

# Quantum Field Theory

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# convention, notation, and units

- 笔记中的度规号差约定为  $(+, -, -, -)$ .
- 使用 natural units, 此时  $\hbar, c, k_B = 1$ , 因此  $1 \text{ m} = \frac{1}{1.97 \times 10^{-16}} \text{ GeV}$  且:

names/dimensions	expressions/values
Planck length ( $L$ )	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} \text{ m}$
Planck time ( $T$ )	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44} \text{ s}$
Planck mass ( $M$ )	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} \text{ kg} \simeq 10^{19} \text{ GeV}$
Planck temperature ( $\Theta$ )	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32} \text{ K}$

- 时空维度用  $d = D + 1$  表示.

**Part I**

**Field Theory**

# Chapter 1

## cross sections and decay rates

### 1.1 cross sections

- cross section 定义为

$$\sigma = \frac{1}{\Phi} \frac{P}{\Delta t}, \quad (1.1.1)$$

其中  $\Phi := nv = \frac{|\vec{v}_1 - \vec{v}_2|}{V}$  是 incoming flux, 是入射粒子数密度乘粒子速度,  $P$  是发生散射的概率.

- 实验上定义 luminosity 为

$$L\Delta t = \frac{dN}{d\sigma}, \quad (1.1.2)$$

其中  $dN$  是  $d\Omega$  内发生散射的粒子数.

- 用 S-matrix elements 来表示 cross section, 有

$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} d\Pi, \quad (1.1.3)$$

其中  $d\Pi$  是末态动量体元

$$d\Pi = \prod_i \delta^{(3)}(\vec{p} = 0) d^3 p_{f,i} = \prod_i \frac{V}{(2\pi)^3} d^3 p_{f,i}, \quad (1.1.4)$$

这保证了无相互作用时  $\int dP = 1$ .

- 对于初末态有

$$\begin{cases} \langle i|i\rangle = \langle p_1, p_2 | p_1, p_2 \rangle = (2\pi)^3 2\omega_{p_1} \delta^{(3)}(0) (2\pi)^3 2\omega_{p_2} \delta^{(3)}(0) = (2\omega_{p_1} V) (2\omega_{p_2} V) \\ \langle f|f\rangle = \prod_i (2\omega_{p_{f,i}} V) \end{cases}. \quad (1.1.5)$$

- 一般将 S-matrix 写为

$$S = I + i\mathcal{T}, \quad \mathcal{T} = (2\pi)^4 \delta^{(4)}(\sum_{i,f} p) \mathcal{M}, \quad (1.1.6)$$

其中  $\mathcal{T}$  称为 transfer matrix, 而  $\mathcal{M}$  才是 S-matrix 的 non-trivial part. 有

$$\langle f|S - I|i\rangle = i(2\pi)^4 \delta^{(4)}(\sum_{i,f} p) \langle f|\mathcal{M}|i\rangle. \quad (1.1.7)$$

- 对于  $|f\rangle \neq |i\rangle$  的情况, 有

$$|\langle f|S|i\rangle|^2 = (2\pi)^4 T V \delta^{(4)}(\sum_{i,f} p) |\langle f|\mathcal{M}|i\rangle|^2, \quad (1.1.8)$$

那么

$$dP = \frac{T}{V} \frac{1}{(2\omega_{p_1})(2\omega_{p_2})} |\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}, \quad (1.1.9)$$

其中 LIPS 表示 Lorentz-invariant phase space,

$$d\Pi_{\text{LIPS}} = (2\pi)^4 \delta^{(4)}(\sum_{i,f} p) \prod_i \frac{d^3 p_{f,i}}{(2\pi)^3 2\omega_{p_{f,i}}}. \quad (1.1.10)$$

- 最终有 (将 (1.1.1) 中的  $\Delta t$  替换为  $T$ )

$$d\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2| (2\omega_{p_1})(2\omega_{p_2})} |\langle f|\mathcal{M}|i\rangle|^2 d\Pi_{\text{LIPS}}. \quad (1.1.11)$$

## 1.2 decay rates

- decay rate,  $\Gamma$ , 是粒子单位时间发生衰变的概率,

$$d\Gamma = \frac{dP}{T}. \quad (1.2.1)$$

- 因为  $|f\rangle \neq |i\rangle$ , 有

$$d\Gamma = \frac{1}{2\omega_p} |\langle f | \mathcal{M} | i \rangle|^2 d\Pi_{\text{LIPS}}. \quad (1.2.2)$$

