Lie Groups and Lie Algebras

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February 29, 2024

Contents

Part I

Finite Groups

Chapter 1

finite groups

- a useful reference: https://sites.ualberta.ca/~vbouchar/MAPH464/notes.html.

- def. of groups (Abelian groups, cyclic groups, symmetry groups, permutation groups).
- order of G denoted by $|G|$, order of element g.

- conjugated element $hgh^{-1} = g'$, conjugacy class.
- subgroup, (left/right) coset of a subgroup (2 theorems + Lagrange theorem).
- conjugacy subgroup hHh^{-1}.
- **normal subgroup** (i.e. invariant subgroup) $N \triangleleft G$, $gNg^{-1} \subseteq N, \forall g$.
  - **center**, $Z(G) = \{z \in G | gzg^{-1} = z, \forall g\}$.
    center is normal, but normal subgroup is not necessarily central.
  - the center of a Lie algebra is $\mathfrak{h} = \{A \in \mathfrak{g} | [A, B] = 0, \forall B\} \equiv \{A \in \mathfrak{g} | \mathrm{ad}_A = 0\}$.
    center is an ideal, but ideal is not necessarily a center.
- groups without nontrivial normal subgroups are **simple**.
- **direct product group** $G \times H$ (Cartesian product, direct product and direct sum).
  def.: $G \times H = \{(g, h) | g \in G, h \in H\}$ with group product defined by $(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_1 \circ h_2)$.
- factor (quotient) group G/H_N.
- isomorphism vs. homomorphism.
  - kernel $K \mapsto \{e\}$ of a homomorphism.

1.1 representation theory

- representation of a group $D(g)$.
- 用 basis of functions 来构建 rep. of G,

$$\Omega_g \psi_i(\vec{x}) = \psi_i(g^{-1}\vec{x}) \tag{1.1.1}$$

- trivial rep. (1 dim.) $D_{11}(\forall g) = 1$.
- regular rep. $D_{ij}(g) = \langle g_i | g g_j \rangle \equiv \delta_{g_i, g g_j}$.

1.1.1 reducibility

- reducible rep. vs. completely reducible (semisimple) rep..
  completely reducible rep.,

$$TD(g)T^{-1} = D^{(1)}(g) \oplus D^{(2)}(g) \oplus \cdots \tag{1.1.2}$$

- completely reducible $\iff$ invariant subspace is trivial.

1.2 unitarity theorem

- any **finite-dim. rep. of a finite group** are equivalent to a **unitary rep.**.

proof:

for a finite-dim. rep. $\Gamma = \{D(g), \cdots\}$, consider $H = \sum_g D^\dagger(g)D(g)$, we have,

$$D^\dagger(h)HD(h) = H \tag{1.2.1}$$

H is a Hermitian matrix which can be diagonalized by a unitary matrix,

$$M \equiv \mathrm{diag}(\lambda_1, \cdots) = UHU^\dagger \tag{1.2.2}$$

then let,

$$B(g) = M^{1/2}UD(g)U^\dagger M^{-1/2} \tag{1.2.3}$$

where $M^{1/2} = \mathrm{diag}(\lambda_1^{1/2}, \cdots)$, we can see that,

$$\begin{aligned} B^\dagger(g)B(g) &= M^{-1/2}UD^\dagger(g)U^\dagger MUD(g)U^\dagger M^{-1/2} \\ &= M^{-1/2}MM^{-1/2} = I \end{aligned} \tag{1.2.4}$$

so $\{B(g), \cdots\}$ is a unitary rep..

- all the **reducible unitary rep.** are **completely reducible**.

proof:

unitary rep. 作用于 $V = W \oplus W^\perp$, 其中 V 是 Hilbert 空间, 内积为 $\langle \cdot, \cdot \rangle$, $W^\perp$ 与 W 正交, W 是表示的不变子空间, 下面证明 $W^\perp$ 也是不变子空间,

$$\langle B(g)w^\perp | w \rangle = \langle w^\perp | B(-g)w \rangle = \langle w^\perp | w' \rangle = 0, \forall w^\perp \in W^\perp, w \in W \tag{1.2.5}$$

其中, $w' \in W$, 可见 $B(g)[W^\perp] \subseteq W^\perp$

其实不需要要求表示幺正, 只需要 B 和 $B^\dagger$ 拥有同一个不变子空间 W 就行.

这对 infinite group 也成立.

1.3 Schur's lemmas

- **Schur's 1st lemma**

for 2 **irreducible real or complex rep.** $\Gamma_1 = \{D^{(1)}(g), \cdots\}$ and $\Gamma_2 = \{D^{(2)}(g), \cdots\}$, $\exists A$ s.t. $\forall g$,

$$AD^{(1)}(g) = D^{(2)}(g)A \tag{1.3.1}$$

then, there are only 2 possibilities:

1. $A = 0$,
2. A is reversible matrix and Γ_1, Γ_2 are equivalent.

proof:

consider,

$$\begin{aligned} &AD^{(1)}(g)[\ker A] = D^{(2)}(g)A[\ker A] = 0 \\ \Longrightarrow &D^{(1)}(g)[\ker A] \subseteq \ker A \end{aligned} \tag{1.3.2}$$

so, $\ker A$ is a invariant subspace of rep. Γ_1
but Γ_1 is irreducible, so $\ker A$ is trivial, i.e. $\ker A$ is either 0 or V, which implies that...

对 infinite group 也成立.

- **Schur's 2nd lemma**

  for a **irreducible complex rep.** $\Gamma = \{D(g), \cdots\}$, if $\forall g$,

$$AD(g) = D(g)A \tag{1.3.3}$$

  then $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

proof:

A must have (at least) one eigenvalue λ, then $\det(A - \lambda I) = 0$ is irreversible matrix,

$$AD(g) = D(g)A \Longrightarrow (A - \lambda I)D(g) = D(g)(A - \lambda I) \tag{1.3.4}$$

by Schur's 1st lemma, irreversible matrix $A - \lambda I$ must be 0.

- **Schur's 3rd lemma**

  for 2 **irreducible complex rep.** $\Gamma_1 = \{D^{(1)}(g), \cdots\}$ and $\Gamma_2 = \{D^{(2)}(g), \cdots\}$, if $\forall g$,

$$\begin{cases} AD^{(1)}(g) = D^{(2)}(g)A \\ BD^{(1)}(g) = D^{(2)}(g)B \end{cases} \tag{1.3.5}$$

  then $B = \lambda A$ for some $\lambda \in \mathbb{C}$.

proof:

$$(A - \lambda B)D^{(1)}(g) = D^{(2)}(g)(A - \lambda B) \tag{1.3.6}$$

choose λ s.t. $\det(A - \lambda B) = 0$, then, according to Schur's 1st lemma, $A - \lambda B = 0$.

1.4 the great orthogonal theorem

- **the great orthogonality theorem**

  for 2 inequivalent irreducible rep. $\Gamma^a = \{D^{(a)}(g), \cdots\}$ where $a = 1, 2$,

$$\frac{1}{|G|} \sum_g D_{ij}^{(a)}(g^{-1}) D_{j'i'}^{(b)}(g) = \frac{1}{d} \delta_{ii'} \delta_{jj'} \delta^{ab} \tag{1.4.1}$$

  or for unitary rep.,

$$\frac{1}{|G|} \sum_g B_{ij}^{(a)*}(g) B_{i'j'}^{(b)}(g) = \frac{1}{d} \delta_{ii'} \delta_{jj'} \delta^{ab} \tag{1.4.2}$$

  where d is the dim. of the rep..

proof:

for $a = b$:
consider $A = \sum_g B^{(a)\dagger}(g) X B^{(a)}(g)$ where $B^{(a)}(g) = TD^{(a)}(g)T^{-1}$ is the unitary rep. equivalent to Γ_a, then,

$$AB^{(a)}(h) = B^{(a)\dagger}(h^{-1})A \Longrightarrow AB^{(a)}(h) = B^{(a)}(h)A \tag{1.4.3}$$

according to Schur's 1st lemma, $A = \lambda I$, then,

$$\begin{aligned} \lambda I &= \sum_g (T^{-1} B^{(a)\dagger}(g) T)(T^{-1} X T)(T^{-1} B^{(a)}(g) T) \\ &= \sum_g D^{(a)}(g^{-1}) X' D^{(a)}(g) \end{aligned} \tag{1.4.4}$$

choose $X'_{\cdot,\cdot} = \delta_{\cdot,j} \delta_{j',\cdot}$ then we have $\lambda I = \sum_g D_{\cdot,j}^{(a)}(g^{-1}) D_{j',\cdot}^{(a)}(g)$, calculate the trace of the matrix,

$$\lambda d_a = \sum_g \delta_{jj'} = |G| \delta_{jj'} \tag{1.4.5}$$

so we can conclude that,

$$\frac{1}{|G|}\sum_g D^{(a)}_{ij}(g^{-1})D^{(a)}_{j'i'}(g) = \frac{1}{d_a}\delta_{ii'}\delta_{jj'} \tag{1.4.6}$$

---

for $a \neq b$:
still consider $A = \sum_g B^{(a)\dagger}(g)XB^{(b)}(g)$ then,

$$AB^{(b)}(h) = B^{(a)}(h)A \tag{1.4.7}$$

according to Schur's 1st lemma, $A = 0$, consequently,

$$\sum_g D^{(a)}_{ij}(g^{-1})D^{(b)}_{j'i'}(g) = 0 \tag{1.4.8}$$

- characters of the rep. Γ_a of group G is the set $\{\chi^{(a)}(g) = \mathrm{tr}D^{(a)}(g)|g \in G\}$
- character table is the matrix $X = \{X^a{}_i = \chi^{(a=1,\cdots,\rho)}(g_{i=1,\cdots,c})\}$.

  where g_i is the rep. of the ith conjugacy class, and ρ is the number of the irreducible inequivalent rep. of G. ($\rho = c$, as to be proved later).

- **1st theorem of the orthogonality of the characters**

  the character of irreducible inequivalent rep. of G are orthogonal to each other, which can be derived easily from the great orthogonality theorem.

$$\frac{1}{|G|}\sum_g \chi^{(a)*}(g)\chi^{(b)}(g) = \delta^{ab} \tag{1.4.9}$$

- **2nd theorem of the orthogonality of the characters**

$$\sum_{a=1}^{\rho} \chi^{(a)*}(g_i)\chi^{(a)}(g_j) = \frac{|G|}{n_i}\delta_{ij} \tag{1.4.10}$$

  where g_i is the rep. of the ith conjugacy class, n_i is the number of elements in this conjugacy class, and ρ is the number of the irreducible inequivalent rep. of G.

proof:

by 1st theorem,

$$X\mathrm{diag}(\frac{n_1}{|G|},\cdots,\frac{n_c}{|G|})X^\dagger = I \tag{1.4.11}$$

then,

$$\Longrightarrow \sum_j \left(X^\dagger X\mathrm{diag}(\frac{n_1}{|G|},\cdots,\frac{n_c}{|G|})\right)_{ij} X^\dagger{}_j{}^a = X^\dagger{}_i{}^a \tag{1.4.12}$$

since vectors $(X^a{}_1,\cdots,X^a{}_c)$ forms an orthogonal basis of the vector space, then we must have,

$$\left(X^\dagger X\mathrm{diag}(\frac{n_1}{|G|},\cdots,\frac{n_c}{|G|})\right)_{ij} = \delta_{ij} \tag{1.4.13}$$

then, finally, we have,

$$\sum_{a=1}^{\rho} \chi^{(a)*}(g_i)\chi^{(a)}(g_j) = \frac{|G|}{n_i}\delta_{ij} \tag{1.4.14}$$

- 群 G 的 irreducible inequivalent rep. 的数量等于其 conjugacy class 的数量 c.

proof:

一个 irreducible inequivalent rep. 由其 characters 表示 $\{\chi^{(a)}(g), \cdots\}$
(根据 theorem of the orthogonality of the characters) 不同的 irreducible inequivalent rep. 的 characters 一定不同.
且 conjugacy class 内的元素的 character 一定相等, 所以一个 rep. 实际上只有 conjugacy class 的数量 c 个不同的 characters, 所以可以将 characters 视为 c 维向量 $\frac{1}{\sqrt{|G|}}(\chi^{(a)}(g), \cdots)$, 那么 c 维向量空间中互相正交归一的向量最多只有 c 个.
利用 2nd theorem of... 可证... 最少有 c 个. 所以... 等于...

- characters of completely reducible rep..
  suppose a completely reducible rep. $\Gamma = \oplus_{a=1}^{c} m_a \Gamma_a$, where $m_a = 0, 1, 2, \cdots$, then,

$$\chi(g) = \sum_a m_a \chi^{(a)}(g) \tag{1.4.15}$$

(e.g. for $D(g) = D^{(1)}(g) \oplus D^{(1)}(g)$, $m_1 = 2$).
and,

$$\frac{1}{|G|} \sum_g \chi^*(g)\chi(g) = \sum_a m_a^2 > 1 \tag{1.4.16}$$

- **Burnside theorem**

$$\sum_{a=1}^{c} d_a^2 = |G| \tag{1.4.17}$$

where d_a is the dim. of the ath inequivalent irreducible rep. of G.

proof:

by 2nd orthogonality theorem of characters,

$$\sum_{a=1}^{c} \chi^{(a)*}(e)\big(\chi^{(a)}(e) = d_a\big) = \frac{|G|}{(n_e = 1)} \Longrightarrow \sum_{a=1}^{c} d_a^2 = |G| \tag{1.4.18}$$

- rep. of **direct product group** $G = H \times F$ is derived from **irreducible rep.** of H and F by $\Gamma = \Gamma_H \times \Gamma_F = \{D(hf) = D_H(h) \otimes D_F(f)\}$, then Γ **is also an irreducible rep.**.

proof:

利用 characters of completely reducible rep. 的性质.

- direct product of group rep.: $\Gamma = \Gamma_a \times \Gamma_b$, then $\chi(g) = \chi^{(a)}(g)\chi^{(b)}(g)$

- projection operator is,

$$P_a = \frac{d_a}{|G|} \sum_g \chi^{(a)*}(g) T^{-1} \begin{pmatrix} \ddots & & \\ & D^{(b)}(g) & \\ & & \ddots \end{pmatrix} T = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} I & \\ & & \ddots \end{pmatrix} T \tag{1.4.19}$$

i.e.,

$$P_a = \frac{d_a}{|G|} \sum_g \chi^{(a)*}(g) D(g) = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} I & \\ & & \ddots \end{pmatrix} T \tag{1.4.20}$$

where $TD(g)T^{-1} = \cdots \oplus D^{(b)}(g) \oplus \cdots$.

notice that P_a is not necessarily a diagonal matrix, unless T consists of orthogonal column vectors.

- how to use a projection operator:

$$P_a D(g) = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} D^{(a)}(g) & \\ & & \ddots \end{pmatrix} T \tag{1.4.21}$$

and $\mathrm{tr}(P_a) = m_a d_a$.

- about 1-dim. rep. $\Gamma_1 = \{D^{(1)}(g), \cdots\}$:

1-dim. rep. must be **irreducible** and **unitary**, so,

$$\chi^{(1)}(g) = D^{(1)}(g) \qquad \chi^{(1)}(g^{-1}) = \chi^{(1)*}(g) \tag{1.4.22}$$

so we can conclude that,

$$|\chi^{(1)}(g)| = |D^{(1)}(g)| = 1 \tag{1.4.23}$$

- Γ_a is a n-dim. irreducible rep., then $\Gamma_1 \times \Gamma_a$ is also an irreducible rep..

proof:

let $\Gamma = \Gamma_1 \times \Gamma_a = \{D^{(1)}(g) \otimes D^{(a)}(g), \cdots\}$, then,

$$\frac{1}{|G|} \sum_g |\chi(g)|^2 = \frac{1}{|G|} \sum_g \underbrace{|\chi^{(1)}(g)|^2}_{=1} |\chi^{(a)}(g)|^2 = 1 \tag{1.4.24}$$

Part II

General Theory

Chapter 2

Lie groups

2.1 Lie groups

- **Lie group** G is a group and a manifold,
  - group multiplication, $G \times G \to G$, is C^∞.
  - inverse, $G \to G$, is C^∞.
- **left transformation**, $L_g : G \to G, L_g(h) = gh$.
  - $L_e = \mathrm{id}$.
  - $L_g L_h = L_{gh}$.
  - $L_g^{-1} = L_{g^{-1}}$.
  - L_g is diffeomorphism, i.e. bijective + C^∞.
- property of elements near e, if $x^i(e) = 0$, then,

$$x^i(gh) = x^i(g) + x^i(h) \tag{2.1.1}$$

proof:

$$\begin{aligned} gh &= \Big(e + x^i(g)\frac{\partial g}{\partial x^i}\Big|_e + \cdots\Big)\Big(e + x^i(h)\frac{\partial g}{\partial x^i}\Big|_e + \cdots\Big) \\ &= e + (x^i(g) + x^i(h))\frac{\partial g}{\partial x^i}\Big|_e + \cdots \end{aligned} \tag{2.1.2}$$

consequently, $x^i(g^{-1}) = -x^i(g)$.

  - for example, GL,

$$x_{ij}(I + \Delta) = \Delta_{ij} \tag{2.1.3}$$

2.2 topological properties

2.2.1 compactness

- compactness is a property that seeks to generalize the notion of a **closed** and **bounded** subset of Euclidean space.

  The idea is that a compact space has no "punctures" or "missing endpoints", i.e. it includes all **limiting** values of points.
- **def.:** compact Lie group:
  - 有限个 $\mathbb{R}^n$ 中的闭集通过坐标映射到 Lie group 上可以覆盖整个 Lie group.
  - 注意, $\mathbb{R}$ 不是闭集, $\mathbb{R} \cup \{\pm\infty\}$ 才是闭集.
- **Heine-Borel theorem:**

  a **matrix** Lie group is compact $\iff$ it is topologically **closed** as a subset of $\mathcal{M}_m(\mathbb{C})$ and **bounded**.

compact	noncompact
$\mathrm{O}(m), \mathrm{SO}(m), \mathrm{U}(m), \mathrm{SU}(m), \mathrm{Sp}(m)$	$\mathrm{SL}(m, \mathbb{R})$ (not bounded)

2.2.2 connectedness

- a topological space is connected if it is not the union of two **disjoint nonempty open sets**.
- **matrix** Lie group is **connected** $\iff$ it is **path-connected**.

- the **identity component** of G, denoted by G_0, is the biggest connected subset containing I.
  - G_0 is a **normal subgroup** of G.

proof:

* G_0 is a subgroup.
  $\forall A, B \in G_0$ there are paths $A(t), B(t)$ connecting to I.
  then $A(t)B(t)$ is a continuous path connecting I and AB.
  $(A(t))^{-1}$ is... I and A^{-1}.
* G_0 is invariant.
  $\forall A \in G_0, B \in G$ there are a path $BA(t)B^{-1}$ connecting BAB^{-1} and I.

2.2.3 simple connectedness

- a topological space is **simply connected** $\iff$ it is **path connected** and every **loop** can be **shrunk continuously into a point**.

more precisely:

for every loop $A(t), t \in [0,1]$ in G, $A(0) = A(1)$.
there exist a function $A(s,t), s, t \in [0,1]$ such that:

- $A(0,t) = A(t)$ is the original loop.
- $A(1,t) = A(1,0)$ is a point.
- $A(s,0) = A(s,1)$ which means $A(s,t)$ is a loop.

- summary:

matrix Lie groups	compactness	components	simple connectedness
$\mathrm{GL}(m, \mathbb{C})$	no	1	no
$\mathrm{GL}(m, \mathbb{R})$	no	2	no
$\mathrm{SL}(m, \mathbb{C})$	no	1	yes
$\mathrm{SL}(m, \mathbb{R})$	no	1	no
$\mathrm{O}(m)$	yes	2	
$\mathrm{SO}(m)$	yes	1	no
$\mathrm{U}(m)$	yes	1	no
$\mathrm{SU}(m)$	yes	1	yes
$\mathrm{O}(m,1)$	yes	4	
$\mathrm{SO}(m,1)$	yes	2	$m=1$, yes; $m \geq 2$, no
$\mathrm{E}(m)$ (Euclidean group)		2	
$\mathrm{P}(m,1)$ (Poincaré group)		4	

2.3 Lie subgroups

- **def.:** a **Lie subgroup** H of a Lie group G is a subgroup which is also a submanifold.

- **closed subgroup theorem:** {closed subgroups} = {Lie subgroups}.

proof:

first, let's prove that a closed subgroup H is a Lie subgroup.

- let,

$$\mathfrak{h} = \{A \in \mathfrak{g} | \exp(tA) \in H, \forall t \in \mathbb{R}\} \tag{2.3.1}$$

  - $\mathfrak{h}$ is a subspace of $\mathfrak{g}$.

$$\begin{aligned}\lim_{n\to\infty}\Big(\exp(\frac{A}{n})\exp(\frac{B}{n})\Big)^n &= \lim_{n\to\infty}\Big(\exp(\frac{A}{n}+\frac{B}{n}+O(\frac{1}{n^2}))\Big)^n \\ &= \exp(A+B) \in H\end{aligned} \tag{2.3.2}$$

    极限存在要求 H 是**闭集**.

- $W \subset \mathfrak{h}$ is a neighborhood of 0, which is small enough that $\exp : W \to H$ is a one-to-one homomorphism (**local diffeomorphism**).
- $\exp^{-1} : \exp[V] \to V$ with $V \cap \mathfrak{h} = W$ is a diffeomorphism, so $(\exp^{-1}, \exp[V], V)$ is a chart on G, which can be extended by left translation. so, H is a submanifold.

---

second, let's prove that Lie subgroups are closed.

- 暂时不会证.

Chapter 3

Lie algebras

3.1 left-invariant vector fields

- vector field $\bar{A}$ is invariant under push-forward, $L_{g*}: V_h \to V_{gh}, \forall h$,

$$(L_{g*}\bar{A})\big|_{gh} = \bar{A}\big|_{gh} \tag{3.1.1}$$

  i.e.,

$$\bar{A}(x^i)\big|_h = \bar{A}(y^i)\big|_{gh} \tag{3.1.2}$$

  where $L_g^* y^i = x^i \iff y^i(gh) = x^i(h)$.

- see appendix B, maps between manifolds.
- the set of all left invariant vector filed is denoted by $\mathfrak{g}$, and $\mathfrak{g} \simeq V_e$.

3.2 Lie algebras

- $A \equiv \bar{A}_e$ and $\bar{A}_g = L_{g*}A, \forall g$.
- a vector space, V, along with Lie bracket, $[,]: V \times V \to V$, is a **Lie algebra**,
  - $[A, B] = -[B, A]$.
  - Jacob identity, $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$.
- for a Lie group G, its Lie bracket is the commutator,

$$[\bar{A}, \bar{B}]^a = \bar{A}^b \nabla_b \bar{B}^a - \bar{B}^b \nabla_b \bar{A}^a \tag{3.2.1}$$

  - $L_{g*}[\bar{A}, \bar{B}] = [L_{g*}\bar{A}, L_{g*}\bar{B}] = [\bar{A}, \bar{B}] \in \mathfrak{g}$.

proof:

$$\begin{aligned} L_{g*}[\bar{A}, \bar{B}] &= L_{g*}\Big(\frac{\partial}{\partial x^i}\Big|_h\Big)\Big(A^j \frac{\partial}{\partial x^j} B^i - B^j \frac{\partial}{\partial x^j} A^i\Big)\Big|_{h,x} \\ &= \Big(\frac{\partial}{\partial y^i}\Big|_{gh}\Big)\Big(A^j \frac{\partial}{\partial x^j} B^i - B^j \frac{\partial}{\partial x^j} A^i\Big)\Big|_{h,x} \end{aligned} \tag{3.2.2}$$

notice that for left-invariant v. f. as a scalar field, $(L_g^* A^i\big|_y)\big|_h = A^i\big|_{gh,y}$ and,

$$\begin{aligned} &\Big(\frac{\partial}{\partial x^j} A^i\Big)\Big|_{h,x} \equiv \Big(\frac{\partial}{\partial x^j}\Big)\Big|_h (L_g^* A^i\big|_y)\Big|_h = L_{g*}\Big(\frac{\partial}{\partial x^j}\Big|_h\Big)(A^i\big|_{gh,y}) \\ \Longrightarrow &\Big(\frac{\partial}{\partial x^j} A^i\Big)\Big|_{h,x} = \Big(\frac{\partial}{\partial y^j} A^i\Big)\Big|_{gh,y} \end{aligned} \tag{3.2.3}$$

so $L_{g*}[\bar{A}, \bar{B}] = [\bar{A}, \bar{B}]$.

  - satisfies the Jacob identity.

proof:

$$
\begin{aligned}
&[A,[B,C]]+[C,[A,B]]+[B,[C,A]] \\
=&A^c\partial_c(B^b\partial_b C^a - C^b\partial_b B^a) - (B^c\partial_c C^b - C^c\partial_c B^b)\partial_b A^a + \cdots \\
=&A^c\partial_c(B^b)\partial_b C^a + A^c B^b\partial_c\partial_b C^a - A^c\partial_c(C^b)\partial_b B^a + A^c C^b\partial_c\partial_b B^a \\
&- B^c\partial_c(C^b)\partial_b A^a + C^c\partial_c(B^b)\partial_b A^a \\
&+ (B\partial C\partial A - B\partial A\partial C - C\partial A\partial B + A\partial C\partial B) \\
&+ (BC\partial\partial A - BA\partial\partial C) \\
&+ (C\partial A\partial B - C\partial B\partial A - A\partial B\partial C + B\partial A\partial C) \\
&+ (CA\partial\partial B - CB\partial\partial A) = 0
\end{aligned}
\tag{3.2.4}
$$

- **def.:** the **Lie algebra direct sum** of two Lie algebras, $\mathfrak{g}_1, \mathfrak{g}_2$, is the **vector space direct sum** (i.e. $\mathfrak{g}_1, \mathfrak{g}_2$ are linearly independent $\iff \mathfrak{g}_1 \cap \mathfrak{g}_2 = \{0\}$), $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with the Lie bracket defined to be,

$$
\begin{aligned}
&[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g} \\
&[A_1 + A_2, B_1 + B_2] = [A_1, B_1] + [A_2, B_2] \quad \forall A_1, B_1 \in \mathfrak{g}_1, A_2, B_2 \in \mathfrak{g}_2
\end{aligned}
\tag{3.2.5}
$$

i.e. we define the Lie bracket in the way that $[\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$.

3.2.1 subalgebras, ideals & simple, solvable, nilpotent Lie algebras

- **def.: subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is a subspace, satisfying that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.
  - **def.:** Abelian subalgebra $\mathfrak{h}$ is a subalgebra, satisfying that $[A, B] = 0, \forall A, B \in \mathfrak{h}$.
- **def.:** invariant subalgebra (i.e. **ideal**) $\mathfrak{h}$ is a subalgebra, satisfying that $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.
  - Abelian ideal.
  - proper invariant subalgebra (also called **proper ideal**) is an ideal that is not $\mathfrak{g}, \{0\}$.
  - trivial subalgebras are $\mathfrak{g}, \{0\}$.

- Lie algebra decomposes as the direct sum of its ideals, $\mathfrak{h}_1, \mathfrak{h}_2, \cdots$, i.e.,

$$
\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \tag{3.2.6}
$$

then $\oplus$ is called **Lie algebra direct sum**.

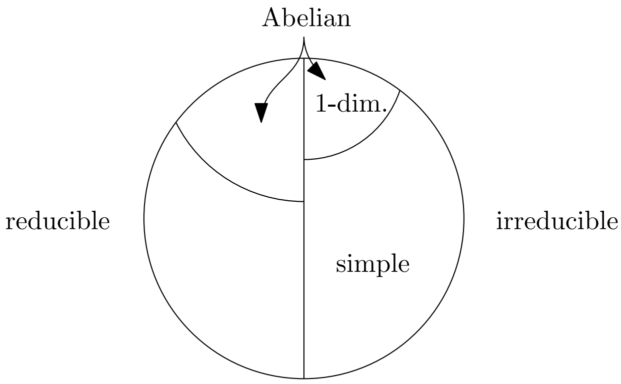
proof:

by def., $[\mathfrak{h}_i, \mathfrak{h}_j] \subseteq \mathfrak{h}_i \cap \mathfrak{h}_j = \{0\}$, if $i \neq j$.

- **def.:** a Lie algebra **without nontrivial ideal** is **irreducible**.
  - all 1-dim. Lie algebras are irreducible.
- **def.:** a **irreducible** Lie algebra with $\dim \mathfrak{g} \geq 2$ is **simple**.
  - equivalent **def.:** irreducible non-Abelian Lie algebras are simple.

proof:

all the subspaces of an Abelian Lie algebra is its ideal $\Longrightarrow$ Abelian Lie algebras aren't irreducible unless dim $= 1$, so,



- **def.:** a Lie algebra $\mathfrak{g}$ is **solvable** if $\mathfrak{g}_i = \{0\}$ for some i, where,

$$\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}_i] \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{g} \tag{3.2.7}$$

  - $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i-1}$, but not necessarily an ideal in $\mathfrak{g}$.

proof:

$\forall A \in \mathfrak{g}_i \subseteq \mathfrak{g}_{i-1}$ and $\forall B \in \mathfrak{g}_{i-1}$, $[A, B] \in \mathfrak{g}_i$, which means $[\mathfrak{g}_i, \mathfrak{g}_{i-1}] \subseteq \mathfrak{g}_i$.

- **def.:** a Lie algebra $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}^i = \{0\}$ for some i, where,

$$\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i] \quad \text{and} \quad \mathfrak{g}^0 = \mathfrak{g} \tag{3.2.8}$$

  - $\mathfrak{g}^{i+1} \subseteq \mathfrak{g}^i$.
  - $\mathfrak{g}^i$ is an ideal in $\mathfrak{g}$.
  - nilpotent Lie algebra is solvable.

3.2.2 structure constants

- structure constants,

$$[X_i, X_j] = i f_{ij}{}^k X_k \iff [X_i, X_j]^a = i f_{bc}{}^a (X_i)^b (X_j)^c \tag{3.2.9}$$

$$[A_i, A_j] = -f_{ij}{}^k A_k \iff [A_i, A_j]^a = -f_{bc}{}^a (A_i)^b (A_j)^c \tag{3.2.10}$$

where $X_i = -iA_i$ are called the generators.

