Quantum Field Theory

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Contents

convention, notation, and units

- 笔记中的**度规号差**约定为 $(+,-,-,-)$.
- 使用 natural units, 此时 $\hbar, c, k_B = 1$, 因此 $1\,\mathrm{m} = \frac{1}{1.97\times10^{-16}\,\mathrm{GeV}}$ 且:

names/dimensions	expressions/values
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

Part I

Field Theory

Chapter 1

cross sections and decay rates

1.1 cross sections

- cross section 定义为
$$\sigma = \frac{1}{\Phi}\frac{P}{\Delta t}, \tag{1.1.1}$$
其中 $\Phi := nv = \frac{|\vec{v}_1 - \vec{v}_2|}{V}$ 是 incoming flux, 是入射粒子数密度乘粒子速度, P 是发生散射的概率.
- 实验上定义 luminosity 为
$$L\Delta t = \frac{dN}{d\sigma}, \tag{1.1.2}$$
其中 dN 是 $d\Omega$ 内发生散射的粒子数.
- 用 S-matrix elements 来表示 cross section, 有
$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}d\Pi, \tag{1.1.3}$$
其中 $d\Pi$ 是末态动量体元
$$d\Pi = \prod_i \delta^{(3)}(\vec{p}=0)d^3p_{f,i} = \prod_i \frac{V}{(2\pi)^3}d^3p_{f,i}, \tag{1.1.4}$$
这保证了无相互作用时 $\int dP = 1$.
- 对于初末态有
$$\begin{cases} \langle i|i\rangle = \langle p_1, p_2|p_1, p_2\rangle = (2\pi)^3 2\omega_{p_1}\delta^{(3)}(0)(2\pi)^3 2\omega_{p_2}\delta^{(3)}(0) = (2\omega_{p_1}V)(2\omega_{p_2}V) \\ \langle f|f\rangle = \prod_i (2\omega_{p_{f,i}}V) \end{cases}. \tag{1.1.5}$$

- 一般将 S-matrix 写为
$$S = I + i\mathcal{T}, \quad \mathcal{T} = (2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\mathcal{M}, \tag{1.1.6}$$
其中 $\mathcal{T}$ 称为 transfer matrix, 而 $\mathcal{M}$ 才是 S-matrix 的 non-trivial part. 有
$$\langle f|S - I|i\rangle = i(2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\langle f|\mathcal{M}|i\rangle. \tag{1.1.7}$$
- 对于 $|f\rangle \neq |i\rangle$ 的情况, 有
$$|\langle f|S|i\rangle|^2 = (2\pi)^4 TV\delta^{(4)}(\textstyle\sum_{i,f} p)|\langle f|\mathcal{M}|i\rangle|^2, \tag{1.1.8}$$
那么
$$dP = \frac{T}{V}\frac{1}{(2\omega_{p_1})(2\omega_{p_2})}|\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}, \tag{1.1.9}$$
其中 LIPS 表示 Lorentz-invariant phase space,
$$d\Pi_{\text{LIPS}} = (2\pi)^4\delta^{(4)}(\textstyle\sum_{i,f} p)\displaystyle\prod_i \frac{d^3p_{f,i}}{(2\pi)^3 2\omega_{p_{f,i}}}. \tag{1.1.10}$$
- 最终有 (将 (1.1.1) 中的 Δt 替换为 T)
$$d\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2|(2\omega_{p_1})(2\omega_{p_2})}|\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}. \tag{1.1.11}$$

1.2 decay rates

- decay rate, Γ, 是粒子单位时间发生衰变的概率,

$$d\Gamma = \frac{dP}{T}. \tag{1.2.1}$$

- 因为 $|f\rangle \neq |i\rangle$, 有

$$d\Gamma = \frac{1}{2\omega_p} |\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}. \tag{1.2.2}$$

Chapter 2

the S-matrix and time-ordered products

2.1 the LSZ reduction formula

- S-matrix element 为

$$\begin{cases} |i\rangle = \sqrt{(2\pi)^3 2\omega_{p_1}}\sqrt{(2\pi)^3 2\omega_{p_2}} a^\dagger_{\vec{p}_1}(-\infty) a^\dagger_{\vec{p}_2}(-\infty)\,|\Omega\rangle \\ |f\rangle = \sqrt{(2\pi)^3 2\omega_{p_3}}\cdots\sqrt{(2\pi)^3 2\omega_{p_n}} a^\dagger_{\vec{p}_3}(+\infty)\cdots a^\dagger_{\vec{p}_n}(+\infty)\,|\Omega\rangle \\ \langle f|S|i\rangle = (2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega|a_{\vec{p}_3}(+\infty)\cdots a_{\vec{p}_n}(+\infty)a^\dagger_{\vec{p}_1}(-\infty)a^\dagger_{\vec{p}_2}(-\infty)|\Omega\rangle \end{cases} . \tag{2.1.1}$$

remark:

注意到

$$\begin{cases} a_{\vec{p}}(t) = U^\dagger(t) a_{\vec{p}}(0) U(t) \\ |\vec{p}(t)\rangle = U(t)\,|\vec{p}(0)\rangle \end{cases} \Longrightarrow |\vec{p}(t)\rangle = a^\dagger_{\vec{p}}(-t)\,|\Omega\rangle\,, \tag{2.1.2}$$

那么, 对于初末态, 有

$$\begin{aligned} &|i(-\infty)\rangle = \sqrt{(2\pi)^3 2\omega_{p_1}}\sqrt{(2\pi)^3 2\omega_{p_2}} a^\dagger_{\vec{p}_1}(0) a^\dagger_{\vec{p}_2}(0)\,|\Omega\rangle \\ \Longrightarrow\ &|i(0)\rangle = \cdots\,, \end{aligned} \tag{2.1.3}$$

可见 (2.1.1) 中的 $|i\rangle\,,|f\rangle$ 都是 $t=0$ 即 Heisenberg picture 中的形式.

- LSZ 需要用到以下公式 (其中 p is on-shell),

$$i\int d^4x\, e^{ip\cdot x}(\partial^2+m^2)\phi(x) = \sqrt{(2\pi)^3 2\omega_p}\left(e^{i\omega_p t} a_{\vec{p}}(t)\right)\Big|_{-\infty}^{\infty}, \tag{2.1.4}$$

注意到, 对于 free field, 等式两边等于零.

proof:

注意到

$$\begin{aligned} \int d^3x\, e^{ip\cdot x}(\partial^2+m^2)\phi(x) &= \int d^3x\, e^{ip\cdot x}(\partial_t^2 - \nabla^2 + m^2)\phi(x) \\ &= \int d^3x\,\left(e^{ip\cdot x}(\partial_t^2+m^2)\phi(x) - \phi(x)\nabla^2 e^{ip\cdot x}\right) \\ &= \int d^3x\, e^{ip\cdot x}(\partial_t^2+\omega_p^2)\phi(x) \\ &= \int d^3x\,\partial_t(e^{ip\cdot x}\dot\phi - i\omega_p e^{ip\cdot x}\phi) \end{aligned} \tag{2.1.5}$$

$$= \frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}}\partial_t\big(e^{i\omega_p t}(\partial_t - i\omega_p)(a_{\vec{p}}(t) + a^\dagger_{-\vec{p}}(t))\big) \tag{2.1.6}$$

其中注意到了

$$e^{i\omega_p t}(\partial_t^2 + \omega_p^2)O(t) = \partial_t(e^{i\omega_p t}(\partial_t - i\omega_p)O(t)), \tag{2.1.7}$$

因此

$$\text{LHS} = i\frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}}(e^{i\omega_p t}(\partial_t - i\omega_p)(a_{\vec{p}}(t) + a^\dagger_{-\vec{p}}(t)))\Big|_{-\infty}^{\infty}, \tag{2.1.8}$$

注意到 $t = \pm\infty$ 时, fields are free, 所以

$$\begin{cases} \lim\limits_{t\to\pm\infty} a_{\vec{p}}(t) \propto e^{-i\omega_p t}a_{\vec{p}} \\ \lim\limits_{t\to\pm\infty} a^\dagger_{-\vec{p}}(t) \propto e^{i\omega_p t}a^\dagger_{-\vec{p}} \end{cases} \Longrightarrow \begin{cases} \lim\limits_{t\to\pm\infty}(\partial_t - i\omega_p)a_{\vec{p}}(t) = -2i\omega_p a_{\vec{p}}(t) \\ \lim\limits_{t\to\pm\infty}(\partial_t - i\omega_p)a^\dagger_{-\vec{p}}(t) = 0 \end{cases}, \tag{2.1.9}$$

代入得到

$$\text{LHS} = \sqrt{(2\pi)^3 2\omega_p}(e^{i\omega_p t}a_{\vec{p}}(t))\Big|_{-\infty}^{\infty}. \tag{2.1.10}$$

