Quantum Field Theory

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convention, notation, and units

- 笔记中的**度规号差**约定为 $(+,-,-,-)$.
- 使用 natural units, 此时 $\hbar, c, k_B = 1$, 因此 $1\,\mathrm{m} = \frac{1}{1.97\times 10^{-16}\,\mathrm{GeV}}$ 且:

names/dimensions	expressions/values
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

Part I

Field Theory

Chapter 1

cross sections and decay rates

1.1 cross sections

- cross section 定义为
$$\sigma = \frac{1}{\Phi}\frac{P}{\Delta t}, \tag{1.1.1}$$
其中 $\Phi := nv = \frac{|\vec{v}_1 - \vec{v}_2|}{V}$ 是 incoming flux, 是入射粒子数密度乘粒子速度, P 是发生散射的概率.
- 实验上定义 luminosity 为
$$L\Delta t = \frac{dN}{d\sigma}, \tag{1.1.2}$$
其中 dN 是 $d\Omega$ 内发生散射的粒子数.
- 用 S-matrix elements 来表示 cross section, 有
$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}d\Pi, \tag{1.1.3}$$
其中 $d\Pi$ 是末态动量体元
$$d\Pi = \prod_i \delta^{(3)}(\vec{p}=0)d^3p_{f,i} = \prod_i \frac{V}{(2\pi)^3}d^3p_{f,i}, \tag{1.1.4}$$
这保证了无相互作用时 $\int dP = 1$.
- 对于初末态有
$$\begin{cases} \langle i|i\rangle = \langle p_1, p_2|p_1, p_2\rangle = (2\pi)^3 2\omega_{p_1}\delta^{(3)}(0)(2\pi)^3 2\omega_{p_2}\delta^{(3)}(0) = (2\omega_{p_1}V)(2\omega_{p_2}V) \\ \langle f|f\rangle = \prod_i (2\omega_{p_{f,i}}V) \end{cases}. \tag{1.1.5}$$

- 一般将 S-matrix 写为
$$S = I + i\mathcal{T}, \quad \mathcal{T} = (2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\mathcal{M}, \tag{1.1.6}$$
其中 $\mathcal{T}$ 称为 transfer matrix, 而 $\mathcal{M}$ 才是 S-matrix 的 non-trivial part. 有
$$\langle f|S-I|i\rangle = i(2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\langle f|\mathcal{M}|i\rangle. \tag{1.1.7}$$
- 对于 $|f\rangle \neq |i\rangle$ 的情况, 有
$$|\langle f|S|i\rangle|^2 = (2\pi)^4 TV\delta^{(4)}(\textstyle\sum_{i,f} p)|\langle f|\mathcal{M}|i\rangle|^2, \tag{1.1.8}$$
那么
$$dP = \frac{T}{V}\frac{1}{(2\omega_{p_1})(2\omega_{p_2})}|\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}, \tag{1.1.9}$$
其中 LIPS 表示 Lorentz-invariant phase space,
$$d\Pi_{\text{LIPS}} = (2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\displaystyle\prod_i \frac{d^3p_{f,i}}{(2\pi)^3 2\omega_{p_{f,i}}}. \tag{1.1.10}$$
- 最终有 (将 (1.1.1) 中的 Δt 替换为 T)
$$d\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2|(2\omega_{p_1})(2\omega_{p_2})}|\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}. \tag{1.1.11}$$

1.2 decay rates

- decay rate, Γ, 是粒子单位时间发生衰变的概率,

$$d\Gamma = \frac{dP}{T}. \tag{1.2.1}$$

- 因为 $|f\rangle \neq |i\rangle$, 有

$$d\Gamma = \frac{1}{2\omega_p} |\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}. \tag{1.2.2}$$

