

Quantum Field Theory

a study note based on A. Zee's textbook

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convention, notation, and units

- 笔记中的度规号差约定为 $(-, +, +, +)$.
- 使用 Planck units, 此时 $G, \hbar, c, k_B = 1$, 因此,

name/dimension	expression/value
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} \text{ m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44} \text{ s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} \text{ kg} \simeq 10^{19} \text{ GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32} \text{ K}$

- 时空维度用 $d = D + 1$ 表示.

Part I

motivation and foundation

Chapter 1

free field theory

1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \quad (1.1.1)$$

A. Zee 说: 在作用量里, 时间的导数项必须是正的, 包括标量场的 $(\partial_0\phi)^2$ 和电磁场的 $(\partial_0 A_i)^2$.

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi e^{i \int d^d x (-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \quad (1.1.2)$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

-
- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (1.1.3)$$

另外, 用 $W(J)$ 表示指数上的部分 (去掉虚数 i).

proof:

注意 $\partial^\mu \phi \partial_\mu \phi = \partial^\mu (\phi \partial_\mu \phi) - \phi \partial^2 \phi$, 忽略全微分项, 那么,

$$Z(J) = \int D\phi e^{i \int d^d x (\frac{1}{2} \phi (\partial^2 - m^2) \phi + J(x)\phi(x))} \quad (1.1.4)$$

代入 (B.1.1), 可知,

$$Z(J) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (1.1.5)$$

其中 $D(x-y)$ 满足,

$$\begin{cases} (\partial^2 - m^2) D(x-y) = \delta^{(d)}(x-y) \\ (-p^2 - m^2) \tilde{D}(p, q) = (2\pi)^d \delta^{(d)}(p-q) \end{cases} \implies D(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot (x-y)}}{-k^2 - m^2} \quad (1.1.6)$$

1.2 free propagator

- 为了使 (1.1.4) 中的积分在 ϕ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon \int d^d x \phi^2}$.
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 ϵ 避免发散,

$$D(x) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot x}}{-k^2 - m^2 + i\epsilon} = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left(\theta(t) e^{i(-\omega_k t + \vec{k} \cdot \vec{x})} + \theta(-t) e^{i(\omega_k t + \vec{k} \cdot \vec{x})} \right) \quad (1.2.1)$$

calculation:

对 k^0 积分, 注意有两个奇点 $k^0 = \pm(\omega_k - i\epsilon)$, 当 $t > 0$ 时, contour 处于下半平面, ... (另外注意到我们可以任意改变 \vec{k} 的符号).

- $D(x)$ 的取值与 x 的类时, 类空性质关系密切.

– 类时区域,

$$D(t, 0) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} \left(\theta(t) e^{-i\omega_k t} + \theta(-t) e^{i\omega_k t} \right) \quad (1.2.2)$$

– 类空区域,

$$D(0, \vec{x}) = -i \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{i\vec{k} \cdot \vec{x}} \sim e^{-m|\vec{x}|} \quad (1.2.3)$$

1.3 from field to particle to force

1.3.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,

$$W(J) = -\frac{1}{2} \int d^d x d^d y J(y) D(x-y) J(y) \quad (1.3.1)$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}(k) \quad (1.3.2)$$

其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k) = \tilde{J}^*(k)$.

- 考虑 $J(x) = J_1(x) + J_2(x)$, 那么 $W(J)$ 共有 4 项, 其中一个交叉项如下,

$$W_{12}(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_1(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_2(k) \quad (1.3.3)$$

可见 $W(J)$ 取值较大的条件是:

1. $\tilde{J}_1(k), \tilde{J}_2(k)$ 有较大重叠,
2. 重叠位置的 k 是 on shell (即 $k^2 = -m^2$).

- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

1.3.2 from particle to force

- 考虑 $J(x) = \delta^{(D)}(\vec{x} - \vec{x}_1) + \delta^{(D)}(\vec{x} - \vec{x}_2) \Rightarrow \tilde{J}_a(k) = 2\pi e^{-i\vec{k} \cdot \vec{x}_a} \delta(k^0)$, 那么,

$$W_{12}(J) + W_{21}(J) = \delta(0) \int \frac{d^D k}{(2\pi)^{D-1}} \frac{1}{|\vec{k}|^2 + m^2 - i\epsilon} \cos(\vec{k} \cdot (\vec{x}_1 - \vec{x}_2))$$

$$\stackrel{D=3}{=} 2\pi \delta(0) \frac{1}{4\pi r} e^{-mr} \quad (1.3.4)$$

($-i\epsilon$ 显然可以舍去), 注意到 $\langle 0 | e^{-iHT} | 0 \rangle = e^{-iET}$, 而时间间隔 $T = \int dx^0 = 2\pi \delta(0)$, 所以,

$$E = -\frac{W(J)}{T} \stackrel{D=3}{=} -\frac{1}{4\pi r} e^{-mr} \quad (1.3.5)$$

calculation:

计算 (1.3.4) 中的积分, 令 $\vec{x}_1 - \vec{x}_2 = \vec{r}$,

$$I_D = \int \frac{d^D k}{(2\pi)^D} \frac{1}{|\vec{k}|^2 + m^2} \overbrace{\cos(\vec{k} \cdot \vec{r})}^{\mapsto e^{i\vec{k} \cdot \vec{r}}}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^D} \int (k \sin \theta_1)^{D-2} d\Omega_{D-2} \int k d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1} \\
&= \frac{S_{D-2}}{(2\pi)^D} \int k^{D-1} \sin^{D-2} \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ikr \cos \theta_1}
\end{aligned} \tag{1.3.6}$$

取 $D = 3$, 那么,

$$\begin{aligned}
I_{D=3} &= \frac{1}{(2\pi)^2} \int k^2 \sin \theta_1 d\theta_1 dk \frac{1}{k^2 + m^2} e^{ik \cos \theta_1} \\
&= \frac{1}{2\pi^2 r} \int_0^\infty \sin(kr) \frac{k dk}{k^2 + m^2} = \frac{-i}{4\pi^2 r} \int_{-\infty}^\infty e^{ikr} \frac{k dk}{k^2 + m^2} \\
&= \frac{-i}{4\pi^2 r} 2\pi i \underbrace{\text{Res}(f, im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r} e^{-mr}
\end{aligned} \tag{1.3.7}$$

Chapter 2

Coulomb and Newton: repulsive and attraction

2.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \quad (2.1.1)$$

其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

- 做路径积分,

$$Z(J) = \int DA e^{i \int d^d x (\mathcal{L} + J_\mu A^\mu)} = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y J_\mu D^{\mu\nu}(x-y) J_\nu(y)} \quad (2.1.2)$$

calculation:

massive photon 的作用量为,

$$\begin{aligned} S(A) &= \int d^d x \frac{1}{2} \left(-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu \right) \\ &= \int d^d x \frac{1}{2} \left(A_\nu \partial^2 A^\nu - A_\nu \partial^\nu \partial_\mu A^\mu - m^2 A_\mu A^\mu \right) + \text{total differential} \\ &= \int d^d x \frac{1}{2} A_\mu \left(-\partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \right) A_\nu + \text{total differential} \\ &= \int \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(-k) \left(k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2) \right) \tilde{A}_\nu(k) + \text{boundary term} \end{aligned} \quad (2.1.3)$$

那么, 需要有,

$$\begin{aligned} (-\partial^\mu \partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2)) D_{\rho\nu}(x-y) &= \delta_\nu^\mu \delta^{(d)}(x-y) \\ \implies \tilde{D}_{\mu\nu}(k) &= \frac{k_\mu k_\nu / m^2 + \eta_{\mu\nu}}{-k^2 - m^2} \end{aligned} \quad (2.1.4)$$

考虑到积分需要收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, (为什么 A_μ 类空, 只知道 \tilde{A}_μ 类空, 见 subsection 2.1.2, 但路径积分中的 A 显然不满足 field equation \implies 路径积分中起主要作用的 \tilde{A} 类空, 因此 $-\epsilon|\tilde{A}|^2 < 0$).

- 因此,

$$W(J) = -\frac{1}{2} \int d^d x d^d y J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) \quad (2.1.5)$$

$$= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}_\mu(-k) \frac{k^\mu k^\nu / m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon} \tilde{J}_\nu(k) \quad (2.1.6)$$

注意到 current conservation, 有 $\partial_\mu J^\mu = 0 \iff k^\mu \tilde{J}_\mu(k) = 0$, 所以,

$$W(J) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{J}^\mu(-k) \frac{1}{-k^2 - m^2 + i\epsilon} \tilde{J}_\mu(k) \quad (2.1.7)$$

观察电荷分量, 可见同性相斥, 异性相吸.

2.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的 x, y, z 方向, 在粒子静止系下, 极化矢量 $(\epsilon^i)_\mu = \delta_\mu^i, i = 1, 2, 3$, 而 $k_\mu = (-m, 0, 0, 0)$, 所以,

$$k^\mu (\epsilon^i)_\mu = 0 \quad (2.1.8)$$

– 注意, 一个粒子的极化方向用 e^i (这不是矢量) 表示, 极化矢量为 $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$.

- 在粒子静止系下, 考虑,

$$\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu} \quad (2.1.9)$$

可见,

$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon} \quad (2.1.10)$$

2.1.2 Maxwell Lagrangian

- 根据 (2.1.1) 中的 Lagrangian density, 得到 field equation 如下,

$$\left(-\partial^\mu \partial^\nu + \eta^{\mu\nu} (\partial^2 - m^2) \right) A_\nu \quad (2.1.11)$$

– spin-1 particle 有 3 个自旋自由度, 而 A_μ 有 4 个分量, 所以需要有一个约束方程,

$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0 \quad (2.1.12)$$

实际上在 (2.1.11) 左右两边作用一个 ∂_μ 即可得到这个约束方程.

2.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = **linearized** Einstein Lagrangian.
- 受 subsection 2.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足,

$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases} \quad (2.2.1)$$

其中下指标 μ, ν 对称, $a = 1, \dots, 5$, (可以验证 $(\epsilon^a)_{\mu\nu}$ 确实有 5 个独立分量).

- 对 $(\epsilon^a)_{\mu\nu}$ 的归一化条件可以定义为 $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$.
- 与 subsection 2.1.1 中提示一样, 粒子的极化方向用 e^a 表示.

- 那么,

$$\sum_{a=1}^5 (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma} \quad (2.2.2)$$

calculation:

首先用 k_μ 和 $\eta_{\mu\nu}$ 构造最一般的关于 $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$ 对称的 4 阶张量, (下式中把 $\frac{k_\mu}{m}$ 略写作 k_μ),

$$\begin{aligned} & A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ & + D \eta_{\mu\nu} \eta_{\rho\sigma} + E(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) \end{aligned} \quad (2.2.3)$$

代入 (2.2.1) 得,

$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4} \quad (2.2.4)$$

因此, 这个 4 阶张量最终确定为,

$$\frac{3}{4}A\left((G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\right) \quad (2.2.5)$$

- 所以,

$$\tilde{D}_{\mu\nu\rho\sigma}(k) = \frac{(G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}) - \frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \quad (2.2.6)$$

- 计算路径积分中的 $W(T)$,

$$W(T) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{\mu\nu}(-k) \frac{(G^{\mu\rho}G^{\nu\sigma} + G^{\mu\sigma}G^{\nu\rho}) - \frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tilde{T}_{\rho\sigma}(k) \quad (2.2.7)$$

注意到 $\partial_\mu T^{\mu\nu}(x) = 0 \iff k_\mu \tilde{T}^{\mu\nu}(k) = 0$, 并考虑到 T 是对称张量, 所以,

$$W(T) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{\mu\nu}(-k) \frac{2\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2 - m^2 + i\epsilon} \tilde{T}_{\rho\sigma}(k) \quad (2.2.8)$$

考虑能量项, 可见质量互相吸引.

2.3 remarks

- 由于 seesaw mechanism (见 subsection C.1.1), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为 $\eta^{00} = -1$.
- 本 chapter 中的计算都是 $m \neq 0$ 的粒子, 与真实世界有差异.

Chapter 3

Feynman diagrams

3.1 a baby problem

- 考虑如下积分,

$$Z(J) = \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4 + Jq} \quad (3.1.1)$$

- Schwinger's way:** 把 integrand 对 λ 展开, 并将 q 用 $\frac{\partial}{\partial J}$ 替代, 得到,

$$\begin{aligned} Z(J) &= e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4} \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 + Jq} \\ &= \sqrt{\frac{2\pi}{m^2}} e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4} e^{\frac{J^2}{2m^2}} \end{aligned} \quad (3.1.2)$$

后面的计算中忽略 $Z(J=0, \lambda=0)$.