  - if the generators are Hermitian, then the structure constants are real,

$$[X_i, X_j]^\dagger = -i f^*_{ij}{}^k X_k = [X_j, X_i] = i \underbrace{f_{ji}{}^k}_{=-f_{ij}{}^k} X_k \Longrightarrow f^*_{ij}{}^k = f_{ij}{}^k \tag{3.2.11}$$

Chapter 4

exponential maps

4.1 one-parameter subgroups

- a C^∞ (Lie group) homomorphism $\gamma : \mathbb{R} \to G$, with $\gamma(s)\gamma(t) = \gamma(s+t)$.
- $\{\gamma(s)|s \in \mathbb{R}\}$ is an **integral curve** (passing through e) of a **left-invariant vector field**.
  - the integral curve of a left-invariant vector field is complete, i.e. it's homomorphism to $\mathbb{R}$.

proof:

notation: $\frac{d}{dt}\gamma(t) \equiv \frac{\partial}{\partial t}(\equiv \frac{dx^i(\mu(t))}{dt}\frac{\partial}{\partial x^i})$
let $\mu : (-\epsilon, \epsilon) \to G$ be an integral curve of $\bar{A}$, with $\mu(0) = e$, then,

$$\frac{d}{dt}\Big|_s \mu(t) = A_{\mu(s)} = L_{\mu(s)*}(A_e) = L_{\mu(s)*}\frac{d}{dt}\Big|_0 \mu(t) = \frac{d}{dt}\Big|_{t=0}(\mu(s)\mu(t)) \tag{4.1.1}$$

---

calculation:

$$\frac{dx^i(\mu(t))}{dt}\Big|_s = \Big(L_{\mu(s)*}\frac{d}{dt}\Big|_0 \mu(t)\Big)x^i\Big|_{\mu(s)} = \Big(\frac{d}{dt}\Big|_0 \mu(t)\Big)y^i\Big|_e \tag{4.1.2}$$

where $y^i\big|_g \equiv L^*_{\mu(s)}x^i\big|_g = x^i\big|_{\mu(s)g}$ so,

$$\Big(\frac{d}{dt}\Big|_0 \mu(t)\Big)y^i\Big|_e = \frac{dy^i(\mu(t))}{dt}\Big|_e = \frac{dx^i(\mu(s)\mu(t))}{dt}\Big|_{t=0} \tag{4.1.3}$$

---

so, as we can see, $\nu : (-\epsilon + s, \epsilon + s) \to G, t \mapsto \mu(s)\mu(t-s)$ is also an integral curve of $\bar{A}$, with at least one intersection with μ, $\nu(s) = \mu(s)$.
since a vector field only has one integral curve through a fixed point,

---

proof:
for a vector field A, the integral curve μ through point p must satisfy,

$$\frac{dx^i(\mu(t))}{dt}\Big|_s = A^i\Big|_{\mu(s)} \tag{4.1.4}$$

which is a linear differential equation of order one, consequently, the solution can be determined by $x^i(\mu(t)) = \text{Const.}$.

---

we can conclude that μ and ν is all part of one complete integral curve through e, $\gamma : \mathbb{R} \to G$.

  - the integral curve of $\bar{A}$ through e is a one-parameter subgroup.

proof:

we have already proved that $\nu(s+t) = \mu(s)\mu(t)$ and $\mu = \nu = \gamma$.
so $\gamma(s+t) = \gamma(s)\gamma(t)$.

- the tangent vector of γ is left-invariant.

proof:

$$\left(L_{\gamma(t_2)*}\frac{d}{dt}\Big|_{t_1}\gamma(t)\right)x^i\Big|_{\gamma(t_2+t_1)} = \frac{dx^i(\gamma(t_2+t))}{dt}\Big|_{t_1} = \left(\frac{d}{dt}\gamma(t)\right)x^i\Big|_{\gamma(t_2+t_1)} \tag{4.1.5}$$

- a useful lemma: for a curve γ on manifold M_1, and a map $\psi : M_1 \to M_2$, then,

$$\psi_*\left(\frac{d}{dt}\Big|_{p\in M_1}\gamma\right) = \frac{d}{dt}\Big|_{\psi(p)\in M_2}\psi\circ\gamma \tag{4.1.6}$$

the proof is in appendix B.1.4.

4.2 exponential maps

- **def.:** exp. map on a **Riemann manifold**, $\exp_p : V_p(\text{or its subspace}) \to M$.
  - $\exp_p(v) = \gamma(1)$, where γ is the geodesic determined by v and p.
- **def.:** exp. map on a **Lie group**, $\exp : V_e \to G$.
  - $\exp(A) = \gamma(1)$ where γ is the one-para. subgroup determined by $\bar{A}$.
  - def. for physicists: $\exp : \mathfrak{g} \to G$, with $\exp(iX) = \exp(A) = \gamma(1)$.
- **theorem:** for **compact** Lie group, the exponential map, $\exp : V_e \to G$, is **onto**.

4.2.1 matrix exponential and logarithm

- properties of exp. function of matrices (in general linear group):
  - $(e^A)^\dagger = e^{A^\dagger}$.
  - if $\det e^A \neq 0$, then $(e^A)^{-1} = e^{-A}$.
  - $\det e^A = e^{\mathrm{tr}A}$.

proof:

* if A is diagonalizable,
  diagonalize A by T, $TAT^{-1} = D = \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$, then,

$$\det e^A = \det(Te^AT^{-1}) = \det e^D = e^{\lambda_1+\cdots+\lambda_m} = e^{\mathrm{tr}A} \tag{4.2.1}$$

* otherwise, it is still can be proved as follow,

$$\frac{d}{dt}\Big|_t\det(e^{tA}) = \frac{d}{ds}\Big|_{s=0}\det(e^{(s+t)A}) = \det(e^{tA})\frac{d}{ds}\Big|_{s=0}\det(e^{sA}) \tag{4.2.2}$$

and,

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\det(e^{sA}) &= \frac{d}{ds}\Big|_{s=0}\det(I+sA)\\ &= \frac{d}{ds}\Big|_{s=0}\epsilon_{ij\cdots k}(\delta^i_1 + sA^i{}_1)\cdots(\delta^k_m + sA^k{}_m)\\ &= \epsilon_{i2\cdots m}A^i{}_1 + \cdots + \epsilon_{12\cdots k}A^k{}_m = \mathrm{tr}A\end{aligned} \tag{4.2.3}$$

so we have,

$$\begin{cases} \dfrac{1}{\det(e^{tA})}\dfrac{d}{dt}\Big|_t \det(e^{tA}) = \mathrm{tr}A \\ \det(e^{tA})\Big|_{t=0} = 1 \end{cases} \Longrightarrow \det(e^{tA}) = e^{t\,\mathrm{tr}A} \tag{4.2.4}$$

– Baker-Campbell-Hausdorff formula,

$$e^A e^B = \exp\Big(A + B + \frac{1}{2}[A,B] + \frac{1}{12}\Big([A,[A,B]] + [B,[B,A]]\Big) + \cdots\Big) \tag{4.2.5}$$

- the Hilbert-Schmidt norm of $A \in \mathcal{M}_m(\mathbb{C})$ is,

$$\|A\| = \Big(\sum_{i,j=1}^{m} |A_{ij}|^2\Big)^{1/2} \tag{4.2.6}$$

- matrix logarithm is,

$$\ln M = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(M-I)^n}{n} \tag{4.2.7}$$

where M is a complex matrix with $\|M - I\| < 1$.

  – $\forall M$ with $\|M - I\| < 1$, $e^{\ln M} = M$.
  – $\forall A$ with $\|A\| < \ln 2$ then $\|e^A - I\| < 1$ and $\ln e^A = A$.

- for a **connected** Lie group G, every element $g \in G$ can be written in the form,

$$g = \exp(A_1)\exp(A_2)\cdots\exp(A_N) \tag{4.2.8}$$

for some $A_1, A_2, \cdots, A_N \in \mathfrak{g}$.

proof:

曲线 $\gamma : [0,1] \to G, \gamma(0) = I, \gamma(1) = g$.
选取 N 足够大, 使得 $\gamma^{-1}(\frac{i-1}{N})\gamma(\frac{i}{N})$ 在 I 的邻域, 那么, 存在 $A_i \in \mathfrak{g}$ 使得,

$$\gamma^{-1}(\frac{i-1}{N})\gamma(\frac{i}{N}) = \exp(A_i) \tag{4.2.9}$$

所以,

$$g = \gamma^{-1}(0)\gamma(1) = \exp(A_1)\cdots\exp(A_N) \tag{4.2.10}$$

错误的推断:

combined with BCH formula, $\exp : \mathfrak{g} \to G$ is onto for connected Lie groups, i.e. $G \neq \exp[\mathfrak{g}]$.

  – onto 仅对 **compact connected** Lie groups 成立,
  – 原因: BCH 公式中的级数展开可能不存在.

4.3 Baker-Campbell-Hausdorff formula

4.3.1 the Campbell's identity

- $\mathrm{Ad}_{\exp(A)} = e^{\mathrm{ad}_A} : V_e \to V_e$.

proof: (maybe not very rigorously)

consider,

$$B(s) = \mathrm{Ad}_{\exp(sA)}(B) = \frac{d}{dt}\Big|_0 \exp(sA)\exp(tB)\exp(-sA) \tag{4.3.1}$$

the derivative of $B(s)$ is,

$$\frac{dB(s)}{ds} = \lim_{\Delta s \to 0} \frac{\text{numerator}}{\Delta s} = [A, \mathrm{Ad}_{\exp(sA)}(B)] = \mathrm{ad}_A B(s) \tag{4.3.2}$$

---

where the numerator is:

$$\begin{aligned}
&\text{numerator} \\
=&\frac{d}{dt}\Big|_0 \exp(sA)(1+\Delta sA)\exp(tB)\exp(-sA)(1-\Delta sA) \\
&-\frac{d}{dt}\Big|_0 \exp(sA)\exp(tB)\exp(-sA) \\
=&\Delta s[A, \mathrm{Ad}_{\exp(sA)}(B)]
\end{aligned} \tag{4.3.3}$$

---

so, the nth derivative is $\frac{d^n}{ds^n}B(s) = (\mathrm{ad}_A)^n B(s)$, then naturally,

$$B(s) = e^{\mathrm{ad}_A} B \tag{4.3.4}$$

4.3.2 BCH formula

- theorem 1 (Campbell's identity in the case of $\mathfrak{gl}(m)$):

$$e^A B e^{-A} = e^{\mathrm{ad}_A} B \tag{4.3.5}$$

proof:

consider $F(t) = e^{tA}Be^{-tA}$, so $F(0) = B$, and,

$$\frac{d}{dt}F(t) = [A, F(t)] = \mathrm{ad}_A F(t) \Longrightarrow \frac{d^n}{dt^n}F(t) = (\mathrm{ad}_A)^n F(t) \tag{4.3.6}$$

so it is clear that $F(t) = e^{\mathrm{ad}_A}B$.

- theorem 2:

$$e^{A(t)}\frac{d}{dt}e^{-A(t)} = -f(\mathrm{ad}_A)\frac{dA(t)}{dt} \tag{4.3.7}$$

where $f(z) = \frac{e^z-1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$.

proof:

consider $F(s,t) = e^{sA(t)}\frac{d}{dt}e^{-sA(t)}$, with $F(0,t) = 0$, and,

$$\begin{aligned}
\frac{d}{ds}F(s,t) &= A(t)F(s,t) - e^{sA(t)}\frac{d}{dt}\left(A(t)e^{-sA(t)}\right) \\
&= -e^{sA(t)}\frac{dA(t)}{dt}e^{-sA(t)} \\
&= -e^{\mathrm{ad}(sA(t))}\frac{dA(t)}{dt}
\end{aligned} \tag{4.3.8}$$

and the nth derivative is,

$$\frac{d^n}{ds^n}F(s,t) = \mathrm{ad}^{n-1}(A(t))\frac{d}{ds}F(s,t) \tag{4.3.9}$$

when $s=0$, $\frac{d^n}{ds^n}\Big|_{s=0}F(s,t) = -\mathrm{ad}^{n-1}(A(t))\frac{dA(t)}{dt}$, so,

$$F(s=1,t) = -\sum_{n=1}^{\infty}\frac{\mathrm{ad}^{n-1}(A(t))}{n!}\frac{dA(t)}{dt} \tag{4.3.10}$$

(the 0th order term is 0)

- theorem 3:

$$\frac{d}{dt}e^{-A(t)} = -\int_0^1 e^{-sA(t)}\frac{dA(t)}{dt}e^{-(1-s)A(t)}ds \tag{4.3.11}$$

proof:

consider the following equation,

$$e^{-A} - e^{-B} = \int_0^1 e^{-sA}(B-A)e^{-(1-s)B}ds \tag{4.3.12}$$

---

proof:

consider the following equation,

$$e^{-sA}(B-A)e^{-(1-s)B} = \frac{d}{ds}\left(e^{-sA}e^{-(1-s)B}\right) \tag{4.3.13}$$

integrate both side of the equation,

$$\int_0^1 \cdots ds = e^{-A} - e^{-B} \tag{4.3.14}$$

---

take $A = A(t), B = A(t-\Delta t)$, with $\Delta t \to 0$, then,

$$\frac{d}{dt}e^{-A(t)} = -\int_0^1 e^{-sA(t)}\frac{dA(t)}{dt}e^{-(1-s)A(t)}ds \tag{4.3.15}$$

- theorem 3 is equivalent to theorem 2.

calculation:

$$\begin{aligned} e^{A(t)}\frac{d}{dt}e^{-A(t)} &= -\int_0^1 e^{(1-s)A(t)}\frac{dA(t)}{dt}e^{-(1-s)A(t)}ds \\ &= -\int_0^1 \underbrace{e^{\mathrm{ad}((1-s)A(t))}}_{=e^{(1-s)\mathrm{ad}_{A(t)}}}\frac{dA(t)}{dt}ds \\ &= -f(\mathrm{ad}_{A(t)})\frac{dA(t)}{dt} \end{aligned} \tag{4.3.16}$$

where $f(z)$ is defined in theorem 2.

- the Baker-Campbell-Hausdorff formula is,

$$\begin{aligned} e^A e^B &= \exp\left(B + \left(\int_0^1 g(e^{t\,\mathrm{ad}_A}e^{\mathrm{ad}_B})dt\right)A\right) \\ &= \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \cdots\right) \end{aligned} \tag{4.3.17}$$

where $g(z) = \frac{\ln z}{z-1} = \sum_{n=0}^{\infty}\frac{(1-z)^n}{n+1}$, for $|z-1| < 1$.

proof:

consider $e^{C(t)} = e^{tA}e^B$, then,

$$e^{\mathrm{ad}_{C(t)}} = e^{t\,\mathrm{ad}_A}e^{\mathrm{ad}_B} \tag{4.3.18}$$

---

proof:

consider the following equation,

$$e^{\mathrm{ad}_{C(t)}}W = e^{C(t)}We^{-C(t)}$$

$$
\begin{aligned}
&= e^{tA} e^{B} W e^{-B} e^{-tA} \\
&= e^{tA} e^{\mathrm{ad}_B} W e^{-tA} \\
&= e^{t\,\mathrm{ad}_A} e^{\mathrm{ad}_B} W
\end{aligned} \tag{4.3.19}
$$

---

then, let's consider, (notice that $\mathrm{ad}_A A = 0$),

$$
\begin{aligned}
e^{C(t)} \frac{d}{dt} e^{-C(t)} &= -f(\mathrm{ad}_{C(t)}) \frac{dC(t)}{dt} \\
&= e^{tA} e^{B} \frac{d}{dt} e^{-B} e^{-tA} \\
&= e^{tA} \frac{d}{dt} e^{-tA} \\
&= -f(t\,\mathrm{ad}_A) A = -A
\end{aligned} \tag{4.3.20}
$$

$$
\Longrightarrow f(\mathrm{ad}_{C(t)}) \frac{dC(t)}{dt} = A \tag{4.3.21}
$$

notice that $g(e^z) = 1/f(z)$, so we have,

$$
\frac{dC(t)}{dt} = g(e^{\mathrm{ad}_{C(t)}}) A \Longrightarrow C(1) - \underbrace{C(0)}_{=B} = \Big(\int_0^1 g(e^{t\,\mathrm{ad}_A} e^{\mathrm{ad}_B}) dt \Big) A \tag{4.3.22}
$$

Chapter 5

basic representation theory

5.1 Lie group and Lie algebra homomorphisms

- $\Phi : G \to H$ is a **Lie group homomorphism**, then there exists a unique real-linear map $\phi = \Phi_* : \mathfrak{g} \to \mathfrak{h}$ s.t.,

$$\Phi \circ \exp(A) = \exp(\phi A) \tag{5.1.1}$$

  ϕ has the following properties:

  1. $\phi \mathrm{Ad}_g(A) = \mathrm{Ad}_{\Phi(g)}(A), \forall A, g,$
  2. ϕ is **Lie algebra homomorphism**,
  3. $\phi(A) = \frac{d}{dt}\Big|_0 \Phi \circ \exp(tA)$.

proof:

let's prove the 3rd identity first,

$$\begin{aligned}\left(\Phi_* \frac{d}{dt}\Big|_s \gamma(t)\right) y^i = \left(\frac{d}{dt}\Big|_s \gamma(t)\right) \Phi^* y^i = \frac{d\Phi^* y^i(\gamma(t))}{dt}\Big|_s = \frac{dy^i(\Phi\gamma(t))}{dt}\Big|_s \\ \Longrightarrow \Phi_* \circ L_{\exp(sA)*} A = \frac{d}{dt}\Big|_s \Phi \exp(tA)\end{aligned} \tag{5.1.2}$$

and,

$$\begin{cases} L_{\Phi(g)*} \circ \Phi_* A = (L_{\Phi(g)} \circ \Phi)_* A \\ L_{\Phi(g)} \circ \Phi = \Phi \circ L_g \end{cases} \Longrightarrow L_{\Phi(g)*} \circ \Phi_* A = \Phi_* \circ L_{g*} A \tag{5.1.3}$$

so,

$$\frac{d}{dt}\Big|_s \Phi \exp(tA) = L_{\Phi \exp(sA)*} \circ \Phi_* A \Longrightarrow \exp(\Phi_* A) = \Phi \exp(A) \tag{5.1.4}$$

---

the 1st identity is easy to prove,

$$\mathrm{Ad}_g \equiv I_{g*} \Longrightarrow \begin{cases} \Phi_* \circ I_{g*} = (\Phi \circ I_g)_* \\ \Phi \circ I_g = I_{\Phi(g)} \circ \Phi \end{cases} \Longrightarrow \cdots \tag{5.1.5}$$

---

now let's prove the 2nd identity,

$$L_{\Phi(g)*} \circ \Phi_* A = \Phi_* \circ L_{g*} A \Longrightarrow (\Phi_* A)_{\Phi(g)} = \Phi_* A_g \tag{5.1.6}$$

$$\Longrightarrow ((\Phi_* A)_{\Phi(g)})^i \Phi_* \frac{\partial}{\partial x^i} = (A_g)^i \Phi_* \frac{\partial}{\partial x^i} \tag{5.1.7}$$

$$\Longrightarrow A^i\Big|_g = \Phi^*((\Phi_* A)^i\Big|_{\Phi(g)}) \tag{5.1.8}$$

where A^i and $(\Phi_* A)^i$ are treated as functions on G and H.
so,

$$(\Phi_*[A, B]_g)^i \Phi_* \frac{\partial}{\partial x^i} = \left((A_g)^j \frac{\partial}{\partial x^j}(B_g)^i - \cdots\right) \Phi_* \frac{\partial}{\partial x^i} \tag{5.1.9}$$

and,

$$([\Phi_* A, \Phi_* B]_{\Phi(g)})^i \Phi_* \frac{\partial}{\partial x^i} = \Big((\Phi_* A)^a \nabla_a (\Phi_* B)^i - \cdots\Big)\Big|_{\Phi(g)} \Phi_* \frac{\partial}{\partial x^i} \tag{5.1.10}$$

where,

$$\begin{aligned}(\Phi_* A)^a \nabla_a (\Phi_* B)^i\Big|_{\Phi(g)} &= (A_g)^i \Phi_* \frac{\partial}{\partial x^i}(\Phi_* B)^i \\ &= (A_g)^i \frac{\partial}{\partial x^i}\Big|_{\Phi(g)} \Phi^*(\Phi_* B)^i \\ &= (A_g)^i \frac{\partial}{\partial x^i}\Big|_g B^i \end{aligned} \tag{5.1.11}$$

so, we proved that $\Phi_*[A, B] = [\Phi_* A, \Phi_* B]$.

- for a Lie group homomorphism $\Phi : G \to H$ and $\phi = \Phi_*$,

$$\mathrm{Lie}(\ker \Phi) = \ker \phi \tag{5.1.12}$$

proof:

- $\ker \Phi = \{g \in G | \Phi(g) = I\}$ is a **closed normal subgroup** of G.
  * $G(\ker \Phi)G^{-1} \subseteq \ker \Phi$.
  * $\{I\}$ is a closed subgroup, and Φ is continuous.
- $\mathrm{Lie}(\ker \Phi) \subseteq \ker \phi$.
  for all $A \in \mathrm{Lie}(\ker \Phi)$,

$$\Phi \exp(tA) \in \Phi(\ker \Phi) = \{I\} \Longrightarrow \phi A = \frac{d}{dt}\Big|_0 \Phi \exp(tA) = 0 \tag{5.1.13}$$

  so, $A \in \ker \phi$.
- $\mathrm{Lie}(\ker \Phi) \supseteq \ker \phi$.
  for all $A \in \ker \phi$,

$$\exp(\phi A) = \Phi \exp(A) = I \Longrightarrow \exp(A) \in \ker \Phi \tag{5.1.14}$$

  so, $A \in \mathrm{Lie}(\ker \Phi)$.

5.1.1 simply connected Lie groups

- Lie algebra homomorphism $\Longrightarrow$ Lie group homomorphism, when G is **simply connected**.
  $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, (if G is simply connected) then there **exist** a **unique** Lie group homomorphism $\Phi : G \to H$ s.t. $\Phi(\exp(A)) = \exp(\phi A)$ and $\phi = \Phi_*$.

proof:

G is **connected**, so, for all $g \in G$ there exists a path $g(t)$ s.t. $g(0) = I, g(1) = g$
N is large enough that,

$$g^{-1}(\frac{i-1}{N})g(\frac{i}{N}) \in U \tag{5.1.15}$$

---

where $U \subset G$ is a neighborhood of I s.t. there exists an isomorphism,

$$\begin{aligned} \ln : U &\to \ln[U] \subset \mathfrak{g} \\ g = \exp(A) &\mapsto A, \forall g \in U \end{aligned} \tag{5.1.16}$$

which implies that there exists a unique local homomorphism,

$$\begin{aligned} f : U &\to H \\ g &\mapsto \exp(\phi \ln g), \forall g \in U \end{aligned} \tag{5.1.17}$$

where,

$$\begin{aligned} f(g_1 g_2) &= \exp(\phi \ln(\exp(A_1)\exp(A_2))) \\ &= \exp\Big(\phi \ln \exp(A + B + \frac{1}{2}[A, B] + \frac{1}{12}\cdots)\Big) \\ &= \exp(\phi A)\exp(\phi B) \\ &= f(g_1)f(g_2) \end{aligned} \tag{5.1.18}$$

---

so, there **exists** a homomorphism,

$$\begin{aligned} \Phi &: G \to H \\ g &\mapsto f\Big(g^{-1}(0)g(\frac{1}{N})\Big)\cdots f\Big(g^{-1}(\frac{N-1}{N})g(1)\Big), \forall g \in G \end{aligned} \tag{5.1.19}$$

---

finally, the **uniqueness**:
Φ is independent from the choice of path $g(t)$ and the choice of partition $0 = t_0 < t_1 < \cdots t_N = 1$.

- independence of the partition:

  for any good partition (partition that guarantees $g^{-1}(t_{i-1})g(t_i) \in U$) insert s between t_{i-1} and t_i, since f is a local homomorphism,

  $$f(g^{-1}(t_{i-1})g(s))f(g^{-1}(s)g(t_i)) = f(g^{-1}(t_{i-1})g(t_i)) \tag{5.1.20}$$

- independence of the path:

  since G is **simply connected**, there exists a continuous map,

  $$\begin{aligned} g &: [0,1] \times [0,1] \to G \\ &g(s,t) = g_s(t) \\ &g(s,0) = I, g(s,1) = g \end{aligned} \tag{5.1.21}$$

  and choose a good partition that $g^{-1}_{s_{j-1}}(t)g_{s_j}(t) \in U$, so,

  $$\begin{cases} \Phi_{s_{j-1}}(g) = \cdots f(g^{-1}_{s_{j-1}}(t_{i-1})g_{s_{j-1}}(t_i))\cdots \\ \Phi_{s_j}(g) = \cdots f(g^{-1}_{s_j}(t_{i-1})g_{s_{j-1}}(t_{i-1})g^{-1}_{s_{j-1}}(t_{i-1})g_{s_{j-1}}(t_i)g^{-1}_{s_{j-1}}(t_i)g_{s_j}(t_i))\cdots \end{cases} \tag{5.1.22}$$

  the red terms will be canceled due to f is homomorphism.

  so $\Phi_{s_{j-1}} = \Phi_{s_j}$ which implies that $\Phi_0 = \Phi_1$.

显然, 根据上述选择,

$$\begin{cases} \Phi \circ \exp(A) = \exp(\phi A) \\ \Phi(g) = \exp(\phi A_1)\cdots\exp(\phi A_N) \end{cases} \tag{5.1.23}$$

---

now, let's prove $\phi = \Phi_*$.
consider,

$$\exp(\Phi_* A) = \exp(\phi A) \tag{5.1.24}$$

and if A is close to 0 enough, exp is one-to-one, moreover, Φ_* and ϕ is linear, so $\phi = \Phi_*$.

- for 2 **simply connected** Lie groups G, H, there exists a Lie algebra **isomorphism** $\phi : \mathfrak{g} \to \mathfrak{h}$, then G, H are **isomorphic** to each other.

  换句话说: simply connected Lie groups are determined by their Lie algebra.

  - but, exponential maps, $\exp : \mathfrak{g} \to G$, are **not** one-to-one even for simply connected Lie groups. e.g. in SU(2), $\exp(4\pi i J_3) = I$.

proof:

let Φ, Ψ correspond to ϕ, ϕ^{-1} respectively, then,

$$\Phi \circ \Psi(\exp(A_1) \cdots \exp(A_N)) = \exp(\phi \circ \phi^{-1} A_1) \cdots \exp(\phi \circ \phi^{-1} A_N) \tag{5.1.25}$$

which means $\Phi \circ \Psi = I$ similarly, $\Psi \circ \Phi = I$.
so Φ is a reversible homomorphism, i.e. an isomorphism.

- for a simply connected Lie group G, its Lie algebra $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, then, there exist 2 **closed, simply connected** subgroups H_1, H_2 corresponded to $\mathfrak{h}_1, \mathfrak{h}_2$ and $G \simeq H_1 \times H_2$.

proof:

consider the projection map $\phi_1 \in \mathrm{End}(\mathfrak{g})$, s.t. $\phi_1(A + B) = A, \forall A \in \mathfrak{h}_1, B \in \mathfrak{h}_2$.

- since G is simply connected, Φ_1 is the corresponding Lie group homomorphism.
- according to (5.1.12), $\ker \phi_1 = \mathfrak{h}_2 = \mathrm{Lie}(\ker \Phi_1)$.
- let H_2 be the identity component of $\ker \Phi_1$, thus H_2 is a **closed connected** Lie subgroup.
- construct H_1 in a similar way.

ϕ_1 is the identity on $\mathfrak{h}_1$, so Φ_1 is the identity on H_1.

- consider a loop $h(t)$ on H_1.
- there is a way to shrink $h(t)$ into a point on G, say $g(s,t)$ with $g(0,t) = h(t)$ and $g(1,t)$ is a point.
- define $h(s,t) = \Phi_1(g(s,t))$, then $h(0,t) = h(t)$ and $h(1,t)$ is a point.

so, H_1 is **simply connected**.

---

finally, let's prove $G \simeq H_1 \times H_2$.

- since $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, $[\mathfrak{h}_1, \mathfrak{h}_2] = \{0\}$, so $h_1 h_2 = h_2 h_1, \forall h_1 \in H_1, h_2 \in H_2$.
- $\Psi : H_1 \times H_2 \to G, (h_1, h_2) \mapsto h_1 h_2$ is a Lie group homomorphism.
  (we don't know $H_1 \times H_2$ is simply connected yet)
- $\psi = \Psi_* : \mathfrak{h}_1 \oplus \mathfrak{h}_2 \to \mathfrak{g}$ is the original isomorphism.

$$\exp(\psi(A + B)) = \Psi \circ \exp(A + B) = \exp(A + B) \Longrightarrow \psi(A + B) = A + B \tag{5.1.26}$$

- so the homomorphism $\Psi' : G \to H_1 \times H_2$ associated with ψ^{-1} is an isomorphism.

5.1.2 universal covers

- G is a **connected** Lie group, H is a **simply connected** Lie group with $\mathfrak{g} \simeq \mathfrak{h}$.

  then, H is the **universal cover** of G and the homomorphism $\Phi : H \to G$ associated to the isomorphism $\phi : \mathfrak{h} \to \mathfrak{g}$ is called the **covering map**.

- the universal cover of $\mathrm{SO}(3)$ is $\mathrm{SU}(2)$, and $\ker \Phi = \{\pm I\}$.
- the universal cover of $\mathrm{SO}(n \geq 3)$ is $\mathrm{Spin}(n)$ and may be constructed as a certain group of invertible elements in the **Clifford algebra** over $\mathbb{R}^n$.
  - the covering map is two-to-one.
  - and $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$.

5.2 basic representation theory

- **def.:** a **finite-dimensional representation** of a Lie group G (or a Lie algebra $\mathfrak{g}$) is a **Lie group** (or a **Lie algebra**) **homomorphism**,

$$\begin{cases} \Pi : G \to \mathrm{GL}(V) \\ \pi : \mathfrak{g} \to \mathfrak{gl}(V) \end{cases} \tag{5.2.1}$$

  where $\mathrm{GL}(V)$ is the group of invertible linear transformations of V and $\mathfrak{gl}(V) = \mathrm{End}(V)$ is the space of all linear operators from V to itself with Lie bracket $[A, B] = AB - BA$.

- for a finite-dimensional representation of G,

$$\pi(A) = \frac{d}{dt}\Big|_0 \Pi(e^{tA}) \tag{5.2.2}$$

  then $\Pi(\exp(A)) = e^{\pi(A)}$ and π is the representation of $\mathfrak{g}$ on the same vector space.

