} \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{ip_1 \cdot x_1 + ip_2 \cdot x_2} \left( \langle \overbrace{(\bar{\Psi}(y_1)\phi(y_1)\Psi(y_1))}^{\quad} \overbrace{(\bar{\Psi}(y_2)\phi(y_2)\Psi(y_2))}^{\quad} \Psi_1 \bar{\Psi}_2 \rangle \right. \\
&\quad \left. + (y_1 \leftrightarrow y_2) \right) \\
&= (ig)^2 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{ip_1 \cdot x_1 + ip_2 \cdot x_2} iS(x_1 - y_1) iS(y_1 - y_2) iS(y_2 - x_2) iD(y_1 - y_2) \\
&= (ig)^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2) \\
&\quad \frac{i}{\not{p}_1 - m + i\epsilon} \int \frac{d^4p_3}{(2\pi)^4} \left( \frac{i}{\not{p}_3 - m + i\epsilon} \frac{i}{(p_1 - p_3)^2 - \mu^2 + i\epsilon} \right) \frac{i}{\not{p}_2 - m + i\epsilon}. \tag{10.2.13}
\end{aligned}$$

**Part III**

**quantum electrodynamics**

# Chapter 11

## Maxwell's equations

### 11.1 the $(1, 0) \oplus (0, 1)$ representation of the Lorentz algebra

- 反对称张量  $F_{[\mu\nu]}$  是  $(1, 0) \oplus (0, 1)$  rep. 中的向量, 详见笔记 [Lie Groups and Lie Algebras](#).

### 11.2 Maxwell's equations

- 电磁场的 Lagrangian 为 (现实中  $\mu = 0$ )

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu, \quad \text{with} \quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}, \quad (11.2.1)$$

其中  $A_\mu = (\phi, -\vec{A})$ , 对作用量变分得到运动方程,

$$\partial^\nu F_{\nu\mu} + \mu^2 A_\mu = 0. \quad (11.2.2)$$

calculation:

场方程还可以写成

$$\left(-\partial^\mu\partial^\nu + \eta^{\mu\nu}(\partial^2 + \mu^2)\right)A_\nu = 0 \iff (k^2 - \mu^2)\tilde{A}_\mu(k) = k_\mu k^\nu \tilde{A}_\nu(k) \quad (11.2.3)$$

和

$$\begin{cases} -\nabla^2 A_0 - \nabla \cdot \frac{\partial \vec{A}}{\partial t} + \mu^2 A_0 = 0 \\ \frac{\partial}{\partial t} \left( -\nabla A_0 - \frac{\partial \vec{A}}{\partial t} \right) + \underbrace{\left( \nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) \right)}_{=-\nabla \times (\nabla \times \vec{A})} - \mu^2 \vec{A} = 0 \end{cases} \quad (11.2.4)$$

- 如果引入 Lorentz gauge condition (如 (2.1.12) 所示, 在  $\mu \neq 0$  时必然成立),

$$\text{field eq. (11.2.2)} \xrightarrow{\mu \neq 0} \begin{cases} (\partial^2 + \mu^2)A_\mu = 0 \\ \partial^\mu A_\mu = 0 \end{cases}. \quad (11.2.5)$$

- 此外,  $F_{\mu\nu}$  满足 Bianchi identity,

$$\nabla_\rho F_{\mu\nu} + \nabla_\nu F_{\rho\mu} + \nabla_\mu F_{\nu\rho} = 0. \quad (11.2.6)$$

calculation:

代入定义式,

$$\begin{aligned} \nabla_\rho \nabla_{[\mu} A_{\nu]} + \dots &= +\rho\mu\nu - \rho\nu\mu \\ &+ \nu\rho\mu - \nu\mu\rho \\ &+ \mu\nu\rho - \mu\rho\nu \end{aligned}$$

$$= \underbrace{(R_{\rho\mu\nu}{}^\sigma + R_{\nu\rho\mu}{}^\sigma + R_{\mu\nu\rho}{}^\sigma)}_{=0} A_\sigma. \quad (11.2.7)$$

• 令

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \iff \begin{cases} F_{0i} = \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \\ -\frac{1}{2}\epsilon^{ijk}F_{jk} = \vec{B} = \nabla \times \vec{A} \end{cases}, \quad (11.2.8)$$

代入场方程 (11.2.2),

$$\begin{cases} \nabla \cdot \vec{E} + \mu^2\phi = 0 \\ -\frac{\partial\vec{E}}{\partial t} + \nabla \times \vec{B} + \mu^2\vec{A} = 0 \end{cases}, \quad (11.2.9)$$

代入 Bianchi identity (11.2.6),

$$\begin{cases} \nabla \cdot \vec{B} = 0 & \rho, \mu, \nu = 1, 2, 3 \\ \nabla \times \vec{E} - \frac{\partial\vec{B}}{\partial t} = 0 & \rho, \mu, \nu = 0, i, j \end{cases}. \quad (11.2.10)$$

• 最后, 电磁场的能动量张量见 subsection D.4.1.

### 11.2.1 gauge symmetry (gauge redundancy)

•  $A_\mu$  有 4 个分量, 但光子只有 2 个自由度 (偏振态).

• 首先, 考虑  $\tilde{A}_\mu$  的场方程, 对于指标  $\mu = 0$  有 (计算过程见 subsection A.2.1)

$$\tilde{A}_0 = \frac{k^0}{\omega_k^2} \vec{k} \cdot \vec{\tilde{A}} \implies A_0(t, \vec{x}) = \int d^3y \frac{e^{-\mu|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} \nabla_y \cdot \frac{\partial \vec{\tilde{A}}(t, \vec{y})}{\partial t}, \quad (11.2.11)$$

这是一个约束条件, 将此式代入剩余的场方程, 得到

$$(k^2 - \mu^2)\vec{\tilde{A}} = (k^2 - \mu^2)\frac{\vec{k}(\vec{k} \cdot \vec{\tilde{A}})}{\omega_k^2}, \quad (11.2.12)$$

因此:

– 当  $\mu = 0$  时,

$$\begin{cases} \text{on shell: } \vec{\tilde{A}} \text{ 取值任意} \\ \text{off shell: } \tilde{A}_0 = \frac{k^0}{|\vec{k}|} |\vec{\tilde{A}}|, \vec{\tilde{A}} = |\vec{\tilde{A}}| \hat{e}_k \end{cases} \implies \tilde{A}_\mu(k) = \tilde{\mathcal{A}}_\mu(k) 2\pi\delta(k^2) - ik_\mu \tilde{\lambda}(k), \quad (11.2.13)$$

其中  $\tilde{\mathcal{A}}(k), \tilde{\lambda}(k) = \frac{|\vec{\tilde{A}}|}{|\vec{k}|}$  是任意函数, 且  $\text{sign}(k^0)|\vec{k}|\tilde{A}_0 - \vec{k} \cdot \vec{\tilde{A}} = 0$ . 因此

$$A_\mu(x) = \partial_\mu \lambda(x) + \int \frac{d^3k}{(2\pi)^3 2|\vec{k}|} \left( \tilde{\mathcal{A}}_\mu(|\vec{k}|, \vec{k}) e^{-i(|\vec{k}|x^0 - \vec{k} \cdot \vec{x})} + \tilde{\mathcal{A}}_\mu^*(|\vec{k}|, \vec{k}) e^{i(|\vec{k}|x^0 - \vec{k} \cdot \vec{x})} \right), \quad (11.2.14)$$

第二项是 on shell 平面波解的叠加, 并且振幅满足  $k^\mu \tilde{\mathcal{A}}_\mu = 0$ .