Chapter 2

the S-matrix and time-ordered products

2.1 the LSZ reduction formula

- S-matrix element 为

$$\begin{cases} |i\rangle = \sqrt{(2\pi)^3 2\omega_{p_1}}\sqrt{(2\pi)^3 2\omega_{p_2}}a^\dagger_{\vec{p}_1}(-\infty)a^\dagger_{\vec{p}_2}(-\infty)\,|\Omega\rangle \\ |f\rangle = \sqrt{(2\pi)^3 2\omega_{p_3}}\cdots\sqrt{(2\pi)^3 2\omega_{p_n}}a^\dagger_{\vec{p}_3}(+\infty)\cdots a^\dagger_{\vec{p}_n}(+\infty)\,|\Omega\rangle \\ \langle f|S|i\rangle = (2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega|a_{\vec{p}_3}(+\infty)\cdots a_{\vec{p}_n}(+\infty)a^\dagger_{\vec{p}_1}(-\infty)a^\dagger_{\vec{p}_2}(-\infty)|\Omega\rangle \end{cases}. \tag{2.1.1}$$

remark:

注意到

$$\begin{cases} a_{\vec{p}}(t) = U^\dagger(t)a_{\vec{p}}(0)U(t) \\ |\vec{p}(t)\rangle = U(t)\,|\vec{p}(0)\rangle \end{cases} \Longrightarrow |\vec{p}(t)\rangle = a^\dagger_{\vec{p}}(-t)\,|\Omega\rangle\,, \tag{2.1.2}$$

那么, 对于初末态, 有

$$\begin{aligned} &|i(-\infty)\rangle = \sqrt{(2\pi)^3 2\omega_{p_1}}\sqrt{(2\pi)^3 2\omega_{p_2}}a^\dagger_{\vec{p}_1}(0)a^\dagger_{\vec{p}_2}(0)\,|\Omega\rangle \\ \Longrightarrow &|i(0)\rangle = \cdots\,, \end{aligned} \tag{2.1.3}$$

可见 (2.1.1) 中的 $|i\rangle\,,|f\rangle$ 都是 $t=0$ 即 Heisenberg picture 中的形式.

- LSZ 需要用到以下公式 (其中 p 是 on-shell),

$$i\int d^4x\,e^{ip\cdot x}(\partial^2+m^2)\phi(x) = \sqrt{(2\pi)^3 2\omega_p}(e^{i\omega_p t}a_{\vec{p}}(t))\Big|_{-\infty}^{\infty}, \tag{2.1.4}$$

注意到, 对于 free field, 等式两边等于零.

proof:

注意到

$$\begin{aligned} &\phi(x) = \int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}(t)e^{-i\vec{p}\cdot\vec{x}}) \\ \Longrightarrow &(\partial^2+m^2)\phi(x) = (\partial_t^2+\omega_p^2)\phi(x), \end{aligned} \tag{2.1.5}$$

代入, 得

$$\begin{aligned} \mathrm{LHS} &= i\int d^4x\,e^{ip\cdot x}(\partial_t^2+\omega_p^2)\phi(x) \\ &= i\int dt\,e^{i\omega_p t}\int d^3x\,e^{-i\vec{p}\cdot\vec{x}}(\partial_t^2+\omega_p^2)\int\frac{d^3q}{(2\pi)^{3/2}\sqrt{2\omega_q}}(a_{\vec{q}}(t)e^{i\vec{q}\cdot\vec{x}} + a^\dagger_{\vec{q}}(t)e^{-i\vec{q}\cdot\vec{x}}) \end{aligned}$$

$$= i\frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}}\int dt\, e^{i\omega_p t}(\partial_t^2+\omega_p^2)(a_{\vec{p}}(t)+a^\dagger_{-\vec{p}}(t)), \tag{2.1.6}$$

注意到

$$e^{i\omega_p t}(\partial_t^2+\omega_p^2)O(t)=\partial_t(e^{i\omega_p t}(\partial_t - i\omega_p)O(t)), \tag{2.1.7}$$