## Chapter 2

# the S-matrix and time-ordered products

### 2.1 the LSZ reduction formula

- S-matrix element 为

$$\begin{cases} |i\rangle = \sqrt{(2\pi)^3 2\omega_{p_1}} \sqrt{(2\pi)^3 2\omega_{p_2}} a_{\vec{p}_1}^\dagger(-\infty) a_{\vec{p}_2}^\dagger(-\infty) |\Omega\rangle \\ |f\rangle = \sqrt{(2\pi)^3 2\omega_{p_3}} \cdots \sqrt{(2\pi)^3 2\omega_{p_n}} a_{\vec{p}_3}^\dagger(+\infty) \cdots a_{\vec{p}_n}^\dagger(+\infty) |\Omega\rangle \\ \langle f|S|i\rangle = (2\pi)^{3n/2} \sqrt{2\omega_{p_1}} \cdots \sqrt{2\omega_{p_n}} \langle \Omega | a_{\vec{p}_3}(+\infty) \cdots a_{\vec{p}_n}(+\infty) a_{\vec{p}_1}^\dagger(-\infty) a_{\vec{p}_2}^\dagger(-\infty) |\Omega\rangle \end{cases} \quad (2.1.1)$$

**remark:**

注意到

$$\begin{cases} a_{\vec{p}}(t) = U^\dagger(t) a_{\vec{p}}(0) U(t) \\ |\vec{p}(t)\rangle = U(t) |\vec{p}(0)\rangle \end{cases} \implies |\vec{p}(t)\rangle = a_{\vec{p}}^\dagger(-t) |\Omega\rangle, \quad (2.1.2)$$

那么, 对于初末态, 有

$$\begin{aligned} |i(-\infty)\rangle &= \sqrt{(2\pi)^3 2\omega_{p_1}} \sqrt{(2\pi)^3 2\omega_{p_2}} a_{\vec{p}_1}^\dagger(0) a_{\vec{p}_2}^\dagger(0) |\Omega\rangle \\ \implies |i(0)\rangle &= \cdots, \end{aligned} \quad (2.1.3)$$

可见 (2.1.1) 中的  $|i\rangle, |f\rangle$  都是  $t=0$  即 Heisenberg picture 中的形式.

- LSZ 需要用到以下公式 (其中  $p$  is on-shell),

$$i \int d^4x e^{ip \cdot x} (\partial^2 + m^2) \phi(x) = \sqrt{(2\pi)^3 2\omega_p} (e^{i\omega_p t} a_{\vec{p}}(t)) \Big|_{-\infty}^{\infty}, \quad (2.1.4)$$

注意到, 对于 free field, 等式两边等于零.

**proof:**

注意到

$$\begin{aligned} \int d^3x e^{ip \cdot x} (\partial^2 + m^2) \phi(x) &= \int d^3x e^{ip \cdot x} (\partial_t^2 - \nabla^2 + m^2) \phi(x) \\ &= \int d^3x (e^{ip \cdot x} (\partial_t^2 + m^2) \phi(x) - \phi(x) \nabla^2 e^{ip \cdot x}) \\ &= \int d^3x e^{ip \cdot x} (\partial_t^2 + \omega_p^2) \phi(x) \\ &= \int d^3x \partial_t (e^{ip \cdot x} \dot{\phi} - i\omega_p e^{ip \cdot x} \phi) \end{aligned} \quad (2.1.5)$$

$$= \frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}} \partial_t (e^{i\omega_p t} (\partial_t - i\omega_p) (a_{\vec{p}}(t) + a_{-\vec{p}}^\dagger(t))) \quad (2.1.6)$$

其中注意到了

$$e^{i\omega_p t} (\partial_t^2 + \omega_p^2) O(t) = \partial_t (e^{i\omega_p t} (\partial_t - i\omega_p) O(t)), \quad (2.1.7)$$

因此

$$\text{LHS} = i \frac{(2\pi)^{3/2}}{\sqrt{2\omega_p}} (e^{i\omega_p t} (\partial_t - i\omega_p) (a_{\vec{p}}(t) + a_{-\vec{p}}^\dagger(t))) \Big|_{-\infty}^{\infty}, \quad (2.1.8)$$

注意到  $t = \pm\infty$  时, fields are free, 所以

$$\begin{cases} \lim_{t \rightarrow \pm\infty} a_{\vec{p}}(t) \propto e^{-i\omega_p t} a_{\vec{p}} \\ \lim_{t \rightarrow \pm\infty} a_{-\vec{p}}^\dagger(t) \propto e^{i\omega_p t} a_{-\vec{p}}^\dagger \end{cases} \Rightarrow \begin{cases} \lim_{t \rightarrow \pm\infty} (\partial_t - i\omega_p) a_{\vec{p}}(t) = -2i\omega_p a_{\vec{p}}(t) \\ \lim_{t \rightarrow \pm\infty} (\partial_t - i\omega_p) a_{-\vec{p}}^\dagger(t) = 0 \end{cases}, \quad (2.1.9)$$