– 当  $\mu \neq 0$  时,

$$\begin{cases} \text{on shell: } \vec{\tilde{A}} \text{ 取值任意} \\ \text{off shell: } \vec{k} \cdot \vec{\tilde{A}} = 0 \implies \vec{\tilde{A}} = \tilde{A}_0 = 0 \end{cases} \implies \tilde{A}_\mu(k) = \tilde{\mathcal{A}}_\mu(k) 2\pi\delta(k^2 - \mu^2). \quad (11.2.15)$$

其中  $\tilde{\mathcal{A}}(k)$  是任意函数, 且  $\text{sign}(k^0)\omega_k \tilde{A}_0 - \vec{k} \cdot \vec{\tilde{A}} = 0$ . 因此

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( \tilde{\mathcal{A}}_\mu(\omega_k, \vec{k}) e^{-i(\omega_k x^0 - \vec{k} \cdot \vec{x})} + \tilde{\mathcal{A}}_\mu^*(\omega_k, \vec{k}) e^{i(\omega_k x^0 - \vec{k} \cdot \vec{x})} \right). \quad (11.2.16)$$

calculation:

对 (11.2.12) 两边同时内积  $\vec{k}$ , 有

$$(k^2 - \mu^2)(\vec{k} \cdot \vec{A}) = 0 \quad \text{or} \quad \omega_k = |\vec{k}|, \quad (11.2.17)$$

因此  $\mu = 0$  时 (11.2.12) 自然成立, 而 massive 情况下  $\tilde{A}_0 = \vec{k} \cdot \vec{A} = 0$  除非 on shell.

- 除此之外还需要一个约束条件 (接下来默认  $\mu = 0$ ).
- 考虑 (11.2.14), 注意到给定初始条件  $A_\mu(t_0, \vec{x}), \partial_\nu A_\mu(t_0, \vec{x})$  无法唯一确定参数  $\lambda(x), \tilde{\mathcal{A}}_\mu(|\vec{k}|, \vec{k})$ .

calculation:

对  $A_\mu$  做平面波分解,

$$\begin{cases} \int d^3x e^{-i\vec{p}\cdot\vec{x}} A_0(x) = \frac{1}{2|\vec{p}|} \left( e^{-i|\vec{p}|x^0} \tilde{\mathcal{A}}_0(|\vec{p}|, \vec{p}) + e^{i|\vec{p}|x^0} \tilde{\mathcal{A}}_0(-|\vec{p}|, \vec{p}) \right) + \int d^3x e^{-i\vec{p}\cdot\vec{x}} \frac{\partial \lambda}{\partial t} \\ \int d^3x e^{-i\vec{p}\cdot\vec{x}} \vec{A}(x) = \frac{1}{2|\vec{p}|} \left( e^{-i|\vec{p}|x^0} \vec{\mathcal{A}}(|\vec{p}|, \vec{p}) + e^{i|\vec{p}|x^0} \vec{\mathcal{A}}(-|\vec{p}|, \vec{p}) \right) + i\vec{p} \int d^3x e^{-i\vec{p}\cdot\vec{x}} \lambda \end{cases}, \quad (11.2.18)$$

对  $\partial_\nu A_\mu$  做平面波分解,

$$\begin{cases} \frac{\partial A_0(x)}{\partial t} \mapsto \frac{-i}{2} \left( e^{-i|\vec{p}|x^0} \tilde{\mathcal{A}}_0(|\vec{p}|, \vec{p}) - e^{i|\vec{p}|x^0} \tilde{\mathcal{A}}_0(-|\vec{p}|, \vec{p}) \right) + \int d^3x e^{-i\vec{p}\cdot\vec{x}} \frac{\partial^2 \lambda}{\partial t^2} \\ \frac{\partial \vec{A}(x)}{\partial t} \mapsto \frac{-i}{2} \left( e^{-i|\vec{p}|x^0} \vec{\mathcal{A}}(|\vec{p}|, \vec{p}) - e^{i|\vec{p}|x^0} \vec{\mathcal{A}}(-|\vec{p}|, \vec{p}) \right) - i\vec{p} \int d^3x e^{-i\vec{p}\cdot\vec{x}} \frac{\partial \lambda}{\partial t} \\ \nabla A_\mu \mapsto i\vec{p} \int d^3x e^{-i\vec{p}\cdot\vec{x}} A_\mu(x) \quad (\text{这是冗余的}) \end{cases}, \quad (11.2.19)$$

可见, 由于  $\lambda$  的存在, 我们无法唯一确定参数  $\tilde{\mathcal{A}}_\mu$ .

也就是说, 给定初始条件, 我们可以求解  $A_\mu(x)$  up to a function  $\partial_\mu \lambda$ .

- gauge redundancy: 将  $A_\mu$  和  $A_\mu + \partial_\mu \lambda$  认为是同一个物理态.
- Lorentz gauge 是

$$\partial_\mu A^\mu = 0, \quad (11.2.20)$$

注意到方程  $\partial^2 \lambda = -f$  总是有解, 它的 Green's function 是

$$G^{(\pm)}(x) = \frac{1}{4\pi|\vec{x}|} \delta(x^0 \mp |\vec{x}|). \quad (11.2.21)$$

– Lorentz gauge 下,  $A_\mu$  的通解是

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2|\vec{k}|} \left( \tilde{\mathcal{A}}_\mu(|\vec{k}|, \vec{k}) e^{-i(|\vec{k}|x^0 - \vec{k}\cdot\vec{x})} + \tilde{\mathcal{A}}_\mu^*(|\vec{k}|, \vec{k}) e^{i(|\vec{k}|x^0 - \vec{k}\cdot\vec{x})} \right). \quad (11.2.22)$$

- Coulomb gauge 是

$$\nabla \cdot \vec{A} = 0, \quad (11.2.23)$$

注意到方程  $\nabla^2 \lambda = -f$  总是有解. 且根据 (11.2.11) 可知  $A_0(x) = 0$ .

– Coulomb gauge 下,  $A_\mu$  的通解是

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2|\vec{k}|} \left( \vec{\mathcal{A}}_\perp(|\vec{k}|, \vec{k}) e^{-i(|\vec{k}|x^0 - \vec{k}\cdot\vec{x})} + \vec{\mathcal{A}}_\perp^*(|\vec{k}|, \vec{k}) e^{i(|\vec{k}|x^0 - \vec{k}\cdot\vec{x})} \right), \quad (11.2.24)$$

其中  $\vec{\mathcal{A}}_\perp = \vec{\mathcal{A}} - \hat{e}_k \hat{e}_k \cdot \vec{\mathcal{A}}$ .

## Chapter 12

# the quantization of the electromagnetic field

- 本章分别在 Lorentz gauge 和 Coulomb gauge 下对电磁场量子化.
- 电磁场的广义动量为

$$\pi^{\mu,\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = -\partial^\mu A^\nu + \partial^\nu A^\mu = F^{\nu\mu}, \quad (12.0.1)$$

简写  $\pi^\mu \equiv \pi^{0\mu}$ , 有

$$\pi^0 = 0, \quad \vec{\pi} = \vec{E}. \quad (12.0.2)$$

- $\vec{E}$  是可观测量, 因此其形式不受 gauges 影响,

$$\vec{\pi} = i \int \frac{d^3k}{(2\pi)^3 2|\vec{k}|} |\vec{k}| \left( \vec{\tilde{A}}_\perp(k) e^{-i(|\vec{k}|x^0 - \vec{k} \cdot \vec{x})} - \vec{\tilde{A}}_\perp^*(k) e^{i(|\vec{k}|x^0 - \vec{k} \cdot \vec{x})} \right). \quad (12.0.3)$$

### 12.1 in Lorentz gauge

- 在 Lorentz gauge 下,

### 12.2 in Coulomb gauge

•

# Appendices



## Appendix A

# Dirac delta function & Fourier transformation

### A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx \delta(x) e^{-ikx}. \quad (\text{A.1.1})$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|}. \quad (\text{A.1.2})$$

- 第二个常用的公式 ([Sokhotski-Plemelj theorem](#)),

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{x + i\epsilon} = \mathcal{P} \frac{1}{x} - i\pi \delta(x), \quad (\text{A.1.3})$$

其中  $\mathcal{P}$  表示复函数的主值 (principal value).

**proof:**

考虑

$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \quad \text{and} \quad \int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \text{Res}(f, i\epsilon) = \pi, \quad (\text{A.1.4})$$

所以...