- subspace $W \subset V$ is **invariant** if $\Pi(g)[W] \subseteq W, \forall g \in G$.
- **def.:** a representation without nontrivial invariant subspaces ($\{0\}, V$) is called **irreducible**.
  对 Lie algebra 的 irreducible rep. 的定义是一样的.
- Π, π are associated representations of **connected** Lie group G and its Lie algebra $\mathfrak{g}$, then:
  - Π is **irreducible** $\iff$ π is **irreducible**.

proof:

* Π is irreducible $\Longrightarrow$ π is irreducible.
  设 $W \subseteq V$ 是 π 的不变子空间, 那么 $\forall g$,

$$\Pi(g)[W] = e^{\pi(A_1)} \cdots e^{\pi(A_N)}[W] \subseteq W \tag{5.2.3}$$

  (其中用到了 (4.2.8) 式), 而 Π 是不可约表示, 所以 $W = \{0\}$ or V
* Π is irreducible $\Longleftarrow$ π is irreducible.
  设 $W \subseteq V$ 是 Π 的不变子空间, 那么 $\forall A$,

$$\pi(A)[W] = \frac{d}{dt}\Big|_0 \Pi(\exp(tA))[W] \subseteq W \tag{5.2.4}$$

  所以...

  - Π_1, Π_2 are **isomorphic** $\iff$ π_1, π_2 are **isomorphic**.
- π is a **irreducible** rep. of $\mathfrak{g}_{\mathbb{C}}$ $\iff$ π is a (complex) **irreducible** rep. of $\mathfrak{g}$.
  where the rep. of $\mathfrak{g}_{\mathbb{C}}$ is $\pi(A + iB) = \pi(A) + i\pi(B)$ which is the unique extension of the rep. of $\mathfrak{g}$, π.

5.2.1 new representations from old

- three ways to obtain new rep. from old:
  1. direct sums,
  2. tensor products,
  3. dual representations.

direct sums

- **def.:** the direct sum of $\Pi_1, \cdots, \Pi_m$ is a rep. of G on $V_1 \oplus \cdots \oplus V_m$, defined by,

$$\Pi_1 \oplus \cdots \oplus \Pi_m(g)(v_1, \cdots, v_m) = (\Pi_1(g)v_1, \cdots, \Pi_m(g)v_m) \tag{5.2.5}$$

  对 Lie algebra rep. $\pi_1, \cdots, \pi_m$ 的直和的定义是一样的.

tensor products

- Π_1, Π_2 are rep. of G, H respectively. then, the tensor product rep. $\Pi_1 \otimes \Pi_2$ of $G \times H$ is defined to be,

$$(\Pi_1 \otimes \Pi_2)(g, h) = \Pi_1(g) \otimes \Pi_2(h) \tag{5.2.6}$$

- the tensor product rep. $\pi_1 \otimes \pi_2$ of $\mathfrak{g} \oplus \mathfrak{h}$ is,

$$(\pi_1 \otimes \pi_2)(A, B) = \pi_1(A) \otimes I + I \otimes \pi_2(B) \tag{5.2.7}$$

proof:

令 $\pi_1 : \mathfrak{g} \to \mathrm{End}(U), \pi_2 : \mathfrak{h} \to \mathrm{End}(V)$, 那么,

$$\begin{aligned}(\pi_1 \otimes \pi_2)(A, B)(u \otimes v) &= \left(\frac{d}{dt}\Big|_0 (\Pi_1 \otimes \Pi_2)(\exp(tA), \exp(tB))\right)(u \otimes v) \\ &= \frac{d}{dt}\Big|_0 \underbrace{\Pi_1(\exp(tA))u}_{=u(t)} \otimes \underbrace{\Pi_2(\exp(tB))v}_{=v(t)}\end{aligned} \tag{5.2.8}$$

其中, $u(t), v(t)$ 是 U, V 中的两条 C^∞ 的曲线,

$$(u + du) \otimes (v + dv) - u \otimes v = du \otimes v + u \otimes dv \tag{5.2.9}$$

代入, 所以,

$$(\pi_1 \otimes \pi_2)(A, B)(u \otimes v) = \pi_1(A)u \otimes v + u \otimes \pi_2(B)v \tag{5.2.10}$$

dual representations

- 对于 $\Pi : G \to \mathrm{End}(V)$, dual rep. 就是 $\Pi^\dagger : G \to \mathrm{End}(V^*)$, 其中 V^* 是 V 的对偶空间.

5.2.2 complete reducibility

- 参见有限群中的定义 (group 和 Lie algebra 的定义都一样).
- a group or Lie algebra is said to have the **complete reducibility property** if every finite-dim. rep. of it is completely reducible.

- **unitary** rep. of $G, \mathfrak{g}$ is **completely reducible**.
  notice, the 'unitary' (skew self-adjoint) rep. of $\mathfrak{g}$ is $\pi^\dagger(A) = -\pi(A)$
  证明参见有限群.
- **compact** Lie groups have the **complete reducibility property**.

proof:

for an n-dim. Lie group G,

$$\epsilon = A^1 \wedge \cdots \wedge A^n \tag{5.2.11}$$

is a **right-invariant n-form** composed of the dual vectors of a basis of $\mathfrak{g}$.
if G is **compact**, we can integrate any smooth function over all G, denoted by,

$$\int_G f(g)\epsilon(g) \tag{5.2.12}$$

and, since ϵ is right-invariant,

$$\int_G f(gh)\epsilon(g) = \int_G f(g)\epsilon(g) \tag{5.2.13}$$

---

for a rep. of G, $\Pi : G \to \mathrm{End}(V)$, define an arbitrary inner product $\langle\cdot,\cdot\rangle$ on V, then define another inner product on V by,

$$\langle\cdot,\cdot\rangle_G : V \times V \to \mathbb{C}$$

$$\langle u, v\rangle_G = \int_G \langle \Pi(g)u|\Pi(g)v\rangle\, \epsilon(g) \tag{5.2.14}$$

then,

$$\langle u, v\rangle_G = \langle \Pi(h)u, \Pi(h)v\rangle_G \tag{5.2.15}$$

and $\langle v, v\rangle_G > 0$ for all $v \neq 0$.
so, $\Pi(g)$ is **unitary** with respect to $\langle \cdot, \cdot\rangle_G$.

- $\mathrm{SU}(m)$ are compact, hence have the complete reducibility property.

5.2.3 Schur's lemma

- **def.:** an **intertwining map** of rep. Π_1, Π_2 (or π_1, π_2) is a linear map $\phi : V \to W$, s.t.,

$$\begin{cases} \phi\Pi_1(g) = \Pi_2(g)\phi \\ \phi\pi_1(A) = \pi_2(A)\phi \end{cases} \in \mathrm{End}(W) \tag{5.2.16}$$

- **Schur's 1st lemma**

  for 2 **irreducible real or complex rep.** Π_1, Π_2 (or π_1, π_2) on V, W, the intertwining map ϕ is either 0 or an isomorphism.
  证明参见有限群.

- **Schur's 2nd lemma**

  for a **irreducible complex rep.** Π (or π) on V, the intertwining map $\phi : V \to V$ is λI for some $\lambda \in \mathbb{C}$.

- **Schur's 3rd lemma**

  for 2 **irreducible complex rep.** Π_1, Π_2 (or π_1, π_2) on V, W, and 2 intertwining map $\phi_1, \phi_2 : V \to V$, then $\phi_1 = \lambda\phi_2$ for some $\lambda \in \mathbb{C}$.

5.3 Lie's third theorem

- **Lie's third theorem:** every **finite-dimensional** Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is associated to a Lie group G.
- every **finite-dimensional** Lie algebra is isomorphic to the Lie algebra of some **matrix** Lie group.

5.4 adjoint representations

5.4.1 adjoint rep. of Lie groups

- consider the adjoint diffeomorphism on G,

$$I_g : G \to G, h \mapsto ghg^{-1} \tag{5.4.1}$$

- $\mathrm{Ad}_g = I_{g*} : V_e \to V_e$ is the pushforward,

$$\mathrm{Ad}_g\Big(\frac{d}{dt}\Big|_0 \gamma(t)\Big)x^i\Big|_e = \frac{dy^i(\gamma(t))}{dt}\Big|_0 \tag{5.4.2}$$

  where $y^i(h) = x^i(ghg^{-1})$, so we have,

$$\mathrm{Ad}_g\Big(\frac{d}{dt}\Big|_0 \gamma(t)\Big) = \frac{d}{dt}\Big|_0 g\gamma(t)g^{-1} \tag{5.4.3}$$

  i.e. $\exp(\mathrm{Ad}_g(A)) = I_g \exp(A)$.

  - as we can see, $\mathrm{Ad}_g \in \mathrm{Aut}(V_e)$ is a linear and reversible automorphism on V_e, since $\mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}} = I$.

- $\mathrm{Ad} : G \to \mathrm{Aut}(V_e) \simeq \mathrm{GL}(m, \mathbb{R})$ is the **adjoint representation of the Lie group**, G.

  - Ad is a homomorphism.

proof:

$$\mathrm{Ad}_g \circ \mathrm{Ad}_h = I_{g*} \circ I_{h*} = (I_g \circ I_h)_* = \mathrm{Ad}_{gh} \tag{5.4.4}$$

5.4.2 adjoint rep. of Lie algebras

- The **structure constants** themselves generate a **representation of the Lie algebra**, called the **adjoint representation**.
- the Jacob identity written in the structure constants is,

$$f_{il}{}^m f_{jk}{}^l + f_{kl}{}^m f_{ij}{}^l + f_{jl}{}^m f_{ki}{}^l = 0 \tag{5.4.5}$$

consider the structure constants as the components of matrices, $-if_{ij}{}^k = (T_i)_j{}^k$, since $f_{ij}{}^k = -f_{ji}{}^k$, the matrices have the property that $(T_i)_j{}^k = -(T_j)_i{}^k$, then,

$$\begin{aligned} &if_{jk}{}^l(T_l)_i{}^m + \underbrace{(T_iT_k)_j{}^m}_{=-(T_jT_k)_i{}^m} + (T_kT_j)_i{}^m = 0 \\ \Longrightarrow &[T_j, T_k]_i{}^m = if_{jk}{}^l(T_l)_i{}^m \end{aligned} \tag{5.4.6}$$

or, more compactly, $[T_i, T_j] = if_{ij}{}^k T_k$.

- $\{(T_i)_j{}^k = -if_{ij}{}^k\}$ is called the adjoint representation of the Lie algebra $\{X_i\}$.

- more formally, adjoint representation is a map, $\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the group G,

$$\mathrm{ad}_A(B) = [A, B] \tag{5.4.7}$$

as one can see, $(\mathrm{ad}_A)^a{}_b = -f_{cb}{}^a A^c \in \mathcal{L}(\mathfrak{g})$, or written in components,

$$(\mathrm{ad}_{A_i})^k{}_j = -f_{ij}{}^k \Longrightarrow \mathrm{ad}_{A_i} = (iT_i)^T \tag{5.4.8}$$

and $[\mathrm{ad}_{A_i}, \mathrm{ad}_{A_j}] = \mathrm{ad}_{[A_i, A_j]} = -f_{ij}{}^k \mathrm{ad}_{A_k}$.

- $\mathrm{ad} : \mathfrak{g} \to \mathcal{L}(\mathfrak{g})$ is a homomorphism, i.e.,

$$\mathrm{ad}_{[A,B]} = [\mathrm{ad}_A, \mathrm{ad}_B] \tag{5.4.9}$$

proof:

$$\begin{aligned} (\mathrm{ad}_A\mathrm{ad}_B - \mathrm{ad}_B\mathrm{ad}_A)C &= [A, [B, C]] - [B, [A, C]] \\ &= [[A, B], C] = \mathrm{ad}_{[A,B]}C \end{aligned} \tag{5.4.10}$$

5.5 Killing forms

- $\forall A, B \in \mathfrak{g}$, the Killing form is,

$$B(A, B) = \mathrm{tr}(\mathrm{ad}_A \circ \mathrm{ad}_B) \tag{5.5.1}$$

which can be written in terms of structure constants,

$$B_{ij} = f_{ik}{}^l f_{jl}{}^k \tag{5.5.2}$$

proof:

$$B(A_i, A_j) = \mathrm{tr}(\mathrm{ad}_{A_i}\mathrm{ad}_{A_j}) = (-f_{ik}{}^l)(-f_{jl}{}^k) \tag{5.5.3}$$

  - $B([A, B], C) = B(A, [B, C])$.

proof:

recall that,

$$\mathrm{ad}_{[A,B]} = [\mathrm{ad}_A, \mathrm{ad}_B] \tag{5.5.4}$$

so,

$$\begin{aligned} B([A,B],C) &= \mathrm{tr}([\mathrm{ad}_A, \mathrm{ad}_B]\mathrm{ad}_C) \\ &= \mathrm{tr}(\mathrm{ad}_A\mathrm{ad}_B\mathrm{ad}_C) - \mathrm{tr}(\mathrm{ad}_A\mathrm{ad}_C\mathrm{ad}_B) \\ &= B(A,[B,C]) \end{aligned} \tag{5.5.5}$$

- two basis-independent properties of the Killing form:
  - the **number** of zero eigenvalues.
  - the **sign** of the non-zero eigenvalues.
- the structure constants with lowered indices are **completely antisymmetric**,

$$f_{ij}{}^{l}B_{lk} = -f_{ijk} = -f_{[ijk]} \tag{5.5.6}$$

proof:

$$f_{ij}{}^{l}B_{lk} = f_{ij}{}^{l}f_{lm}{}^{n}f_{kn}{}^{m} \tag{5.5.7}$$

notice that, according to Jacob identity, $f_{ij}{}^{l}f_{lm}{}^{n} = 2f_{[i|l}{}^{n}f_{|j]m}{}^{l}$, then,

$$f_{ijk} = -2f_{[i|l}{}^{n}f_{|j]m}{}^{l}f_{kn}{}^{m} \tag{5.5.8}$$

we can see that the equation holds under index permutation like $(i,j,k) \to (k,i,j) \to (j,k,i)$, and consequently, all three indices of f_{ijk} are antisymmetric.

Part III

Semisimple Lie Algebras

Chapter 6

semisimple Lie algebras

6.1 semisimple and reductive Lie algebras

- **def.:** a complex Lie algebra is **reductive** if there exists a **compact** Lie group K s.t.,

$$\mathfrak{g} \simeq \mathfrak{k}_{\mathbb{C}} \tag{6.1.1}$$

  - an alternate def. from Wikipedia: a Lie algebra is reductive if its adjoint rep. is completely reducible.

proof of equivalence:

$\Longrightarrow$, complexification of a compact Lie group is reductive:

- the adjoint rep. of a compact Lie group is completely reducible, so is its complexification (they have the same invariant subspaces, $W, W^{\perp}$, only complexificated).

$\Longleftarrow$, reductive is isomorphic to the complexification of some compact Lie group:

- the invariant subspaces of the adjoint representation are the ideals of $\mathfrak{g}$, especially, the kernel of the adjoint rep. is the center, $\mathfrak{z}$.
- $\mathfrak{g}$ decomposes as $\mathfrak{z} \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots$, where $\mathfrak{h}_1, \cdots$ are the smallest ideals of $\mathfrak{g}$, i.e. they don't have nontrivial ideals themselves $\Longrightarrow$ irreducible.
- moreover, if $\dim \mathfrak{h}_i = 1$, then,

$$[\mathfrak{h}_i, \mathfrak{z} \oplus \bigoplus_{j \neq i} \mathfrak{h}_j] = [\mathfrak{h}_i, \bigoplus_{j \neq i} \mathfrak{h}_j] \subseteq \mathfrak{h}_i \cap \bigoplus_{j \neq i} \mathfrak{h}_j = \{0\} \tag{6.1.2}$$

  then $\mathfrak{h}_i$ is just part of the center.
- so, $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}_1 \cdots$, where $\mathfrak{h}_1, \cdots$ are simple Lie subalgebras.
- according to the converse of (6.1.13) (?), $\mathfrak{h}_1 \oplus \cdots$ is a semisimple Lie algebra.
- according to the converse of (6.1.6) (?), a Lie algebra decomposes as its center and a semisimple Lie algebra is compact.

- **def.:** a complex Lie algebra is **semisimple** if it is reductive and the center of $\mathfrak{g}$ is trivial, i.e. $\mathfrak{z} = \{A \in \mathfrak{g} | \mathrm{ad}_A = 0\} = \{0\}$.
- **def.:** $\mathfrak{k}$ in (6.1.1) is the **compact real form** of the semisimple Lie algebra.
- some semisimple Lie algebras:

Lie algebras	reductive	semisimple	compact real forms
$\mathfrak{sl}(m \geq 2, \mathbb{C})$	yes	yes	$\mathfrak{su}(m)$
$\mathfrak{so}(m \geq 3, \mathbb{C})$	yes	yes	$\mathfrak{so}(m)$
$\mathfrak{so}(2, \mathbb{C})$	yes	no	$\mathfrak{so}(2)$
$\mathfrak{sp}(m \geq 1, \mathbb{C})$	yes	yes	$\mathfrak{sp}(m, \mathbb{R})$
$\mathfrak{gl}(m, \mathbb{C})$	yes	no	$\mathfrak{u}(m)$

6.1.1 some properties of reductive and semisimple Lie algebras

- let $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ be a **reductive** Lie algebra, then there exists an inner product s.t.,

$$\langle \mathrm{ad}_X A, B \rangle = -\langle A, \mathrm{ad}_X B \rangle \tag{6.1.3}$$

for all $A, B \in \mathfrak{g}, X \in \mathfrak{k}$.

proof:

Ad : $K \to \mathrm{End}(\mathfrak{k})$ is a unitary representation under the inner product chosen in (5.2.14) (which requires **compactness**),

$$\langle A, B \rangle = \int_K (\mathrm{Ad}_g A, \mathrm{Ad}_g B) \epsilon(g) \tag{6.1.4}$$

where (A, B) is some real positive definite inner product on $\mathfrak{k}$, and ϵ is the volume form composed by right invariant dual vector fields.
so, the associated Lie algebra rep. ad : $\mathfrak{k} \to \mathrm{End}(\mathfrak{k})$ satisfies $\mathrm{ad}_X^\dagger = -\mathrm{ad}_X$ (skew self-adjoint).

- for a **reductive** Lie algebra $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$, $\mathfrak{h}$ is one of its ideals, then,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \tag{6.1.5}$$

where $\mathfrak{h}^\perp$ is orthogonal to $\mathfrak{h}$ with respect to the inner product in (6.1.3), and it is also an **ideal**.

proof:

- if $\mathfrak{h}$ ($\mathrm{ad}_A[\mathfrak{h}] \subseteq \mathfrak{h}, \forall A$) is an ideal of $\mathfrak{g}$, then it is also an ideal of $\mathfrak{k}$ (obviously).
- unitary rep. is completely reducible, so both $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are its invariant subspace, i.e. ideals.
- $[\mathfrak{h}, \mathfrak{h}^\perp] \subseteq \mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$.
- so, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$.

- every **complex reductive** Lie algebra, $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$, decomposes as,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{z} \tag{6.1.6}$$

where $\mathfrak{g}_1$ is **semisimple** and $\mathfrak{z}$ is its **center**.
moreover,

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}' \tag{6.1.7}$$

where $\mathfrak{z}'$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_1$ is the compact real form of $\mathfrak{g}_1$.

proof:

center is an ideal, so,

$$\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z} \tag{6.1.8}$$

now we have to prove $\mathfrak{g}_1 = \mathfrak{z}^\perp$ is semisimple,

- first, the **center** of $\mathfrak{z}^\perp$ is **trivial**, for obvious reasons.

- $A \in \mathfrak{z} \iff \mathrm{ad}_A[\mathfrak{k}] = \{0\}$, so, for all $A = X + iY \in \mathfrak{z}, X, Y \in \mathfrak{k}$,

$$A^* := X - iY \in \mathfrak{z} \tag{6.1.9}$$

i.e. $\mathfrak{z}$ is closed under conjugation $* : X + iY \mapsto X - iY$
so, $\mathfrak{g}_1$ is also closed under conjugation.
    - 注意, 这里的定义和 Hall 书上的不一样, Hall 的定义是 $A^* = -X + iY, \bar{A} = X - iY$.
- so, for $\mathfrak{z}' := \mathfrak{z} \cap \mathfrak{k}, \mathfrak{k}_1 := \mathfrak{g}_1 \cap \mathfrak{k}$,

$$\mathfrak{z} = \mathfrak{z}'_{\mathbb{C}} \quad \mathfrak{g}_1 = \mathfrak{k}_{1\mathbb{C}} \tag{6.1.10}$$

- consider the adjoint representation of K and $\mathfrak{k}$,

$$\mathrm{Lie}(\mathrm{Ad}[K]) = \mathrm{ad}[\mathfrak{k}] \simeq \mathfrak{k}/\ker(\mathrm{ad}) = \mathfrak{k}/\mathfrak{z}' = \mathfrak{k}_1 \tag{6.1.11}$$

  Ad is a continuous map, so $\mathrm{Ad}[K]$ is a **compact** Lie group as K.

- so, $\mathfrak{k}_1$ is the **compact real form** of $\mathfrak{g}_1$.

- if K is a **simply connected compact** Lie group, then $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is **semisimple**.

proof:

since K is simply connected and $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}'$, so K decomposes as,

$$K = K_1 \times Z' \tag{6.1.12}$$

where K_1, Z' are closed simply connected subgroup associated with $\mathfrak{k}_1, \mathfrak{z}'$.
simply connected Lie group Z' is isomorphic to $\mathbb{R}^n$ for some n, but Z' is closed subgroup of a compact group, it is also compact, which means $n = 0$, i.e. $\mathfrak{z}' = \{0\} = \mathfrak{z}$, the center is trivial.

- an important **theorem:**
semisimple Lie algebra $\mathfrak{g}$ decomposes as,

$$\mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i \tag{6.1.13}$$

where $\mathfrak{g}_i$ are **simple** (see 3.2.1) and **unique** up to order (the converse of the theorem is also true (?)).

proof:

first, let's prove $\mathfrak{g}_i$ are simple,

- according to (6.1.5), semisimple Lie algebra with ideal $\mathfrak{h}$ decomposes as,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp} \tag{6.1.14}$$

  suppose $\mathfrak{h}'$ is an ideal of $\mathfrak{h}$, notice that $[\mathfrak{h}, \mathfrak{h}^{\perp}] = \{0\}$, so $\mathfrak{h}'$ is also an ideal of $\mathfrak{g}$.

- let $\mathfrak{h}'' = \mathfrak{h}'^{\perp} \cap \mathfrak{h}$, and $[\mathfrak{h}'', \mathfrak{h}' \oplus \mathfrak{h}^{\perp}] = \{0\}$, so it is also an ideal, then,

$$\mathfrak{g} = \mathfrak{h}'' \oplus \mathfrak{h}' \oplus \mathfrak{h}^{\perp} \tag{6.1.15}$$

- proceeding on the same way,

$$\mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i \tag{6.1.16}$$

  where $\mathfrak{g}_i$ are ideals without nontrivial ideals, i.e. **irreducible**.

- if $\dim \mathfrak{g}_i = 1$, then $\mathfrak{g}_i$ is Abelian, moreover,

$$[\mathfrak{g}_i, \bigoplus_{j \neq i} \mathfrak{g}_j] = \{0\} \tag{6.1.17}$$

  $\mathfrak{g}_i \subseteq \mathfrak{z}$ which contradicts to semisimpleness (without nontrivial center). so, $\dim \mathfrak{g}_i \geq 2$.

---

now, let's prove uniqueness,

- $\boldsymbol{\pi_i := \mathrm{ad}\big|_{\mathfrak{g}_i}} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}_i)$ is an **irreducible rep.**, since the nontrivial invariant subspace of π_i is $\{$an ideal of $\mathfrak{g}\} \cap \mathfrak{g}_i$, and consider (6.1.17), it is also an ideal of $\mathfrak{g}_i$, which doesn't exist.

- since $\pi_i[\mathfrak{g}_{j \neq i}] = \{0\}$ while $\pi_i[\mathfrak{g}_i] \neq \{0\}$ (simple Lie algebras are non-Abelian) $\Longrightarrow$ these rep. are **not isomorphic** to each other.

– for a simple ideal $\mathfrak{h}$ of $\mathfrak{g}$, $\pi_{\mathfrak{h}} := \mathrm{ad}\big|_{\mathfrak{h}} : \mathfrak{g} \to \mathrm{End}(\mathfrak{h})$ is an irreducible rep..

– the projection map $p_i : \mathfrak{g} \to \mathfrak{g}_i$ is an intertwining map,

$$p_i\Big|_{\mathfrak{g}_j} \pi_j(A) = \pi_i(A) p_i\Big|_{\mathfrak{g}_j} \begin{cases} = 0 & i \neq j \text{ or } A \notin \mathfrak{g}_{i=j} \\ \neq 0 & i = j, A \in \mathfrak{g}_{i=j} \end{cases} \tag{6.1.18}$$

and,

$$p_i\Big|_{\mathfrak{h}} \pi_{\mathfrak{h}}(A) = \pi_i(A) p_i\Big|_{\mathfrak{h}} \tag{6.1.19}$$

according to Schur's lemma, $p_i\big|_{\mathfrak{h}} = 0$ or isomorphism.

– $p_i\big|_{\mathfrak{h}}$ is a projection map, so there must be some i so that $p_i\big|_{\mathfrak{h}} \neq 0$, so $\mathfrak{h} = \mathfrak{g}_i$ for some i.

6.2 Cartan subalgebra

- **def.:** $\mathfrak{g}$ is a complex semisimple Lie algebra, its subalgebra $\mathfrak{h}$ is called **Cartan subalgebra** if:

  1. it is Abelian,
  2. if for some $A \in \mathfrak{g}$ and $[A, H] = 0, \forall H \in \mathfrak{h}$, then $A \in \mathfrak{h}$, (make sure it is maximal),
  3. $\forall H \in \mathfrak{h}, \mathrm{ad}_H$ is diagonalizable.

  some remark:

  – condition 1 and 2 say that $\mathfrak{h}$ is a **maximal Abelian subalgebra** (not contained in a larger Abelian subalgebra) of $\mathfrak{g}$ (there may be more than one maximal Abelian subalgebra).

  – $[\mathrm{ad}_{H_1}, \mathrm{ad}_{H_2}] = \mathrm{ad}_{[H_1,H_2]} = 0$, so they are **simultaneously diagonalizable**.

  – the def. makes sense in any Lie algebra, but if $\mathfrak{g}$ is not semisimple, it may not have any Cartan subalgebra.

- now, let's prove **Cartan subalgebra exists in semisimple Lie algebras**.

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is a complex semisimple Lie algebra, $\mathfrak{t}$ is a **maximal Abelian subalgebra of $\mathfrak{k}$**, then, the **Cartan subalgebra** of $\mathfrak{g}$ is,

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \tag{6.2.1}$$

proof:

first, let's prove $\mathfrak{h}$ is maximal Abelian,

– $\mathfrak{h}$ is obviously Abelian.

– if $[A, \mathfrak{h}] = \{0\}$, for some $A = X + iY \in \mathfrak{g}$, then $[X, \mathfrak{h}] = [Y, \mathfrak{h}] = \{0\}$, which means $\mathfrak{t}$ is not maximal.

---

now, let's show that $\mathrm{ad}_H, \forall H \in \mathfrak{h}$ are diagonalizable,

– choose inner product shown in (5.2.14), so ad_X is skew self-adjoint for all $X \in \mathfrak{k}$, which means it is diagonalizable.

– $\mathrm{ad}_T, \forall T \in \mathfrak{t}$ is diagonalizable, and $[\mathrm{ad}_T, \mathrm{ad}_H] = 0, \forall H \in \mathfrak{h}$, so $\mathrm{ad}_H, \forall H \in \mathfrak{h}$ are simultaneously diagonalizable.

- **def.:** the **rank**, $r = \dim \mathfrak{h}$, of a semisimple Lie algebra is the dimension of any of its Cartan subalgebras.

  – any two Cartan subalgebra $\mathfrak{h}_1, \mathfrak{h}_2$ of a semisimple Lie algebra are isomorphic to each other (?).

6.3 roots and root spaces

- from now on, we only consider the Cartan subalgebra in (6.2.1), $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$.
- **def.:** a **nonzero** element $\alpha \in \mathfrak{h}$ (because $\langle\alpha| \in \mathfrak{h}^*$) is called a **root** if there exists a nonzero $A \in \mathfrak{g}$ s.t.,

$$[H, A] = \langle\alpha, H\rangle A \tag{6.3.1}$$

  for all $H \in \mathfrak{h}$.
  - the inner product (on $\mathfrak{h}$) is arbitrarily chosen.
  - the set of all root is denoted as $R = \{\alpha\}$.
- if we choose the inner product in (5.2.14), then, for all root $\alpha \in i\mathfrak{t}$.

proof:

- choose $H \in \mathfrak{t}$, ad_H is skew self-adjoint under the chosen inner product.
- the eigenvalue $\langle\alpha, H\rangle$ is pure imaginary (and nonzero).
- the inner product is real on $\mathfrak{k}$.
- so, $\alpha \in i\mathfrak{k} \cap \mathfrak{h} = i\mathfrak{t}$.

- **def.:** for a root α, the **root space** is,

$$\mathfrak{g}_\alpha = \{A \in \mathfrak{g} | [H, A] = \langle\alpha, H\rangle A, \forall H \in \mathfrak{h}\} \tag{6.3.2}$$

  a nonzero element of $\mathfrak{g}_\alpha$ is called a **root vector**.
  - more generally, for any element $\alpha \in \mathfrak{h}$, we can define $\mathfrak{g}_\alpha$ as in (6.3.2), but we don't call it a root space unless α is a root.
    * if α is not a root, then, $\mathfrak{g}_\alpha$ is either $\{0\}$ $(\alpha \neq 0)$ or $\mathfrak{h}$ $(\alpha = 0)$.
    * by def. $[\mathfrak{h}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$.
- the complex semisimple Lie algebra decomposes as,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \tag{6.3.3}$$

  and $\mathfrak{h} \cap \mathfrak{g}_\alpha = \mathfrak{g}_\alpha \cap \mathfrak{g}_\beta = \{0\}$, furthermore, $\mathfrak{h}$ and $\mathfrak{g}_\alpha, \forall\alpha \in R$ are linearly independent.
  note that $\oplus$ is **not Lie algebra direct sum**, as that $\mathfrak{h}, \mathfrak{g}_\alpha$ are not ideals.

proof:

$\mathrm{ad}_H, H \in \mathfrak{h}$ can be simultaneously diagonalized, so, according to (A.3.9) in appendix A.3,

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}} \mathfrak{g}_\alpha \tag{6.3.4}$$

and $\mathfrak{g}_\alpha \cap \mathfrak{g}_\beta = \{0\}, \forall\alpha \neq \beta \in \mathfrak{h}$.
but if $\alpha = 0$, $\mathfrak{g}_0 = \mathfrak{h}$ and if $\alpha \neq 0$ and not a root, $\mathfrak{g}_\alpha = \{0\}$, so...

