# Quantum Field Theory

## a study note based on A. Zee's textbook

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https://github.com/siyang03/my-note---A.-Zee-QFT

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# Contents

# convention, notation, and units

- 笔记中的**度规号差**约定为 $(-,+,+,+)$.
- 使用 Planck units, 此时 $G, \hbar, c, k_B = 1$, 因此,

| name/dimension | expression/value |
|---|---|
| Planck length ($L$) | $l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$ |
| Planck time ($T$) | $t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$ |
| Planck mass ($M$) | $m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$ |
| Planck temperature ($\Theta$) | $T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$ |

- 时空维度用 $d = D + 1$ 表示.

# Part I

# motivation and foundation

# Chapter 1

# free field theory

## 1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{1.1.1}$$

A. Zee 说: 在作用量里, 时间的导数项必须是正的, 包括标量场的 $(\partial_0\phi)^2$ 和电磁场的 $(\partial_0 A_i)^2$.

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi\, e^{i\int d^d x\,(-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \tag{1.1.2}$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

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- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^d x d^d y\, J(x)D(x-y)J(y)} \tag{1.1.3}$$

另外, 用 $W(J)$ 表示指数上的部分 (去除掉虚数 $i$).

**proof:**

注意 $\partial^\mu\phi\partial_\mu\phi = \partial^\mu(\phi\partial_\mu\phi) - \phi\partial^2\phi$, 忽略全微分项, 那么,

$$Z(J) = \int D\phi\, e^{i\int d^d x\,(\frac{1}{2}\phi(\partial^2 - m^2)\phi + J(x)\phi(x))} \tag{1.1.4}$$

代入 (B.1.1), 可知,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^d x d^d y\, J(x)D(x-y)J(y)} \tag{1.1.5}$$

其中 $D(x-y)$ 满足,

$$\begin{cases}(\partial^2 - m^2)D(x-y) = \delta^{(d)}(x-y)\\ (-p^2 - m^2)\tilde{D}(p,q) = (2\pi)^d\delta^{(d)}(p-q)\end{cases} \Longrightarrow D(x-y) = \int \frac{d^d k}{(2\pi)^d}\frac{e^{ik\cdot(x-y)}}{-k^2 - m^2} \tag{1.1.6}$$

## 1.2 free propagator

- 为了使 (1.1.4) 中的积分在 $\phi$ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon\int d^d x\phi^2}$.
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 $\epsilon$ 避免发散,

$$D(x) = \int \frac{d^d k}{(2\pi)^d}\frac{e^{ik\cdot x}}{-k^2 - m^2 + i\epsilon} = -i\int \frac{d^D k}{(2\pi)^D 2\omega_k}\Big(\theta(t)e^{i(-\omega_k t + \vec{k}\cdot\vec{x})} + \theta(-t)e^{i(\omega_k t + \vec{k}\cdot\vec{x})}\Big) \tag{1.2.1}$$

**calculation:**

对 $k^0$ 积分, 注意有两个奇点 $k^0 = \pm(\omega_k - i\epsilon)$, 当 $t > 0$ 时, contour 处于下半平面, ... (另外注意到我们可以任意改变 $\vec{k}$ 的符号).

- $D(x)$ 的取値与 $x$ 的类时, 类空性质关系密切.
  - 类时区域,
$$D(t,0) = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\left(\theta(t)e^{-i\omega_k t} + \theta(-t)e^{i\omega_k t}\right) \tag{1.2.2}$$
  - 类空区域,
$$D(0,\vec{x}) = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \sim e^{-m|\vec{x}|} \tag{1.2.3}$$

## 1.3 from field to particle to force

### 1.3.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,

$$\begin{aligned} W(J) &= -\frac{1}{2}\int d^dx d^dy\, J(y)D(x-y)J(y) && (1.3.1)\\ &= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}(-k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}(k) && (1.3.2)\end{aligned}$$

其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k) = \tilde{J}^*(k)$.

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- 考虑 $J(x) = J_1(x) + J_2(x)$, 那么 $W(J)$ 共有 4 项, 其中一个交叉项如下,

$$W_{12}(J) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}_1(-k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}_2(k) \tag{1.3.3}$$

可见 $W(J)$ 取値较大的条件是:

1. $\tilde{J}_1(k), \tilde{J}_2(k)$ 有较大重叠,
2. 重叠位置的 $k$ 是 on shell (即 $k^2 = -m^2$).

- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

### 1.3.2 from particle to force

- 考虑 $J(x) = \delta^{(D)}(\vec{x}-\vec{x}_1) + \delta^{(D)}(\vec{x}-\vec{x}_1) \Longrightarrow \tilde{J}_a(k) = 2\pi e^{-i\vec{k}\cdot\vec{x}_a}\delta(k^0)$, 那么,

$$\begin{aligned} W_{12}(J) + W_{21}(J) &= \delta(0)\int \frac{d^Dk}{(2\pi)^{D-1}}\frac{1}{|\vec{k}|^2+m^2-i\epsilon}\cos(\vec{k}\cdot(\vec{x}_1-\vec{x}_2))\\ &\overset{D=3}{=} 2\pi\delta(0)\frac{1}{4\pi r}e^{-mr} && (1.3.4)\end{aligned}$$

($-i\epsilon$ 显然可以舍去), 注意到 $\langle 0|e^{-iHT}|0\rangle = e^{-iET}$, 而时间间隔 $T = \int dx^0 = 2\pi\delta(0)$, 所以,

$$E = -\frac{W(J)}{T} \overset{D=3}{=} -\frac{1}{4\pi r}e^{-mr} \tag{1.3.5}$$

**calculation:**

计算 (1.3.4) 中的积分, 令 $\vec{x}_1 - \vec{x}_2 = \vec{r}$,

$$I_D = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{|\vec{k}|^2+m^2}\overbrace{\cos(\vec{k}\cdot\vec{r})}^{\mapsto e^{i\vec{k}\cdot\vec{r}}}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^D} \int (k\sin\theta_1)^{D-2} d\Omega_{D-2} \int k d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ikr\cos\theta_1} \\
&= \frac{S_{D-2}}{(2\pi)^D} \int k^{D-1} \sin^{D-2}\theta_1 d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ikr\cos\theta_1}
\end{aligned} \tag{1.3.6}
$$

取 $D=3$, 那么,

$$
\begin{aligned}
I_{D=3} &= \frac{1}{(2\pi)^2} \int k^2 \sin\theta_1 d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ik\cos\theta_1} \\
&= \frac{1}{2\pi^2 r} \int_0^\infty \sin(kr) \frac{kdk}{k^2+m^2} = \frac{-i}{4\pi^2 r} \int_{-\infty}^\infty e^{ikr} \frac{kdk}{k^2+m^2} \\
&= \frac{-i}{4\pi^2 r} 2\pi i \underbrace{\mathrm{Res}(f, im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r} e^{-mr}
\end{aligned} \tag{1.3.7}
$$

# Chapter 2

# Coulomb and Newton: repulsive and attraction

## 2.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{2.1.1}$$

其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

- 做路径积分,

$$Z(J) = \int DA\, e^{i\int d^dx(\mathcal{L}+J_\mu A^\mu)} = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J_\mu D^{\mu\nu}(x-y)J_\nu(y)} \tag{2.1.2}$$

**calculation:**

massive photon 的作用量为,

$$\begin{aligned}
S(A) &= \int d^dx\,\frac{1}{2}\Big(-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu\Big) \\
&= \int d^dx\,\frac{1}{2}\Big(A_\nu\partial^2 A^\nu - A_\nu\partial^\nu\partial_\mu A^\mu - m^2 A_\mu A^\mu\Big) + \text{total differential} \\
&= \int d^dx\,\frac{1}{2}A_\mu\Big(-\partial^\mu\partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2)\Big)A_\nu + \text{total differential} \\
&= \int \frac{d^dk}{(2\pi)^d}\tilde{A}_\mu(-k)\Big(k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2)\Big)\tilde{A}_\nu(k) + \text{boundary term}
\end{aligned} \tag{2.1.3}$$

那么, 需要有,

$$\begin{aligned}
&(-\partial^\mu\partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2))D_{\rho\nu}(x-y) = \delta^\mu_\nu\delta^{(d)}(x-y) \\
\Longrightarrow &\tilde{D}_{\mu\nu}(k) = \frac{k_\mu k_\nu/m^2 + \eta_{\mu\nu}}{-k^2 - m^2}
\end{aligned} \tag{2.1.4}$$

考虑到积分需要收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, (为什么 $A_\mu$ 类空, 只知道 $\tilde{A}_\mu$ 类空, 见 subsection 2.1.2, 但路径积分中的 $A$ 显然不满足 field equation $\Longrightarrow$ 路径积分中起主要作用的 $\tilde{A}$ 类空, 因此 $-\epsilon|\tilde{A}|^2 < 0$).

- 因此,

$$\begin{aligned}
W(J) &= -\frac{1}{2}\int d^dx d^dy\, J_\mu(x)D^{\mu\nu}(x-y)J_\nu(y) && (2.1.5) \\
&= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}_\mu(-k)\frac{k^\mu k^\nu/m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon}\tilde{J}_\nu(k) && (2.1.6)
\end{aligned}$$

注意到 current conservation, 有 $\partial_\mu J^\mu = 0 \iff k^\mu\tilde{J}_\mu(k) = 0$, 所以,

$$W(J) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}^\mu(-k)\frac{1}{-k^2 - m^2 + i\epsilon}\tilde{J}_\mu(k) \tag{2.1.7}$$

观察电荷分量, 可见同性相斥, 异性相吸.

### 2.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的 $x, y, z$ 方向, 在粒子静止系下, 极化矢量 $(\epsilon^i)_\mu = \delta^i_\mu, i = 1, 2, 3$, 而 $k_\mu = (-m, 0, 0, 0)$, 所以,

$$k^\mu (\epsilon^i)_\mu = 0 \tag{2.1.8}$$

  - 注意, 一个粒子的极化方向用 $e^i$ (这不是矢量) 表示, 极化矢量为 $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$.

- 在粒子静止系下, 考虑,

$$\sum_{i=1}^{3} (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu} \tag{2.1.9}$$

  可见,

$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon} \tag{2.1.10}$$

### 2.1.2 Maxwell Lagrangian

- 根据 (2.1.1) 中的 Lagrangian density, 得到 field equation 如下,

$$\Big( -\partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \Big) A_\nu \tag{2.1.11}$$

  - spin-1 particle 有 3 个自旋自由度, 而 $A_\mu$ 有 4 个分量, 所以需要一个约束方程,

$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0 \tag{2.1.12}$$

    实际上在 (2.1.11) 左右两边作用一个 $\partial_\mu$ 即可得到这个约束方程.

## 2.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = linearized Einstein Lagrangian.
- 受 subsection 2.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足,

$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases} \tag{2.2.1}$$

  其中下指标 $\mu, \nu$ 对称, $a = 1, \cdots, 5$, (可以验证 $(\epsilon^a)_{\mu\nu}$ 确实有 5 个独立分量).

  - 对 $(\epsilon^a)_{\mu\nu}$ 的归一化条件可以定义为 $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$.
  - 与 subsection 2.1.1 中提示一样, 粒子的极化方向用 $e^a$ 表示.

- 那么,

$$\sum_{a=1}^{5} (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma} \tag{2.2.2}$$

**calculation:**

首先用 $k_\mu$ 和 $\eta_{\mu\nu}$ 构造最一般的关于 $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$ 对称的 4 阶张量, (下式中把 $\frac{k_\mu}{m}$ 略写作 $k_\mu$),

$$\begin{aligned} & A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ & + D\eta_{\mu\nu}\eta_{\rho\sigma} + E(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \end{aligned} \tag{2.2.3}$$

代入 (2.2.1) 得,

$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4} \tag{2.2.4}$$

因此, 这个 4 阶张量最终确定为,

$$\frac{3}{4}A\Big((G_{\mu\rho}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\rho})-\frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\Big) \tag{2.2.5}$$

- 所以,

$$\tilde{D}_{\mu\nu\rho\sigma}(k)=\frac{(G_{\mu\rho}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\rho})-\frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2-m^2+i\epsilon} \tag{2.2.6}$$

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- 计算路径积分中的 $W(T)$,

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{(G^{\mu\rho}G^{\nu\sigma}+G^{\mu\sigma}G^{\nu\rho})-\frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2-m^2+i\epsilon}\tilde{T}_{\rho\sigma}(k) \tag{2.2.7}$$

注意到 $\partial_\mu T^{\mu\nu}(x)=0 \iff k_\mu\tilde{T}^{\mu\nu}(k)=0$, 并考虑到 $T$ 是对称张量, 所以,

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{2\eta^{\mu\rho}\eta^{\nu\sigma}-\frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2-m^2+i\epsilon}\tilde{T}_{\rho\sigma}(k) \tag{2.2.8}$$

考虑能量项, 可见质量互相吸引.

## 2.3 remarks

- 由于 seesaw mechanism (见 subsection C.1.1), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为 $\eta^{00}=-1$.
- 本 chapter 中的计算都是 $m\neq 0$ 的粒子, 与真实世界有差异.

# Chapter 3

# Feynman diagrams

## 3.1 a baby problem

- 考虑如下积分,

$$Z(J) = \int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq} \tag{3.1.1}$$

- **Schwinger's way:** 把 integrand 对 $\lambda$ 展开, 并将 $q$ 用 $\frac{\partial}{\partial J}$ 替代, 得到,

$$\begin{aligned} Z(J) &= e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4} \int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2 + Jq} \\ &= \sqrt{\frac{2\pi}{m^2}}\, e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4} e^{\frac{J^2}{2m^2}} \end{aligned} \tag{3.1.2}$$

  后面的计算中忽略 $Z(J=0, \lambda=0)$.

- 每个 vertex 带有 $-\lambda$, 每个 line 带有 $\frac{1}{m^2}$, 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里 $q$ 无法区分, 所以不方便算, 先略了),

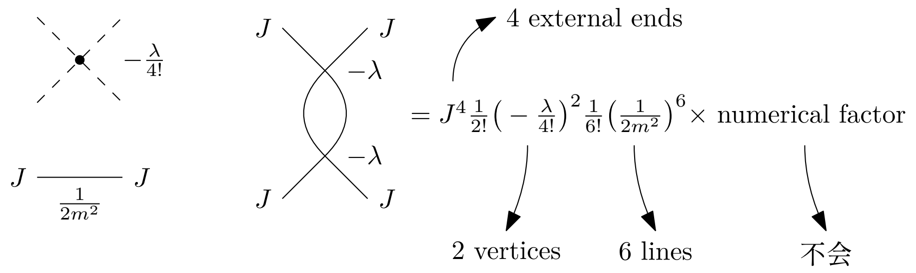
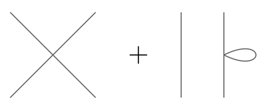



Figure 3.1: baby problem - Feynman diagram

**calculation:**

在这里计算 $\lambda J^4$ 项,



$$+ \quad \text{具体暂时不会算} \tag{3.1.3}$$

但是直接计算 (3.1.2) 的展开项, 得到的结果与 (3.1.5) 一样.

### 3.1.1 Wick contraction and Green's functions

- 把积分 (3.1.1) 对 $J$ 展开,

$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} J^n \underbrace{\int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4} q^n}_{=Z(0,0)G^{(n)}} \tag{3.1.4}$$

  其中 Green's function $G^{(n)}$ 对 $\lambda$ 展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

**calculation:**

计算 $\lambda J^4$ 项, 它来自 $G^{(4)}$ 对 $\lambda$ 展开的一阶项,

$$\begin{aligned}-\frac{\lambda}{4!}\int dq\, e^{-\frac{1}{2}m^2q^2}q^8 &= -\frac{\lambda}{4!}\langle q^8\rangle \\ &= -\frac{\lambda}{4!}\sum_{\text{Wick}}\left(\frac{1}{m^2}\right)^4 \\ &= -\frac{\lambda}{4!}\frac{7\times 5\times 3\times 1}{m^8}\end{aligned} \tag{3.1.5}$$

所以 $\lambda J^4$ 项等于 $\frac{105}{(4!)^2}\frac{-\lambda J^4}{m^8}$.