因此

$$\text{LHS} = i\frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}}(e^{i\omega_p t}(\partial_t - i\omega_p)(a_{\vec{p}}(t)+a^\dagger_{-\vec{p}}(t)))\Big|_{-\infty}^{\infty}, \tag{2.1.8}$$

注意到 $t=\pm\infty$ 时, fields are free, 所以

$$\begin{cases}\lim\limits_{t\to\pm\infty} a_{\vec{p}}(t)\propto e^{-i\omega_p t}a_{\vec{p}} \\ \lim\limits_{t\to\pm\infty} a^\dagger_{-\vec{p}}(t)\propto e^{i\omega_p t}a^\dagger_{-\vec{p}}\end{cases} \Longrightarrow \begin{cases}\lim\limits_{t\to\pm\infty}(\partial_t - i\omega_p)a_{\vec{p}}(t) = -2i\omega_p a_{\vec{p}}(t) \\ \lim\limits_{t\to\pm\infty}(\partial_t - i\omega_p)a^\dagger_{-\vec{p}}(t)=0\end{cases}, \tag{2.1.9}$$

代入得到

$$\text{LHS} = \sqrt{(2\pi)^3 2\omega_p}(e^{i\omega_p t}a_{\vec{p}}(t))\Big|_{-\infty}^{\infty}. \tag{2.1.10}$$

- 那么 (up to an infinite phase)

$$\begin{aligned}
&\langle f|S|i\rangle \\
=&(2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega|a_{\vec{p}_3}(+\infty)\cdots a_{\vec{p}_n}(+\infty)a^\dagger_{\vec{p}_1}(-\infty)a^\dagger_{\vec{p}_2}(-\infty)|\Omega\rangle \\
=&(2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega|T((a_{\vec{p}_3}(+\infty)-a_{\vec{p}_3}(-\infty))\cdots(a^\dagger_{\vec{p}_2}(-\infty)-a^\dagger_{\vec{p}_2}(+\infty)))|\Omega\rangle \\
=&\left(i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)\right)\cdots\left(i\int d^4x_n\, e^{ip_n\cdot x_n}(\partial_n^2+m^2)\right)\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \\
=&(2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}} \\
&\langle\Omega|T((e^{i\omega_p(+\infty)}a_{\vec{p}_3}(+\infty)-e^{i\omega_p(-\infty)}a_{\vec{p}_3}(-\infty))\cdots(e^{i\omega_p(-\infty)}a^\dagger_{\vec{p}_2}(-\infty)-e^{i\omega_p(+\infty)}a^\dagger_{\vec{p}_2}(+\infty)))|\Omega\rangle
\end{aligned} \tag{2.1.11}$$

得到 LSZ reduction formula.

calculation:

$$\begin{aligned}
&\left(i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)\right)\cdots\left(i\int d^4x_n\, e^{ip_n\cdot x_n}(\partial_n^2+m^2)\right)\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \\
=&n!\,\langle\Omega|\int_{-\infty}^{\infty}dt_1\, i\int d^3x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)\phi_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n\cdots|\Omega\rangle
\end{aligned} \tag{2.1.12}$$

并注意到 (其中 $f(t)=F'(t)$, 且一般地 $[F(t_1),F(t_2)]\neq 0$)

$$\begin{aligned}
\int_a^b dt_1\, f(t_1)\int_a^{t_1}dt_2\, f(t_2) &= \int_a^b dt_1\, f(t_1)(F(t_1)-F(a)) \\
&= \int_a^b dt_1\, f(t_1)F(t_1) - (F(b)-F(a))F(a) \\
&= -\int_a^b dt_1\, F(t_1)f(t_1) + F(b)(F(b)-F(a)) \\
&= \cdots \\
&\overset{?}{=} \frac{1}{2}(F(b)(F(b)-F(a))-(F(b)-F(a))F(a)) \\
&= \frac{1}{2}T((F(b)-F(a))^2),
\end{aligned} \tag{2.1.13}$$