代入得到

$$\text{LHS} = \sqrt{(2\pi)^3 2\omega_p} (e^{i\omega_p t} a_{\vec{p}}(t)) \Big|_{-\infty}^{\infty}. \quad (2.1.10)$$

- 考虑  $p_1 \neq p_2 \neq \dots \neq p_n$  的情况.
- 考虑

$$\begin{aligned} & \frac{d}{dt} T\{A_1(t_1) \cdots A_{n-1}(t_{n-1}) B(t)\} \\ &= T\{A_1(t_1) \cdots A_n(t_n) \frac{dB(t)}{dt}\} + \sum_{i=1}^n \delta(t - t_i) T\{A_1(t_1) \cdots [B(t_i), A_i(t_i)] \cdots A_n(t_n)\}, \end{aligned} \quad (2.1.11)$$

那么

$$\begin{aligned} & \partial_{t_i}^2 T\{\phi_1 \cdots \phi_n\} \\ &= \partial_{t_i} T\{\cdots \partial_{t_i} \phi_i \cdots\} + \sum_{j \neq i} \delta(\cdots) T\{\cdots \underbrace{[\phi(t_j, \vec{x}_i), \phi(t_j, \vec{x}_j)]}_{=0} \cdots\} \\ &= T\{\cdots \partial_{t_i}^2 \phi_i \cdots\} + \sum_{j \neq i} \delta(t_j - t_i) T\{\cdots [\dot{\phi}(t_j, \vec{x}_i), \phi(t_j, \vec{x}_j)] \cdots\} \\ &= T\{\cdots \partial_{t_i}^2 \phi_i \cdots\} - i \sum_{j \neq i} \delta^{(4)}(x_j - x_i) T\{\cdots \phi_{j-1} \phi_{j+1} \cdots\}, \end{aligned} \quad (2.1.12)$$

所以

$$\begin{aligned} & i \int d^4 x_i e^{ip_i \cdot x_i} (\partial_i^2 + m^2) T\{\phi_1 \cdots \phi_n\} \\ &= i \int d^4 x_i e^{ip_i \cdot x_i} \left( T\{\cdots (\partial_i^2 + m^2) \phi_i \cdots\} - i \sum_{j \neq i} \delta^{(4)}(x_j - x_i) T\{\cdots \phi_{j-1} \phi_{j+1} \cdots\} \right) \\ &= i \int dt_i T\{\cdots \int d^3 x_i \partial_{t_i} (e^{ip_i \cdot x_i} \dot{\phi}_i - i\omega_{p_i} e^{ip_i \cdot x_i} \phi_i) \cdots\} \\ & \quad + \sum_{j \neq i} e^{ip_i \cdot x_j} T\{\cdots \phi_{j-1} \phi_{j+1} \cdots\} \\ &= i \int dt_i \left( \partial_{t_i} T\{\cdots \int d^3 x_i (e^{ip_i \cdot x_i} \dot{\phi}_i - i\omega_{p_i} e^{ip_i \cdot x_i} \phi_i) \cdots\} \right. \\ & \quad \left. - \sum_{j \neq i} \delta(t_i - t_j) T\{\cdots \underbrace{\left[ \int d^3 x_i e^{i(\omega_{p_i} \cdot t_j - \vec{p}_i \cdot \vec{x}_i)} (\dot{\phi}(t_j, \vec{x}_i) - i\omega_{p_i} \phi(t_j, \vec{x}_i)), \phi_j \right]}_{=-ie^{ip_i \cdot x_j}} \cdots\} \right) \\ & \quad + \sum_{j \neq i} e^{ip_i \cdot x_j} T\{\cdots \phi_{j-1} \phi_{j+1} \cdots\} \\ &= i \int dt_i \partial_{t_i} T\{\cdots \int d^3 x_i (e^{ip_i \cdot x_i} \dot{\phi}_i - i\omega_{p_i} e^{ip_i \cdot x_i} \phi_i) \cdots\} \end{aligned} \quad (2.1.13)$$



$$\begin{aligned}
&= (2\pi)^{3/2} \sqrt{2\omega_{p_i}} \left( e^{i\omega_{p_i}(+\infty)} a_{\vec{p}_i}(+\infty) T\{\cdots \phi_{i-1} \phi_{i+1} \cdots\} \right. \\
&\quad \left. - T\{\cdots \phi_{i-1} \phi_{i+1} \cdots\} e^{i\omega_{p_i}(-\infty)} a_{\vec{p}_i}(-\infty) \right), \tag{2.1.14}
\end{aligned}$$

其中, (2.1.13) 将对易子抵消掉是很重要的一步.