- for any $\alpha, \beta \in \mathfrak{h}$, we have,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \tag{6.3.5}$$

proof:

for all $A \in \mathfrak{g}_\alpha, B \in \mathfrak{g}_\beta$,

$$[H, [A, B]] = -[B, [H, A]] - [A, [B, H]] = \langle\alpha + \beta, H\rangle [A, B] \tag{6.3.6}$$

- two useful propositions:
  - if α is a root, so does $-\alpha$, and for all $A = X + iY \in \mathfrak{g}_\alpha$, $A^* = X - iY \in \mathfrak{g}_{-\alpha}$ (where $X, Y \in \mathfrak{k}$).

proof:

for any $H \in \mathfrak{t}$,

$$[H, A^*] = ([H, A])^* = (\langle \alpha, H \rangle)^* A^* \tag{6.3.7}$$

and because $\alpha \in i\mathfrak{t}$, so $(\langle \alpha, H \rangle)^* = -\langle \alpha, H \rangle$.

– $\text{span}(R) = \mathfrak{h}$.

proof:

if the root doesn't span $\mathfrak{h}$, then there nonzero exists $H \in \mathfrak{h}$ s.t.,

$$\langle \alpha, H \rangle = 0, \forall \alpha \in R \Longrightarrow [H, A] = 0, \forall A \in \mathfrak{g} \tag{6.3.8}$$

i.e. H is in the center of $\mathfrak{g}$, which contradicts to semisimpleness of $\mathfrak{g}$ (without nontrivial center).

6.3.1 subalgebras isomorphic to $\mathfrak{su}(2)_{\mathbb{C}}$

- for each root $\alpha \in R$, we have the **coroot**,

$$H_\alpha = 2\frac{\alpha}{\langle \alpha, \alpha \rangle} \in \mathfrak{h} \tag{6.3.9}$$

associated to it, and $\forall A_\alpha \in \mathfrak{g}_\alpha, B_\alpha \in \mathfrak{g}_{-\alpha}$ there is,

$$\begin{cases} [H_\alpha, A_\alpha] = 2A_\alpha \\ [H_\alpha, B_\alpha] = -2B_\alpha \\ [A_\alpha, B_\alpha] = H_\alpha \qquad \text{(with } \textbf{normalization}\text{)} \end{cases} \tag{6.3.10}$$

and $B_\alpha = -A_\alpha^*$ (as part of the normalization).

proof:

for all $A \in \mathfrak{g}_\alpha, B \in \mathfrak{g}_{-\alpha}, H \in \mathfrak{h}$, then $[A, B] \in \mathfrak{h}$ and,

$$[A, B] = \langle -A^*, B \rangle \alpha \tag{6.3.11}$$

---

proof:

– $[A, B] \in \mathfrak{h}$ because $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and $\mathfrak{g}_0 = \mathfrak{h}$.

– and,

$$\begin{aligned} \langle H, [A, B] \rangle &= \langle \text{ad}_A^\dagger H, B \rangle = \langle \text{ad}_{-A^*} H, B \rangle \\ &= \langle [H, A^*], B \rangle = \langle \langle -\alpha, H \rangle A^*, B \rangle \\ &= \langle H, \alpha \rangle \langle -A^*, B \rangle \end{aligned} \tag{6.3.12}$$

for all $H \in \mathfrak{h}$, so,

$$[A, B] = \langle -A^*, B \rangle \alpha \tag{6.3.13}$$

---

choose the **normalization**,

$$\begin{cases} B_\alpha = -A_\alpha^* \\ \langle A_\alpha, A_\alpha \rangle^* \langle \alpha, \alpha \rangle = 2 \end{cases} \Longleftrightarrow \begin{cases} H = [A, -A^*] = \langle A, A \rangle^* \alpha \\ H_\alpha = \dfrac{2}{\langle \alpha, H \rangle} H \\ A_\alpha = \sqrt{\dfrac{2}{\langle \alpha, H \rangle}} A \\ B_\alpha = -A_\alpha^* \qquad\qquad \text{notice } \langle \alpha, H \rangle \in \mathbb{R} \end{cases} \tag{6.3.14}$$

$\forall A \in \mathfrak{g}_\alpha$ (notice that $\langle \alpha, \alpha \rangle \in \mathbb{R}^-$ and $\langle A, A \rangle = \langle X, X \rangle + \langle Y, Y \rangle - 2\text{Im} \langle X, Y \rangle \in \mathbb{R}, \forall A \in \mathfrak{g}$).

- compare $\mathrm{span}(H_\alpha, A_\alpha, B_\alpha)_{\mathbb{C}}$ with $\mathfrak{su}(2)_{\mathbb{C}}$, we have,

$$H_\alpha \mapsto 2J_3 \quad A_\alpha \mapsto \sqrt{2}J_+ \quad B_\alpha \mapsto \sqrt{2}J_- \tag{6.3.15}$$

- from the complex subalgebra $\mathfrak{s}^\alpha = \mathrm{span}(H_\alpha, A_\alpha, B_\alpha)$, we can conclude that,
  1. if α and $c\alpha$ are both roots, then $c = \pm 1$,
  2. $\dim \mathfrak{g}_\alpha = 1$ for all root spaces.

proof:

consider $A_{c\alpha} \in \mathfrak{g}_{c\alpha}$,

$$[H_\alpha, A_{c\alpha}] = \underbrace{\langle c\alpha, H_\alpha\rangle}_{=2c^*} A \tag{6.3.16}$$

$2c^*$ is an eigenvalue of $\mathrm{ad}_{H_\alpha} \in \mathrm{End}(\mathfrak{g})$, which is a finite-dim. rep. of $\mathfrak{su}(2)_{\mathbb{C}}$, so the eigenvalue must be an integer, i.e.,

$$2c^*, 2\frac{1}{c^*} \in \mathbb{Z} \Longrightarrow c = \pm 1, \pm 2, \pm\frac{1}{2} \tag{6.3.17}$$

let $\pm\alpha, \pm 2\alpha$ (notice $\pm 4\alpha$ are not roots) be all the roots $\propto \alpha$, then let,

$$V^\alpha = \mathrm{span}(H_\alpha) \oplus \bigoplus_{\beta = \pm\alpha, \pm 2\alpha} \mathfrak{g}_\beta \tag{6.3.18}$$

where $\oplus$ is not Lie algebra direct sum.
$V^\alpha \supseteq \mathfrak{s}^\alpha$ is a subalgebra of $\mathfrak{g}$.

---

proof:
for all $\beta, \beta' = \pm\alpha, \pm 2\alpha$, we have,

- according to (6.3.11), $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \propto \alpha \propto H_\alpha$.
- $[H_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_\beta$.
- $[\mathfrak{g}_\beta, \mathfrak{g}_{\beta'}] \subseteq \mathfrak{g}_{\beta+\beta'} = \mathfrak{g}_{\pm 2^i \alpha}$ or $\{0\}$ (where $\beta + \beta' \neq 0$).

---

now, let's prove $V^\alpha = \mathfrak{s}^\alpha$,

- consider the 'unitary' (skew self-adjoint) rep. (ad, V^α) of $\mathrm{span}(H_\alpha, A_\alpha, B_\alpha) \simeq \mathfrak{su}(2)_{\mathbb{C}}$, $\mathfrak{s}^\alpha$ is the invariant subspace of the rep., and the rep. is completely reducible, so $\mathfrak{s}^{\alpha\perp}$ is also an invariant subspace.
- the eigenvalues of ad_{H_α} in V^α are 0 and $\langle \beta, H_\alpha\rangle = \pm 2, \pm 4$.
- recall the property of the eigenvalues of $\pi(H)$, 0 must be one of the eigenvalues of ad_{H_α} in the rep. $(\mathrm{ad}, \mathfrak{s}^{\alpha\perp})$, which is **impossible** since $H_\alpha \in \mathfrak{s}^\alpha$ is the only vector with eigenvalue 0.
- so, $\mathfrak{s}^{\alpha\perp} = \{0\}$, i.e. the only roots $\propto \alpha$ are $\pm\alpha$, and,

$$\mathrm{span}(H_\alpha) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{s}^\alpha \equiv \mathrm{span}(H_\alpha, A_\alpha, B_\alpha) \tag{6.3.19}$$

  i.e. $\mathfrak{g}_\alpha = \mathrm{span}(A_\alpha)$ or $\dim \mathfrak{g}_\alpha = 1$.

- a rephrase of (6.3.3): for all $A \in \mathfrak{g}$, A is either a root or in a root space, and,

$$\begin{cases} \mathfrak{s}^\alpha \cap \mathfrak{s}^\beta = \{0\} & \alpha \neq \pm\beta \\ \mathfrak{s}^\alpha = \mathfrak{s}^{-\alpha} & H_\alpha = -H_{-\alpha} \quad A_\alpha = B_{-\alpha} \quad B_\alpha = A_{-\alpha} \end{cases} \tag{6.3.20}$$

- $\mathfrak{s}^\alpha, \mathfrak{h}, \mathfrak{g}_\alpha, \forall \alpha \in R$ are not ideals.
- the set of roots, R, may not be linearly independent.
  - the maximal set of linearly independent roots is called the **simple root**.
  - but $\mathfrak{g}_\alpha, \forall\alpha \in R$ are linearly independent, as stated in (6.3.3).

6.3.2 root systems

- for all roots $\alpha, \beta \in R \subset i\mathfrak{t}$, we have,

$$\langle \alpha, H_\beta \rangle = 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \tag{6.3.21}$$

proof:

consider $\mathfrak{s}^\beta = \mathrm{span}(H_\beta, A_\beta, B_\beta)$, and its adjoint representation $\mathrm{ad} : \mathfrak{s}^\beta \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ (which is finite dimensional),

$$[H_\beta, A_\alpha] = \langle \alpha, H_\beta \rangle\, A_\alpha \tag{6.3.22}$$

the eigenvalue of ad_{H_β} must be an integer, according to (9.1.6), so,

$$\langle \alpha, H_\beta \rangle \in \mathbb{Z} \tag{6.3.23}$$

  - the **projection** of α to β ($\alpha \cdot \hat{e}_\beta$) is a (half-)integer multiple of $|\beta|$,

$$\frac{\langle \alpha, \beta \rangle}{\sqrt{\langle \beta, \beta \rangle}} = (0, \pm\frac{1}{2}, \pm 1, \cdots)|\beta| \tag{6.3.24}$$

- summary:
  - the roots span $i\mathfrak{t}$.
  - if $\alpha \in R$, the only multiples of α in R is $-\alpha$.
  - $\alpha \in R$, then $s_\beta \alpha \in R$, where $s_\beta = I - 2\frac{|\beta\rangle\langle\beta|}{\langle\beta,\beta\rangle}$ (see (6.5.2)).
  - for all $\alpha, \beta \in R$, their inner product $2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$.

  any such collection of vectors is called a **root system**.

6.4 Cartan's criterion

- **Cartan's criterion for simplicity**:
  complex Lie algebra $\mathfrak{g}$ is semisimple $\iff$ its Killing form is non-degenerate.

proof:

first, let's prove $\Longrightarrow$,

  - consider,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \tag{6.4.1}$$

    (where $\oplus$ is the vector space direct sum) and the adjoint representation is $\mathrm{ad} : \mathfrak{s}^\alpha \to \mathrm{End}(\mathfrak{s}^\alpha)$. and notice $\mathfrak{h} = \mathrm{span}(R)$.

  - so, for any $\alpha \in R$, we have,

$$\begin{cases} H_\alpha & B(H_\alpha, H_\alpha) = 8 \\ A_\alpha \text{ or } B_\alpha & B(A_\alpha, B_\alpha) = 4 \end{cases} \tag{6.4.2}$$

  - so, for all $A \neq 0 \in \mathfrak{g}$, there exists some $B \in \mathfrak{g}$ s.t. $B(A, B) \neq 0$, i.e. the Killing form is non-degenerate.

---

now, let's prove $\Longleftarrow$,

  - first, the center $\mathfrak{z} = \{0\}$, otherwise, there exists some $A \in \mathfrak{g}$ s.t. $\mathrm{ad}_A = 0$, which contradicts to the non-degeneracy.

  - second, the adjoint rep. of $\mathfrak{g}$ is completely reducible, otherwise, the Killing form is degenerate (?).

6.5 the Weyl group (from the Lie algebra approach)

- **def.:** for each root $\alpha \in R$, define a linear map,

$$s_\alpha = I - \overbrace{|\alpha\rangle \langle H_\alpha|}^{=2\frac{|\alpha\rangle\langle\alpha|}{\langle\alpha,\alpha\rangle}} : \mathfrak{h} \to \mathfrak{h} \text{ or } i\mathfrak{t} \to i\mathfrak{t}$$
$$H \mapsto H - \alpha \langle H_\alpha, H\rangle \tag{6.5.1}$$

notice s_α is the reflection about the hyperplane orthogonal to α, i.e.,

  - $s_\alpha |H\rangle = |H\rangle$ for all $|H\rangle$ orthogonal to α.
  - $s_\alpha |\alpha\rangle = -|\alpha\rangle$.

also notice $s_\alpha = s_{-\alpha}$ and $s_\alpha^2 = I$.

- **def.:** the **Weyl group** is $W = \langle\{s_\alpha, \alpha \in R\}\rangle$, i.e. every element in W can be expressed as a combination of finite $s_\alpha, \alpha \in R$.
  - W is a subgroup of the orthogonal group $\mathrm{O}(i\mathfrak{t})$.

- for all $\alpha \in R, w \in W$,

$$w|\alpha\rangle \in R \tag{6.5.2}$$

proof:

equivalently, we need to prove for all $\alpha, \beta \in R$,

$$s_\alpha |\beta\rangle \in R \tag{6.5.3}$$

notice that for all $H \in \mathfrak{h}$,

$$\begin{cases} \mathrm{Ad}_{S_\alpha} H = s_\alpha |H\rangle \implies \mathrm{Ad}_{S_\alpha} \mathrm{ad}_H \mathrm{Ad}_{S_\alpha}^{-1} = \mathrm{ad}_{s_\alpha|H\rangle} \\ \mathrm{Ad}_{S_\alpha}^{-1} H = s_\alpha |H\rangle \implies \mathrm{Ad}_{S_\alpha}^{-1} \mathrm{ad}_H \mathrm{Ad}_{S_\alpha} = \mathrm{ad}_{s_\alpha|H\rangle} \end{cases} \tag{6.5.4}$$

where $\mathrm{Ad}_{S_\alpha} = e^{\mathrm{ad}_{A_\alpha}} e^{-\mathrm{ad}_{B_\alpha}} e^{\mathrm{ad}_{A_\alpha}} \in \mathrm{End}(\mathfrak{g})$.

---

proof:
notice that if $\langle\alpha, H\rangle = 0$, then $[H, A_\alpha] = [H, B_\alpha] = 0$, which implies $[\mathrm{ad}_H, \mathrm{ad}_{A_\alpha \text{ or } B_\alpha}] = 0$, so,

$$\begin{cases} \mathrm{Ad}_{S_\alpha}^{-1} H = e^{-\mathrm{ad}_{A_\alpha}} e^{\mathrm{ad}_{B_\alpha}} e^{-\mathrm{ad}_{A_\alpha}} H = H & \langle\alpha, H\rangle = 0 \\ \mathrm{Ad}_{S_\alpha}^{-1} H = -H & H \propto \alpha \end{cases} \tag{6.5.5}$$

---

consider any $H \in \mathfrak{h}$ and $A_\beta \in \mathfrak{g}_\beta$ with $\beta \in R$,

$$\mathrm{Ad}_{S_\alpha} A_\beta \in \mathfrak{g} \tag{6.5.6}$$

and,

$$\begin{aligned} [H, \mathrm{Ad}_{S_\alpha} A_\beta] &= \mathrm{ad}_H \mathrm{Ad}_{S_\alpha} A_\beta \\ &= \mathrm{Ad}_{S_\alpha} (\mathrm{Ad}_{S_\alpha}^{-1} \mathrm{ad}_H \mathrm{Ad}_{S_\alpha}) A_\beta \\ &= \mathrm{Ad}_{S_\alpha} [s_\alpha H, A_\beta] = \langle\beta, s_\alpha H\rangle \mathrm{Ad}_{S_\alpha} A_\beta \end{aligned} \tag{6.5.7}$$

and notice that $\alpha \in i\mathfrak{t} \implies s_\alpha^\dagger = s_\alpha$, so,

$$[H, \mathrm{Ad}_{S_\alpha} A_\beta] = \langle s_\alpha\beta, H\rangle \mathrm{Ad}_{S_\alpha} A_\beta \tag{6.5.8}$$

which means $s_\alpha\beta \in R$ and $\mathrm{Ad}_{S_\alpha} A_\beta \in \mathfrak{g}_{s_\alpha\beta}$.

- the Weyl group is **finite**.

proof:

since there are only finite roots, s_α (which is reversible) is nothing but a **permutation** of the roots, so is every element in the Weyl group.

6.6 simple Lie algebras

- recall the def. of simple Lie algebra in section 3.2.1.
- see (6.1.13), $\mathfrak{g}$ is simple $\Longrightarrow$ $\mathfrak{g}$ is semisimple (不会证).

- $\mathfrak{g}_{\mathbb{C}}$ is simple $\Longrightarrow$ $\mathfrak{g}$ is also simple.
  but, $\mathfrak{g}$ is simple $\not\Longrightarrow \mathfrak{g}_{\mathbb{C}}$ is not necessarily simple.

proof:

- $\dim \mathfrak{g} = \dim \mathfrak{g}_{\mathbb{C}} \geq 2$.
- if $\mathfrak{g}$ has a nontrivial ideal, $\mathfrak{h}$, then $\mathfrak{h}_{\mathbb{C}}$ is a nontrivial ideal of $\mathfrak{g}_{\mathbb{C}}$.

- **def.:** a real Lie algebra, $\mathfrak{g}$, is said to **admit a complex structure** if it is isomorphic to a complex Lie algebra, $\mathfrak{h}$,

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{h} \\ A &\mapsto \phi_1(A) + i\phi_2(A) \end{aligned} \tag{6.6.1}$$

and,

$$\phi([A,B]) = [\phi(A), \phi(B)] \Longrightarrow \begin{cases} \phi_1([A,B]) = [\phi_1(A), \phi_1(B)] - [\phi_2(A), \phi_2(B)] \\ \phi_2([A,B]) = [\phi_1(A), \phi_2(B)] + [\phi_2(A), \phi_1(B)] \end{cases} \tag{6.6.2}$$

and ϕ_1, ϕ_2 are not one-to-one.

  - equivalently, there exists a "multiplication by i" map on $\mathfrak{g}$, $J : \mathfrak{g} \to \mathfrak{g}$, s.t.,

$$J^2 = -I \quad \text{and} \quad [A, B + JC] = [A,B] + J[A,C] \tag{6.6.3}$$

proof:

let's prove def. 1. $\Longrightarrow$ there exits a J on $\mathfrak{g}$,

- let $J = (\phi^{-1} \circ iI \circ \phi) \in \mathrm{End}(\mathfrak{g})$.
- for all $X \in \mathfrak{h}$, there exists some $A = \phi^{-1}X$, so,

$$\begin{aligned} &(\phi \circ J)A = (\phi \circ J \circ \phi^{-1})X = iX = i\phi(A) \\ \Longrightarrow &\phi([\boldsymbol{A}, \boldsymbol{JB}]) = [\phi(A), i\phi(B)] = i\phi([A,B]) = \phi(\boldsymbol{J}[\boldsymbol{A}, \boldsymbol{B}]) \end{aligned} \tag{6.6.4}$$

- a **non-Abelian compact** Lie algebra, $\mathfrak{k}$, **doesn't admit a complex structure**.

proof:

- if $\mathfrak{k}$ admits a complex structure, it has a "multiplication by i" map, $J \in \mathrm{End}(\mathfrak{k})$.
- choose the inner product on $\mathfrak{k}$, so that $\mathrm{ad}_X, \forall X \in \mathfrak{k}$ are skew self-adjoint, hence diagonalizable in $\mathfrak{k}_{\mathbb{C}}$, with pure-imaginary (not all-zero) eigenvalues.
  * $\mathfrak{k} \simeq \mathfrak{h}$ where $\mathfrak{h}$ is a complex Lie algebra.
  * there exists $H = \phi(X) \in \mathfrak{h}$ and $A = \phi(Y) \in \mathfrak{h}$, s.t.,

$$\phi([X,Y]) = ia\phi(Y) \Longrightarrow [X,Y] = JaY \tag{6.6.5}$$

where $a \in \mathrm{R}$ since ad_X has pure imaginary eigenvalues.

* which is **impossible**, because ad_{JX} has real eigenvalue,

$$[JX, Y] = -aY \tag{6.6.6}$$

- $\mathfrak{k}$ is the Lie algebra of a compact Lie group, then, $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is simple $\iff$ $\mathfrak{k}$ is simple.

proof:

we only need to prove $\Longleftarrow$,

- $\mathfrak{k}$ is simple $\Longrightarrow$ without a nontrivial center $\Longrightarrow$ $\mathfrak{g}$ is semisimple $\Longrightarrow$ is a direct sum of simple Lie algebras (and the decomposition is unique up to ordering, see (6.1.13)),

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{g} = \bigoplus_i \mathfrak{g}_i \tag{6.6.7}$$

- if $\mathfrak{g}_i$ is an simple ideal of $\mathfrak{g}$, so is $\mathfrak{g}_i^* = \{A^* | A \in \mathfrak{g}_i\}$, which (together with the uniqueness of decomposition) implies $\mathfrak{g}_i^* = \mathfrak{g}_j$ for some j
  * if $\mathfrak{g}_i^* = \mathfrak{g}_i$, then $\mathfrak{g}_i \cap \mathfrak{k}$ is a nontrivial ideal of $\mathfrak{k}$, contradicts to simpleness.
  * if $\mathfrak{g}_i^* = \mathfrak{g}_j$ with $i \neq j$, then let $\mathfrak{g}' = \mathfrak{g}_i \cup \mathfrak{g}_i^*$, we have $\mathfrak{g}'^* = \mathfrak{g}'$, thus $\mathfrak{g}' \cap \mathfrak{k}$ is a nontrivial ideal of $\mathfrak{k}$, unless $\mathfrak{g}' = \mathfrak{g}$.

now, let's discuss what happens if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^*$, where $\mathfrak{g}_1, \mathfrak{g}_1^*$ are both simple ideals of $\mathfrak{g}$.

- define a linear map (projection),

$$\begin{aligned} \phi : \mathfrak{g}_1 &\to \mathfrak{k} \\ A &\mapsto \frac{1}{2}(A + A^*) \end{aligned} \tag{6.6.8}$$

notice that for all $A \in \mathfrak{g}_1$, we have $A^* \in \mathfrak{g}_1^*$, thus $[A, A^*] = 0$, so,

$$\boldsymbol{\phi([A,B])} = \frac{1}{2}([A,B] + [A^*, B^*]) = \frac{1}{2}([A + A^*, B + B^*]) = \boldsymbol{[\phi(A), \phi(B)]} \tag{6.6.9}$$

  * furthermore, ϕ is **one-to-one**, because,

$$\boldsymbol{A + A^* = B + B^*} \Longrightarrow A - B = B^* - A^* \in \mathfrak{g}_1 \cap \mathfrak{g}_1^* = \{0\} \Longrightarrow \boldsymbol{A = B} \tag{6.6.10}$$

  * ϕ is also **on-to**, because as a complex Lie algebra, $\mathfrak{g}_1$ has the same dimension of the real Lie algebra, $\mathfrak{k}$, thus for every $X \in \mathfrak{k}$, there exists some $A \in \mathfrak{g}_1$, s.t. $X = \phi(A)$.

- so, $\mathfrak{k}$ is isomorphic to a complex Lie algebra $\mathfrak{g}_1$, i.e. it **admits a complex structure**, which contradicts to compactness.
- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is simple.

- $\mathfrak{g}$ is not simple $\iff$ $\mathfrak{h}$ decomposes into $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $\mathfrak{h}_1 \perp \mathfrak{h}_2$ (orthogonal direct sum), and every root is either in $\mathfrak{h}_1$ or $\mathfrak{h}_2$.

where,

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is a complex semisimple Lie algebra.
- $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ is the complexification of the maximal Abelian subalgebra of $\mathfrak{k}$, $\mathfrak{t}$, i.e. the Cartan subalgebra.

proof:

first, let's prove $\Longrightarrow$,

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is not simple $\Longrightarrow$ $\mathfrak{k}$ is not simple (form the theorem above) $\Longrightarrow$ $\mathfrak{k}_1$ is the nontrivial ideal of $\mathfrak{k}$, i.e. an invariant subspace of $\mathrm{ad} : \mathfrak{k} \to \mathrm{End}(\mathfrak{k})$.

- notice the adjoint representation on $\mathfrak{k}$ is completely reducible, there is another ideal $\mathfrak{k}_2$ s.t. $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$.
  * if we choose the inner product so that the adjoint rep. on $\mathfrak{k}$ is unitary, then $\mathfrak{h}_1 \perp \mathfrak{h}_2$ (see section 1.2).
- now, we have $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_1^{\perp}$, which implies $\mathfrak{k}_{\mathbb{C}} = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_i = \mathfrak{k}_{i\mathbb{C}}$, and, of course, $\mathfrak{g}_1 \perp \mathfrak{g}_2$.
- the maximal Abelian subalgebra, $\mathfrak{t}$, decomposes as $\mathfrak{t}_1 \oplus \mathfrak{t}_2$, where $\mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{k}_i$.

---

proof:

* consider $T = X + Y \in \mathfrak{t}$ with $X \in \mathfrak{k}_1$ and $Y \in \mathfrak{k}_2$, then,

$$[T_1, T_2] = \underbrace{[X_1, X_2]}_{\in \mathfrak{k}_1} + \underbrace{[Y_1, Y_2]}_{\in \mathfrak{k}_2} = 0 \tag{6.6.11}$$

  notice that $\mathfrak{k}_1, \mathfrak{k}_2$ are linearly independent, so, $[X_1, X_2] = [Y_1, Y_2] = 0$.
* which means $[X, \mathfrak{t}] = \{0\}$, but $\mathfrak{t}$ is maximal, so $X \in \mathfrak{t} \cap \mathfrak{k}_1$, similarly, $Y \in \mathfrak{t} \cap \mathfrak{k}_2$.
* so, $\mathfrak{t}_1 = \mathfrak{t} \cap \mathfrak{k}_1$ and $\mathfrak{t}_2 = \mathfrak{t} \cap \mathfrak{k}_2$, then, we have the Lie algebra direct sum, $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$.

- consequently, the Cartan subalgebra decomposes as $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, with $\mathfrak{h}_i = \mathfrak{t}_{i\mathbb{C}}$, and, of course, $\mathfrak{h}_1 \perp \mathfrak{h}_2$.
- every root is either in $\mathfrak{h}_1$ or $\mathfrak{h}_2$.

---

proof:

* let R_i be the roots for $\mathfrak{g}_i$ in $\mathfrak{h}_i$.
  (i.e., excuse the sloppy notation, there exists a nonzero $A \in \mathfrak{g}_i$ s.t. $[\mathfrak{h}_i, A] = \langle R_i, \mathfrak{h}_i \rangle A$).
* now, we claim $R_{i=1,2} \subset R$, because for all $\alpha \in R_1$,

$$[H_1 + H_2, A] = \langle \alpha, H_1 \rangle A + 0 = \langle \alpha, H_1 + H_2 \rangle A \tag{6.6.12}$$

  where we noticed that the root vector $A \in \mathfrak{g}_1 = \mathfrak{k}_{1\mathbb{C}}$ and $H_2 \in \mathfrak{t}_{2\mathbb{C}}$ commutes with A, and $\alpha \in \mathfrak{h}_1 \perp \mathfrak{h}_2$.
* notice that $R - (R_1 \cup R_2)$ are the roots associated to root vectors neither in $\mathfrak{g}_1$ nor $\mathfrak{g}_2$.
  · consider $A = A_1 + A_2$, with $A_i \in \mathfrak{g}_i$, is a root vector of $\alpha \in R$, then, consider,

$$\begin{aligned} &[H_1, A_1 + A_2] = [H_1, A_1] = \langle \alpha, H_1 \rangle A_1 \propto A_1 + A_2 \\ \Longrightarrow &\text{either } A_2 = 0 \text{ or } \langle \alpha, H_1 \rangle = 0 \end{aligned} \tag{6.6.13}$$

  so, if $A_2 = 0$, then $\alpha \in R_1$, else, $\alpha \in \mathfrak{h}_2$, which means $\alpha \in R_2$.
* so, either α is in R_1 or in R_2.

- $\Longrightarrow$ is proved.

now, let's prove $\Longleftarrow$,

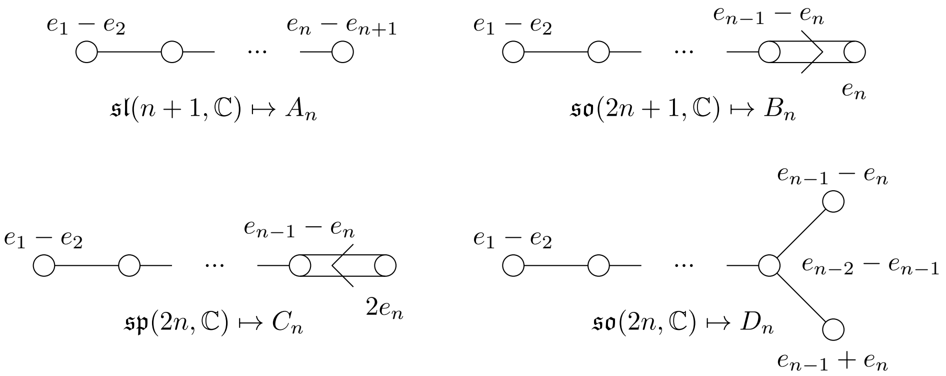
- $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_1 \perp \mathfrak{h}_2$, and $R_i = R \cap \mathfrak{h}_i$.
- then, $\mathfrak{g}$ decomposes as,

$$\mathfrak{g} = \overbrace{\left(\mathfrak{h}_1 \oplus \bigoplus_{\alpha \in R_1} \mathfrak{g}_\alpha\right)}^{=\mathfrak{g}_1} \oplus \overbrace{\left(\mathfrak{h}_2 \oplus \bigoplus_{\beta \in R_2} \mathfrak{g}_\beta\right)}^{=\mathfrak{g}_2} \tag{6.6.14}$$

  where $\mathfrak{g}_\alpha, \forall \alpha \in R$ are linearly independent (see (6.3.3)).
  * and it is easy to see that $[\mathfrak{g}_\alpha, \mathfrak{h}_2] = \{0\}, \alpha \in R_1$ since $\alpha \perp \mathfrak{h}_2$, and, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} = \{0\}$ if $\alpha \in R_1, \beta \in R_2$ $(\alpha + \beta \notin R)$.
- so, $\mathfrak{g}$ decomposes as the Lie algebra direct sum, $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, i.e. it is not simple.