- 每个 vertex 带有 $-\lambda$, 每个 line 带有 $\frac{1}{m^2}$, 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里 q 无法区分, 所以不方便算, 先略了),

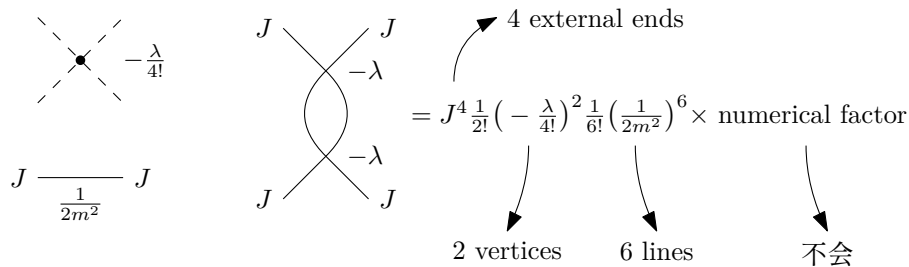


Figure 3.1: baby problem - Feynman diagram

calculation:

在这里计算 λJ^4 项,



具体暂时不会算

(3.1.3)

但是直接计算 (3.1.2) 的展开项, 得到的结果与 (3.1.5) 一样.

3.1.1 Wick contraction and Green's functions

- 把积分 (3.1.1) 对 J 展开,

$$Z(J) = \sum_{n=0}^{\infty} \frac{1}{n!} J^n \underbrace{\int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4} q^n}_{=Z(0,0)G^{(n)}} \quad (3.1.4)$$

其中 Green's function $G^{(n)}$ 对 λ 展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

calculation:

计算 λJ^4 项, 它来自 $G^{(4)}$ 对 λ 展开的一阶项,

$$\begin{aligned}
-\frac{\lambda}{4!} \int dq e^{-\frac{1}{2}m^2 q^2} q^8 &= -\frac{\lambda}{4!} \langle q^8 \rangle \\
&= -\frac{\lambda}{4!} \sum_{\text{Wick}} \left(\frac{1}{m^2} \right)^4 \\
&= -\frac{\lambda}{4!} \frac{7 \times 5 \times 3 \times 1}{m^8}
\end{aligned} \tag{3.1.5}$$

所以 λJ^4 项等于 $\frac{105}{(4!)^2} \frac{-\lambda J^4}{m^8}$.

3.1.2 connected vs. disconnected

- 考虑,

$$Z(J, \lambda) = Z(J=0, \lambda) e^{W(J, \lambda)} \tag{3.1.6}$$

其中 $Z(J=0, \lambda)$ 由 diagrams with no external source J 组成, 而 $W(J, \lambda)$ 则由 connected diagrams 组成(?).

- 我们希望计算的是 W , 而不是 Z (?).

3.2 a child problem

- 考虑如下积分,

$$Z(J) = \int dq_1 \cdots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q - \frac{\lambda}{4!} q^4 + J^T \cdot q} \tag{3.2.1}$$

其中 $q^4 = \sum_i q_i^4$.

- Schwinger's way:** 对 λ 展开并把 q 替换为 $\frac{\partial}{\partial J}$, 得到,

$$Z(J) = \sqrt{\frac{(2\pi)^N}{\det A}} e^{-\frac{\lambda}{4!} \left(\frac{\partial}{\partial J}\right)^4} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \tag{3.2.2}$$

其中 $\left(\frac{\partial}{\partial J}\right)^4 = \sum_i \left(\frac{\partial}{\partial J_i}\right)^4$.

3.2.1 n -point Green's function

- Wick's way:** 对 J 展开获得带 Green's function 的表达式,

$$\begin{aligned}
Z(J) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N J_{i_1} \cdots J_{i_n} \underbrace{\int dq_1 \cdots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q - \frac{\lambda}{4!} q^4} q_{i_1} \cdots q_{i_n}}_{=Z(0,0)G_{i_1 \cdots i_n}^{(n)}} \\
&= Z(0,0) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N J_{i_1} \cdots J_{i_n} G_{i_1 \cdots i_n}^{(n)}
\end{aligned} \tag{3.2.3}$$

其中 $G_{i_1 \cdots i_n}^{(n)}$ 称为 n -point Green's function.

Taylor expansion:

多元函数的 Taylor 展开如下,

$$\begin{aligned}
f(x_1, \cdots, x_N) &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_N^{n_N}}{n_N!} \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_N}}{\partial x_N^{n_N}} f(x=0) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N x_{i_1} \cdots x_{i_n} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}} f(x=0)
\end{aligned} \tag{3.2.4}$$

这两种求和方法, $x_1^{n_1} \cdots x_N^{n_N}$ 项的 numerical factor 都等于,

$$\frac{1}{n!} \times \frac{n!}{n_1! \cdots n_N!} = \frac{1}{n_1! \cdots n_N!} \tag{3.2.5}$$

其中 $n = n_1 + \dots + n_N$.

- 在 $\lambda = 0$ 时, 2-point Green's function 就是 propagator,

$$\begin{aligned} G_{ij}^{(2)}(\lambda = 0) &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} q_i q_j \\ &= \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J} \Big|_{J=0} = A_{ij}^{-1} \end{aligned} \quad (3.2.6)$$

- 来计算 2, 3, 4-point Green's functions,

$$\begin{cases} G_{ij}^{(2)} = A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \\ G_{ijk}^{(3)} = 0 \\ G_{ijkl}^{(4)} = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \\ \quad - \frac{\lambda}{4!} \sum_m (A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} A_{kl}^{-1} + \dots + 4! A_{im}^{-1} A_{jm}^{-1} A_{km}^{-1} A_{lm}^{-1}) + O(\lambda^2) \end{cases} \quad (3.2.7)$$

calculation:

2-point Green's function 计算如下,

$$\begin{aligned} G_{ij}^{(2)} &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j \\ &= A_{ij}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j \rangle + O(\lambda^2) \\ &= A_{ij}^{-1} - \frac{\lambda}{4!} \sum_m (3A_{mm}^{-1} A_{mm}^{-1} A_{ij}^{-1} + 12A_{mm}^{-1} A_{mi}^{-1} A_{mj}^{-1}) + O(\lambda^2) \end{aligned} \quad (3.2.8)$$

3-point Green's function 计算如下,

$$G_{ijk}^{(3)} = \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j q_k = 0 \quad (3.2.9)$$

4-point Green's function 计算如下,

$$\begin{aligned} G_{ijkl}^{(4)} &= \frac{1}{Z(0,0)} \int dq_1 \dots dq_N e^{-\frac{1}{2} q^T \cdot A \cdot q} \left(1 - \frac{\lambda}{4!} q^4 + O(\lambda^2)\right) q_i q_j q_k q_l \\ &= A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} - \frac{\lambda}{4!} \langle q^4 q_i q_j q_k q_l \rangle + O(\lambda^2) \end{aligned} \quad (3.2.10)$$

3.3 perturbative field theory

- 做如下替换即可,

$$\begin{cases} A \mapsto -i(\partial^2 - m^2) \\ J \mapsto iJ \end{cases} \quad (3.3.1)$$

- Schwinger's way:** ϕ^4 theory 的路径积分,

$$Z(J) = \int D\phi e^{i \int d^d x \left(\frac{1}{2} \phi (\partial^2 - m^2) \phi - \frac{\lambda}{4!} \phi^4 + J(x) \phi(x) \right)} \quad (3.3.2)$$

$$= Z(0,0) e^{-i \frac{\lambda}{4!} \int d^d z \left(\frac{\delta}{i \delta J(z)} \right)^4} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \quad (3.3.3)$$

其中 $D(x-y)$ 是自由场的 propagator, 见 (1.2.1).

- **Wick's way:** 同样, 对 J 展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^d x_1 \cdots d^d x_n J(x_1) \cdots J(x_n) G^{(n)}(x_1, \cdots, x_n) \quad (3.3.4)$$

其中,

$$G^{(n)}(x_1, \cdots, x_n) = \frac{1}{Z(0,0)} \int D\phi e^{i \int d^d x (\frac{1}{2} \phi (\partial^2 - m^2) \phi - \frac{\lambda}{4!} \phi^4)} \phi(x_1) \cdots \phi(x_n) \quad (3.3.5)$$

有时 $Z(J)$ 被称为 generating functional, 因为它能生成 Green's functions.

3.3.1 collision between particles

- 通过 Wick's way, 考虑 $J(x_1)J(x_2)J(x_3)J(x_4)$ 项, 实际上就是要计算 $G^{(4)}(x_1, x_2, x_3, x_4)$, 它的 0 阶项为,

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4, \lambda = 0) &= \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta J(x_2)} \frac{\delta}{i\delta J(x_3)} \frac{\delta}{i\delta J(x_4)} e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \\ &= -(D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}) \end{aligned} \quad (3.3.6)$$

其中 D_{ij} 是 $D(x_i - x_j)$ 的简写, 可见, 传播子实际上是 $(-i)^3 D = iD$.

- $G_{1234}^{(4)}$ 的 1 阶项为,

$$\begin{aligned} \text{1st order term} &= -\frac{i\lambda}{4!} \int d^d z \langle \phi_1 \cdots \phi_4 \phi^4(z) \rangle \\ &= -\frac{i\lambda}{4!} \int d^d z \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_4} \left(\frac{\delta}{i\delta J(z)} \right)^4 e^{-\frac{i}{2} \int d^d x d^d y J(x) D(x-y) J(y)} \\ &= -\frac{i\lambda}{4!} \int d^d z \left(4! D_{1z} D_{2z} D_{3z} D_{4z} \right. \\ &\quad \left. + 4 \times 3 D_{12} D_{3z} D_{4z} D_{zz} + \cdots + 3 D_{12} D_{34} D_{zz} D_{zz} + \cdots \right) \end{aligned} \quad (3.3.7)$$

其中各项分别对应如下 Feynman diagrams,

$$-i\lambda \int d^d z D_{1z} D_{2z} D_{3z} D_{4z} \quad \frac{-i\lambda}{2!} \int d^d z D_{13} D_{2z} D_{4z} D_{zz} \quad \frac{-i\lambda}{8} \int d^d z D_{13} D_{24} D_{zz} D_{zz}$$

Figure 3.2: position space - Feynman diagrams

其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,

$$= (4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!} \right)^2 \int d^d z_1 d^d z_2 D_{1z_1} D_{2z_1} D_{3z_2} D_{4z_2} D_{z_1 z_2} D_{z_1 z_2} \quad (3.3.8)$$

3.3.2 in momentum space

- 本 subsection 将 (3.3.5) 转换到 momentum space, 注意到 $\tilde{J}(k)$ 和 $\tilde{J}(-k)$ 并不独立, 所以 $\frac{\partial}{\partial i\tilde{J}}$ 不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\tilde{G}^{(n)}(k_1, \cdots, k_n) = \int d^d x_1 \cdots d^d x_n e^{-i(k_1 \cdot x_1 + \cdots)} G^{(n)}(x_1, \cdots, x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \cdots d^d x_n e^{-i(k_1 \cdot x_1 + \cdots)} \langle \left(-\frac{i\lambda}{4!} \int d^d z \phi_z^4 \right)^n \phi_1 \cdots \phi_n \rangle \quad (3.3.9)$$

– propagator 的 Fourier transformation 是,

$$\tilde{D}_{pq} = \int d^d x d^d y e^{-i(p \cdot x + q \cdot y)} D(x - y) = \frac{(2\pi)^d \delta^{(d)}(p + q)}{-p^2 - m^2 + i\epsilon} \quad (3.3.10)$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ 的 1 阶项为,

$$\text{1st order term} = -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \langle \phi_z^4 \phi_1 \cdots \phi_4 \rangle \quad (3.3.11)$$

考虑第 1 项,

$$\begin{aligned} & -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z 4! D_{1z} \cdots D_{4z} \\ &= -i\lambda \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - z) + \cdots)} \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\ &= -i\lambda \underbrace{\int d^d z e^{-iz \cdot (k_1 + \cdots + k_4)}}_{=(2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4)} \prod_{i=1}^4 \frac{1}{-k_i^2 - m^2 + i\epsilon} \end{aligned} \quad (3.3.12)$$

– 出射粒子不一定 on-shell (?).