## 2.2 LSZ for operators

- $|\Omega\rangle$  的具体形式见 subsection 3.2.1.
- 考虑

$$\phi(x) = e^{iP \cdot x} \phi(0) e^{-iP \cdot x}, \tag{2.2.1}$$

且  $P|\Omega\rangle = 0$ , 因此

$$\begin{cases} \langle \Omega | \phi(x) | \Omega \rangle = \langle \Omega | \phi(0) | \Omega \rangle \\ \langle p | \phi(x) | \Omega \rangle = e^{ip \cdot x} \langle p | \phi(0) | \Omega \rangle \end{cases}. \tag{2.2.2}$$

- LSZ reduction formula 的前提是场算符满足

$$\begin{cases} \langle \Omega | \phi(x) | \Omega \rangle = 0 \\ \langle p | \phi(x) | \Omega \rangle = e^{ip \cdot x} \end{cases}. \tag{2.2.3}$$

**remark:**

前提是场算符具有以下形式,

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^{3/2} \sqrt{2\omega_p}} (a_{\vec{p}}(t) e^{i\vec{p} \cdot \vec{x}} + a_{\vec{p}}^\dagger(t) e^{-i\vec{p} \cdot \vec{x}}), \tag{2.2.4}$$

和  $a_{\vec{p}}^\dagger(\pm\infty) |\Omega\rangle \in \text{span}(|p\rangle)$  是一个单粒子态, 且  $\langle \Omega | a_{\vec{p}}^\dagger(+\infty) = a_{\vec{p}}(-\infty) |\Omega\rangle = 0$ .

- 注意, 在 Heisenberg picture 中,

$$\phi(t=0, \vec{x}) = \phi_0(\vec{x}) = \int \frac{d^3 p}{(2\pi)^{3/2} \sqrt{2\omega_p}} (a_{\vec{p}} e^{i\vec{p} \cdot \vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p} \cdot \vec{x}}). \tag{2.2.5}$$

- (2.1.9) 和  $H|\Omega\rangle = 0$  (要求  $H$  不含时) 矛盾 (?), 因为 LSZ 考虑的 Hamiltonian 是

$$H = H_0 + V\theta(t-T)\theta(t+T), T \gg 0, \tag{2.2.6}$$

所以 (2.2.3) 只在  $-T < t < T$  成立.

## Chapter 3

# Feynman rules

### 3.1 Lagrangian derivation

- 假设存在相互作用时, 场算符依然满足

$$\begin{cases} [\phi(t, \vec{x}), \phi(t, \vec{y})] = 0 \\ [\phi(t, \vec{x}), \partial_t \phi(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \end{cases}, \quad (3.1.1)$$

即 causality and canonical commutation relation.

- 一个重要的中间公式为

$$(\partial_x^2 + m^2) \langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle = \langle \Omega | T \{ (\partial_x^2 + m^2) \phi(x) \phi(y) \} | \Omega \rangle - i\delta^{(4)}(x - y). \quad (3.1.2)$$

**proof:**

考虑

$$\begin{aligned} \partial_t \langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle &= \partial_t (\langle \Omega | \phi(x) \phi(y) | \Omega \rangle \theta(t - t') + \langle \Omega | \phi(y) \phi(x) | \Omega \rangle \theta(t' - t)) \\ &= \langle \Omega | \partial_t \phi(x) \phi(y) | \Omega \rangle \theta(t - t') + \langle \Omega | \phi(x) \phi(y) | \Omega \rangle \delta(t - t') \\ &\quad + \langle \Omega | \phi(y) \partial_t \phi(x) | \Omega \rangle \theta(t' - t) - \langle \Omega | \phi(y) \phi(x) | \Omega \rangle \delta(t' - t) \\ &= \langle \Omega | T \{ \partial_t \phi(x) \phi(y) \} | \Omega \rangle + \langle \Omega | [\phi(t, \vec{x}), \phi(t, \vec{y})] | \Omega \rangle \delta(t - t') \\ &= \langle \Omega | T \{ \partial_t \phi(x) \phi(y) \} | \Omega \rangle, \end{aligned} \quad (3.1.3)$$

