Quantum Field Theory

a study note based on A. Zee's textbook

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convention, notation, and units

- 笔记中的**度规号差**约定为 $(-,+,+,+)$.
- 使用 Planck units, 此时 $G,\hbar,c,k_B=1$, 因此,

name/dimension	expression/value
Planck length (L)	$l_P=\sqrt{\frac{\hbar G}{c^3}}=1.616\times10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P=\frac{l_P}{c}=5.391\times10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P=\sqrt{\frac{\hbar c}{G}}=2.176\times10^{-8}\,\mathrm{kg}\simeq10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P=\sqrt{\frac{\hbar c^5}{Gk_B^2}}=1.417\times10^{32}\,\mathrm{K}$

- 时空维度用 $d=D+1$ 表示.

Part I

motivation and foundation

Chapter 1

free field theory

1.1 partition function

- 考虑如下标量场,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{1.1.1}$$

A. Zee 说: 在作用量里, 时间的导数项必须是正的, 包括标量场的 $(\partial_0\phi)^2$ 和电磁场的 $(\partial_0 A_i)^2$.

- 含有 source function 的路径积分为,

$$Z(J) = \int D\phi\, e^{i\int d^dx\,(-\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))} \tag{1.1.2}$$

- 当 $V(\phi) = \frac{1}{2}m^2\phi^2$ 时, 称作 free or Gaussian theory.

- 计算 free theory 的 partition function, 得到,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)} \tag{1.1.3}$$

另外, 用 $W(J)$ 表示指数上的部分 (去除掉虚数 i).

proof:

注意 $\partial^\mu\phi\partial_\mu\phi = \partial^\mu(\phi\partial_\mu\phi) - \phi\partial^2\phi$, 忽略全微分项, 那么,

$$Z(J) = \int D\phi\, e^{i\int d^dx\,(\frac{1}{2}\phi(\partial^2 - m^2)\phi + J(x)\phi(x))} \tag{1.1.4}$$

代入 (B.1.1), 可知,

$$Z(J) = \mathcal{C}e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)} \tag{1.1.5}$$

其中 $D(x-y)$ 满足,

$$\begin{cases}(\partial^2 - m^2)D(x-y) = \delta^{(d)}(x-y) \\ (-p^2 - m^2)\tilde{D}(p,q) = (2\pi)^d\delta^{(d)}(p-q)\end{cases} \Longrightarrow D(x-y) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot(x-y)}}{-k^2-m^2} \tag{1.1.6}$$

1.2 free propagator

- 为了使 (1.1.4) 中的积分在 ϕ 较大时收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, 这样被积函数中会出现一项 $e^{-\epsilon\int d^dx\phi^2}$.
- 注意 (1.1.6) 中的积分会遇到奇点, 必须加入正无穷小量 ϵ 避免发散,

$$D(x) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot x}}{-k^2-m^2+i\epsilon} = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\left(e^{i(-\omega_k t + \vec{k}\cdot\vec{x})}\theta(t) + e^{i(\omega_k t + \vec{k}\cdot\vec{x})}\theta(-t)\right) \tag{1.2.1}$$

calculation:

对 k^0 积分, 注意有两个奇点 $k^0 = \pm(\omega_k - i\epsilon)$, 当 $t > 0$ 时, contour 处于下半平面, ...

- $D(x)$ 的取值与 x 的类时, 类空性质关系密切.
  - 类时区域,
$$D(t,0) = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\left(e^{-i\omega_k t}\theta(t) + e^{i\omega_k t}\theta(-t)\right) \tag{1.2.2}$$
  - 类空区域,
$$D(0,\vec{x}) = -i\int \frac{d^Dk}{(2\pi)^D 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \sim e^{-m|\vec{x}|} \tag{1.2.3}$$

1.3 from field to particle to force

1.3.1 from field to particle

- 考虑 (1.1.3) 中的 $W(J)$,
$$\begin{aligned} W(J) &= -\frac{1}{2}\int d^dx d^dy\, J(y)D(x-y)J(y) && (1.3.1)\\ &= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}(-k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}(k) && (1.3.2)\end{aligned}$$
  其中, 如果 $J(x)$ 是实函数, 那么 $\tilde{J}(-k) = \tilde{J}^*(k)$.

- 考虑 $J(x) = J_1(x) + J_2(x)$, 那么 $W(J)$ 共有 4 项, 其中一个交叉项如下,
$$W_{12}(J) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}_1(-k)\frac{1}{-k^2-m^2+i\epsilon}\tilde{J}_2(k) \tag{1.3.3}$$
  可见 $W(J)$ 取值较大的条件是:
  1. $\tilde{J}_1(k), \tilde{J}_2(k)$ 有较大重叠,
  2. 重叠位置的 k 是 on shell (即 $k^2 = -m^2$).
- 可以看出来, 这里有一个粒子从 1 传递到 2 (?).

1.3.2 from particle to force

- 考虑 $J(x) = \delta^{(D)}(\vec{x}-\vec{x}_1) + \delta^{(D)}(\vec{x}-\vec{x}_1) \Longrightarrow \tilde{J}_a(k) = 2\pi e^{-i\vec{k}\cdot\vec{x}_a}\delta(k^0)$, 那么,
$$\begin{aligned} W_{12}(J) + W_{21}(J) &= \delta(0)\int \frac{d^Dk}{(2\pi)^{D-1}}\frac{1}{|\vec{k}|^2+m^2-i\epsilon}\cos(\vec{k}\cdot(\vec{x}_1-\vec{x}_2)) \\ &\overset{D=3}{=} 2\pi\delta(0)\frac{1}{4\pi r}e^{-mr} && (1.3.4)\end{aligned}$$
  ($-i\epsilon$ 显然可以舍去), 注意到 $\langle 0|e^{-iHT}|0\rangle = e^{-iET}$, 而时间间隔 $T = \int dx^0 = 2\pi\delta(0)$, 所以,
$$E = -\frac{W(J)}{T} \overset{D=3}{=} -\frac{1}{4\pi r}e^{-mr} \tag{1.3.5}$$

calculation:

计算 (1.3.4) 中的积分, 令 $\vec{x}_1 - \vec{x}_2 = \vec{r}$,

$$I_D = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{|\vec{k}|^2+m^2}\overbrace{\cos(\vec{k}\cdot\vec{r})}^{\mapsto e^{i\vec{k}\cdot\vec{r}}}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^D} \int (k\sin\theta_1)^{D-2} d\Omega_{D-2} \int k d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ikr\cos\theta_1} \\
&= \frac{S_{D-2}}{(2\pi)^D} \int k^{D-1} \sin^{D-2}\theta_1 d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ikr\cos\theta_1} \qquad (1.3.6)
\end{aligned}
$$

取 $D = 3$, 那么,

$$
\begin{aligned}
I_{D=3} &= \frac{1}{(2\pi)^2} \int k^2 \sin\theta_1 d\theta_1 dk \, \frac{1}{k^2+m^2} e^{ik\cos\theta_1} \\
&= \frac{1}{2\pi^2 r} \int_0^\infty \sin(kr) \frac{kdk}{k^2+m^2} = \frac{-i}{4\pi^2 r} \int_{-\infty}^{\infty} e^{ikr} \frac{kdk}{k^2+m^2} \\
&= \frac{-i}{4\pi^2 r} 2\pi i \underbrace{\mathrm{Res}(f, im)}_{=\frac{1}{2}e^{-mr}} = \frac{1}{4\pi r} e^{-mr} \qquad (1.3.7)
\end{aligned}
$$

Chapter 2

Coulomb and Newton: repulsive and attraction

2.1 massive spin-1 particle & QED

- 构造有质量的光子的 Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{2.1.1}$$

其中 $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

- 做路径积分,

$$Z(J) = \int DA\, e^{i\int d^dx(\mathcal{L}+J_\mu A^\mu)} = \mathcal{C}e^{-\frac{i}{2}\int d^dx d^dy\, J_\mu D^{\mu\nu}(x-y)J_\nu(y)} \tag{2.1.2}$$

calculation:

massive photon 的作用量为,

$$\begin{aligned}S(A) &= \int d^dx\,\frac{1}{2}\Big(-(\partial_\mu A_\nu)(\partial^\mu A^\nu) + (\partial_\mu A_\nu)(\partial^\nu A^\mu) - m^2 A_\mu A^\mu\Big)\\ &= \int d^dx\,\frac{1}{2}\Big(A_\nu\partial^2 A^\nu - A_\nu\partial^\nu\partial_\mu A^\mu - m^2 A_\mu A^\mu\Big) + \text{total differential}\\ &= \int d^dx\,\frac{1}{2}A_\mu\Big(-\partial^\mu\partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2)\Big)A_\nu + \text{total differential}\\ &= \int \frac{d^dk}{(2\pi)^d}\tilde{A}_\mu(-k)\Big(k^\mu k^\nu + \eta^{\mu\nu}(-k^2 - m^2)\Big)\tilde{A}_\nu(k) + \text{boundary term}\end{aligned} \tag{2.1.3}$$

那么, 需要有,

$$\begin{aligned}&(-\partial^\mu\partial^\rho + \eta^{\mu\rho}(\partial^2 - m^2))D_{\rho\nu}(x-y) = \delta^\mu_\nu\delta^{(d)}(x-y)\\ \Longrightarrow &\tilde{D}_{\mu\nu}(k) = \frac{k_\mu k_\nu/m^2 + \eta_{\mu\nu}}{-k^2 - m^2}\end{aligned} \tag{2.1.4}$$

考虑到积分需要收敛, 作替换 $m^2 \mapsto m^2 - i\epsilon$, (为什么 A_μ 类空, 只知道 $\tilde{A}_\mu$ 类空, 见 subsection 2.1.2, 但路径积分中的 A 显然不满足 field equation $\Longrightarrow$ 路径积分中起主要作用的 $\tilde{A}$ 类空, 因此 $-\epsilon|\tilde{A}|^2 < 0$).