因此

$$\int_a^b dt_1\, f(t_1)\int_a^{t_1}dt_2\, f(t_2)\cdots\int_a^{t_{n-1}}dt_n\, f(t_n) \tag{2.1.14}$$

2.2 LSZ for operators

- $|\Omega\rangle$ 的具体形式见 subsection 3.2.1.
- 考虑
$$\phi(x)=e^{iP\cdot x}\phi(0)e^{-iP\cdot x}, \tag{2.2.1}$$
且 $P|\Omega\rangle=0$, 因此
$$\begin{cases}\langle\Omega|\phi(x)|\Omega\rangle=\langle\Omega|\phi(0)|\Omega\rangle\\ \langle p|\phi(x)|\Omega\rangle=e^{ip\cdot x}\langle p|\phi(0)|\Omega\rangle\end{cases}. \tag{2.2.2}$$
- LSZ reduction formula 的前提是场算符满足
$$\begin{cases}\langle\Omega|\phi(x)|\Omega\rangle=0\\ \langle p|\phi(x)|\Omega\rangle=e^{ip\cdot x}\end{cases}. \tag{2.2.3}$$

remark:

前提是场算符具有以下形式,
$$\phi(x)=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}}+a^\dagger_{\vec{p}}(t)e^{-i\vec{p}\cdot\vec{x}}), \tag{2.2.4}$$
和 $a^\dagger_{\vec{p}}(\pm\infty)|\Omega\rangle\in\mathrm{span}(|p\rangle)$ 是一个单粒子态, 且 $\langle\Omega|a^\dagger_{\vec{p}}(+\infty)=a_{\vec{p}}(-\infty)|\Omega\rangle=0$.

- 注意, 在 Heisenberg picture 中,
$$\phi(t=0,\vec{x})=\phi_0(\vec{x})=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}+a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}). \tag{2.2.5}$$
- (2.1.9) 和 $H|\Omega\rangle=0$ (要求 H 不含时) 矛盾 (?), 因为 LSZ 考虑的 Hamiltonian 是
$$H=H_0+V\theta(t-T)\theta(t+T),T\gg 0, \tag{2.2.6}$$
所以 (2.2.3) 只在 $-T<t<T$ 成立.

Chapter 3

Feynman rules

3.1 Lagrangian derivation

- 假设存在相互作用时, 场算符依然满足

$$\begin{cases}[\phi(t,\vec{x}),\phi(t,\vec{y})]=0\\ [\phi(t,\vec{x}),\partial_t\phi(t,\vec{y})]=i\delta^{(3)}(\vec{x}-\vec{y})\end{cases}, \tag{3.1.1}$$

即 causality and canonical commutation relation.

- 一个重要的中间公式为

$$(\partial_x^2+m^2)\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle=\langle\Omega|T((\partial_x^2+m^2)\phi(x)\phi(y))|\Omega\rangle-i\delta^{(4)}(x-y). \tag{3.1.2}$$

proof:

考虑

$$\begin{aligned}\partial_t\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle=&\partial_t(\langle\Omega|\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(y)\phi(x)|\Omega\rangle\theta(t'-t))\\ =&\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(x)\phi(y)|\Omega\rangle\delta(t-t')\\ &+\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\theta(t'-t)-\langle\Omega|\phi(y)\phi(x)|\Omega\rangle\delta(t'-t)\\ =&\langle\Omega|T(\partial_t\phi(x)\phi(y))|\Omega\rangle+\langle\Omega|[\phi(t,\vec{x}),\phi(t,\vec{y})]|\Omega\rangle\delta(t-t')\\ =&\langle\Omega|T(\partial_t\phi(x)\phi(y))|\Omega\rangle,\end{aligned} \tag{3.1.3}$$