那么

$$\begin{aligned} \partial_t^2 \langle \Omega | T \{ \phi(x) \phi(y) \} | \Omega \rangle &= \partial_t \langle \Omega | T \{ \partial_t \phi(x) \phi(y) \} | \Omega \rangle \\ &= \partial_t (\langle \Omega | \partial_t \phi(x) \phi(y) | \Omega \rangle \theta(t - t') + \langle \Omega | \phi(y) \partial_t \phi(x) | \Omega \rangle \theta(t' - t)) \\ &= \langle \Omega | \partial_t^2 \phi(x) \phi(y) | \Omega \rangle \theta(t - t') + \langle \Omega | \partial_t \phi(x) \phi(y) | \Omega \rangle \delta(t - t') \\ &\quad + \langle \Omega | \phi(y) \partial_t^2 \phi(x) | \Omega \rangle \theta(t' - t) - \langle \Omega | \phi(y) \partial_t \phi(x) | \Omega \rangle \delta(t' - t) \\ &= \langle \Omega | T \{ \partial_t^2 \phi(x) \phi(y) \} | \Omega \rangle + \langle \Omega | [\partial_t \phi(t, \vec{x}), \phi(t, \vec{y})] | \Omega \rangle \delta(t - t') \\ &= \langle \Omega | T \{ \partial_t^2 \phi(x) \phi(y) \} | \Omega \rangle - i\delta^{(4)}(x - y). \end{aligned} \quad (3.1.4)$$

- (3.1.2) 可以推广为 (其中  $\phi_i$  是  $\phi(x_i)$  的简写)

$$\begin{aligned} &(\partial_1^2 + m^2) \langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle \\ &= \langle \Omega | T \{ (\partial_1^2 + m^2) \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle - i \sum_{i=2}^n \delta^{(4)}(x_i - x_1) \langle \Omega | \phi_2 \cdots \phi_{i-1} \phi_{i+1} \cdots \phi_n | \Omega \rangle. \end{aligned} \quad (3.1.5)$$

**proof:**

首先

$$\partial_{t_1} \langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle = \langle \Omega | T \{ \partial_{t_1} \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle, \quad (3.1.6)$$

那么

$$\partial_{t_1}^2 \langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle = \partial_{t_1} \langle \Omega | T \{ \partial_{t_1} \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle = \cdots \quad (3.1.7)$$

- canonical commutation relation 保证了 quantum field 满足 Euler-Lagrange equation, 因此

$$(\partial^2 + m^2)\phi = -\frac{\delta}{\delta\phi} V(\phi). \quad (3.1.8)$$

- 结合 (3.1.5) 和 (3.1.8) 得到 Schwinger-Dyson equations,

$$\begin{aligned} & (\partial_1^2 + m^2) \langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle \\ &= \langle \Omega | T \left\{ -\frac{\delta}{\delta\phi_1} V(\phi_1) \cdots \phi(x_n) \right\} | \Omega \rangle - i \sum_{i=2}^n \delta^{(4)}(x_i - x_1) \langle \Omega | \phi_2 \cdots \phi_{i-1} \phi_{i+1} \cdots \phi_n | \Omega \rangle. \end{aligned} \quad (3.1.9)$$

### 3.1.1 position-space Feynman rules

- Feynman propagator 为  $\langle \phi_0(x) \phi_0(y) \rangle$  是 free field, 本 chapter 不做特别说明都是存在相互作用的 Heisenberg picture)

$$D_F(x-y) := \langle 0 | T \{ \phi_0(x) \phi_0(y) \} | 0 \rangle = \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{ik \cdot (x-y)}, \quad (3.1.10)$$

满足

$$(\partial^2 + m^2)D_F(x-y) = -i\delta^{(4)}(x-y). \quad (3.1.11)$$

- 2-point correlation function 可以重写作 (注意到  $\lim_{x \rightarrow \infty} D_{xy} = \lim_{x \rightarrow \infty} \partial_x D_{xy} = 0$ )

$$\begin{aligned} \langle \Omega | T \{ \phi_1 \phi_2 \} | \Omega \rangle &= i \int d^4 x ((\partial_x^2 + m^2)D_{x1}) \langle \Omega | T \{ \phi_x \phi_2 \} | \Omega \rangle \\ &= i \int d^4 x D_{x1} (\partial_x^2 + m^2) \langle \Omega | T \{ \phi_x \phi_2 \} | \Omega \rangle \\ &= D_{12} + i \int d^4 x D_{x1} \langle \Omega | T \left\{ -\frac{\delta}{\delta\phi_x} V(\phi_x) \phi_2 \right\} | \Omega \rangle, \end{aligned} \quad (3.1.12)$$