取  $\epsilon = 0.1$  时, 复变函数的实部, 虚部分别如下:

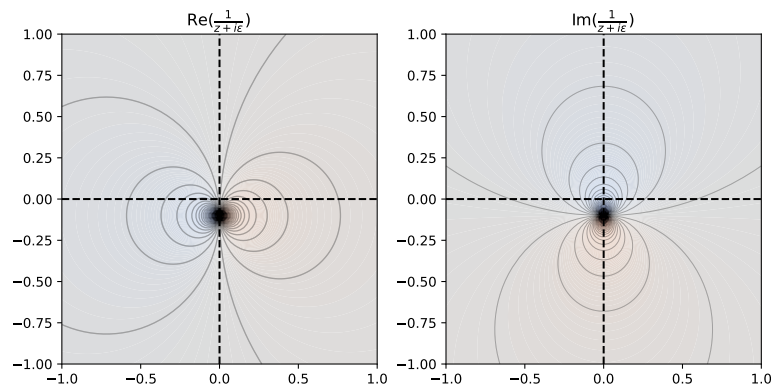


Figure A.1: graph of  $\frac{1}{z + i\epsilon}$

- 另外,  $\delta(x - a)\delta(x - b) = \delta(b - a)\delta(x - a)$ .

## A.2 Fourier transformation

- $d$ -dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^d x e^{-ik \cdot x} \phi(x) \end{cases}, \quad (\text{A.2.1})$$

- 因此

$$\partial_\mu \phi(x) \mapsto i k_\mu \tilde{\phi}(k). \quad (\text{A.2.2})$$

- 对于实函数, Fourier transformation 是正交变换, 其 Jacobi determinant 为

$$\left| \frac{\partial \phi(x) \cdots}{\partial \text{Re} \tilde{\phi}(k) \cdots \partial \text{Im} \tilde{\phi}(k) \cdots} \right| = \left( \frac{2}{V} \right)^{(2N+1)^d} \det A = \left( \frac{2(2N)^d}{V^2} \right)^{\frac{(2N+1)^d}{2}}. \quad (\text{A.2.3})$$

**proof:**

position space 和 momentum space 的格点分别为

$$\begin{cases} x_i^\mu = i^\mu \epsilon \in \{0, \pm\epsilon, \dots, \frac{L}{2}\} \\ k_n^\mu = n^\mu \frac{2\pi}{L} \in \{0, \pm\frac{2\pi}{L}, \dots, \frac{\pi}{\epsilon}\} \end{cases} \iff i^\mu, n^\mu \in \{0, \pm 1, \dots, \pm N\}, \quad (\text{A.2.4})$$

$x^\mu, k^\mu$  分别有  $2N+1$  个取值, 其中  $N\epsilon = \frac{L}{2}$ , 时空总体积为  $V = L^d$ , momentum space 的总体积为  $\tilde{V} = \frac{(4\pi N)^d}{V}$ .

将 (A.2.1) 写成格点求和的形式,

$$\begin{cases} \phi(x_i) = \frac{1}{(2\pi)^d} \left( \frac{2\pi}{L} \right)^d \sum_n e^{ik_n \cdot x_i} \tilde{\phi}(k_n) \\ \quad = \frac{2}{V} \sum_{n^0 > 0} \left( \cos(k_n \cdot x_i) \text{Re} \tilde{\phi}(k_n) - \sin(k_n \cdot x_i) \text{Im} \tilde{\phi}(k_n) \right) \\ \tilde{\phi}(k_n) = \epsilon^d \sum_i e^{-ik_n \cdot x_i} \phi(x_i) \\ \quad = \frac{V}{(2N)^d} \sum_i \left( \cos(k_n \cdot x_i) - i \sin(k_n \cdot x_i) \right) \phi(x_i) \end{cases}. \quad (\text{A.2.5})$$

**proof:**

$\phi(x_i)$  的变换需要做一些说明. 注意到  $\tilde{\phi}$  的分量的数量是  $\phi$  的两倍 (考虑到实部与虚部), 但在  $\phi \in \mathbb{R}^{(2N+1)^d}$  时,

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k), \quad (\text{A.2.6})$$

可见  $\tilde{\phi}$  的分量并不独立, 取  $k^0 > 0$  的部分为独立分量, 那么...

将 (A.2.5) 写成矩阵的形式,

$$\begin{cases} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} = \frac{2}{V} \overbrace{\begin{pmatrix} \cos k_0 \cdot x_0 & \cdots & \cos k_{\max} \cdot x_0 & -\sin k_0 \cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A} \begin{pmatrix} \text{Re} \tilde{\phi}(k_0) \\ \vdots \\ \text{Im} \tilde{\phi}(k_0) \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \text{Re} \tilde{\phi}(k_0) \\ \vdots \\ \text{Im} \tilde{\phi}(k_0) \\ \vdots \end{pmatrix} = \frac{V}{(2N)^d} \begin{pmatrix} \cos k_0 \cdot x_0 & \cdots & \cos k_0 \cdot x_{\max} \\ \vdots & \ddots & \vdots \\ -\sin k_0 \cdot x_0 & \cdots & -\sin k_0 \cdot x_{\max} \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} \end{cases}, \quad (\text{A.2.7})$$

观察可见  $\tilde{\phi}$  的变换中的矩阵是  $A^T$ , 所以

$$\frac{2}{V} \frac{V}{(2N)^d} A A^T = I \implies \det A = \left( \frac{(2N)^d}{2} \right)^{\frac{(2N+1)^d}{2}}, \quad (\text{A.2.8})$$

因此...

– 顺便,

$$\int d^d x f(x) g(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{f}(-k) \tilde{g}(k). \quad (\text{A.2.9})$$

### A.2.1 an important example

- 考虑如下 PDE,

$$(\nabla^2 - \mu^2) \phi(\vec{x}) = f(\vec{x}), \quad (\text{A.2.10})$$

其 Green's function 为

$$G(\vec{x}) = -\frac{1}{4\pi} \frac{e^{-\mu r}}{r}, \quad (\text{A.2.11})$$

其中  $r = |\vec{x}|$ .

#### calculation:

Green's function 满足

$$(\nabla^2 - \mu^2) G(\vec{x}) = \delta^{(3)}(\vec{x}) \implies \tilde{G}(\vec{k}) = -\frac{1}{|\vec{k}|^2 + \mu^2}, \quad (\text{A.2.12})$$

因此

$$\begin{aligned} G(\vec{x}) &= -\int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot \vec{x}}}{|\vec{k}|^2 + \mu^2} \\ &= -\int \frac{k^2 \sin \theta d\theta d\phi dk}{(2\pi)^3} \frac{e^{i \cos \theta kr}}{k^2 + \mu^2} = -\frac{1}{(2\pi)^3} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \int_0^\infty dk \frac{k^2 e^{i \cos \theta kr}}{k^2 + \mu^2} \\ &= -\frac{1}{(2\pi)^2} \int_0^\infty \frac{k^2}{k^2 + \mu^2} \frac{2 \sin kr}{kr} dk, \end{aligned} \quad (\text{A.2.13})$$

注意到 (在复平面上考虑以下积分并使用 residue theorem)

$$\begin{aligned} \int_0^\infty \frac{k \sin kr}{k^2 + \mu^2} dk &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{k \sin kr}{k^2 + \mu^2} dk = \frac{1}{2i} \int_{-\infty}^{+\infty} \frac{k e^{ikr}}{k^2 + \mu^2} dk \\ &= \frac{1}{2i} \int_{-\infty}^{+\infty} \frac{k e^{ikr}}{(k + i\mu)(k - i\mu)} dk = \frac{1}{2i} 2\pi i \text{Res}(f, k = i\mu) = \frac{\pi}{2} e^{-\mu r}. \end{aligned} \quad (\text{A.2.14})$$

## Appendix B

# Gaussian integrals and Gaussian-Berezin integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \quad (\text{B.0.1})$$

$$\langle x^{2n} \rangle = \frac{\int dx e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n} (2n-1)!!, \quad (\text{B.0.2})$$

其中  $(2n-1)!! = 1 \cdot 3 \cdots (2n-3)(2n-1)$ .