那么

$$\begin{aligned}\partial_t^2\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle=&\partial_t\langle\Omega|T(\partial_t\phi(x)\phi(y))|\Omega\rangle\\ =&\partial_t(\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\theta(t'-t))\\ =&\langle\Omega|\partial_t^2\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\delta(t-t')\\ &+\langle\Omega|\phi(y)\partial_t^2\phi(x)|\Omega\rangle\theta(t'-t)-\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\delta(t'-t)\\ =&\langle\Omega|T(\partial_t^2\phi(x)\phi(y))|\Omega\rangle+\langle\Omega|[\partial_t\phi(t,\vec{x}),\phi(t,\vec{y})]|\Omega\rangle\delta(t-t')\\ =&\langle\Omega|T(\partial_t^2\phi(x)\phi(y))|\Omega\rangle-i\delta^{(4)}(x-y).\end{aligned} \tag{3.1.4}$$

- (3.1.2) 可以推广为 (其中 ϕ_i 是 $\phi(x_i)$ 的简写)

$$\begin{aligned}&(\partial_1^2+m^2)\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle\\ =&\langle\Omega|T((\partial_1^2+m^2)\phi(x_1)\cdots\phi(x_n))|\Omega\rangle-i\sum_{i=2}^{n}\delta^{(4)}(x_i-x_1)\langle\Omega|\phi_2\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_n|\Omega\rangle.\end{aligned} \tag{3.1.5}$$

proof:

首先

$$\partial_{t_1}\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle=\langle\Omega|T(\partial_{t_1}\phi(x_1)\cdots\phi(x_n))|\Omega\rangle, \tag{3.1.6}$$

那么

$$\partial_{t_1}^2 \langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \partial_{t_1}\langle\Omega|T(\partial_{t_1}\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \cdots \tag{3.1.7}$$

- canonical commutation relation 保证了 quantum field 满足 Euler–Lagrange equation, 因此

$$(\partial^2 + m^2)\phi = -\frac{\delta}{\delta\phi}V(\phi). \tag{3.1.8}$$

- 结合 (3.1.5) 和 (3.1.8) 得到 Schwinger–Dyson equations,

$$\begin{aligned}&(\partial_1^2 + m^2)\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \\ =&\langle\Omega|T(-\frac{\delta}{\delta\phi_1}V(\phi_1)\cdots\phi(x_n))|\Omega\rangle - i\sum_{i=2}^{n}\delta^{(4)}(x_i - x_1)\langle\Omega|\phi_2\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_n|\Omega\rangle .\end{aligned} \tag{3.1.9}$$

3.1.1 position-space Feynman rules

- Feynman propagator 为 ($\phi_0(x)$ 是 free field, 本 chapter 不做特别说明都是存在相互作用的 Heisenberg picture)

$$D_F(x-y) := \langle 0|T(\phi_0(x)\phi_0(y))|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}e^{ik\cdot(x-y)}, \tag{3.1.10}$$

满足

$$(\partial^2 + m^2)D_F(x-y) = -i\delta^{(4)}(x-y). \tag{3.1.11}$$

- 2-point correlation function 可以重写作 (注意到 $\lim_{x\to\infty} D_{xy} = \lim_{x\to\infty}\partial_x D_{xy} = 0$)

$$\begin{aligned}\langle\Omega|T(\phi_1\phi_2)|\Omega\rangle &= i\int d^4x\,((\partial_x^2 + m^2)D_{x1})\langle\Omega|T(\phi_x\phi_2)|\Omega\rangle \\ &= i\int d^4x\, D_{x1}(\partial_x^2 + m^2)\langle\Omega|T(\phi_x\phi_2)|\Omega\rangle \\ &= D_{12} + i\int d^4x\, D_{x1}\langle\Omega|T(-\frac{\delta}{\delta\phi_x}V(\phi_x)\phi_2)|\Omega\rangle ,\end{aligned} \tag{3.1.12}$$