类似地, 4-point function 可以写作

$$\begin{aligned} \langle \Omega | T \{ \phi_1 \phi_2 \phi_3 \phi_4 \} | \Omega \rangle &= D_{12} D_{34} + D_{13} D_{24} + D_{14} D_{23} \\ &\quad + i \int d^4 x D_{x1} \langle \Omega | T \left\{ -\frac{\delta}{\delta\phi_x} V(\phi_x) \phi_2 \phi_3 \phi_4 \right\} | \Omega \rangle \\ &\quad + D_{12} i \int d^4 y D_{3y} \langle \Omega | T \left\{ -\frac{\delta}{\delta\phi_y} V(\phi_y) \phi_4 \right\} | \Omega \rangle + \cdots \end{aligned} \quad (3.1.13)$$

- 考虑  $V(\phi) = -\frac{g}{3!} \phi^3$ , 那么 2-point function 为

$$\begin{aligned} \langle \Omega | T \{ \phi_1 \phi_2 \} | \Omega \rangle &= D_{12} + i \int d^4 x D_{x1} \langle \Omega | T \left\{ \frac{g}{2!} \phi_x^2 \phi_2 \right\} | \Omega \rangle \\ &= D_{12} + i \int d^4 x D_{x1} \left( i \int d^4 y D_{y2} \left\langle \frac{g}{2!} \phi_x^2 \frac{g}{2!} \phi_y^2 \right\rangle + 2! D_{x2} \left\langle \frac{g}{2!} \phi_x \right\rangle \right), \end{aligned} \quad (3.1.14)$$

其中

$$\begin{cases} \langle \phi_x \rangle = i \int d^4 y D_{yx} (\partial_y^2 + m^2) \langle \phi_y \rangle = i \int d^4 y D_{yx} \left\langle \frac{g}{2!} \phi_y^2 \right\rangle \\ \quad = i \frac{g}{2!} \int d^4 y D_{yx} D_{yy} + O(g^2) \\ \langle \phi_x^2 \phi_y^2 \rangle = D_{xx} D_{yy} + 2 D_{xy} D_{xy} + O(g) \end{cases}, \quad (3.1.15)$$

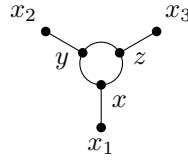
因此

$$\langle \phi_1 \phi_2 \rangle = \begin{array}{c} x_2 \\ | \\ x_1 \end{array} + \begin{array}{c} x_2 \\ | \\ \text{loop} \\ | \\ x_1 \end{array} + \begin{array}{c} x_2 \\ | \\ \text{triangle} \\ | \\ x_1 \end{array} + \begin{array}{c} x_2 \\ | \\ \text{bubble} \\ | \\ x_1 \end{array} + O(g^3)$$

$$= D_{12} + \frac{(ig)^2}{4} D_{1x} D_{xx} D_{yy} D_{y2} + \frac{(ig)^2}{2} D_{1x} D_{xy} D_{xy} D_{y2} + \frac{(ig)^2}{2} D_{1x} D_{xy} D_{yy} D_{x2} + O(g^3), \quad (3.1.16)$$

其中 4, 2 是 symmetry factors.

- 下图的 symmetry factor 为 1,



$$= (ig)^3 D_{1x} D_{xy} D_{yz} D_{zx} D_{y2} D_{z3}. \quad (3.1.17)$$

## 3.2 Hamiltonian derivation

- interaction picture...

### 3.2.1 vacuum matrix elements

- section 2.2 中注意到了  $|\Omega\rangle$  的定义 (in Heisenberg picture),

$$\begin{cases} a_{\vec{p}}(-\infty) |\Omega\rangle = U^\dagger(-\infty) a_{\vec{p}} U(-\infty) |\Omega\rangle = 0 \\ a_{\vec{p}}(+\infty) |\Omega\rangle = U^\dagger(+\infty) a_{\vec{p}} U(+\infty) |\Omega\rangle = 0 \end{cases} \implies U(\pm\infty) |\Omega\rangle \propto |0\rangle, \quad (3.2.1)$$

那么 Schrodinger picture 和 interaction picture 中分别有

$$\begin{cases} |\Omega(t)\rangle = \mathcal{N}_i U(t, -\infty) |0\rangle = \mathcal{N}_f U(t, +\infty) |0\rangle & \text{Schrodinger picture} \\ |\Omega(t)\rangle_I = \mathcal{N}_i U_I(t, -\infty) |0\rangle = \mathcal{N}_f U_I(t, +\infty) |0\rangle & \text{interaction picture} \end{cases}. \quad (3.2.2)$$

**calculation:**

令

$$|\Omega\rangle \equiv |\Omega(0)\rangle = \mathcal{N}_i U(0, -\infty) |0\rangle = \mathcal{N}_f U(0, +\infty) |0\rangle, \quad (3.2.3)$$