### 3.1.2 connected vs. disconnected

- 考虑,
$$Z(J,\lambda)=Z(J=0,\lambda)e^{W(J,\lambda)} \tag{3.1.6}$$
其中 $Z(J=0,\lambda)$ 由 diagrams with no external source $J$ 组成, 而 $W(J,\lambda)$ 则由 connected diagrams 组成 (?).

- 我们希望计算的是 $W$, 而不是 $Z$ (?).

## 3.2 a child problem

- 考虑如下积分,
$$Z(J)=\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q-\frac{\lambda}{4!}q^4+J^T\cdot q} \tag{3.2.1}$$
其中 $q^4=\sum_i q_i^4$.

- **Schwinger's way:** 对 $\lambda$ 展开并把 $q$ 替换为 $\frac{\partial}{\partial J}$, 得到,
$$Z(J)=\sqrt{\frac{(2\pi)^N}{\det A}}e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \tag{3.2.2}$$
其中 $\left(\frac{\partial}{\partial J}\right)^4=\sum_i\left(\frac{\partial}{\partial J_i}\right)^4$.

### 3.2.1 $n$-point Green's function

- **Wick's way:** 对 $J$ 展开获得带 Green's function 的表达式,
$$Z(J)=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}J_{i_1}\cdots J_{i_n}\underbrace{\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q-\frac{\lambda}{4!}q^4}q_{i_1}\cdots q_{i_n}}_{=Z(0,0)G^{(n)}_{i_1\cdots i_n}} \tag{3.2.3}$$
其中 $G^{(n)}_{i_1\cdots i_n}$ 称为 $n$-point Green's function.

**Taylor expansion:**

多元函数的 Taylor 展开如下,

$$\begin{aligned}f(x_1,\cdots,x_N) &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_N^{n_N}}{n_N!}\frac{\partial^{n_1}}{\partial x_1^{n_1}}\cdots\frac{\partial^{n_N}}{\partial x_N^{n_N}}f(x=0) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}x_{i_1}\cdots x_{i_n}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_N}}f(x=0)\end{aligned} \tag{3.2.4}$$

这两种求和方法, $x_1^{n_1}\cdots x_N^{n_N}$ 项的 numerical factor 都等于,

$$\frac{1}{n!}\times\frac{n!}{n_1!\cdots n_N!}=\frac{1}{n_1!\cdots n_N!} \tag{3.2.5}$$

其中 $n=n_1+\cdots+n_N$.

- 在 $\lambda=0$ 时, 2-point Green's function 就是 propagator,

$$\begin{aligned}G_{ij}^{(2)}(\lambda=0)&=\frac{1}{Z(0,0)}\int dq_1\cdots dq_N\,e^{-\frac{1}{2}q^T\cdot A\cdot q}q_iq_j\\&=\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J}\Big|_{J=0}=A_{ij}^{-1}\end{aligned}\tag{3.2.6}$$

- 来计算 $2,3,4$-point Green's functions,

$$\begin{cases}G_{ij}^{(2)}=A_{ij}^{-1}-\frac{\lambda}{4!}\sum_m(3A_{mm}^{-1}A_{mm}^{-1}A_{ij}^{-1}+12A_{mm}^{-1}A_{mi}^{-1}A_{mj}^{-1})+O(\lambda^2)\\G_{ijk}^{(3)}=0\\G_{ijkl}^{(4)}=A_{ij}^{-1}A_{kl}^{-1}+A_{ik}^{-1}A_{jl}^{-1}+A_{il}^{-1}A_{jk}^{-1}\\\qquad-\frac{\lambda}{4!}\sum_m(A_{mm}^{-1}A_{mm}^{-1}A_{ij}^{-1}A_{kl}^{-1}+\cdots+4!A_{im}^{-1}A_{jm}^{-1}A_{km}^{-1}A_{lm}^{-1})+O(\lambda^2)\end{cases}\tag{3.2.7}$$

**calculation:**

2-point Green's function 计算如下,

$$\begin{aligned}G_{ij}^{(2)}&=\frac{1}{Z(0,0)}\int dq_1\cdots dq_N\,e^{-\frac{1}{2}q^T\cdot A\cdot q}\Big(1-\frac{\lambda}{4!}q^4+O(\lambda^2)\Big)q_iq_j\\&=A_{ij}^{-1}-\frac{\lambda}{4!}\langle q^4q_iq_j\rangle+O(\lambda^2)\\&=A_{ij}^{-1}-\frac{\lambda}{4!}\sum_m(3A_{mm}^{-1}A_{mm}^{-1}A_{ij}^{-1}+12A_{mm}^{-1}A_{mi}^{-1}A_{mj}^{-1})+O(\lambda^2)\end{aligned}\tag{3.2.8}$$

3-point Green's function 计算如下,

$$G_{ijk}^{(32)}=\frac{1}{Z(0,0)}\int dq_1\cdots dq_N\,e^{-\frac{1}{2}q^T\cdot A\cdot q}\Big(1-\frac{\lambda}{4!}q^4+O(\lambda^2)\Big)q_iq_jq_k=0\tag{3.2.9}$$

4-point Green's function 计算如下,

$$\begin{aligned}G_{ijkl}^{(4)}&=\frac{1}{Z(0,0)}\int dq_1\cdots dq_N\,e^{-\frac{1}{2}q^T\cdot A\cdot q}\Big(1-\frac{\lambda}{4!}q^4+O(\lambda^2)\Big)q_iq_jq_kq_l\\&=A_{ij}^{-1}A_{kl}^{-1}+A_{ik}^{-1}A_{jl}^{-1}+A_{il}^{-1}A_{jk}^{-1}-\frac{\lambda}{4!}\langle q^4q_iq_jq_kq_l\rangle+O(\lambda^2)\end{aligned}\tag{3.2.10}$$

## 3.3 perturbative field theory

- 做如下替换即可,

$$\begin{cases}A\mapsto -i(\partial^2-m^2)\\J\mapsto iJ\end{cases}\tag{3.3.1}$$

- **Schwinger's way:** $\phi^4$ theory 的路径积分,

$$Z(J)=\int D\phi\,e^{i\int d^dx\,(\frac{1}{2}\phi(\partial^2-m^2)\phi-\frac{\lambda}{4!}\phi^4+J(x)\phi(x))}\tag{3.3.2}$$

$$=Z(0,0)e^{-i\frac{\lambda}{4!}\int d^dz\,(\frac{\delta}{i\delta J(z)})^4}e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)}\tag{3.3.3}$$

其中 $D(x-y)$ 是自由场的 propagator, 见 (1.2.1).

- **Wick's way:** 同样, 对 $J$ 展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)}=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^dx_1\cdots d^dx_n\,J(x_1)\cdots J(x_n)G^{(n)}(x_1,\cdots,x_n) \tag{3.3.4}$$

  其中,

$$G^{(n)}(x_1,\cdots,x_n)=\frac{1}{Z(0,0)}\int D\phi\,e^{i\int d^dx\,(\frac{1}{2}\phi(\partial^2-m^2)\phi-\frac{\lambda}{4!}\phi^4)}\phi(x_1)\cdots\phi(x_n) \tag{3.3.5}$$

  有时 $Z(J)$ 被称为 generating functional, 因为它能生成 Green's functions.

### 3.3.1 collision between particles

- 通过 Wick's way, 考虑 $J(x_1)J(x_2)J(x_3)J(x_4)$ 项, 实际上就是要计算 $G^{(4)}(x_1,x_2,x_3,x_4)$, 它的 0 阶项为,

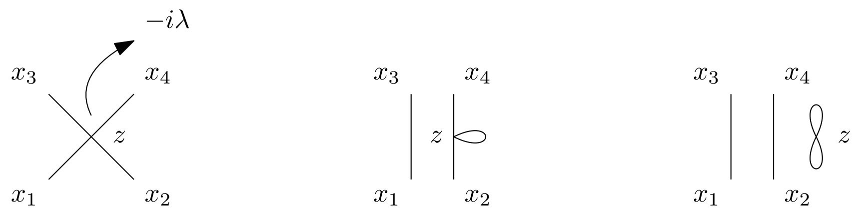
$$\begin{aligned}G^{(4)}(x_1,x_2,x_3,x_4,\lambda=0)&=\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)}\\&=-(D_{12}D_{34}+D_{13}D_{24}+D_{14}D_{23})\end{aligned} \tag{3.3.6}$$

  其中 $D_{ij}$ 是 $D(x_i-x_j)$ 的简写, 可见, 传播子实际上是 $(-i)^3D=iD$.

- $G^{(4)}_{1234}$ 的 1 阶项为,

$$\begin{aligned}\text{1st order term}&=-\frac{i\lambda}{4!}\int d^dz\,\langle\phi_1\cdots\phi_4\phi^4(z)\rangle\\&=-\frac{i\lambda}{4!}\int d^dz\,\frac{\delta}{i\delta J_1}\cdots\frac{\delta}{i\delta J_4}\Big(\frac{\delta}{i\delta J(z)}\Big)^4e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)}\\&=-\frac{i\lambda}{4!}\int d^dz\,\Big(4!D_{1z}D_{2z}D_{3z}D_{4z}\\&\quad+4\times3D_{12}D_{3z}D_{4z}D_{zz}+\cdots+3D_{12}D_{34}D_{zz}D_{zz}+\cdots\Big)\end{aligned} \tag{3.3.7}$$

  其中各项分别对应如下 Feynman diagrams,

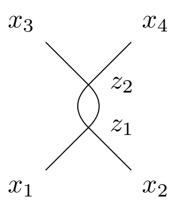



$-i\lambda\int d^dz\,D_{1z}D_{2z}D_{3z}D_{4z}$ $\quad\frac{-i\lambda}{2!}\int d^dzD_{13}D_{2z}D_{4z}D_{zz}$ $\quad\frac{-i\lambda}{8}\int d^dzD_{13}D_{24}D_{zz}D_{zz}$

Figure 3.2: position space - Feynman diagrams

  其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,




$$=(4\times3)^2\times2\times\big(\frac{-i\lambda}{4!}\big)^2\int d^dz_1d^dz_2\,D_{1z_1}D_{2z_1}D_{3z_2}D_{4z_2}D_{z_1z_2}D_{z_1z_2} \tag{3.3.8}$$

### 3.3.2 in momentum space

- 本 subsection 将 (3.3.5) 转换到 momentum space, 注意到 $\tilde{J}(k)$ 和 $\tilde{J}(-k)$ 并不独立, 所以 $\frac{\partial}{\partial i\tilde{J}}$ 不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\tilde{G}^{(n)}(k_1,\cdots,k_n)=\int d^dx_1\cdots d^dx_n\,e^{-i(k_1\cdot x_1+\cdots)}G^{(n)}(x_1,\cdots,x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^dx_1 \cdots d^dx_n \, e^{-i(k_1\cdot x_1+\cdots)} \langle \Big( -\frac{i\lambda}{4!} \int d^dz \, \phi_z^4 \Big)^n \phi_1 \cdots \phi_n \rangle \tag{3.3.9}$$

– propagator 的 Fourier transformation 是,

$$\tilde{D}_{pq} = \int d^dx d^dy \, e^{-i(p\cdot x+q\cdot y)} D(x-y) = \frac{(2\pi)^d \delta^{(d)}(p+q)}{-p^2-m^2+i\epsilon} \tag{3.3.10}$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1,k_2,k_3,k_4)$ 的 1 阶项为,

$$\text{1st order term} = -\frac{i\lambda}{4!} \int d^dx_1 \cdots d^dx_4 \, e^{-i(k_1\cdot x_1+\cdots)} \int d^dz \, \langle \phi_z^4 \phi_1 \cdots \phi_4 \rangle \tag{3.3.11}$$

考虑第 1 项,

$$\begin{aligned}
&-\frac{i\lambda}{4!} \int d^dx_1 \cdots d^dx_4 \, e^{-i(k_1\cdot x_1+\cdots)} \int d^dz \, 4! D_{1z} \cdots D_{4z} \\
&= -i\lambda \int d^dx_1 \cdots d^dx_4 d^dz \, e^{-i(k_1\cdot x_1+\cdots)} e^{i(p_1\cdot(x_1-z)+\cdots)} \prod_{i=1}^{4} \int \frac{d^dp_i}{(2\pi)^d} \frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -i\lambda \underbrace{\int d^dz \, e^{-iz\cdot(k_1+\cdots+k_4)}}_{=(2\pi)^d\delta^{(d)}(k_1+\cdots+k_4)} \prod_{i=1}^{4} \frac{1}{-k_i^2-m^2+i\epsilon}
\end{aligned} \tag{3.3.12}$$

– 出射粒子不一定 on-shell (?).

- 得到这些 Feynman diagrams,

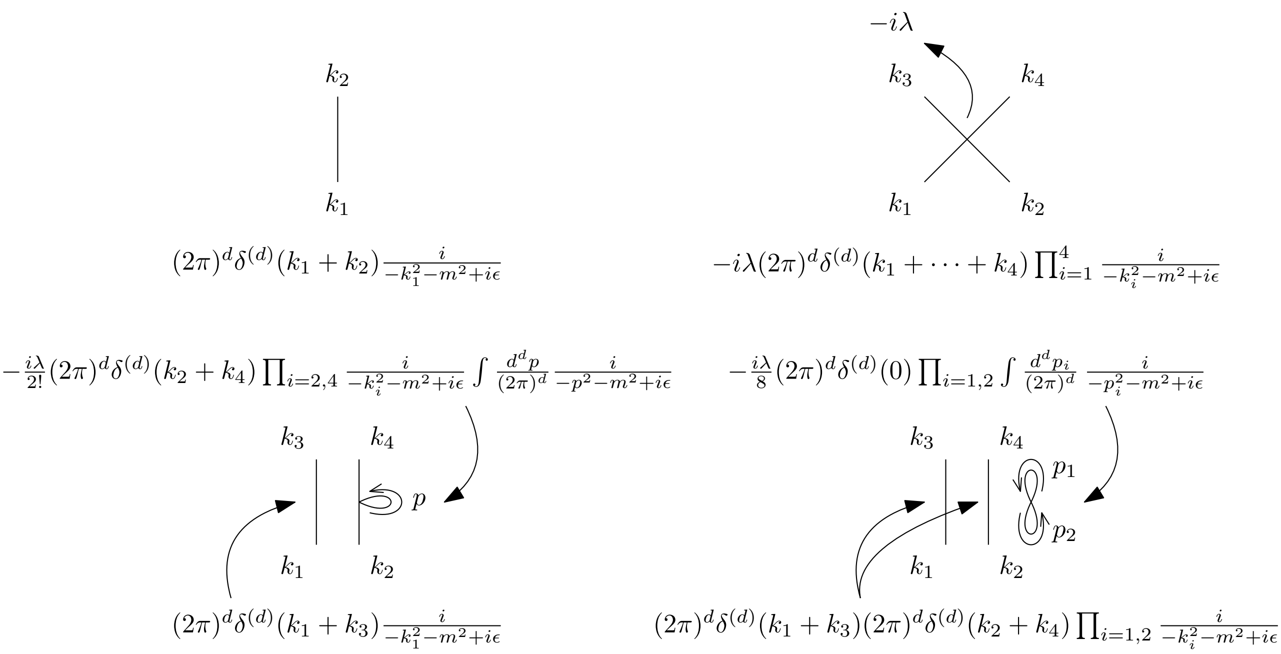



Figure 3.3: momentum space - Feynman diagrams

**calculation:**

第 3 幅图的计算如下,

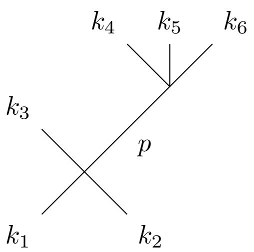
$$-\frac{i\lambda}{2!} \int d^dx_1 \cdots d^dx_4 \, e^{-i(k_1\cdot x_1+\cdots)} \int d^dz \, D_{13}D_{2z}D_{4z}D_{zz}$$

$$
\begin{aligned}
&= -\frac{i\lambda}{2!}\int d^dx_1\cdots d^dx_4 d^dz\, e^{-i(k_1\cdot x_1+\cdots)}e^{i(p_1\cdot(x_1-x_3)+p_2\cdot(x_2-z)+p_4\cdot(x_4-z)+p_4\cdot 0)} \\
&\quad \prod_{i=1}^4\int \frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{2!}\int d^dz\, e^{-iz\cdot(p_2+p_4)}\delta^{(d)}(p_1-k_1)\delta^{(d)}(p_2-k_2)\delta^{(d)}(p_1+k_3)\delta^{(d)}(p_4-k_4) \\
&\quad \prod_{i=1}^4\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{2!}(2\pi)^d\delta^{(d)}(k_1+k_3)\delta^{(d)}(k_2+k_4)\prod_{i=1,2,4}\frac{1}{-k_i^2-m^2+i\epsilon}\int\frac{d^dp}{-p^2-m^2+i\epsilon}
\end{aligned} \tag{3.3.13}
$$