- 因此,

$$W(J) = -\frac{1}{2}\int d^dx d^dy\, J_\mu(x)D^{\mu\nu}(x-y)J_\nu(y) \tag{2.1.5}$$

$$= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}_\mu(-k)\frac{k^\mu k^\nu/m^2 + \eta^{\mu\nu}}{-k^2 - m^2 + i\epsilon}\tilde{J}_\nu(k) \tag{2.1.6}$$

注意到 current conservation, 有 $\partial_\mu J^\mu = 0 \iff k^\mu\tilde{J}_\mu(k) = 0$, 所以,

$$W(J) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\tilde{J}^\mu(-k)\frac{1}{-k^2 - m^2 + i\epsilon}\tilde{J}_\mu(k) \tag{2.1.7}$$

观察电荷分量, 可见同性相斥, 异性相吸.

2.1.1 spin & polarization vector

- spin-1 particle 可以有 3 个极化方向, 即空间的 x, y, z 方向, 在粒子静止系下, 极化矢量 $(\epsilon^i)_\mu = \delta^i_\mu, i = 1, 2, 3$, 而 $k_\mu = (-m, 0, 0, 0)$, 所以,
$$k^\mu (\epsilon^i)_\mu = 0 \tag{2.1.8}$$
  - 注意, 一个粒子的极化方向用 e^i (这不是矢量) 表示, 极化矢量为 $\sum_{i=1}^3 e^i (\epsilon^i)_\mu$.
- 在粒子静止系下, 考虑,
$$\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \frac{k_\mu k_\nu}{m^2} + \eta_{\mu\nu} := -G_{\mu\nu} \tag{2.1.9}$$
可见,
$$\tilde{D}_{\mu\nu}(k) = \frac{\sum_{i=1}^3 (\epsilon^i)_\mu (\epsilon^i)_\nu}{-k^2 - m^2 + i\epsilon} \tag{2.1.10}$$

2.1.2 Maxwell Lagrangian

- 根据 (2.1.1) 中的 Lagrangian density, 得到 field equation 如下,
$$\Big(-\partial^\mu \partial^\nu + \eta^{\mu\nu}(\partial^2 - m^2) \Big) A_\nu \tag{2.1.11}$$
  - spin-1 particle 有 3 个自旋自由度, 而 A_μ 有 4 个分量, 所以需要一个约束方程,
$$\partial^\mu A_\mu = 0 \iff k^\mu \tilde{A}_\mu(k) = 0 \tag{2.1.12}$$
  实际上在 (2.1.11) 左右两边作用一个 ∂_μ 即可得到这个约束方程.

2.2 massive spin-2 particle & gravity

- Lagrangian for spin-2 particle = linearized Einstein Lagrangian.
- 受 subsection 2.1.1 启发, 对于 spin-2 particle, 其极化矢量有 5 个方向, 满足,
$$\begin{cases} k^\mu (\epsilon^a)_{(\mu\nu)} = 0 \\ \eta^{\mu\nu} (\epsilon^a)_{(\mu\nu)} = 0 \end{cases} \tag{2.2.1}$$
其中下指标 μ, ν 对称, $a = 1, \cdots, 5$, (可以验证 $(\epsilon^a)_{\mu\nu}$ 确实有 5 个独立分量).
  - 对 $(\epsilon^a)_{\mu\nu}$ 的归一化条件可以定义为 $\sum_{a=1}^5 (\epsilon^a)_{12} (\epsilon^a)_{12} = 1$.
  - 与 subsection 2.1.1 中提示一样, 粒子的极化方向用 e^a 表示.
- 那么,
$$\sum_{a=1}^5 (\epsilon^a)_{\mu\nu} (\epsilon^a)_{\rho\sigma} = (G_{\mu\rho} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\rho}) - \frac{2}{3} G_{\mu\nu} G_{\rho\sigma} \tag{2.2.2}$$

calculation:

首先用 k_μ 和 $\eta_{\mu\nu}$ 构造最一般的关于 $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma, \mu\nu \leftrightarrow \rho\sigma$ 对称的 4 阶张量, (下式中把 $\frac{k_\mu}{m}$ 略写作 k_μ),
$$\begin{aligned} &A k_\mu k_\nu k_\rho k_\sigma + B(k_\mu k_\nu \eta_{\rho\sigma} + k_\rho k_\sigma \eta_{\mu\nu}) + C(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}) \\ &+ D \eta_{\mu\nu} \eta_{\rho\sigma} + E(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) \end{aligned} \tag{2.2.3}$$
代入 (2.2.1) 得,
$$\begin{cases} 0 = -A + B + 2C = -B + D = -C + E \\ 0 = -A + 4B + 4C = -B + 4D + 2E \end{cases} \implies \frac{B = D, C = E}{A} = -\frac{1}{2}, \frac{3}{4} \tag{2.2.4}$$

因此, 这个 4 阶张量最终确定为,

$$\frac{3}{4}A\Big((G_{\mu\rho}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\rho})-\frac{2}{3}G_{\mu\nu}G_{\rho\sigma}\Big) \tag{2.2.5}$$

- 所以,

$$\tilde{D}_{\mu\nu\rho\sigma}(k)=\frac{(G_{\mu\rho}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\rho})-\frac{2}{3}G_{\mu\nu}G_{\rho\sigma}}{-k^2-m^2+i\epsilon} \tag{2.2.6}$$

- 计算路径积分中的 $W(T)$,

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{(G^{\mu\rho}G^{\nu\sigma}+G^{\mu\sigma}G^{\nu\rho})-\frac{2}{3}G^{\mu\nu}G^{\rho\sigma}}{-k^2-m^2+i\epsilon}\tilde{T}_{\rho\sigma}(k) \tag{2.2.7}$$

注意到 $\partial_\mu T^{\mu\nu}(x)=0 \iff k_\mu\tilde{T}^{\mu\nu}(k)=0$, 并考虑到 T 是对称张量, 所以,

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{\mu\nu}(-k)\frac{2\eta^{\mu\rho}\eta^{\nu\sigma}-\frac{2}{3}\eta^{\mu\nu}\eta^{\rho\sigma}}{-k^2-m^2+i\epsilon}\tilde{T}_{\rho\sigma}(k) \tag{2.2.8}$$

考虑能量项, 可见质量互相吸引.

2.3 remarks

- 由于 seesaw mechanism (见 subsection C.1.1), 引入扰动一般会降低基态能量, 因此大多数相互作用表现为吸引, 而 spin-1 表现为同性相斥是因为 $\eta^{00}=-1$.
- 本 chapter 中的计算都是 $m\neq 0$ 的粒子, 与真实世界有差异.

Chapter 3

Feynman diagrams

3.1 a baby problem

- 考虑如下积分,

$$Z(J)=\int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2-\frac{\lambda}{4!}q^4+Jq} \tag{3.1.1}$$

- **Schwinger's way:** 把 integrand 对 λ 展开, 并将 q 用 $\frac{\partial}{\partial J}$ 替代, 得到,

$$\begin{aligned} Z(J) &= e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4}\int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2+Jq} \\ &= \sqrt{\frac{2\pi}{m^2}}e^{-\frac{\lambda}{4!}\left(\frac{\partial}{\partial J}\right)^4}e^{\frac{J^2}{2m^2}} \end{aligned} \tag{3.1.2}$$

  后面的计算中忽略 $Z(J=0,\lambda=0)$.

- 每个 vertex 带有 $-\lambda$, 每个 line 带有 $\frac{1}{m^2}$, 剩下的系数通过展开项算, 如下 (numerical factors 最好通过 Wick's way 算, 不过 baby problem 里 q 无法区分, 所以不方便算, 先略了),

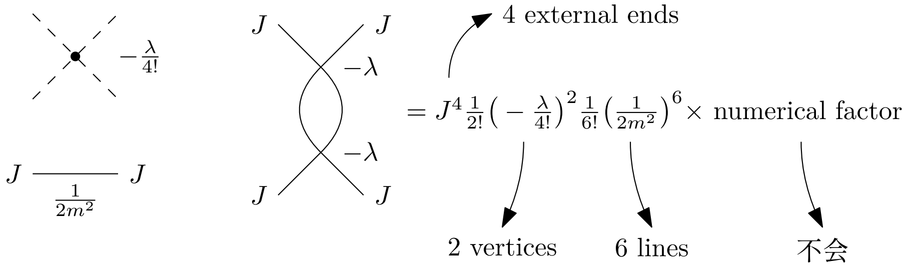
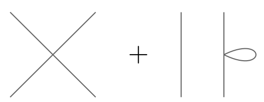


Figure 3.1: baby problem - Feynman diagram

calculation:

在这里计算 λJ^4 项,



具体暂时不会算 (3.1.3)

但是直接计算 (3.1.2) 的展开项, 得到的结果与 (3.1.5) 一样.

3.1.1 Wick contraction and Green's functions

- 把积分 (3.1.1) 对 J 展开,

$$Z(J)=\sum_{n=0}^{\infty}\frac{1}{n!}J^n\underbrace{\int_{-\infty}^{+\infty} dq\, e^{-\frac{1}{2}m^2q^2-\frac{\lambda}{4!}q^4}q^n}_{=Z(0,0)G^{(n)}} \tag{3.1.4}$$

  其中 Green's function $G^{(n)}$ 对 λ 展开后, 可以用 Wick contraction 计算 (见 (B.1.5)), 这就是 **Wick's way**.

calculation:

计算 λJ^4 项, 它来自 $G^{(4)}$ 对 λ 展开的一阶项,

$$\begin{aligned}-\frac{\lambda}{4!}\int dq\, e^{-\frac{1}{2}m^2q^2}q^8 &= -\frac{\lambda}{4!}\langle q^8\rangle \\ &= -\frac{\lambda}{4!}\sum_{\text{Wick}}\left(\frac{1}{m^2}\right)^4 \\ &= -\frac{\lambda}{4!}\frac{7\times5\times3\times1}{m^8}\end{aligned} \tag{3.1.5}$$

所以 λJ^4 项等于 $\frac{105}{(4!)^2}\frac{-\lambda J^4}{m^8}$.