- 得到这些 Feynman diagrams,

$$\begin{aligned} & (2\pi)^d \delta^{(d)}(k_1 + k_2) \frac{i}{-k_1^2 - m^2 + i\epsilon} & -i\lambda (2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4) \prod_{i=1}^4 \frac{i}{-k_i^2 - m^2 + i\epsilon} \\ & -\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=2,4} \frac{i}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} & -\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \prod_{i=1,2} \int \frac{d^d p_i}{(2\pi)^d} \frac{i}{-p_i^2 - m^2 + i\epsilon} \\ & (2\pi)^d \delta^{(d)}(k_1 + k_3) \frac{i}{-k_1^2 - m^2 + i\epsilon} & (2\pi)^d \delta^{(d)}(k_1 + k_3) (2\pi)^d \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{i}{-k_i^2 - m^2 + i\epsilon} \end{aligned}$$

Figure 3.3: momentum space - Feynman diagrams

calculation:

第 3 幅图的计算如下,

$$-\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z D_{13} D_{2z} D_{4z} D_{zz}$$

$$\begin{aligned}
&= -\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - z) + p_4 \cdot (x_4 - z) + p_4 \cdot 0)} \\
&\quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{2!} \int d^d z e^{-iz \cdot (p_2 + p_4)} \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_4 - k_4) \\
&\quad \prod_{i=1}^4 \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{2!} (2\pi)^d \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2,4} \frac{1}{-k_i^2 - m^2 + i\epsilon} \int \frac{d^d p}{-p^2 - m^2 + i\epsilon} \quad (3.3.13)
\end{aligned}$$

第 4 幅图的计算如下,

$$\begin{aligned}
&-\frac{i\lambda}{8} \int d^d x_1 \cdots d^d x_4 e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z D_{13} D_{24} D_{zz} D_{zz} \\
&= -\frac{i\lambda}{8} \int d^d x_1 \cdots d^d x_4 d^d z e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - x_3) + p_2 \cdot (x_2 - x_4) + p_3 \cdot 0 + p_4 \cdot 0)} \\
&\quad \prod_{i=1}^4 \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{8} \int d^d z \delta^{(d)}(p_1 - k_1) \delta^{(d)}(p_2 - k_2) \delta^{(d)}(p_1 + k_3) \delta^{(d)}(p_2 + k_4) \\
&\quad \prod_{i=1}^4 \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \\
&= -\frac{i\lambda}{8} (2\pi)^d \delta^{(d)}(0) \delta^{(d)}(k_1 + k_3) \delta^{(d)}(k_2 + k_4) \prod_{i=1,2} \frac{1}{-k_i^2 - m^2 + i\epsilon} \\
&\quad \prod_{i=1,2} \int d^d p_i \frac{1}{-p_i^2 - m^2 + i\epsilon} \quad (3.3.14)
\end{aligned}$$

- 再举一个例子 (略去了 $\prod_{i=1}^6 \frac{i}{-k_i^2 - m^2 + i\epsilon}$),



$$\begin{aligned}
&= (4!)^2 \times \left(-\frac{i\lambda}{4!}\right)^2 (2\pi)^{2d} \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \delta^{(d)}(k_1 + k_2 + k_3 + p) \delta^{(d)}(k_4 + k_5 + k_6 - p) \\
&= (-i\lambda)^2 (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6) \frac{i}{-(k_1 + k_2 + k_3)^2 - m^2 + i\epsilon} \quad (3.3.15)
\end{aligned}$$

3.3.3 loops and a first look at divergence

- subsection 3.3.2 里的 loop diagrams 出现了如下积分,

$$\int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} = \int \frac{d^D p}{(2\pi)^D 2\omega_p} \sim \int \frac{d^D p}{|p|} \quad (3.3.16)$$

积分发散.

- 再举一个例子 (略去了 $\prod_{i=1}^4 \frac{i}{-k_i^2 - m^2 + i\epsilon}$),



$$\begin{aligned}
&= (4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!} \right)^2 \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{i}{-q^2 - m^2 + i\epsilon} \\
&\quad (2\pi)^d \delta^{(d)}(k_1 + k_2 + p - q) (2\pi)^d \delta^{(d)}(k_3 + k_4 - p + q) \tag{3.3.17}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(-i\lambda)^2}{2} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int \frac{d^d p}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \frac{i}{-(k_1 + k_2 + p)^2 - m^2 + i\epsilon} \\
&= \frac{(-i\lambda)^2}{2} (2\pi)^d \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int \frac{d^D p}{(2\pi)^D} \left(\frac{1}{2\omega_p} \frac{i}{(k_1^0 + k_2^0 - \omega_p)^2 - \omega_{k_1+k_2+p}^2} \right. \\
&\quad \left. + \frac{i}{(\omega_{k_1+k_2+p} - k_1^0 - k_2^0)^2 - \omega_p^2} \frac{1}{2\omega_{k_1+k_2+p}} \right) \tag{3.3.18}
\end{aligned}$$

$$\sim \int \frac{d^D p}{p^3} \tag{3.3.19}$$

同样, 积分发散.

Chapter 4

canonical quantization

- A. Zee: the canonical and the path integral formalisms often appear complementary, in the sense that results difficult to see in one are clear in the other.
- **nobody is perfect:**
 - **canonical quantization:** 如何定义场算符乘积的顺序.
 - **path integral:** integration measure.

4.1 Heisenberg and Dirac

4.1.1 quantum mechanics

- 单粒子的 classical Lagrangian 为,

$$L = \frac{1}{2}\dot{q}^2 - V(q) \implies \begin{cases} p = \dot{q} \\ H = p\dot{q} - L = \frac{1}{2}p^2 + V(q) \end{cases} \quad (4.1.1)$$

- canonical commutation relation 如下,

$$[p, q] = -i \quad (4.1.2)$$

因此, 算符的演化方程为,

$$\begin{cases} \frac{dp}{dt} = i[H, p] = -V'(q) \\ \frac{dq}{dt} = i[H, q] = p \end{cases} \quad (4.1.3)$$

calculation:

$$\begin{cases} [p, q] = -i \\ [p, q^2] = -2iq \\ \vdots \\ [p, q^n] = -iq^{n-1} + q[p, q^{n-1}] \end{cases} \implies [p, q^n] = -inq^{n-1} \implies [p, V(q)] = -iV'(q) \quad (4.1.4)$$

- follow Dirac's approach,

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \iff \begin{cases} q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \\ p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \end{cases} \implies [a, a^\dagger] = 1 \quad (4.1.5)$$

算符 a 的演化方程为,

$$\frac{da}{dt} = -i\sqrt{\frac{\omega}{2}}\left(\frac{1}{\omega}V'(q) + ip\right) \quad (4.1.6)$$

4.1.2 scalar field

- 标量场的 Lagrangian 为,

$$L = \int d^D x \left(-\frac{1}{2}((\partial\phi)^2 + m^2\phi^2) - u(\phi) \right) \quad (4.1.7)$$

canonical commutation relation 为,

$$\pi(\vec{x}, t) = \frac{\delta L(t)}{\delta \partial_0 \phi(\vec{x}, t)} = \partial_0 \phi(\vec{x}, t) \quad \text{and} \quad [\pi(\vec{x}, t), \phi(\vec{y}, t)] = -i\delta^{(D)}(\vec{x} - \vec{y}) \quad (4.1.8)$$

标量场的 Hamiltonian 为,

$$H = \int d^D x (\pi\phi - \mathcal{L}) = \int d^D x \left(\frac{1}{2}(\pi^2 + |\vec{\nabla}\phi|^2 + m^2\phi^2) + u(\phi) \right) \quad (4.1.9)$$

- 算符的演化方程为,

$$\begin{cases} \partial_0 \phi = i[H, \phi] = \pi \\ \partial_0 \pi = i[H, \pi] = (-\vec{\nabla}^2 + m^2)\phi + \frac{du}{d\phi} \end{cases} \implies (\partial^2 - m^2)\phi - \frac{du}{d\phi} = 0 \quad (4.1.10)$$

- 当 $u(\phi) = 0$ 时, 求解场方程 (4.1.10) 和 canonical commutation relation (4.1.8) 得到,

$$\phi(\vec{x}, t) = \int \frac{d^D k}{(2\pi)^D 2\omega_k} (\alpha_k(t) e^{i\vec{k}\cdot\vec{x}} + \alpha_k^\dagger(t) e^{-i\vec{k}\cdot\vec{x}}) \quad (4.1.11)$$

其中,

$$\alpha_k(t) = \sqrt{(2\pi)^D 2\omega_k} a_{\vec{k}} e^{-i\omega_k t} \quad \text{and} \quad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = \delta^{(D)}(\vec{p} - \vec{q}) \quad (4.1.12)$$

另外, 在后面的笔记中使用简记 $\sqrt{(2\pi)^D 2\omega_k} = \rho(k)$.

calculation:

求解场方程 (4.1.10), 得到,

$$\phi(\vec{x}, t) = \int \frac{d^D k}{(2\pi)^D} (\alpha_{\vec{k}} e^{i(-\omega_k t + \vec{k}\cdot\vec{x})} + \alpha_{\vec{k}}^\dagger e^{-i(-\omega_k t + \vec{k}\cdot\vec{x})}) \quad (4.1.13)$$

代入 canonical commutation relation (4.1.8), 有 (其中 $x^0 = y^0 = t, k^0 = \omega_k$),

$$\begin{aligned} & \int \frac{d^D k_2}{(2\pi)^D} \left(-i\omega_{k_1} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] e^{i(k_1 \cdot x + k_2 \cdot y)} + i\omega_{k_1} [\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}^\dagger] e^{-i(k_1 \cdot x + k_2 \cdot y)} \right. \\ & \quad \left. - i\omega_{k_1} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] e^{i(k_1 \cdot x - k_2 \cdot y)} + i\omega_{k_1} [\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}] e^{-i(k_1 \cdot x - k_2 \cdot y)} \right) = -ie^{i\vec{k}_1 \cdot (\vec{x} - \vec{y})} \\ \implies & \begin{cases} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] = \frac{1}{2\omega_{k_1}} \delta^{(D)}(\vec{k}_1 + \vec{k}_2) \implies [\alpha_{\vec{k}}, \alpha_{\vec{k}}] \neq 0 & \text{wrong} \\ [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] = \frac{1}{2\omega_{k_1}} \delta^{(D)}(\vec{k}_1 - \vec{k}_2) & \text{right} \end{cases} \end{aligned} \quad (4.1.14)$$

- 代入 (4.1.9) 可得 (依然是 $u(\phi) = 0$ 的情况下),

$$H = \int d^D k \omega_k \frac{a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^\dagger}{2} = \int d^D k \omega_k \left(a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2} \delta^{(D)}(0) \right) \implies \langle 0|H|0 \rangle = V \int \frac{d^D k}{(2\pi)^D} \frac{1}{2} \omega_k \quad (4.1.15)$$

其中, $V = \int d^D x = (2\pi)^D \delta^{(D)}(0)$.

- vacuum state 定义为 $a_{\vec{k}}|0\rangle = 0$, 有,

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{ik \cdot (x-y)} \quad (4.1.16)$$

其中 $k^0 = \omega_k$. 因此, 对比 (1.2.1), 有,

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = iD(x-y) \quad (4.1.17)$$

4.2 interaction picture

- 注意, 在 $u(\phi) \neq 0$ 的情况下, (即便在 Schrödinger's picture 里, $t = 0$ 时) (4.1.11) 不再成立, 因此无法通过 Schrödinger's picture or Heisenberg's picture 求解存在相互作用的场论.
- 将 Hamiltonian 分成两个部分,

$$H = H_0 + H' \quad (4.2.1)$$

- operators 以自由场的 Hamiltonian 演化,

$$O_I(t) = U_0^\dagger(t, 0)O(0)U_0(t, 0) \quad \text{where} \quad U_0(t_2, t_1) = \text{Texp}\left(-i \int_{t_1}^{t_2} dt H_0\right) \quad (4.2.2)$$

states 以如下方式演化,

$$|\psi(t)\rangle_I = U_0^\dagger(t, 0)U(t, 0)|\psi(0)\rangle \quad \text{where} \quad U(t_2, t_1) = \text{Texp}\left(-i \int_{t_1}^{t_2} dt H\right) \quad (4.2.3)$$

因此,

$$|\psi(t_2)\rangle_I = U_I(t_2, t_1)|\psi(t_1)\rangle_I \quad \text{where} \quad U_I(t_2, t_1) = \text{Texp}\left(-i \int_{t_1}^{t_2} dt H_I(t)\right) \quad (4.2.4)$$

注意, (4.2.2) 和 (4.2.3) 中, Texp 里的 H, H_0 都是 Schrödinger's picture 里的算符.

calculation:

首先有,

$$U_I(t_2, t_1) = U_0^\dagger(t_2, 0)U(t_2, t_1)U_0(t_1, 0) \quad (4.2.5)$$

因此,

$$\begin{aligned} \frac{d}{dt}U_I(t, t_0) &= iH_0U_I(t, t_0) - iU_0^\dagger(t, 0)HU(t, t_0)U_0(t_0, 0) \\ &= -i \underbrace{U_0^\dagger(t, 0)H'U_0(t, 0)}_{=H_I(t)}U_I(t, t_0) \end{aligned} \quad (4.2.6)$$

4.3 scattering amplitude

- 最一般的过程是 $p_1, \dots, p_m \rightarrow q_1, \dots, q_n$, 其 scattering amplitude 为,

$$\langle q_1, \dots, q_n | U_0^\dagger(-\infty, 0)U_I(+\infty, -\infty)U_0(-\infty, 0) | p_1, \dots, p_m \rangle \quad (4.3.1)$$

一般会忽略掉 U_0 产生的相位.