第 4 幅图的计算如下,

$$
\begin{aligned}
& -\frac{i\lambda}{8}\int d^dx_1\cdots d^dx_4\, e^{-i(k_1\cdot x_1+\cdots)}\int d^dz\, D_{13}D_{24}D_{zz}D_{zz} \\
&= -\frac{i\lambda}{8}\int d^dx_1\cdots d^dx_4 d^dz\, e^{-i(k_1\cdot x_1+\cdots)}e^{i(p_1\cdot(x_1-x_3)+p_2\cdot(x_2-x_4)+p_3\cdot 0+p_4\cdot 0)} \\
&\quad \prod_{i=1}^4\int\frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{8}\int d^dz\,\delta^{(d)}(p_1-k_1)\delta^{(d)}(p_2-k_2)\delta^{(d)}(p_1+k_3)\delta^{(d)}(p_2+k_4) \\
&\quad \prod_{i=1}^4\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{8}(2\pi)^d\delta^{(d)}(0)\delta^{(d)}(k_1+k_3)\delta^{(d)}(k_2+k_4)\prod_{i=1,2}\frac{1}{-k_i^2-m^2+i\epsilon} \\
&\quad \prod_{i=1,2}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon}
\end{aligned} \tag{3.3.14}
$$

- 再举一个例子 (略去了 $\prod_{i=1}^6\frac{i}{-k_i^2-m^2+i\epsilon}$),




$$
\begin{aligned}
&=(4!)^2\times\Big(-\frac{i\lambda}{4!}\Big)^2(2\pi)^{2d}\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\delta^{(d)}(k_1+k_2+k_3+p)\delta^{(d)}(k_4+k_5+k_6-p) \\
&=(-i\lambda)^2(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4+k_5+k_6)\frac{i}{-(k_1+k_2+k_3)^2-m^2+i\epsilon}
\end{aligned} \tag{3.3.15}
$$

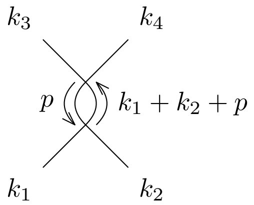
### 3.3.3 loops and a first look at divergence

- subsection 3.3.2 里的 loop diagrams 出现了如下积分,

$$
\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}=\int\frac{d^Dp}{(2\pi)^D 2\omega_p}\sim\int\frac{d^Dp}{|p|} \tag{3.3.16}
$$

积分发散.

- 再举一个例子 (略去了 $\prod_{i=1}^4 \frac{i}{-k_i^2-m^2+i\epsilon}$),




$$
\begin{aligned}
=&(4\times 3)^2\times 2\times\left(\frac{-i\lambda}{4!}\right)^2\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\int\frac{d^dq}{(2\pi)^d}\frac{i}{-q^2-m^2+i\epsilon}\\
&(2\pi)^d\delta^{(d)}(k_1+k_2+p-q)(2\pi)^d\delta^{(d)}(k_3+k_4-p+q) \qquad (3.3.17)\\
=&\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4)\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\frac{i}{-(k_1+k_2+p)^2-m^2+i\epsilon}\\
=&\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4)\int\frac{d^Dp}{(2\pi)^D}\Big(\frac{1}{2\omega_p}\frac{i}{(k_1^0+k_2^0-\omega_p)^2-\omega_{k_1+k_2+p}^2}\\
&+\frac{i}{(\omega_{k_1+k_2+p}-k_1^0-k_2^0)^2-\omega_p^2}\frac{1}{2\omega_{k_1+k_2+p}}\Big) \qquad (3.3.18)\\
\sim&\int\frac{d^Dp}{p^3} \qquad (3.3.19)
\end{aligned}
$$

同样, 积分发散.

# Chapter 4

# canonical quantization

- A. Zee: the canonical and the path integral formalisms often appear complementary, in the sense that results difficult to see in one are clear in the other.
- **nobody is perfect:**
  - **canonical quantization:** 如何定义场算符乘积的顺序.
  - **path integral:** integration measure.

## 4.1 Heisenberg and Dirac

### 4.1.1 quantum mechanics

- 单粒子的 classical Lagrangian 为,

$$L = \frac{1}{2}\dot{q}^2 - V(q) \Longrightarrow \begin{cases} p = \dot{q} \\ H = p\dot{q} - L = \frac{1}{2}p^2 + V(q) \end{cases} \tag{4.1.1}$$

- canonical commutation relation 如下,

$$[p, q] = -i \tag{4.1.2}$$

因此, 算符的演化方程为,

$$\begin{cases} \frac{dp}{dt} = i[H, p] = -V'(q) \\ \frac{dq}{dt} = i[H, q] = p \end{cases} \tag{4.1.3}$$

**calculation:**

$$\begin{cases} [p, q] = -i \\ [p, q^2] = -2iq \\ \qquad \vdots \\ [p, q^n] = -iq^{n-1} + q[p, q^{n-1}] \end{cases} \Longrightarrow [p, q^n] = -inq^{n-1} \Longrightarrow [p, V(q)] = -iV'(q) \tag{4.1.4}$$

---

- follow Dirac's approach,

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \iff \begin{cases} q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \\ p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \end{cases} \Longrightarrow [a, a^\dagger] = 1 \tag{4.1.5}$$

算符 $a$ 的演化方程为,

$$\frac{da}{dt} = -i\sqrt{\frac{\omega}{2}}\left(\frac{1}{\omega}V'(q) + ip\right) \tag{4.1.6}$$

### 4.1.2 scalar field

- 标量场的 Lagrangian 为,
$$L=\int d^Dx\Big(-\frac{1}{2}((\partial\phi)^2+m^2\phi^2)-u(\phi)\Big) \tag{4.1.7}$$
canonical commutation relation 为,
$$\pi(\vec{x},t)=\frac{\delta L(t)}{\delta\partial_0\phi(\vec{x},t)}=\partial_0\phi(\vec{x},t) \quad \text{and} \quad [\pi(\vec{x},t),\phi(\vec{y},t)]=-i\delta^{(D)}(\vec{x}-\vec{y}) \tag{4.1.8}$$
标量场的 Hamiltonian 为,
$$H=\int d^Dx\,(\pi\phi-\mathcal{L})=\int d^Dx\Big(\frac{1}{2}(\pi^2+|\vec{\nabla}\phi|^2+m^2\phi^2)+u(\phi)\Big) \tag{4.1.9}$$

---

- 算符的演化方程为,
$$\begin{cases}\partial_0\phi=i[H,\phi]=\pi\\ \partial_0\pi=i[H,\pi]=(-\vec{\nabla}^2+m^2)\phi+\dfrac{du}{d\phi}\end{cases}\Longrightarrow(\partial^2-m^2)\phi-\frac{du}{d\phi}=0 \tag{4.1.10}$$

- 当 $u(\phi)=0$ 时, 求解场方程 (4.1.10) 和 canonical commutation relation (4.1.8) 得到,
$$\phi(\vec{x},t)=\int\frac{d^Dk}{(2\pi)^D2\omega_k}(\alpha_k(t)e^{i\vec{k}\cdot\vec{x}}+\alpha_k^\dagger(t)e^{-i\vec{k}\cdot\vec{x}}) \tag{4.1.11}$$
其中,
$$\alpha_k(t)=\sqrt{(2\pi)^D2\omega_k}\,a_{\vec{k}}e^{-i\omega_kt} \quad \text{and} \quad [a_{\vec{p}},a_{\vec{q}}^\dagger]=\delta^{(D)}(\vec{p}-\vec{q}) \tag{4.1.12}$$
另外, 在后面的笔记中使用简记 $\sqrt{(2\pi)^D2\omega_k}=\rho(k)$.

**calculation:**

求解场方程 (4.1.10), 得到,
$$\phi(\vec{x},t)=\int\frac{d^Dk}{(2\pi)^D}(\alpha_{\vec{k}}e^{i(-\omega_kt+\vec{k}\cdot\vec{x})}+\alpha_{\vec{k}}^\dagger e^{-i(-\omega_kt+\vec{k}\cdot\vec{x})}) \tag{4.1.13}$$
代入 canonical commutation relation (4.1.8), 有 (其中 $x^0=y^0=t,k^0=\omega_k$),
$$\begin{aligned}&\int\frac{d^Dk_2}{(2\pi)^D}\Big(-i\omega_{k_1}[\alpha_{\vec{k}_1},\alpha_{\vec{k}_2}]e^{i(k_1\cdot x+k_2\cdot y)}+i\omega_{k_1}[\alpha_{\vec{k}_1}^\dagger,\alpha_{\vec{k}_2}^\dagger]e^{-i(k_1\cdot x+k_2\cdot y)}\\&-i\omega_{k_1}[\alpha_{\vec{k}_1},\alpha_{\vec{k}_2}^\dagger]e^{i(k_1\cdot x-k_2\cdot y)}+i\omega_{k_1}[\alpha_{\vec{k}_1}^\dagger,\alpha_{\vec{k}_2}]e^{-i(k_1\cdot x-k_2\cdot y)}\Big)=-ie^{i\vec{k}_1\cdot(\vec{x}-\vec{y})}\\&\Longrightarrow\begin{cases}[\alpha_{\vec{k}_1},\alpha_{\vec{k}_2}]=\dfrac{1}{2\omega_{k_1}}\delta^{(D)}(\vec{k}_1+\vec{k}_2)\Longrightarrow[\alpha_{\vec{k}},\alpha_{\vec{k}}]\neq0 \quad \text{wrong}\\ [\alpha_{\vec{k}_1},\alpha_{\vec{k}_2}^\dagger]=\dfrac{1}{2\omega_{\vec{k}_1}}\delta^{(D)}(\vec{k}_1-\vec{k}_2) \quad \text{right}\end{cases}\end{aligned} \tag{4.1.14}$$

- 代入 (4.1.9) 可得 (依然是 $u(\phi)=0$ 的情况下),
$$H=\int d^Dk\,\omega_k\frac{a_{\vec{k}}^\dagger a_{\vec{k}}+a_{\vec{k}}a_{\vec{k}}^\dagger}{2}=\int d^Dk\,\omega_k\Big(a_{\vec{k}}^\dagger a_{\vec{k}}+\frac{1}{2}\delta^{(D)}(0)\Big)\Longrightarrow\langle0|H|0\rangle=V\int\frac{d^Dk}{(2\pi)^D}\frac{1}{2}\omega_k \tag{4.1.15}$$
其中, $V=\int d^Dx=(2\pi)^D\delta^{(D)}(0)$.

---

- vacuum state 定义为 $a_{\vec{k}}|0\rangle=0$, 有,
$$\langle0|\phi(x)\phi(y)|0\rangle=\int\frac{d^Dk}{(2\pi)^D2\omega_k}e^{ik\cdot(x-y)} \tag{4.1.16}$$
其中 $k^0=\omega_k$. 因此, 对比 (1.2.1), 有,
$$\langle0|T(\phi(x)\phi(y))|0\rangle=iD(x-y) \tag{4.1.17}$$

## 4.2 interaction picture

- 注意, 在 $u(\phi) \neq 0$ 的情况下, (即便在 Schrödinger's picture 里, $t=0$ 时) (4.1.11) 不再成立, 因此无法通过 Schrödinger's picture or Heisenberg's picture 求解存在相互作用的场论.
- 将 Hamiltonian 分成两个部分,
$$H = H_0 + H' \tag{4.2.1}$$
- operators 以自由场的 Hamiltonian 演化,
$$O_I(t) = U_0^\dagger(t,0) O(0) U_0(t,0) \quad \text{where} \quad U_0(t_2,t_1) = \text{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H_0\Big) \tag{4.2.2}$$
states 以如下方式演化,
$$|\psi(t)\rangle_I = U_0^\dagger(t,0) U(t,0)\, |\psi(0)\rangle \quad \text{where} \quad U(t_2,t_1) = \text{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H\Big) \tag{4.2.3}$$
因此,
$$|\psi(t_2)\rangle_I = U_I(t_2,t_1)\, |\psi(t_1)\rangle_I \quad \text{where} \quad U_I(t_2,t_1) = \text{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H_I(t)\Big) \tag{4.2.4}$$
注意, (4.2.2) 和 (4.2.3) 中, Texp 里的 $H, H_0$ 都是 Schrödinger's picture 里的算符.

**calculation:**

首先有,
$$U_I(t_2,t_1) = U_0^\dagger(t_2,0) U(t_2,t_1) U_0(t_1,0) \tag{4.2.5}$$
因此,
$$\begin{aligned} \frac{d}{dt} U_I(t,t_0) &= iH_0 U_I(t,t_0) - iU_0^\dagger(t,0) H U(t,t_0) U_0(t_0,0) \\ &= -i\, \underbrace{U_0^\dagger(t,0) H' U_0(t,0)}_{=H_I(t)}\, U_I(t,t_0) \end{aligned} \tag{4.2.6}$$

## 4.3 scattering amplitude

- 最一般的过程是 $p_1, \cdots, p_m \to q_1, \cdots, q_n$, 其 scattering amplitude 为,
$$\langle q_1, \cdots, q_n | U_0^\dagger(-\infty,0) U_I(+\infty,-\infty) U_0(-\infty,0) | p_1, \cdots, p_m\rangle \tag{4.3.1}$$
一般会忽略掉 $U_0$ 产生的相位.