3.1.2 connected vs. disconnected

- 考虑,

$$Z(J,\lambda) = Z(J=0,\lambda)e^{W(J,\lambda)} \tag{3.1.6}$$

其中 $Z(J=0,\lambda)$ 由 diagrams with no external source J 组成, 而 $W(J,\lambda)$ 则由 connected diagrams 组成 (?).

- 我们希望计算的是 W, 而不是 Z (?).

3.2 a child problem

- 考虑如下积分,

$$Z(J) = \int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q-\frac{\lambda}{4!}q^4+J^T\cdot q} \tag{3.2.1}$$

其中 $q^4=\sum_i q_i^4$.

- **Schwinger's way:** 对 λ 展开并把 q 替换为 $\frac{\partial}{\partial J}$, 得到,

$$Z(J) = \sqrt{\frac{(2\pi)^N}{\det A}}e^{-\frac{\lambda}{4!}(\frac{\partial}{\partial J})^4}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J} \tag{3.2.2}$$

其中 $\left(\frac{\partial}{\partial J}\right)^4=\sum_i\left(\frac{\partial}{\partial J_i}\right)^4$.

3.2.1 n-point Green's function

- **Wick's way:** 对 J 展开获得带 Green's function 的表达式,

$$Z(J) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}J_{i_1}\cdots J_{i_n}\underbrace{\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q-\frac{\lambda}{4!}q^4}q_{i_1}\cdots q_{i_n}}_{=Z(0,0)G^{(n)}_{i_1\cdots i_n}} \tag{3.2.3}$$

其中 $G^{(n)}_{i_1\cdots i_n}$ 称为 n-point Green's function.

Taylor expansion:

多元函数的 Taylor 展开如下,

$$\begin{aligned}f(x_1,\cdots,x_N) &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_N^{n_N}}{n_N!}\frac{\partial^{n_1}}{\partial x_1^{n_1}}\cdots\frac{\partial^{n_N}}{\partial x_N^{n_N}}f(x=0) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1=1}^{N}\cdots\sum_{i_n=1}^{N}x_{i_1}\cdots x_{i_n}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_N}}f(x=0)\end{aligned} \tag{3.2.4}$$

这两种求和方法, $x_1^{n_1}\cdots x_N^{n_N}$ 项的 numerical factor 都等于,

$$\frac{1}{n!}\times\frac{n!}{n_1!\cdots n_N!} = \frac{1}{n_1!\cdots n_N!} \tag{3.2.5}$$

其中 $n=n_1+\cdots+n_N$.

- 在 $\lambda=0$ 时, 2-point Green's function 就是 propagator,

$$G_{ij}^{(2)}(\lambda=0)=\frac{1}{Z(0,0)}\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q}q_iq_j$$
$$=\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}e^{\frac{1}{2}J^T\cdot A^{-1}\cdot J}\Big|_{J=0}=A_{ij}^{-1} \tag{3.2.6}$$

- 来计算 $2,3,4$-point Green's functions,

$$\begin{cases} G_{ij}^{(2)}=A_{ij}^{-1}-\frac{\lambda}{4!}\sum_m(3A_{mm}^{-1}A_{mm}^{-1}A_{ij}^{-1}+12A_{mm}^{-1}A_{mi}^{-1}A_{mj}^{-1})+O(\lambda^2) \\ G_{ijk}^{(3)}=0 \\ G_{ijkl}^{(4)}=A_{ij}^{-1}A_{kl}^{-1}+A_{ik}^{-1}A_{jl}^{-1}+A_{il}^{-1}A_{jk}^{-1} \\ \qquad -\frac{\lambda}{4!}\sum_m(A_{mm}^{-1}A_{mm}^{-1}A_{ij}^{-1}A_{kl}^{-1}+\cdots+4!A_{im}^{-1}A_{jm}^{-1}A_{km}^{-1}A_{lm}^{-1})+O(\lambda^2)\end{cases} \tag{3.2.7}$$

calculation:

2-point Green's function 计算如下,

$$G_{ij}^{(2)}=\frac{1}{Z(0,0)}\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q}\Big(1-\frac{\lambda}{4!}q^4+O(\lambda^2)\Big)q_iq_j$$
$$=A_{ij}^{-1}-\frac{\lambda}{4!}\langle q^4q_iq_j\rangle+O(\lambda^2)$$
$$=A_{ij}^{-1}-\frac{\lambda}{4!}\sum_m(3A_{mm}^{-1}A_{mm}^{-1}A_{ij}^{-1}+12A_{mm}^{-1}A_{mi}^{-1}A_{mj}^{-1})+O(\lambda^2) \tag{3.2.8}$$

3-point Green's function 计算如下,

$$G_{ijk}^{(32)}=\frac{1}{Z(0,0)}\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q}\Big(1-\frac{\lambda}{4!}q^4+O(\lambda^2)\Big)q_iq_jq_k=0 \tag{3.2.9}$$

4-point Green's function 计算如下,

$$G_{ijkl}^{(4)}=\frac{1}{Z(0,0)}\int dq_1\cdots dq_N\, e^{-\frac{1}{2}q^T\cdot A\cdot q}\Big(1-\frac{\lambda}{4!}q^4+O(\lambda^2)\Big)q_iq_jq_kq_l$$
$$=A_{ij}^{-1}A_{kl}^{-1}+A_{ik}^{-1}A_{jl}^{-1}+A_{il}^{-1}A_{jk}^{-1}-\frac{\lambda}{4!}\langle q^4q_iq_jq_kq_l\rangle+O(\lambda^2) \tag{3.2.10}$$

3.3 perturbative field theory

- 做如下替换即可,

$$\begin{cases} A\mapsto -i(\partial^2-m^2) \\ J\mapsto iJ\end{cases} \tag{3.3.1}$$

- **Schwinger's way:** ϕ^4 theory 的路径积分,

$$Z(J)=\int D\phi\, e^{i\int d^dx\,(\frac{1}{2}\phi(\partial^2-m^2)\phi-\frac{\lambda}{4!}\phi^4+J(x)\phi(x))} \tag{3.3.2}$$
$$=Z(0,0)e^{-i\frac{\lambda}{4!}\int d^dz\,(\frac{\delta}{i\delta J(z)})^4}e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)} \tag{3.3.3}$$

其中 $D(x-y)$ 是自由场的 propagator, 见 (1.2.1).

- **Wick's way:** 同样, 对 J 展开得到含 Green's functions 的表达式,

$$\frac{Z(J)}{Z(0,0)}=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^dx_1\cdots d^dx_n\,J(x_1)\cdots J(x_n)G^{(n)}(x_1,\cdots,x_n) \tag{3.3.4}$$

其中,

$$G^{(n)}(x_1,\cdots,x_n)=\frac{1}{Z(0,0)}\int D\phi\, e^{i\int d^dx\,(\frac{1}{2}\phi(\partial^2-m^2)\phi-\frac{\lambda}{4!}\phi^4)}\phi(x_1)\cdots\phi(x_n) \tag{3.3.5}$$

有时 $Z(J)$ 被称为 generating functional, 因为它能生成 Green's functions.

3.3.1 collision between particles

- 通过 Wick's way, 考虑 $J(x_1)J(x_2)J(x_3)J(x_4)$ 项, 实际上就是要计算 $G^{(4)}(x_1,x_2,x_3,x_4)$, 它的 0 阶项为,

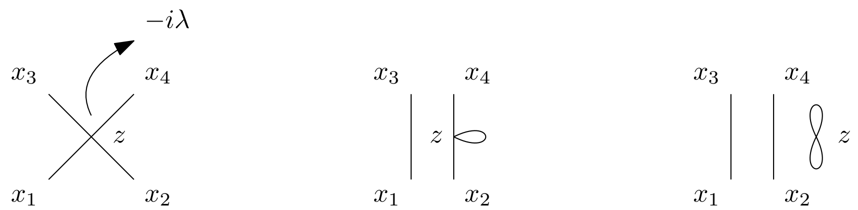
$$\begin{aligned}G^{(4)}(x_1,x_2,x_3,x_4,\lambda=0)&=\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}\frac{\delta}{i\delta J(x_3)}\frac{\delta}{i\delta J(x_4)}e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)}\\&=-(D_{12}D_{34}+D_{13}D_{24}+D_{14}D_{23})\end{aligned} \tag{3.3.6}$$

其中 D_{ij} 是 $D(x_i-x_j)$ 的简写, 可见, 传播子实际上是 $(-i)^3D=iD$.

- $G^{(4)}_{1234}$ 的 1 阶项为,

$$\begin{aligned}\text{1st order term}&=-\frac{i\lambda}{4!}\int d^dz\,\langle\phi_1\cdots\phi_4\phi^4(z)\rangle\\&=-\frac{i\lambda}{4!}\int d^dz\,\frac{\delta}{i\delta J_1}\cdots\frac{\delta}{i\delta J_4}\Big(\frac{\delta}{i\delta J(z)}\Big)^4e^{-\frac{i}{2}\int d^dxd^dy\,J(x)D(x-y)J(y)}\\&=-\frac{i\lambda}{4!}\int d^dz\,\Big(4!D_{1z}D_{2z}D_{3z}D_{4z}\\&\quad+4\times3D_{12}D_{3z}D_{4z}D_{zz}+\cdots+3D_{12}D_{34}D_{zz}D_{zz}+\cdots\Big)\end{aligned} \tag{3.3.7}$$

其中各项分别对应如下 Feynman diagrams,

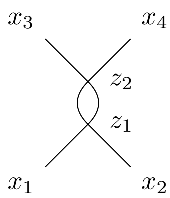


$-i\lambda\int d^dz\,D_{1z}D_{2z}D_{3z}D_{4z}$ $\quad\frac{-i\lambda}{2!}\int d^dzD_{13}D_{2z}D_{4z}D_{zz}$ $\quad\frac{-i\lambda}{8}\int d^dzD_{13}D_{24}D_{zz}D_{zz}$

Figure 3.2: position space - Feynman diagrams

其中 numerical factor 可以从 vertex 的四个 external end 的对称性得出.