- 一个重要的变体如下,

$$\int dx e^{-\frac{a}{2}x^2 + Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}, \quad (\text{B.0.3})$$

另外, 将  $a, J$  分别替换为  $-ia, iJ$  也是重要的变体.

## B.1 generalize to $N$ -dim.

- 考虑如下积分,

$$Z(A, J) = \int dx_1 \cdots dx_N e^{-\frac{1}{2}x^T \cdot A \cdot x + J^T \cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J}, \quad (\text{B.1.1})$$

其中  $x, J$  是  $N$ -dim. 列向量,  $A$  是  $N \times N$  实对称矩阵.

### calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得

$$A = O^{-1} \cdot D \cdot O, \quad (\text{B.1.2})$$

其中  $D$  是一个 diagonal matrix. 令  $y = O \cdot x$ , 那么

$$\begin{aligned} Z(A, J) &= \int dy_1 \cdots dy_N e^{-\frac{1}{2}y^T \cdot D \cdot y + (OJ)^T \cdot y} \\ &= \prod_{i=1}^N \sqrt{\frac{2\pi}{D_{ii}}} e^{\frac{1}{2D_{ii}}(OJ)_i^2} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J}, \end{aligned} \quad (\text{B.1.3})$$

其中, 注意到了  $\frac{1}{D_{ii}} = (O \cdot A^{-1} \cdot O^{-1})_{ii}$  以及  $\text{tr } D = \det A$ .

- 一个重要的变体是  $A \mapsto -iA, J \mapsto iJ$ .
- 考虑 (B.0.2) 的变体, (注意  $A$  是对称的),

$$\langle x_i x_j \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} Z(A, J) \Big|_{J=0} = A_{ij}^{-1}, \quad (\text{B.1.4})$$

$$\langle x_i x_j \cdots x_k x_l \rangle = \sum_{\text{Wick}} A_{i'j'}^{-1} \cdots A_{k'l'}^{-1}, \quad (\text{B.1.5})$$

其中 (B.1.5) 中有偶数个  $x$ , 否则等于零.

**calculation:**

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A, J) \Big|_{J=0} = \cdots \quad (\text{B.1.6})$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1}, \quad (\text{B.1.7})$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle \overbrace{x_i x_j x_k x_l} \rangle = A_{ik}^{-1} A_{jl}^{-1}. \quad (\text{B.1.8})$$

## B.2 Grassmann number and Grassmann integrals

- 对于 Grassmann number  $\theta_1, \theta_2$ , 有反对易关系,

$$\theta_1 \theta_2 = -\theta_2 \theta_1, \quad (\text{B.2.1})$$

因此  $\theta^2 = 0$ , 且关于 Grassmann number 最一般的函数为

$$f(\theta) = a\theta + b, \quad (\text{B.2.2})$$

其中  $a, b \in \mathbb{C}$ .

- 注意到  $(\theta_1 \theta_2) \theta_3 = \theta_3 (\theta_1 \theta_2)$ , (但是  $(\theta_1 \theta_2)^2 = 0$ , 所以  $\theta_1 \theta_2 \notin \mathbb{C}$ ), 且有

$$(\theta_1 \theta_2)(\theta_3 \theta_4) = \theta_3 (\theta_1 \theta_2) \theta_4 = (\theta_3 \theta_4)(\theta_1 \theta_2). \quad (\text{B.2.3})$$

- 定义 Grassmann integral (也称作 Berezin integral),

$$\int d\theta \theta = 1, \quad \int d\theta = 0, \quad (\text{B.2.4})$$

并且具有 linearity.

**comment:**

我们希望积分在 integration variable been shifted 之后 ( $\theta \mapsto \theta + \eta$ ) 保持不变,

$$\int d\theta (a\theta + b) = \int d\theta (a\theta + a\eta + b), \quad (\text{B.2.5})$$

因此, 积分结果应该与常数无关, 只与斜率有关, 所以直接定义

$$\int d\theta (a\theta + b) = a. \quad (\text{B.2.6})$$

– 另外, 对于  $f(\theta) = \eta\theta + b$ , 有

$$\int d\theta (\eta\theta + b) = \int d\theta (-\theta\eta + b) = -\eta. \quad (\text{B.2.7})$$

### B.2.1 Gaussian-Berezin integrals

- 回顾 section 1.4 和 (8.4.2), 我们希望 Gauss 积分中出现正号而不是符号, 即

$$\int dx e^{-\frac{1}{2}ax^2} = \sqrt{2\pi} e^{-\frac{1}{2}\ln a} \mapsto \propto e^{+\frac{1}{2}\ln a}. \quad (\text{B.2.8})$$

- 对于两个独立的 Grassmann number  $\theta, \bar{\theta}$ , 有 Gauss 积分,

$$\int d\theta \int d\bar{\theta} e^{\bar{\theta}a\theta} = \int d\theta \int d\bar{\theta} (1 + \bar{\theta}a\theta) = a = e^{+\ln a}. \quad (\text{B.2.9})$$

- 推广以上积分, 对于  $\theta = (\theta_1, \dots, \theta_N) \in V, \bar{\theta} = (\bar{\theta}_1, \dots, \bar{\theta}_N) \in V^*$ , 有

$$\int d\theta \int d\bar{\theta} e^{\bar{\theta}A\theta} = \det A, \quad (\text{B.2.10})$$

其中  $A$  是  $N \times N$  normal matrix.

#### calculation:

对向量做么正变换,  $\eta = U\theta, \bar{\eta} = \bar{\theta}U^\dagger$ , 使得  $A$  对角化  $D = UAU^\dagger$ , (注意对积分顺序的定义),

$$I = \int d\eta \int d\bar{\eta} e^{\bar{\eta}D\eta} = \sum_{n=0}^{\infty} \int d\eta_N \cdots d\eta_1 \int d\bar{\eta}_1 \cdots d\bar{\eta}_N \frac{(\sum_{i=1}^N \bar{\eta}_i D_i \eta_i)^n}{n!}, \quad (\text{B.2.11})$$

其中, 唯一不为零的项是  $\propto \prod_{i=1}^N (\bar{\eta}_i D_i \eta_i)$ , 并且注意到  $(\bar{\eta}_i D_i \eta_i)$  互相对易, 所以

$$\begin{aligned} I &= \int d\eta_N \cdots d\eta_1 \int d\bar{\eta}_1 \cdots d\bar{\eta}_N \frac{n! \prod_{i=1}^N (\bar{\eta}_i D_i \eta_i)}{n!} \\ &= \int d\eta_N \cdots d\eta_1 \int d\bar{\eta}_1 \cdots d\bar{\eta}_N (\bar{\eta}_N D_N \eta_N) \cdots (\bar{\eta}_1 D_1 \eta_1) \\ &= \int d\eta_N \cdots d\eta_1 \int d\bar{\eta}_1 \cdots d\bar{\eta}_{N-1} \overbrace{(\bar{\eta}_{N-1} D_{N-1} \eta_{N-1}) \cdots (\bar{\eta}_1 D_1 \eta_1)}^{\text{commutes with } \eta_N} D_N \eta_N \\ &= \cdots = \int d\eta_N \cdots d\eta_1 D_1 \eta_1 \cdots D_N \eta_N = \prod_{i=1}^N D_i = \det A, \end{aligned} \quad (\text{B.2.12})$$

注意到, 由于  $(\bar{\eta}_i D_i \eta_i)$  互相对易, 所以  $\eta, \bar{\eta}$  的积分顺序并不重要, 唯一的要求是  $\eta$  和  $\bar{\eta}$  的积分顺序互相对应 (顺序正好相反), 即  $d\eta_j d\eta_i \leftrightarrow d\bar{\eta}_i d\bar{\eta}_j$ , (Coleman 对积分顺序的定义是  $d\eta d\bar{\eta} = d\eta_1 d\bar{\eta}_1 \cdots d\eta_N d\bar{\eta}_N$ , 这与我们的定义是等效的).