---

- 考虑 $\phi^4$ 理论中的 $k_1, k_2 \to k_3, k_4$ 过程,
$$\langle k_3, k_4 | e^{-i\int d^dx\, \frac{\lambda}{4!}\phi^4} | k_1, k_2\rangle \tag{4.3.2}$$
对 $\lambda$ 展开, 0 阶项为,
$$\begin{aligned} \text{0th order term} &= \langle k_3, k_4 | k_1, k_2 \rangle \\ &= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\, \langle 0| a_{\vec{k}_3} a_{\vec{k}_4} a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle \\ &= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4) \Big( \underbrace{\langle 0| a_{\vec{k}_3} a_{\vec{k}_4} a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle}_{=\delta^{(D)}_{31}\delta^{(D)}_{42}} + \underbrace{\langle 0| a_{\vec{k}_3} a_{\vec{k}_4} a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle}_{=\delta^{(D)}_{32}\delta^{(D)}_{41}} \Big) \\ &= (2\pi)^{2D} 4\omega_{k_1}\omega_{k_2} (\delta^{(D)}(\vec{k}_1 - \vec{k}_3)\delta^{(D)}(\vec{k}_2 - \vec{k}_4) + \delta^{(D)}(\vec{k}_1 - \vec{k}_4)\delta^{(D)}(\vec{k}_2 - \vec{k}_3)) \end{aligned} \tag{4.3.3}$$
1 阶项为 (其中 $k^0 = \omega_k$),
$$\text{1st order term} = \frac{-i\lambda}{4!} \int d^dx\, \langle k_3, k_4 | \phi^4(x) | k_1, k_2 \rangle$$

$$
= \overbrace{4! \times \frac{-i\lambda}{4!} \int d^d x \, e^{i(k_1+k_2-k_3-k_4)\cdot x}}^{=-i\lambda(2\pi)^d \delta^{(d)}(k_1+k_2-k_3-k_4)} + \rho(k_1)\rho(k_4)\delta^{(D)}_{14} \times 12 \times \frac{-i\lambda}{4!}(2\pi)^d \delta^{(d)}_{23} \int \frac{d^D p}{\rho^2(p)}
$$

$$
+ \cdots + \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\delta^{(D)}_{13}\delta^{(D)}_{24} \times 3 \times \frac{-i\lambda}{4!} \int d^d x \int \frac{d^D p_1}{\rho^2(p_1)} \frac{d^D p_2}{\rho^2(p_2)} + \cdots \tag{4.3.4}
$$

分别对应如下 Feynman diagrams,

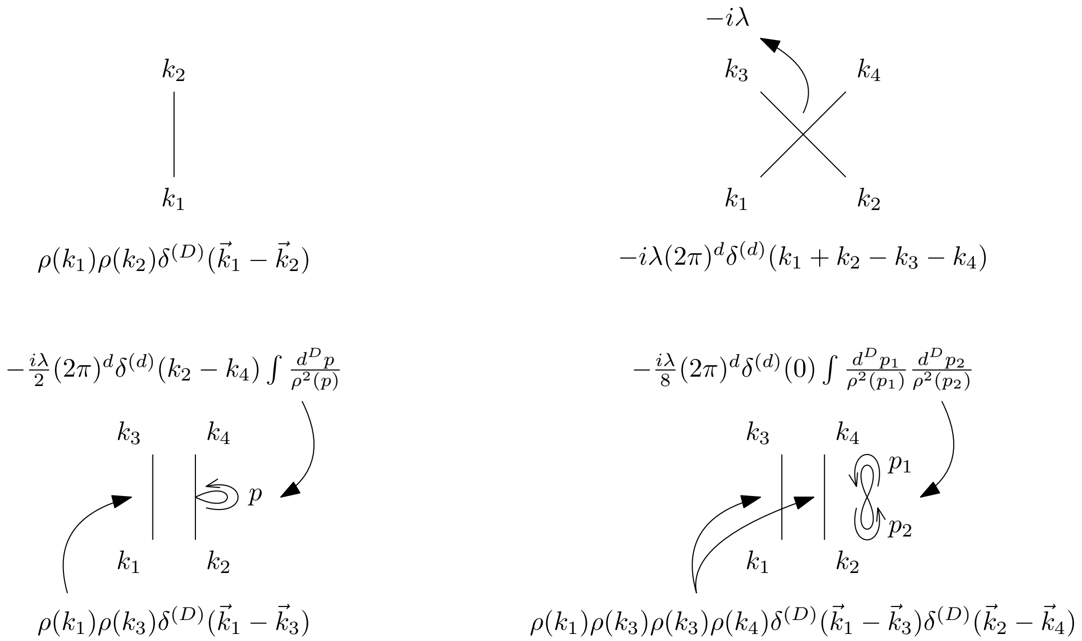
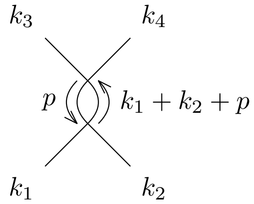



Figure 4.1: canonical quantization - Feynman diagrams

观察可见, 上图和 figure 3.3 有对应关系.

- 再举一个例子,




$$
\begin{aligned}
=&(4\times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!}\right)^2 \rho(k_1)\cdots \int d^d x_1 d^d x_2 \int \frac{d^D p_1 \cdots}{\rho(p_1)\cdots} \frac{d^D q_1 \cdots}{\rho(q_1)\cdots} e^{i(p_1+p_2-p_3-p_4)\cdot x_1} e^{i(q_1+q_2-q_3-q_4)\cdot x_2} \\
&\left(\theta(t_2 - t_1)\langle 0| a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{q}_1} a_{\vec{q}_2} a^\dagger_{\vec{q}_3} a^\dagger_{\vec{q}_4} a_{\vec{p}_1} a_{\vec{p}_2} a^\dagger_{\vec{p}_3} a^\dagger_{\vec{p}_4} a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle + \cdots\right) \\
=&\frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 \int \frac{d^D p_3}{\rho^2(p_3)} \frac{d^D p_4}{\rho^2(p_4)} \Big(\theta(t_2 - t_1) e^{i(k_1+k_2-p_3-p_4)\cdot x_1} e^{i(p_3+p_4-k_3-k_4)\cdot x_2} \\
&+ \theta(t_1 - t_2) e^{i(k_1+k_2+p_3+p_4)\cdot x_1} e^{i(-p_3-p_4-k_3-k_4)\cdot x_2}\Big) \\
=&\frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 \, e^{i((k_1+k_2)\cdot x_1 - (k_3+k_4)\cdot x_2)} \int \frac{d^D p_3}{\rho^2(p_3)} \frac{d^D p_4}{\rho^2(p_4)} \Big(\theta(t_2 - t_1) e^{i(p_3+p_4)\cdot(x_2 - x_1)} \\
&+ \theta(t_1 - t_2) e^{i(p_3+p_4)\cdot(x_1 - x_2)}\Big)
\end{aligned} \tag{4.3.5}
$$

同样, 与 (3.3.18) 有对应关系, (注意按时间排序 $\langle k_3 k_4 | T(\phi^4(x_1)\phi^4(x_2)) | k_1 k_2 \rangle$).

**calculation:**

从 (3.3.17) 开始 (与 (1.2.1) 类似, $\vec{p},\vec{q}$ 的符号可以任意改变),

$$
\begin{aligned}
&\int d^dx_1d^dx_2\, e^{i(k_1+k_2+p-q)\cdot x_1}e^{i(k_3+k_4-p+q)\cdot x_2}\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\frac{i}{-q^2-m^2+i\epsilon}\\
&=\int d^dx_1d^dx_2\, e^{i((k_1+k_2)\cdot x_1+(k_3+k_4)\cdot x_2)}\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{ie^{ip\cdot(x_1-x_2)}}{-p^2-m^2+i\epsilon}\frac{ie^{ie^{iq\cdot(x_2-x_1)}}}{-q^2-m^2+i\epsilon}\\
&=\int d^dx_1d^dx_2\, e^{i((k_1+k_2)\cdot x_1+(k_3+k_4)\cdot x_2)}\int\frac{d^Dp}{(2\pi)^d}\frac{d^Dq}{(2\pi)^d}\Big(\theta(t_2-t_1)\frac{2\pi i^2e^{-ip\cdot(x_1-x_2)}}{-2\omega_p}\\
&\quad\frac{-2\pi i^2e^{iq\cdot(x_2-x_1)}}{2\omega_q}+\theta(t_1-t_2)\frac{-2\pi i^2e^{ip\cdot(x_1-x_2)}}{2\omega_p}\frac{2\pi i^2e^{-iq\cdot(x_2-x_1)}}{-2\omega_q}\Big)\\
&=\int d^dx_1d^dx_2\, e^{i((k_1+k_2)\cdot x_1+(k_3+k_4)\cdot x_2)}\int\frac{d^Dp}{\rho^2(p)}\frac{d^Dq}{\rho^2(q)}\Big(\theta(t_2-t_1)e^{i(p+q)\cdot(x_2-x_1)}\\
&\quad+\theta(t_1-t_2)e^{i(p+q)\cdot(x_1-x_2)}\Big)
\end{aligned}
\tag{4.3.6}
$$

结果与 (4.3.5) 对应.

## 4.4 complex scalar field

- complex scalar field 的 Lagrangian 为,

$$\mathcal{L}=-(\partial\psi^\dagger)(\partial\psi)-m^2\psi^\dagger\psi \tag{4.4.1}$$

实际上, complex scalar field 可以视为 2 个 real scalar fields 的和,

$$\psi=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)\Longrightarrow\left|\frac{\partial\phi_1,\phi_2}{\partial\psi,\psi^\dagger}\right|=i \tag{4.4.2}$$

因此, 也可以把 $\psi,\psi^\dagger$ 视为两个独立的场.

- 其 canonical momentum 为,

$$\pi(x)=\frac{\delta\mathcal{L}}{\delta\partial_0\psi}=\partial_0\psi^\dagger\quad \pi^\dagger=\partial_0\psi \tag{4.4.3}$$

其 Hamiltonian 为,

$$\mathcal{H}=\pi^\dagger\pi+(\vec{\nabla}\psi^\dagger)\cdot(\vec{\nabla}\psi)+m^2\psi^\dagger\psi \tag{4.4.4}$$

$$\Longrightarrow\begin{cases}\partial_0\pi=i[H,\pi]=\vec{\nabla}^2\psi^\dagger-m^2\psi^\dagger\\ \partial_0\psi=i[H,\psi]=\pi^\dagger\end{cases}\Longrightarrow(-\partial^2-m^2)\psi=0 \tag{4.4.5}$$

- 求解得到 (其中 $k^0=\omega_k$),

$$\psi(x)=\int\frac{d^Dk}{\rho(k)}(a_{\vec{k}}e^{ik\cdot x}+b^\dagger_{\vec{k}}e^{-ik\cdot x}) \tag{4.4.6}$$

---

- 从 path integral 的角度,

$$Z(J,J^\dagger)=\int D\psi D\psi^\dagger\, e^{i\int d^dx\,(\psi^\dagger(\partial^2-m^2)\psi+J^\dagger\psi+\psi^\dagger J)} \tag{4.4.7}$$

$$=\mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\,2J^\dagger(x)D(x-y)J(y)} \tag{4.4.8}$$

**calculation:**

转换为 $\phi_1,\phi_2$ 后计算路径积分,

$$Z(J,J^\dagger)=\mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\,(J_1(x)D(x-y)J_1(y)+J_2(x)D(x-y)J_2(y))}$$

$$= \mathcal{C} e^{-\frac{i}{2}\int d^dx d^dy\, 2J^\dagger(x) D(x-y) J(y)} \tag{4.4.9}$$

### 4.4.1 charge

- 对场算符做如下变换,

$$\psi(x,\lambda) = e^{i\lambda}\psi(x) \Longrightarrow D_\lambda \mathcal{L} = 0 \tag{4.4.10}$$

- 因此, 得到 conserved current,

$$J^\mu = \pi^\mu D_\lambda \psi + \pi^{\dagger\mu} D_\lambda \psi^\dagger = i(\psi\partial^\mu\psi^\dagger - \psi^\dagger\partial^\mu\psi) \tag{4.4.11}$$

其 0 分量对空间积分就是 charge,

$$\begin{aligned} Q &= \int d^D x\, J^0 = \int d^D x\, i(\psi^\dagger\partial_0\psi - \psi\partial_0\psi^\dagger) \\ &= \int d^D k\, (a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}}) \end{aligned} \tag{4.4.12}$$

**calculation:**

$$\begin{aligned} Q = \int d^D x \int \frac{d^D p}{\rho(p)}\frac{d^D q}{\rho(q)} i\Big( &(a^\dagger_{\vec{p}} e^{-ip\cdot x} + b_{\vec{p}} e^{ip\cdot x})(-i\omega_q)(a_{\vec{q}} e^{iq\cdot x} - b^\dagger_{\vec{q}} e^{-iq\cdot x}) \\ &- (a_{\vec{q}} e^{iq\cdot x} + b^\dagger_{\vec{q}} e^{-iq\cdot x})(i\omega_p)(a^\dagger_{\vec{p}} e^{-ip\cdot x} - b_{\vec{p}} e^{ip\cdot x})\Big) \\ = \int d^D x \int \frac{d^D p}{\rho(p)}\frac{d^D q}{\rho(q)} \Big( &(\omega_p a_{\vec{q}} a^\dagger_{\vec{p}} + \omega_q a^\dagger_{\vec{p}} a_{\vec{q}}) e^{-i(p-q)\cdot x} - (\omega_p b^\dagger_{\vec{q}} b_{\vec{p}} + \omega_q b_{\vec{p}} b^\dagger_{\vec{q}}) e^{i(p-q)\cdot x} \\ &+ a^\dagger_{\vec{p}} b^\dagger_{\vec{q}} (\omega_p - \omega_q) e^{-i(p+q)\cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{i(p+q)\cdot x}\Big) \\ = \int \frac{d^D p}{\rho(p)}\frac{d^D q}{\rho(q)} & \\ \bigg(\Big((\omega_p a_{\vec{q}} a^\dagger_{\vec{p}} &+ \omega_q a^\dagger_{\vec{p}} a_{\vec{q}}) e^{i(\omega_p-\omega_q)\cdot t} - (\omega_p b^\dagger_{\vec{q}} b_{\vec{p}} + \omega_q b_{\vec{p}} b^\dagger_{\vec{q}}) e^{-i(\omega_p-\omega_q)\cdot t}\Big)(2\pi)^D \delta^{(D)}(\vec{p}-\vec{q}) \\ &+ \Big(a^\dagger_{\vec{p}} b^\dagger_{\vec{q}} (\omega_p - \omega_q) e^{i(\omega_p+\omega_q)\cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{-i(\omega_p+\omega_q)\cdot x}\Big)(2\pi)^D \delta^{(D)}(\vec{p}+\vec{q})\bigg) \\ = \int \frac{d^D k}{2} (a_{\vec{k}} a^\dagger_{\vec{k}} &+ a^\dagger_{\vec{k}} a_{\vec{k}} - b_{\vec{k}} b^\dagger_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}}) = \int d^D k\, (a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}}) \end{aligned} \tag{4.4.13}$$

- 代入 (D.3.2), 有 $i[Q,\psi] = -i\psi$, 所以,

$$e^{-i\lambda Q}\psi e^{i\lambda Q} = e^{i\lambda}\psi \tag{4.4.14}$$

# Chapter 5

# disturbing the vacuum: Casimir effect

- 考虑一个沿 $x^1$ 方向满足 periodic b.c. 的空间, 在垂直于 $x^1$ 方向有两个 plates, s.t. 在 plates 上 $\phi(x) = 0$, 如下图,

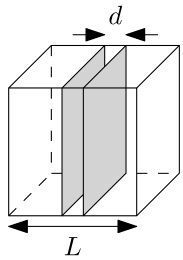



Figure 5.1: Casimir effect

- 平板内外, 标量场的波矢的取值为,

$$\begin{cases} (n\frac{\pi}{d}, k_2, k_3) & \text{平板内} \\ (n\frac{\pi}{L-d}, k_2, k_3) & \text{平板外} \end{cases} \tag{5.0.1}$$

其中 $n \in \mathbb{Z}^+$.

- 因此, 代入真空能公式 (4.1.15), 平板内的能量为,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2} \tag{5.0.2}$$

而总能量为 $E = E(d) + E(L-d)$.

---

- 为解决能量发散的问题, 引入 ultra-violet (UV) cut-off,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2}\, e^{-a\sqrt{(n\frac{\pi}{d})^2 + k_2^2 + k_3^2}} \tag{5.0.3}$$

for some $a \ll d$.

- 为了简化问题, 考虑 $d = 1+1$ 的情况,

$$E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-\frac{a\pi}{d}n} = \frac{\pi}{2d} \frac{e^{\frac{a\pi}{d}}}{(e^{\frac{a\pi}{d}} - 1)^2} = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O(a^2) \tag{5.0.4}$$

因此,

$$E_{1+1} = E_{1+1}(d) + E_{1+1}(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24}\left(\frac{1}{d} + \frac{1}{L-d}\right) + O(a^2) \tag{5.0.5}$$

得到 Casimir force,

$$F_{1+1} = -\frac{\partial E_{1+1}}{\partial d} = -\frac{\pi}{24}\left(\frac{1}{d^2} - \frac{1}{(L-d)^2}\right) + O(a^2) \overset{L\to\infty, a\to 0}{=} -\frac{\pi}{24d^2} \tag{5.0.6}$$

- 问题中, $a$ 引入了 UV cut-off, $L$ 引入了 infrared cut-off.