- 再举一个例子,



$$=(4\times3)^2\times2\times\Big(\frac{-i\lambda}{4!}\Big)^2\int d^dz_1d^dz_2\,D_{1z_1}D_{2z_1}D_{3z_2}D_{4z_2}D_{z_1z_2}D_{z_1z_2} \tag{3.3.8}$$

3.3.2 in momentum space

- 本 subsection 将 (3.3.5) 转换到 momentum space, 注意到 $\tilde{J}(k)$ 和 $\tilde{J}(-k)$ 并不独立, 所以 $\frac{\partial}{\partial i\tilde{J}}$ 不适用. 最方便的办法是直接对 position space 下的结果做 Fourier transformation,

$$\tilde{G}^{(n)}(k_1,\cdots,k_n)=\int d^dx_1\cdots d^dx_n\,e^{-i(k_1\cdot x_1+\cdots)}G^{(n)}(x_1,\cdots,x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \cdots d^d x_n \, e^{-i(k_1 \cdot x_1 + \cdots)} \langle \Big(-\frac{i\lambda}{4!} \int d^d z \, \phi_z^4 \Big)^n \phi_1 \cdots \phi_n \rangle \tag{3.3.9}$$

– propagator 的 Fourier transformation 是,

$$\tilde{D}_{pq} = \int d^d x d^d y \, e^{-i(p\cdot x + q \cdot y)} D(x-y) = \frac{(2\pi)^d \delta^{(d)}(p+q)}{-p^2 - m^2 + i\epsilon} \tag{3.3.10}$$

但似乎没有用.

- $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$ 的 1 阶项为,

$$\text{1st order term} = -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 \, e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \, \langle \phi_z^4 \phi_1 \cdots \phi_4 \rangle \tag{3.3.11}$$

考虑第 1 项,

$$\begin{aligned} & -\frac{i\lambda}{4!} \int d^d x_1 \cdots d^d x_4 \, e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \, 4! D_{1z} \cdots D_{4z} \\ = & -i\lambda \int d^d x_1 \cdots d^d x_4 d^d z \, e^{-i(k_1 \cdot x_1 + \cdots)} e^{i(p_1 \cdot (x_1 - z) + \cdots)} \prod_{i=1}^{4} \int \frac{d^d p_i}{(2\pi)^d} \frac{1}{-p_i^2 - m^2 + i\epsilon} \\ = & -i\lambda \underbrace{\int d^d z \, e^{-iz\cdot(k_1 + \cdots + k_4)}}_{=(2\pi)^d \delta^{(d)}(k_1 + \cdots + k_4)} \prod_{i=1}^{4} \frac{1}{-k_i^2 - m^2 + i\epsilon} \end{aligned} \tag{3.3.12}$$

– 出射粒子不一定 on-shell (?).

- 得到这些 Feynman diagrams,

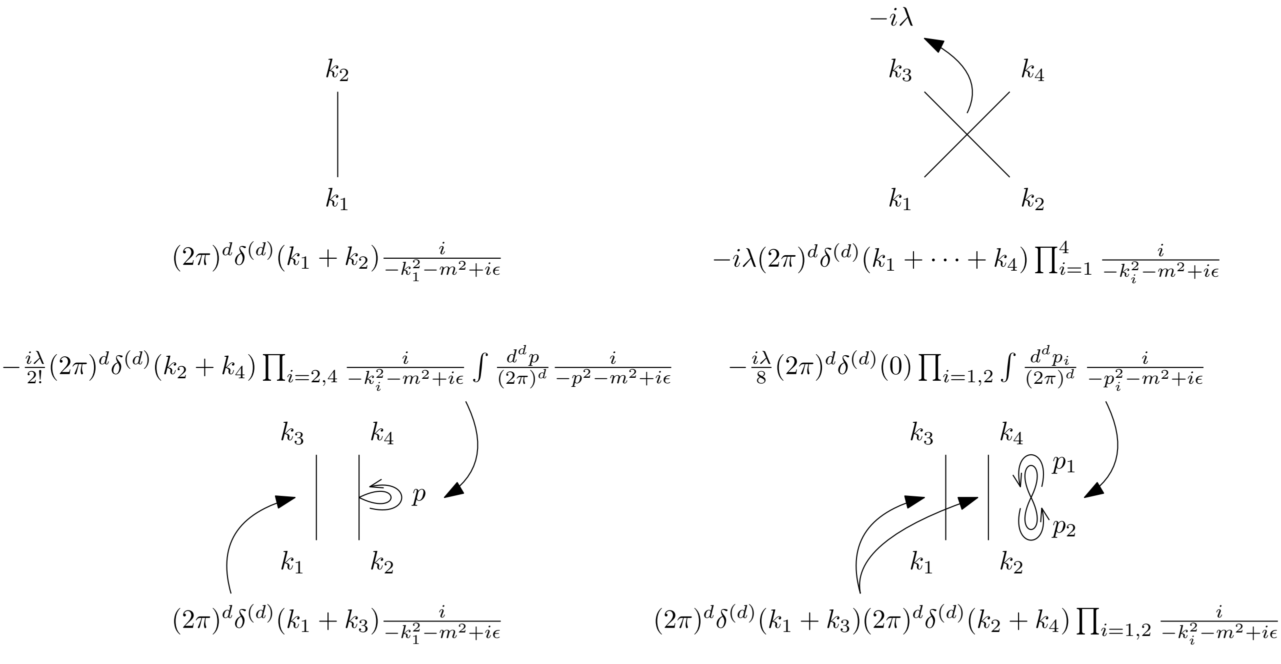


Figure 3.3: momentum space - Feynman diagrams

calculation:

第 3 幅图的计算如下,

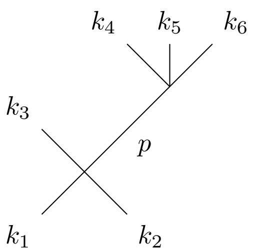
$$-\frac{i\lambda}{2!} \int d^d x_1 \cdots d^d x_4 \, e^{-i(k_1 \cdot x_1 + \cdots)} \int d^d z \, D_{13} D_{2z} D_{4z} D_{zz}$$

$$
\begin{aligned}
&= -\frac{i\lambda}{2!}\int d^dx_1\cdots d^dx_4 d^dz\, e^{-i(k_1\cdot x_1+\cdots)}e^{i(p_1\cdot(x_1-x_3)+p_2\cdot(x_2-z)+p_4\cdot(x_4-z)+p_4\cdot 0)} \\
&\quad \prod_{i=1}^{4}\int\frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{2!}\int d^dz\, e^{-iz\cdot(p_2+p_4)}\delta^{(d)}(p_1-k_1)\delta^{(d)}(p_2-k_2)\delta^{(d)}(p_1+k_3)\delta^{(d)}(p_4-k_4) \\
&\quad \prod_{i=1}^{4}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{2!}(2\pi)^d\delta^{(d)}(k_1+k_3)\delta^{(d)}(k_2+k_4)\prod_{i=1,2,4}\frac{1}{-k_i^2-m^2+i\epsilon}\int\frac{d^dp}{-p^2-m^2+i\epsilon}
\end{aligned}
\tag{3.3.13}
$$

第 4 幅图的计算如下,

$$
\begin{aligned}
&-\frac{i\lambda}{8}\int d^dx_1\cdots d^dx_4\, e^{-i(k_1\cdot x_1+\cdots)}\int d^dz\, D_{13}D_{24}D_{zz}D_{zz} \\
&= -\frac{i\lambda}{8}\int d^dx_1\cdots d^dx_4 d^dz\, e^{-i(k_1\cdot x_1+\cdots)}e^{i(p_1\cdot(x_1-x_3)+p_2\cdot(x_2-x_4)+p_3\cdot 0+p_4\cdot 0)} \\
&\quad \prod_{i=1}^{4}\int\frac{d^dp_i}{(2\pi)^d}\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{8}\int d^dz\,\delta^{(d)}(p_1-k_1)\delta^{(d)}(p_2-k_2)\delta^{(d)}(p_1+k_3)\delta^{(d)}(p_2+k_4) \\
&\quad \prod_{i=1}^{4}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon} \\
&= -\frac{i\lambda}{8}(2\pi)^d\delta^{(d)}(0)\delta^{(d)}(k_1+k_3)\delta^{(d)}(k_2+k_4)\prod_{i=1,2}\frac{1}{-k_i^2-m^2+i\epsilon} \\
&\quad \prod_{i=1,2}\int d^dp_i\frac{1}{-p_i^2-m^2+i\epsilon}
\end{aligned}
\tag{3.3.14}
$$

- 再举一个例子 (略去了 $\prod_{i=1}^{6}\frac{i}{-k_i^2-m^2+i\epsilon}$),



$$
\begin{aligned}
=&(4!)^2\times\Big(-\frac{i\lambda}{4!}\Big)^2(2\pi)^{2d}\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\delta^{(d)}(k_1+k_2+k_3+p)\delta^{(d)}(k_4+k_5+k_6-p) \\
=&(-i\lambda)^2(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4+k_5+k_6)\frac{i}{-(k_1+k_2+k_3)^2-m^2+i\epsilon}
\end{aligned}
\tag{3.3.15}
$$

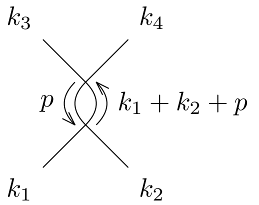
3.3.3 loops and a first look at divergence

- subsection 3.3.2 里的 loop diagrams 出现了如下积分,

$$
\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}=\int\frac{d^Dp}{(2\pi)^D2\omega_p}\sim\int\frac{d^Dp}{|p|}
\tag{3.3.16}
$$

积分发散.