类似地, 4-point function 可以写作

$$\begin{aligned}\langle\Omega|T(\phi_1\phi_2\phi_3\phi_4)|\Omega\rangle =&D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23} \\ &+ i\int d^4x\, D_{x1}\langle\Omega|T(-\frac{\delta}{\delta\phi_x}V(\phi_x)\phi_2\phi_3\phi_4)|\Omega\rangle \\ &+ D_{12}i\int d^4y\, D_{3y}\langle\Omega|T(-\frac{\delta}{\delta\phi_y}V(\phi_y)\phi_4)|\Omega\rangle + \cdots\end{aligned} \tag{3.1.13}$$

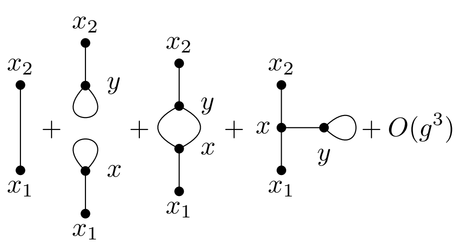
- 考虑 $V(\phi) = -\frac{g}{3!}\phi^3$, 那么 2-point function 为

$$\begin{aligned}\langle\Omega|T(\phi_1\phi_2)|\Omega\rangle &= D_{12} + i\int d^4x\, D_{x1}\langle\Omega|T(\frac{g}{2!}\phi_x^2\phi_2)|\Omega\rangle \\ &= D_{12} + i\int d^4x\, D_{x1}\Big(i\int d^4y\, D_{y2}\langle\frac{g}{2!}\phi_x^2\frac{g}{2!}\phi_y^2\rangle + 2!D_{x2}\langle\frac{g}{2!}\phi_x\rangle\Big),\end{aligned} \tag{3.1.14}$$

其中

$$\begin{cases}\langle\phi_x\rangle = i\int d^4y\, D_{yx}(\partial_y^2 + m^2)\langle\phi_y\rangle = i\int d^4y\, D_{yx}\langle\frac{g}{2!}\phi_y^2\rangle \\ \qquad = i\frac{g}{2!}\int d^4y\, D_{yx}D_{yy} + O(g^2) \\ \langle\phi_x^2\phi_y^2\rangle = D_{xx}D_{yy} + 2D_{xy}D_{xy} + O(g)\end{cases}, \tag{3.1.15}$$

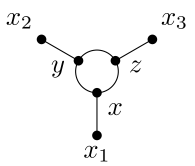
因此

$\langle\phi_1\phi_2\rangle =$ 

$$= D_{12} + \frac{(ig)^2}{4} D_{1x} D_{xx} D_{yy} D_{y2} + \frac{(ig)^2}{2} D_{1x} D_{xy} D_{xy} D_{y2} + \frac{(ig)^2}{2} D_{1x} D_{xy} D_{yy} D_{x2} + O(g^3), \quad (3.1.16)$$

其中 $4, 2$ 是 symmetry factors.

- 下图的 symmetry factor 为 1,



$$= (ig)^3 D_{1x} D_{xy} D_{yz} D_{zx} D_{y2} D_{z3}. \quad (3.1.17)$$

3.2 Hamiltonian derivation

- interaction picture...

3.2.1 vacuum matrix elements

- section 2.2 中注意到了 $|\Omega\rangle$ 的定义 (in Heisenberg picture),

$$\begin{cases} a_{\vec{p}}(-\infty)\,|\Omega\rangle = U^\dagger(-\infty) a_{\vec{p}} U(-\infty)\,|\Omega\rangle = 0 \\ a_{\vec{p}}(+\infty)\,|\Omega\rangle = U^\dagger(+\infty) a_{\vec{p}} U(+\infty)\,|\Omega\rangle = 0 \end{cases} \Longrightarrow U(\pm\infty)\,|\Omega\rangle \propto |0\rangle\,, \quad (3.2.1)$$

那么 Schrodinger picture 和 interaction picture 中分别有

$$\begin{cases} |\Omega(t)\rangle = \mathcal{N}_i U(t,-\infty)\,|0\rangle = \mathcal{N}_f U(t,+\infty)\,|0\rangle & \text{Schrodinger picture} \\ |\Omega(t)\rangle_I = \mathcal{N}_i U_I(t,-\infty)\,|0\rangle = \mathcal{N}_f U_I(t,+\infty)\,|0\rangle & \text{interaction picture} \end{cases}. \quad (3.2.2)$$

calculation:

令

$$|\Omega\rangle \equiv |\Omega(0)\rangle = \mathcal{N}_i U(0,-\infty)\,|0\rangle = \mathcal{N}_f U(0,+\infty)\,|0\rangle\,, \quad (3.2.3)$$

且

$$\begin{cases} U_I(0,-\infty) U_0(0,-\infty) = U(0,-\infty) \\ U_I(0,+\infty) U_0(0,+\infty) = U(0,+\infty) \end{cases} \quad \text{and} \quad U_0(0,\pm\infty)\,|0\rangle = |0\rangle\,, \quad (3.2.4)$$

因此

$$|\Omega\rangle_I = \begin{cases} \mathcal{N}_i U_I(t,0) U(0,-\infty)\,|0\rangle = \mathcal{N}_i U_I(t,-\infty)\,|0\rangle \\ \mathcal{N}_f U_I(t,0) U(0,+\infty)\,|0\rangle = \mathcal{N}_f U_I(t,+\infty)\,|0\rangle \end{cases}. \quad (3.2.5)$$

  - 归一化要求 $\langle\Omega|\Omega\rangle = \mathcal{N}_f^* \mathcal{N}_i \langle 0|U(+\infty,-\infty)|0\rangle = \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty,-\infty)|0\rangle = |\mathcal{N}_i|^2 = |\mathcal{N}_f|^2 = 1$.
  - $H\,|\Omega\rangle = 0$ 因为 $[H, U(t_1,t_2)] = 0$ (前提是 Hamiltonian 不含时).

- 因此

$$\langle\Omega|T(\phi_1\cdots\phi_n)|\Omega\rangle = \frac{\langle 0|T(\phi_{0,1}\cdots\phi_{0,n} U_I(+\infty,-\infty))|0\rangle}{\langle 0|U_I(+\infty,-\infty)|0\rangle}. \quad (3.2.6)$$

calculation:

$$\begin{aligned} &\langle\Omega|T(\phi_1\cdots\phi_n)|\Omega\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U(+\infty,0) T((U(0,t_1)\phi_{S,1} U(t_1,0))(U(0,t_2)\phi_{S,2} U(t_2,0))\cdots) U(0,-\infty)|0\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty,0) T((U_I(0,t_1) U_0(0,t_1)\phi_{S,1} U_0(t_1,0) U_I(t_1,0))\cdots) U_I(0,-\infty)|0\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty,0) T((U_I(0,t_1)\phi_{0,1} U_I(t_1,0))\cdots) U_I(0,-\infty)|0\rangle = \cdots \end{aligned} \quad (3.2.7)$$

另外注意到

$$\mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(-\infty,+\infty)|0\rangle = 1. \quad (3.2.8)$$

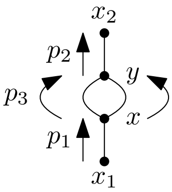
3.2.2 time-ordered products and contractions

- 用 Wick contraction 计算 (3.2.6), 得到

$$\langle\Omega|T(\phi_1\cdots\phi_n)|\Omega\rangle = \langle 0|T(\phi_{0,1}\cdots\phi_{0,n}U_I(+\infty,-\infty))|0\rangle_{\text{no bubbles}}\,. \tag{3.2.9}$$

3.3 momentum-space Feynman rules

- 考虑如下 Feynman 图,

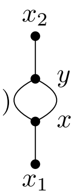


$$= \frac{(ig)^2}{2}\int d^4x\int d^4y\, D_{1x}D_{xy}^2D_{y2}$$

$$= \frac{(ig)^2}{2}\int\frac{d^4p_1}{(2\pi)^4}\frac{ie^{ip_1\cdot x_1}}{p_1^2-m^2+i\epsilon}\int\frac{d^4p_2}{(2\pi)^4}\frac{ie^{-ip_2\cdot x_2}}{p_2^2-m^2+i\epsilon}$$
$$\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{(p_1-p_3)^2-m^2+i\epsilon}(2\pi)^4\delta^{(4)}(p_1-p_2), \tag{3.3.1}$$

