Quantum Field Theory

- based on A. Zee's textbook -

万思扬

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Contents

convention, notation, and units

- 笔记中的**度规号差**约定为 $(-,+,+,+)$.
- 使用 Planck units, 此时 $G, \hbar, c, k_B = 1$, 因此,

name/dimension	expression/value
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

Part I

motivation and foundation

Chapter 1

free field theory

1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{1.1.1}$$

A. Zee 说: 在作用量里, 时间的导数项必须是正的, 包括标量场的 $(\partial_0\phi)^2$ 和电磁场的 $(\partial_0 A_i)^2$.

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi\, e^{i\int d^dx\,(-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \tag{1.1.2}$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)} \tag{1.1.3}$$

另外, 用 $W(J)$ 表示指数上的部分 (去除掉虚数 i).

proof:

注意 $\partial^\mu\phi\partial_\mu\phi = \partial^\mu(\phi\partial_\mu\phi) - \phi\partial^2\phi$, 忽略全微分项, 那么,

$$Z(J) = \int D\phi\, e^{i\int d^dx\,\frac{1}{2}(\phi(\partial^2 - m^2)\phi + J(x)\phi(x))} \tag{1.1.4}$$

代入 (B.1.1), 可知,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)} \tag{1.1.5}$$

其中 $D(x-y)$ 满足,

$$\begin{cases}(\partial^2 - m^2)D(x-y) = \delta^{(d)}(x-y)\\ (-p^2 - m^2)\tilde{D}(p,q) = (2\pi)^d\delta^{(d)}(p-q)\end{cases} \Longrightarrow D(x-y) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot(x-y)}}{-k^2 - m^2} \tag{1.1.6}$$

1.2 free propagator

- 为了使 (1.1.4) 中的积分在 ϕ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon\int d^dx\phi^2}$.
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 ϵ 避免发散,

$$D(x) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot x}}{-k^2 - m^2 + i\epsilon} = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\Big(e^{i(-\omega_k t + \vec{k}\cdot\vec{x})}\theta(t) + e^{i(\omega_k t + \vec{k}\cdot\vec{x})}\theta(-t)\Big) \tag{1.2.1}$$

calculation:

对 k^0 积分, 注意有两个奇点 $k^0 = \pm(\omega_k - i\epsilon)$, 当 $t > 0$ 时, contour 处于下半平面, ...

- $D(x)$ 的取值与 x 的类时, 类空性质关系密切.
  - 类时区域,
$$D(t,0) = -i\int \frac{d^D k}{(2\pi)^D 2\omega_k}\left(e^{-i\omega_k t}\theta(t) + e^{i\omega_k t}\theta(-t)\right) \tag{1.2.2}$$
  - 类空区域,
$$D(0,\vec{x}) = -i\int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \sim e^{-m|\vec{x}|} \tag{1.2.3}$$

Chapter 2

from field to particle to force

2.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,

$$W(J) = -\frac{1}{2}\int d^dx d^dy\, J(y)D(x-y)J(y) \tag{2.1.1}$$

$$= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}(-k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}(k) \tag{2.1.2}$$

其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k)=\tilde{J}^*(k)$.

- 考虑 $J(x)=J_1(x)+J_2(x)$, 那么 $W(J)$ 共有 4 项, 其中一个交叉项如下,

$$W_{12}(J) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}_1(-k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}_2(k) \tag{2.1.3}$$

可见 $W(J)$ 取値较大的条件是:

1. $\tilde{J}_1(k), \tilde{J}_2(k)$ 有较大重叠,
2. 重叠位置的 k 是 on shell (即 $k^2=-m^2$).

- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

2.2 from particle to force

- 考虑 $J(x)=\delta^{(D)}(\vec{x}-\vec{x}_1)+\delta^{(D)}(\vec{x}-\vec{x}_1) \Longrightarrow \tilde{J}_a(k)=2\pi e^{-i\vec{k}\cdot\vec{x}_a}\delta(k^0)$, 那么,

$$\begin{aligned}W_{12}(J)+W_{21}(J) &= \delta(0)\int\frac{d^Dk}{(2\pi)^{D-1}}\frac{1}{|\vec{k}|^2+m^2-i\epsilon}\cos(\vec{k}\cdot(\vec{x}_1-\vec{x}_2)) \\ &\overset{D=3}{=} 2\pi\delta(0)\frac{1}{4\pi r}e^{-mr}\end{aligned} \tag{2.2.1}$$

($-i\epsilon$ 显然可以舍去), 注意到 $\langle 0|e^{-iHT}|0\rangle = e^{-iET}$, 而时间间隔 $T=\int dx^0 = 2\pi\delta(0)$, 所以,

$$E = -\frac{W(J)}{T} \overset{D=3}{=} -\frac{1}{4\pi r}e^{-mr} \tag{2.2.2}$$

calculation:

计算 (2.2.1) 中的积分, 令 $\vec{x}_1-\vec{x}_2=\vec{r}$,

$$I_D = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{|\vec{k}|^2+m^2}\overbrace{\cos(\vec{k}\cdot\vec{r})}^{\mapsto e^{i\vec{k}\cdot\vec{r}}}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^D} \int (k \sin\theta_1)^{D-2} d\Omega_{D-2} \int k d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ikr\cos\theta_1} \\
&= \frac{S_{D-2}}{(2\pi)^D} \int k^{D-1} \sin^{D-2}\theta_1 d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ikr\cos\theta_1}
\end{aligned} \tag{2.2.3}
$$

取 $D=3$, 那么,

$$
\begin{aligned}
I_{D=3} &= \frac{1}{(2\pi)^2} \int k^2 \sin\theta_1 d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ik\cos\theta_1} \\
&= \frac{1}{2\pi^2 r} \int_0^\infty \sin(kr) \frac{kdk}{k^2+m^2} = \frac{-i}{4\pi^2 r} \int_{-\infty}^{\infty} e^{ikr} \frac{kdk}{k^2+m^2} \\
&= \frac{-i}{4\pi^2 r} 2\pi i \underbrace{\mathrm{Res}(f, im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r} e^{-mr}
\end{aligned} \tag{2.2.4}
$$

Chapter 3

Coulomb and Newton: repulsive and attraction

3.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{3.1.1}$$

其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

- 做路径积分,

$$Z(J) = \int DA\, e^{i\int d^dx(\mathcal{L}+J_\mu A^\mu)} = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J_\mu D^{\mu\nu}(x-y)J_\nu(y)} \tag{3.1.2}$$

calculation:

massive photon 的作用量为,

$$\begin{aligned} S(A) &= \int d^dx\,\frac{1}{2}\Big(-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu\Big) \\ &= \int d^dx\,\frac{1}{2}\Big(A_\nu\partial^2 A^\nu - A_\nu\partial^\nu\partial_\mu A^\mu - m^2 A_\mu A^\mu\Big) + \text{total differential} \\ &= \int d^dx\,\frac{1}{2}A_\mu\Big(-\partial^\mu\partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2)\Big)A_\nu + \text{total differential} \end{aligned} \tag{3.1.3}$$

那么, 需要有,

$$\begin{aligned} &(-\partial^\mu\partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2))D_{\rho\nu}(x-y) = \delta^\mu_\nu\delta^{(d)}(x-y) \\ \Longrightarrow &\tilde{D}_{\mu\nu}(k) = \frac{k_\mu k_\nu/m^2 + \eta_{\mu\nu}}{-k^2 - m^2} \end{aligned} \tag{3.1.4}$$

考虑到积分需要收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, (为什么 A_μ 类空 (?)).

- 因此,

$$W(J) = -\frac{1}{2}\int d^dx d^dy\, J_\mu(x)D^{\mu\nu}(x-y)J_\nu(y) \tag{3.1.5}$$

$$= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}_\mu(-k)\frac{k^\mu k^\nu/m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon}\tilde{J}_\nu(k) \tag{3.1.6}$$

注意到 current conservation, 有 $\partial_\mu J^\mu = 0 \iff k^\mu \tilde{J}_\mu(k) = 0$, 所以,

$$W(J) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}_\mu(-k)\frac{1}{-k^2 - m^2 + i\epsilon}\tilde{J}^\mu(k) \tag{3.1.7}$$

3.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的 x, y, z 方向, 在粒子静止系下, 极化矢量 $(\epsilon^i)_\mu = \delta^i_\mu, i = 1, 2, 3$, 而 $k_\mu = (-m, 0, 0, 0)$, 所以,
$$k^\mu (\epsilon^i)_\mu = 0 \tag{3.1.8}$$
  - 注意, 一个粒子的极化方向用 e^i (这不是矢量) 表示, 极化矢量为 $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$.
- 在粒子静止系下, 考虑,
$$\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu} \tag{3.1.9}$$
  可见,
$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon} \tag{3.1.10}$$

3.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = linearized Einstein Lagrangian.
- 受 subsection 3.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足,
$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases} \tag{3.2.1}$$
  其中下指标 μ, ν 对称, $a = 1, \cdots, 5$, (可以验证 $(\epsilon^a)_{\mu\nu}$ 确实有 5 个独立分量).
  - 对 $(\epsilon^a)_{\mu\nu}$ 的归一化条件可以定义为 $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$.
  - 与 subsection 3.1.1 中提示一样, 粒子的极化方向用 e^a 表示.
- 那么,
$$\sum_{a=1}^5 (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma} \tag{3.2.2}$$

calculation:

首先用 k_μ 和 $\eta_{\mu\nu}$ 构造最一般的关于 $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$ 对称的 4 阶张量, (下式中把 $\frac{k_\mu}{m}$ 略写作 k_μ),
$$\begin{aligned} &A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ &+ D\eta_{\mu\nu}\eta_{\rho\sigma} + E(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \end{aligned} \tag{3.2.3}$$
代入 (3.2.1) 得,
$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4} \tag{3.2.4}$$
因此, 这个 4 阶张量最终确定为,
$$\frac{3}{4} A \Big((G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma} \Big) \tag{3.2.5}$$

- 所以,
$$\tilde{D}_{\mu\nu\rho\sigma}(k) = \frac{(G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tag{3.2.6}$$

Appendices

Appendix A

Dirac delta function & Fourier transformation

A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx\, \delta(x) e^{-ikx} \tag{A.1.1}$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \tag{A.1.2}$$

- 第二个常用的公式 (Sokhotski-Plemelj theorem),

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{A.1.3}$$

其中 $\mathcal{P}$ 表示复函数的主值 (principal value).

proof:

考虑,

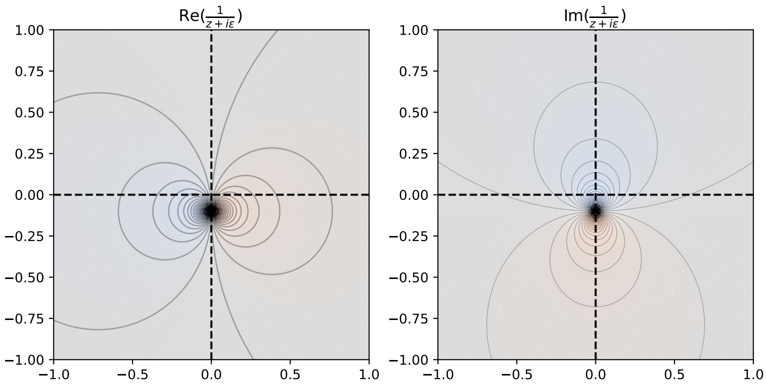
$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \tag{A.1.4}$$

且注意到,

$$\int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \,\mathrm{Res}(f, i\epsilon) = \pi \tag{A.1.5}$$

所以...

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,



A.2 Fourier transformation

- d-dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \int \dfrac{d^d k}{(2\pi)^d} e^{ik\cdot x} \tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^d x \, e^{-ik\cdot x} \phi(x) \end{cases} \tag{A.2.1}$$

- 因此,

$$\partial_\mu \phi(x) \mapsto i k_\mu \tilde{\phi}(k) \tag{A.2.2}$$

Appendix B

Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{B.0.1}$$

$$\langle x^{2n}\rangle = \frac{\int dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!! \tag{B.0.2}$$

  其中 $(2n-1)!! = 1\cdot 3\cdots(2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx\, e^{-\frac{a}{2}x^2+Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \tag{B.0.3}$$

  另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 N-dim. generalization

- 考虑如下积分,

$$Z(A,J) = \int dx_1\cdots dx_N\, e^{-\frac{1}{2}x^T\cdot A\cdot x + J^T\cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \tag{B.1.1}$$

  其中 x, J 是 N-dim. 列向量, A 是 $N\times N$ 实对称矩阵.

calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1}\cdot D\cdot O \tag{B.1.2}$$

其中 D 是一个 diagonal matrix. 令 $y = O\cdot x$, 那么,

$$\begin{aligned} Z(A,J) &= \int dy_1\cdots dy_N\, e^{-\frac{1}{2}y^T\cdot D\cdot y + (OJ)^T\cdot y} \\ &= \prod_{i=1}^{N}\sqrt{\frac{2\pi}{D_{ii}}} e^{\frac{1}{2D_{ii}}(OJ)_i{}^2} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \end{aligned} \tag{B.1.3}$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O\cdot A^{-1}\cdot O^{-1})_{ii}$ 以及 $\operatorname{tr} D = \det A$.

- 一个重要的变体是 $A\mapsto -iA, J\mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j\rangle = \frac{1}{Z(A,0)}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}Z(A,J)\Big|_{J=0} = A^{-1}_{ij} \tag{B.1.4}$$

$$\langle x_i x_j\cdots x_k x_l\rangle = \sum_{\text{Wick}} A^{-1}_{i'j'}\cdots A^{-1}_{k'l'} \tag{B.1.5}$$

  其中 (B.1.5) 中有偶数个 x, 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A,J) \Big|_{J=0} = \cdots \tag{B.1.6}$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} \tag{B.1.7}$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ik} A^{-1}_{jl} \tag{B.1.8}$$