- 考虑 ϕ^4 理论中的 $k_1, k_2 \rightarrow k_3, k_4$ 过程,

$$\langle k_3, k_4 | e^{-i \int d^d x \frac{\lambda}{4!} \phi^4} | k_1, k_2 \rangle \quad (4.3.2)$$

对 λ 展开, 0 阶项为,

$$\begin{aligned} \text{0th order term} &= \langle k_3, k_4 | k_1, k_2 \rangle \\ &= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4) \langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle \\ &= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4) \left(\underbrace{\langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle}_{=\delta_{31}^{(D)}\delta_{42}^{(D)}} + \underbrace{\langle 0 | a_{\vec{k}_3} a_{\vec{k}_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger | 0 \rangle}_{=\delta_{32}^{(D)}\delta_{41}^{(D)}} \right) \\ &= (2\pi)^{2D} 4\omega_{k_1}\omega_{k_2} (\delta^{(D)}(\vec{k}_1 - \vec{k}_3)\delta^{(D)}(\vec{k}_2 - \vec{k}_4) + \delta^{(D)}(\vec{k}_1 - \vec{k}_4)\delta^{(D)}(\vec{k}_2 - \vec{k}_3)) \end{aligned} \quad (4.3.3)$$

1 阶项为 (其中 $k^0 = \omega_k$),

$$\text{1st order term} = \frac{-i\lambda}{4!} \int d^d x \langle k_3, k_4 | \phi^4(x) | k_1, k_2 \rangle$$

$$\begin{aligned}
& \overbrace{= -i\lambda(2\pi)^d \delta^{(d)}(k_1 + k_2 - k_3 - k_4)} \\
& = 4! \times \frac{-i\lambda}{4!} \int d^d x e^{i(k_1 + k_2 - k_3 - k_4) \cdot x} + \rho(k_1) \rho(k_4) \delta_{14}^{(D)} \times 12 \times \frac{-i\lambda}{4!} (2\pi)^d \delta_{23}^{(d)} \int \frac{d^D p}{\rho^2(p)} \\
& \quad + \cdots + \rho(k_1) \rho(k_2) \rho(k_3) \rho(k_4) \delta_{13}^{(D)} \delta_{24}^{(D)} \times 3 \times \frac{-i\lambda}{4!} \int d^d x \int \frac{d^D p_1}{\rho^2(p_1)} \frac{d^D p_2}{\rho^2(p_2)} + \cdots \quad (4.3.4)
\end{aligned}$$

分别对应如下 Feynman diagrams,

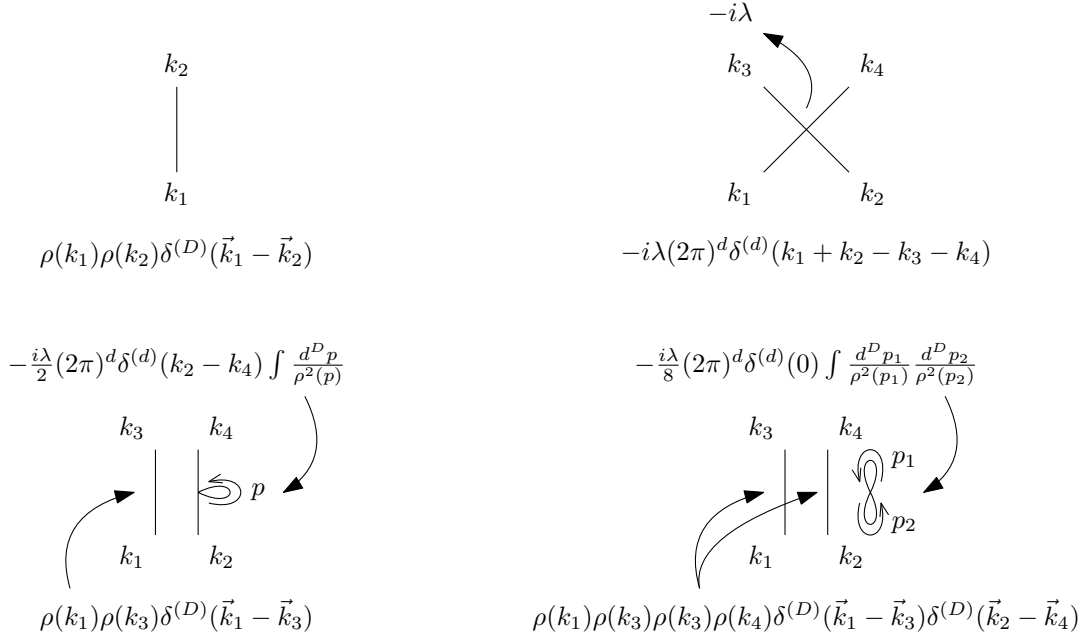


Figure 4.1: canonical quantization - Feynman diagrams

观察可见, 上图和 figure 3.3 有对应关系.

- 再举一个例子,

$$\begin{aligned}
& \text{Diagram: A crossing with external momenta } k_1, k_2, k_3, k_4. \text{ A loop with momentum } p \text{ is attached to the crossing. The loop is labeled } k_1 + k_2 + p. \\
& = (4 \times 3)^2 \times 2 \times \left(\frac{-i\lambda}{4!} \right)^2 \rho(k_1) \cdots \int d^d x_1 d^d x_2 \int \frac{d^D p_1 \cdots d^D q_1 \cdots}{\rho(p_1) \cdots \rho(q_1) \cdots} e^{i(p_1 + p_2 - p_3 - p_4) \cdot x_1} e^{i(q_1 + q_2 - q_3 - q_4) \cdot x_2} \\
& \quad \left(\theta(t_2 - t_1) \langle 0 | a_{\vec{k}_3}^- a_{\vec{k}_4}^- a_{\vec{q}_1}^- a_{\vec{q}_2}^- a_{\vec{q}_3}^\dagger a_{\vec{q}_4}^\dagger a_{\vec{p}_1}^- a_{\vec{p}_2}^- a_{\vec{p}_3}^\dagger a_{\vec{p}_4}^\dagger a_{\vec{k}_1}^- a_{\vec{k}_2}^- | 0 \rangle + \cdots \right) \\
& = \frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 \int \frac{d^D p_3}{\rho^2(p_3)} \frac{d^D p_4}{\rho^2(p_4)} \left(\theta(t_2 - t_1) e^{i(k_1 + k_2 - p_3 - p_4) \cdot x_1} e^{i(p_3 + p_4 - k_3 - k_4) \cdot x_2} \right. \\
& \quad \left. + \theta(t_1 - t_2) e^{i(k_1 + k_2 + p_3 + p_4) \cdot x_1} e^{i(-p_3 - p_4 - k_3 - k_4) \cdot x_2} \right) \\
& = \frac{(-i\lambda)^2}{2} \int d^d x_1 d^d x_2 e^{i((k_1 + k_2) \cdot x_1 - (k_3 + k_4) \cdot x_2)} \int \frac{d^D p_3}{\rho^2(p_3)} \frac{d^D p_4}{\rho^2(p_4)} \left(\theta(t_2 - t_1) e^{i(p_3 + p_4) \cdot (x_2 - x_1)} \right. \\
& \quad \left. + \theta(t_1 - t_2) e^{i(p_3 + p_4) \cdot (x_1 - x_2)} \right) \quad (4.3.5)
\end{aligned}$$

同样, 与 (3.3.18) 有对应关系, (注意按时间排序 $\langle k_3 k_4 | T(\phi^4(x_1) \phi^4(x_2)) | k_1 k_2 \rangle$).

calculation:

从 (3.3.17) 开始 (与 (1.2.1) 类似, \vec{p}, \vec{q} 的符号可以任意改变),

$$\begin{aligned}
& \int d^d x_1 d^d x_2 e^{i((k_1+k_2+p-q)\cdot x_1 + (k_3+k_4-p+q)\cdot x_2)} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{i}{-p^2 - m^2 + i\epsilon} \frac{i}{-q^2 - m^2 + i\epsilon} \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{ie^{ip\cdot(x_1-x_2)}}{-p^2 - m^2 + i\epsilon} \frac{ie^{iq\cdot(x_2-x_1)}}{-q^2 - m^2 + i\epsilon} \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^D p}{(2\pi)^d} \frac{d^D q}{(2\pi)^d} \left(\theta(t_2 - t_1) \frac{2\pi i^2 e^{-ip\cdot(x_1-x_2)}}{-2\omega_p} \right. \\
&\quad \left. \frac{-2\pi i^2 e^{iq\cdot(x_2-x_1)}}{2\omega_q} + \theta(t_1 - t_2) \frac{-2\pi i^2 e^{ip\cdot(x_1-x_2)}}{2\omega_p} \frac{2\pi i^2 e^{-iq\cdot(x_2-x_1)}}{-2\omega_q} \right) \\
&= \int d^d x_1 d^d x_2 e^{i((k_1+k_2)\cdot x_1 + (k_3+k_4)\cdot x_2)} \int \frac{d^D p}{\rho^2(p)} \frac{d^D q}{\rho^2(q)} \left(\theta(t_2 - t_1) e^{i(p+q)\cdot(x_2-x_1)} \right. \\
&\quad \left. + \theta(t_1 - t_2) e^{i(p+q)\cdot(x_1-x_2)} \right) \tag{4.3.6}
\end{aligned}$$

结果与 (4.3.5) 对应.

4.4 complex scalar field

- complex scalar field 的 Lagrangian 为,

$$\mathcal{L} = -(\partial\psi^\dagger)(\partial\psi) - m^2\psi^\dagger\psi \tag{4.4.1}$$

实际上, complex scalar field 可以视为 2 个 real scalar fields 的和,

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \implies \left| \frac{\partial\phi_1, \phi_2}{\partial\psi, \psi^\dagger} \right| = i \tag{4.4.2}$$

因此, 也可以把 ψ, ψ^\dagger 视为两个独立的场.

- 其 canonical momentum 为,

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi} = \partial_0\psi^\dagger \quad \pi^\dagger = \partial_0\psi \tag{4.4.3}$$

其 Hamiltonian 为,

$$\mathcal{H} = \pi^\dagger\pi + (\vec{\nabla}\psi^\dagger) \cdot (\vec{\nabla}\psi) + m^2\psi^\dagger\psi \tag{4.4.4}$$

$$\implies \begin{cases} \partial_0\pi = i[H, \pi] = \vec{\nabla}^2\psi^\dagger - m^2\psi^\dagger \\ \partial_0\psi = i[H, \psi] = \pi^\dagger \end{cases} \implies (-\partial^2 - m^2)\psi = 0 \tag{4.4.5}$$

- 求解得到 (其中 $k^0 = \omega_k$),

$$\psi(x) = \int \frac{d^D k}{\rho(k)} (a_{\vec{k}} e^{ik\cdot x} + b_{\vec{k}}^\dagger e^{-ik\cdot x}) \tag{4.4.6}$$

- 从 path integral 的角度,

$$Z(J, J^\dagger) = \int D\psi D\psi^\dagger e^{i \int d^d x (\psi^\dagger(\partial^2 - m^2)\psi + J^\dagger\psi + \psi^\dagger J)} \tag{4.4.7}$$

$$= \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y 2J^\dagger(x) D(x-y) J(y)} \tag{4.4.8}$$

calculation:

转换为 ϕ_1, ϕ_2 后计算路径积分,

$$Z(J, J^\dagger) = \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y (J_1(x) D(x-y) J_1(y) + J_2(x) D(x-y) J_2(y))}$$

$$= \mathcal{C} e^{-\frac{i}{2} \int d^d x d^d y 2J^\dagger(x) D(x-y) J(y)} \quad (4.4.9)$$

4.4.1 charge

- 对场算符做如下变换,

$$\psi(x, \lambda) = e^{i\lambda} \psi(x) \implies D_\lambda \mathcal{L} = 0 \quad (4.4.10)$$

- 因此, 得到 conserved current,

$$J^\mu = \pi^\mu D_\lambda \psi + \pi^{\dagger\mu} D_\lambda \psi^\dagger = i(\psi \partial^\mu \psi^\dagger - \psi^\dagger \partial^\mu \psi) \quad (4.4.11)$$