且

$$\begin{cases} U_I(0, -\infty) U_0(0, -\infty) = U(0, -\infty) \\ U_I(0, +\infty) U_0(0, +\infty) = U(0, +\infty) \end{cases} \quad \text{and} \quad U_0(0, \pm\infty) |0\rangle = |0\rangle, \quad (3.2.4)$$

因此

$$|\Omega\rangle_I = \begin{cases} \mathcal{N}_i U_I(t, 0) U(0, -\infty) |0\rangle = \mathcal{N}_i U_I(t, -\infty) |0\rangle \\ \mathcal{N}_f U_I(t, 0) U(0, +\infty) |0\rangle = \mathcal{N}_f U_I(t, +\infty) |0\rangle \end{cases}. \quad (3.2.5)$$

– 归一化要求  $\langle\Omega|\Omega\rangle = \mathcal{N}_f^* \mathcal{N}_i \langle 0|U(+\infty, -\infty)|0\rangle = \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty, -\infty)|0\rangle = |\mathcal{N}_i|^2 = |\mathcal{N}_f|^2 = 1$ .

–  $H|\Omega\rangle = 0$  因为  $[H, U(t_1, t_2)] = 0$  (前提是 Hamiltonian 不含时).

- 因此

$$\langle\Omega|T\{\phi_1 \cdots \phi_n\}|\Omega\rangle = \frac{\langle 0|T\{\phi_{0,1} \cdots \phi_{0,n} U_I(+\infty, -\infty)\}|0\rangle}{\langle 0|U_I(+\infty, -\infty)|0\rangle}. \quad (3.2.6)$$

**calculation:**

$$\begin{aligned} & \langle\Omega|T\{\phi_1 \cdots \phi_n\}|\Omega\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U(+\infty, 0) T\{(U(0, t_1) \phi_{S,1} U(t_1, 0)) (U(0, t_2) \phi_{S,2} U(t_2, 0)) \cdots\} U(0, -\infty)|0\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty, 0) T\{(U_I(0, t_1) U_0(0, t_1) \phi_{S,1} U_0(t_1, 0) U_I(t_1, 0)) \cdots\} U_I(0, -\infty)|0\rangle \\ &= \mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(+\infty, 0) T\{(U_I(0, t_1) \phi_{0,1} U_I(t_1, 0)) \cdots\} U_I(0, -\infty)|0\rangle = \cdots \end{aligned} \quad (3.2.7)$$

另外注意到

$$\mathcal{N}_f^* \mathcal{N}_i \langle 0|U_I(-\infty, +\infty)|0\rangle = 1. \quad (3.2.8)$$

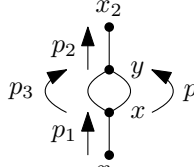
### 3.2.2 time-ordered products and contractions

- 用 Wick contraction 计算 (3.2.6), 得到

$$\langle \Omega | T \{ \phi_1 \cdots \phi_n \} | \Omega \rangle = \langle 0 | T \{ \phi_{0,1} \cdots \phi_{0,n} U_I(+\infty, -\infty) \} | 0 \rangle_{\text{no bubbles}}. \quad (3.2.9)$$

### 3.3 momentum-space Feynman rules

- 考虑如下 Feynman 图,

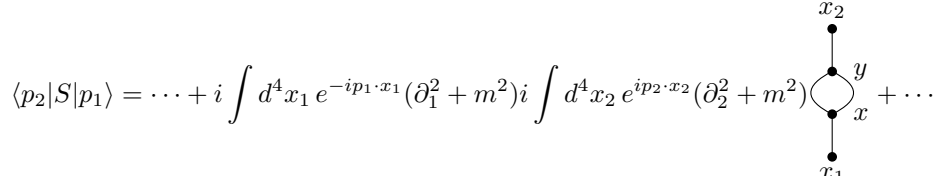


$$p_1 - p_3 = \frac{(ig)^2}{2} \int d^4x \int d^4y D_{1x} D_{xy}^2 D_{y2}$$

$$= \frac{(ig)^2}{2} \int \frac{d^4p_1}{(2\pi)^4} \frac{ie^{ip_1 \cdot x_1}}{p_1^2 - m^2 + i\epsilon} \int \frac{d^4p_2}{(2\pi)^4} \frac{ie^{-ip_2 \cdot x_2}}{p_2^2 - m^2 + i\epsilon}$$

$$\int \frac{d^4p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} \frac{i}{(p_1 - p_3)^2 - m^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(p_1 - p_2), \quad (3.3.1)$$