- 进一步推广,

$$Z(A, \eta, \bar{\eta}) = \int d\theta \int d\bar{\theta} e^{\bar{\theta}A\theta + \bar{\eta}\theta + \bar{\theta}\eta} = \det A e^{-\bar{\eta}A^{-1}\eta}, \quad (\text{B.2.13})$$

只需要注意到  $(\bar{\theta} + \bar{\eta}A^{-1})A(\theta + A^{-1}\eta) = \bar{\theta}A\theta + \bar{\eta}\theta + \bar{\theta}\eta + \bar{\eta}A^{-1}\eta$ , 其中  $\eta \in V, \bar{\eta} \in V^*$  都是 Grassmann number 组成的向量.

- 最后, 考虑 (B.1.5) 的变体,

$$\langle \theta_i \rangle = \langle \bar{\theta}_j \rangle = 0, \quad (\text{B.2.14})$$

$$\begin{aligned} \langle \cdots \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \cdots \rangle &= \overbrace{\langle \cdots \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \cdots \rangle}^{\text{contraction}} + \overbrace{\langle \cdots \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \cdots \rangle}^{\text{contraction}} + \cdots, \\ &= \cdots (-A_{jk}^{-1})(-A_{il}^{-1}) \cdots = \cdots (-A_{ik}^{-1})(-A_{jl}^{-1}) \cdots \end{aligned} \quad (\text{B.2.15})$$

技巧在于先把  $\cdots \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \cdots$  的顺序调整到  $\theta, \bar{\theta}$  互相对应 (像 (B.2.15) 等号右边第一项), 然后再做 contraction.

#### calculation:

考虑

$$\begin{aligned} \langle \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \rangle &= \frac{\partial}{\partial \bar{\eta}_i} \frac{\partial}{\partial \bar{\eta}_j} \left( -\frac{\partial}{\partial \eta_k} \right) \left( -\frac{\partial}{\partial \eta_l} \right) e^{-\bar{\eta}A^{-1}\eta} = \frac{\partial}{\partial \bar{\eta}_i} \frac{\partial}{\partial \bar{\eta}_j} (-\eta_{j'} A_{j'k}^{-1})(-\eta_{i'} A_{i'l}^{-1}) \\ &= \overbrace{\langle \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \rangle}^{\text{contraction}} + \overbrace{\langle \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \rangle}^{\text{contraction}} \\ &= (-A_{jk}^{-1})(-A_{il}^{-1}) = -(-A_{ik}^{-1})(-A_{jl}^{-1}) \end{aligned} \quad (\text{B.2.16})$$

# Appendix C

## perturbation theory in QM

- this chapter is based on MIT OpenCourseWare [Quantum Physics III Chapter 1: Perturbation Theory](#).

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda \delta H, \quad (\text{C.0.1})$$

其中  $\lambda \in [0, 1]$ .

- 考虑  $H^{(0)}$  的本征态为

$$H^{(0)} |k^{(0)}\rangle = E_k^{(0)} |k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)} | l^{(0)} \rangle = \delta_{kl} \\ E_0^{(0)} \leq E_1^{(0)} \leq E_2^{(0)} \leq \dots \end{cases} \quad (\text{C.0.2})$$

### C.1 non-degenerate perturbation theory

- 考虑 non-degenerate 能级  $k$ , 有  $\dots \leq E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \leq \dots$ , 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \dots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \dots \end{cases} \quad (\text{C.1.1})$$

– 注意, 我们可以选取修正项满足

$$\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \dots \quad (\text{C.1.2})$$

**proof:**

假设我们求解得到的修正项不满足  $\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \dots$ , 考虑

$$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n |k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)} | k^{(n)} \rangle' = 0, \quad (\text{C.1.3})$$

那么 (注意到态矢量可以乘一个常数,  $\frac{1}{1-a_1\lambda-a_2\lambda^2-\dots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \dots$ )

$$\begin{aligned} |k\rangle_\lambda &= (1 - a_1\lambda - a_2\lambda^2 - \dots) |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \dots \\ |k\rangle_\lambda' &= |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \dots} (\lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \dots) \\ &= |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 (a_1 |k^{(1)}\rangle' + |k^{(2)}\rangle') + \dots, \end{aligned} \quad (\text{C.1.4})$$

可见修正项都与  $|k^{(0)}\rangle$  正交.

– 注意, 不能要求  $_\lambda \langle k | k \rangle_\lambda = 1$ , 否则  $|k^{(n)}\rangle$  将与  $\lambda$  相关 (包括  $|k^{(0)}\rangle$ ),

$$\begin{aligned} _\lambda \langle k | k \rangle_\lambda &= \langle k^{(0)} | k^{(0)} \rangle \\ &\quad + \lambda (\langle k^{(1)} | k^{(0)} \rangle + \langle k^{(0)} | k^{(1)} \rangle) \\ &\quad + \lambda^2 (\langle k^{(2)} | k^{(0)} \rangle + \langle k^{(1)} | k^{(1)} \rangle + \langle k^{(0)} | k^{(2)} \rangle) \end{aligned}$$

$$\begin{aligned} & \vdots \\ & + \lambda^n (\langle k^{(n)} | k^{(0)} \rangle + \langle k^{(n-1)} | k^{(1)} \rangle + \cdots + \langle k^{(0)} | k^{(n)} \rangle). \end{aligned} \quad (\text{C.1.5})$$

- 将 (C.1.1) 代入 Schrödinger's eq., 得到:

---


$$\begin{array}{ll} \lambda^0 & (H^{(0)} - E_k^{(0)}) |k^{(0)}\rangle = 0 \\ \lambda^1 & (H^{(0)} - E_k^{(0)}) |k^{(1)}\rangle = (E_k^{(1)} - \delta H) |k^{(0)}\rangle \\ \lambda^2 & (H^{(0)} - E_k^{(0)}) |k^{(2)}\rangle = (E_k^{(1)} - \delta H) |k^{(1)}\rangle + E_k^{(2)} |k^{(0)}\rangle \\ \vdots & \vdots \\ \lambda^n & (H^{(0)} - E_k^{(0)}) |k^{(n)}\rangle = (E_k^{(1)} - \delta H) |k^{(n-1)}\rangle + E_k^{(2)} |k^{(n-2)}\rangle + \cdots + E_k^{(n)} |k^{(0)}\rangle \end{array}$$


---

**calculation:**

Schrödinger's eq. 为

$$(H^{(0)} + \lambda \delta H - E_k(\lambda)) |k\rangle_\lambda = 0, \quad (\text{C.1.6})$$

展开为

$$\left( (H^{(0)} - E_k^{(0)}) + \lambda(\delta H - E_k^{(1)}) - \lambda^2 E_k^{(2)} - \cdots \right) (|k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \cdots) = 0. \quad (\text{C.1.7})$$

- 现在来计算  $\langle l^{(0)} | k^{(n)} \rangle$ , 有

$$\begin{cases} (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(1)} \rangle = E_k^{(1)} \delta_{lk} - \delta H_{lk} \\ (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(2)} \rangle = E_k^{(1)} \langle l^{(0)} | k^{(1)} \rangle - \langle l^{(0)} | \delta H | k^{(1)} \rangle + E_k^{(2)} \delta_{lk} \\ \vdots \\ (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(n)} \rangle = E_k^{(1)} \langle l^{(0)} | k^{(n-1)} \rangle - \langle l^{(0)} | \delta H | k^{(n-1)} \rangle \\ \quad + E_k^{(2)} \langle l^{(0)} | k^{(n-2)} \rangle + \cdots + E_k^{(n)} \delta_{lk} \end{cases}, \quad (\text{C.1.8})$$

其中  $\delta H_{lk} = \langle l^{(0)} | \delta H | k^{(0)} \rangle$ , 对于满足 (C.1.2) 的解, 有

$$E_k^{(n)} = \langle k^{(0)} | \delta H | k^{(n-1)} \rangle, n = 1, 2, \cdots, \quad (\text{C.1.9})$$