其 0 分量对空间积分就是 charge,

$$\begin{aligned} Q &= \int d^D x J^0 = \int d^D x i(\psi^\dagger \partial_0 \psi - \psi \partial_0 \psi^\dagger) \\ &= \int d^D k (a_k^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}}) \end{aligned} \quad (4.4.12)$$

calculation:

$$\begin{aligned} Q &= \int d^D x \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} i \left((a_{\vec{p}}^\dagger e^{-ip \cdot x} + b_{\vec{p}} e^{ip \cdot x}) (-i\omega_q) (a_{\vec{q}} e^{iq \cdot x} - b_{\vec{q}}^\dagger e^{-iq \cdot x}) \right. \\ &\quad \left. - (a_{\vec{q}} e^{iq \cdot x} + b_{\vec{q}}^\dagger e^{-iq \cdot x}) (i\omega_p) (a_{\vec{p}}^\dagger e^{-ip \cdot x} - b_{\vec{p}} e^{ip \cdot x}) \right) \\ &= \int d^D x \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} \left((\omega_p a_{\vec{q}} a_{\vec{p}}^\dagger + \omega_q a_{\vec{p}}^\dagger a_{\vec{q}}) e^{-i(p-q) \cdot x} - (\omega_p b_{\vec{q}}^\dagger b_{\vec{p}} + \omega_q b_{\vec{p}} b_{\vec{q}}^\dagger) e^{i(p-q) \cdot x} \right. \\ &\quad \left. + a_{\vec{p}}^\dagger b_{\vec{q}}^\dagger (\omega_p - \omega_q) e^{-i(p+q) \cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{i(p+q) \cdot x} \right) \\ &= \int \frac{d^D p}{\rho(p)} \frac{d^D q}{\rho(q)} \\ &\quad \left(\left((\omega_p a_{\vec{q}} a_{\vec{p}}^\dagger + \omega_q a_{\vec{p}}^\dagger a_{\vec{q}}) e^{i(\omega_p - \omega_q) \cdot t} - (\omega_p b_{\vec{q}}^\dagger b_{\vec{p}} + \omega_q b_{\vec{p}} b_{\vec{q}}^\dagger) e^{-i(\omega_p - \omega_q) \cdot t} \right) (2\pi)^D \delta^{(D)}(\vec{p} - \vec{q}) \right. \\ &\quad \left. + \left(a_{\vec{p}}^\dagger b_{\vec{q}}^\dagger (\omega_p - \omega_q) e^{i(\omega_p + \omega_q) \cdot x} - a_{\vec{q}} b_{\vec{p}} (\omega_p - \omega_q) e^{-i(\omega_p + \omega_q) \cdot x} \right) (2\pi)^D \delta^{(D)}(\vec{p} + \vec{q}) \right) \\ &= \int \frac{d^D k}{2} (a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}}^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}}) = \int d^D k (a_{\vec{k}}^\dagger a_{\vec{k}} - b_{\vec{k}}^\dagger b_{\vec{k}}) \end{aligned} \quad (4.4.13)$$

- 代入 (D.3.2), 有 $i[Q, \psi] = -i\psi$, 所以,

$$e^{-i\lambda Q} \psi e^{i\lambda Q} = e^{i\lambda} \psi \quad (4.4.14)$$

Chapter 5

disturbing the vacuum: Casimir effect

- 考虑一个沿 x^1 方向满足 periodic b.c. 的空间, 在垂直于 x^1 方向有两个 plates, s.t. 在 plates 上 $\phi(x) = 0$, 如下图,



Figure 5.1: Casimir effect

- 平板内外, 标量场的波矢的取值为,

$$\begin{cases} (n\frac{\pi}{d}, k_2, k_3) & \text{平板内} \\ (n\frac{\pi}{L-d}, k_2, k_3) & \text{平板外} \end{cases} \quad (5.0.1)$$

其中 $n \in \mathbb{Z}^+$.

- 因此, 代入真空能公式 (4.1.15), 平板内的能量为,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2} \quad (5.0.2)$$

而总能量为 $E = E(d) + E(L-d)$.

- 为解决能量发散的问题, 引入 ultra-violet (UV) cut-off,

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dk_2 dk_3}{(2\pi)^2} \frac{1}{2} \sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2} e^{-a\sqrt{\left(n\frac{\pi}{d}\right)^2 + k_2^2 + k_3^2}} \quad (5.0.3)$$

for some $a \ll d$.

- 为了简化问题, 考虑 $d = 1 + 1$ 的情况,

$$E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-\frac{a\pi}{d}n} = \frac{\pi}{2d} \frac{e^{\frac{a\pi}{d}}}{(e^{\frac{a\pi}{d}} - 1)^2} = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O(a^2) \quad (5.0.4)$$

因此,

$$E_{1+1} = E_{1+1}(d) + E_{1+1}(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24} \left(\frac{1}{d} + \frac{1}{L-d} \right) + O(a^2) \quad (5.0.5)$$

得到 Casimir force,

$$F_{1+1} = -\frac{\partial E_{1+1}}{\partial d} = -\frac{\pi}{24} \left(\frac{1}{d^2} - \frac{1}{(L-d)^2} \right) + O(a^2) \stackrel{L \rightarrow \infty, a \rightarrow 0}{=} -\frac{\pi}{24d^2} \quad (5.0.6)$$

- 问题中, a 引入了 UV cut-off, L 引入了 infrared cut-off.

Part II

Dirac and spinor

Chapter 6

the Dirac equation

- 整个 Part II 中, 我们使用 $(+, -, -, -)$ 号差, 因为 $\text{Cl}_{1,3}(\mathbb{R}) \cong \text{Cl}_{3,1}(\mathbb{R})$.
- 本笔记中的算符的定义与 A. Zee 的定义不同, 存在如下对应关系,

A. Zee's def.	my def.
$\omega_{\mu\nu}$	$\omega_{\mu\nu}$
$-iJ^{\mu\nu}$	$J^{\mu\nu}$
$-i\sigma^{\mu\nu}$	$\sigma^{\mu\nu}$

- $\Pi(\Lambda)$ 的写法可能不准确, (要考虑 universal cover, $\text{Spin}(1,3) \simeq \text{Spin}(3,1)$), 因为 Lorentz transform 对 spinor 的操作是"path dependent", 因此本 chapter 中的 Λ 都默认沿着以下的 path 做变换,

$$\Lambda(\lambda) = e^{\frac{\lambda}{2}\omega_{\mu\nu}J^{\mu\nu}}, \lambda \in [0, 1] \quad (6.0.1)$$

6.1 gamma matrices

- Pauli 矩阵如下,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (6.1.1)$$

- gamma 矩阵 (also called Dirac matrices) 如下 (其中 $i = 1, 2, 3$),

$$\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} = I \otimes \tau_3 \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = i\sigma_i \otimes \tau_2 \quad \Omega = \gamma^0\gamma^1\gamma^2\gamma^3 = -i \begin{pmatrix} & I \\ I & \end{pmatrix} = -iI \otimes \tau_1 \quad (6.1.2)$$

其中 $\tau_{2,3}$ 也是 Pauli 矩阵, 最后, 按照惯例, 定义 $\gamma^5 = i\Omega = I \otimes \tau_1$.

– 另外,

$$\begin{cases} \gamma^0\gamma^i = \sigma_i \otimes \tau_1 \\ \gamma^i\gamma^j = -(\sigma_i\sigma_j) \otimes I \end{cases} \quad \begin{cases} \Omega\gamma^0 = -I \otimes \sigma_2 \\ \Omega\gamma^i = i\sigma_i \otimes \tau_3 \end{cases} \quad (6.1.3)$$

- gamma 矩阵满足,

$$\begin{cases} (\gamma^\mu)^2 = \eta^{\mu\mu} \\ \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \mu \neq \nu \end{cases} \implies \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (6.1.4)$$

- 且存在如下关系,

$$\begin{aligned} \Omega\gamma^0 &= -\gamma^1\gamma^2\gamma^3 & \Omega\gamma^1 &= -\gamma^0\gamma^2\gamma^3 & \Omega\gamma^2 &= \gamma^0\gamma^1\gamma^3 & \Omega\gamma^3 &= -\gamma^0\gamma^1\gamma^2 \\ \iff -\epsilon^{\mu\nu\rho} \sigma_\sigma \Omega\gamma^\sigma &= \gamma^\mu\gamma^\nu\gamma^\rho & \text{when } \mu \neq \nu \neq \rho \end{aligned} \quad (6.1.5)$$

并且有 (注意到 $\Omega^2 = -1$),

$$\{\Omega, \gamma^\mu\} = 0 \quad \{\Omega, \Omega\gamma^\mu\} = 0 \quad [\Omega, \gamma^\mu\gamma^\nu] = 0 \quad (6.1.6)$$

- 定义 $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ (注意, 我们的定义中没有虚数 i , 与 A. Zee 的定义不同),

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} + \sigma^{\mu\nu} \Rightarrow \begin{cases} \sigma^{0i} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \sigma_i \otimes \tau_1 \\ \sigma^{ij} = -i\epsilon^{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = -i\epsilon^{ijk} \sigma_k \otimes I \end{cases} \quad (6.1.7)$$

6.2 Lorentz transformation and the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation

- Lorentz 变换可以写成如下形式,

$$\Lambda = e^{\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}} \quad (6.2.1)$$

其中 $\omega_{\mu\nu}$ 反对称, J^{0i} generate boosts and J^{ij} generate rotations, (详见笔记 [Lie Groups and Lie Algebras](#)).

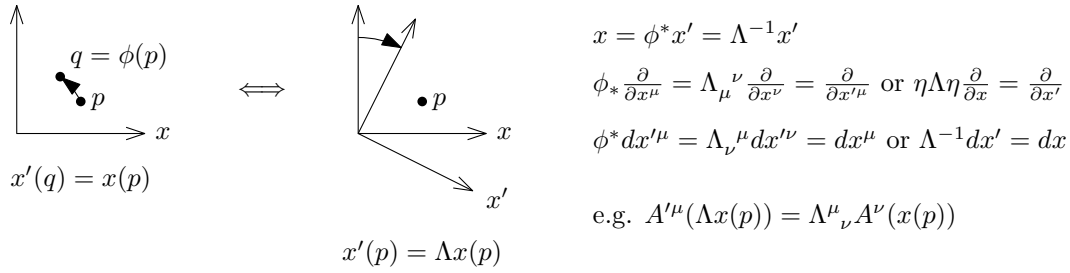


Figure 6.1: Lorentz transformation

- 有 $\pi_{(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})}(J^{\mu\nu}) = \frac{1}{2}\sigma^{\mu\nu}$ (up to a similarity transformation).

calculation:

做如下相似变换,

$$S = \frac{\sqrt{2}}{2} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \iff S^{-1} = \frac{\sqrt{2}}{2} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \quad (6.2.2)$$

得到,

$$S^{-1}\sigma^{0i}S = \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix} \quad S^{-1}\sigma^{ij}S = -i\epsilon^{ijk} \begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} \quad (6.2.3)$$

得到的结果和笔记 [Lie Groups and Lie Algebras](#) 中 $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ 表示是完全一样的.

- Dirac spinor 是 $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ rep. 的 vector space 中的元素,

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad \text{with} \quad S^{-1}\psi = \frac{\sqrt{2}}{2} \begin{pmatrix} \phi - \chi \\ \phi + \chi \end{pmatrix} \quad (6.2.4)$$

其中,

$$\frac{\sqrt{2}}{2} \begin{pmatrix} \phi \\ -\phi \end{pmatrix} \in (\frac{1}{2}, 0) \quad \text{and} \quad \frac{\sqrt{2}}{2} \begin{pmatrix} \chi \\ \chi \end{pmatrix} \in (0, \frac{1}{2}) \quad (6.2.5)$$

- 对于 gamma 矩阵, 有,

$$\Pi(\Lambda)\gamma^\rho\Pi^{-1}(\Lambda) = e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = (\Lambda^{-1})^\rho_\sigma \gamma^\sigma \quad (6.2.6)$$

calculation:

利用 Campbell's identity,

$$e^{\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{1}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} = e^{\frac{1}{4}\omega_{\mu\nu}\text{ad}_{\sigma^{\mu\nu}}}\gamma^\rho \quad (6.2.7)$$

其中 (注意 $(J^{\mu\nu})^\rho_\sigma = 2\eta^{[\mu|\rho}\delta^{|\nu]}_\sigma$, 其中度规号差与笔记 [Lie Groups and Lie Algebras](#) 中的不同),

$$\begin{cases} \rho \neq \mu, \nu & [\sigma^{\mu\nu}, \gamma^\rho] = \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu) \\ & = -\frac{1}{2}(\underbrace{\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\nu\mu\rho\sigma} - \epsilon^{\rho\mu\nu\sigma} + \epsilon^{\rho\nu\mu\sigma}}_{=0})\Omega\gamma_\sigma = 0 \\ \rho = \mu \text{ or } \nu \text{ and } \mu \neq \nu & [\sigma^{\mu\nu}, \gamma^\rho] = 2(\eta^{\mu\rho}\gamma^\mu - \eta^{\nu\rho}\gamma^\nu) \end{cases}$$

$$\Rightarrow [\sigma^{\mu\nu}, \gamma^\rho] = 2(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu) = -2(J^{\mu\nu})^\rho_\sigma\gamma^\sigma \quad (6.2.8)$$