代入 LSZ, 有



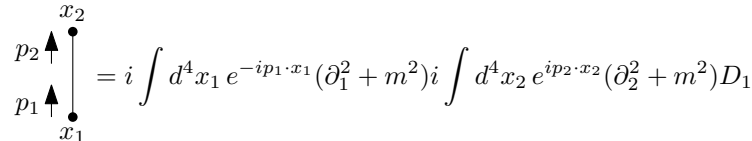
$$\langle p_2 | S | p_1 \rangle = \cdots + i \int d^4x_1 e^{-ip_1 \cdot x_1} (\partial_1^2 + m^2) i \int d^4x_2 e^{ip_2 \cdot x_2} (\partial_2^2 + m^2) \text{ (bubble) } + \cdots$$

$$= \cdots + \frac{(ig)^2}{2} (2\pi)^4 \delta^{(4)}(p_1 - p_2) \int \frac{d^4p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} \frac{i}{(p_1 - p_3)^2 - m^2 + i\epsilon} + \cdots, \quad (3.3.2)$$

所以

$$i \langle p_2 | \mathcal{M} | p_1 \rangle = \cdots + \frac{(ig)^2}{2} \int \frac{d^4p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} \frac{i}{(p_1 - p_3)^2 - m^2 + i\epsilon} + \cdots \quad (3.3.3)$$

- 再考虑 (LSZ 中的  $p_1, p_2$  都 on-shell)



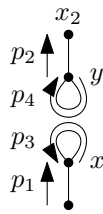
$$= i \int d^4x_1 e^{-ip_1 \cdot x_1} (\partial_1^2 + m^2) i \int d^4x_2 e^{ip_2 \cdot x_2} (\partial_2^2 + m^2) D_{12}$$

$$= i(2\pi)^4 \delta^{(4)}(p_1 - p_2) (-p_1^2 + m^2) = 0, \quad (3.3.4)$$

符合预期 (见 (2.1.4)).

#### 3.3.1 disconnected graphs

- disconnected graphs are never important (?).
- 考虑



$$= \frac{(ig)^2}{4} \int d^4x \int d^4y D_{1x} D_{xx} D_{yy} D_{y2}$$

$$= \frac{(ig)^2}{4} \frac{i}{-m^2 + i\epsilon} \int \frac{d^4p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} \frac{i}{-m^2 + i\epsilon} \int \frac{d^4p_4}{(2\pi)^4} \frac{i}{p_4^2 - m^2 + i\epsilon}, \quad (3.3.5)$$

那么

$$\begin{aligned}
\langle p_2 | S | p_1 \rangle &= \cdots + \frac{(ig)^2}{4} (2\pi)^4 \delta^{(4)}(p_1) \int \frac{d^4 p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} \\
&\quad (2\pi)^4 \delta^{(4)}(p_2) \int \frac{d^4 p_4}{(2\pi)^4} \frac{i}{p_4^2 - m^2 + i\epsilon} + \cdots \\
&= \cdots + \frac{(ig)^2}{4} (2\pi)^4 \delta^{(4)}(p_1 - p_2) \\
&\quad \left( (2\pi)^4 \delta^{(4)}(p_1) \int \frac{d^4 p_3}{(2\pi)^4} \frac{i}{p_3^2 - m^2 + i\epsilon} \int \frac{d^4 p_4}{(2\pi)^4} \frac{i}{p_4^2 - m^2 + i\epsilon} \right) + \cdots \quad (3.3.6)
\end{aligned}$$

• 再考虑

$$\begin{aligned}
\begin{array}{c} x_3 \quad x_4 \\ \uparrow \quad \uparrow \\ p_3 \quad p_4 \\ \uparrow \quad \uparrow \\ p_1 \quad p_2 \\ \uparrow \quad \uparrow \\ x_1 \quad x_2 \end{array} &= i \int d^4 x_1 e^{-ip_1 \cdot x_1} (\partial_1^2 + m^2) i \int d^4 x_2 e^{-ip_2 \cdot x_2} (\partial_2^2 + m^2) \\
&\quad i \int d^4 x_3 e^{ip_3 \cdot x_3} (\partial_3^2 + m^2) i \int d^4 x_4 e^{ip_4 \cdot x_4} (\partial_4^2 + m^2) D_{13} D_{24} \\
&= i (2\pi)^4 \delta^{(4)}(p_1 - p_3) (-p_1^2 + m^2) i (2\pi)^4 \delta^{(4)}(p_2 - p_4) (-p_2^2 + m^2) \\
&= i (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) (i (2\pi)^4 \delta^{(4)}(p_1 - p_3) (-p_1^2 + m^2) (-p_2^2 + m^2)). \quad (3.3.7)
\end{aligned}$$

### 3.4 Mandelstam variables

• 适用于  $2 \rightarrow 2$  散射.

**Part II**

**Quantum Electrodynamics**

## Chapter 4

### spin 1 and gauge invariance



# Appendices