- 考虑 $p_1 \neq p_2 \neq \cdots \neq p_n$ 的情况.
- 考虑

$$\begin{aligned} &\frac{d}{dt}T\{A_1(t_1)\cdots A_{n-1}(t_{n-1})B(t)\} \\ =&T\{A_1(t_1)\cdots A_n(t_n)\frac{dB(t)}{dt}\} + \sum_{i=1}^n \delta(t-t_i)T\{A_1(t_1)\cdots[B(t_i), A_i(t_i)]\cdots A_n(t_n)\}, \end{aligned} \tag{2.1.11}$$

那么

$$\begin{aligned} &\partial_{t_i}^2 T\{\phi_1\cdots\phi_n\} \\ =&\partial_{t_i}T\{\cdots\partial_{t_i}\phi_i\cdots\} + \sum_{j\neq i}\delta(\cdots)T\{\cdots\underbrace{[\phi(t_j,\vec{x}_i), \phi(t_j,\vec{x}_j)]}_{=0}\cdots\} \\ =&T\{\cdots\partial_{t_i}^2\phi_i\cdots\} + \sum_{j\neq i}\delta(t_j - t_i)T\{\cdots[\dot{\phi}(t_j,\vec{x}_i), \phi(t_j,\vec{x}_j)]\cdots\} \\ =&T\{\cdots\partial_{t_i}^2\phi_i\cdots\} - i\sum_{j\neq i}\delta^{(4)}(x_j - x_i)T\{\cdots\phi_{j-1}\phi_{j+1}\cdots\}, \end{aligned} \tag{2.1.12}$$

所以

$$\begin{aligned} &i\int d^4x_i\, e^{ip_i\cdot x_i}(\partial_i^2 + m^2)T\{\phi_1\cdots\phi_n\} \\ =&i\int d^4x_i\, e^{ip_i\cdot x_i}\Big(T\{\cdots(\partial_i^2 + m^2)\phi_i\cdots\} - i\sum_{j\neq i}\delta^{(4)}(x_j - x_i)T\{\cdots\phi_{j-1}\phi_{j+1}\cdots\}\Big) \\ =&i\int dt_i\, T\{\cdots\int d^3x_i\,\partial_{t_i}(e^{ip_i\cdot x_i}\dot{\phi}_i - i\omega_{p_i}e^{ip_i\cdot x_i}\phi_i)\cdots\} \\ &+ \sum_{j\neq i}e^{ip_i\cdot x_j}T\{\cdots\phi_{j-1}\phi_{j+1}\cdots\} \\ =&i\int dt_i\,\Big(\partial_{t_i}T\{\cdots\int d^3x_i\,(e^{ip_i\cdot x_i}\dot{\phi}_i - i\omega_{p_i}e^{ip_i\cdot x_i}\phi_i)\cdots\} \\ &- \sum_{j\neq i}\delta(t_i - t_j)T\{\cdots\underbrace{[\int d^3x_i\, e^{i(\omega_{p_i}\cdot t_j - \vec{p}_i\cdot\vec{x}_i)}(\dot{\phi}(t_j,\vec{x}_i) - i\omega_{p_i}\phi(t_j,\vec{x}_i)), \phi_j]}_{=-ie^{ip_i\cdot x_j}}\cdots\}\Big) \\ &+ \sum_{j\neq i}e^{ip_i\cdot x_j}T\{\cdots\phi_{j-1}\phi_{j+1}\cdots\} \\ =&i\int dt_i\,\partial_{t_i}T\{\cdots\int d^3x_i\,(e^{ip_i\cdot x_i}\dot{\phi}_i - i\omega_{p_i}e^{ip_i\cdot x_i}\phi_i)\cdots\} \end{aligned} \tag{2.1.13}$$

$$
\begin{aligned}
=&(2\pi)^{3/2}\sqrt{2\omega_{p_i}}\Big(e^{i\omega_{p_i}(+\infty)}a_{\vec{p}_i}(+\infty)T\{\cdots\phi_{i-1}\phi_{i+1}\cdots\}\\
&-T\{\cdots\phi_{i-1}\phi_{i+1}\cdots\}e^{i\omega_{p_i}(-\infty)}a_{\vec{p}_i}(-\infty)\Big),
\end{aligned}
\tag{2.1.14}
$$

其中, (2.1.13) 将对易子抵消掉是很重要的一步.

remark:

注意 (2.1.13) 中用到的对易关系

$$
[\underbrace{\int d^3y\, e^{i(\omega_p t-\vec{p}\cdot\vec{y})}(\dot{\phi}(t,\vec{y})-i\omega_p\phi(t,\vec{y}))}_{\overset{t\to\pm\infty}{=}-i(2\pi)^{3/2}\sqrt{2\omega_p}e^{i\omega_p t}a_{\vec{p}}(t)},\phi(t,\vec{x})]=-ie^{ip\cdot x},
\tag{2.1.15}
$$

对比

$$
[-i(2\pi)^{3/2}\sqrt{2\omega_p}e^{i\omega_p t}a_{\vec{p}}(t),\phi(x)]=-ie^{i\omega_p t}U^\dagger(t)[a_{\vec{p}},\phi(0,\vec{x})]U(t)=-ie^{ip\cdot x}.
\tag{2.1.16}
$$

- 那么 (其中 $p_1,\cdots,p_m$ 是入射粒子, $p_{m+1},\cdots,p_n$ 是出射粒子)

$$
\begin{aligned}
&\left(i\int d^4x_1\, e^{ip_1\cdot x_1}(\partial_1^2+m^2)\right)\cdots\left(i\int d^4x_n\, e^{-ip_n\cdot x_n}(\partial_n^2+m^2)\right)\langle\Omega|T\{\phi_1\cdots\phi_n\}|\Omega\rangle\\
=&(2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega|e^{i\omega_{p_1}(+\infty)}a_{\vec{p}_1}(+\infty)\cdots e^{-i\omega_{p_n}(-\infty)}a^\dagger_{\vec{p}_n}(-\infty)|\Omega\rangle\\
&-(2\pi)^{3n/2}\sqrt{2\omega_{p_1}\cdots 2\omega_{p_n}}\,\langle\Omega|a_{\vec{p}_1}(+\infty)\cdots a^\dagger_{\vec{p}_n}(+\infty)|\Omega\rangle\\
=&\langle f|S-I|i\rangle\,.
\end{aligned}
\tag{2.1.17}
$$

2.2 LSZ for operators

- $|\Omega\rangle$ 的具体形式见 subsection 3.2.1.
- 考虑

$$
\phi(x)=e^{iP\cdot x}\phi(0)e^{-iP\cdot x},
\tag{2.2.1}
$$

且 $P|\Omega\rangle=0$, 因此

$$
\begin{cases}
\langle\Omega|\phi(x)|\Omega\rangle=\langle\Omega|\phi(0)|\Omega\rangle\\
\langle p|\phi(x)|\Omega\rangle=e^{ip\cdot x}\langle p|\phi(0)|\Omega\rangle
\end{cases}.
\tag{2.2.2}
$$

- LSZ reduction formula 的前提是场算符满足

$$
\begin{cases}
\langle\Omega|\phi(x)|\Omega\rangle=0\\
\langle p|\phi(x)|\Omega\rangle=e^{ip\cdot x}
\end{cases}.
\tag{2.2.3}
$$

remark:

前提是场算符具有以下形式,

$$
\phi(x)=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(a_{\vec{p}}(t)e^{i\vec{p}\cdot\vec{x}}+a^\dagger_{\vec{p}}(t)e^{-i\vec{p}\cdot\vec{x}}),
\tag{2.2.4}
$$

和 $a^\dagger_{\vec{p}}(\pm\infty)|\Omega\rangle\in\mathrm{span}(|p\rangle)$ 是一个单粒子态, 且 $\langle\Omega|a^\dagger_{\vec{p}}(+\infty)=a_{\vec{p}}(-\infty)|\Omega\rangle=0$.

- 注意, 在 Heisenberg picture 中,

$$
\phi(t=0,\vec{x})=\phi_0(\vec{x})=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}+a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}).
\tag{2.2.5}
$$

- (2.1.9) 和 $H|\Omega\rangle=0$ (要求 H 不含时) 矛盾 (?), 因为 LSZ 考虑的 Hamiltonian 是

$$
H=H_0+V\theta(t-T)\theta(t+T),T\gg 0,
\tag{2.2.6}
$$

所以 (2.2.3) 只在 $-T<t<T$ 成立.

Chapter 3

Feynman rules

3.1 Lagrangian derivation

- 假设存在相互作用时, 场算符依然满足

$$\begin{cases}[\phi(t,\vec{x}),\phi(t,\vec{y})]=0\\ [\phi(t,\vec{x}),\partial_t\phi(t,\vec{y})]=i\delta^{(3)}(\vec{x}-\vec{y})\end{cases}, \tag{3.1.1}$$

  即 causality and canonical commutation relation.