并且

$$|k^{(1)}\rangle = - \sum_{l \neq k} \frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} |l^{(0)}\rangle \implies E_k^{(2)} = - \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}}. \quad (\text{C.1.10})$$

**calculation:**

将 (C.1.10) 代入 (C.1.8), 得到 ( $l \neq k$ )

$$(E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(2)} \rangle = -E_k^{(1)} \frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} + \sum_{m \neq k} \frac{\delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}, \quad (\text{C.1.11})$$

所以

$$\begin{cases} |k^{(2)}\rangle = \sum_{l \neq k} \left( - \frac{\delta H_{00} \delta H_{lk}}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m \neq k} \frac{\delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \right) |l^{(0)}\rangle \\ E_k^{(3)} = \sum_{l \neq k} \left( - \frac{\delta H_{00} |\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m \neq k} \frac{\delta H_{kl} \delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \right) \end{cases}. \quad (\text{C.1.12})$$

计算归一化系数

$${}_\lambda \langle k | k \rangle_\lambda = 1 + \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + O(\lambda^3). \quad (\text{C.1.13})$$



### C.1.1 level repulsion or the seesaw mechanism

- 能量的展开式为

$$E_k(\lambda) = E_k^{(0)} + \lambda \delta H_{kk} - \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3), \quad (\text{C.1.14})$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

### C.1.2 validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即

$$|\lambda V| < \frac{1}{2} \Delta E^{(0)}, \quad (\text{C.1.15})$$

因此, 对于能级简并的情况,  $\Delta E^{(0)} = 0$ , 情况会更复杂.

**calculation:**

对于两能级系统

$$H(\lambda) = H^{(0)} + \lambda \hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix}, \quad (\text{C.1.16})$$

$H(\lambda)$  的本征值可以直接计算,

$$E_{\pm}(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)}) \sqrt{1 + \left( \frac{\lambda |V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})} \right)^2}, \quad (\text{C.1.17})$$

考虑  $\sqrt{1+z^2}$  的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1} \frac{(2n-3)!!}{2^n n!} z^{2n} + \cdots, \quad (\text{C.1.18})$$

注意到  $\sqrt{1+z^2}$  在  $z = \pm i$  有 branch cut, 因此  $z = 0$  附近的 Taylor expansion 只有在  $|z| < 1$  内才收敛.

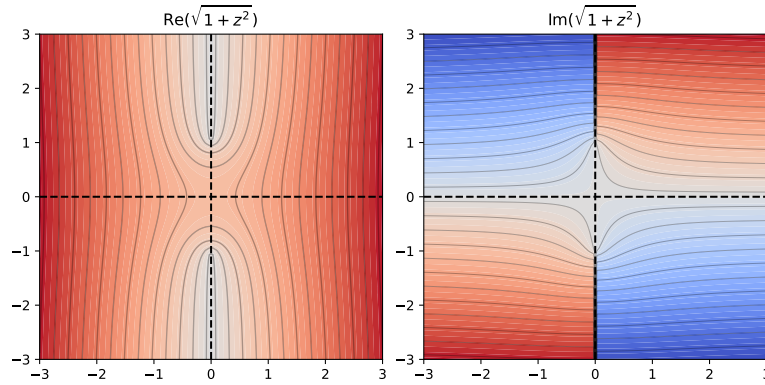


Figure C.1: graph of  $\sqrt{1+z^2}$

## C.2 degenerate perturbation theory

- 暂时先跳过.

## Appendix D

# classical field theory and Noether's theorem

### D.1 classical field theory

#### D.1.1 Lagrangian density and the action

- Lagrangian density,  $\mathcal{L}$ , 是  $\phi^a(x), \partial_\mu \phi^a(x), t$  的函数.
- 对作用量变分得到 Euler-Lagrangian equation of motion,

$$\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) = 0. \quad (\text{D.1.1})$$

calculation:

对作用量进行变分,

$$\begin{aligned} \delta S &= \int d^4x \left( \frac{\delta \mathcal{L}}{\delta \phi^a} \delta \phi^a + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \delta \partial_\mu \phi^a \right) \\ &= \int d^4x \left( \left( \frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) \right) \delta \phi^a + \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \delta \phi^a \right) \right), \end{aligned} \quad (\text{D.1.2})$$

由于边界变分为零...

#### D.1.2 canonical momentum and the Hamiltonian

- **def.:** 定义一个叫  $\pi_a^\mu$  的量,

$$\pi_a^\mu = \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)}, \quad (\text{D.1.3})$$

其中  $\pi_a \equiv \pi_a^0$  称作 canonical momentum of the field.

- **def.:** the Hamiltonian density is

$$\mathcal{H} = \pi_a \partial_0 \phi^a - \mathcal{L}. \quad (\text{D.1.4})$$

- the Hamilton's equations are

$$\begin{cases} \partial_0 \phi^a = \frac{\delta \mathcal{H}}{\delta \pi_a} \\ -\partial_0 \pi_a = \frac{\delta \mathcal{H}}{\delta \phi^a} - \partial_i \left( \frac{\delta \mathcal{H}}{\delta (\partial_i \phi^a)} \right) \end{cases}. \quad (\text{D.1.5})$$

– 第二个方程可以写成更紧凑的形式

$$\partial_\mu \pi_a^\mu = \frac{\delta \mathcal{H}}{\delta \phi^a}. \quad (\text{D.1.6})$$

## D.2 Noether's theorem

- Noether 定理中的对称性是 global symmetry, 不是 gauge symmetry.
- strictly speaking, gauge symmetry is not a symmetry but a **redundancy**.

### D.2.1 in classical particle mechanics

- 系统的 Lagrangian 为  $L(q^a, \dot{q}^a, t)$ .
- 系统通过以下形式变换,

$$q^a(t) \mapsto q^a(\lambda, t) \quad \text{and} \quad q^a(t, 0) = q^a(t), \quad (\text{D.2.1})$$

并定义

$$D_\lambda q^a = \left. \frac{\partial q^a}{\partial \lambda} \right|_{\lambda=0}. \quad (\text{D.2.2})$$

- **Noether's theorem:** the continuous transform  $\lambda$  is a **continuous symmetry** iff.

$$D_\lambda L = \frac{dF(q^a, \dot{q}^a, t)}{dt}, \quad (\text{D.2.3})$$

for some  $F(q^a, \dot{q}^a, t)$ , and the corresponding **conserved quantity** is

$$Q = p_a D_\lambda q^a - F(q^a, \dot{q}^a, t). \quad (\text{D.2.4})$$

**proof:**

$$D_\lambda L = \frac{\partial L}{\partial q^a} D_\lambda q^a + \frac{\partial L}{\partial \dot{q}^a} \frac{dD_\lambda q^a}{dt} = \frac{d}{dt}(p_a D_\lambda q^a). \quad (\text{D.2.5})$$

- 几个例子如下,

- **空间平移**,  $\vec{x}(t) \mapsto \vec{x}(t) + \hat{e}_i \lambda$ , 相应地,  $D_\lambda \vec{x} = \hat{e}_i$ , 且

$$D_\lambda L = \frac{\partial L}{\partial x^i}, \quad (\text{D.2.6})$$

如果  $\frac{\partial L}{\partial x^i} = 0$ , 那么, 有守恒量  $p_i$ .

- **时间平移**,  $q^a(t) \mapsto q^a(t + \lambda)$ , 相应地,  $D_\lambda q^a = \dot{q}^a$ , 且

$$D_\lambda L = \frac{dL}{dt} - \frac{\partial L}{\partial t}, \quad (\text{D.2.7})$$

如果  $\frac{\partial L}{\partial t} = 0$ , 那么, 有守恒量  $H = p_a \dot{q}^a - L$ .