6.7 the root systems of the classical Lie algebras

- 四个 root systems 的 Dynkin diagrams (见 section 7.6) 如下,



  - B_2 and C_2, A_3 and D_3 are isomorphic to each other.
  - D_2 的 Dynkin diagram is not connected $\Longrightarrow$ D_2 is reducible $\Longrightarrow$ $\mathfrak{so}(4,\mathbb{C})$ is not simple,

$$\mathfrak{so}(4,\mathbb{C}) = \Big(\mathrm{span}(e_1 - e_2) \oplus \mathfrak{g}_{\pm(e_1-e_2)}\Big) \oplus \Big(\mathrm{span}(e_1 + e_2) \oplus \mathfrak{g}_{\pm(e_1+e_2)}\Big) \tag{6.7.1}$$

    中间粗体的 $\oplus$ 是 Lie algebra direct sum, (两个 $\mathfrak{su}(2)_{\mathbb{C}}$).
  - $A_n, B_n, C_n, n \geq 1$ 和 $D_n, n \geq 3$ 都对应 simple Lie algebra,

$$\underset{n\geq 1}{\mathfrak{sl}(n+1,\mathbb{C}) \mapsto A_n} \quad \underset{n\geq 1}{\mathfrak{so}(2n+1,\mathbb{C}) \mapsto B_n} \quad \underset{n\geq 1}{\mathfrak{sp}(2n,\mathbb{C}) \mapsto C_n} \quad \underset{n\geq 3}{\mathfrak{so}(2n,\mathbb{C}) \mapsto D_n} \tag{6.7.2}$$

6.7.1 the special linear algebras, $\mathfrak{sl}(n+1,\mathbb{C}) = \mathfrak{su}(n+1)_{\mathbb{C}}$, and A_n

- $\mathfrak{su}(n+1) = \{A \in \mathcal{M}_{n+1}(\mathbb{C}) | A^\dagger = -A \text{ and } \mathrm{tr}A = 0\}$, 它的 maximal commutative subalgebra 是,

$$\mathfrak{t} = \{\mathrm{diag}(ia_1, \cdots, ia_{n+1}) | a_i \in \mathbb{R} \text{ and } a_1 + \cdots + a_{n+1} = 0\} \tag{6.7.3}$$

  从而得到 Cartan subalgebra, $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(\lambda_1, \cdots, \lambda_{n+1}) | \lambda_i \in \mathbb{C} \text{ and } \lambda_1 + \cdots + \lambda_{n+1} = 0\}$.

- 令 $E_{ij}, i \neq j \in \{1, \cdots, n+1\}$ 是第 i 行第 j 列的分量为 1, 其余位置为零的矩阵, $H = \mathrm{diag}(\lambda_1, \cdots) \in \mathfrak{h}$, 那么,

$$[H, E_{ij}] = (\lambda_i - \lambda_j) E_{ij} \tag{6.7.4}$$

- 选择一个内积, 使得 $\mathrm{ad}_X, \forall X \in \mathfrak{su}(n+1)$ 是 skew self-adjoint,

$$\langle A, B \rangle = \mathrm{tr}(A^\dagger B), \forall A, B \in \mathfrak{su}(n+1)_{\mathbb{C}} \tag{6.7.5}$$

proof:

注意这个内积在任何李代数中都保证 $\mathrm{ad}_X, \forall X \in \mathfrak{k}$ 是 skew self-adjoint, 但是根据 Cartan's criterion, 只有 semisimple 才能保证它 non-degenerate.

$$\mathrm{tr}(A^\dagger \mathrm{ad}_X B) = \mathrm{tr}(A^\dagger XB - A^\dagger BX) = \mathrm{tr}(A^\dagger XB - XA^\dagger B) = \mathrm{tr}(-\mathrm{ad}_X AB) \tag{6.7.6}$$

注意, 对于 $H, H' \in \mathfrak{h}$, 有 $\langle H, H' \rangle = \sum_i \lambda_i^* \lambda_i'$.

- 可见 E_{ij} 对应的 root 为,

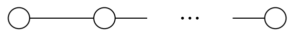
$$[H, E_{ij}] = \langle \underbrace{e_i - e_j}_{=\alpha_{ij}}, H \rangle E_{ij}, i \neq j \tag{6.7.7}$$

- $\mathfrak{sl}(n+1,\mathbb{C})$ 对应的 root system 用 A_n 表示,
  - $E = \{v \in \mathbb{R}^{n+1} | v_1 + \cdots + v_n = 0\}$, 所以 $\dim E = n$.

- $R = \{\alpha_{ij} = e_i - e_j | i \neq j \in \{1, \cdots, n+1\}\}$, 共有 $n(n+1)$ 个根. ($\dim \mathfrak{sl}(n+1, \mathbb{C}) = (n+1)^2 - 1$)
- $\Delta = \{e_1 - e_2, \cdots, e_n - e_{n+1}\}$ is a base, and $R^+ = \{e_i - e_j | i < j\}$, with,

$$e_i - e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{j-1} - e_j) \tag{6.7.8}$$

- 所有根的长度为 $\sqrt{2}$, 因此 $\langle \alpha, \beta \rangle = \langle \alpha, H_\beta \rangle$.
- $\langle \alpha, \beta \rangle = 0, \pm 1$ (when $\alpha \neq \pm\beta$).
- 两个 roots ($\alpha \neq \pm\beta$) 之间的夹角可能是 $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}$.
- 对于 base 中的根, 相邻 (consecutive) 的根夹角为 $\frac{2\pi}{3}$, 不相邻的互相垂直, 所以其 Dynkin 图如下,



A_n

- $s_{\alpha_{ij}}$ 作用到向量 $|v\rangle$ 使其 i, j 分量的位置交换, 因此 A_n 的 Weyl 群是 $n+1$ 个元素的 permutation group.

6.7.2 the orthogonal algebras, $\mathfrak{so}(2n, \mathbb{C})$, and D_n

- $\mathfrak{so}(2n, \mathbb{R}) = \mathfrak{o}(2n, \mathbb{R}) = \{A \in \mathcal{M}_{2n}(\mathbb{R}) | A^T = -A\}$, 它的 maximal commutative subalgebra 是,

$$\mathfrak{t} = \{H_a = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} | a = \mathrm{diag}(a_1, \cdots, a_n) \text{ with } a_i \in \mathbb{R}\} \tag{6.7.9}$$

proof:

任何 $\mathfrak{so}(2n, \mathbb{C})$ 中的元素都可以展开成 $\mathfrak{h} = \mathfrak{t}_C$ 和 D^α_{ij} (见下文) 的叠加, 那么, 与 $\mathfrak{h}$ 对易的元素一定不含有 D^α_{ij} 分量, 所以... 是 maximal. (总共有 $2n^2 - 2n$ 个根, 且 rank 为 n, 所以总维数为 $2n^2 - n = \frac{2n(2n-1)}{2}$)
另外, 注意如果 $n = 2$, $D^1_{11} = D^2_{11} = 0$ 而,

$$D^3_{11} = -D^4_{11} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \in \mathfrak{h} \tag{6.7.10}$$

也即 $\mathfrak{so}(2, \mathbb{C}) = \mathfrak{h}$, 与不存在 nontrivial center 的对应不符, 不是 semisimple.

- the root vectors are $D^\alpha_{ij} = C^\alpha_{ij} - (C^\alpha_{ij})^T$, where $\alpha = 1, 2, 3, 4$ and,

$$\begin{aligned} C^1_{ij} &= \begin{pmatrix} E_{ij} & iE_{ij} \\ iE_{ij} & -E_{ij} \end{pmatrix} & C^2_{ij} &= \begin{pmatrix} E_{ij} & -iE_{ij} \\ -iE_{ij} & -E_{ij} \end{pmatrix} \\ C^3_{ij} &= \begin{pmatrix} E_{ij} & -iE_{ij} \\ iE_{ij} & E_{ij} \end{pmatrix} & C^4_{ij} &= \begin{pmatrix} E_{ij} & iE_{ij} \\ -iE_{ij} & E_{ij} \end{pmatrix} \end{aligned} \tag{6.7.11}$$

where $i \neq j \in \{1, \cdots, n\}$ (如果 $i = j$, 那么 $D^{1,2}_{ii} = 0, D^{3,4}_{ii} \in \mathfrak{h}$), and we have,

$$\begin{aligned} [H_a, D^1_{ij}] &= i(a_i + a_j) D^1_{ij} & [H_a, D^2_{ij}] &= -i(a_i + a_j) D^2_{ij} \\ [H_a, D^3_{ij}] &= i(a_i - a_j) D^3_{ij} & [H_a, D^4_{ij}] &= -i(a_i - a_j) D^4_{ij} \end{aligned} \tag{6.7.12}$$

calculation:

we have $D^1_{ij} = C^1_{ij} - C^1_{ji}, D^2_{ij} = C^2_{ij} - C^2_{ji}, D^3_{ij} = C^3_{ij} - C^4_{ji}, D^4_{ij} = C^4_{ij} - C^3_{ji}$, and,

$$\begin{aligned} [H_a, C^1_{ij}] &= i(a_i + a_j) C^1_{ij} & [H_a, C^2_{ij}] &= -i(a_i + a_j) C^2_{ij} \\ [H_a, C^3_{ij}] &= i(a_i - a_j) C^3_{ij} & [H_a, C^4_{ij}] &= -i(a_i - a_j) C^4_{ij} \end{aligned} \tag{6.7.13}$$

- 内积定义为 $\langle A,B\rangle=\frac{1}{2}\mathrm{tr}(A^\dagger B)$, 那么,

$$\langle H_a,H_b\rangle=-\sum_{i=1}^{n}a_i^*b_i \tag{6.7.14}$$

  所以, 可以将 H_a 视作 $i(a_1,\cdots,a_n)$.

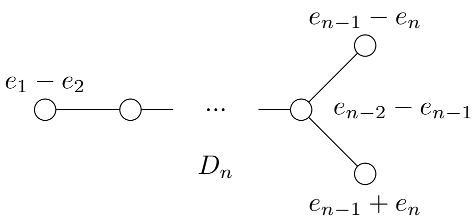
- 可见 root vectors 和 roots 的对应关系为 $(i\neq j\in\{1,\cdots,n\})$,

$$D_{ij}^1\mapsto\alpha_{ij}=e_i+e_j\quad D_{ij}^2\mapsto-\alpha_{ij}\quad D_{ij}^3\mapsto\beta_{ij}=e_i-e_j\quad D_{ij}^4\mapsto-\beta_{ij} \tag{6.7.15}$$

- $\mathfrak{so}(2n,\mathbb{C})$ 对应的 root system 用 D_n 表示,
  - $E=\mathbb{R}^n$.
  - $R=\{\pm e_i\pm e_j|i\neq j\in\{1,\cdots,n\}\}$, 共有 $\frac{n(n-1)}{2}\times 4=2n^2-2n$ 个根. ($\dim\mathfrak{so}(2n,\mathbb{C})=\frac{2n(2n-1)}{2}$)
  - $\Delta=\{e_1-e_2,\cdots,e_{n-1}-e_n\}\cup\{e_{n-1}+e_n\}$ is a base, and $R^+=\{e_i-e_j|i<j\}\cup\{e_i+e_j\}$, with,

$$e_i+e_j=\underbrace{(e_i-e_{i+1})+\cdots+(e_{n-1}+e_n)}_{=e_i+e_n}+\underbrace{(e_j-e_{j+1})+\cdots+(e_{n-1}-e_n)}_{=e_j-e_n} \tag{6.7.16}$$

  - 所有根的长度为 $\sqrt{2}$, 因此也有 $\langle\alpha,\beta\rangle=\langle\alpha,H_\beta\rangle$.
  - $\langle\alpha,\beta\rangle=0,\pm1$ (when $\alpha\neq\pm\beta$), 所以两个根之间的夹角可能是 $\frac{\pi}{2}$ 或 $\frac{\pi}{3},\frac{2\pi}{3}$.
  - D_n 的 Dynkin 图如下,



  - $s_\alpha=s_{-\alpha},\alpha\in R$ 分别为,

$$\begin{cases}s_{\alpha_{ij}}:(\cdots,v_i,\cdots,v_j,\cdots)\mapsto(\cdots,-v_j,\cdots,-v_i,\cdots)\\ s_{\beta_{ij}}:(\cdots,v_i,\cdots,v_j,\cdots)\mapsto(\cdots,v_j,\cdots,v_i,\cdots)\end{cases} \tag{6.7.17}$$

6.7.3 the orthogonal algebras, $\mathfrak{so}(2n+1,\mathbb{C})$, and B_n

- its maximal commutative subalgebra is,

$$\mathfrak{t}=\{\left(\begin{array}{cc|c}0&a&\\-a&0&\\\hline&&0\end{array}\right)|a=\mathrm{diag}(a_1,\cdots,a_n)\text{ with }a_i\in\mathbb{R}\} \tag{6.7.18}$$

  both $\mathfrak{so}(2n+1,\mathbb{C})$ and $\mathfrak{so}(2n,\mathbb{C})$ have rank n.

- every root in $\mathfrak{so}(2n,\mathbb{C})$ is a root in $\mathfrak{so}(2n+1,\mathbb{C})$, but there are $2n$ additional roots in $\mathfrak{so}(2n+1,\mathbb{C})$.

- the additional root vectors are,

$$B_k^1=\left(\begin{array}{c|c|c}&&\vdots\\&&1\\&&\vdots\\\hline&&\vdots\\&&i\\&&\vdots\\\hline\cdots\ -1\ \cdots&\cdots\ -i\ \cdots&0\end{array}\right)\quad B_k^2=\left(\begin{array}{c|c|c}&&\vdots\\&&1\\&&\vdots\\\hline&&\vdots\\&&-i\\&&\vdots\\\hline\cdots\ -1\ \cdots&\cdots\ i\ \cdots&0\end{array}\right) \tag{6.7.19}$$

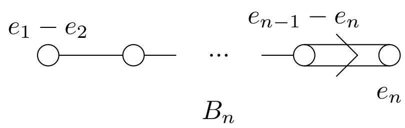
其中 $B_k^{1,2}$ 的非零元素位于 $(k,2n+1),(n+k,2n+1)$ 和通过转置相对应的位置, 有对易关系,

$$[H_a, B_k^1] = ia_k B_k^1 \quad [H_a, B_k^2] = -ia_k B_k^2 \tag{6.7.20}$$

- 选取与上一 subsection 一样的内积, 那么 root vectors 和 roots 的对应关系为,

$$B_k^1 \mapsto e_k \quad B_k^2 \mapsto -e_k \tag{6.7.21}$$

- $\mathfrak{so}(2n+1,\mathbb{C})$ 对应的 root system 用 B_n 表示,
  - $E = \mathbb{R}^n$.
  - $R = \{\pm e_i \pm e_j \text{ and } \pm e_k | i \neq j, k \in \{1,\cdots,n\}\}$, 共有 $2n^2$ 个根. $(\dim \mathfrak{so}(2n+1,\mathbb{C}) = \frac{(2n+1)2n}{2})$
  - $\Delta = \{e_1 - e_2, \cdots, e_{n-1} - e_n\} \cup \{e_n\}$ is a base, and $R^+ = \{e_i - e_j | i < j\} \cup \{e_i + e_j\} \cup \{e_k\}$.
  - $\langle \alpha, \beta \rangle = 0, \pm 1$ (when $\alpha \neq \pm\beta$), 所以两个根之间的夹角可能为 $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}$.
  - B_n 的 Dynkin 图如下,



B_n

6.7.4 the symplectic algebras, $\mathfrak{sp}(2n,\mathbb{C})$, and C_n

- $\mathfrak{sp}(2n,\mathbb{C}) = \{A \in \mathcal{M}_{2n}(\mathbb{C}) | \Omega A^T \Omega = A\}$, where,

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{6.7.22}$$

$\mathfrak{sp}(2n,\mathbb{C})$ 中的矩阵可以写成如下形式,

$$A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \tag{6.7.23}$$

where $a, b, c \in \mathcal{M}_n(\mathbb{C})$, and b, c are symmetric.

- 可以认为 $\mathfrak{k} = \mathfrak{sp}(2n,\mathbb{C}) \cap \mathfrak{u}(2n)$ 是其 compact real form,

$$\mathfrak{sp}(2n,\mathbb{C}) \cap \mathfrak{u}(2n) = \{\begin{pmatrix} a & b \\ -b^\dagger & -a^T \end{pmatrix} | a^\dagger = -a, b^T = b\} \tag{6.7.24}$$

- the maximal commutative subalgebra of $\mathfrak{k}$ is,

$$\mathfrak{t} = \{H_a = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} | a = \mathrm{diag}(a_1, \cdots, a_n), ia_i \in \mathbb{R}\} \tag{6.7.25}$$

- the root vectors are $(i \neq j)$,

$$\begin{gathered} A_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix} \quad B_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix} \quad C_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \\ F_k = \begin{pmatrix} 0 & E_{kk} \\ 0 & 0 \end{pmatrix} \quad G_k = \begin{pmatrix} 0 & 0 \\ E_{kk} & 0 \end{pmatrix} \end{gathered} \tag{6.7.26}$$

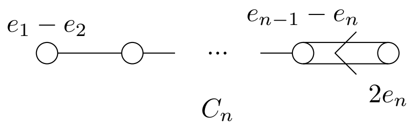
对易关系为,

$$\begin{gathered} [H_a, A_{ij}] = (a_i + a_j) A_{ij} \quad [H_a, B_{ij}] = -(a_i + a_j) B_{ij} \quad [H_a, C_{ij}] = (a_i - a_j) C_{ij} \\ [H_a, F_k] = 2a_k F_k \quad [H_a, G_k] = -2a_k G_k \end{gathered} \tag{6.7.27}$$

- 选取内积为 $\langle A, B \rangle = \frac{1}{2}\mathrm{tr}(A^\dagger B)$, 所以 H_a 可以视为 $(a_1, \cdots, a_n)$, 那么 root vectors 和 roots 的对应关系为,

$$A_{ij} \mapsto e_i + e_j \quad B_{ij} \mapsto -e_i - e_j \quad C_{ij} \mapsto e_i - e_j \quad F_k \mapsto 2e_k \quad G_k \mapsto -2e_k \tag{6.7.28}$$

- $\mathfrak{sp}(2n,\mathbb{C})$ 对应的 root system 用 C_n 表示,
  - $E = \mathbb{R}^n$.
  - $R = \{\pm e_i \pm e_j \text{ and } \pm 2e_k | i \neq j, k \in \{1, \cdots, n\}\}$, 与 B_n 相似 (区别是 $\pm e_k$ 前的系数 2), 共有 $2n^2$ 个根. $(\dim \mathfrak{sp}(2n,\mathbb{C}) = n(2n+1))$
  - $\Delta = \{e_1 - e_2, \cdots, e_{n-1} - e_n\} \cup \{2e_n\}$ and $R = \{e_i - e_j | i < j\} \cup \{e_i + e_j\} \cup \{2e_k\}$.
  - $\langle \alpha, \beta \rangle = 0, \pm 1, \pm 2$ (when $\alpha \neq \pm\beta$), 所以两个根之间夹角可能为 $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}$.
  - C_n 的 Dynkin 图如下,



C_n

Chapter 7

root systems

7.1 abstract root systems

- **def.:** a **root system** (E, R) is a finite-dimensional vector space $E = \mathrm{span}(R)$, with a finite collection of non-zero vectors R, and an inner product $\langle \cdot, \cdot \rangle$, and,

  1. $E = \mathrm{span}(R)$,
  2. if $\alpha \in R$, then $c\alpha \in R \iff c = \pm 1$,
  3. if $\alpha, \beta \in R$, then $s_\alpha |\beta\rangle \in R$, where $s_\alpha = 1 - 2\frac{|\alpha\rangle\langle\alpha|}{\langle\alpha,\alpha\rangle}$,
  4. for all $\alpha, \beta \in R$, $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

  $\dim E$ is called the **rank** of the system, elements in R are called **roots**.

- **def.:** the **Weyl group**, W, of R is the finite subgroup of the orthogonal group of E generated by $s_\alpha, \forall \alpha \in R$.

- **def.:** (E, R) and (F, S) are two root systems, then $(E \oplus F, R \cup S)$ is a root system, and $R \cup S$ is called the **direct sum** of R and S.

  (it is easy to see the direct sum root system satisfies the def. of root systems)

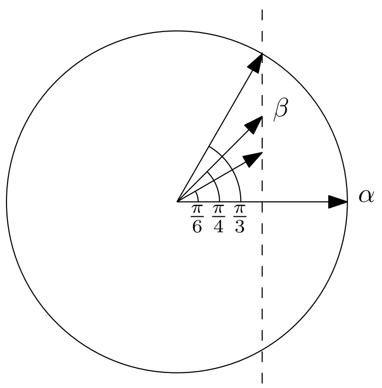
- **def.:** a root system is called **reducible** if there exists an orthogonal decomposition $E = E_2 \oplus E_2$ with $E_1 \perp E_2$ and $\dim E_i > 0$, and every root is either in E_1 or E_2.

  - the root system of a semisimple Lie algebra is irreducible $\iff$ the semisimple Lie algebra is simple (见 section 6.6 最后一个定理).

- **def.:** an **isomorphism** is a linear map that **preserves the reflection**, not the inner product,

$$A : E \to F \quad \text{s.t.} \quad A s_\alpha |\beta\rangle = s_{A\alpha} |A\beta\rangle \tag{7.1.1}$$

- 对于 $\langle\beta,\beta\rangle \leq \langle\alpha,\alpha\rangle$, 且 $\beta \not\propto \alpha$, 根 α, β 之间可能的关系如下,
  - $\beta \perp \alpha$.
  - or, $\langle\alpha,\alpha\rangle = 1, 2, 3\,\langle\beta,\beta\rangle$ (图中没有画出 $\beta \mapsto -\beta$ 的情况, 那时夹角是图中夹角的补角).



- 如果根 α, β 之间夹锐角, 那么 $\pm(\alpha-\beta)$ 也是根；如果夹钝角, 那么 $\pm(\alpha+\beta)$ 也是根.

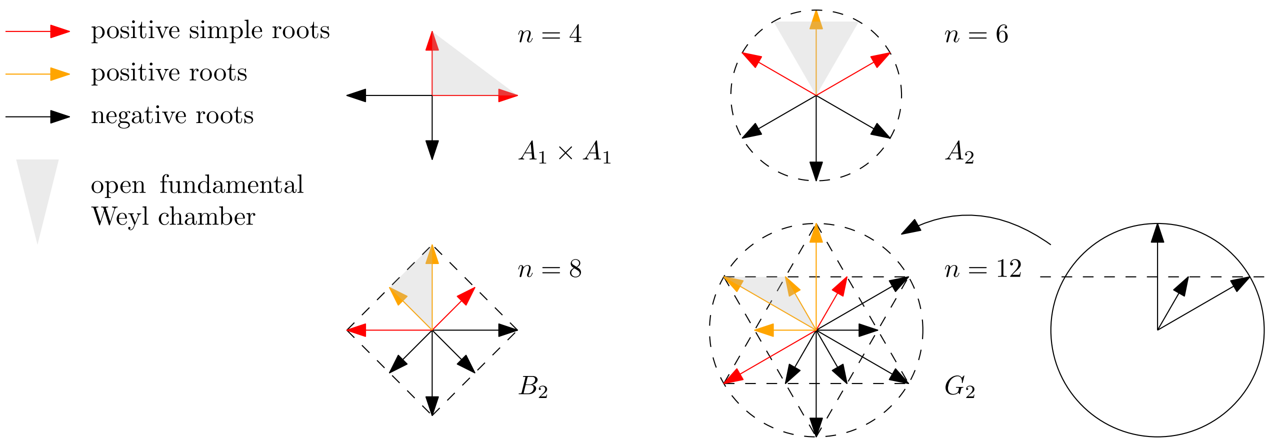
proof:

假设 $\langle\alpha, \alpha\rangle \geq \langle\beta, \beta\rangle$, 考虑夹锐角的情况, 此时, $\beta-\alpha=s_\alpha|\beta\rangle$；对于夹钝角的情况, 令 $\beta'=-\beta$ 即可.

7.2 rank-two systems

- if rank is one, the roots are $R=\{-\alpha, \alpha\}$.
- **every rank-two system** is isomorphic to one of the systems below,

minimum angle $\theta=\frac{2\pi}{n}$



分别考虑两个根之间最小夹角为 $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ 的情况, 然后使用 $s_\alpha|\beta\rangle$ 生成整个 R.
for positive simple roots, positive roots, negative roots and Weyl chambers, see section 7.4.

- the **Weyl group** of a rank-two root system, R, with minimum angle $\theta=\frac{2\pi}{n}$ is the symmetry group of a regular $\frac{n}{2}$-gon (正 $\frac{n}{2}$ 边形).
  - 群元素包括 $\frac{n}{2}$ 个镜面反射和 2θ 转动.

7.3 duality

- **def.:** for a root $\alpha \in R$ in a root system (E, R), its **coroot** is,

$$H_\alpha=2\frac{\alpha}{\langle\alpha, \alpha\rangle} \quad \text{with} \quad \begin{cases} s_{H_\alpha}=s_\alpha \\ \dfrac{\langle H_\alpha, H_\beta\rangle}{\langle H_\alpha, H_\alpha\rangle}=\dfrac{\langle\alpha, \beta\rangle}{\langle\beta, \beta\rangle} \end{cases} \tag{7.3.1}$$

and the **dual root system** to R is $R^\vee=\{H_\alpha|\alpha \in R\}$.

  - $R^\vee$ is also a root system, with the same Weyl group as R (because $s_{H_\alpha}=s_\alpha$).
  - $H_{H_\alpha}=\alpha$ and $(R^\vee)^\vee=R$.
  - note that although $H_{s_\alpha|\beta\rangle}=s_{H_\alpha}|H_\beta\rangle$, the map H is not linear, so $R^\vee$ and R are not necessarily isomorphic to each other.

7.4 bases and Weyl chambers

- **def.:** for a root system (E, R), a subset $\Delta \subset R$ is called a **base** if,

1. Δ is a basis of E,
2. each root $\alpha \in R$ can be expressed as a linear combination of basis vectors in Δ with non-negative (positive roots, R^+) or non-positive (negative roots, R^-) integer coefficients, $R=R^+\cup R^-$.

elements in Δ are called **positive simple roots**.

- $\alpha \neq \beta \in \Delta$, then $\langle \alpha, \beta \rangle \leq 0$.

proof:

如果 α, β 之间夹锐角, 那么 $\pm(\alpha - \beta)$ 也是根, 不满足系数同时非负 (或非正) 的要求.

- for a root system (E, R), there exists a hyperplane V through the origin in E, s.t. V does not contain any root.

proof:

考虑一个向量 $H \in E$, 它不在任何一个垂直于某个根向量的超平面 (这样的超平面有限多, 所以 H 存在) 上, 那么 $V \perp H$ 就是我们要找的超平面.

- **def.:** choose one side of V to be R^+, the other side to be R^-, an element $\alpha \in R^+$ is **decomposable** if $\alpha = \beta + \gamma$ for some $\beta, \gamma \in R^+$, otherwise, α is **indecomposable**.
- the indecomposable roots in R^+ form the base Δ, and Δ exists.

proof:

let Δ denote the set of indecomposable elements in R^+, now we will prove Δ is the base:

- every $\alpha \in R^+$ can be expressed as a linear combination of elements in Δ with non-negative integer coefficients.

---

proof:

* 考虑 $H \perp V$, 且 $\langle \alpha, H \rangle > 0, \forall \alpha \in R^+$.
* 考虑 Δ' 是不能表示成 Δ 的元素的非负整数系数的线性叠加的 R^+ 元素的集合, 那么一定有 $\Delta' \cap \Delta = \emptyset$.
* 考虑 $\alpha \in \Delta'$ 且 $\langle \alpha, H \rangle$ 是 Δ' 中元素里最小的, 而且 $\alpha = \beta_1 + \beta_2$ (且 $\beta_1, \beta_2 \in R^+$), 那么 β_1, β_2 至少有一个是 Δ' 的元素, 但是 $\langle \alpha, H \rangle = \langle \beta_1, H \rangle + \langle \beta_2, H \rangle$ 这与 $\langle \alpha, H \rangle$ 最小矛盾.
* 可见 $\beta_1, \beta_2 \notin \Delta'$, α 一定可以表示为 Δ 的元素的... 的线性叠加.

- elements in Δ are linearly independent.

---

proof:

如果,

$$\sum_{\alpha \in \Delta} c'_\alpha \alpha = 0 \Longrightarrow \sum_\alpha c_\alpha \alpha = \sum_\beta d_\beta \beta = u \in R^+ \tag{7.4.1}$$

其中 $c_\alpha \geq 0, -d_\beta < 0$ 分别是 $\{c'_\alpha\}$ 中非负和负的系数, 等号两边对 Δ 的两个无交集的子集求和.

考虑,

$$\langle u, u \rangle = \langle \sum_\alpha c_\alpha \alpha, \sum_\beta d_\beta \beta \rangle = \sum_{\alpha, \beta} c_\alpha d_\beta \langle \alpha, \beta \rangle \tag{7.4.2}$$

但是, 对于 $\alpha \neq \beta \in \Delta$, 一定有 $\langle \alpha, \beta \rangle \leq 0$, 所以 $\langle u, u \rangle = 0$, 即 $u = 0$, 这与 $\alpha \in R^+$ 矛盾.

---

proof of $\langle \alpha, \beta \rangle \leq 0, \forall \alpha \neq \beta \in \Delta$:

如果 α, β 呈锐角, 那么 $\pm(\alpha - \beta)$ 也是根, 且其中一个属于 R^+, 比如 $\alpha - \beta \in R^+$, 那么 $\alpha = (\alpha - \beta) + \beta$, 与 indecomposable 矛盾.

最后, 注意到 indecomposable root 一定存在. 只需考虑 $\langle \alpha, H \rangle$ 值最小的 $\alpha \in R^+$ 即可证明存在.

- for any base Δ for R, there exists a hyperplane V, s.t. Δ arises as in the theorem above.

proof:

Δ 是一组基底, 张成向量空间中的一个锥形, 存在一个区域, 这个区域中的每个向量都与基底夹锐角 (这个区域就是 fundamental Weyl chamber), 那么 V 就是垂直于这个区域中的某个矢量的超平面.
由于基向量线性无关, 所以任何基向量都不可分解 (indecomposable).