- 再举一个例子 (略去了 $\prod_{i=1}^4 \frac{i}{-k_i^2-m^2+i\epsilon}$),



$$
\begin{aligned}
&=(4\times 3)^2\times 2\times\left(\frac{-i\lambda}{4!}\right)^2\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\int\frac{d^dq}{(2\pi)^d}\frac{i}{-q^2-m^2+i\epsilon}\\
&\quad(2\pi)^d\delta^{(d)}(k_1+k_2+p-q)(2\pi)^d\delta^{(d)}(k_3+k_4-p+q)\\
&=\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4)\int\frac{d^dp}{(2\pi)^d}\frac{i}{-p^2-m^2+i\epsilon}\frac{i}{-(k_1+k_2+p)^2-m^2+i\epsilon}\\
&=\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4)\int\frac{d^Dp}{(2\pi)^D}\Big(\frac{1}{2\omega_p}\frac{i}{(k_1^0+k_2^0-\omega_p)^2-\omega_{k_1+k_2+p}^2}\\
&\quad+\frac{i}{(\omega_{k_1+k_2+p}-k_1^0-k_2^0)^2-\omega_p^2}\frac{1}{2\omega_{k_1+k_2+p}}\Big) \qquad (3.3.17)\\
&\sim\int\frac{d^Dp}{p^3} \qquad (3.3.18)
\end{aligned}
$$

同样, 积分发散.

Chapter 4

canonical quantization

- A. Zee: the canonical and the path integral formalisms often appear complementary, in the sense that results difficult to see in one are clear in the other.

4.1 Heisenberg and Dirac

4.1.1 quantum mechanics

- 单粒子的 classical Lagrangian 为,

$$L = \frac{1}{2}\dot{q}^2 - V(q) \Longrightarrow \begin{cases} p = \dot{q} \\ H = p\dot{q} - L = \dfrac{1}{2}p^2 + V(q) \end{cases} \tag{4.1.1}$$

- canonical commutation relation 如下,

$$[p, q] = -i \tag{4.1.2}$$

因此, 算符的演化方程为,

$$\begin{cases} \dfrac{dp}{dt} = i[H, p] = -V'(q) \\ \dfrac{dq}{dt} = i[H, q] = p \end{cases} \tag{4.1.3}$$

calculation:

$$\begin{cases} [p, q] = -i \\ [p, q^2] = -2iq \\ \qquad\vdots \\ [p, q^n] = -iq^{n-1} + q[p, q^{n-1}] \end{cases} \Longrightarrow [p, q^n] = -inq^{n-1} \Longrightarrow [p, V(q)] = -iV'(q) \tag{4.1.4}$$

- follow Dirac's approach,

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \Longleftrightarrow \begin{cases} q = \dfrac{1}{\sqrt{2\omega}}(a + a^\dagger) \\ p = -i\sqrt{\dfrac{\omega}{2}}(a - a^\dagger) \end{cases} \Longrightarrow [a, a^\dagger] = 1 \tag{4.1.5}$$

算符 a 的演化方程为,

$$\frac{da}{dt} = -i\sqrt{\frac{\omega}{2}}\Big(\frac{1}{\omega}V'(q) + ip\Big) \tag{4.1.6}$$

4.1.2 scalar field

- 标量场的 Lagrangian 为,

$$L = \int d^D x \Big(-\frac{1}{2}((\partial\phi)^2 + m^2\phi^2) - u(\phi)\Big) \tag{4.1.7}$$

canonical commutation relation 为,

$$\pi(\vec{x},t) = \frac{\delta L(t)}{\delta \partial_0 \phi(\vec{x},t)} = \partial_0\phi(\vec{x},t) \quad \text{and} \quad [\pi(\vec{x},t), \phi(\vec{y},t)] = -i\delta^{(D)}(\vec{x}-\vec{y}) \tag{4.1.8}$$

标量场的 Hamiltonian 为,

$$H = \int d^D x \, (\pi\phi - \mathcal{L}) = \int d^D x \, \Big(\frac{1}{2}(\pi^2 + |\vec{\nabla}\phi|^2 + m^2\phi^2) + u(\phi)\Big) \tag{4.1.9}$$

- 算符的演化方程为,

$$\begin{cases} \partial_0\phi = i[H,\phi] = \pi \\ \partial_0\pi = i[H,\pi] = (-\vec{\nabla}^2 + m^2)\phi + \dfrac{du}{d\phi} \end{cases} \Longrightarrow (\partial^2 - m^2)\phi - \frac{du}{d\phi} = 0 \tag{4.1.10}$$

- 当 $u(\phi) = 0$ 时, 求解场方程 (4.1.10) 和 canonical commutation relation (4.1.8) 得到,

$$\phi(\vec{x},t) = \int \frac{d^D k}{(2\pi)^D 2\omega_k}(\alpha_k(t)e^{i\vec{k}\cdot\vec{x}} + \alpha_k^\dagger(t)e^{-i\vec{k}\cdot\vec{x}}) \tag{4.1.11}$$

其中,

$$\alpha_k(t) = \sqrt{(2\pi)^D 2\omega_k}\, a_{\vec{k}}e^{-i\omega_k t} \quad \text{and} \quad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = \delta^{(D)}(\vec{p}-\vec{q}) \tag{4.1.12}$$

另外, 在后面的笔记中使用简记 $\sqrt{(2\pi)^D 2\omega_k} = \rho(k)$.

calculation:

求解场方程 (4.1.10), 得到,

$$\phi(\vec{x},t) = \int \frac{d^D k}{(2\pi)^D}(\alpha_{\vec{k}}e^{i(-\omega_k t + \vec{k}\cdot\vec{x})} + \alpha_{\vec{k}}^\dagger e^{-i(-\omega_k t + \vec{k}\cdot\vec{x})}) \tag{4.1.13}$$

代入 canonical commutation relation (4.1.8), 有 (其中 $x^0 = y^0 = t, k^0 = \omega_k$),

$$\begin{aligned} &\int \frac{d^D k_2}{(2\pi)^D}\Big(-i\omega_{k_1}[\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}]e^{i(k_1\cdot x + k_2\cdot y)} + i\omega_{k_1}[\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}^\dagger]e^{-i(k_1\cdot x + k_2\cdot y)} \\ &- i\omega_{k_1}[\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger]e^{i(k_1\cdot x - k_2\cdot y)} + i\omega_{k_1}[\alpha_{\vec{k}_1}^\dagger, \alpha_{\vec{k}_2}]e^{-i(k_1\cdot x - k_2\cdot y)}\Big) = -ie^{i\vec{k}_1\cdot(\vec{x}-\vec{y})} \end{aligned}$$

$$\Longrightarrow \begin{cases} [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}] = \dfrac{1}{2\omega_{k_1}}\delta^{(D)}(\vec{k}_1 + \vec{k}_2) \Longrightarrow [\alpha_{\vec{k}}, \alpha_{\vec{k}}] \neq 0 \quad \text{wrong} \\ [\alpha_{\vec{k}_1}, \alpha_{\vec{k}_2}^\dagger] = \dfrac{1}{2\omega_{\vec{k}_1}}\delta^{(D)}(\vec{k}_1 - \vec{k}_2) \qquad\qquad\qquad\quad \text{right} \end{cases} \tag{4.1.14}$$

- 代入 (4.1.9) 可得 (依然是 $u(\phi) = 0$ 的情况下),

$$H = \int d^D k \, \omega_k \frac{a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^\dagger}{2} = \int d^D k \, \omega_k \Big(a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2}\delta^{(D)}(0)\Big) \tag{4.1.15}$$

- vacuum state 定义为 $a_{\vec{k}}|0\rangle = 0$, 有,

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^D k}{(2\pi)^D 2\omega_k}e^{ik\cdot(x-y)} \tag{4.1.16}$$

其中 $k^0 = \omega_k$. 因此, 对比 (1.2.1), 有,

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = iD(x-y) \tag{4.1.17}$$

4.2 interaction picture

- 注意, 在 $u(\phi) \neq 0$ 的情况下, (即便在 Schrödinger's picture 里, $t=0$ 时) (4.1.11) 不再成立, 因此无法通过 Schrödinger's picture or Heisenberg's picture 求解存在相互作用的场论.

- 将 Hamiltonian 分成两个部分,

$$H = H_0 + H' \tag{4.2.1}$$

- operators 以自由场的 Hamiltonian 演化,

$$O_I(t) = U_0^\dagger(t,0)O(0)U_0(t,0) \quad \text{where} \quad U_0(t_2,t_1) = \mathrm{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H_0\Big) \tag{4.2.2}$$

states 以如下方式演化,

$$|\psi(t)\rangle_I = U_0^\dagger(t,0)U(t,0)\,|\psi(0)\rangle \quad \text{where} \quad U(t_2,t_1) = \mathrm{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H\Big) \tag{4.2.3}$$

因此,

$$|\psi(t_2)\rangle_I = U_I(t_2,t_1)\,|\psi(t_1)\rangle_I \quad \text{where} \quad U_I(t_2,t_1) = \mathrm{Texp}\Big(-i\int_{t_1}^{t_2} dt\, H_I(t)\Big) \tag{4.2.4}$$

注意, (4.2.2) 和 (4.2.3) 中, Texp 里的 H, H_0 都是 Schrödinger's picture 里的算符.

calculation:

首先有,

$$U_I(t_2,t_1) = U_0^\dagger(t_2,0)U(t_2,t_1)U_0(t_1,0) \tag{4.2.5}$$

因此,

$$\begin{aligned}\frac{d}{dt}U_I(t,t_0) &= iH_0U_I(t,t_0) - iU_0^\dagger(t,0)HU(t,t_0)U_0(t_0,0) \\ &= -i\,\underbrace{U_0^\dagger(t,0)H'U_0(t,0)}_{=H_I(t)}\,U_I(t,t_0)\end{aligned} \tag{4.2.6}$$

4.3 scattering amplitude

- 最一般的过程是 $p_1,\cdots,p_m \to q_1,\cdots,q_n$, 其 scattering amplitude 为,

$$\langle q_1,\cdots,q_n|U_0^\dagger(-\infty,0)U_I(+\infty,-\infty)U_0(-\infty,0)|p_1,\cdots,p_m\rangle \tag{4.3.1}$$

一般会忽略掉 U_0 产生的相位.