# Part II

# Dirac and spinor

# Chapter 6

# the Dirac spinor

- 整个 Part II 中, 我们使用 $(+,-,-,-)$ 号差, 因为 $\mathrm{Cl}_{1,3}(\mathbb{R}) \not\simeq \mathrm{Cl}_{3,1}(\mathbb{R})$.
- 本笔记中的算符的定义与 A. Zee 的定义不同, 存在如下对应关系,

| A. Zee's def. | my def. |
|---|---|
| $\omega_{\mu\nu}$ | $\omega_{\mu\nu}$ |
| $-iJ^{\mu\nu}$ | $J^{\mu\nu}$ |
| $-i\sigma^{\mu\nu}$ | $\sigma^{\mu\nu}$ |

- $\Pi(\Lambda)$ 的写法可能不准确, (要考虑 universal cover, $\mathrm{Spin}(1,3) \simeq \mathrm{Spin}(3,1)$), 因为 Lorentz transform 对 spinor 的操作是"path dependent", 因此本 chapter 中的 $\Lambda$ 都默认沿着以下的 path 做变换,

$$\Lambda(\lambda) = e^{\frac{\lambda}{2}\omega_{\mu\nu}J^{\mu\nu}}, \lambda \in [0,1] \tag{6.0.1}$$

## 6.1 gamma matrices

- Pauli 矩阵如下,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.1.1}$$

- gamma 矩阵 (also called Dirac matrices) 如下 (其中 $i = 1,2,3$),

$$\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} = I \otimes \tau_3 \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = i\sigma_i \otimes \tau_2 \quad \Omega = \gamma^0\gamma^1\gamma^2\gamma^3 = -i\begin{pmatrix} & I \\ I & \end{pmatrix} = -iI \otimes \tau_1 \tag{6.1.2}$$

其中 $\tau_{2,3}$ 也是 Pauli 矩阵, 最后, 按照惯例, 定义 $\gamma^5 = i\Omega = I \otimes \tau_1$.

  - 另外,

$$\begin{cases} \gamma^0\gamma^i = \sigma_i \otimes \tau_1 \\ \gamma^i\gamma^j = -(\sigma_i\sigma_j) \otimes I \end{cases} \quad \begin{cases} \Omega\gamma^0 = -I \otimes \sigma_2 \\ \Omega\gamma^i = i\sigma_i \otimes \tau_3 \end{cases} \tag{6.1.3}$$

---

- gamma 矩阵满足,

$$\begin{cases} (\gamma^\mu)^2 = \eta^{\mu\mu} \\ \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \mu \neq \nu \end{cases} \Longrightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{6.1.4}$$

- 且存在如下关系,

$$\begin{aligned} &\Omega\gamma^0 = -\gamma^1\gamma^2\gamma^3 \quad \Omega\gamma^1 = -\gamma^0\gamma^2\gamma^3 \quad \Omega\gamma^2 = \gamma^0\gamma^1\gamma^3 \quad \Omega\gamma^3 = -\gamma^0\gamma^1\gamma^2 \\ \Longleftrightarrow &-\epsilon^{\mu\nu\rho}{}_\sigma\Omega\gamma^\sigma = \gamma^\mu\gamma^\nu\gamma^\rho \quad \text{when} \quad \mu \neq \nu \neq \rho \end{aligned} \tag{6.1.5}$$

并且有 (注意到 $\Omega^2 = -1$),

$$\{\Omega, \gamma^\mu\} = 0 \quad \{\Omega, \Omega\gamma^\mu\} = 0 \quad [\Omega, \gamma^\mu\gamma^\nu] = 0 \tag{6.1.6}$$

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

- 定义 $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ (注意, 我们的定义中没有虚数 $i$, 与 A. Zee 的定义不同),

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} + \sigma^{\mu\nu} \Longrightarrow \begin{cases} \sigma^{0i} = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix} = \sigma_i \otimes \tau_1 \\ \sigma^{ij} = -i\epsilon^{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk}\sigma_k \otimes I \end{cases} \tag{6.1.7}$$

### 6.1.1 gamma matrices under Weyl basis

- 做 (6.2.2) 中的相似变换,

$$\gamma^0 = \begin{pmatrix} & I \\ I & \end{pmatrix} \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -I & \\ & I \end{pmatrix} \tag{6.1.8}$$

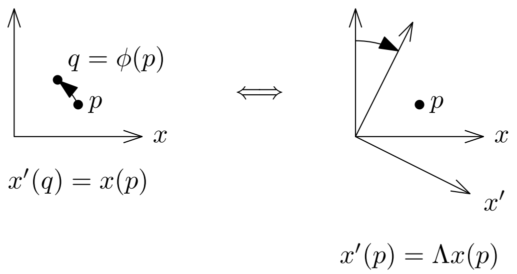
有时候使用符号 $\sigma^\mu = (I, \vec{\sigma}), \bar{\sigma}^\mu = (I, -\vec{\sigma})$.

## 6.2 Lorentz transformation and the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation

- Lorentz 变换可以写成如下形式,

$$\Lambda = e^{\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{6.2.1}$$

其中 $\omega_{\mu\nu}$ 反对称, $J^{0i}$ generate boosts and $J^{ij}$ generate rotations, (详见笔记 Lie Groups and Lie Algebras).




$$x = \phi^* x' = \Lambda^{-1} x'$$

$$\phi_* \frac{\partial}{\partial x^\mu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x'^\mu} \text{ or } \eta\Lambda\eta \frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$$

$$\phi^* dx'^\mu = \Lambda_\nu{}^\mu dx'^\nu = dx^\mu \text{ or } \Lambda^{-1} dx' = dx$$

e.g. $A'^\mu(\Lambda x(p)) = \Lambda^\mu{}_\nu A^\nu(x(p))$

Figure 6.1: Lorentz transformation

---

- 有 $\pi_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(J^{\mu\nu}) = \frac{1}{2}\sigma^{\mu\nu}$ (up to a similarity transformation).

**calculation:**

做如下相似变换,

$$S = \frac{\sqrt{2}}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix} \Longleftrightarrow S^{-1} = \frac{\sqrt{2}}{2}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} \tag{6.2.2}$$

得到,

$$S^{-1}\sigma^{0i}S = \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix} \quad S^{-1}\sigma^{ij}S = -i\epsilon^{ijk}\begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} \tag{6.2.3}$$

得到的结果和笔记 Lie Groups and Lie Algebras 中 $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ 表示是完全一样的.

- Dirac spinor 是 $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ rep. 的 vector space 中的元素,

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad \text{with} \quad S^{-1}\Psi = \frac{\sqrt{2}}{2}\begin{pmatrix} \phi - \chi \\ \phi + \chi \end{pmatrix} \tag{6.2.4}$$

其中,

$$\Psi_L = \frac{\sqrt{2}}{2}\begin{pmatrix} \psi_L \\ -\psi_L \end{pmatrix} \in (\frac{1}{2}, 0) \quad \text{and} \quad \Psi_R = \frac{\sqrt{2}}{2}\begin{pmatrix} \psi_R \\ \psi_R \end{pmatrix} \in (0, \frac{1}{2}) \tag{6.2.5}$$

在 Weyl basis 下很容易看出 $\Psi_L = \frac{1}{2}(1 - \gamma^5)\Psi, \Psi_R = \frac{1}{2}(1 + \gamma^5)\Psi$.

---

- 对于 gamma 矩阵, 有,

$$\Pi(\Lambda)\gamma^\rho\Pi^{-1}(\Lambda) = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = (\Lambda^{-1})^\rho{}_\sigma\gamma^\sigma \tag{6.2.6}$$

**calculation:**

利用 Campbell's identity,

$$e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = e^{\frac{1}{4}\omega_{\mu\nu}\mathrm{ad}_{\sigma^{\mu\nu}}}\gamma^\rho \tag{6.2.7}$$

其中 (注意 $(J^{\mu\nu})^\rho{}_\sigma = 2\eta^{[\mu|\rho}\delta^{|\nu]}{}_\sigma$, 其中度规号差与笔记 Lie Groups and Lie Algebras 中的不同),

$$\begin{cases} \rho \neq \mu,\nu & [\sigma^{\mu\nu},\gamma^\rho] = \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu) \\ & \qquad = -\frac{1}{2}\underbrace{(\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\nu\mu\rho\sigma} - \epsilon^{\rho\mu\nu\sigma} + \epsilon^{\rho\nu\mu\sigma})}_{=0}\Omega\gamma_\sigma = 0 \\ \rho = \mu \text{ or } \nu \text{ and } \mu\neq\nu & [\sigma^{\mu\nu},\gamma^\rho] = 2(\eta^{\mu\rho}\gamma^\mu - \eta^{\nu\rho}\gamma^\nu) \end{cases}$$

$$\Longrightarrow [\sigma^{\mu\nu},\gamma^\rho] = 2(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu) = -2(J^{\mu\nu})^\rho{}_\sigma\gamma^\sigma \tag{6.2.8}$$

代入, 得到,

$$e^{\frac{1}{4}\omega_{\mu\nu}\mathrm{ad}_{\sigma^{\mu\nu}}}\gamma^\rho = \sum_{n=0}^\infty \frac{1}{n!}\Big(\Big(-\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}\Big)^n\Big)^\rho{}_\sigma\gamma^\sigma = (\Lambda^{-1})^\rho{}_\sigma\gamma^\sigma \tag{6.2.9}$$

---

可以用" 无穷小" Lorentz 变换验证以上计算,

$$\begin{aligned}\Pi(1+\delta\omega^\mu{}_\nu)\gamma^\rho\Pi^{-1}(1+\delta\omega^\mu{}_\nu) &= \gamma^\rho + \frac{1}{4}\delta\omega_{\mu\nu}[\sigma^{\mu\nu},\gamma^\rho] \\ &= (1-\delta\omega^\rho{}_\sigma)\gamma^\sigma \end{aligned}\tag{6.2.10}$$

### 6.2.1 Dirac spinor

- 对于 Dirac spinor,

$$\Pi(\Lambda)\Psi(x) = \Psi'(\Lambda x) \tag{6.2.11}$$

注意 $\partial'_\mu = \Lambda_\mu{}^\nu\partial_\nu$, 所以,

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \iff (i\gamma^\mu\partial'_\mu - m)\Psi'(\Lambda x) = 0 \tag{6.2.12}$$

  – 关键部分在于,

$$\gamma^\mu\Psi'(\Lambda x) = \gamma^\mu\Pi(\Lambda)\Psi(x) = \Pi(\Lambda)\Lambda^\mu{}_\nu\gamma^\nu\Psi(x) \tag{6.2.13}$$

**calculation:**

首先,

$$\Lambda^T\eta\Lambda = \eta \Longrightarrow (\Lambda^{-1})^\mu{}_\nu = (\eta\Lambda^T\eta)^\mu{}_\nu = \Lambda_\nu{}^\mu \tag{6.2.14}$$

考虑,

$$\Pi^{-1}(\Lambda)\gamma^\mu\Pi(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu \Longrightarrow \gamma^\mu\Pi(\Lambda) = \Lambda^\mu{}_\nu\Pi(\Lambda)\gamma^\nu \tag{6.2.15}$$

代入,

$$\begin{aligned}(i\gamma^\mu\partial'_\mu - m)\Psi'(\Lambda x) &= (i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - m)\Pi(\Lambda)\Psi(x) \\ &= \Pi(\Lambda)(i\gamma^\rho\underbrace{\Lambda^\mu{}_\rho\Lambda_\mu{}^\nu}_{=\delta^\nu_\rho}\partial_\nu - m)\Psi(x) \\ &= \Pi(\Lambda)(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \end{aligned}\tag{6.2.16}$$

### 6.2.2 Dirac bilinears

- $\gamma^0$ 是 Hermitian 矩阵, 而 $\gamma^i$ 不是, 有,

$$\gamma^{i\dagger} = -\gamma^i = \gamma^0\gamma^i\gamma^0 \tag{6.2.17}$$

可以统一写作 $\gamma^{\mu\dagger}=\gamma^0\gamma^\mu\gamma^0$, 并且有,

$$\sigma^{\mu\nu\dagger}=-\gamma^0\sigma^{\mu\nu}\gamma^0 \quad \Pi^\dagger(\Lambda)=\gamma^0\Pi(\Lambda^{-1})\gamma^0 \tag{6.2.18}$$

**calculation:**

对于 $\sigma^{\mu\nu}$,

$$\sigma^{\mu\nu\dagger}=\frac{1}{2}(\gamma^{\nu\dagger}\gamma^{\mu\dagger}-\gamma^{\mu\dagger}\gamma^{\nu\dagger})=\gamma^0\sigma^{\nu\mu}\gamma^0=-\gamma^0\sigma^{\mu\nu}\gamma^0 \tag{6.2.19}$$

所以,

$$((\omega_{\mu\nu}\sigma^{\mu\nu})^\dagger)^n=\gamma^0(-\omega_{\mu\nu}\sigma^{\mu\nu})^n\gamma^0 \Longrightarrow \Pi^\dagger(\Lambda)=\gamma^0\Pi(\Lambda^{-1})\gamma^0 \tag{6.2.20}$$

- 所以,

$$\begin{cases}\bar\Psi'(\Lambda x)\Psi'(\Lambda x)=\bar\Psi\Psi & \text{scalar field}\\ \bar\Psi'\gamma^\mu\Psi'=\Lambda^\mu{}_\nu\bar\Psi\gamma^\nu\Psi & \text{vector field}\end{cases} \tag{6.2.21}$$

其中 $\bar\Psi=\Psi^\dagger\gamma^0$.

**calculation:**

$$\begin{cases}\Psi'^\dagger(\Lambda x)\gamma^0\Psi'(\Lambda x)=\Psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\Pi(\Lambda)\Psi(x)=\Psi^\dagger\gamma^0\Psi\\ \Psi'^\dagger\gamma^0\gamma^\mu\Psi'=\Psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\gamma^\mu\Pi(\Lambda)\Psi(x)=\Lambda^\mu{}_\nu\Psi^\dagger\gamma^0\gamma^\nu\Psi\end{cases} \tag{6.2.22}$$

---

此外,

$$\begin{cases}\bar\Psi'\sigma^{\mu\nu}\Psi'=\Psi^\dagger\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\sigma^{\mu\nu}\Pi(\Lambda)\Psi=\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar\Psi\sigma^{\rho\sigma}\Psi & \text{order 2 tensor}\\ \bar\Psi'\Omega\gamma^\mu\Psi'=\bar\Psi\Pi(\Lambda^{-1})\Omega\gamma^\mu\Pi(\Lambda)\Psi=\det(\Lambda)\Lambda^\mu{}_\nu\bar\Psi\Omega\gamma^\nu\Psi & \text{pseudovector}\\ \bar\Psi'\Omega\Psi'=\bar\Psi\Pi(\Lambda^{-1})\Omega\Pi(\Lambda)\Psi=\det(\Lambda)\bar\Psi\Omega\Psi & \text{4-form (pseudoscalar)}\end{cases} \tag{6.2.23}$$

其中 (注意到下面的计算中, 第二个等号后, 含 $\eta$ 的项都等于零; 由此可以看出, 对 $\mu_i$ 求和的过程中, 任何两个 $\mu_i,\mu_j$ 相等的项求和之后都等于零),

$$\begin{aligned}\Pi(\Lambda^{-1})\Omega\Pi(\Lambda)&=\prod_{i=0}^{3}\Lambda^i{}_{\mu_i}\gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\\ &=\prod_{i=0}^{3}\Lambda^i{}_{\mu_i}(\eta^{\mu_0\mu_1}+\sigma^{\mu_0\mu_1})(\eta^{\mu_2\mu_3}+\sigma^{\mu_2\mu_3})\\ &=\prod_{i=0}^{3}\Lambda^i{}_{\mu_i}\gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\quad \text{with}\quad \mu_0\neq\mu_1\neq\mu_2\neq\mu_3\\ &=\det(\Lambda)\Omega\end{aligned} \tag{6.2.24}$$

### 6.2.3 parity and time reversal

- 这里沿用笔记 Lie Groups and Lie Algebras 中的记号, 选择 $\mathrm{O}(3,1)$ 而非 $\mathrm{O}(1,3)$, 因为他们没有区别.
- $\mathrm{O}(3,1)$ 有 4 个联通分支,

$$I\in \mathrm{SO}_+(3,1) \quad PT\in \mathrm{SO}_-(3,1) \quad P\in \mathrm{O}'_+(3,1) \quad T\in \mathrm{O}'_-(3,1) \tag{6.2.25}$$

其中,

$$P=\mathrm{diag}(+1,-1,-1,-1) \quad T=\mathrm{diag}(-1,+1,+1,+1) \tag{6.2.26}$$

另外, $\eta P\eta=P, \eta T\eta=T$.