- 一个重要的中间公式为

$$(\partial_x^2+m^2)\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle=\langle\Omega|T\{(\partial_x^2+m^2)\phi(x)\phi(y)\}|\Omega\rangle-i\delta^{(4)}(x-y). \tag{3.1.2}$$

proof:

考虑

$$\begin{aligned}\partial_t\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle=&\partial_t(\langle\Omega|\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(y)\phi(x)|\Omega\rangle\theta(t'-t))\\ =&\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(x)\phi(y)|\Omega\rangle\delta(t-t')\\ &+\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\theta(t'-t)-\langle\Omega|\phi(y)\phi(x)|\Omega\rangle\delta(t'-t)\\ =&\langle\Omega|T\{\partial_t\phi(x)\phi(y)\}|\Omega\rangle+\langle\Omega|[\phi(t,\vec{x}),\phi(t,\vec{y})]|\Omega\rangle\delta(t-t')\\ =&\langle\Omega|T\{\partial_t\phi(x)\phi(y)\}|\Omega\rangle,\end{aligned} \tag{3.1.3}$$

那么

$$\begin{aligned}\partial_t^2\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle=&\partial_t\langle\Omega|T\{\partial_t\phi(x)\phi(y)\}|\Omega\rangle\\ =&\partial_t(\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\theta(t'-t))\\ =&\langle\Omega|\partial_t^2\phi(x)\phi(y)|\Omega\rangle\theta(t-t')+\langle\Omega|\partial_t\phi(x)\phi(y)|\Omega\rangle\delta(t-t')\\ &+\langle\Omega|\phi(y)\partial_t^2\phi(x)|\Omega\rangle\theta(t'-t)-\langle\Omega|\phi(y)\partial_t\phi(x)|\Omega\rangle\delta(t'-t)\\ =&\langle\Omega|T\{\partial_t^2\phi(x)\phi(y)\}|\Omega\rangle+\langle\Omega|[\partial_t\phi(t,\vec{x}),\phi(t,\vec{y})]|\Omega\rangle\delta(t-t')\\ =&\langle\Omega|T\{\partial_t^2\phi(x)\phi(y)\}|\Omega\rangle-i\delta^{(4)}(x-y).\end{aligned} \tag{3.1.4}$$

- (3.1.2) 可以推广为 (其中 ϕ_i 是 $\phi(x_i)$ 的简写)

$$\begin{aligned}&(\partial_1^2+m^2)\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle\\ =&\langle\Omega|T\{(\partial_1^2+m^2)\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle-i\sum_{i=2}^n\delta^{(4)}(x_i-x_1)\langle\Omega|\phi_2\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_n|\Omega\rangle.\end{aligned} \tag{3.1.5}$$

proof:

首先

$$\partial_{t_1}\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle=\langle\Omega|T\{\partial_{t_1}\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle, \tag{3.1.6}$$

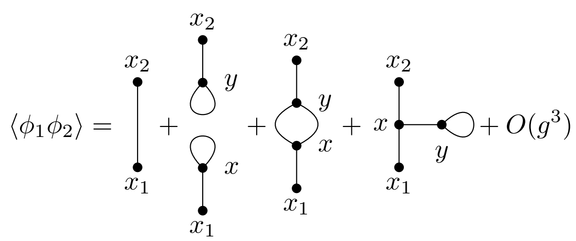
那么

$$\partial_{t_1}^2 \langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \partial_{t_1}\langle\Omega|T\{\partial_{t_1}\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \cdots \tag{3.1.7}$$

- canonical commutation relation 保证了 quantum field 满足 Euler–Lagrange equation, 因此
$$(\partial^2+m^2)\phi = -\frac{\delta}{\delta\phi}V(\phi). \tag{3.1.8}$$
- 结合 (3.1.5) 和 (3.1.8) 得到 Schwinger–Dyson equations,
$$\begin{aligned}&(\partial_1^2+m^2)\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle\\ =&\langle\Omega|T\{-\frac{\delta}{\delta\phi_1}V(\phi_1)\cdots\phi(x_n)\}|\Omega\rangle - i\sum_{i=2}^{n}\delta^{(4)}(x_i-x_1)\langle\Omega|\phi_2\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_n|\Omega\rangle .\end{aligned} \tag{3.1.9}$$