- $\alpha \in \Delta$ cannot be expressed as a linear combination of $R^+ - \Delta$ with non-negative real (not integer) coefficients.

proof:

let $\Delta = \{\alpha_1, \cdots, \alpha_r\}$, suppose,

$$\alpha_1 = \sum_{\beta \in R^+ - \Delta} c_\beta \beta = \sum_{\beta, i} c_\beta d_{\beta,i} \alpha_i \tag{7.4.3}$$

where $d_{\beta,i}$ are non-negative integers.
if c_β are non-negative, it will contradict to the linear independence.

- $\{H_\alpha | \alpha \in \Delta\}$ is the base of $R^\vee$.

proof:

  - 首先, 选取 Δ 对应的 V, 并以这个平面推出 $\Delta^\vee$ (这个 base 存在), 那么 $H_\alpha \in R^{\vee +} \iff \alpha \in R^+$.
  - 考虑 $\alpha \in R^+ - \Delta$, 那么 α 是 $\alpha_1, \cdots, \alpha_r$ 的非负整数的线性叠加, 那么 H_α 是 $H_{\alpha_1}, \cdots, H_{\alpha_r}$ 的非负实数的线性叠加.
  - 根据上一个 theorem 可知 $H_\alpha \notin \Delta^\vee$ 且 $H_{\alpha_1}, \cdots, H_{\alpha_r}$ 是 E 的基底, 所以一定有 $\Delta^\vee = \{H_{\alpha_1}, \cdots, H_{\alpha_r}\}$.

- **def.:** the **open Weyl chambers** for a root system (E, R) are connected components of,

$$E - \bigcup_{\alpha \in R} V_\alpha \tag{7.4.4}$$

where $V_\alpha \perp \alpha$ is a hyperplane through the origin.

- **def.:** the **open fundamental Weyl chamber** (relative to Δ) is $\{H | \langle \alpha, H \rangle > 0, \forall \alpha \in \Delta\}$.
  - open fundamental Weyl chamber is connected (consider $\langle H, \beta \rangle > \langle H, \alpha \rangle, \alpha \in \Delta, \beta \in R^+ - \Delta, H \perp V$).
  - every elements in the open fundamental Weyl chamber has a positive inner product with root in R^+, and negative inner product with root in R^-, so open fundamental Weyl chamber is an open Weyl chamber.

- for each open Weyl chamber C, there exists a unique base Δ_C, s.t. C is the open fundamental Weyl chamber relative to Δ_C.
  - there is a one-to-one correspondence between bases and Weyl chambers.

proof:

考虑 $H \in C$, 以 $V \perp H$ 建立起的 base 就是 Δ_C.
考虑 Δ, Δ' 都对应同一个 C, 它们的 $R^+ = R'^+$, 且可以选取 $V = V'$, 那么一定有 $\Delta = \Delta'$ (都是不可分解的根).

- every root is an element of some base.

proof:

任何一个根 α 对应的 $V_\alpha \perp \alpha$ 都包含某个 open Weyl chamber C 的边界.
考虑 $H \in V_\alpha$ 且 $H + \epsilon\alpha \in C$, 选取 $V \perp H' = H + \epsilon\alpha$, 显然 $\langle \alpha, H' \rangle$ 是 R^+ 中最小的, 所以一定有 $\alpha \in \Delta_C$.

7.5 Weyl chambers and Weyl group

- the Weyl group act **transitively** on the set of Weyl chambers, i.e. for every open Weyl chamber C, we have,

$$\{w(C)|w \in W\} = E - \bigcup_{\alpha \in R} V_\alpha \tag{7.5.1}$$

proof:

consider chamber C with its base Δ_C, we want to prove that $wH' \in C$ for all $H' \in E - \bigcup_{\alpha \in R} V_\alpha$ ($H' \in C$ case is trivial) and $w \in W'$ where W' is generated by $s_\alpha, \alpha \in \Delta_C$.

- in the case when $H' \notin C$, there exists some $\alpha \in \Delta_C$ that $\langle \alpha, H' \rangle < 0$ (夹钝角).
- since W' is a finite group, there exists a $w \in W'$ that bring H' closest to some $H \in C$.
- if $wH' \notin C$, then there exists $\alpha \in \Delta_C$ that $\langle \alpha, wH' \rangle < 0$, then,

$$\begin{aligned} |wH' - H|^2 - |s_\alpha wH' - H|^2 &= 2 \langle wH' | s_\alpha - 1 | H \rangle \\ &= -4 \frac{\langle wH' | \alpha \rangle \langle \alpha | H \rangle}{\langle \alpha, \alpha \rangle} > 0 \end{aligned} \tag{7.5.2}$$

  which contradicts to the closest-ness.
- so, we must have $wH' \in C$.

- W is generated by $s_\alpha, \alpha \in \Delta$.

proof:

we want to prove that for all α, there exists some $w \in W'$ (generated by $s_\beta, \beta \in \Delta_C$) s.t.,

$$s_{w|\alpha\rangle} = w s_\alpha w^{-1} \in W' \tag{7.5.3}$$

- let $\alpha \in \Delta_D$ where D is some chamber.
- we already proved that there is some $w \in W'$ that $w[D] = C$, since w preserves inner product, $w[\Delta_D] = \Delta_C$.
- so, $w\,|\alpha\rangle \in \Delta_C$, i.e. $s_{w|\alpha\rangle} \in W'$.

- **def.:** the **minimal expression** of $w \in W$ is the expression of w in terms of $s_\alpha, \alpha \in \Delta$ with the minimal number of s_α (the minimal expression need not be unique).
- $\bar{C}$ is the closure of a Weyl chamber C, if $H, H' \in \bar{C}$ and $w\,|H\rangle = H'$, then $H = H'$.
  i.e. two distinct elements of $\bar{C}$ cannot be in the same orbit of W.

proof:

we proceed by induction on the number of the minimal expression of w in terms of $s_\alpha, \alpha \in \Delta_C$.

- if the minimal number is zero, i.e. $w = I$, the result holds.
- if the result holds when the minimal number is $k-1$, then, consider $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$.
- C and $w[C]$ lie on opposite sides of hyperplane V_{α_1}, i.e. $\bar{C} \cap w[\bar{C}] \subset V_{\alpha_1}$.

---

proof:

let's prove by induction. for $w = s_{\alpha_1}$, the result holds, consider $w = us_{\alpha_k}$, where $u = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{k-1}}$,

* C and $u[C]$ lie on opposite sides of V_{α_1} (by induction).
* if C and $w[C]$ lie on the same side, then $w[C] = u \circ s_{\alpha_k}[C]$ lies on the opposite side of $u[C]$, i.e. C and $s_{\alpha_k}[C]$ lie on opposite sides of $V_{u^{-1}|\alpha_1\rangle}$.
* notice that $\alpha_k \in \Delta_C$, consider $H \in V_{\alpha_k}$ which also lies on the boundary of C, then, $s_{\alpha_k}H = H$ also lies on the boundary of $s_{\alpha_k}[C]$, which implies $V_{u^{-1}|\alpha_1\rangle} = V_{\alpha_k}$, so,

$$u^{-1}s_{\alpha_1}u = s_{u^{-1}|\alpha_1\rangle} = s_{\alpha_k} \Longrightarrow w = s_{\alpha_1}u = s_{\alpha_2}\cdots s_{\alpha_k} \tag{7.5.4}$$

which contradicts to the minimal expression assumption.

- since $w\,|H\rangle = H' \in w[\bar{C}] \cap \bar{C} \subset V_{\alpha_1}$, which implies,

$$s_{\alpha_1}H' = H' = s_{\alpha_2}\cdots s_{\alpha_k}H \tag{7.5.5}$$

by induction, $H = H'$.

- if $H \in C$ for some chamber C, and $w\,|H\rangle = H$, then, $w = I$ (W acts **freely**).

proof:

since $w\,|H\rangle \in C$, and w is a continuous map, so we must have $w[C] = C$, i.e. for all $H' \in C$, we have $w\,|H'\rangle \in C \Longrightarrow w\,|H'\rangle = H'$ (according to the theorem above), then $w = I$.

- W acts **freely** and **transitively** on Weyl chambers, the same is true for bases, i.e. for two bases Δ_1, Δ_2, there exits (transitiveness) a unique (free-ness) w, s.t. $w[\Delta_1] = \Delta_2$.
- C is a Weyl chamber, $H \in E$, then there is exactly one point in the W-orbit of H that lies in $\bar{C}$ (but the w that $w\,|H\rangle \in C$ is not necessarily unique).

proof:

- H is in the closure of some chamber D, and there exists a w that $w[\bar{D}] = \bar{C}$, so $w\,|H\rangle \in \bar{C}$.
- if $H', H'' \in \bar{C}$ are point in the W-orbit of H, then $H' = H''$.

- for all $\alpha \in \Delta, \beta \in R^+$, and $\beta \neq \alpha$, we have $s_\alpha\,|\beta\rangle \in R^+$.

proof:

- write $\beta = \sum_{\gamma\in\Delta} c_\gamma \gamma$ with $c_\gamma \in \mathbb{Z}^+$.
- notice that $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$, so, $s_\alpha\,|\beta\rangle = \beta - n\alpha$ for some integer n.
- in the expansion,

$$s_\alpha\,|\beta\rangle = \sum_{\gamma\in\Delta-\{\alpha\}} c_\gamma\gamma + (c_\alpha - n)\alpha \tag{7.5.6}$$

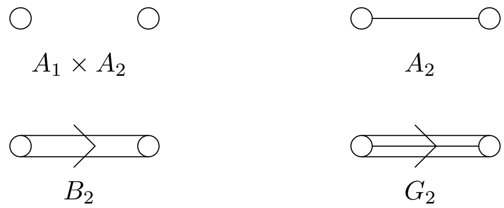
only the coefficient c_α changes.

- if one coefficient is positive in the expansion, all other coefficients must be positive, so $s_\alpha\,|\beta\rangle \in R^+$.

7.6 Dynkin diagrams

- **def.:** $\Delta = \{\alpha_1, \cdots, \alpha_r\}$ is the base of R, the **Dynkin diagram** for R is:

1. 图中有 r 个**结点**,
2. 节点 v_i, v_j 之间根据 α_i, α_j 之间的夹角决定连线的**条数**, $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ 分别对于 $0, 1, 2, 3$ 条连线,
3. 如果 α_i, α_j 长度不同, 连线上画出一条**指向更短的根的箭头** (可以将箭头视作**大于符号**).



- 注意, 夹角为 $\frac{2\pi}{3}, \frac{3\pi}{4}$ 的根长度一定不相等, 即, $2, 3$ 条线上一定有箭头；相反, 一条线上一定没有箭头.
- 同一个 root system 的两个 Δ_1, Δ_2 的 Dynkin 图一定完全相同 (isomorphic).

proof:

there exists $w \in W$ s.t. $w[\Delta_1] = \Delta_2$, and w preserves angles and lengths.

- a root system is irreducible (see section 7.1) $\iff$ its Dynkin diagram is connected.
  - semisimple Lie algebra $\mathfrak{g}$ is **simple** $\iff$ the Dynkin diagram of $R \subset it$ is **connected**.

proof:

如果 R 是 reducible, 那么 $\Delta = \Delta_1 \cup \Delta_2$ 且 $\Delta_1 \perp \Delta_2$, 则 Dynkin 图一定 not connected.

反之, Dynkin 图 not connected $\Longrightarrow$ $\Delta = \Delta_1 \cup \Delta_2$ 且 $\Delta_1 \perp \Delta_2$, 那么 $E = E_1 \oplus E_2$ with $E_i = \text{span}(\Delta_i)$.
Weyl 群由 $s_\alpha, \alpha \in \Delta$ 生成, 而 $s_\alpha, \alpha \in \Delta_1$ 在 E_2 上是单位映射, 可见 $W = W_1 \times W_2$, 因此, $R = W[\Delta] = W_1[\Delta_1] \cup W_2[\Delta_2] = R_1 \cup R_2$, 即根要么属于 E_1 要么属于 E_2.

- Dynkin diagrams are isomorphic $\iff$ root systems are isomorphic.

7.7 integral and dominant integral elements

- **def.:** an element $\mu \in E$ is an **integral element** if for all $\alpha \in R$,

$$\langle \mu, H_\alpha \rangle = 2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \tag{7.7.1}$$

μ is **dominant** (relative to Δ) if $\langle \mu, \alpha \rangle \geq 0, \forall \alpha \in \Delta$, and **strictly dominant** if $\langle \mu, \alpha \rangle > 0, \forall \alpha \in \Delta$.

  - μ is (strictly) dominant (relative to Δ_C) $\iff$ $\mu \in \bar{C}$ (or C).
  - for all μ, there exists $w \in W$ s.t. $w\,|\mu\rangle \in \bar{C}$.
  - every integer linear combination of roots (e.g. $2\alpha + 3\beta + 5\gamma$) is an integral element. 但一般不是所有 integral elements 都是根的整数线性组合.

- 注意 $\{H_\alpha | \alpha \in \Delta\}$ 是 $R^\vee$ 的 base (见 section 7.4), 所以 $\langle \mu, H_\alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta \Longrightarrow \mu$ 是 integral element.
- **def.:** the **fundamental weights** (relative to $\Delta = \{\alpha_1, \cdots, \alpha_l\}$) are $\mu_1, \cdots, \mu_r$ s.t.,

$$\langle \mu_i, H_{\alpha_j} \rangle = \delta_{ij} \tag{7.7.2}$$

i.e. the dual basis of $\Delta^\vee$.

  - $\Delta^{\vee *}$ 的非负 (正) 整数的线性组合是 (strictly) dominant integral element.
  - $\Delta^{\vee *}$ 的整数线性组合的集合 $=$ integral elements 的集合.

- **def.: half the sum of the positive roots** (relative to Δ) is,

$$\delta = \frac{1}{2}\sum_{\alpha\in R^+}\alpha \tag{7.7.3}$$

- δ is a strictly dominant integral element, and,

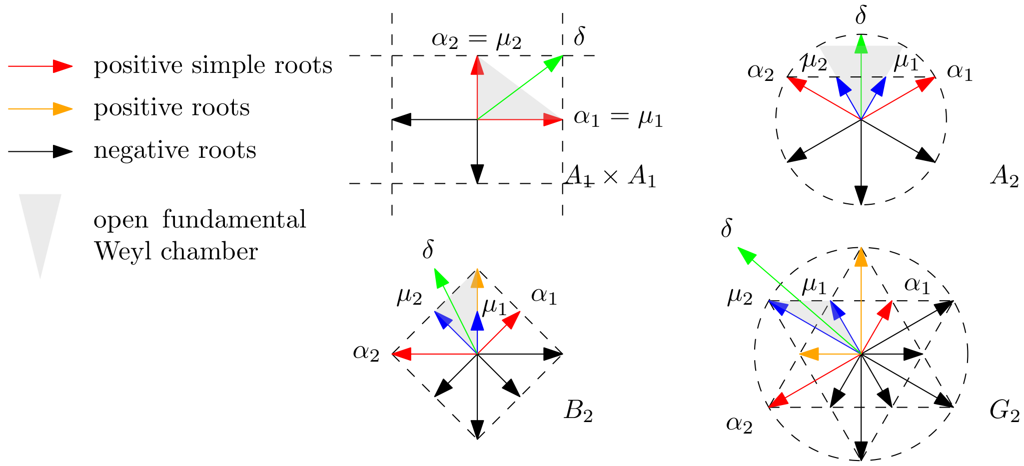
$$\langle\delta, H_\alpha\rangle = 1, \forall\alpha\in\Delta \iff \delta = \sum_{i=1}^{r}\mu_i \tag{7.7.4}$$

proof:

注意 section 7.5 最后一个定理, $s_\alpha[R^+ - \{\alpha\}] = R^+ - \{\alpha\}$, 所以 $R^+ - \{\alpha\} = \{\beta_1, s_\alpha\beta_1, \beta_2, s_\alpha\beta_2, \cdots\}$. 且有 $\langle\beta_1 + s_\alpha\beta_1, H_\alpha\rangle = 0$, 所以,

$$\langle\delta, H_\alpha\rangle = \langle\frac{1}{2}\alpha, H_\alpha\rangle = 1 \tag{7.7.5}$$

- fundamental wights and half the sum of the positive roots in rank-two systems 见下图,



7.8 the partial ordering

- **def.:** relative to $\Delta = \{\alpha_1, \cdots, \alpha_r\}$, $\mu \succeq \nu$ (μ is **higher** than ν) if,

$$\mu - \nu = c_1\alpha_1 + \cdots + c_r\alpha_r \tag{7.8.1}$$

其中 $c_1, \cdots, c_r \geq 0$, 类似地, 可以定义 $\nu \preceq \mu$ (... **lower** than...).

  - $\succeq$ 定义了一个 partial ordering on E, 但两个矢量之间可能既不存在 $\succeq$ 也不存在 $\preceq$ 的关系.

- $\mu \in E$ is dominant $\Longrightarrow \mu \succeq 0$.

proof:

考虑 Δ 的 dual basis $\Delta^* = \{\alpha_1^*, \cdots, \alpha_r^*\}$, 有,

$$c_i = \langle\alpha_i^*, \mu\rangle = \sum_{j=1}^{r}\langle\alpha_i^*, \alpha_j^*\rangle\langle\alpha_j, \mu\rangle \tag{7.8.2}$$

Δ 中的任何两个向量夹钝角 (见 section 7.4 定义后的第一条定理), 那么它的对偶基底中的任意两个向量夹锐角 (见 appendix A.4), 所以 $\langle\alpha_i^*, \alpha_j^*\rangle \geq 0, \langle\alpha_j, \mu\rangle \geq 0$, 所以 $c_i \geq 0$.

- if μ is dominant (i.e. $\mu \in \bar{C}$), then $w\,|\mu\rangle \preceq \mu$ for all $w \in W$.

proof:

O is the Weyl-group orbit of μ. 考虑到 O 是有限集合, 令 $\nu \in O$ 使得没有其它元素高于 ν, 那么一定有 $\nu \in \bar{C}$ (即 dominant), 否则, 如果 $\langle \nu, \alpha \rangle < 0, \exists \alpha \in \Delta_C$, 那么,

$$s_\alpha |\nu\rangle = \nu - 2\frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \succeq \nu \tag{7.8.3}$$

考虑到 section 7.5 的第四个结论, 可知 $\nu = \mu$.

现在证明 O 中没有元素既不高于也不低于 μ.
考虑所有既不... 也不... 的元素的集合 O', $\xi \in O'$ 且没有 O' 中的元素高于它, 那么,

– 如果 $o \in O - O'$, 那么一定有 $\mu \succeq o$, 且如果 $o \succeq \xi$, 那么 $\mu \succeq o \succeq \xi$, 与 $\xi \in O'$ 矛盾.

所以 O 中没有元素高于 ξ, 可知 $\xi \in \bar{C}$, 矛盾.

- if μ is a strictly dominant ($\mu \in C$) integral element, then $\mu \succeq \delta$ (δ is half the sum of positive roots).

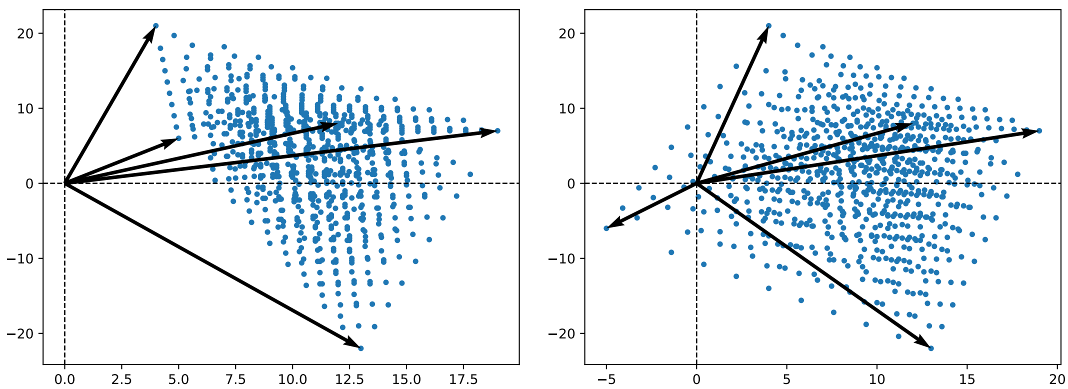
proof:

μ is a strictly dominant integral element $\Longrightarrow \langle \mu, \alpha \rangle \in \mathbb{Z}^+ - \{0\}, \forall \alpha \in \Delta_C$; $\langle \delta, \alpha \rangle = 1, \forall \alpha \in \Delta_C$. 所以 $\mu - \delta \in \bar{C} \Longrightarrow \mu \succeq \delta$.

- **def.:** the **convex hull** of vectors $v_1, \cdots, v_N$ is the set,

$$\mathrm{Conv}(v_1, \cdots, v_N) = \{c_1 v_1 + \cdots + c_N v_N | c_1 + \cdots + c_N = 1 \text{ and } c_i \in \mathrm{R}^+\} \tag{7.8.4}$$

两个例子如下图,



– K is a compact, convex subset of E, and $\lambda \in E - K$, then there is an element $\gamma \in E$ s.t.,

$$\langle \gamma, \lambda \rangle > \langle \gamma, \kappa \rangle, \forall \kappa \in K \tag{7.8.5}$$

proof:

由于 K 是紧致的, 存在 $\kappa_0 \in K$ 使得 $|\lambda - \kappa_0|$ 最小, 令 $\gamma = \lambda - \kappa_0$, 那么,

$$\langle \gamma, \lambda - \kappa_0 \rangle > 0 \Longrightarrow \langle \gamma, \lambda \rangle > \langle \gamma, \kappa_0 \rangle \tag{7.8.6}$$

对于 K 中的任意元素 κ, $\kappa(s) = s\kappa + (1-s)\kappa_0, s \in [0,1]$ 属于 K, 那么,

$$|\lambda - \kappa(s)|^2 \geq |\lambda - \kappa_0|^2 \Longrightarrow s^2|\kappa - \kappa_0|^2 - 2s\langle \lambda - \kappa_0, \kappa - \kappa_0 \rangle \geq 0 \tag{7.8.7}$$

考虑 $s \ll 1$ 的情况, 可见,

$$\langle \underbrace{\lambda - \kappa_0}_{=\gamma}, \kappa - \kappa_0 \rangle \leq 0 \Longrightarrow \langle \gamma, \lambda \rangle > \langle \gamma, \kappa_0 \rangle \geq \langle \gamma, \kappa \rangle \tag{7.8.8}$$

– μ, ν are dominant ($\in \bar{C}$) and $\nu \notin \mathrm{Conv}(W\,|\mu\rangle)$, then there exists a dominant element $\gamma \in \bar{C}$ s.t.,

$$\langle \gamma, \nu \rangle > \langle \gamma, w\mu \rangle, \forall w \in W \tag{7.8.9}$$

meaning that $\nu \not\preceq w\mu, \forall w \in W$.

proof:

根据上一个定理, 存在 $\gamma' \in E$ 使得 $\langle \gamma', \nu \rangle > \langle \gamma', \kappa \rangle, \forall \kappa \in \mathrm{Conv}(W\,|\mu\rangle)$, 特别地, $\langle \gamma', \nu \rangle > \langle \gamma', w\mu \rangle, \forall w \in W$.
考虑 $\{\gamma\} = W\,|\gamma'\rangle \cap \bar{C}$, 这个 $\gamma = w_0\gamma'$ 是唯一的, 且 $\gamma \succeq \gamma'$. 所以,

$$\gamma - \gamma' \in \bar{C} \Longrightarrow \langle \gamma - \gamma', \nu \rangle \geq 0 \Longrightarrow \langle \gamma, \nu \rangle > \langle w_0\gamma, w\mu \rangle, \forall w \in W \Longrightarrow \cdots \tag{7.8.10}$$

($\gamma - \gamma'$ 与 positive simple root 的内积为正, 且 ν 可以展开成 positive simple root 的正系数叠加)

- 两个定理:
  – if μ, ν are dominant, then $\nu \in \mathrm{Conv}(W\,|\mu\rangle) \iff \nu \preceq \mu$.
  – μ is dominant and $\nu \in E$, then $\nu \in \mathrm{Conv}(W\,|\mu\rangle) \iff w\,|\nu\rangle \preceq \mu, \forall w \in W$.

proof:

上一个定理已经证明了 $\Longleftarrow$, 我们现在来证明 $\Longrightarrow$. μ 是 dominant, 那么 $w\mu \preceq \mu, \forall w \in W$, 所以,

$$\left(\sum_{i=1}^{|W|} c_i w_i\,|\mu\rangle \right) - \mu = \sum_{i=1}^{|W|} c_i \underbrace{(w_i\,|\mu\rangle - \mu)}_{\preceq 0} \preceq 0 \tag{7.8.11}$$

所以 $\mathrm{Conv}(W\,|\mu\rangle) \preceq \mu$.

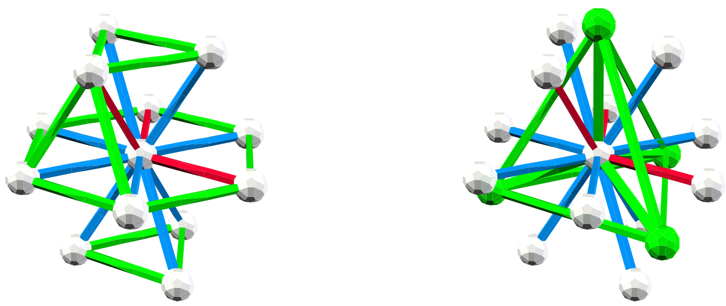
首先, 显然有 $\nu \in \mathrm{Conv}(W\,|\mu\rangle) \iff w\,|\nu\rangle \in \mathrm{Conv}(W\,|\mu\rangle), \forall w \in W$. 那么考虑 $\nu' = w_0\nu \in \bar{C}$, 有,

$$\nu \in \mathrm{Conv}(W\,|\mu\rangle) \iff \nu' \in \mathrm{Conv}(W\,|\mu\rangle) \iff \nu' \preceq \mu \tag{7.8.12}$$

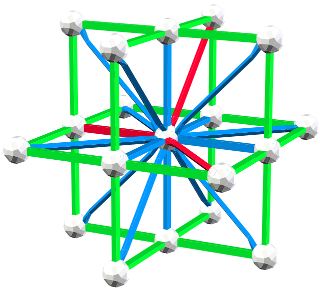
而 $w\,|\nu\rangle \preceq \nu' \preceq \mu, \forall w \in W$, 得证.

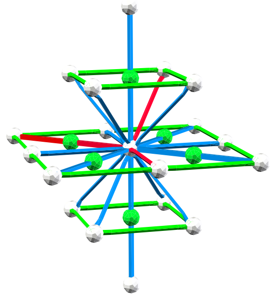
7.9 rank-three systems

- 本 section 只考虑 irreducible rank-three systems, 总共有三种, 分别是 A_3, B_3, C_3, 它们分别来自 $\mathfrak{sl}(4, \mathbb{C})$, $\mathfrak{so}(7, \mathbb{C})$ 和 $\mathfrak{sp}(3, \mathbb{C})$.
- A_3 root system 见下图, 其中, base 由红色向量组成, Weyl 群是右图中绿色正四面体的对称群,

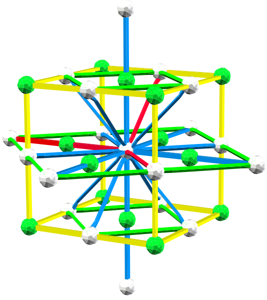


- B_3, C_3 root systems 分别见下图,





它们的 Weyl 群显然相同, 是下图中黄色立方体的对称群,

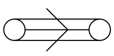


7.10 the classical root systems

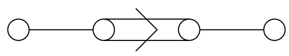
- 见 section 6.7.

7.11 the classification

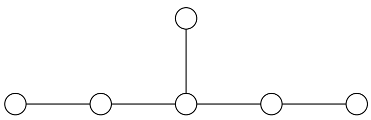
- every irreducible root system is either the root system of a classical Lie algebra(types $A_n, B_n, C_n, n \geq 1$ and $D_n, n \geq 3$, with $B_2 \simeq C_2, A_3 \simeq D_3$) or one of five **exceptional root systems**.
- the **exceptional root systems** are G_2, F_4, E_6, E_7, E_8, 它们的 Dynkin 图如下,



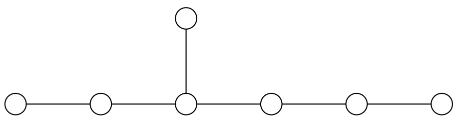
G_2



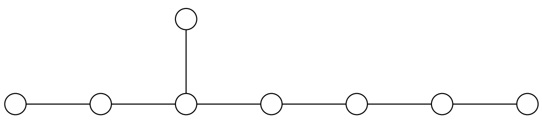
F_4



E_6



E_7



E_8

- 三个有用的定理:
  - $\mathfrak{h}_1, \mathfrak{h}_2$ are Cartan subalgebras of the semisimple Lie algebra $\mathfrak{g}$, then there exists a automorphism (自同构) $\phi : \mathfrak{g} \to \mathfrak{g}$ s.t. $\phi[\mathfrak{h}_1] = \mathfrak{h}_2$. (见 section 6.2 末尾)
  - the root systems associated to $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are isomorphic $\Longrightarrow \mathfrak{g}_1, \mathfrak{g}_2$ are isomorphic.
  - for every root system R, there exists a root system associated to $(\mathfrak{g}, \mathfrak{h})$ isomorphic to R.

因此, 所有 simple Lie algebra 都与下表中的某个 classical Lie algebra,

$\mathfrak{sl}(n+1,\mathbb{C}) \mapsto A_n$	$\mathfrak{so}(2n+1,\mathbb{C}) \mapsto B_n$	$\mathfrak{sp}(2n,\mathbb{C}) \mapsto C_n$	$\mathfrak{so}(2n,\mathbb{C}) \mapsto D_n$
$n \geq 1$	$n \geq 2$	$n \geq 3$	$n \geq 4$
$n = 1$	$B_1 \simeq A_1$	$C_1 \simeq A_1$	$n \neq 1$
$n = 2$		$C_2 \simeq B_2$	$n \neq 2$
$n = 3$			$D_3 \simeq A_3$

或 $G_2, F_4, E_{6,7,8}$ 中的某个 exceptional Lie algebra 相 isomorphic.