- 另外, Lorentz algebra 的 representation 不能自然的生成对 $P,T$ 的表示, 因为本质上它只能生成 spin group 的表示, 是 $\mathrm{SO}_+(3,1)$ 的 universal cover, 与 Lorentz group 的其它三个连通分支没有直接联系.
- 因此, 对 $P,T$ 的表示要从物理的角度定义, (可能) 无法单纯靠数学的方法给出, 所以这部分放在下一章.

# Chapter 7

# the Dirac equation

## 7.1 Dirac equation

- A. Zee: our discussion provides a unified view of the equations of motion in relativistic physics: they just project out the unphysical components.
- the Dirac equation is,
$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \iff (\gamma^\mu p_\mu - m)\tilde{\Psi} = 0 \tag{7.1.1}$$
首先可以看出 $\Psi$ 满足 Klein-Gordan equation,
$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\Psi &= \Big(-\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu - 2im\gamma^\mu\partial_\mu + m^2\Big)\Psi = 0\\ \Longrightarrow &(-\partial^2 - m^2)\Psi = 0\end{aligned} \tag{7.1.2}$$
  - 在粒子静止系下 $p_\mu = (m,0,0,0)$, Dirac 方程给出,
  $$(\gamma^0 - 1)\tilde{\Psi} = 0 \Longrightarrow \begin{pmatrix}0 & \\ & I\end{pmatrix}\tilde{\Psi} = 0 \tag{7.1.3}$$
  因此, $\tilde{\Psi}$ 的后两个分量为零 $\Longrightarrow$ $\Psi$ 只有两个自由度.
- Dirac 方程的 Lorentz covariance 见 (6.2.12).

## 7.2 Dirac Lagrangian

- 根据 (6.2.21) 以及之前标量场的计算经验, 可知,
$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = (-i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi})\Psi + \text{total diff.} \tag{7.2.1}$$
其中, 与复标量场论中类似, 可以把 $\Psi, \Psi^\dagger$ 或 $\Psi, \bar{\Psi}$ 视为独立变量.

## 7.3 chirality or handedness

- 本 section 使用 Weyl basis.
- parity transformation 会把 left spinor 变成 right spinor and vice versa,
$$\gamma^0\Psi_L = \begin{pmatrix}0\\ \psi_L\end{pmatrix} \quad \gamma^0\Psi_R = \begin{pmatrix}\psi_R\\ 0\end{pmatrix} \tag{7.3.1}$$
- 把 Lagrangian 中的 $\Psi$ 拆开,
$$\begin{aligned}\mathcal{L} &= \bar{\Psi}_L(i\partial\!\!\!/)\Psi_L + \bar{\Psi}_R(i\partial\!\!\!/)\Psi_R - m(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L)\\ &= \psi_L^\dagger i\bar{\sigma}^\mu\partial_\mu\psi_L + \psi_R^\dagger i\sigma^\mu\partial_\mu\psi_R - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)\end{aligned} \tag{7.3.2}$$
其中注意到了 $\gamma^0\gamma^\mu$ 的非对角分块为零.

### 7.3.1 internal vector symmetry

- 做变换 $\Psi \mapsto e^{i\theta}\Psi$, Lagrangian 保持不变, 利用 Noether's theorem 得到守恒流 (见 section D.2),

$$J_V^\mu = \bar{\Psi}\gamma^\mu\Psi \tag{7.3.3}$$

其中, 按照惯例省略了虚数 $i$.

**calculation:**

计算广义动量,

$$\begin{cases} \pi_\Psi^\mu = \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\Psi} = \bar{\Psi}i\gamma^\mu \\ \pi_{\bar{\Psi}}^\mu = 0 \end{cases} \quad \text{or} \quad \begin{cases} \pi_\Psi^\mu = 0 \\ \pi_{\bar{\Psi}}^\mu = \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}} = -i\gamma^\mu\Psi \end{cases} \tag{7.3.4}$$

---

这里看起来有点奇怪, 需要再说明一下. 对于 (7.2.1) 第一个等号后边,

$$\begin{cases} \pi_\Psi^\mu = \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\Psi} = \bar{\Psi}i\gamma^\mu & \dfrac{\delta\mathcal{L}}{\delta\Psi} = -m\bar{\Psi} \\ \pi_{\bar{\Psi}}^\mu = 0 & \dfrac{\delta\mathcal{L}}{\delta\bar{\Psi}} = (i\gamma^\mu\partial_\mu - m)\Psi \end{cases} \Longrightarrow \begin{cases} -(\partial_\mu\bar{\Psi})i\gamma^\mu - m\bar{\Psi} = 0 \\ (i\gamma^\mu\partial_\mu - m)\Psi = 0 \end{cases} \tag{7.3.5}$$

对于 (7.2.1) 第二个等号后边, 忽略掉全微分项,

$$\begin{cases} \pi_\Psi^\mu = 0 & \dfrac{\delta\mathcal{L}}{\delta\Psi} = -i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} \\ \pi_{\bar{\Psi}}^\mu = \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\Psi}} = -i\gamma^\mu\Psi & \dfrac{\delta\mathcal{L}}{\delta\bar{\Psi}} = -m\Psi \end{cases} \Longrightarrow \begin{cases} -i\partial_\mu\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \\ (i\gamma^\mu\partial_\mu - m)\Psi = 0 \end{cases} \tag{7.3.6}$$

### 7.3.2 axial symmetry

- 做变换,

$$\Psi \mapsto e^{i\theta\gamma^5}\Psi = \begin{pmatrix} e^{-i\theta}\Psi_L \\ e^{i\theta}\Psi_R \end{pmatrix} \tag{7.3.7}$$

在 $m = 0$ 时 Lagrangian 保持不变, 对应的守恒流为,

$$J_A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi \tag{7.3.8}$$

根据 (6.2.23), 是一个 pseudovector.

## 7.4 energy-momentum tensor and angular momentum

- Dirac 场的 energy-momentum tensor 为,

$$T_{\mu\nu} = i\bar{\Psi}\gamma_\mu\partial_\nu\Psi - \eta_{\mu\nu}\mathcal{L} \tag{7.4.1}$$

其中, 对于满足运动方程的 Dirac 场, $\mathcal{L} = 0$.

- Dirac 场的 angular momentum 为,

$$M^{\mu\nu\rho} = \frac{i}{2}\bar{\Psi}\gamma^\mu\sigma^{\nu\rho}\Psi(x) + (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}) \tag{7.4.2}$$

**calculation:**

做变换 $x \mapsto e^{\frac{1}{2}\lambda\omega_{\mu\nu}J^{\mu\nu}}x$, 那么,

$$\begin{aligned} &\Psi(x) \mapsto \Psi'(x') = e^{\frac{1}{4}\lambda\omega_{\mu\nu}\sigma^{\mu\nu}}\Psi(x) \\ \Longrightarrow &D_\lambda\Psi'(x) = \frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\Psi(x) - \frac{1}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma\partial_\rho\Psi(x) \end{aligned} \tag{7.4.3}$$

所以,

$$J^\mu = \frac{i}{4}\omega_{\nu\rho}\bar{\Psi}\gamma^\mu\sigma^{\nu\rho}\Psi(x) + \cdots \Longrightarrow M^{\mu\nu\rho} = \frac{i}{2}\bar{\Psi}\gamma^\mu\sigma^{\nu\rho}\Psi(x) + (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}) \tag{7.4.4}$$

## 7.5 charge conjugation, parity and time reversal

### 7.5.1 charge conjugation and antimatter

- 定义矩阵 $C$,

$$C = -\gamma^0\gamma^2 \Longrightarrow C\gamma^0 = -i\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} = \gamma^2 \Longrightarrow \begin{cases} (\gamma^2)^{-1}\gamma^\mu\gamma^2 = -\gamma^{\mu *} \\ C^{-1}\gamma^\mu C = -(\gamma^\mu)^T \end{cases} \tag{7.5.1}$$

因此 $-\gamma^{\mu *}$ 同样满足 Clifford algebra.

  – 另外, 有 $(\gamma^2)^{-1} = \gamma^{2*} = -\gamma^2$ 和 $C^{-1} = C$.

**calculation:**

$$\gamma^0 C^{-1}\gamma^0 C\gamma^0 = -\gamma^{\mu *} \Longrightarrow C^{-1}\gamma^0 C = -\gamma^0\gamma^{\mu *}\gamma^0 = -(\underbrace{\gamma^0\gamma^\mu\gamma^0}_{=\gamma^{\mu\dagger}})^* \tag{7.5.2}$$

- $\Psi_c = \gamma^2\Psi^*$ 满足如下方程,

$$(-i\gamma^{\mu *}(\partial_\mu + ieA_\mu) - m)\Psi^* = 0 \Longrightarrow (\gamma^2)^{-1}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\Psi_c = 0 \tag{7.5.3}$$

可见 $\Psi_c$ 满足 $e \mapsto -e$ 后的 Dirac 方程, $\Psi_c$ is the field of positron.

---

- 对于 Lorentz 变换, $e^{\frac{1}{2}\lambda\omega_{\mu\nu}J^{\mu\nu}}, \lambda \in [0, 1]$, 有,

$$\begin{cases} \Psi \mapsto \Psi'(x') = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\Psi \\ \Psi_c \mapsto \gamma^2\underbrace{(\gamma^2)^{-1}e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^2\Psi^*}_{=(\Psi'(x'))^*} = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\Psi_c \end{cases} \tag{7.5.4}$$

### 7.5.2 parity

- 对于 parity, 有 $x \to x' = (x^0, -\vec{x})$, 在 Dirac eq. 中,

$$\gamma^0\gamma^\mu = P^\mu{}_\nu\gamma^\nu\gamma^0 \Longrightarrow (i\gamma^\mu\partial'_\mu - m)\gamma^0\Psi(x) = 0 \tag{7.5.5}$$

因此,

$$P : \Psi(x) \mapsto \Psi'(x') = \gamma^0\Psi(x) \tag{7.5.6}$$

### 7.5.3 time reversal

- 

## 7.6 interaction in QED

- QED 的 Lagrangian 为,

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\mu^2 A^\mu A_\mu \tag{7.6.1}$$

其中,

$$D_\mu = \partial_\mu - ieA_\mu \tag{7.6.2}$$

可见电子和电磁场耦合项为 $eA_\mu J^\mu_V$, 其中 $J^\mu_V$ 是 internal vector symmetry 的守恒流, 见 (7.3.3).

- QED 里的 Dirac 方程为,

$$(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\Psi = 0 \quad \text{and} \quad -i(\partial_\mu + ieA_\mu)\bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \tag{7.6.3}$$

## 7.7 Majorana neutrino

- 因为在 Lorentz 变换下, $\Psi, \Psi_c$ 行为相同, 因此 Majorana 方程同样满足 Lorentz covariance,

$$i\partial\!\!\!/\Psi - m\Psi_c = 0 \quad \text{and} \quad i\partial\!\!\!/\Psi_c - m\Psi = 0 \tag{7.7.1}$$

因此,

$$(-\partial^2 - m^2)\Psi = 0 \tag{7.7.2}$$

满足 Klein-Gordon 方程.

**calculation:**

$$-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\Psi = m(i\partial\!\!\!/)\Psi_c = m^2\Psi \tag{7.7.3}$$

- Majorana 方程对应的 Lagrangian 为,

$$\mathcal{L} = \bar{\Psi}i\partial\!\!\!/\Psi - \frac{1}{2}m(\Psi^T C\Psi + \bar{\Psi}C\bar{\Psi}^T) \tag{7.7.4}$$

相应的广义动量为,

$$\begin{cases} \pi^\mu_\Psi = \bar{\Psi}i\gamma^\mu & \dfrac{\delta\mathcal{L}}{\delta\Psi} = -m\Psi^T C \\ \pi^\mu_{\bar{\Psi}} = 0 & \dfrac{\delta\mathcal{L}}{\delta\bar{\Psi}} = i\partial\!\!\!/\Psi - mC\bar{\Psi}^T = i\partial\!\!\!/\Psi - m\Psi_c \end{cases} \tag{7.7.5}$$

  - 注意, $\Psi$ 应该被当作 Grassmann numbers, 因此, 对于反对称矩阵 $C$, 有 $\Psi^T C\Psi, \bar{\Psi}C\bar{\Psi}^T \neq 0$.

**calculation:**

对 $\Psi$ 变分得到,

$$\begin{aligned} 0 &= \frac{\delta\mathcal{L}}{\delta\Psi} - \partial_\mu\pi^\mu_\Psi \\ &= -m\Psi^T C - i\partial_\mu\bar{\Psi}\gamma^\mu \\ &= (-m\Psi^T - i\partial_\mu\bar{\Psi}\gamma^\mu C)C \\ &= (-m\Psi + iC(\gamma^\mu)^T\gamma^0\partial_\mu\Psi^*)^T C \end{aligned} \tag{7.7.6}$$

其中,

$$C(\gamma^\mu)^T\gamma^0 = C(-C^{-1}\gamma^\mu C)\gamma^0 = -\gamma^\mu C\gamma^0 = -\gamma^\mu\gamma^2 \tag{7.7.7}$$

代入, 得到 (?),

$$-i\partial\!\!\!/\Psi_c - m\Psi = 0 \tag{7.7.8}$$

- Majorana eq. v.s. Dirac eq.:
  - Majorana eq. 只适用于 electrically neutral fields (?).
  - Majorana eq. preserves handedness (?).

# Appendices

# Appendix A

# Dirac delta function & Fourier transformation

## A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx\, \delta(x) e^{-ikx} \tag{A.1.1}$$

---

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \tag{A.1.2}$$

- 第二个常用的公式 (Sokhotski-Plemelj theorem),

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{A.1.3}$$

其中 $\mathcal{P}$ 表示复函数的主値 (principal value).

**proof:**

考虑,

$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \quad \text{and} \quad \int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \,\mathrm{Res}(f, i\epsilon) = \pi \tag{A.1.4}$$

所以...

---

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,

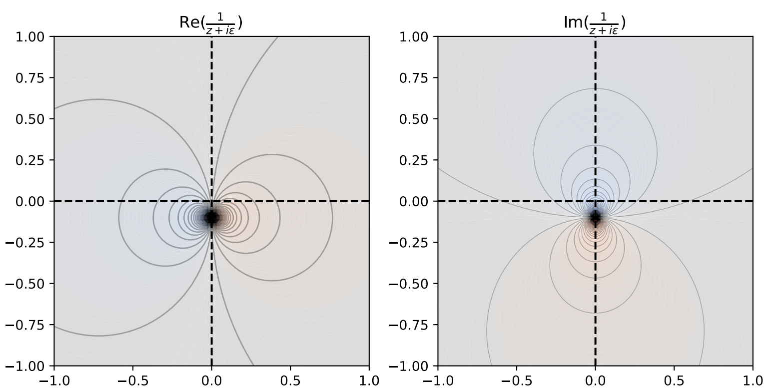



Figure A.1: graph of $\frac{1}{z+i\epsilon}$

- 另外, $\delta(x-a)\delta(x-b) = \delta(b-a)\delta(x-a)$.