代入, 得到,

$$e^{\frac{1}{4}\omega_{\mu\nu}\text{ad}_{\sigma^{\mu\nu}}}\gamma^\rho = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left(-\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} \right)^n \right)^\rho_\sigma \gamma^\sigma = (\Lambda^{-1})^\rho_\sigma \gamma^\sigma \quad (6.2.9)$$

可以用”无穷小” Lorentz 变换验证以上计算,

$$\begin{aligned} \Pi(1 + \delta\omega^\mu_\nu)\gamma^\rho\Pi^{-1}(1 + \delta\omega^\mu_\nu) &= \gamma^\rho + \frac{1}{4}\delta\omega_{\mu\nu}[\sigma^{\mu\nu}, \gamma^\rho] \\ &= (1 - \delta\omega^\rho_\sigma)\gamma^\sigma \end{aligned} \quad (6.2.10)$$

6.2.1 Dirac spinor

- 对于 Dirac spinor,

$$\Pi(\Lambda)\psi(x) = \psi'(\Lambda x) \quad (6.2.11)$$

注意 $\partial'_\mu = \Lambda_\mu^\nu \partial_\nu$, 所以,

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \iff (i\gamma^\mu\partial'_\mu - m)\psi'(\Lambda x) = 0 \quad (6.2.12)$$

calculation:

首先,

$$\Lambda^T\eta\Lambda = \eta \implies (\Lambda^{-1})^\mu_\nu = (\eta\Lambda^T\eta)^\mu_\nu = \Lambda_\nu^\mu \quad (6.2.13)$$

考虑,

$$\Pi^{-1}(\Lambda)\gamma^\mu\Pi(\Lambda) = \Lambda^\mu_\nu\gamma^\nu \implies \gamma^\mu\Pi(\Lambda) = \Lambda^\mu_\nu\Pi(\Lambda)\gamma^\nu \quad (6.2.14)$$

代入,

$$\begin{aligned} (i\gamma^\mu\partial'_\mu - m)\psi'(\Lambda x) &= (i\gamma^\mu\Lambda_\mu^\nu\partial_\nu - m)\Pi(\Lambda)\psi(x) \\ &= \Pi(\Lambda)(i\underbrace{\gamma^\rho\Lambda^\mu_\rho\Lambda_\mu^\nu}_{=\delta^\nu_\rho}\partial_\nu - m)\psi(x) \\ &= \Pi(\Lambda)(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \end{aligned} \quad (6.2.15)$$

– 关键部分在于,

$$\gamma^\mu\psi'(\Lambda x) = \gamma^\mu\Pi(\Lambda)\psi(x) = \Pi(\Lambda)\Lambda^\mu_\nu\gamma^\nu\psi(x) \quad (6.2.16)$$

6.2.2 Dirac bilinears

- γ^0 是 Hermitian 矩阵, 而 γ^i 不是, 有,

$$\gamma^{i\dagger} = -\gamma^i = \gamma^0\gamma^i\gamma^0 \quad (6.2.17)$$

可以统一写作 $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, 并且有,

$$\sigma^{\mu\nu\dagger} = -\gamma^0\sigma^{\mu\nu}\gamma^0 \quad \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0 \quad (6.2.18)$$

calculation:

对于 $\sigma^{\mu\nu}$,

$$\sigma^{\mu\nu\dagger} = \frac{1}{2}(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger}) = \gamma^0\sigma^{\nu\mu}\gamma^0 = -\gamma^0\sigma^{\mu\nu}\gamma^0 \quad (6.2.19)$$

所以,

$$((\omega_{\mu\nu}\sigma^{\mu\nu})^\dagger)^n = \gamma^0(-\omega_{\mu\nu}\sigma^{\mu\nu})^n\gamma^0 \implies \Pi^\dagger(\Lambda) = \gamma^0\Pi(\Lambda^{-1})\gamma^0 \quad (6.2.20)$$

- 所以,

$$\begin{cases} \bar{\psi}'(\Lambda x)\psi'(\Lambda x) = \bar{\psi}\psi & \text{scalar field} \\ \bar{\psi}'\gamma^\mu\psi' = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi & \text{vector field} \end{cases} \quad (6.2.21)$$

其中 $\bar{\psi} = \psi^\dagger\gamma^0$.

calculation:

$$\begin{cases} \psi'^\dagger(\Lambda x)\gamma^0\psi'(\Lambda x) = \psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\Pi(\Lambda)\psi(x) = \psi^\dagger\gamma^0\psi \\ \psi'^\dagger\gamma^0\gamma^\mu\psi' = \psi^\dagger(x)\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\gamma^\mu\Pi(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\psi^\dagger\gamma^0\gamma^\nu\psi \end{cases} \quad (6.2.22)$$

此外,

$$\begin{cases} \bar{\psi}'\sigma^{\mu\nu}\psi' = \psi^\dagger\gamma^0\Pi(\Lambda^{-1})(\gamma^0)^2\sigma^{\mu\nu}\Pi(\Lambda)\psi = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\psi}\sigma^{\rho\sigma}\psi & \text{order 2 tensor} \\ \bar{\psi}'\Omega\gamma^\mu\psi' = \bar{\psi}\Pi(\Lambda^{-1})\Omega\gamma^\mu\Pi(\Lambda)\psi = \det(\Lambda)\Lambda^\mu{}_\nu\bar{\psi}\Omega\gamma^\nu\psi & \text{pseudovector} \\ \bar{\psi}'\Omega\psi' = \bar{\psi}\Pi(\Lambda^{-1})\Omega\Pi(\Lambda)\psi = \det(\Lambda)\bar{\psi}\Omega\psi & \text{4-form, aka. pseudoscalar} \end{cases} \quad (6.2.23)$$

其中 (注意到下面的计算中, 第二个等号后, 含 η 的项都等于零; 由此可以看出, 对 μ_i 求和的过程中, 任何两个 μ_i, μ_j 相等的项求和之后都等于零),

$$\begin{aligned} \Pi(\Lambda^{-1})\Omega\Pi(\Lambda) &= \prod_{i=0}^3 \Lambda^i{}_{\mu_i} \gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \\ &= \prod_{i=0}^3 \Lambda^i{}_{\mu_i} (\eta^{\mu_0\mu_1} + \sigma^{\mu_0\mu_1})(\eta^{\mu_2\mu_3} + \sigma^{\mu_2\mu_3}) \\ &= \prod_{i=0}^3 \Lambda^i{}_{\mu_i} \gamma^{\mu_0}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \quad \text{with } \mu_0 \neq \mu_1 \neq \mu_2 \neq \mu_3 \\ &= \det(\Lambda)\Omega \end{aligned} \quad (6.2.24)$$

6.2.3 parity and time reversal

- 这里沿用笔记 [Lie Groups and Lie Algebras](#) 中的记号, 选择 $O(3,1)$ 而非 $O(1,3)$, 因为他们没有区别.
- $O(3,1)$ 有 4 个联通分支,

$$I \in \text{SO}_+(3,1) \quad PT \in \text{SO}_-(3,1) \quad P \in O'_+(3,1) \quad T \in O'_-(3,1) \quad (6.2.25)$$

其中,

$$P = \text{diag}(+1, -1, -1, -1) \quad T = \text{diag}(-1, +1, +1, +1) \quad (6.2.26)$$

另外, $\eta P \eta = P, \eta T \eta = T$.

- 另外, Lorentz algebra 的 representation 不能自然的生成对 P, T 的表示, 因为本质上它只能生成 spin group 的表示, 是 $\text{SO}_+(3,1)$ 的 universal cover, 与 Lorentz group 的其它三个连通分支没有直接联系.
- 因此, 对 P, T 的表示要从物理的角度定义, (可能) 无法单纯靠数学的方法给出.

parity

- 对于 parity, 有,

$$\Pi(P) = \gamma^0 \gamma^0 \gamma^\mu = P^\mu{}_\nu \gamma^\nu \gamma^0 \quad (6.2.27)$$

6.3 Dirac equation

- A. Zee: our discussion provides a unified view of the equations of motion in relativistic physics: they just project out the unphysical components.
- the Dirac equation is,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \iff (\gamma^\mu p_\mu - m)\tilde{\psi} = 0 \quad (6.3.1)$$

首先可以看出 ψ 满足 Klein-Gordon equation,

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\psi &= \left(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu - 2im\gamma^\mu \partial_\mu + m^2 \right)\psi = 0 \\ \implies (-\partial^2 - m^2)\psi &= 0 \end{aligned} \quad (6.3.2)$$

– 在粒子静止系下 $p_\mu = (m, 0, 0, 0)$, Dirac 方程给出,

$$(\gamma^0 - 1)\tilde{\psi} = 0 \implies \begin{pmatrix} 0 & \\ & I \end{pmatrix} \tilde{\psi} = 0 \quad (6.3.3)$$

因此, $\tilde{\psi}$ 的后两个分量为零 $\implies \psi$ 只有两个自由度.

Appendices

Appendix A

Dirac delta function & Fourier transformation

A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx \delta(x) e^{-ikx} \quad (\text{A.1.1})$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x))g(x)dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \quad (\text{A.1.2})$$

- 第二个常用的公式 ([Sokhotski-Plemelj theorem](#)),

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{x + i\epsilon} = \mathcal{P} \frac{1}{x} - i\pi\delta(x) \quad (\text{A.1.3})$$

其中 \mathcal{P} 表示复函数的主值 (principal value).

proof:

考虑,

$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \quad \text{and} \quad \int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \text{Res}(f, i\epsilon) = \pi \quad (\text{A.1.4})$$

所以...

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,

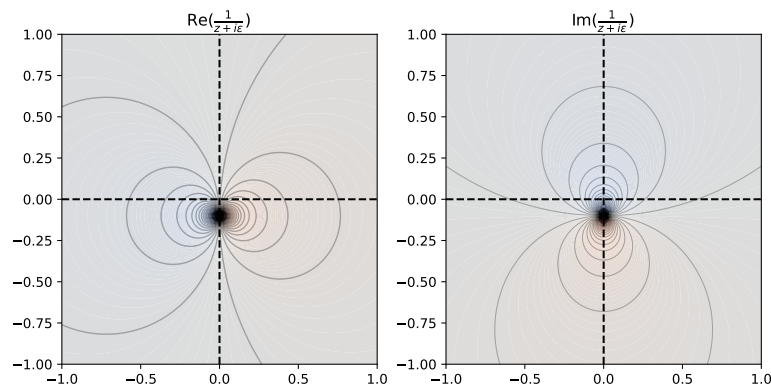


Figure A.1: graph of $\frac{1}{z + i\epsilon}$

- 另外, $\delta(x - a)\delta(x - b) = \delta(b - a)\delta(x - a)$.

A.2 Fourier transformation

- d -dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \tilde{\phi}(k) \\ \tilde{\phi}(k) = \int d^d x e^{-ik \cdot x} \phi(x) \end{cases} \quad (\text{A.2.1})$$

- 因此,

$$\partial_\mu \phi(x) \mapsto ik_\mu \tilde{\phi}(k) \quad (\text{A.2.2})$$

- 对于**实函数**, Fourier transformation 是正交变换, 其 Jacobi determinant 为,

$$\left| \frac{\partial \phi(x) \cdots}{\partial \text{Re} \tilde{\phi}(k) \cdots \partial \text{Im} \tilde{\phi}(k) \cdots} \right| = \left(\frac{2}{V} \right)^{(2N+1)^d} \det A = \left(\frac{2(2N)^d}{V^2} \right)^{\frac{(2N+1)^d}{2}} \quad (\text{A.2.3})$$

proof:

position space 和 momentum space 的格点分别为,

$$\begin{cases} x_i^\mu = i^\mu \epsilon \in \{0, \pm\epsilon, \dots, \frac{L}{2}\} \\ k_n^\mu = n^\mu \frac{2\pi}{L} \in \{0, \pm \frac{2\pi}{L}, \dots, \frac{\pi}{\epsilon}\} \end{cases} \iff i^\mu, n^\mu \in \{0, \pm 1, \dots, N\} \quad (\text{A.2.4})$$

x^μ, k^μ 分别有 $2N+1$ 个取值, 其中 $N\epsilon = \frac{L}{2}$, 时空总体积为 $V = L^d$, momentum space 的总体积为 $\tilde{V} = \frac{(4\pi N)^d}{V}$.