- 考虑 ϕ^4 理论中的 $k_1,k_2 \to k_3,k_4$ 过程,

$$\langle k_3,k_4|e^{-i\int d^dx\,\frac{\lambda}{4!}\phi^4}|k_1,k_2\rangle \tag{4.3.2}$$

对 λ 展开, 0 阶项为,

$$\begin{aligned}\text{0th order term} &= \langle k_3,k_4|k_1,k_2\rangle \\ &= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\,\langle 0|a_{\vec k_3}a_{\vec k_4}a^\dagger_{\vec k_1}a^\dagger_{\vec k_2}|0\rangle \\ &= \rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\Big(\underbrace{\langle 0|a_{\vec k_3}a_{\vec k_4}a^\dagger_{\vec k_1}a^\dagger_{\vec k_2}|0\rangle}_{=\delta^{(D)}_{31}\delta^{(D)}_{42}} + \underbrace{\langle 0|a_{\vec k_3}a_{\vec k_4}a^\dagger_{\vec k_1}a^\dagger_{\vec k_2}|0\rangle}_{=\delta^{(D)}_{32}\delta^{(D)}_{41}}\Big) \\ &= (2\pi)^{2D}4\omega_{k_1}\omega_{k_2}(\delta^{(D)}(\vec k_1-\vec k_3)\delta^{(D)}(\vec k_2-\vec k_4)+\delta^{(D)}(\vec k_1-\vec k_4)\delta^{(D)}(\vec k_2-\vec k_3))\end{aligned} \tag{4.3.3}$$

1 阶项为 (其中 $k^0 = \omega_k$),

$$\text{1st order term} = \frac{-i\lambda}{4!}\int d^dx\,\langle k_3,k_4|\phi^4(x)|k_1,k_2\rangle$$

$$
\begin{aligned}
=&\overbrace{4!\times\frac{-i\lambda}{4!}\int d^dx\, e^{i(k_1+k_2-k_3-k_4)\cdot x}}^{=-i\lambda(2\pi)^d\delta^{(d)}(k_1+k_2-k_3-k_4)}+\rho(k_1)\rho(k_4)\delta^{(D)}_{14}\times 12\times\frac{-i\lambda}{4!}(2\pi)^d\delta^{(d)}_{23}\int\frac{d^Dp}{\rho^2(p)}\\
&+\cdots+\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)\delta^{(D)}_{13}\delta^{(D)}_{24}\times 3\times\frac{-i\lambda}{4!}\int d^dx\int\frac{d^Dp_1}{\rho^2(p_1)}\frac{d^Dp_2}{\rho^2(p_2)}+\cdots
\end{aligned}
\tag{4.3.4}
$$

分别对应如下 Feynman diagrams,

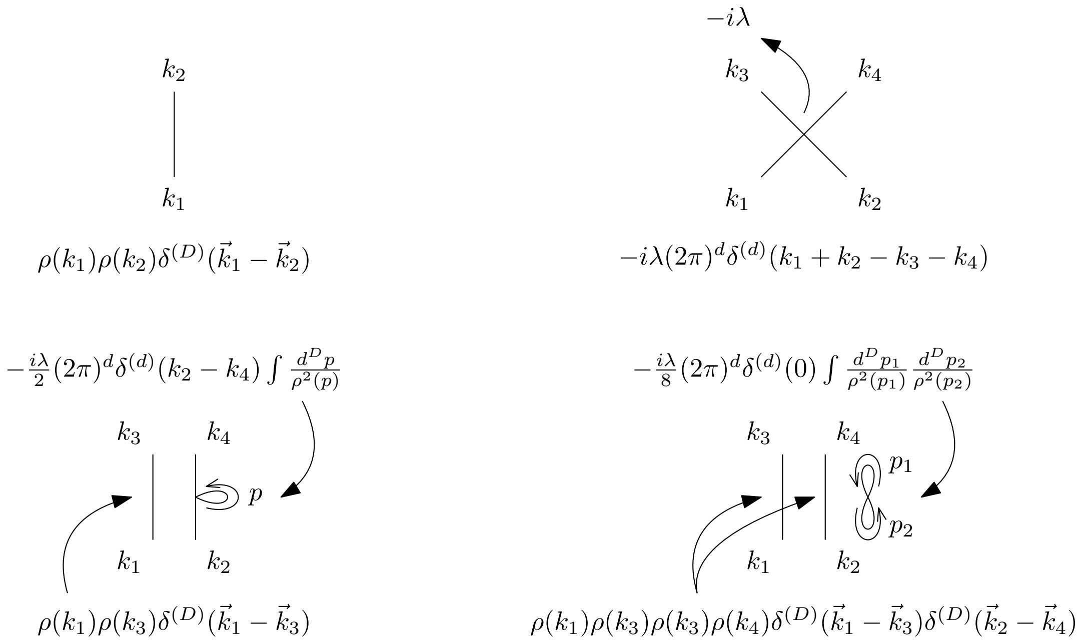
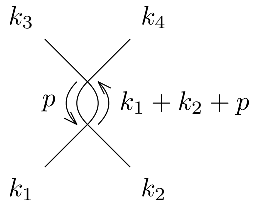


Figure 4.1: canonical quantization - Feynman diagrams

观察可见, 上图和 figure 3.3 有对应关系.

- 再举一个例子,



$$
\begin{aligned}
=&(4\times 3)^2\times 2\times\left(\frac{-i\lambda}{4!}\right)^2\rho(k_1)\cdots\int d^dx_1d^dx_2\int\frac{d^Dp_1\cdots}{\rho(p_1)\cdots}\frac{d^Dq_1\cdots}{\rho(q_1)\cdots}e^{i(p_1+p_2-p_3-p_4)\cdot x_1}e^{i(q_1+q_2-q_3-q_4)\cdot x_2}\\
&\left(\langle 0|a_{\vec{k}_3}a_{\vec{k}_4}a_{\vec{q}_1}a_{\vec{q}_2}a^\dagger_{\vec{q}_3}a^\dagger_{\vec{q}_4}a_{\vec{p}_1}a_{\vec{p}_2}a^\dagger_{\vec{p}_3}a^\dagger_{\vec{p}_4}a^\dagger_{\vec{k}_1}a^\dagger_{\vec{k}_2}|0\rangle+\cdots\right)\\
=&\frac{(-i\lambda)^2}{2}\int d^dx_1d^dx_2\int\frac{d^Dp_3}{\rho^2(p_3)}\frac{d^Dp_4}{\rho^2(p_4)}e^{i(k_1+k_2-p_3-p_4)\cdot x_1}e^{i(p_3+p_4-k_3-k_4)\cdot x_2}\\
=&\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2-k_3-k_4)\int\frac{d^Dp_3}{\rho^2(p_3)}\frac{d^Dp_4}{\rho^2(p_4)}(2\pi)^d\delta^{(d)}(k_1+k_2-p_3-p_4)\\
=&\frac{(-i\lambda)^2}{2}(2\pi)^d\delta^{(d)}(k_1+k_2-k_3-k_4)\int d^Dp\,\frac{(2\pi)^d\delta(\omega_{k_1}+\omega_{k_2}-\omega_p-\omega_{k_1+k_2-p})}{\rho^2(p)\rho^2(k_1+k_2-p)}
\end{aligned}
\tag{4.3.5}
$$

同样, 与 (3.3.17) 有对应关系 (?).

Appendices

Appendix A

Dirac delta function & Fourier transformation

A.1 Delta function

- 可以认为以下是定义式,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \iff \tilde{\delta}(k) = 1 = \int dx\, \delta(x) e^{-ikx} \tag{A.1.1}$$

- 第一个常用的公式,

$$\int_{-\infty}^{+\infty} \delta(f(x)) g(x) dx = \sum_{\{i, f(x_i)=0\}} \frac{g(x_i)}{|f'(x_i)|} \tag{A.1.2}$$

- 第二个常用的公式 (Sokhotski-Plemelj theorem),

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \tag{A.1.3}$$

其中 $\mathcal{P}$ 表示复函数的主值 (principal value).

proof:

考虑,

$$\frac{1}{x + i\epsilon} = \frac{x - i\epsilon}{x^2 + \epsilon^2} \quad \text{and} \quad \int \frac{\epsilon}{x^2 + \epsilon^2} dx = 2\pi i \operatorname{Res}(f, i\epsilon) = \pi \tag{A.1.4}$$

所以...

取 $\epsilon = 0.1$ 时, 复变函数的实部, 虚部分别如下,

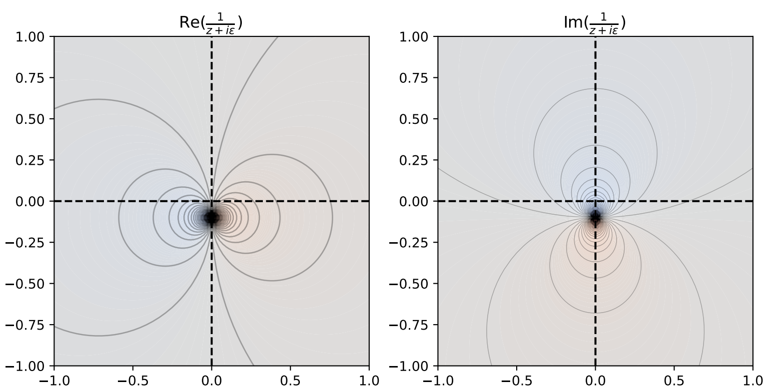


Figure A.1: graph of $\frac{1}{z+i\epsilon}$

- 另外, $\delta(x-a)\delta(x-b) = \delta(b-a)\delta(x-a)$.