## A.2 Fourier transformation

- $d$-dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x}\tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^dx\, e^{-ik\cdot x}\phi(x) \end{cases} \tag{A.2.1}$$

- 因此,

$$\partial_\mu \phi(x) \mapsto ik_\mu \tilde{\phi}(k) \tag{A.2.2}$$

---

- 对于**实函数**, Fourier transformation 是正交变换, 其 Jacobi determinant 为,

$$\left|\frac{\partial\phi(x)\cdots}{\partial\mathrm{Re}\tilde{\phi}(k)\cdots\partial\mathrm{Im}\tilde{\phi}(k)\cdots}\right| = \left(\frac{2}{V}\right)^{(2N+1)^d} \det A = \left(\frac{2(2N)^d}{V^2}\right)^{\frac{(2N+1)^d}{2}} \tag{A.2.3}$$

**proof:**

position space 和 momentum space 的格点分别为,

$$\begin{cases} x_i^\mu = i^\mu\epsilon \in \{0, \pm\epsilon, \cdots, \frac{L}{2}\} \\ k_n^\mu = n^\mu \frac{2\pi}{L} \in \{0, \pm\frac{2\pi}{L}, \cdots, \frac{\pi}{\epsilon}\} \end{cases} \iff i^\mu, n^\mu \in \{0, \pm 1, \cdots, N\} \tag{A.2.4}$$

$x^\mu, k^\mu$ 分别有 $2N+1$ 个取值, 其中 $N\epsilon = \frac{L}{2}$, 时空总体积为 $V = L^d$, momentum space 的总体积为 $\tilde{V} = \frac{(4\pi N)^d}{V}$.

---

将 (A.2.1) 写成格点求和的形式,

$$\begin{cases} \phi(x_i) = \frac{1}{(2\pi)^d}\left(\frac{2\pi}{L}\right)^d \sum_n e^{ik_n\cdot x_i}\tilde{\phi}(k_n) \\ \qquad = \frac{2}{V}\sum_{n^0>0}\Big(\cos(k_n\cdot x_i)\mathrm{Re}\tilde{\phi}(k_n) - \sin(k_n\cdot x_i)\mathrm{Im}\tilde{\phi}(k_n)\Big) \\ \tilde{\phi}(k_n) = \epsilon^d \sum_i e^{-ik_n\cdot x_i}\phi(x_i) \\ \qquad = \frac{V}{(2N)^d}\sum_i \Big(\cos(k_n\cdot x_i) - i\sin(k_n\cdot x_i)\Big)\phi(x_i) \end{cases} \tag{A.2.5}$$

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

**proof:**

$\phi(x_i)$ 的变换需要做一些说明. 注意到 $\tilde{\phi}$ 的分量的数量是 $\phi$ 的两倍 (考虑到实部与虚部), 但在 $\phi \in \mathbb{R}^{(2N+1)^d}$ 时,

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k) \tag{A.2.6}$$

可见 $\tilde{\phi}$ 的分量并不独立, 取 $k^0 > 0$ 的部分为独立分量, 那么...

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

将 (A.2.5) 写成矩阵的形式,

$$\begin{cases} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} = \frac{2}{V} \overbrace{\begin{pmatrix} \cos k_0\cdot x_0 & \cdots & \cos k_{\max}\cdot x_0 & -\sin k_0\cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A} \begin{pmatrix} \mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots \end{pmatrix} = \frac{V}{(2N)^d} \begin{pmatrix} \cos k_0\cdot x_0 & \cdots & \cos k_0\cdot x_{\max} \\ \vdots & \ddots & \\ -\sin k_0\cdot x_0 & \cdots & -\sin k_0\cdot x_{\max} \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} \end{cases} \tag{A.2.7}$$

观察可见 $\tilde{\phi}$ 的变换中的矩阵是 $A^T$, 所以,

$$\frac{2}{V}\frac{V}{(2N)^d}AA^T = I \Longrightarrow \det A = \left(\frac{(2N)^d}{2}\right)^{\frac{(2N+1)^d}{2}} \tag{A.2.8}$$

因此...

– 顺便,

$$\int d^dx\, f(x)g(x) = \int \frac{d^dk}{(2\pi)^d}\tilde{f}(-k)\tilde{g}(k) \tag{A.2.9}$$

# Appendix B

# Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{B.0.1}$$

$$\langle x^{2n}\rangle = \frac{\int dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!! \tag{B.0.2}$$

其中 $(2n-1)!! = 1\cdot 3\cdots(2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx\, e^{-\frac{a}{2}x^2+Jx} = \sqrt{\frac{2\pi}{a}}e^{\frac{J^2}{2a}} \tag{B.0.3}$$

另外, 将 $a, J$ 分别替换为 $-ia, iJ$ 也是重要的变体.

## B.1 $N$-dim. generalization

- 考虑如下积分,

$$Z(A,J) = \int dx_1\cdots dx_N\, e^{-\frac{1}{2}x^T\cdot A\cdot x + J^T\cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \tag{B.1.1}$$

其中 $x, J$ 是 $N$-dim. 列向量, $A$ 是 $N\times N$ 实对称矩阵.

**calculation:**

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1}\cdot D\cdot O \tag{B.1.2}$$

其中 $D$ 是一个 diagonal matrix. 令 $y = O\cdot x$, 那么,

$$\begin{aligned} Z(A,J) &= \int dy_1\cdots dy_N\, e^{-\frac{1}{2}y^T\cdot D\cdot y + (OJ)^T\cdot y} \\ &= \prod_{i=1}^{N}\sqrt{\frac{2\pi}{D_{ii}}}e^{\frac{1}{2D_{ii}}(OJ)_i{}^2} = \sqrt{\frac{(2\pi)^N}{\det A}}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \end{aligned} \tag{B.1.3}$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O\cdot A^{-1}\cdot O^{-1})_{ii}$ 以及 $\operatorname{tr} D = \det A$.

- 一个重要的变体是 $A\mapsto -iA, J\mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 $A$ 是对称的),

$$\langle x_i x_j\rangle = \frac{1}{Z(A,0)}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}Z(A,J)\Big|_{J=0} = A^{-1}_{ij} \tag{B.1.4}$$

$$\langle x_i x_j\cdots x_k x_l\rangle = \sum_{\text{Wick}} A^{-1}_{i'j'}\cdots A^{-1}_{k'l'} \tag{B.1.5}$$

其中 (B.1.5) 中有偶数个 $x$, 否则等于零.

**calculation:**

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A,J) \Big|_{J=0} = \cdots \tag{B.1.6}$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} \tag{B.1.7}$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ik} A^{-1}_{jl} \tag{B.1.8}$$

# Appendix C

# perturbation theory in QM

- this chapter is based on MIT OpenCourseWare Quantum Physics III Chapter 1: Perturbation Theory.

---

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda \delta H \tag{C.0.1}$$

其中 $\lambda \in [0, 1]$.

- 考虑 $H^{(0)}$ 的本征态为,

$$H^{(0)} |k^{(0)}\rangle = E_k^{(0)} |k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)} | l^{(0)} \rangle = \delta_{kl} \\ E_0^{(0)} \leq E_1^{(0)} \leq E_2^{(0)} \leq \cdots \end{cases} \tag{C.0.2}$$

## C.1 non-degenerate perturbation theory

- 考虑 non-degenerate 能级 $k$, 有 $\cdots \leq E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \leq \cdots$, 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \cdots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \cdots \end{cases} \tag{C.1.1}$$

  - 注意, 我们可以选取修正项满足,

$$\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \cdots \tag{C.1.2}$$

**proof:**

假设我们求解得到的修正项不满足 $\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \cdots$, 考虑,

$$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n |k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)} | k^{(n)} \rangle' = 0 \tag{C.1.3}$$

那么, (注意到态矢量可以乘一个常数, $\frac{1}{1-a_1\lambda-a_2\lambda^2-\cdots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \cdots$),

$$\begin{aligned} |k\rangle_\lambda &= (1 - a_1\lambda - a_2\lambda^2 - \cdots) |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \cdots \\ |k\rangle'_\lambda &= |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \cdots} \Big( \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \cdots \Big) \\ &= |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 (a_1 |k^{(1)}\rangle' + |k^{(2)}\rangle') + \cdots \end{aligned} \tag{C.1.4}$$

可见修正项都与 $|k^{(0)}\rangle$ 正交.

  - 注意, 不能要求 ${}_\lambda\langle k | k \rangle_\lambda = 1$, 否则 $|k^{(n)}\rangle$ 将与 $\lambda$ 相关 (包括 $|k^{(0)}\rangle$),

$$\begin{aligned} {}_\lambda\langle k | k \rangle_\lambda = &\langle k^{(0)} | k^{(0)} \rangle \\ &+ \lambda(\langle k^{(1)} | k^{(0)} \rangle + \langle k^{(0)} | k^{(1)} \rangle) \\ &+ \lambda^2(\langle k^{(2)} | k^{(0)} \rangle + \langle k^{(1)} | k^{(1)} \rangle + \langle k^{(0)} | k^{(2)} \rangle) \end{aligned}$$

$$
\vdots
$$

$$
+\lambda^n(\langle k^{(n)}|k^{(0)}\rangle+\langle k^{(n-1)}|k^{(1)}\rangle+\cdots+\langle k^{(0)}|k^{(n)}\rangle) \tag{C.1.5}
$$

- 将 (C.1.1) 代入 Schrödinger's eq., 得到,

| | |
|---|---|
| $\lambda^0$ | $(H^{(0)}-E_k^{(0)})\,\lvert k^{(0)}\rangle=0$ |
| $\lambda^1$ | $(H^{(0)}-E_k^{(0)})\,\lvert k^{(1)}\rangle=(E_k^{(1)}-\delta H)\,\lvert k^{(0)}\rangle$ |
| $\lambda^2$ | $(H^{(0)}-E_k^{(0)})\,\lvert k^{(2)}\rangle=(E_k^{(1)}-\delta H)\,\lvert k^{(1)}\rangle+E_k^{(2)}\,\lvert k^{(0)}\rangle$ |
| $\vdots$ | $\vdots$ |
| $\lambda^n$ | $(H^{(0)}-E_k^{(0)})\,\lvert k^{(n)}\rangle=(E_k^{(1)}-\delta H)\,\lvert k^{(n-1)}\rangle+E_k^{(2)}\,\lvert k^{(n-2)}\rangle+\cdots+E_k^{(n)}\,\lvert k^{(0)}\rangle$ |

**calculation:**

Schrödinger's eq. 为,

$$
(H^{(0)}+\lambda\delta H-E_k(\lambda))\,|k\rangle_\lambda=0 \tag{C.1.6}
$$

展开为,

$$
\Big((H^{(0)}-E_k^{(0)})+\lambda(\delta H-E_k^{(1)})-\lambda^2E_k^{(2)}-\cdots\Big)(|k^{(0)}\rangle+\lambda\,|k^{(1)}\rangle+\lambda^2\,|k^{(2)}\rangle+\cdots)=0 \tag{C.1.7}
$$

- 现在来计算 $\langle l^{(0)}|k^{(n)}\rangle$, 有,

$$
\begin{cases}
(E_l^{(0)}-E_k^{(0)})\,\langle l^{(0)}|k^{(1)}\rangle = E_k^{(1)}\delta_{lk}-\delta H_{lk}\\
(E_l^{(0)}-E_k^{(0)})\,\langle l^{(0)}|k^{(2)}\rangle = E_k^{(1)}\,\langle l^{(0)}|k^{(1)}\rangle-\langle l^{(0)}|\delta H|k^{(1)}\rangle+E_k^{(2)}\delta_{lk}\\
\qquad\vdots \qquad\qquad\qquad \vdots\\
(E_l^{(0)}-E_k^{(0)})\,\langle l^{(0)}|k^{(n)}\rangle = E_k^{(1)}\,\langle l^{(0)}|k^{(n-1)}\rangle-\langle l^{(0)}|\delta H|k^{(n-1)}\rangle\\
\qquad\qquad\qquad +E_k^{(2)}\,\langle l^{(0)}|k^{(n-2)}\rangle+\cdots+E_k^{(n)}\delta_{lk}
\end{cases} \tag{C.1.8}
$$

其中 $\delta H_{lk}=\langle l^{(0)}|\delta H|k^{(0)}\rangle$, 对于满足 (C.1.2) 的解, 有,

$$
E_k^{(n)}=\langle k^{(0)}|\delta H|k^{(n-1)}\rangle\,, n=1,2,\cdots \tag{C.1.9}
$$

并且,

$$
|k^{(1)}\rangle=-\sum_{l\neq k}\frac{\delta H_{lk}}{E_l^{(0)}-E_k^{(0)}}\,|l^{(0)}\rangle \Longrightarrow E_k^{(2)}=-\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{E_l^{(0)}-E_k^{(0)}} \tag{C.1.10}
$$

**calculation:**

将 (C.1.10) 代入 (C.1.8), 得到 $(l\neq k)$,

$$
(E_l^{(0)}-E_k^{(0)})\,\langle l^{(0)}|k^{(2)}\rangle=-E_k^{(1)}\frac{\delta H_{lk}}{E_l^{(0)}-E_k^{(0)}}+\sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)}-E_k^{(0)}} \tag{C.1.11}
$$

所以,

$$
\begin{cases}
|k^{(2)}\rangle=\displaystyle\sum_{l\neq k}\Big(-\frac{\delta H_{00}\delta H_{lk}}{(E_l^{(0)}-E_k^{(0)})^2}+\sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)}-E_k^{(0)}}\Big)\,|l^{(0)}\rangle\\
E_k^{(3)}=\displaystyle\sum_{l\neq k}\Big(-\frac{\delta H_{00}|\delta H_{lk}|^2}{(E_l^{(0)}-E_k^{(0)})^2}+\sum_{m\neq k}\frac{\delta H_{kl}\delta H_{lm}\delta H_{mk}}{E_m^{(0)}-E_k^{(0)}}\Big)
\end{cases} \tag{C.1.12}
$$

---

计算归一化系数,

$$
{}_\lambda\langle k|k\rangle_\lambda=1+\lambda^2\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{(E_l^{(0)}-E_k^{(0)})^2}+O(\lambda^3) \tag{C.1.13}
$$

### C.1.1 level repulsion or the seesaw mechanism

- 能量的展开式为,

$$E_k(\lambda) = E_k^{(0)} + \lambda \delta H_{kk} - \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3) \tag{C.1.14}$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

### C.1.2 validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即,

$$|\lambda V| < \frac{1}{2} \Delta E^{(0)} \tag{C.1.15}$$

因此, 对于能级简并的情况, $\Delta E^{(0)} = 0$, 情况会更复杂.

**calculation:**

对于两能级系统,

$$H(\lambda) = H^{(0)} + \lambda \hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix} \tag{C.1.16}$$

$H(\lambda)$ 的本征値可以直接计算,

$$E_\pm(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)}) \sqrt{1 + \left( \frac{\lambda |V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})} \right)^2} \tag{C.1.17}$$

考虑 $\sqrt{1+z^2}$ 的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1} \frac{(2n-3)!!}{2^n n!} z^{2n} + \cdots \tag{C.1.18}$$

注意到 $\sqrt{1+z^2}$ 在 $z = \pm i$ 有 branch cut, 因此 $z = 0$ 附件的 Taylor expansion 只有在 $|z| < 1$ 内才收敛.