将 (A.2.1) 写成格点求和的形式,

$$\begin{cases} \phi(x_i) = \frac{1}{(2\pi)^d} \left(\frac{2\pi}{L} \right)^d \sum_n e^{ik_n \cdot x_i} \tilde{\phi}(k_n) \\ \quad = \frac{2}{V} \sum_{n^0 > 0} \left(\cos(k_n \cdot x_i) \text{Re} \tilde{\phi}(k_n) - \sin(k_n \cdot x_i) \text{Im} \tilde{\phi}(k_n) \right) \\ \tilde{\phi}(k_n) = \epsilon^d \sum_i e^{-ik_n \cdot x_i} \phi(x_i) \\ \quad = \frac{V}{(2N)^d} \sum_i \left(\cos(k_n \cdot x_i) - i \sin(k_n \cdot x_i) \right) \phi(x_i) \end{cases} \quad (\text{A.2.5})$$

proof:

$\phi(x_i)$ 的变换需要做一些说明. 注意到 $\tilde{\phi}$ 的分量的数量是 ϕ 的两倍 (考虑到实部与虚部), 但在 $\phi \in \mathbb{R}^{(2N+1)^d}$ 时,

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k) \quad (\text{A.2.6})$$

可见 $\tilde{\phi}$ 的分量并不独立, 取 $k^0 > 0$ 的部分为独立分量, 那么...

将 (A.2.5) 写成矩阵的形式,

$$\begin{cases} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} = \frac{2}{V} \overbrace{\begin{pmatrix} \cos k_0 \cdot x_0 & \cdots & \cos k_{\max} \cdot x_0 & -\sin k_0 \cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A} \begin{pmatrix} \text{Re} \tilde{\phi}(k_0) \\ \vdots \\ \text{Im} \tilde{\phi}(k_0) \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \text{Re} \tilde{\phi}(k_0) \\ \vdots \\ \text{Im} \tilde{\phi}(k_0) \\ \vdots \end{pmatrix} = \frac{V}{(2N)^d} \begin{pmatrix} \cos k_0 \cdot x_0 & \cdots & \cos k_0 \cdot x_{\max} \\ \vdots & \ddots & \vdots \\ -\sin k_0 \cdot x_0 & \cdots & -\sin k_0 \cdot x_{\max} \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} \end{cases} \quad (\text{A.2.7})$$

观察可见 $\tilde{\phi}$ 的变换中的矩阵是 A^T , 所以,

$$\frac{2}{V} \frac{V}{(2N)^d} A A^T = I \implies \det A = \left(\frac{(2N)^d}{2} \right)^{\frac{(2N+1)^d}{2}} \quad (\text{A.2.8})$$

因此...

– 顺便,

$$\int d^d x f(x) g(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{f}(-k) \tilde{g}(k) \quad (\text{A.2.9})$$

Appendix B

Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \quad (\text{B.0.1})$$

$$\langle x^{2n} \rangle = \frac{\int dx e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n} (2n-1)!! \quad (\text{B.0.2})$$

其中 $(2n-1)!! = 1 \cdot 3 \cdots (2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx e^{-\frac{a}{2}x^2 + Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \quad (\text{B.0.3})$$

另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 N -dim. generalization

- 考虑如下积分,

$$Z(A, J) = \int dx_1 \cdots dx_N e^{-\frac{1}{2}x^T \cdot A \cdot x + J^T \cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \quad (\text{B.1.1})$$

其中 x, J 是 N -dim. 列向量, A 是 $N \times N$ 实对称矩阵.

calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1} \cdot D \cdot O \quad (\text{B.1.2})$$

其中 D 是一个 diagonal matrix. 令 $y = O \cdot x$, 那么,

$$\begin{aligned} Z(A, J) &= \int dy_1 \cdots dy_N e^{-\frac{1}{2}y^T \cdot D \cdot y + (OJ)^T \cdot y} \\ &= \prod_{i=1}^N \sqrt{\frac{2\pi}{D_{ii}}} e^{\frac{1}{2D_{ii}}(OJ)_i^2} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \end{aligned} \quad (\text{B.1.3})$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O \cdot A^{-1} \cdot O^{-1})_{ii}$ 以及 $\text{tr } D = \det A$.

- 一个重要的变体是 $A \mapsto -iA, J \mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} Z(A, J) \Big|_{J=0} = A_{ij}^{-1} \quad (\text{B.1.4})$$

$$\langle x_i x_j \cdots x_k x_l \rangle = \sum_{\text{Wick}} A_{i'j'}^{-1} \cdots A_{k'l'}^{-1} \quad (\text{B.1.5})$$

其中 (B.1.5) 中有偶数个 x , 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A, 0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A, J) \Big|_{J=0} = \cdots \quad (\text{B.1.6})$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1} \quad (\text{B.1.7})$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle \overbrace{x_i x_j x_k x_l} \rangle = A_{ik}^{-1} A_{jl}^{-1} \quad (\text{B.1.8})$$

Appendix C

perturbation theory in QM

- this chapter is based on MIT OpenCourseWare [Quantum Physics III Chapter 1: Perturbation Theory](#).

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda \delta H \quad (\text{C.0.1})$$

其中 $\lambda \in [0, 1]$.

- 考虑 $H^{(0)}$ 的本征态为,

$$H^{(0)} |k^{(0)}\rangle = E_k^{(0)} |k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)} | l^{(0)} \rangle = \delta_{kl} \\ E_0^{(0)} \leq E_1^{(0)} \leq E_2^{(0)} \leq \dots \end{cases} \quad (\text{C.0.2})$$

C.1 non-degenerate perturbation theory

- 考虑 non-degenerate 能级 k , 有 $\dots \leq E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \leq \dots$, 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \dots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \dots \end{cases} \quad (\text{C.1.1})$$

– 注意, 我们可以选取修正项满足,

$$\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \dots \quad (\text{C.1.2})$$

proof:

假设我们求解得到的修正项不满足 $\langle k^{(0)} | k^{(n)} \rangle = 0, n = 1, 2, \dots$, 考虑,

$$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n |k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)} | k^{(n)} \rangle' = 0 \quad (\text{C.1.3})$$

那么, (注意到态矢量可以乘一个常数, $\frac{1}{1-a_1\lambda-a_2\lambda^2-\dots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \dots$),

$$\begin{aligned} |k\rangle_\lambda &= (1 - a_1\lambda - a_2\lambda^2 - \dots) |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \dots \\ |k\rangle_\lambda' &= |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \dots} (\lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \dots) \\ &= |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 (a_1 |k^{(1)}\rangle' + |k^{(2)}\rangle') + \dots \end{aligned} \quad (\text{C.1.4})$$

可见修正项都与 $|k^{(0)}\rangle$ 正交.

– 注意, 不能要求 ${}_\lambda \langle k | k \rangle_\lambda = 1$, 否则 $|k^{(n)}\rangle$ 将与 λ 相关 (包括 $|k^{(0)}\rangle$),

$$\begin{aligned} {}_\lambda \langle k | k \rangle_\lambda &= \langle k^{(0)} | k^{(0)} \rangle \\ &\quad + \lambda (\langle k^{(1)} | k^{(0)} \rangle + \langle k^{(0)} | k^{(1)} \rangle) \\ &\quad + \lambda^2 (\langle k^{(2)} | k^{(0)} \rangle + \langle k^{(1)} | k^{(1)} \rangle + \langle k^{(0)} | k^{(2)} \rangle) \end{aligned}$$

$$\begin{aligned} & \vdots \\ & + \lambda^n (\langle k^{(n)} | k^{(0)} \rangle + \langle k^{(n-1)} | k^{(1)} \rangle + \dots + \langle k^{(0)} | k^{(n)} \rangle) \end{aligned} \quad (\text{C.1.5})$$

- 将 (C.1.1) 代入 Schrödinger's eq., 得到,

$$\begin{array}{ll} \lambda^0 & (H^{(0)} - E_k^{(0)}) |k^{(0)}\rangle = 0 \\ \lambda^1 & (H^{(0)} - E_k^{(0)}) |k^{(1)}\rangle = (E_k^{(1)} - \delta H) |k^{(0)}\rangle \\ \lambda^2 & (H^{(0)} - E_k^{(0)}) |k^{(2)}\rangle = (E_k^{(1)} - \delta H) |k^{(1)}\rangle + E_k^{(2)} |k^{(0)}\rangle \\ \vdots & \vdots \\ \lambda^n & (H^{(0)} - E_k^{(0)}) |k^{(n)}\rangle = (E_k^{(1)} - \delta H) |k^{(n-1)}\rangle + E_k^{(2)} |k^{(n-2)}\rangle + \dots + E_k^{(n)} |k^{(0)}\rangle \end{array}$$

calculation:

Schrödinger's eq. 为,

$$(H^{(0)} + \lambda \delta H - E_k(\lambda)) |k\rangle_\lambda = 0 \quad (\text{C.1.6})$$

展开为,

$$\left((H^{(0)} - E_k^{(0)}) + \lambda(\delta H - E_k^{(1)}) - \lambda^2 E_k^{(2)} - \dots \right) (|k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \dots) = 0 \quad (\text{C.1.7})$$

- 现在来计算 $\langle l^{(0)} | k^{(n)} \rangle$, 有,

$$\begin{cases} (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(1)} \rangle = E_k^{(1)} \delta_{lk} - \delta H_{lk} \\ (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(2)} \rangle = E_k^{(1)} \langle l^{(0)} | k^{(1)} \rangle - \langle l^{(0)} | \delta H | k^{(1)} \rangle + E_k^{(2)} \delta_{lk} \\ \vdots \\ (E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(n)} \rangle = E_k^{(1)} \langle l^{(0)} | k^{(n-1)} \rangle - \langle l^{(0)} | \delta H | k^{(n-1)} \rangle \\ \quad + E_k^{(2)} \langle l^{(0)} | k^{(n-2)} \rangle + \dots + E_k^{(n)} \delta_{lk} \end{cases} \quad (\text{C.1.8})$$

其中 $\delta H_{lk} = \langle l^{(0)} | \delta H | k^{(0)} \rangle$, 对于满足 (C.1.2) 的解, 有,

$$E_k^{(n)} = \langle k^{(0)} | \delta H | k^{(n-1)} \rangle, n = 1, 2, \dots \quad (\text{C.1.9})$$

并且,

$$|k^{(1)}\rangle = - \sum_{l \neq k} \frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} |l^{(0)}\rangle \implies E_k^{(2)} = - \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} \quad (\text{C.1.10})$$

calculation:

将 (C.1.10) 代入 (C.1.8), 得到 ($l \neq k$),

$$(E_l^{(0)} - E_k^{(0)}) \langle l^{(0)} | k^{(2)} \rangle = -E_k^{(1)} \frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} + \sum_{m \neq k} \frac{\delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \quad (\text{C.1.11})$$

所以,

$$\begin{cases} |k^{(2)}\rangle = \sum_{l \neq k} \left(- \frac{\delta H_{00} \delta H_{lk}}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m \neq k} \frac{\delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \right) |l^{(0)}\rangle \\ E_k^{(3)} = \sum_{l \neq k} \left(- \frac{\delta H_{00} |\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m \neq k} \frac{\delta H_{kl} \delta H_{lm} \delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \right) \end{cases} \quad (\text{C.1.12})$$

计算归一化系数,

$${}_l \langle k | k \rangle_\lambda = 1 + \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + O(\lambda^3) \quad (\text{C.1.13})$$

C.1.1 level repulsion or the seesaw mechanism

- 能量的展开式为,

$$E_k(\lambda) = E_k^{(0)} + \lambda \delta H_{kk} - \lambda^2 \sum_{l \neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3) \quad (\text{C.1.14})$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

C.1.2 validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即,

$$|\lambda V| < \frac{1}{2} \Delta E^{(0)} \quad (\text{C.1.15})$$

因此, 对于能级简并的情况, $\Delta E^{(0)} = 0$, 情况会更复杂.

calculation:

对于两能级系统,

$$H(\lambda) = H^{(0)} + \lambda \hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix} \quad (\text{C.1.16})$$

$H(\lambda)$ 的本征值可以直接计算,

$$E_{\pm}(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)}) \sqrt{1 + \left(\frac{\lambda |V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})} \right)^2} \quad (\text{C.1.17})$$

考虑 $\sqrt{1+z^2}$ 的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1} \frac{(2n-3)!!}{2^n n!} z^{2n} + \cdots \quad (\text{C.1.18})$$

注意到 $\sqrt{1+z^2}$ 在 $z = \pm i$ 有 branch cut, 因此 $z = 0$ 附近的 Taylor expansion 只有在 $|z| < 1$ 内才收敛.