Chapter 8

representations of semisimple Lie algebras

8.1 weights of representations

- **def.:** (π, V) is a (possibly infinite dimensional) rep. of semisimple Lie algebra $\mathfrak{g}$, then $\lambda \in \mathfrak{h}$ is the **weight** of π if there **exists** a $v \neq 0 \in V$ s.t.,

$$\pi(H)v = \langle \lambda, H \rangle v, \forall H \in \mathfrak{h} \iff \det(\pi(H) - \langle \lambda, H \rangle I) = 0, \forall H \in \mathfrak{h} \tag{8.1.1}$$

the **weight space** of λ (denoted by V_λ) is the set of all $v \in V$ satisfying (8.1.1), and the dimension of the weight space is called the (geometric) **multiplicity**. (more about weights, see appendix A.3)

- (π, V) is finite-dimensional $\Longrightarrow$ every weight of π is an **integral element**.

proof:

$\pi\big|_{\mathfrak{s}^\alpha}$ 可以视为 $\mathfrak{s}^\alpha = \mathrm{span}(H_\alpha, A_\alpha, B_\alpha) \simeq \mathfrak{su}(2)_{\mathbb{C}}$ 的表示, 那么根据 (9.1.6), $\pi(H_\alpha) \equiv \pi(2J_3)$ 的 eigenvalue 是整数, 所以,

$$\langle \lambda, H_\alpha \rangle \in \mathbb{Z} \tag{8.1.2}$$

- for finite-dimensional rep., for a weight λ of π, $\boldsymbol{w} \left|\boldsymbol{\lambda}\right\rangle, \forall w \in W$ is still a weight and $V_{w|\lambda\rangle} \simeq V_\lambda$.

proof:

注意, 令 $S_\alpha = e^{A_\alpha} e^{-B_\alpha} e^{A_\alpha}$, 那么,

$$\mathrm{Ad}_{S_\alpha} H_\alpha = -H_\alpha \Longrightarrow \mathrm{Ad}_{S_\alpha} = s_\alpha \tag{8.1.3}$$

证明见 (9.1.7). 所以, 考虑 $s_\alpha \left|\lambda\right\rangle$ (注意到 $s_\alpha^{-1} = s_\alpha$),

$$\begin{aligned} &\begin{cases} \pi(s_\alpha^{-1} H)v = \langle \lambda, s_\alpha^{-1} H \rangle v \quad \forall v \in V_\lambda \\ \pi(s_\alpha^{-1} H) = \pi(\mathrm{Ad}_{S_\alpha} H) = \Pi(S_\alpha)\pi(H)\Pi^{-1}(S_\alpha) \end{cases} \\ \Longrightarrow & \pi(H)(\Pi^{-1}(S_\alpha)v) = \langle s_\alpha \lambda, H \rangle (\Pi^{-1}(S_\alpha)v) \\ \Longrightarrow & \Pi^{-1}(S_\alpha)[V_\lambda] = V_{s_\alpha|\lambda\rangle} \end{aligned} \tag{8.1.4}$$

($\Pi(S_\alpha)$ 一定是可逆矩阵, 否则不存在逆元, Π 就根本不是一个表示)

- 考虑半单李代数的正根为 $R^+ = \{\alpha_1, \cdots, \alpha_N\}$, 李代数的基底是 $\Delta \cup \{A_1, \cdots, A_N\} \cup \{B_1, \cdots, B_N\}$, 其中 $\Delta = \{\alpha_1, \cdots, \alpha_r\}$, 且 $A_i \in \mathfrak{g}_{\alpha_i}, B_i \in \mathfrak{g}_{-\alpha_i}$.
  - 那么, $\forall \alpha \in R$,

$$\begin{cases} \pi(H)\pi(A_\alpha)v = \langle \lambda + \alpha, H \rangle \pi(A_\alpha)v \\ \pi(H)\pi(B_\alpha)v = \langle \lambda - \alpha, H \rangle \pi(B_\alpha)v \end{cases} \Longrightarrow \begin{cases} \pi(A_\alpha)[V_\lambda] \subseteq V_{\lambda+\alpha} \\ \pi(B_\alpha)[V_\lambda] \subseteq V_{\lambda-\alpha} \end{cases} \tag{8.1.5}$$

– 对于所有的不可约表示, $\pi(H), \forall H \in \mathfrak{h}$ 都可以被对角化, 因此也可以被同时对角化.

proof:

U 是 V 的子空间, 由 $\mathfrak{h}$ 的 simultaneous eigenvectors 构成, 根据 (8.1.5), $\pi(A_\alpha)[U] \subseteq U$, 所以 U 是不变子空间 (且不为零, 因为 $\mathfrak{h}$ 是 Abelian, 至少存在一个权, 见 appendix A.3). 又因为 (π, V) 不可约, 所以 $V = U = \bigoplus_\lambda V_\lambda$.

- 三个关于 **highest weight** 的定理:
  - every irreducible, finite-dim. rep. of $\mathfrak{g}$ has a highest weight. (最高权存在)
  - two irreducible, finite-dim. rep. with the same highest weight are isomorphic. (一一对应)
  - the **highest weight** $\boldsymbol{\mu}$ of a irreducible, finite-dim. rep. is a **dominant integral element**.

proof:

reordering lemma: 考虑李代数 $\mathfrak{g}$ 及其表示 π, $\{A_1, \cdots, A_n\}$ 是李代数的一组基底, 那么下式,

$$\pi(A_{i_1}) \cdots \pi(A_{i_N}) \tag{8.1.6}$$

可以表示成,

$$\pi(A_n)^{j_n} \cdots \pi(A_1)^{j_1} \tag{8.1.7}$$

的线性组合, 其中 $j_1 + \cdots + j_n \leq N$.

---

proof:
用数学归纳法证明, $N = 1$ 时显然成立, 假设 $N - 1$ 时成立, 那么 N 时,

$$\pi(A_{i_1}) \cdots \pi(A_{i_N}) = \pi(A_{i_1}) \Big(\sum_{j_1 + \cdots + j_N \leq N-1} C_{j_1, \cdots, j_N} \pi(A_n)^{j_n} \cdots \pi(A_1)^{j_1} \Big) \tag{8.1.8}$$

用对易关系改变 $\pi(A_{i_1})$ 的位置,

$$\pi(A_{i_1})\pi(A_k) = \pi(A_k)\pi(A_{i_1}) + \underbrace{\pi([A_{i_1}, A_k])}_{= \sum_l -f_{i_1 k}{}^l A_l} \tag{8.1.9}$$

右边的一项最多含 $N - 1$ 个基矢, 所以命题得证.

- 令 (dominant) integral element μ 为 (π, V) 的 **highest weight**, 那么 (根据 (8.1.5)) 一定有 $\pi(A_{\alpha_i})[V_\mu] = \{0\}, \forall \alpha_i \in R^+$.
- 选取 $\{B_1, \cdots, B_N\} \cup \Delta \cup \{A_1, \cdots, A_N\}$ 为 $\mathfrak{g}$ 的基底 (其中 N 是正根的个数), 那么考虑 some $v \in V_\mu$,

  $$\pi(B_{i_1}) \cdots \pi(B_{i_M}) v = \text{linear combination of } \pi(B_N)^{j_N} \cdots \pi(B_1)^{j_1} v \tag{8.1.10}$$

  (注意到 v 是 $\pi(H_i)$ 的本征向量, 而 $\pi(A_i)v = 0$)
  另外, 一定有 $\mu - j_1\alpha_1 - \cdots - j_N\alpha_N \in \text{Conv}(W\,|\mu\rangle)$, 否则 $\pi(B_N)^{j_N} \cdots \pi(B_1)^{j_1} v = 0$.
- 考虑,

  $$\text{linear combinations of } \pi(B_{i_1}) \cdots \pi(B_{i_M}) v \text{ with } M \geq 0, \text{ for some } v \in V_\mu \tag{8.1.11}$$

  这是 V 的不变子空间, 考虑到 irreducibility, (8.1.11) 等于 V. 同时也证明了 $\dim V_\mu = 1$, 且 μ 是唯一的最高权, 因此它一定是 dominant.

- if μ is a **dominant integral element**, there exists an irreducible, finite-dim. rep. of $\mathfrak{g}$ with **highest weight** μ.
  本 chapter 的剩余部分将用来证明这个定理.

8.2 the highest weight cyclic representations & an introduction to Verma modules

- **def.:** for a (maybe infinite-dim.) rep. (π, V) of $\mathfrak{g}$ with highest weight $\mu \in \mathfrak{h}$ (不一定是 integral), if there **exists** $v \neq 0 \in V$ s.t.,

  1. $\pi(H)v = \langle \mu, H \rangle v, \forall H \in \mathfrak{h}$ (simultaneously diagonalizable, 见 appendix A.3.2),
  2. $\pi(A)v = 0, \forall A \in \mathfrak{g}_\alpha$, with $\alpha \in R^+$,
  3. the smallest invariant subspace (见 section 5.2 第三点, $\pi(A)[W] \subseteq W, \forall A \in \mathfrak{g}$) containing v is V,

  then it is said to be **highest weight cyclic**.

  – **有限维**情况下, highest weight cyclic rep. 是 irreducible, 且最高权相同 μ 的... 互相 isomorphic.

- 下面初步介绍构造 Verma module (π_μ, V^μ) 的思路 (V^μ 选择上标, 以区分 weight space V_μ).
- 依旧是选取,

$$\{B_1, \cdots, B_N\} \cup \Delta \cup \{A_1, \cdots, A_N\} \quad \text{with} \quad \begin{cases} R^+ = \{\underbrace{\alpha_1, \cdots, \alpha_r}_{=\Delta}, \alpha_{r+1}, \cdots, \alpha_N\} \\ A_i \in \mathfrak{g}_{\alpha_i} \quad i = 1, \cdots, N \\ B_i \in \mathfrak{g}_{-\alpha_i} \quad i = 1, \cdots, N \end{cases} \tag{8.2.1}$$

  作为 $\mathfrak{g}$ 的基底.

- 由于对于 (π_μ, V^μ), μ 是最高权, 所以一定存在,

$$v_0 \in V^\mu, \text{ s.t. } \pi_\mu(A)v_0 = 0, \forall A \in \mathfrak{g}_\alpha, \text{ with } \alpha \in R^+ \tag{8.2.2}$$

- 根据 (8.1.11), 考虑具有以下形式的向量,

$$\pi_\mu(B_1)^{n_1} \cdots \pi_\mu(B_N)^{n_N} v_0 \in V_{\mu - \sum_{i=1}^N n_i \alpha_i} \subset V^\mu, \text{ with } n_i \in \mathbb{Z}^+ \tag{8.2.3}$$

  它们的线性组合张成 V^μ.

  – Verma module 中的 weights 仅具有如下形式,

$$\mu - \sum_{i=1}^N n_i \alpha_i \tag{8.2.4}$$

  其中 n_i 是非负整数.

  – 这样定义后, 我们就能 (通过对易关系) 计算 $\mathfrak{g}$ 中每个元素的表示如何作用于任何一个 V^μ 中的向量.

8.3 universal enveloping algebras

- **def.: 李代数 $\mathfrak{g}$ 嵌入的 associative algebra** (对 algebra 的一般定义见 appendix A 开头), $\mathcal{A}$, 是:

  – 存在乘法单位元 e, 且满足结合律 (unital, associative algebra).

  – $\mathfrak{g}$ 嵌入于 $\mathcal{A}$ ($\hat{\jmath} : \mathfrak{g} \to \mathcal{A}$).
  (例如: 对于矩阵李群 $G \subseteq \mathrm{GL}(n, \mathbb{C})$, 那么 $\mathfrak{g}$ 就是 $\mathcal{M}_n(\mathbb{C})$ 的子空间)

  – 李括号简化为,

$$\hat{\jmath}([A, B]) = \hat{\jmath}(A) \cdot \hat{\jmath}(B) - \hat{\jmath}(B) \cdot \hat{\jmath}(A) \tag{8.3.1}$$

  – $\mathcal{A}$ 由单位元 e 和如下元素张成,

$$\hat{\jmath}(A_1) \cdots \hat{\jmath}(A_k) \tag{8.3.2}$$

  其中 $k \geq 1$.

  另外, 对于 $\mathfrak{g}$ 一般来说 $\mathcal{A}$ 不唯一.

- **def.:** a pair $(U(\mathfrak{g}), \hat{i})$ (需要满足结合律) with the following properties is called a **universal enveloping algebra**,
  1. $\hat{i}([A,B]) = \hat{i}(A)\cdot\hat{i}(B) - \hat{i}(B)\cdot\hat{i}(A), \forall A, B$,
  2. the **smallest subalgebra** with **identity** $e \in U(\mathfrak{g})$ **containing** $\{\hat{i}(A), A \in \mathfrak{g}\}$ is $U(\mathfrak{g})$, (这个条件称为 $U(\mathfrak{g})$ 由 $\hat{i}(A), A \in \mathfrak{g}$ 生成)
  3. 考虑 $\mathfrak{g}$ 嵌入的某个 associative algebra $\mathcal{A}$ with identity, 那么 $U(\mathfrak{g})$ 和 $\mathcal{A}$ 之间存在 a **unique** algebra homomorphism $\phi : U(\mathfrak{g}) \to \mathcal{A}$, s.t.,
  $$\begin{cases} \phi(e) = e' \in \mathcal{A} \\ \phi \circ \hat{i} = \hat{j} : \mathfrak{g} \to \mathcal{A} \end{cases} \tag{8.3.3}$$
  即 $\mathcal{A} \simeq U(\mathfrak{g})/\ker(\phi)$, (只需要说明这个 $\ker(\phi)$ 是唯一的就行).
- $\mathfrak{g}$ 的任意两个 universal enveloping algebras 互相同构.
  (由于 $U(\mathfrak{g})$ 本身也是 associated algebra, 再利用性质 3)
- **theorem:** 任何李代数都存在一个 universal enveloping algebra.

proof:

- **def.:** the **tensor algebra** $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$, (notation $\mathfrak{g}^{\otimes k} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$).
  - $T(\mathfrak{g})$ 是对于 $B(\cdot,\cdot) = \otimes$ 满足**结合律**的代数.
  - 且存在**单位元** $1 \in \mathbb{C} \equiv \mathfrak{g}^{\otimes 0}$.

$(T(\mathfrak{g}), \otimes)$ 满足 $U(\mathfrak{g})$ 的第两个条件, 但是, 对于第三个条件, 考虑,

$$\begin{cases} \psi(1) = e \in \mathcal{A} \\ \psi : T(\mathfrak{g}) \to \mathcal{A},\ \ A \mapsto \hat{j}(A) \end{cases} \tag{8.3.4}$$

显然, 这样的 homomorphism ψ 不唯一, 实际上 $U(\mathfrak{g})$ 是 $T(\mathfrak{g})$ 的一个商空间 (见下文).

现在, 我们来构造 $U(\mathfrak{g})$. 考虑双向不变子空间 (two-sided ideal) J,

$$J = \left\{ \sum_i \alpha_i \otimes (A_i \otimes B_i - B_i \otimes A_i - [A_i, B_i]) \otimes \beta_i \middle| A_i, B_i \in \mathfrak{g}, \alpha_i, \beta_i \in T(\mathfrak{g}) \right\} \tag{8.3.5}$$

那么 $U(\mathfrak{g}) = T(\mathfrak{g})/J$.

- 注意, J 是一个 **two-sided ideal**, 即 $\forall \alpha \in T(\mathfrak{g}), \beta \in J$, 有 $\alpha \otimes \beta, \beta \otimes \alpha \in J$.
- 且 J 是包含形如 $A \otimes B - B \otimes A - [A,B]$ 的元素的最小的 two-sided ideal.
- 注意, the kernel of an algebra homomorphism is always a two-sided ideal. 考虑 $\phi : U \to \mathcal{A}$, 那么, $\forall \alpha \in \ker(\phi), \beta \in U$,
  $$\phi(\beta \cdot \alpha) = \phi(\beta)\cdot 0 = 0 \tag{8.3.6}$$

---

proof:

- 第一条 ($T(\mathfrak{g})$ 不满足, 但 $T(\mathfrak{g})/J$ 满足),
  $$[A,B] \sim A \otimes B - B \otimes A \tag{8.3.7}$$
- 第二条成立 ($T(\mathfrak{g})$ 和 $T(\mathfrak{g})/J$ 都满足).
- 第三条 ($T(\mathfrak{g})$ 和 $T(\mathfrak{g})/J$ 都满足), 考虑 algebra homomorphism $\psi : T(\mathfrak{g}) \to \mathcal{A}$ s.t.,
  $$\begin{cases} \psi(1) = e \in \mathcal{A} \\ \psi(A_1 \otimes \cdots \otimes A_k) = \hat{j}(A_1)\cdots\hat{j}(A_k) \end{cases} \tag{8.3.8}$$
  那么, (考虑到 kernel 一定是 two-sided ideal), 必然有 $J \subset \ker(\psi)$.
  (令 $\phi = \psi\big|_{U(\mathfrak{g})}$, 有 $\ker(\psi) = J \oplus \ker(\phi)$, 即 $\mathcal{A} = T(\mathfrak{g})/\ker(\psi) = U(\mathfrak{g})/\ker(\phi)$.)

注意, $\mathcal{A}$ 由 e 和 (8.3.2) 中的元素张成, ψ 必须满足 $\psi(1)=e$, 考虑第二个条件 $\phi\circ\hat{\imath}=\hat{\jmath}$, 考虑 $\forall A\in\mathfrak{g}$,

$$\phi(A)=\hat{\jmath}(A) \tag{8.3.9}$$

且 $U(\mathfrak{g})$ 由 $A_1\oplus\cdots\oplus A_k, k\geq 0$ 张成, 所以 ϕ 的选取是唯一的.

- (π,V) 是李代数 $\mathfrak{g}$ 的一个表示 (不一定是有限维), 那么存在一个 unique algebra homomorphism,

$$\tilde{\pi}:U(\mathfrak{g})\to\mathrm{End}(V)\quad \text{s.t.}\quad \begin{cases}\tilde{\pi}(1)=I\\ \tilde{\pi}(A)=\pi(A),\forall A\in\mathfrak{g}\subset U(\mathfrak{g})\end{cases} \tag{8.3.10}$$

proof:

可以认为 $\mathcal{A}=\mathrm{End}(V),\hat{\jmath}=\pi$, 那么, 存在 unique $\tilde{\pi}=\phi:U(\mathfrak{g})\to\mathcal{A}$, ...

8.4 Poincaré-Birkhoff-Witt theorem

- **PBW theorem:** 对于有限维李代数 $\mathfrak{g}$ (不一定半单), 其基矢为 $\{A_1,\cdots,A_k\}$, 那么,

$$\hat{\imath}(A_1)^{n_1}\cdots\hat{\imath}(A_k)^{n_k} \tag{8.4.1}$$

其中 n_i 是非负整数, 构成 $U(\mathfrak{g})$ 的基矢 (张成并线性独立).

  – 同时意味着 $\hat{\imath}:\mathfrak{g}\to U(\mathfrak{g})$ 是 injective (one-to-one).

proof:

证明方法类似于 reordering lemma (见 (8.1.7)).

首先 (8.4.1) 中的向量显然能张成 $U(\mathfrak{g})$, 我需要证明它们线性独立, 方法如下:
考虑一个向量空间 D, 其基底为 $\{v_{i_1,\cdots,i_N}\}$, 其中 $1\leq i_1\leq\cdots\leq i_N\leq k$. 我们的目标是证明存在一个线性映射 $\gamma:U(\mathfrak{g})\to D$, (这个映射不必是同构), 使得,

$$\hat{\imath}(A_{i_1})\cdots\hat{\imath}(A_{i_N})\mapsto v_{i_1,\cdots,i_N} \tag{8.4.2}$$

为此, 我们希望能构造一个线性映射 $\delta:T(\mathfrak{g})\to D$, s.t.,

1. $\delta(A_{i_1}\otimes\cdots\otimes A_{i_N})=v_{i_1,\cdots,i_N}$ if $1\leq i_1\leq\cdots\leq i_N\leq k$,
2. $\delta[J]=\{0\}$, 因此 δ 自然能给出线性映射 $\gamma:U(\mathfrak{g})\to D$.

构造方法如下.

考虑 n 阶单项式 $A_{j_1}\otimes\cdots\otimes A_{j_n}$, 令逆序的下标对数为其 index, (显然 $0,1$ 阶的单项式的 index 都是零), $n\leq k$ 阶单项式的 index 最高为 $\frac{n(n-1)}{2}$. 下面用归纳法来确定 δ.

– 假设 δ 的定义 (已经在 index 小于等于 p, 或者阶数小于等于 $n-1$ 下做出了定义) 使得, 下式在: 等号左边两相的 index 都不超过 $p\geq 1$ 时, 且 $n\leq N$ 时, 成立,

$$\delta(A_{i_1}\cdots(A_{i_j}A_{i_{j+1}}-A_{i_{j+1}}A_{i_j})\cdots A_{i_n})=\delta(A_{i_1}\cdots[A_{i_j},A_{i_{j+1}}]\cdots A_{i_n}) \tag{8.4.3}$$

($p=0$ 一定成立, 因为 $i_j=i_{j+1}$, 等号两边为零)

– 考虑等号左侧第一项的 index 为 $p+1$, 且 $i_j>i_{j+1}$ 是逆序, 那么, 定义 δ 在 (8.4.3) 下依然成立. 这样我们就把 δ 的定义拓展到了 n 阶, index 为 $p+1$ 的情况,

$$\delta(A_{i_1}\cdots\underbrace{A_{i_j}A_{i_{j+1}}}_{\text{逆序}}\cdots A_{i_n})=\delta(A_{i_1}\cdots A_{i_{j+1}}A_{i_j}\cdots A_n)+\delta(\cdots[A_{i_j},A_{i_{j+1}}]\cdots) \tag{8.4.4}$$

– 由于 (8.4.4) 左侧至少有两处逆序 (假设另一个逆序对为 $i_l>i_{l+1}$ 且 $j<l$), 那么还需要证明等式右侧与逆序对的选取无关, 我们通过分类讨论证明这一点.

分类讨论:

- 如果 $j+1 \le l-1$.
  考虑,

$$\begin{aligned}
&\delta(\cdots A_{i_j} A_{i_{j+1}} \cdots A_{i_l} A_{i_{l+1}} \cdots) \\
=&\delta(\cdots A_{i_j} A_{i_{j+1}} \cdots A_{i_{l+1}} A_{i_l} \cdots) + \delta(\cdots A_{i_j} A_{i_{j+1}} \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) \\
=&\delta(\cdots A_{i_{j+1}} A_{i_j} \cdots A_{i_{l+1}} A_{i_l} \cdots) + \delta(\cdots [A_{i_j}, A_{i_{j+1}}] \cdots A_{i_{l+1}} A_{i_l} \cdots) \\
&+ \delta(\cdots A_{i_{j+1}} A_{i_j} \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) + \delta(\cdots [A_{i_j}, A_{i_{j+1}}] \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) \\
=&\cdots
\end{aligned} \tag{8.4.5}$$

  最后一个等号右侧的第一, 三项和第二, 四项结合, 就得到 (8.4.4) 右侧.
  (要注意, 证明过程中每一个单项式的 index 都小于等于 p, 或者阶数小于等于 $n-1$)

- 如果 $j+1 = l$.
  为了简洁, 用 $A = A_{i_j}, B = A_{i_{j+1=l}}, C = A_{i_{l+1}}$, 那么,

$$\begin{aligned}
&\delta(\cdots BAC \cdots) + \delta(\cdots [A,B]C \cdots) \\
=&\delta(\cdots CBA \cdots) + \delta(\cdots [B,C]A \cdots) + \delta(\cdots B[A,C] \cdots) + \delta(\cdots [A,B]C \cdots)
\end{aligned} \tag{8.4.6}$$

  同时,

$$\begin{aligned}
&\delta(\cdots ACB \cdots) + \delta(\cdots A[B,C] \cdots) \\
=&\delta(\cdots CBA \cdots) + \delta(\cdots [A,C]B \cdots) + \delta(\cdots C[A,B] \cdots) + \delta(\cdots A[B,C] \cdots)
\end{aligned} \tag{8.4.7}$$

  那么, 只需要证明,

$$[[B,C],A] + \underbrace{[B,[A,C]]}_{=[[C,A],B]} + [[A,B],C] = 0 \tag{8.4.8}$$

  而这就是 Jacobi identity.

8.5 construction of Verma modules

- **def.:** a **left ideal** of $U(\mathfrak{g})$ generated by $\{\alpha_i\}$ is,

$$I = \left\{ \sum_i \beta_i \alpha_i \middle| \forall \beta_i \in U(\mathfrak{g}) \right\} \tag{8.5.1}$$

- 用 I_μ 表示一个 left ideal generated by,

$$\{H - \langle \mu, H \rangle, \forall H \in \mathfrak{h}\} \cup \bigcup_{\alpha \in R^+} \mathfrak{g}_\alpha \tag{8.5.2}$$

  (第一个集合中的元素是一个一阶向量减一个零阶向量)

- **def.:** the **Verma module** with highest weight μ is,

$$W_\mu = U(\mathfrak{g})/I_\mu \tag{8.5.3}$$

  用 $[\alpha]$ 表示 $\alpha \in U(\mathfrak{g})$ 在 W_μ 中的像 (等价类).

  - (π_μ, W_μ) 是 universal enveloping algebra 的一个表示,

$$\pi_\mu(\alpha)[\beta] = [\alpha\beta] \tag{8.5.4}$$

proof:

$$\pi_\mu(\alpha_1)\pi_\mu(\alpha_2)[\beta] = [\alpha_1\alpha_2\beta] = \pi_\mu(\alpha_1\alpha_2)[\beta] \tag{8.5.5}$$

且如果 $\beta \sim \beta'$, 那么 $\alpha\beta \sim \alpha\beta'$.

- 所以, (其中 $A \in \mathfrak{g}_{\alpha\in R^+}$),

$$\begin{cases} \pi_\mu(H)[1] = \langle \mu, H\rangle [1] \\ \pi_\mu(A)[1] = 0 \end{cases} \tag{8.5.6}$$

但要注意, 一般 $[A\alpha] \neq 0$, 所以 $\pi_\mu(A) \neq [A] = 0$, (不过 $[\alpha A] = 0$).

- $\mathfrak{n}^\pm = \bigoplus_{\alpha\in R^\pm} \mathfrak{g}_\alpha$, 由于 $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, 所以 $\mathfrak{n}^+, \mathfrak{n}^-$ 都是 $\mathfrak{g}$ 的子代数.

- **theorem:**
  - (π_μ, W_μ) 是一个 highest weight cyclic rep. (定义见 section 8.2 开头), 且最高权为 μ (不过, 由于 W_μ 一定是无限维, 最高权不一定是 dominant), 最高权向量为 $v_0 = [1]$.
  - $\{B_1, \cdots, B_k\}$ 是 $\mathfrak{n}^-$ 的一组基底, 那么,

$$\pi_\mu(B_1)^{n_1} \cdots \pi_\mu(B_k)^{n_k} v_0 \tag{8.5.7}$$

(其中 $n_i \in \mathbb{Z}^+$), 组成 W_μ 的一组基底.

结合 PBW theorem, 可见有向量空间同构 $W_\mu \simeq U(\mathfrak{n}^-)$, 且 $\alpha \mapsto \pi_\mu(\alpha)v_0$.

proof:

lemma: 令 J_μ 是 $U(\mathfrak{n}^+ \oplus \mathfrak{h})$ 上的, 由 (8.5.2) 中的元素生成的 left ideal, 那么 $v_0 = [1] \notin J_\mu$.

---

proof:
考虑一维表示,

$$\sigma_\mu : \mathfrak{n}^+ \oplus \mathfrak{h} \to \underbrace{\mathrm{End}(\mathbb{C})}_{=\mathbb{C}} \quad \text{s.t.} \quad \begin{cases} \sigma_\mu(A) = 0 & A \in \mathfrak{n}^+ \\ \sigma_\mu(H) = \langle \mu, H\rangle & H \in \mathfrak{h} \end{cases} \tag{8.5.8}$$

对比 (8.3.10), 可知存在一个唯一的 $\tilde{\sigma}_\mu : U(\mathfrak{n}^+ \oplus \mathfrak{h}) \to \mathbb{C}$, s.t.,

$$\begin{cases} \tilde{\sigma}_\mu(1) = 1 \\ \tilde{\sigma}_\mu(A + H) = \langle \mu, H\rangle \end{cases} \quad \text{and} \quad \ker(\tilde{\sigma}_\mu) \supset \{0\} \cup \mathfrak{n}^+ \cup \{H \perp \mu\} \cup \{H - \langle \mu, H\rangle\} \tag{8.5.9}$$

且 $\ker(\tilde{\sigma}_\mu)$ 是 $U(\mathfrak{n}^+ \oplus \mathfrak{h})$ 上的一个 two-sided ideal, 所以 $J_\mu \subset \ker(\tilde{\sigma}_\mu)$, 所以...

含有 v_0 的不变子空间 $U = W_\mu$, 因为 $\pi_\mu(\alpha)v_0 = [\alpha]$, 那么证明第一个 theorem 只需要再说明 $[1] \neq [0]$, (highest weight cyclic rep. 的前两个性质见 (8.5.6)).

要说明 $[1] \neq [0]$, 只需要证明 $1 \notin I_\mu$.
考虑 I_μ 中的元素按照 PBW theorem 展开,

$$\begin{aligned} I_\mu \ni \alpha &= \sum \overbrace{\beta_1}^{\in U(\mathfrak{g})} (H - \langle \mu, H\rangle) + \overbrace{\beta_2}^{\in U(\mathfrak{g})} A_\alpha \\ &= \sum (B_{\alpha_1})^{n_1} \cdots (B_{\alpha_N})^{n_N} \underbrace{\gamma_{n_1,\cdots,n_N}}_{\in U(\mathfrak{n}^+)} (H - \langle \mu, H\rangle) + \cdots \\ &= \sum (B_{\alpha_1})^{n_1} \cdots (B_{\alpha_N})^{n_N} \underbrace{\delta_{n_1,\cdots,n_N}}_{\in J_\mu} \end{aligned} \tag{8.5.10}$$

如果 $\alpha = 1 \in I_\mu$, 那么 $n_1 = \cdots = n_N = 0$, 且 $\alpha = 1 = \delta_{0,\cdots,0} \in J_\mu$, 与引理的结论矛盾, 所以 $1 \notin I_\mu$.

现在来证明第二个 theorem. 已经说明了 W_μ 是含 v_0 的最小的不变子空间, 所以 (8.5.7) 中的向量一定张成 W_μ, 我们还需要证明它们线性独立.

考虑, 如果它们线性相关,

$$\sum \overbrace{C_{n_1,\cdots,n_k}}^{\in\mathbb{C}}[(B_1)^{n_1}\cdots(B_k)^{n_k}]=0$$
$$\Longrightarrow \alpha=\sum C_{n_1,\cdots,n_k}(B_1)^{n_1}\cdots(B_k)^{n_k}\in I_\mu \tag{8.5.11}$$

但是, 对照 (8.5.10) (注意, 利用 PBW theorem 得到的展开式是唯一的), 可见 $C_{n_1,\cdots,n_k}\in J_\mu$, 而这不成立.