A.2 Fourier transformation

- d-dim. Fourier transformation 如下,

$$\begin{cases} \phi(x) = \displaystyle\int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x}\tilde{\phi}(k) \\ \tilde{\phi}(k) = \displaystyle\int d^d x\, e^{-ik\cdot x}\phi(x) \end{cases} \tag{A.2.1}$$

- 因此,

$$\partial_\mu \phi(x) \mapsto ik_\mu \tilde{\phi}(k) \tag{A.2.2}$$

- 对于**实函数**, Fourier transformation 是正交变换, 其 Jacobi determinant 为,

$$\left|\frac{\partial\phi(x)\cdots}{\partial\mathrm{Re}\tilde{\phi}(k)\cdots\partial\mathrm{Im}\tilde{\phi}(k)\cdots}\right| = \left(\frac{2}{V}\right)^{(2N+1)^d}\det A = \left(\frac{2(2N)^d}{V^2}\right)^{\frac{(2N+1)^d}{2}} \tag{A.2.3}$$

proof:

position space 和 momentum space 的格点分别为,

$$\begin{cases} x_i^\mu = i^\mu\epsilon \in \{0, \pm\epsilon, \cdots, \dfrac{L}{2}\} \\ k_n^\mu = n^\mu \dfrac{2\pi}{L} \in \{0, \pm\dfrac{2\pi}{L}, \cdots, \dfrac{\pi}{\epsilon}\} \end{cases} \iff i^\mu, n^\mu \in \{0, \pm 1, \cdots, N\} \tag{A.2.4}$$

x^μ, k^μ 分别有 $2N+1$ 个取值, 其中 $N\epsilon = \frac{L}{2}$, 时空总体积为 $V = L^d$, momentum space 的总体积为 $\tilde{V} = \frac{(4\pi N)^d}{V}$.

将 (A.2.1) 写成格点求和的形式,

$$\begin{cases} \phi(x_i) = \dfrac{1}{(2\pi)^d}\left(\dfrac{2\pi}{L}\right)^d \displaystyle\sum_n e^{ik_n\cdot x_i}\tilde{\phi}(k_n) \\ \qquad = \dfrac{2}{V}\displaystyle\sum_{n^0>0}\Big(\cos(k_n\cdot x_i)\mathrm{Re}\tilde{\phi}(k_n) - \sin(k_n\cdot x_i)\mathrm{Im}\tilde{\phi}(k_n)\Big) \\ \tilde{\phi}(k_n) = \epsilon^d \displaystyle\sum_i e^{-ik_n\cdot x_i}\phi(x_i) \\ \qquad = \dfrac{V}{(2N)^d}\displaystyle\sum_i \Big(\cos(k_n\cdot x_i) - i\sin(k_n\cdot x_i)\Big)\phi(x_i) \end{cases} \tag{A.2.5}$$

- -

proof:
$\phi(x_i)$ 的变换需要做一些说明. 注意到 $\tilde{\phi}$ 的分量的数量是 ϕ 的两倍 (考虑到实部与虚部), 但在 $\phi \in \mathbb{R}^{(2N+1)^d}$ 时,

$$\tilde{\phi}^*(k) = \tilde{\phi}(-k) \tag{A.2.6}$$

可见 $\tilde{\phi}$ 的分量并不独立, 取 $k^0 > 0$ 的部分为独立分量, 那么...

- -

将 (A.2.5) 写成矩阵的形式,

$$\begin{cases} \begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} = \dfrac{2}{V}\overbrace{\begin{pmatrix} \cos k_0\cdot x_0 & \cdots & \cos k_{\max}\cdot x_0 & -\sin k_0\cdot x_0 & \cdots \\ \vdots & & \ddots & & \end{pmatrix}}^{=A}\begin{pmatrix} \mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \mathrm{Re}\tilde{\phi}(k_0) \\ \vdots \\ \mathrm{Im}\tilde{\phi}(k_0) \\ \vdots \end{pmatrix} = \dfrac{V}{(2N)^d}\begin{pmatrix} \cos k_0\cdot x_0 & \cdots & \cos k_0\cdot x_{\max} \\ \vdots & \ddots & \\ -\sin k_0\cdot x_0 & \cdots & -\sin k_0\cdot x_{\max} \\ \vdots & & \ddots \end{pmatrix}\begin{pmatrix} \phi(x_0) \\ \vdots \\ \phi(x_{\max}) \end{pmatrix} \end{cases} \tag{A.2.7}$$

观察可见 $\tilde{\phi}$ 的变换中的矩阵是 A^T, 所以,

$$\frac{2}{V}\frac{V}{(2N)^d}AA^T = I \Longrightarrow \det A = \left(\frac{(2N)^d}{2}\right)^{\frac{(2N+1)^d}{2}} \tag{A.2.8}$$

因此...

– 顺便,

$$\int d^dx\, f(x)g(x) = \int \frac{d^dk}{(2\pi)^d}\tilde{f}(-k)\tilde{g}(k) \tag{A.2.9}$$

Appendix B

Gaussian integrals

- 最基本的几个 Gaussian integral 如下,

$$\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \tag{B.0.1}$$

$$\langle x^{2n} \rangle = \frac{\int dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!! \tag{B.0.2}$$

其中 $(2n-1)!! = 1 \cdot 3 \cdots (2n-3)(2n-1)$.

- 一个重要的变体如下,

$$\int dx\, e^{-\frac{a}{2}x^2 + Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \tag{B.0.3}$$

另外, 将 a, J 分别替换为 $-ia, iJ$ 也是重要的变体.

B.1 N-dim. generalization

- 考虑如下积分,

$$Z(A,J) = \int dx_1 \cdots dx_N\, e^{-\frac{1}{2}x^T \cdot A \cdot x + J^T \cdot x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \tag{B.1.1}$$

其中 x, J 是 N-dim. 列向量, A 是 $N \times N$ 实对称矩阵.

calculation:

根据 spectral theorem for normal matrices (对称矩阵是厄密矩阵在实数域上的对应), 可知存在 orthogonal transformation 使得,

$$A = O^{-1} \cdot D \cdot O \tag{B.1.2}$$

其中 D 是一个 diagonal matrix. 令 $y = O \cdot x$, 那么,

$$\begin{aligned} Z(A,J) &= \int dy_1 \cdots dy_N\, e^{-\frac{1}{2}y^T \cdot D \cdot y + (OJ)^T \cdot y} \\ &= \prod_{i=1}^{N} \sqrt{\frac{2\pi}{D_{ii}}} e^{\frac{1}{2D_{ii}}(OJ)_i{}^2} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}J^T \cdot A^{-1} \cdot J} \end{aligned} \tag{B.1.3}$$

其中, 注意到了 $\frac{1}{D_{ii}} = (O \cdot A^{-1} \cdot O^{-1})_{ii}$ 以及 $\operatorname{tr} D = \det A$.

- 一个重要的变体是 $A \mapsto -iA, J \mapsto iJ$.
- 考虑 (B.0.2) 的变体, (注意 A 是对称的),

$$\langle x_i x_j \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} Z(A,J)\Big|_{J=0} = A^{-1}_{ij} \tag{B.1.4}$$

$$\langle x_i x_j \cdots x_k x_l \rangle = \sum_{\text{Wick}} A^{-1}_{i'j'} \cdots A^{-1}_{k'l'} \tag{B.1.5}$$

其中 (B.1.5) 中有偶数个 x, 否则等于零.

calculation:

$$\langle x_i x_j \cdots x_k x_l \rangle = \frac{1}{Z(A,0)} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \cdots \frac{\partial}{\partial J_k} \frac{\partial}{\partial J_l} Z(A,J)\Big|_{J=0} = \cdots \tag{B.1.6}$$

例如,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk} \tag{B.1.7}$$

其中, 可以用 Wick contraction 计算上式, 如下,

$$\langle x_i x_j x_k x_l \rangle = A^{-1}_{ik} A^{-1}_{jl} \tag{B.1.8}$$

Appendix C

perturbation theory in QM

- this chapter is based on MIT OpenCourseWare Quantum Physics III Chapter 1: Perturbation Theory.

- 研究的 Hamiltonian 与 well studied Hamiltonian 有微小差异时, 使用 perturbation theory,

$$H(\lambda) = H^{(0)} + \lambda \delta H \tag{C.0.1}$$

  其中 $\lambda \in [0,1]$.