3.1.1 position-space Feynman rules

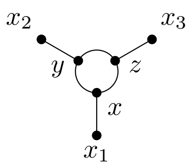
- Feynman propagator 为 ($\phi_0(x)$ 是 free field, 本 chapter 不做特别说明都是存在相互作用的 Heisenberg picture)
$$D_F(x-y) := \langle 0|T\{\phi_0(x)\phi_0(y)\}|0\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik\cdot(x-y)}, \tag{3.1.10}$$
满足
$$(\partial^2+m^2)D_F(x-y) = -i\delta^{(4)}(x-y). \tag{3.1.11}$$
- 2-point correlation function 可以重写作 (注意到 $\lim_{x\to\infty}D_{xy} = \lim_{x\to\infty}\partial_x D_{xy} = 0$)
$$\begin{aligned}\langle\Omega|T\{\phi_1\phi_2\}|\Omega\rangle &= i\int d^4x\,((\partial_x^2+m^2)D_{x1})\langle\Omega|T\{\phi_x\phi_2\}|\Omega\rangle\\ &= i\int d^4x\,D_{x1}(\partial_x^2+m^2)\langle\Omega|T\{\phi_x\phi_2\}|\Omega\rangle\\ &= D_{12}+i\int d^4x\,D_{x1}\langle\Omega|T\{-\frac{\delta}{\delta\phi_x}V(\phi_x)\phi_2\}|\Omega\rangle ,\end{aligned} \tag{3.1.12}$$
类似地, 4-point function 可以写作
$$\begin{aligned}\langle\Omega|T\{\phi_1\phi_2\phi_3\phi_4\}|\Omega\rangle =&D_{12}D_{34}+D_{13}D_{24}+D_{14}D_{23}\\ &+i\int d^4x\,D_{x1}\langle\Omega|T\{-\frac{\delta}{\delta\phi_x}V(\phi_x)\phi_2\phi_3\phi_4\}|\Omega\rangle\\ &+D_{12}i\int d^4y\,D_{3y}\langle\Omega|T\{-\frac{\delta}{\delta\phi_y}V(\phi_y)\phi_4\}|\Omega\rangle+\cdots\end{aligned} \tag{3.1.13}$$
- 考虑 $V(\phi) = -\frac{g}{3!}\phi^3$, 那么 2-point function 为
$$\begin{aligned}\langle\Omega|T\{\phi_1\phi_2\}|\Omega\rangle &= D_{12}+i\int d^4x\,D_{x1}\langle\Omega|T\{\frac{g}{2!}\phi_x^2\phi_2\}|\Omega\rangle\\ &= D_{12}+i\int d^4x\,D_{x1}\Big(i\int d^4y\,D_{y2}\langle\frac{g}{2!}\phi_x^2\frac{g}{2!}\phi_y^2\rangle + 2!D_{x2}\langle\frac{g}{2!}\phi_x\rangle\Big),\end{aligned} \tag{3.1.14}$$
其中
$$\begin{cases}\langle\phi_x\rangle = i\int d^4y\,D_{yx}(\partial_y^2+m^2)\langle\phi_y\rangle = i\int d^4y\,D_{yx}\langle\frac{g}{2!}\phi_y^2\rangle\\ \qquad = i\frac{g}{2!}\int d^4y\,D_{yx}D_{yy}+O(g^2)\\ \langle\phi_x^2\phi_y^2\rangle = D_{xx}D_{yy}+2D_{xy}D_{xy}+O(g)\end{cases}, \tag{3.1.15}$$
因此



$$= D_{12} + \frac{(ig)^2}{4} D_{1x} D_{xx} D_{yy} D_{y2} + \frac{(ig)^2}{2} D_{1x} D_{xy} D_{xy} D_{y2} + \frac{(ig)^2}{2} D_{1x} D_{xy} D_{yy} D_{x2} + O(g^3), \quad (3.1.16)$$

其中 $4, 2$ 是 symmetry factors.

- 下图的 symmetry factor 为 1,



$$= (ig)^3 D_{1x} D_{xy} D_{yz} D_{zx} D_{y2} D_{z3}. \quad (3.1.17)$$

3.2 Hamiltonian derivation

- interaction picture...