代入 LSZ, 有

$$\langle p_2|S|p_1\rangle = \cdots + i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)i\int d^4x_2\, e^{ip_2\cdot x_2}(\partial_2^2+m^2)$$



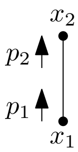
$$+\cdots$$

$$= \cdots + \frac{(ig)^2}{2}(2\pi)^4\delta^{(4)}(p_1-p_2)\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{(p_1-p_3)^2-m^2+i\epsilon}+\cdots, \tag{3.3.2}$$

所以

$$i\langle p_2|\mathcal{M}|p_1\rangle = \cdots + \frac{(ig)^2}{2}\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{(p_1-p_3)^2-m^2+i\epsilon}+\cdots \tag{3.3.3}$$

- 再考虑 (LSZ 中的 p_1,p_2 都 on-shell)

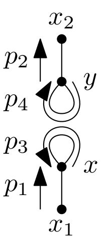


$$= i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)i\int d^4x_2\, e^{ip_2\cdot x_2}(\partial_2^2+m^2)D_{12}$$
$$= i(2\pi)^4\delta^{(4)}(p_1-p_2)(-p_1^2+m^2) = 0, \tag{3.3.4}$$

这是因为 LSZ 算的是 $\langle p_2|i\mathcal{T}|p_1\rangle$.

3.3.1 disconnected graphs

- disconnected graphs are never important (?).
- 考虑



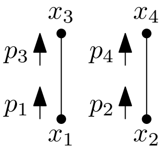
$$= \frac{(ig)^2}{4}\int d^4x\int d^4y\, D_{1x}D_{xx}D_{yy}D_{y2}$$

$$= \frac{(ig)^2}{4}\frac{i}{-m^2+i\epsilon}\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{-m^2+i\epsilon}\int\frac{d^4p_4}{(2\pi)^4}\frac{i}{p_4^2-m^2+i\epsilon}, \tag{3.3.5}$$

那么

$$
\begin{aligned}
\langle p_2|S|p_1\rangle =& \cdots + \frac{(ig)^2}{4}(2\pi)^4\delta^{(4)}(p_1)\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon} \\
&(2\pi)^4\delta^{(4)}(p_2)\int\frac{d^4p_4}{(2\pi)^4}\frac{i}{p_4^2-m^2+i\epsilon}+\cdots \\
=& \cdots + \frac{(ig)^2}{4}(2\pi)^4\delta^{(4)}(p_1-p_2) \\
&\left((2\pi)^4\delta^{(4)}(p_1)\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\int\frac{d^4p_4}{(2\pi)^4}\frac{i}{p_4^2-m^2+i\epsilon}\right)+\cdots
\end{aligned}
\tag{3.3.6}
$$

- 再考虑



$$
\begin{aligned}
&= i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)i\int d^4x_2\, e^{-ip_2\cdot x_2}(\partial_2^2+m^2) \\
&\quad i\int d^4x_3\, e^{ip_3\cdot x_3}(\partial_3^2+m^2)i\int d^4x_4\, e^{ip_4\cdot x_4}(\partial_4^2+m^2)D_{13}D_{24} \\
&= i(2\pi)^4\delta^{(4)}(p_1-p_3)(-p_1^2+m^2)i(2\pi)^4\delta^{(4)}(p_2-p_4)(-p_2^2+m^2) \\
&= i(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)(i(2\pi)^4\delta^{(4)}(p_1-p_3)(-p_1^2+m^2)(-p_2^2+m^2)).
\end{aligned}
\tag{3.3.7}
$$

3.4 Mandelstam variables

-

Appendices