- 考虑 $H^{(0)}$ 的本征态为,

$$H^{(0)} |k^{(0)}\rangle = E_k^{(0)} |k^{(0)}\rangle \quad \text{and} \quad \begin{cases} \langle k^{(0)}|l^{(0)}\rangle = \delta_{kl} \\ E_0^{(0)} \leq E_1^{(0)} \leq E_2^{(0)} \leq \cdots \end{cases} \tag{C.0.2}$$

C.1 non-degenerate perturbation theory

- 考虑 non-degenerate 能级 k, 有 $\cdots \leq E_{k-1}^{(0)} < E_k^{(0)} < E_{k+1}^{(0)} \leq \cdots$, 在 perturbation theory 适用的情况下,

$$\begin{cases} |k\rangle_\lambda = |k^{(0)}\rangle + \lambda |k^{(1)}\rangle + \lambda^2 |k^{(2)}\rangle + \cdots \\ E_k(\lambda) = E_k^{(0)} + \lambda E_k^{(1)} + \lambda^2 E_k^{(2)} + \cdots \end{cases} \tag{C.1.1}$$

  - 注意, 我们可以选取修正项满足,

$$\langle k^{(0)}|k^{(n)}\rangle = 0, n = 1, 2, \cdots \tag{C.1.2}$$

proof:

假设我们求解得到的修正项不满足 $\langle k^{(0)}|k^{(n)}\rangle = 0, n = 1, 2, \cdots$, 考虑,

$$|k^{(n)}\rangle' = |k^{(n)}\rangle + a_n |k^{(0)}\rangle \quad \text{with} \quad \langle k^{(0)}|k^{(n)}\rangle' = 0 \tag{C.1.3}$$

那么, (注意到态矢量可以乘一个常数, $\frac{1}{1-a_1\lambda-a_2\lambda^2-\cdots} = 1 + a_1\lambda + (a_1^2 + a_2)\lambda^2 + \cdots$),

$$\begin{aligned} |k\rangle_\lambda &= (1 - a_1\lambda - a_2\lambda^2 - \cdots) |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \cdots \\ |k\rangle'_\lambda &= |k^{(0)}\rangle + \frac{1}{1 - a_1\lambda - a_2\lambda^2 - \cdots} \left(\lambda |k^{(1)}\rangle' + \lambda^2 |k^{(2)}\rangle' + \cdots\right) \\ &= |k^{(0)}\rangle + \lambda |k^{(1)}\rangle' + \lambda^2(a_1 |k^{(1)}\rangle' + |k^{(2)}\rangle') + \cdots \end{aligned} \tag{C.1.4}$$

可见修正项都与 $|k^{(0)}\rangle$ 正交.

  - 注意, 不能要求 ${}_\lambda\langle k|k\rangle_\lambda = 1$, 否则 $|k^{(n)}\rangle$ 将与 λ 相关 (包括 $|k^{(0)}\rangle$),

$$\begin{aligned} {}_\lambda\langle k|k\rangle_\lambda = &\langle k^{(0)}|k^{(0)}\rangle \\ &+ \lambda(\langle k^{(1)}|k^{(0)}\rangle + \langle k^{(0)}|k^{(1)}\rangle) \\ &+ \lambda^2(\langle k^{(2)}|k^{(0)}\rangle + \langle k^{(1)}|k^{(1)}\rangle + \langle k^{(0)}|k^{(2)}\rangle) \end{aligned}$$

$$\vdots$$
$$+\lambda^n(\langle k^{(n)}|k^{(0)}\rangle + \langle k^{(n-1)}|k^{(1)}\rangle + \cdots + \langle k^{(0)}|k^{(n)}\rangle) \tag{C.1.5}$$

- 将 (C.1.1) 代入 Schrodinger's eq., 得到,

λ^0	$(H^{(0)} - E_k^{(0)})\,\|k^{(0)}\rangle = 0$
λ^1	$(H^{(0)} - E_k^{(0)})\,\|k^{(1)}\rangle = (E_k^{(1)} - \delta H)\,\|k^{(0)}\rangle$
λ^2	$(H^{(0)} - E_k^{(0)})\,\|k^{(2)}\rangle = (E_k^{(1)} - \delta H)\,\|k^{(1)}\rangle + E_k^{(2)}\,\|k^{(0)}\rangle$
$\vdots$	$\vdots$
λ^n	$(H^{(0)} - E_k^{(0)})\,\|k^{(n)}\rangle = (E_k^{(1)} - \delta H)\,\|k^{(n-1)}\rangle + E_k^{(2)}\,\|k^{(n-2)}\rangle + \cdots + E_k^{(n)}\,\|k^{(0)}\rangle$

calculation:

Schrodinger's eq. 为,

$$(H^{(0)} + \lambda\delta H - E_k(\lambda))\,|k\rangle_\lambda = 0 \tag{C.1.6}$$

展开为,

$$\Big((H^{(0)} - E_k^{(0)}) + \lambda(\delta H - E_k^{(1)}) - \lambda^2 E_k^{(2)} - \cdots\Big)(|k^{(0)}\rangle + \lambda\,|k^{(1)}\rangle + \lambda^2\,|k^{(2)}\rangle + \cdots) = 0 \tag{C.1.7}$$

- 现在来计算 $\langle l^{(0)}|k^{(n)}\rangle$, 有,

$$\begin{cases} (E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(1)}\rangle = E_k^{(1)}\delta_{lk} - \delta H_{lk} \\ (E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(2)}\rangle = E_k^{(1)}\,\langle l^{(0)}|k^{(1)}\rangle - \langle l^{(0)}|\delta H|k^{(1)}\rangle + E_k^{(2)}\delta_{lk} \\ \qquad\vdots \qquad\qquad\qquad \vdots \\ (E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(n)}\rangle = E_k^{(1)}\,\langle l^{(0)}|k^{(n-1)}\rangle - \langle l^{(0)}|\delta H|k^{(n-1)}\rangle \\ \qquad\qquad + E_k^{(2)}\,\langle l^{(0)}|k^{(n-2)}\rangle + \cdots + E_k^{(n)}\delta_{lk} \end{cases} \tag{C.1.8}$$

其中 $\delta H_{lk} = \langle l^{(0)}|\delta H|k^{(0)}\rangle$, 对于满足 (C.1.2) 的解, 有,

$$E_k^{(n)} = \langle k^{(0)}|\delta H|k^{(n-1)}\rangle\,, n = 1, 2, \cdots \tag{C.1.9}$$

并且,

$$|k^{(1)}\rangle = -\sum_{l\neq k}\frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}}\,|l^{(0)}\rangle \Longrightarrow E_k^{(2)} = -\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} \tag{C.1.10}$$

calculation:

将 (C.1.10) 代入 (C.1.8), 得到 $(l\neq k)$,

$$(E_l^{(0)} - E_k^{(0)})\,\langle l^{(0)}|k^{(2)}\rangle = -E_k^{(1)}\frac{\delta H_{lk}}{E_l^{(0)} - E_k^{(0)}} + \sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}} \tag{C.1.11}$$

所以,

$$\begin{cases} |k^{(2)}\rangle = \sum_{l\neq k}\Big(-\frac{\delta H_{00}\delta H_{lk}}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m\neq k}\frac{\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}\Big)\,|l^{(0)}\rangle \\ E_k^{(3)} = \sum_{l\neq k}\Big(-\frac{\delta H_{00}|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + \sum_{m\neq k}\frac{\delta H_{kl}\delta H_{lm}\delta H_{mk}}{E_m^{(0)} - E_k^{(0)}}\Big) \end{cases} \tag{C.1.12}$$

计算归一化系数,

$${}_\lambda\langle k|k\rangle_\lambda = 1 + \lambda^2\sum_{l\neq k}\frac{|\delta H_{lk}|^2}{(E_l^{(0)} - E_k^{(0)})^2} + O(\lambda^3) \tag{C.1.13}$$

C.1.1 level repulsion or the seesaw mechanism

- 能量的展开式为,

$$E_k(\lambda) = E_k^{(0)} + \lambda\delta H_{kk} - \lambda^2 \sum_{l\neq k} \frac{|\delta H_{lk}|^2}{E_l^{(0)} - E_k^{(0)}} + O(\lambda^3) \tag{C.1.14}$$

二阶项的效果是使能级间距增大, 对于基态能级, 二阶项使其能量减小.

C.1.2 validity of the perturbation expansion

- 考虑两能级系统, 可以得出微扰展开收敛的条件, 即,

$$|\lambda V| < \frac{1}{2}\Delta E^{(0)} \tag{C.1.15}$$

因此, 对于能级简并的情况, $\Delta E^{(0)} = 0$, 情况会更复杂.

calculation:

对于两能级系统,

$$H(\lambda) = H^{(0)} + \lambda\hat{V} = \begin{pmatrix} E_1^{(0)} & \lambda V \\ \lambda V^* & E_2^{(0)} \end{pmatrix} \tag{C.1.16}$$

$H(\lambda)$ 的本征值可以直接计算,

$$E_\pm(\lambda) = \frac{1}{2}(E_1^{(0)} + E_2^{(0)}) \pm \frac{1}{2}(E_1^{(0)} - E_2^{(0)})\sqrt{1 + \left(\frac{\lambda|V|}{\frac{1}{2}(E_1^{(0)} - E_2^{(0)})}\right)^2} \tag{C.1.17}$$

考虑 $\sqrt{1+z^2}$ 的 Taylor 展开,

$$\sqrt{1+z^2} = 1 + \frac{z^2}{2} - \frac{z^4}{8} + \cdots + (-1)^{n+1}\frac{(2n-3)!!}{2^n n!}z^{2n} + \cdots \tag{C.1.18}$$

注意到 $\sqrt{1+z^2}$ 在 $z = \pm i$ 有 branch cut, 因此 $z = 0$ 附件的 Taylor expansion 只有在 $|z| < 1$ 内才收敛.

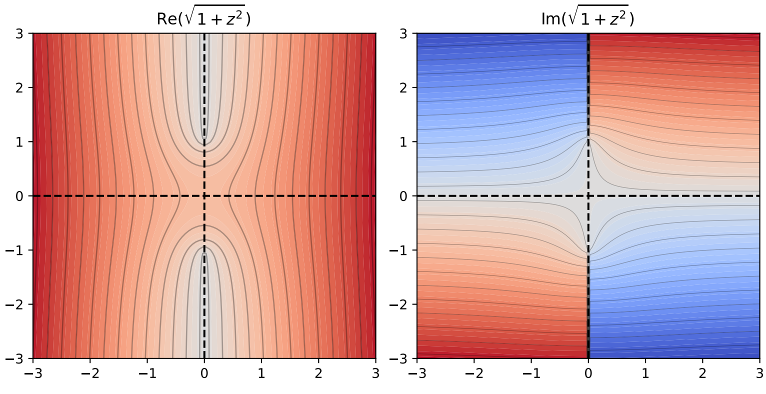


Figure C.1: graph of $\sqrt{1+z^2}$

C.2 degenerate perturbation theory

- 暂时先跳过.