3.2.1 vacuum matrix elements

- section 2.2 中注意到了 $|\Omega\rangle$ 的定义 (in Heisenberg picture),

$$\begin{cases} a_{\vec{p}}(-\infty)\,|\Omega\rangle = U^\dagger(-\infty) a_{\vec{p}} U(-\infty)\,|\Omega\rangle = 0 \\ a_{\vec{p}}(+\infty)\,|\Omega\rangle = U^\dagger(+\infty) a_{\vec{p}} U(+\infty)\,|\Omega\rangle = 0 \end{cases} \Longrightarrow U(\pm\infty)\,|\Omega\rangle \propto |0\rangle\,, \quad (3.2.1)$$

那么 Schrodinger picture 和 interaction picture 中分别有

$$\begin{cases} |\Omega(t)\rangle = \mathcal{N}_i U(t,-\infty)\,|0\rangle = \mathcal{N}_f U(t,+\infty)\,|0\rangle & \text{Schrodinger picture} \\ |\Omega(t)\rangle_I = \mathcal{N}_i U_I(t,-\infty)\,|0\rangle = \mathcal{N}_f U_I(t,+\infty)\,|0\rangle & \text{interaction picture} \end{cases}. \quad (3.2.2)$$

calculation:

令

$$|\Omega\rangle \equiv |\Omega(0)\rangle = \mathcal{N}_i U(0,-\infty)\,|0\rangle = \mathcal{N}_f U(0,+\infty)\,|0\rangle\,, \quad (3.2.3)$$

且

$$\begin{cases} U_I(0,-\infty) U_0(0,-\infty) = U(0,-\infty) \\ U_I(0,+\infty) U_0(0,+\infty) = U(0,+\infty) \end{cases} \quad \text{and} \quad U_0(0,\pm\infty)\,|0\rangle = |0\rangle\,, \quad (3.2.4)$$

因此

$$|\Omega\rangle_I = \begin{cases} \mathcal{N}_i U_I(t,0) U(0,-\infty)\,|0\rangle = \mathcal{N}_i U_I(t,-\infty)\,|0\rangle \\ \mathcal{N}_f U_I(t,0) U(0,+\infty)\,|0\rangle = \mathcal{N}_f U_I(t,+\infty)\,|0\rangle \end{cases}. \quad (3.2.5)$$

  - 归一化要求 $\langle\Omega|\Omega\rangle = \mathcal{N}_f^* \mathcal{N}_i \langle 0|U(+\infty,-\infty)|0\rangle = \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty,-\infty)|0\rangle = |\mathcal{N}_i|^2 = |\mathcal{N}_f|^2 = 1$.
  - $H\,|\Omega\rangle = 0$ 因为 $[H, U(t_1,t_2)] = 0$ (前提是 Hamiltonian 不含时).

- 因此

$$\langle\Omega|T\{\phi_1 \cdots \phi_n\}|\Omega\rangle = \frac{\langle 0|T\{\phi_{0,1} \cdots \phi_{0,n} U_I(+\infty,-\infty)\}|0\rangle}{\langle 0|U_I(+\infty,-\infty)|0\rangle}. \quad (3.2.6)$$

calculation:

$$\begin{aligned} &\langle\Omega|T\{\phi_1 \cdots \phi_n\}|\Omega\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U(+\infty,0) T\{(U(0,t_1)\phi_{S,1} U(t_1,0))(U(0,t_2)\phi_{S,2} U(t_2,0)) \cdots\} U(0,-\infty)|0\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty,0) T\{(U_I(0,t_1) U_0(0,t_1)\phi_{S,1} U_0(t_1,0) U_I(t_1,0)) \cdots\} U_I(0,-\infty)|0\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty,0) T\{(U_I(0,t_1)\phi_{0,1} U_I(t_1,0)) \cdots\} U_I(0,-\infty)|0\rangle = \cdots \end{aligned} \quad (3.2.7)$$

另外注意到

$$\mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(-\infty,+\infty)|0\rangle = 1. \quad (3.2.8)$$

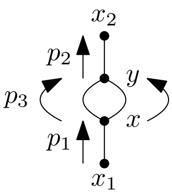
3.2.2 time-ordered products and contractions

- 用 Wick contraction 计算 (3.2.6), 得到

$$\langle\Omega|T\{\phi_1\cdots\phi_n\}|\Omega\rangle = \langle 0|T\{\phi_{0,1}\cdots\phi_{0,n}U_I(+\infty,-\infty)\}|0\rangle_{\text{no bubbles}}\,. \tag{3.2.9}$$