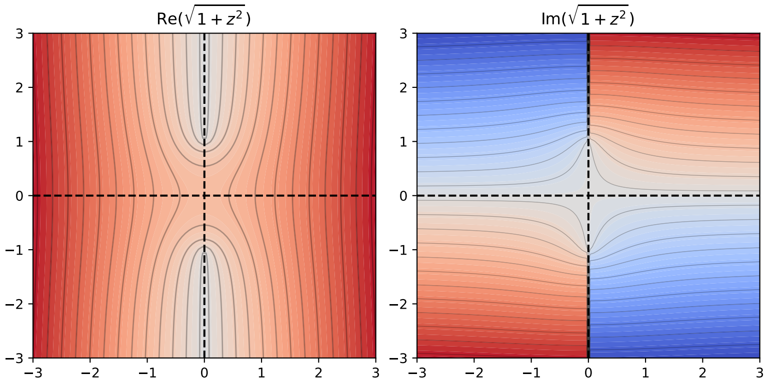



Figure C.1: graph of $\sqrt{1+z^2}$

## C.2 degenerate perturbation theory

- 暂时先跳过.

# Appendix D

# classical field theory and Noether's theorem

## D.1 classical field theory

### D.1.1 Lagrangian density and the action

- Lagrangian density, $\mathcal{L}$, 是 $\phi^a(x), \partial_\mu\phi^a(x), t$ 的函数.
- 对作用量变分得到 Euler-Lagrangian equation of motion,

$$\frac{\delta\mathcal{L}}{\delta\phi^a} - \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\Big) = 0 \tag{D.1.1}$$

**calculation:**

对作用量进行变分,

$$\begin{aligned}\delta S &= \int d^4x\Big(\frac{\delta\mathcal{L}}{\delta\phi^a}\delta\phi^a + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\delta\partial_\mu\phi^a\Big) \\ &= \int d^4x\Big(\Big(\frac{\delta\mathcal{L}}{\delta\phi^a} - \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\Big)\Big)\delta\phi^a + \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\delta\phi^a\Big)\Big)\end{aligned} \tag{D.1.2}$$

由于边界变分为零...

### D.1.2 canonical momentum and the Hamiltonian

- **def.:** 定义一个叫 $\pi_a^\mu$ 的量,

$$\pi_a^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)} \tag{D.1.3}$$

其中 $\pi_a \equiv \pi_a^0$ 称作 canonical momentum of the field.

- **def.:** the Hamiltonian density is,

$$\mathcal{H} = \pi_a\partial_0\phi^a - \mathcal{L} \tag{D.1.4}$$

---

- the Hamilton's equations are,

$$\begin{cases}\partial_0\phi^a = \dfrac{\delta\mathcal{H}}{\delta\pi_a} \\ -\partial_0\pi^a = \dfrac{\delta\mathcal{H}}{\delta\phi^a} - \partial_i\Big(\dfrac{\delta\mathcal{H}}{\delta(\partial_i\phi^a)}\Big)\end{cases} \tag{D.1.5}$$

  – 第二个方程可以写成更紧凑的形式,

$$\partial_\mu\pi_a^\mu = \frac{\delta\mathcal{H}}{\delta\phi^a} \tag{D.1.6}$$

## D.2 Noether's theorem

### D.2.1 in classical particle mechanics

- 系统的 Lagrangian 为 $L(q^a, \dot{q}^a, t)$.
- 系统通过以下形式变换,

$$q^a(t) \mapsto q^a(\lambda, t) \quad \text{and} \quad q^a(t, 0) = q^a(t) \tag{D.2.1}$$

  并定义,

$$D_\lambda q^a = \frac{\partial q^a}{\partial \lambda}\Big|_{\lambda=0} \tag{D.2.2}$$

- **Noether's theorem:** the continuous transform $\lambda$ is a **continuous symmetry** iff.,

$$D_\lambda L = \frac{dF(q^a, \dot{q}^a, t)}{dt} \tag{D.2.3}$$

  for some $F(q^a, \dot{q}^a, t)$, and the corresponding **conserved quantity** is,

$$Q = p_a D_\lambda q^a - F(q^a, \dot{q}^a, t) \tag{D.2.4}$$

**proof:**

$$D_\lambda L = \frac{\partial L}{\partial q^a} D_\lambda q^a + \frac{\partial L}{\partial \dot{q}^a} \frac{dD_\lambda q^a}{dt} = \frac{d}{dt}(p_a D_\lambda q^a) \tag{D.2.5}$$

- 几个例子如下,
  - **空间平移**, $\vec{x}(t) \mapsto \vec{x}(t) + \hat{e}_i \lambda$, 相应地, $D_\lambda \vec{x} = \hat{e}_i$, 且,

$$D_\lambda L = \frac{\partial L}{\partial x^i} \tag{D.2.6}$$

    如果 $\frac{\partial L}{\partial x^i} = 0$, 那么, 有守恒量 $p_i$.
  - **时间平移**, $q^a(t) \mapsto q^a(t + \lambda)$, 相应地, $D_\lambda q^a = \dot{q}^a$, 且,

$$D_\lambda L = \frac{dL}{dt} - \frac{\partial L}{\partial t} \tag{D.2.7}$$

    如果 $\frac{\partial L}{\partial t} = 0$, 那么, 有守恒量 $H = p_a \dot{q}^a - L$.
  - **转动**, $\vec{x}(t) \mapsto R(\lambda, \hat{e}) \cdot \vec{x}(t)$, 相应地, $D_\lambda \vec{x} = \hat{e} \times \vec{x}$, 且,

$$D_\lambda L = \vec{x} \cdot \left( \frac{\partial L}{\partial \vec{x}} \times \hat{e} \right) + \hat{e}(\dot{\vec{x}} \times \vec{p}) \tag{D.2.8}$$

    如果上式中两个括号内的项都为零, 那么, 有守恒量 $\hat{e} \cdot \vec{J} = \hat{e} \cdot (\vec{x} \times \vec{p})$.

### D.2.2 in classical field theory

- 类似地, 系统通过以下形式变换,

$$\phi^a(x) \mapsto \phi^a(x, \lambda) \quad \text{and} \quad \phi^a(x, 0) = \phi^a(x) \tag{D.2.9}$$

  并定义,

$$D_\lambda \phi^a = \frac{\partial \phi^a}{\partial \lambda}\Big|_{\lambda=0} \tag{D.2.10}$$

- **Noether's theorem:** the continuous transform $\lambda$ is a **continuous symmetry** iff.,

$$D_\lambda \mathcal{L} = \partial_\mu F^\mu(\phi^a, \partial_\mu \phi^a, t) \tag{D.2.11}$$

  for some $F^\mu(\phi^a, \partial_\mu \phi^a, t)$, and the **conserved current** is,

$$J^\mu = \pi^\mu_a D_\lambda \phi^a - F^\mu \tag{D.2.12}$$

**proof:**

$$\begin{aligned} D_\lambda \mathcal{L} &= \frac{\delta \mathcal{L}}{\delta \phi^a} D_\lambda \phi^a + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \partial_\mu D_\lambda \phi^a \\ &= \Big( \frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \Big( \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)} \Big) \Big) D_\lambda \phi^a + \partial_\mu \Big( \underbrace{\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)}}_{=\pi^\mu_a} D_\lambda \phi^a \Big) \end{aligned} \tag{D.2.13}$$

代入 (D.1.1), 得...

---

- 注意, conserved current 并不是唯一确定的, 考虑如下变换,

$$F^\mu \mapsto F'^\mu = F^\mu + \partial_\nu A^{\mu\nu} \quad \text{with} \quad A^{\mu\nu} = A^{[\mu\nu]} \tag{D.2.14}$$

新 $F'^\mu$ 依然能满足 (D.2.11).

- 但是, 守恒荷是唯一确定的.

**proof:**

$$Q' = \int d^3x J^0 = \int d^3x (\pi_a D_\lambda \phi^a - F^0) - \int d^3x\, \partial_\mu A^{0\mu} \tag{D.2.15}$$

考虑到边界值为零, 且 $A^{00} = 0$, 所以 $Q' = Q$.

### D.2.3 spacetime translations and the energy-momentum tensor

- 时空平移变换为,

$$\phi^a(x) \mapsto \phi^a(x + \lambda e) \tag{D.2.16}$$

- 所以,

$$D_\lambda \phi^a = e^\mu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = e^\mu \partial_\mu \mathcal{L} \tag{D.2.17}$$

代入 (D.2.12),

$$J^\mu = e^\nu \underbrace{(\pi^\mu_a \partial_\nu \phi^a - \delta^\mu_\nu \mathcal{L})}_{=T^\mu{}_\nu} \tag{D.2.18}$$

---

- 并且有,

$$\partial_\mu T^{\mu\nu} = 0 \implies P^\mu = \int d^3x\, T^{0\mu} = \text{Const.} \tag{D.2.19}$$

来自守恒流散度为零.

### D.2.4 Lorentz transformations, angular momentum and something else

- Lorentz transformation 下坐标做变换 $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, 其中 $\Lambda$ 满足,

$$\eta = \Lambda^T \eta \Lambda \tag{D.2.20}$$

- infinitesimal Lorentz transformation 是,

$$\Lambda = I + \epsilon \tag{D.2.21}$$

其中 $\{\epsilon^{\mu\nu}\} = \epsilon\eta$ 是反对称矩阵.

**proof:**

考虑,

$$\eta = (\Lambda\eta)^T \eta (\Lambda\eta) = (\eta + \epsilon\eta)^T \eta (\eta + \epsilon\eta)$$

$$= \eta + \eta\epsilon^T + \epsilon\eta + O(\epsilon^2) \tag{D.2.22}$$

- 标量场在 Lorentz transform 下的变换为,

$$\Lambda : \phi^a(x) \mapsto \phi^a(\Lambda^{-1}x') \tag{D.2.23}$$

---

- 有,

$$D_\lambda \phi^a = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\epsilon_{\mu\nu}\partial^\mu(x^\nu \mathcal{L}) \tag{D.2.24}$$

代入 (D.2.12),

$$J^\mu = \frac{1}{2}\epsilon_{\nu\rho}M^{\mu\nu\rho} \quad \text{where} \quad M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{D.2.25}$$

且有,

$$\partial_\mu M^{\mu\nu\rho} = 0 \tag{D.2.26}$$

- 对全空间积分, 得到 6 个守恒量,

$$J^{\mu\nu} = \int d^3x\, M^{0\mu\nu} = \text{Const.} \tag{D.2.27}$$

不难发现 $J^{ij}$ 对应角动量, 现在来讨论 $J^{0i}$ 的物理意义,

$$0 = \frac{d}{dt}J^{0i} = \frac{d}{dt}\int d^3x(tT^{0i} - x^i T^{00}) = P^i - \frac{d}{dt}\int d^3x\, x^i T^{00} \tag{D.2.28}$$

其中, 用到了 $\frac{dP^i}{dt} = 0$ (见 (D.2.19)), 可以将上式的第二项理解为质心运动的动量.

## D.3 charge as generators

- the charge associated with the conserved current is,

$$Q = \int d^D\, J^0 = \int d^D x\,(\pi_a D_\lambda \phi^a - F^0) \tag{D.3.1}$$

在 $F^\mu = 0$ 且 $[D_\lambda\phi^a, \phi^a] = 0$ 的情况下,

$$i[Q, \phi^a] = D_\lambda \phi^a \tag{D.3.2}$$

## D.4 what the graviton listens to: energy-momentum tensor

- the energy-momentum tensor is defined as (其中 $g = |\det\{g_{\mu\nu}\}|$),

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M \tag{D.4.1}$$

- 如果将 $\mathcal{L}_M$ 对 $g^{\mu\nu}$ 做展开 $\mathcal{L}_M = A + g^{\mu\nu}B_{\mu\nu} + g^{\mu\nu}g^{\rho\sigma}C_{\mu\nu\rho\sigma} + \cdots$, 那么,

$$T_{\mu\nu} = -2(B_{\mu\nu} + 2g^{\rho\sigma}C_{\mu\nu\rho\sigma} + 3\cdots) + g_{\mu\nu}\mathcal{L}_M \tag{D.4.2}$$

另外, the trace of the energy-momentum tensor is,

$$T = g^{\mu\nu}T_{\mu\nu} = d \times A + (d-2)g^{\mu\nu}B_{\mu\nu} + (d-4)g^{\mu\nu}g^{\rho\sigma}C_{\mu\nu\rho\sigma} \tag{D.4.3}$$

可见 $d = 4$ 时, $T$ 与 $C_{\mu\nu\rho\sigma}$ 无关.

---

- 以 electromagnetic field 为例, $d = 4$,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A^\mu A_\mu \Longrightarrow \begin{cases} T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + m^2 A_{\mu\nu} + g_{\mu\nu}\mathcal{L}_M \\ T = -m^2 A^\mu A_\mu \end{cases} \tag{D.4.4}$$

可见 the energy-momentum tensor of electromagnetic field (when $m = 0$) is traceless.

- $\mathcal{L} = -\frac{1}{2}((\partial\phi)^2 - m^2\phi^2)$ 和 $\mathcal{L} = \frac{1}{2}\phi(\partial^2 - m^2)\phi$ 对应的 energy-momentum tensor 一样吗 (?).

# Appendix E

# antiunitary operator and time reversal

## E.1 complex conjugation operator

- complex conjugation operator, $K$, is an antiunitary operator on the complex plane,

$$\begin{cases} Kz = z^* \\ zK^* = z^* \end{cases} \Longrightarrow K^2 = K^{*2} = 1 \tag{E.1.1}$$

- $K^* I : V^* \to V^*$ 是 dual space 上的算符.
- 对于一组 orthonormal basis, 有,

$$\langle i|K^* IK|j\rangle = \delta_{ij} \tag{E.1.2}$$

并且可以证明在基矢变换后这个等式依然成立.

**proof:**

- 对基矢做 unitary transformation,

$$|i'\rangle = U|i\rangle = \sum_j |j\rangle U_{ji} \quad \text{where} \quad U_{ji} = \langle j|U|i\rangle \tag{E.1.3}$$

那么,

$$\langle i'|K^* IK|j'\rangle = \sum_{kl} \langle k|U^*_{ki} K^* IKU_{lj}|l\rangle = \sum_{kl} U_{ki} U^*_{lj} \delta_{kl} = \delta_{ij} \tag{E.1.4}$$

- 对基矢做 antiunitary transformation, 只需要证明 $|i'\rangle = K|i\rangle$ 的情况, 此时,

$$\langle i'|K^* IK|j'\rangle = \langle i|j\rangle = \delta_{ij} \tag{E.1.5}$$

## E.2 antiunitary operator

- 对于一个 unitary operator, $U$, $\Omega = UK$ 是一个 antiunitary operator.
- 定义其 Hermitian conjugate,

$$\Omega^\dagger = K^* U^\dagger \iff \langle i|\Omega j\rangle = \langle j|\Omega^\dagger i\rangle^* \tag{E.2.1}$$

那么,

$$\begin{cases} \langle \phi|\Omega\psi\rangle = \langle \psi|\Omega^\dagger \phi\rangle^* \\ \langle \Omega\phi|\Omega\psi\rangle = \langle \psi|\phi\rangle \end{cases} \tag{E.2.2}$$

**proof:**

首先,

$$\langle \phi|\Omega\psi\rangle = \sum_{ij} \langle i|\phi_i^* UK\psi_j|j\rangle$$

$$
\begin{aligned}
&= \sum_{ij} \phi_i^* \psi_j^* \langle i|UK|j\rangle \\
&= \Big( \sum_{ij} \langle j|K^* U^\dagger|i\rangle \, \phi_i \psi_j \Big)^* \\
&= \Big( \sum_{ij} \langle j|\psi_j^* K^* U^\dagger \phi_i|i\rangle \Big)^* = \langle \psi|K^* U^\dagger|\phi\rangle^*
\end{aligned} \tag{E.2.3}
$$

其次,

$$
\begin{aligned}
\langle \Omega\phi|\Omega\psi\rangle &= \langle \phi|\Omega^\dagger \Omega\psi\rangle = \langle \phi|K^* I K|\psi\rangle \\
&= \sum_{ij} \langle i|\phi_i^* K^* I K \psi_j|j\rangle \\
&= \sum_{ij} \phi_i \psi_j^* \langle i|K^* I K|j\rangle = \langle \psi|\phi\rangle
\end{aligned} \tag{E.2.4}
$$