- **转动**,  $\vec{x}(t) \mapsto R(\lambda, \hat{e}) \cdot \vec{x}(t)$ , 相应地,  $D_\lambda \vec{x} = \hat{e} \times \vec{x}$ , 且

$$D_\lambda L = \vec{x} \cdot \left( \frac{\partial L}{\partial \vec{x}} \times \hat{e} \right) + \hat{e} \cdot (\dot{\vec{x}} \times \vec{p}), \quad (\text{D.2.8})$$

如果上式中两个括号内的项都为零, 那么, 有守恒量  $\hat{e} \cdot \vec{J} = \hat{e} \cdot (\vec{x} \times \vec{p})$ .

### D.2.2 in classical field theory

- 类似地, 系统通过以下形式变换,

$$\phi^a(x) \mapsto \phi^a(x, \lambda) \quad \text{and} \quad \phi^a(x, 0) = \phi^a(x), \quad (\text{D.2.9})$$

并定义

$$D_\lambda \phi^a = \left. \frac{\partial \phi^a}{\partial \lambda} \right|_{\lambda=0}. \quad (\text{D.2.10})$$

- **Noether's theorem:** the continuous transform  $\lambda$  is a **continuous symmetry** iff.

$$D_\lambda \mathcal{L} = \partial_\mu F^\mu(\phi^a, \partial_\mu \phi^a, t), \quad (\text{D.2.11})$$

for some  $F^\mu(\phi^a, \partial_\mu \phi^a, t)$ , and the **conserved current** is

$$J^\mu = \pi_a^\mu D_\lambda \phi^a - F^\mu. \quad (\text{D.2.12})$$

proof:

$$\begin{aligned} D_\lambda \mathcal{L} &= \frac{\delta \mathcal{L}}{\delta \phi^a} D_\lambda \phi^a + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \partial_\mu D_\lambda \phi^a \\ &= \left( \frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) \right) D_\lambda \phi^a + \partial_\mu \underbrace{\left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} D_\lambda \phi^a \right)}_{=\pi_a^\mu}, \end{aligned} \quad (\text{D.2.13})$$

代入 (D.1.1), 得...

- 注意, conserved current 并不是唯一确定的, 考虑如下变换,

$$F^\mu \mapsto F'^\mu = F^\mu + \partial_\nu A^{\mu\nu} \quad \text{with} \quad A^{\mu\nu} = A^{[\mu\nu]}, \quad (\text{D.2.14})$$

新  $F'^\mu$  依然能满足 (D.2.11).

- 但是, 守恒荷是唯一确定的.

proof:

$$Q' = \int d^3x J^0 = \int d^3x (\pi_a D_\lambda \phi^a - F^0) - \int d^3x \partial_\mu A^{0\mu}, \quad (\text{D.2.15})$$

考虑到边界值为零, 且  $A^{00} = 0$ , 所以  $Q' = Q$ .

### D.2.3 spacetime translations and the energy-momentum tensor

- 时空平移变换为

$$\phi^a(x) \mapsto \phi^a(x + \lambda e). \quad (\text{D.2.16})$$

- 所以

$$D_\lambda \phi^a = e^\mu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = e^\mu \partial_\mu \mathcal{L}, \quad (\text{D.2.17})$$

代入 (D.2.12),

$$J^\mu = e^\nu \underbrace{(\pi_a^\mu \partial_\nu \phi^a - \delta_\nu^\mu \mathcal{L})}_{=T^\mu{}_\nu}. \quad (\text{D.2.18})$$

- 并且有

$$\partial_\mu T^{\mu\nu} = 0 \implies P^\mu = \int d^3x T^{0\mu} = \text{Const.}, \quad (\text{D.2.19})$$

来自守恒流散度为零.

### D.2.4 Lorentz transformations, angular momentum and something else

- Lorentz transformation 下坐标做变换  $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ , 其中  $\Lambda$  满足

$$\eta = \Lambda^T \eta \Lambda. \quad (\text{D.2.20})$$

- infinitesimal Lorentz transformation 是

$$\Lambda = I + \epsilon, \quad (\text{D.2.21})$$

其中  $\{\epsilon^{\mu\nu}\} = \epsilon \eta$  是反对称矩阵.

proof:

考虑

$$\eta = (\Lambda \eta)^T \eta (\Lambda \eta) = (\eta + \epsilon \eta)^T \eta (\eta + \epsilon \eta)$$

$$= \eta + \eta \epsilon^T + \epsilon \eta + O(\epsilon^2). \quad (\text{D.2.22})$$

- 标量场在 Lorentz transform 下的变换为

$$\Lambda : \phi^a(x) \mapsto \phi^a(\Lambda^{-1}x'). \quad (\text{D.2.23})$$

- 有

$$D_\lambda \phi^a = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\epsilon_{\mu\nu} \partial^\mu (x^\nu \mathcal{L}), \quad (\text{D.2.24})$$

代入 (D.2.12),

$$J^\mu = \frac{1}{2} \epsilon_{\nu\rho} M^{\mu\nu\rho} \quad \text{where} \quad M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}, \quad (\text{D.2.25})$$

且有

$$\partial_\mu M^{\mu\nu\rho} = 0. \quad (\text{D.2.26})$$

- 对全空间积分, 得到 6 个守恒量,

$$J^{\mu\nu} = \int d^3x M^{0\mu\nu} = \text{Const.}, \quad (\text{D.2.27})$$

不难发现  $J^{ij}$  对应角动量. 现在来讨论  $J^{0i}$  的物理意义,

$$0 = \frac{d}{dt} J^{0i} = \frac{d}{dt} \int d^3x (x^i T^{00} - t T^{0i}) = P^i - \frac{d}{dt} \int d^3x x^i T^{00}, \quad (\text{D.2.28})$$

其中, 用到了  $\frac{dP^i}{dt} = 0$  (见 (D.2.19)), 可以将上式的第二项理解为质心运动的动量.

### D.3 charge as generators

- the charge associated with the conserved current is

$$Q = \int d^D x J^0 = \int d^D x (\pi_a D_\lambda \phi^a - F^0), \quad (\text{D.3.1})$$

在  $F^\mu = 0$  且  $[D_\lambda \phi^a, \phi^a] = 0$  的情况下,

$$i[Q, \phi^a] = D_\lambda \phi^a. \quad (\text{D.3.2})$$

### D.4 what the graviton listens to: energy-momentum tensor

- the energy-momentum tensor is defined as (其中  $g = |\det\{g_{\mu\nu}\}|$ )

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2 \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M. \quad (\text{D.4.1})$$

- 如果将  $\mathcal{L}_M$  对  $g^{\mu\nu}$  做展开  $\mathcal{L}_M = A + g^{\mu\nu} B_{\mu\nu} + g^{\mu\nu} g^{\rho\sigma} C_{\mu\nu\rho\sigma} + \dots$ , 那么

$$T_{\mu\nu} = -2(B_{\mu\nu} + 2g^{\rho\sigma} C_{\mu\nu\rho\sigma} + 3\cdots) + g_{\mu\nu} \mathcal{L}_M, \quad (\text{D.4.2})$$

另外, the trace of the energy-momentum tensor is

$$T = g^{\mu\nu} T_{\mu\nu} = d \times A + (d-2)g^{\mu\nu} B_{\mu\nu} + (d-4)g^{\mu\nu} g^{\rho\sigma} C_{\mu\nu\rho\sigma}, \quad (\text{D.4.3})$$

可见  $d=4$  时,  $T$  与  $C_{\mu\nu\rho\sigma}$  无关.

- $\mathcal{L} = -\frac{1}{2}((\partial\phi)^2 - m^2\phi^2)$  和  $\mathcal{L} = \frac{1}{2}\phi(\partial^2 - m^2)\phi$  对应的 energy-momentum tensor 一样吗 (?).

### D.4.1 example: energy-momentum tensor of the electromagnetic field

- 在  $(+, -, -, -)$  号差下, 定义  $A_\mu = (\phi, -\vec{A})$ , 这很容易让人误以为 4-potential 是 vector, 而事实上它是 covector, 按照这种定义

$$\frac{\delta F_{\rho\sigma}}{\delta g^{\mu\nu}} = 0. \quad (\text{D.4.4})$$

– Wikipedia: [Maxwell's equations in curved spacetime, Electromagnetic potential](#).