3.3 momentum-space Feynman rules

- 考虑如下 Feynman 图,



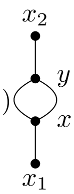
$$= \frac{(ig)^2}{2}\int d^4x\int d^4y\, D_{1x}D_{xy}^2D_{y2}$$

$$= \frac{(ig)^2}{2}\int\frac{d^4p_1}{(2\pi)^4}\frac{ie^{ip_1\cdot x_1}}{p_1^2-m^2+i\epsilon}\int\frac{d^4p_2}{(2\pi)^4}\frac{ie^{-ip_2\cdot x_2}}{p_2^2-m^2+i\epsilon}$$

$$\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{(p_1-p_3)^2-m^2+i\epsilon}(2\pi)^4\delta^{(4)}(p_1-p_2), \tag{3.3.1}$$

代入 LSZ, 有

$$\langle p_2|S|p_1\rangle = \cdots + i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)i\int d^4x_2\, e^{ip_2\cdot x_2}(\partial_2^2+m^2)$$



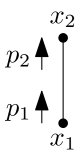
$$+\cdots$$

$$= \cdots + \frac{(ig)^2}{2}(2\pi)^4\delta^{(4)}(p_1-p_2)\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{(p_1-p_3)^2-m^2+i\epsilon}+\cdots, \tag{3.3.2}$$

所以

$$i\langle p_2|\mathcal{M}|p_1\rangle = \cdots + \frac{(ig)^2}{2}\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{(p_1-p_3)^2-m^2+i\epsilon}+\cdots \tag{3.3.3}$$

- 再考虑 (LSZ 中的 p_1, p_2 都 on-shell)



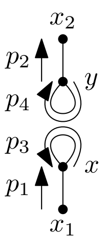
$$= i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)i\int d^4x_2\, e^{ip_2\cdot x_2}(\partial_2^2+m^2)D_{12}$$

$$= i(2\pi)^4\delta^{(4)}(p_1-p_2)(-p_1^2+m^2) = 0, \tag{3.3.4}$$

符合预期 (见 (2.1.4)).

3.3.1 disconnected graphs

- disconnected graphs are never important (?).
- 考虑



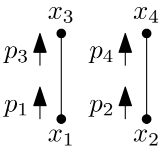
$$= \frac{(ig)^2}{4}\int d^4x\int d^4y\, D_{1x}D_{xx}D_{yy}D_{y2}$$

$$= \frac{(ig)^2}{4}\frac{i}{-m^2+i\epsilon}\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\frac{i}{-m^2+i\epsilon}\int\frac{d^4p_4}{(2\pi)^4}\frac{i}{p_4^2-m^2+i\epsilon}, \tag{3.3.5}$$

那么

$$\begin{aligned}\langle p_2|S|p_1\rangle =& \cdots + \frac{(ig)^2}{4}(2\pi)^4\delta^{(4)}(p_1)\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon} \\ & (2\pi)^4\delta^{(4)}(p_2)\int\frac{d^4p_4}{(2\pi)^4}\frac{i}{p_4^2-m^2+i\epsilon}+\cdots \\ =& \cdots + \frac{(ig)^2}{4}(2\pi)^4\delta^{(4)}(p_1-p_2) \\ & \left((2\pi)^4\delta^{(4)}(p_1)\int\frac{d^4p_3}{(2\pi)^4}\frac{i}{p_3^2-m^2+i\epsilon}\int\frac{d^4p_4}{(2\pi)^4}\frac{i}{p_4^2-m^2+i\epsilon}\right)+\cdots \end{aligned} \tag{3.3.6}$$

- 再考虑



$$\begin{aligned} =& i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2+m^2)i\int d^4x_2\, e^{-ip_2\cdot x_2}(\partial_2^2+m^2) \\ & i\int d^4x_3\, e^{ip_3\cdot x_3}(\partial_3^2+m^2)i\int d^4x_4\, e^{ip_4\cdot x_4}(\partial_4^2+m^2)D_{13}D_{24} \\ =& i(2\pi)^4\delta^{(4)}(p_1-p_3)(-p_1^2+m^2)i(2\pi)^4\delta^{(4)}(p_2-p_4)(-p_2^2+m^2) \\ =& i(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)(i(2\pi)^4\delta^{(4)}(p_1-p_3)(-p_1^2+m^2)(-p_2^2+m^2)). \end{aligned} \tag{3.3.7}$$

3.4 Mandelstam variables

- 适用于 $2\to 2$ 散射.

Part II

Quantum Electrodynamics

Chapter 4

spin 1 and gauge invariance

Appendices