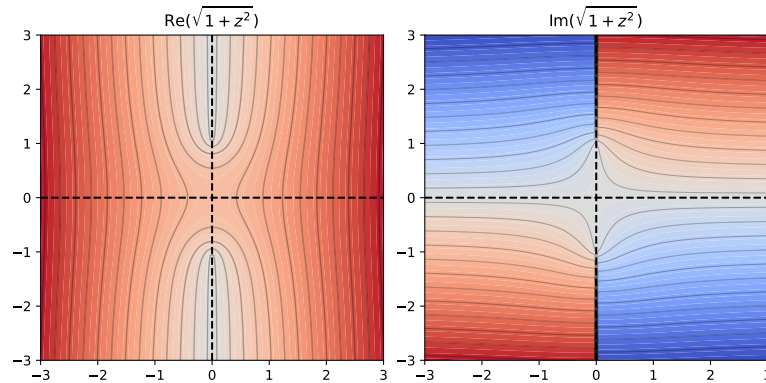


Figure C.1: graph of $\sqrt{1+z^2}$

C.2 degenerate perturbation theory

- 暂时先跳过.

Appendix D

classical field theory and Noether's theorem

D.1 classical field theory

D.1.1 Lagrangian density and the action

- Lagrangian density, \mathcal{L} , 是 $\phi^a(x), \partial_\mu \phi^a(x), t$ 的函数.
- 对作用量变分得到 Euler-Lagrangian equation of motion,

$$\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) = 0 \quad (\text{D.1.1})$$

calculation:

对作用量进行变分,

$$\begin{aligned} \delta S &= \int d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^a} \delta \phi^a + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \delta \partial_\mu \phi^a \right) \\ &= \int d^4x \left(\left(\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) \right) \delta \phi^a + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \delta \phi^a \right) \right) \end{aligned} \quad (\text{D.1.2})$$

由于边界变分为零...

D.1.2 canonical momentum and the Hamiltonian

- **def.:** 定义一个叫 π_a^μ 的量,

$$\pi_a^\mu = \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \quad (\text{D.1.3})$$

其中 $\pi_a \equiv \pi_a^0$ 称作 canonical momentum of the field.

- **def.:** the Hamiltonian density is,

$$\mathcal{H} = \pi_a \partial_0 \phi^a - \mathcal{L} \quad (\text{D.1.4})$$

- the Hamilton's equations are,

$$\begin{cases} \partial_0 \phi^a = \frac{\delta \mathcal{H}}{\delta \pi_a} \\ -\partial_0 \pi^a = \frac{\delta \mathcal{H}}{\delta \phi^a} - \partial_i \left(\frac{\delta \mathcal{H}}{\delta (\partial_i \phi^a)} \right) \end{cases} \quad (\text{D.1.5})$$

- 第二个方程可以写成更紧凑的形式,

$$\partial_\mu \pi_a^\mu = \frac{\delta \mathcal{H}}{\delta \phi^a} \quad (\text{D.1.6})$$

D.2 Noether's theorem

D.2.1 in classical particle mechanics

- 系统的 Lagrangian 为 $L(q^a, \dot{q}^a, t)$.
- 系统通过以下形式变换,

$$q^a(t) \mapsto q^a(\lambda, t) \quad \text{and} \quad q^a(t, 0) = q^a(t) \quad (\text{D.2.1})$$

并定义,

$$D_\lambda q^a = \left. \frac{\partial q^a}{\partial \lambda} \right|_{\lambda=0} \quad (\text{D.2.2})$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.,

$$D_\lambda L = \frac{dF(q^a, \dot{q}^a, t)}{dt} \quad (\text{D.2.3})$$

for some $F(q^a, \dot{q}^a, t)$, and the corresponding **conserved quantity** is,

$$Q = p_a D_\lambda q^a - F(q^a, \dot{q}^a, t) \quad (\text{D.2.4})$$

proof:

$$D_\lambda L = \frac{\partial L}{\partial q^a} D_\lambda q^a + \frac{\partial L}{\partial \dot{q}^a} \frac{dD_\lambda q^a}{dt} = \frac{d}{dt} (p_a D_\lambda q^a) \quad (\text{D.2.5})$$

- 几个例子如下,

- **空间平移**, $\vec{x}(t) \mapsto \vec{x}(t) + \hat{e}_i \lambda$, 相应地, $D_\lambda \vec{x} = \hat{e}_i$, 且,

$$D_\lambda L = \frac{\partial L}{\partial x^i} \quad (\text{D.2.6})$$

如果 $\frac{\partial L}{\partial x^i} = 0$, 那么, 有守恒量 p_i .

- **时间平移**, $q^a(t) \mapsto q^a(t + \lambda)$, 相应地, $D_\lambda q^a = \dot{q}^a$, 且,

$$D_\lambda L = \frac{dL}{dt} - \frac{\partial L}{\partial t} \quad (\text{D.2.7})$$

如果 $\frac{\partial L}{\partial t} = 0$, 那么, 有守恒量 $H = p_a \dot{q}^a - L$.

- **转动**, $\vec{x}(t) \mapsto R(\lambda, \hat{e}) \cdot \vec{x}(t)$, 相应地, $D_\lambda \vec{x} = \hat{e} \times \vec{x}$, 且,

$$D_\lambda L = \vec{x} \cdot \left(\frac{\partial L}{\partial \vec{x}} \times \hat{e} \right) + \hat{e} \cdot (\dot{\vec{x}} \times \vec{p}) \quad (\text{D.2.8})$$

如果上式中两个括号内的项都为零, 那么, 有守恒量 $\hat{e} \cdot \vec{J} = \hat{e} \cdot (\vec{x} \times \vec{p})$.

D.2.2 in classical field theory

- 类似地, 系统通过以下形式变换,

$$\phi^a(x) \mapsto \phi^a(x, \lambda) \quad \text{and} \quad \phi^a(x, 0) = \phi^a(x) \quad (\text{D.2.9})$$

并定义,

$$D_\lambda \phi^a = \left. \frac{\partial \phi^a}{\partial \lambda} \right|_{\lambda=0} \quad (\text{D.2.10})$$

- **Noether's theorem:** the continuous transform λ is a **continuous symmetry** iff.,

$$D_\lambda \mathcal{L} = \partial_\mu F^\mu(\phi^a, \partial_\mu \phi^a, t) \quad (\text{D.2.11})$$

for some $F^\mu(\phi^a, \partial_\mu \phi^a, t)$, and the **conserved current** is,

$$J^\mu = \pi_a^\mu D_\lambda \phi^a - F^\mu \quad (\text{D.2.12})$$

proof:

$$\begin{aligned} D_\lambda \mathcal{L} &= \frac{\delta \mathcal{L}}{\delta \phi^a} D_\lambda \phi^a + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \partial_\mu D_\lambda \phi^a \\ &= \left(\frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} \right) \right) D_\lambda \phi^a + \partial_\mu \underbrace{\left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)} D_\lambda \phi^a \right)}_{=\pi_a^\mu} \end{aligned} \quad (\text{D.2.13})$$

代入 (D.1.1), 得...

- 注意, conserved current 并不是唯一确定的, 考虑如下变换,

$$F^\mu \mapsto F'^\mu = F^\mu + \partial_\nu A^{\mu\nu} \quad \text{with} \quad A^{\mu\nu} = A^{[\mu\nu]} \quad (\text{D.2.14})$$

新 F'^μ 依然能满足 (D.2.11).

- 但是, 守恒荷是唯一确定的.

proof:

$$Q' = \int d^3x J^0 = \int d^3x (\pi_a D_\lambda \phi^a - F^0) - \int d^3x \partial_\mu A^{0\mu} \quad (\text{D.2.15})$$

考虑到边界值为零, 且 $A^{00} = 0$, 所以 $Q' = Q$.

D.2.3 spacetime translations and the energy-momentum tensor

- 时空平移变换为,

$$\phi^a(x) \mapsto \phi^a(x + \lambda e) \quad (\text{D.2.16})$$

- 所以,

$$D_\lambda \phi^a = e^\mu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = e^\mu \partial_\mu \mathcal{L} \quad (\text{D.2.17})$$

代入 (D.2.12),

$$J^\mu = e^\nu \underbrace{(\pi_a^\mu \partial_\nu \phi^a - \delta_\nu^\mu \mathcal{L})}_{=T^\mu_\nu} \quad (\text{D.2.18})$$

- 并且有,

$$\partial_\mu T^{\mu\nu} = 0 \implies P^\mu = \int d^3x T^{0\mu} = \text{Const.} \quad (\text{D.2.19})$$

来自守恒流散度为零.

D.2.4 Lorentz transformations, angular momentum and something else

- Lorentz transformation 下坐标做变换 $x'^\mu = \Lambda^\mu_\nu x^\nu$, 其中 Λ 满足,

$$\eta = \Lambda^T \eta \Lambda \quad (\text{D.2.20})$$

- infinitesimal Lorentz transformation 是,

$$\Lambda = I + \epsilon \quad (\text{D.2.21})$$

其中 $\{\epsilon^{\mu\nu}\} = \epsilon \eta$ 是反对称矩阵.

proof:

考虑,

$$\eta = (\Lambda \eta)^T \eta (\Lambda \eta) = (\eta + \epsilon \eta)^T \eta (\eta + \epsilon \eta)$$

$$= \eta + \eta \epsilon^T + \epsilon \eta + O(\epsilon^2) \quad (\text{D.2.22})$$

- 标量场在 Lorentz transform 下的变换为,

$$\Lambda : \phi^a(x) \mapsto \phi^a(\Lambda^{-1}x') \quad (\text{D.2.23})$$

- 有,

$$D_\lambda \phi^a = -\epsilon^\mu_{\ \nu} x^\nu \partial_\mu \phi^a \quad \text{and} \quad D_\lambda \mathcal{L} = -\epsilon^\mu_{\ \nu} x^\nu \partial_\mu \mathcal{L} = -\epsilon_{\mu\nu} \partial^\mu (x^\nu \mathcal{L}) \quad (\text{D.2.24})$$

代入 (D.2.12),

$$J^\mu = \frac{1}{2} \epsilon_{\nu\rho} M^{\mu\nu\rho} \quad \text{where} \quad M^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \quad (\text{D.2.25})$$

且有,

$$\partial_\mu M^{\mu\nu\rho} = 0 \quad (\text{D.2.26})$$

- 对全空间积分, 得到 6 个守恒量,

$$J^{\mu\nu} = \int d^3x M^{0\mu\nu} = \text{Const.} \quad (\text{D.2.27})$$

不难发现 J^{ij} 对应角动量, 现在来讨论 J^{0i} 的物理意义,

$$0 = \frac{d}{dt} J^{0i} = \frac{d}{dt} \int d^3x (x^i T^{00} - t T^{0i}) = P^i - \frac{d}{dt} \int d^3x x^i T^{00} \quad (\text{D.2.28})$$

其中, 用到了 $\frac{dP^i}{dt} = 0$ (见 (D.2.19)), 可以将上式的第二项理解为质心运动的动量.

D.3 charge as generators

- the charge associated with the conserved current is,

$$Q = \int d^D x J^0 = \int d^D x (\pi_a D_\lambda \phi^a - F^0) \quad (\text{D.3.1})$$

在 $F^\mu = 0$ 且 $[D_\lambda \phi^a, \phi^a] = 0$ 的情况下,

$$i[Q, \phi^a] = D_\lambda \phi^a \quad (\text{D.3.2})$$

D.4 what the graviton listens to: energy-momentum tensor

- the energy-momentum tensor is defined as (其中 $g = |\det\{g_{\mu\nu}\}|$),

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2 \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M \quad (\text{D.4.1})$$

- 如果将 \mathcal{L}_M 对 $g^{\mu\nu}$ 做展开 $\mathcal{L}_M = A + g^{\mu\nu} B_{\mu\nu} + g^{\mu\nu} g^{\rho\sigma} C_{\mu\nu\rho\sigma} + \dots$, 那么,

$$T_{\mu\nu} = -2(B_{\mu\nu} + 2g^{\rho\sigma} C_{\mu\nu\rho\sigma} + 3\dots) + g_{\mu\nu} \mathcal{L}_M \quad (\text{D.4.2})$$

另外, the trace of the energy-momentum tensor is,

$$T = g^{\mu\nu} T_{\mu\nu} = d \times A + (d-2)g^{\mu\nu} B_{\mu\nu} + (d-4)g^{\mu\nu} g^{\rho\sigma} C_{\mu\nu\rho\sigma} \quad (\text{D.4.3})$$

可见 $d=4$ 时, T 与 $C_{\mu\nu\rho\sigma}$ 无关.

- 以 electromagnetic field 为例, $d=4$,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^\mu A_\mu \implies \begin{cases} T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho + m^2 A_{\mu\nu} + g_{\mu\nu} \mathcal{L}_M \\ T = -m^2 A^\mu A_\mu \end{cases} \quad (\text{D.4.4})$$

可见 the energy-momentum tensor of electromagnetic field (when $m=0$) is traceless.

- $\mathcal{L} = -\frac{1}{2}((\partial\phi)^2 - m^2\phi^2)$ 和 $\mathcal{L} = \frac{1}{2}\phi(\partial^2 - m^2)\phi$ 对应的 energy-momentum tensor 一样吗 (?).