- 代入电磁场的 Lagrangian, 见 (11.2.1), 所以

$$T_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \mu^2 A_\mu A_\nu + g_{\mu\nu} \mathcal{L} \quad (\text{D.4.5})$$

– 在 Minkowski 时空中,

$$\begin{aligned} T_{\mu\nu} &= \begin{pmatrix} -\mathcal{E} - \frac{1}{2}\mu^2(\phi^2 + |\vec{A}|^2) & (\vec{E} \times \vec{B})_i - \mu^2\phi A_i \\ \vdots & -\delta_{ij}(\mathcal{E} + \frac{1}{2}\mu^2(\phi^2 - |\vec{A}|^2)) - \mu^2 A_i A_j + E_i E_j + B_i B_j \end{pmatrix} \\ &= \begin{pmatrix} -\mathcal{E} & (\vec{E} \times \vec{B})_i \\ \vdots & -\delta_{ij}\mathcal{E} + E_i E_j + B_i B_j \end{pmatrix} - \mu^2 \begin{pmatrix} \frac{1}{2}(\phi^2 + |\vec{A}|^2) & \phi A_i \\ \vdots & \frac{\delta_{ij}}{2}(\phi^2 - |\vec{A}|^2) + A_i A_j \end{pmatrix}, \end{aligned} \quad (\text{D.4.6})$$

其中  $\mathcal{E} = \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2)$ .

**calculate:**

在 Minkowski 时空中

$$\begin{cases} \mathcal{L} = \frac{1}{2}(|\vec{E}|^2 - |\vec{B}|^2) + \frac{1}{2}\mu^2(\phi^2 - |\vec{A}|^2) \\ F_0{}^\mu F_{0\mu} = -|\vec{E}|^2 \\ F_i{}^\mu F_{0\mu} = (\vec{E} \times \vec{B})_i \\ F_i{}^\mu F_{j\mu} = E_i E_j - \delta_j^i |\vec{B}|^2 + B_i B_j \end{cases}. \quad (\text{D.4.7})$$

– 另外注意到

$$T = -\mu^2 A^\mu A_\mu, \quad (\text{D.4.8})$$

可见 the energy-momentum tensor of electromagnetic field (when  $m = 0$ ) is traceless.

## Appendix E

# antiunitary operator and time reversal

### E.1 complex conjugation operator

- complex conjugation operator,  $K$ , is an antiunitary operator on the complex plane,

$$\begin{cases} Kz = z^* \\ zK^* = z^* \end{cases} \implies K^2 = K^{*2} = 1. \quad (\text{E.1.1})$$

- $K^*I : V^* \rightarrow V^*$  是 dual space 上的算符.
- 对于一组 orthonormal basis, 有

$$\langle i | K^* I K | j \rangle = \delta_{ij}, \quad (\text{E.1.2})$$

并且可以证明在基矢变换后这个等式依然成立.

**proof:**

- 对基矢做 unitary transformation,

$$|i'\rangle = U |i\rangle = \sum_j |j\rangle U_{ji} \quad \text{where} \quad U_{ji} = \langle j | U | i \rangle, \quad (\text{E.1.3})$$

那么

$$\langle i' | K^* I K | j' \rangle = \sum_{kl} \langle k | U_{ki}^* K^* I K U_{lj} | l \rangle = \sum_{kl} U_{ki} U_{lj}^* \delta_{kl} = \delta_{ij}. \quad (\text{E.1.4})$$

- 对基矢做 antiunitary transformation, 只需要证明  $|i'\rangle = K |i\rangle$  的情况, 此时

$$\langle i' | K^* I K | j' \rangle = \langle i | j \rangle = \delta_{ij}. \quad (\text{E.1.5})$$

### E.2 antiunitary operator

- 对于一个 unitary operator,  $U$ ,  $\Omega = UK$  是一个 antiunitary operator.
- 定义其 Hermitian conjugate

$$\Omega^\dagger = K^* U^\dagger \iff \langle i | \Omega j \rangle = \langle j | \Omega^\dagger i \rangle^*, \quad (\text{E.2.1})$$

那么

$$\begin{cases} \langle \phi | \Omega \psi \rangle = \langle \psi | \Omega^\dagger \phi \rangle^* \\ \langle \Omega \phi | \Omega \psi \rangle = \langle \psi | \phi \rangle \end{cases}. \quad (\text{E.2.2})$$

**proof:**

首先,

$$\langle \phi | \Omega \psi \rangle = \sum_{ij} \langle i | \phi_i^* U K \psi_j | j \rangle$$

$$\begin{aligned}
&= \sum_{ij} \phi_i^* \psi_j^* \langle i|UK|j\rangle \\
&= \left( \sum_{ij} \langle j|K^*U^\dagger|i\rangle \phi_i \psi_j \right)^* \\
&= \left( \sum_{ij} \langle j|\psi_j^* K^*U^\dagger \phi_i|i\rangle \right)^* = \langle \psi|K^*U^\dagger|\phi\rangle^*, \tag{E.2.3}
\end{aligned}$$

其次,

$$\begin{aligned}
\langle \Omega\phi|\Omega\psi\rangle &= \langle \phi|\Omega^\dagger\Omega\psi\rangle = \langle \phi|K^*IK|\psi\rangle \\
&= \sum_{ij} \langle i|\phi_i^* K^*IK\psi_j|j\rangle \\
&= \sum_{ij} \phi_i \psi_j^* \langle i|K^*IK|j\rangle = \langle \psi|\phi\rangle. \tag{E.2.4}
\end{aligned}$$

### E.3 time reversal in QM

- 在量子力学中,

$$\mathcal{T}: |\psi\rangle \mapsto |\psi'(t')\rangle = \int d^D x |x\rangle K \langle x|\psi(t)\rangle, \quad \text{where } t' = -t. \tag{E.3.1}$$

- 因此, 对于动量本征态,

$$T|p\rangle = \int d^D x |x\rangle K e^{i\vec{p}\cdot\vec{x}} = |-p\rangle. \tag{E.3.2}$$

- 对于动量算符,

$$TPT^\dagger = \int d^D p |-p\rangle p \langle -p| = -P. \tag{E.3.3}$$

- 对于角动量算符,

$$TLT^\dagger = T(X \times P)T^{-1} = -L. \tag{E.3.4}$$

- 对于平面波,

$$\psi(t) = e^{i(\vec{k}\cdot\vec{x}-Et)} \mapsto \psi'(t') = \langle x|K^*IK|\psi(t)\rangle = e^{-i(\vec{k}\cdot\vec{x}-Et)}, \tag{E.3.5}$$

注意到  $t' = -t$ , 代入,

$$\psi'(t) = e^{i(-\vec{k}\cdot\vec{x}-Et)}. \tag{E.3.6}$$

#### E.3.1 spin- $\frac{1}{2}$ non-relativistic electron

- 时间反演算符作用到 spin-up state 应该得到 spin-down state, 所以

$$T = \sigma_2 K. \tag{E.3.7}$$

- 因此

$$T^2 = \sigma_2 K \sigma_2 K = \sigma_2^* \sigma_2 = -1. \tag{E.3.8}$$

- 具体地,

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix}. \tag{E.3.9}$$

- Kramer's degeneracy:** 含有奇数个电子的时间反演不变系统, 其能级是 twofold degenerate.

**proof:**

因为系统时间反演不变, 所以  $\psi$  和  $T\psi$  有相同的能级, 且  $T\psi \neq e^{i\alpha}\psi, \forall \alpha$ .

考虑  $T\psi = e^{i\alpha}\psi$ , 那么

$$T^2\psi = T e^{i\alpha}\psi = e^{-i\alpha} e^{i\alpha}\psi = \psi, \tag{E.3.10}$$

与  $T^2 = -1$  矛盾.