8.6 irreducible quotient modules

- 本节我们将证明 Verma module W_μ 有一个 largest nonzero invariant subspace U_μ, 而商空间 $V^\mu=W_\mu/U_\mu$ 是最高权为 μ 的不可约表示. 且如果 μ 是 dominant integral, 那么 V^μ 是有限维.

- **def.:** U_μ 由如下向量 $v\in W_\mu$ 组成 (注意, (8.5.7) 是 W_μ 的一组基底):
  1. v 的 $v_0=[1]$ 分量为零,
     - 注意, 并不是所有由低于 μ 的权对应的权向量组成的矢量都属于 U_μ, 例如 $[B_\alpha]\notin U_\mu,\alpha\in R^+$, 因为 $\pi_\mu(A_\alpha)[B_\alpha]=\langle\mu,H_\alpha\rangle\, v_0$, 见第二个条件.
  2. $\pi_\mu(A_1)\cdots\pi(A_k)v, k\geq 1$ 的 v_0 分量也为零, 其中 $A_1,\cdots,A_k\in\mathfrak{n}^+$,

  也就是所有通过升算符无法达到 v_0 的向量.

- U_μ 是一个不变子空间.

proof:

- 首先 $\pi_\mu(A)[U_\mu]\subseteq U_\mu,\forall A\in\mathfrak{n}^+$.
- $\pi_\mu(A_1)\cdots\pi(A_k)v, k\geq 0$ 是由低于 μ 的权对应的权向量组成, 考虑,

$$\pi_\mu(A_1)\cdots\pi(A_k)\pi_\mu(C)v \tag{8.6.1}$$

其中 $C\in\mathfrak{h}\oplus\mathfrak{n}^-$, reordering lemma 告诉我们 (8.6.1) 等于下列形式的向量的线性组合,

$$\pi_\mu^{n_1}(B_1)\cdots\pi_\mu^{n_N}(B_N)\pi_\mu^{n_1'}(H_1)\cdots\pi_\mu^{n_r'}(H_r)\pi_\mu^{n_1''}(A_1)\cdots\pi_\mu^{n_N''}(A_N)v \tag{8.6.2}$$

只能让这些权向量对应的权保持不变或降低, 所以...

- 商空间 $V^\mu=W_\mu/U_\mu$ 构成 $\mathfrak{g}$ 的一个**不可约**表示.

proof:

8.7 Casimir operators

- **def.:** the 2nd-order Casimir operator is,

$$C_2=-B^{ij}A_i\otimes A_j \tag{8.7.1}$$

where $B^{ij}=B^{-1}_{ij}$.

  - the 2nd-order Casimir operator commutes with all the generators.

proof:

$$
\begin{aligned}
[C_2, A_k] &= -B^{ij}[A_i A_j, A_k] \\
&= -B^{ij}(-f_{jk}{}^{l} A_i A_l - f_{ik}{}^{l} A_l A_j)
\end{aligned} \tag{8.7.2}
$$

notice that B^{ij} is symmetric, so,

$$
\begin{aligned}
[C_2, A_k] &= -B^{ij}(-f_{ik}{}^{l} A_j A_l - f_{ik}{}^{l} A_l A_j) \\
&= B^{ij} f_{ik}{}^{l}(A_j A_l + A_l A_j) \\
&= \underbrace{B^{ij} B^{lm}(A_j A_l + A_l A_j)}_{\text{symmetric about } (i,m)} f_{ikm} = 0
\end{aligned} \tag{8.7.3}
$$

Chapter 9

$\mathfrak{su}(2)_{\mathbb{C}}$ algebra

- $\mathfrak{su}(2) = \{A \in \mathcal{M}_2(\mathbb{C}) | A^\dagger = -A \text{ and } \mathrm{tr}A = 0\}$.
  - $\dim \mathfrak{su}(2) = 2^2 - 1 = 3$.
  - $\mathfrak{su}(2) = \mathrm{span}\{iJ_1, iJ_2, iJ_3\}$ is a real vector space.
- its structure is,

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{9.0.1}$$

  where $i, j, k = 1, 2, 3$.

- ladder operators,

$$\begin{cases} J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \in \mathfrak{su}(2)_{\mathbb{C}} \\ [J_3, J_\pm] = \pm J_\pm \\ [J_+, J_-] = J_3 \\ J^2 = J_+J_- + J_-J_+ + J_3^2 \end{cases} \tag{9.0.2}$$

- another basis is $H = 2J_3, A = \sqrt{2}J_+, B = \sqrt{2}J_-$, and,

$$\begin{cases} [H, A] = 2A \\ [H, B] = -2B \\ [A, B] = H \end{cases} \quad \mathrm{ad}_H = \begin{pmatrix} 0 & & \\ & 2 & \\ & & -2 \end{pmatrix} \quad \mathrm{ad}_A = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathrm{ad}_B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \tag{9.0.3}$$

  so, the Killing form is,

$$B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \tag{9.0.4}$$

- its Killing form is $B_{ij} = \epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$.
- its 2nd order Casimir operator is,

$$C_2 = -B^{ij} A_i A_j = \frac{1}{2}\delta_{ij} J_i J_j = \frac{1}{2} J^2 \tag{9.0.5}$$

9.1 representations of $\mathfrak{su}(2)_{\mathbb{C}}$ algebra

- for each (half-)integer j, there exits an $2j+1$ dimensional **irreducible** complex rep.,

$$\pi_j : \mathfrak{su}(2)_{\mathbb{C}} \to \mathrm{span}(|j, m\rangle, m = -j, \cdots, j) \tag{9.1.1}$$

  and any two irreducible rep. with the same dimension are isomorphic.

proof:

let π be an irreducible rep. of $\mathfrak{su}(2)_{\mathbb{C}}$ on a finite-dimensional complex vector space V, and $|u\rangle$ is a eigenvector of $\pi(J_3)$,

$$\begin{cases} \pi(J_3)\,|u\rangle = \alpha\,|u\rangle \\ \pi(J_3)\pi^k(J_\pm)\,|u\rangle = (\alpha \pm k)\pi^k(J_\pm)\,|u\rangle \end{cases} \tag{9.1.2}$$

since V is finite-dimensional, so there is some $N_\pm \geq 0$, s.t.,

$$\pi^{N_\pm}(J_\pm)\,|u\rangle \neq 0 \quad \text{but} \quad \pi^{N_\pm+1}(J_\pm)\,|u\rangle = 0 \tag{9.1.3}$$

let's set $|u_0\rangle = \pi^{N_-}(J_-)\,|u\rangle$ and $\lambda_0 = \alpha - N_-$, $|u_k\rangle = \pi^k(J_+)\,|u_0\rangle$, then,

$$\pi(J_3)\,|u_k\rangle = (\lambda_0 + k)\,|u_k\rangle\,, k = 0, \cdots, 2j \tag{9.1.4}$$

where $j = \frac{N_+ + N_-}{2}$, and,

$$\begin{aligned} \pi(J_-)\,|u_k\rangle &= -k(\lambda_0 + \frac{k-1}{2})\,|u_{k-1}\rangle \\ \xRightarrow{k-1=2j} 0 &= -(2j+1)(\lambda_0 + j)\,|u_{2j-1}\rangle \Longrightarrow \lambda_0 = -j \end{aligned} \tag{9.1.5}$$

so, for any **finite-dimensional** rep. of $\mathfrak{su}(2)_{\mathbb{C}}$, $\lambda_0 = -j$ must be a **(half-)integer**.

- according to appendix A.1, $|u_0\rangle, \cdots, |u_{2j}\rangle$ are **linearly independent**.
- $\text{span}(|u_0\rangle, \cdots, |u_{2j}\rangle)$ is **invariant** under $\pi(J_3), \pi(J_\pm)$, hence invariant under all $\pi(A), A \in \mathfrak{su}(2)_{\mathbb{C}}$.
- so every irreducible rep. is of the form as $\text{span}(|u_0\rangle, \cdots, |u_{2j}\rangle)$.

- for any finite-dim. (not necessarily irreducible) rep. (π, V) of $\mathfrak{su}(2)_{\mathbb{C}}$,
  1. all eigenvalues of $\pi(J_3)$ are (half-)integer,
  $$-j, -j+1, \cdots, j \tag{9.1.6}$$
  2. $\pi(J_\pm)$ are nilpotent,
  3. let $S = e^A e^{-B} e^A \Longrightarrow \Pi(S) = e^{\pi(A)} e^{-\pi(B)} e^{\pi(A)}$, then,
  $$\text{Ad}_S H = -H \Longrightarrow \Pi(S)\pi(H)\Pi(S^{-1}) = -\pi(H) \tag{9.1.7}$$

calculation:

use the Campbell's identity,

$$\begin{aligned} \text{Ad}_{\Pi(S)}\pi(H) &= \pi(\text{Ad}_{e^A}\text{Ad}_{e^{-B}}\text{Ad}_{e^A}H) \\ &= \pi(e^{\text{ad}_A}e^{-\text{ad}_B}e^{\text{ad}_A}H) \end{aligned} \tag{9.1.8}$$

and,

$$\begin{aligned} e^{\text{ad}_A}H &= H - 2A \\ e^{-\text{ad}_B}(H-2A) &= H - 2B - 2(A + H - B) = -H - 2A \\ e^{\text{ad}_A}(-H-2A) &= -(H-2A) - 2A = -H \end{aligned} \tag{9.1.9}$$

---

and,

$$\begin{aligned} \text{Ad}_S^{-1}H &= e^{-\text{ad}_A}e^{\text{ad}_B}e^{-\text{ad}_A}H \\ &= e^{-\text{ad}_A}e^{\text{ad}_B}(H+2A) \\ &= e^{-\text{ad}_A}\underbrace{((H+2B) + 2(A - H - B))}_{=-H+2A} = -H \end{aligned} \tag{9.1.10}$$

but,

$$
\begin{aligned}
e^{\mathrm{ad}_{J_+}} J_3 &= J_3 - J_+ \\
e^{-\mathrm{ad}_{J_-}} (J_3 + J_+) &= (J_3 - J_-) - (J_+ + J_3 - \frac{1}{2}J_-) = -J_+ - \frac{1}{2}J_- \\
e^{\mathrm{ad}_{J_+}} (-J_+ - \frac{1}{2}J_-) &= -J_+ - \frac{1}{2}(J_- + J_3 - \frac{1}{2}J_+)
\end{aligned}
\tag{9.1.11}
$$

- the eigenstates $|j,m\rangle$ of the operators J_3, J^2 are,

$$
\begin{cases}
J_3 \,|j,m\rangle = m\,|j,m\rangle \\
J^2 \,|j,m\rangle = j(j+1)\,|j,m\rangle \\
J_\pm \,|j,m\rangle = \dfrac{1}{\sqrt{2}}\sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle
\end{cases}
\tag{9.1.12}
$$

when $J_1 = \frac{1}{\sqrt{2}}(J_+ + J_-)$ and $J_2 = \frac{1}{i\sqrt{2}}(J_+ - J_-)$ act on $|s,m\rangle$,

$$
\begin{cases}
J_1 \,|j,m\rangle = \lambda_+(j,m)\,|j,m+1\rangle + \lambda_-(j,m)\,|j,m-1\rangle \\
J_2 \,|j,m\rangle = -i\lambda_+(j,m)\,|j,m+1\rangle + i\lambda_-(j,m)\,|j,m-1\rangle
\end{cases}
\tag{9.1.13}
$$

where $\lambda_\pm(j,m) = \sqrt{\frac{j(j+1)-m(m\pm1)}{2}}$.

- **spin-$\mathbf{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots}$** rep. are **faithful**, and **spin-$\mathbf{0, 1, 2, \cdots}$** rep. are **not faithful**.

9.1.1 spin-$\frac{1}{2}$ representation

- choose $s = 1/2$, and $|\frac{1}{2},\frac{1}{2}\rangle = (1,0)^T, |\frac{1}{2},-\frac{1}{2}\rangle = (0,1)^T$, then $J_i = \frac{1}{2}\sigma_i$, where,

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\tag{9.1.14}
$$

and the ladder operators are,

$$
J_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\tag{9.1.15}
$$

9.1.2 spin-1 representation

- choose $s = 1$, and $|1,1\rangle = (1,0,0)^T, |1,0\rangle = (0,1,0)^T, |1,-1\rangle = (0,0,1)^T$, then,

$$
J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\tag{9.1.16}
$$

9.2 direct product representation

- the direct product representation of the SU(2) group is,

$$
D^{1\otimes 2}_{ii'jj'}(g) = D^1_{ij}(g) D^2_{i'j'}(g)
\tag{9.2.1}
$$

- consider a group element near the identity,

$$
\begin{aligned}
(1 + i\alpha_i J_i^{1\otimes 2})_{ii'jj'} &= (\delta^1_{ij} + i\alpha_i (J_i^1)_{ij})(\delta^2_{i'j'} + i\alpha_i (J_i^2)_{i'j'}) \\
&= \delta^1_{ij}\delta^2_{i'j'} + i\alpha_i (J_i^{1\otimes 2})_{ii'jj'}
\end{aligned}
\tag{9.2.2}
$$

where $(J_i^{1\otimes 2})_{ii'jj'} = (J_i^1)_{ij}\delta^2_{i'j'} + \delta^1_{ij}(J_i^2)_{i'j'}$ or more compactly,

$$
J_i^{1\otimes 2} = J_i^1 \otimes I^2 + I^1 \otimes J_i^2
\tag{9.2.3}
$$

- the eigenstates are,

$$J_3^{1\otimes 2}|j_1,m_1\rangle\otimes|j_2,m_2\rangle=(m_1+m_2)|j_1,m_1\rangle\otimes|j_2,m_2\rangle \tag{9.2.4}$$

- the $(J^2)^{j_1\otimes j_2}$ is,

$$\begin{aligned}(J^2)^{j_1\otimes j_2}&=\sum_i (J_i^{j_1}\otimes I^{j_2}+I^{j_1}\otimes J_i^{j_2})^2\\&=(J^2)^{j_1}\otimes I^{j_2}+I^{j_1}\otimes(J^2)^{j_2}+2\sum_i J_i^{j_1}\otimes J_i^{j_2}\end{aligned} \tag{9.2.5}$$

when $(J^2)^{j_1\otimes j_1}$ acts on $|j_1,m_1\rangle\otimes|j_2,m_2\rangle$:

$$\begin{aligned}&(J^2)^{j_1\otimes j_1}|j_1,m_1\rangle\otimes|j_2,m_2\rangle\\=&(j_1(j_1+1)+j_2(j_2+1)+2m_1m_2)|j_1,m_1\rangle\otimes|j_2,m_2\rangle\\&+2(J_1^{j_1}\otimes J_1^{j_2}+J_2^{j_1}\otimes J_2^{j_2})|j_1,m_1\rangle\otimes|j_2,m_2\rangle\end{aligned} \tag{9.2.6}$$

where,

$$\begin{aligned}&2(J_1^{j_1}\otimes J_1^{j_2}+J_2^{j_1}\otimes J_2^{j_2})|j_1,m_1\rangle\otimes|j_2,m_2\rangle\\=&4\lambda_+(j_1,m_1)\lambda_-(j_2,m_2)|j_1,m_1+1\rangle\otimes|j_2,m_2-1\rangle\\&+4\lambda_-(j_1,m_1)\lambda_+(j_2,m_2)|j_1,m_1-1\rangle\otimes|j_2,m_2+1\rangle\end{aligned} \tag{9.2.7}$$

9.2.1 Clebsch-Gordan coefficients

- direct product representation and direct sum representation,

$$\{j_1\}\otimes\{j_2\}=\bigoplus_{j=|j_1-j_2|}^{j_1+j_2}\{j\} \tag{9.2.8}$$

where $\{j\}$ means spin-j representation.

proof:

the eigenvalue and corresponded eigenspace of $J_3^{j_1\otimes j_2}$ is (assuming $j_1\geq j_2$),

eigenvalue	basis of the eigenspace	dimension
j_1+j_2	$\|j_1,j_1,j_2,j_2\rangle$	1
j_1+j_2-1	$\|j_1,j_1-1,j_2,j_2\rangle,\|j_1,j_1,j_2,j_2-1\rangle$	2
$\vdots$	$\vdots$	$\vdots$
$j_1+j_2-2j_2$	$\|j_1,j_1-2j_2,j_2,j_2\rangle,\cdots,\|j_1,j_1,j_2,-j_2\rangle$	$1+2j_2$
j_1-j_2-1	$\|j_1,j_1-2j_2-1,j_2,j_2\rangle,\cdots,\|j_1,j_1-1,j_2,-j_2\rangle$	$1+2j_2$
$\vdots$	$\vdots$	$\vdots$
$j_1+j_2-2j_1$	$\|j_1,-j_1,j_2,j_2\rangle,\cdots,\|j_1,-j_1+2j_2,j_2,-j_2\rangle$	$1+2j_2$
$-j_1+j_2-1$	$\|j_1,-j_1,j_2,j_2-1\rangle,\cdots,\|j_1,-j_1+2j_2-1,j_2,-j_2\rangle$	$2j_2$
$\vdots$	$\vdots$	$\vdots$
$-j_1-j_2$	$\|j_1,-j_1,j_2,-j_2\rangle$	1

so, it is clear that we can use $|j_1,j_1,j_2,j_2\rangle$ and $J_-^{j_1\otimes j_2}$ to produce $\{j_1+j_2\}$, and among the rest of the vectors, the highest eigenvalue of $J_3^{j_1\otimes j_2}$ is j_1+j_2-1 and there is only one vector with this eigenvalue is remained.
hence,

$$\{j_1\}\otimes\{j_2\}=\bigoplus_{j=|j_1-j_2|}^{j_1+j_2}\{j\} \tag{9.2.9}$$

– example: $\{\frac{1}{2}\} \otimes \{\frac{1}{2}\} = \underbrace{\{1\}}_{\text{spin triplet}} \oplus \underbrace{\{0\}}_{\text{spin singlet}}$

- the Clebsch-Gordan coefficients are,

$$\langle j_1, m_1, j_2, m_2 | j_1, j_2, j, m\rangle \tag{9.2.10}$$

where $|j_1, j_2, j, m\rangle$ (it is common to write $|j, m\rangle$ for short) are the coupled eigenstates of $J_3^{j_1\otimes j_2}$ and $(J^2)^{j_1\otimes j_2}$.

- the recursion relations are,

$$\begin{aligned}&\lambda_\pm(j_1, m_1 \mp 1)\,\langle j_1, m_1 \mp 1, j_2, m_2|j, m\rangle\\ &+ \lambda_\pm(j_2, m_2 \mp 1)\,\langle j_2, m_2, j_2, m_2 \mp 1|j, m\rangle\\ &= \lambda_\pm(j, m)\,\langle j_1, m_1, j_2, m_2|j, m \mp 1\rangle\end{aligned} \tag{9.2.11}$$

proof:

just consider the ladder operators $J_\pm^{j_1\otimes j_2} = J_\pm^{j_1} \otimes I^{j_2} + I^{j_1} \otimes J_\pm^{j_2}$,

$$\sum_{j_1,m_1,j_2,m_2} J_\pm^{j_1\otimes j_2}\,|j_1, m_1, j_2, m_2\rangle\,\langle j_1, m_1, j_2, m_2|j, m\rangle = \cdots \tag{9.2.12}$$

taking $m = j$ gives the initial recursion relation,

$$\begin{aligned}&\lambda_+(j_1, m_1 - 1)\,\langle j_1, m_1 - 1, j_2, m_2|j, j\rangle\\ &+ \lambda_+(j_2, m_2 - 1)\,\langle j_2, m_2, j_2, m_2 - 1|j, j\rangle = 0\end{aligned} \tag{9.2.13}$$

- use the phase convention that $\langle j_1, m_1, j_2, m_2|j, j\rangle \in \mathbb{R}$ and > 0, combined with the recursion relations, we can conclude that $\langle j_1, m_1, j_2, m_2|j, m\rangle \in \mathbb{R}$.

Part IV

Applications

Chapter 10

some examples of Lie groups and Lie algebras

10.1 the representations of $\mathfrak{sl}(3,\mathbb{C})$

- this section, we are going to discuss the classification of the irreducible rep. of SU(3) and $\mathfrak{sl}(2,\mathbb{C})$.
- $\mathfrak{sl}(3,\mathbb{C}) \simeq \mathfrak{su}(3)_{\mathbb{C}}$.
- $\mathrm{SU}(m)$ are **simply connected**, **compact** Lie groups.
  - according to section 5.1.1, 单连通李群 (的表示) 完全由其李代数 (的表示) 决定.
    rep. of $\mathfrak{sl}(3,\mathbb{C}) \overset{\text{restrict to}}{\Longrightarrow}$ rep. of $\mathfrak{su}(3) \overset{\text{simple connectedness}}{\Longrightarrow}$ rep. of SU(3).
  - according to section 5.2, Π is irreducible $\iff$ π is irreducible.
    and SU(3) is **compact**, so it has complete reducibility property $\Longrightarrow$ rep. of $\mathfrak{sl}(3,\mathbb{C})$ is **completely reducible**. 可见, 半单李代数的表示都是 completely reducible.

Chapter 11

the spin groups, $\mathrm{Spin}(n)$

- Wikipedia: https://en.wikipedia.org/wiki/Spin_group.
- 关于 universal cover & $\mathrm{Spin}(n)$ 与 $\mathrm{SO}(n \geq 3)$ 和 Clifford algebra 的关系, 见 subsection 5.1.2.

Appendices

Appendix A

linear algebra review

- **def.:** an **algebra** (over a field K) is a vector space + bilinear product $B: A\times A\to A$ (简写做 $\cdot$), 几个主要特征如下,
  1. 双线性形式 $B(\cdot,\cdot)$ 满足左, 右分配律和 (A.0.2),
  2. 可能存在单位元 (不是零向量),
  $$B(e,x)=x,\forall x \tag{A.0.1}$$
  存在单位元的代数称为 **unital algebra**.

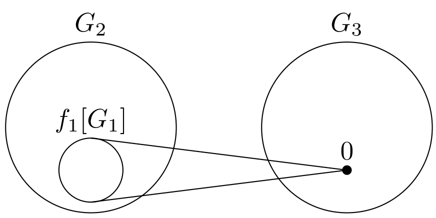
- 注意区分 bilinear form 和 sesquilinear form,
$$\begin{cases} B(ax,by)=abB(x,y) & \text{双线性} \\ S(ax,by)=a^*bS(x,y) & \text{半双线性, 有复共轭} \end{cases} \tag{A.0.2}$$
  一般用 $(\cdot,\cdot)$ 和 $\langle\cdot,\cdot\rangle$ 区分.

- 李代数 $\mathfrak{g}$ 一定不存在单位元, (因为一定有 $[E,E]=0\implies E=0$ 与单位元性质矛盾).

- 另外,
$$\begin{array}{rcl} \text{injective} & \leftrightarrow & \text{one-to-one function} \\ \text{surjective} & \leftrightarrow & \text{onto} \\ \text{bijective} & \leftrightarrow & \text{one-to-one correspondence} \end{array} \tag{A.0.3}$$

- a exact sequence (其中 f_i 都是 homomorphism),
$$G_1\overset{f_1}{\to}G_2\overset{f_2}{\to}G_3\cdots \tag{A.0.4}$$
  表示 $f_1[G_1]=\ker(f_2)$. 例如,
  - $G\to H\to 0$ 表示 $f[G]=\ker(f_2)=H$, 即 f 是 onto.
  - $0\to G\to H$ 表示 $\{0\}=\ker(f)$, 即 f 是 one-to-one.

- a short exact sequence,
$$0\to G_1\overset{f_1}{\to}G_2\overset{f_2}{\to}G_3\to 0 \tag{A.0.5}$$
  表示 f_1 是 one-to-one, f_2 是 onto, 且 $\ker(f_2)=f_1[G_1]$, 所以,



  注意到 f_1,f_2 都是 homomorphism, 所以,
$$G_3=G_2/f_1[G_1] \tag{A.0.6}$$

A.1 eigenvalues and eigenspaces

- eigenvectors associated to different eigenvalues are linearly independent.

proof:

if $v_1, \cdots, v_k$ are linearly independent eigenvectors with different eigenvalues, and v_{k+1} is a linear combination of them and is also an eigenvector, then,

$$\begin{aligned} v_{k+1} = \sum_{i=1}^{k} c^i v_i \Longrightarrow \lambda_{k+1} v_{k+1} = \sum_i c^i \lambda_i v_i \\ \Longrightarrow 0 = \sum_i c^i (\lambda_i - \lambda_{k+1}) v_i \end{aligned} \tag{A.1.1}$$

which contradicts to the linear independence.

A.2 spectral theorem for normal matrices

A.2.1 diagonalization

- we want to use an **reversible matrix** to **diagonalize** a diagonalizable matrix $A \in \mathrm{End}(\mathbb{C}^n)$,

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \iff A = P\mathrm{diag}(\lambda_1, \ldots, \lambda_n)P^{-1} \tag{A.2.1}$$

we can see that:

 - $\det A = \prod_i \lambda_i$.
 - $\mathrm{tr} A = \sum_i \lambda_i$.

method to find P:

consider,

$$AP = P\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \tag{A.2.2}$$

let the column-vector be $P_{ij} = \xi_i^{(j)}$, then,

$$\sum_j A_{ij} \xi_j^{(k)} = \xi_i^{(k)} \lambda_k \quad \text{or} \quad A\xi^{(k)} = \lambda_k \xi^{(k)} \tag{A.2.3}$$

it is clear that $\{\xi^{(i)}\}$ are the eigenvectors of A with corresponding eigenvalues $\{\lambda_i\}$.

- A is diagonalizable $\iff$ the eigenspace of A is n-dimensional.

A.2.2 geometric multiplicity & algebraic multiplicity

- the **dimension theorem:** let $T : V \to W$, then,

$$\dim V = \dim \ker T + \dim T(V) \tag{A.2.4}$$

where $T(\ker T) = 0 \in W$.

proof:

let $U \cap \ker T =$ and $V = U \oplus \ker T$, so,

$$\dim V = \dim \ker T + \dim U \tag{A.2.5}$$

$\forall \left|b_1\right\rangle, \left|b_2\right\rangle \in U$, if $\left|b_1\right\rangle \neq \left|b_2\right\rangle$ then $T\left|b_1\right\rangle \neq T\left|b_2\right\rangle$, so,

$$T(U) \simeq U \Longrightarrow \dim U = \dim T(U) \tag{A.2.6}$$

and notice that $T(V) = T(U)$, so we have $\dim U = \dim T(V)$.

- some times we use $\dim T \equiv \dim T(V)$ for convenience.

- **def.:** the **geometric multiplicity** (of eigenvalue λ_i), $\boldsymbol{\gamma_A(\lambda_i)}$, is defined to be,

$$\gamma_A(\lambda_i) = \dim(\ker(A - \lambda_i I)) \equiv n - \dim(A - \lambda_i I) \tag{A.2.7}$$

- **def.:** the **algebraic multiplicity**, $\boldsymbol{\mu_A(\lambda_i)}$, is defined to be the multiplicity (重根数) of root λ_i in the polynomial $\det(A - \lambda I) = 0$.

- theorem of geometric multiplicity & algebraic multiplicity:

$$1 \leq \gamma_A(\lambda_i) \leq \mu_A(\lambda_i) \leq n \tag{A.2.8}$$

proof:

let $\{v_{i=1,\ldots,\gamma_A(\lambda_i)}\}$ to be the orthogonal basis of the eigenspace of λ_i,

$$A\,|v_j, j \in \{1, \ldots, \gamma_A(\lambda_i)\}\rangle = \lambda_i\,|v_j\rangle \tag{A.2.9}$$

and let $\{v_1, \ldots, v_{\gamma_A(\lambda_i)}, v_{\gamma_A(\lambda_i)+1}, \ldots, v_n\}$ to be the orthogonal basis of the vector space V, (note that $\{v_{\gamma_A(\lambda_i)+1}, \ldots, v_n\}$ are not necessarily eigenvectors), then,

$$\langle v_j|\,A\,|v_k\rangle \equiv A'_{jk} = \begin{pmatrix} \lambda_i & & & *** \\ & \ddots & & *** \\ & & \lambda_i & *** \\ & & & *** \end{pmatrix} \tag{A.2.10}$$

then we have,

$$\det(A - \lambda I) = \det(A' - \lambda I) = (\lambda - \lambda_i)^{\gamma_A(\lambda_i)} \mathcal{P}^c_{n-\gamma_A(\lambda_i)}(\lambda) \tag{A.2.11}$$

so, it is clear that $\mu_A(\lambda_i) \geq \gamma_A(\lambda_i)$.

A.2.3 Schur decomposition

- **Schur decomposition**: for any complex matrix M,

$$M = U(\text{upper triangle matrix})U^\dagger \tag{A.2.12}$$

proof:

let $\lambda \in \mathbb{C}$ to be an eigenvalue of U with corresponding orthonormal eigenvectors $\{v_1, \ldots, v_{\gamma_M(\lambda)}\}$, then use the eigenvectors to construct an orthonormal basis,

$$\langle v_i|\,M\,|v_j\rangle = \begin{pmatrix} \lambda I_{\gamma_M(\lambda)\times\gamma_M(\lambda)} & M_{12} \\ 0 & M_{22} \end{pmatrix} \tag{A.2.13}$$

apply the exact procedure to M_{22} until M is completely trianglized.

A.2.4 spectral theorem for normal matrices

- **def.:** matrix A is **normal** if and only if $[A, A^\dagger] = 0$.

- **spectral theorem** for normal matrices:
  there is an orthogonal basis consisting of eigenvectors of A.

proof:

- **normal triangle** matrix must be **diagonal**.

---

proof:

assume A is an upper triangle normal matrix, then $A^\dagger A$ is upper triangle and $AA^\dagger$ is lower triangle, which implies both of them are diagonal.

$A^\dagger A$ is diagonal $\Longrightarrow$ matrix A is also diagonal (draw A and $A^\dagger$ and it will become obvious).

– A is similar to an upper triangle matrix which is also normal $\Longrightarrow$ similar to a diagonal matrix.

for Hermitian matrices

- for a Hermitian matrix H, $\lambda_i \in \mathbb{R}$.
- if $\lambda_i \neq \lambda_j$ then their eigenvectors are orthogonal.

proof:

$$\langle v_i| H |v_j\rangle = \lambda_j \langle v_i|v_j\rangle = (\langle v_j| H |v_i\rangle)^* = \lambda_i^* \langle v_i|v_j\rangle \Longrightarrow \begin{cases} i = j & \lambda_i \in \mathbb{R} \\ i \neq j & \langle v_i|v_j\rangle = 0 \end{cases} \tag{A.2.14}$$

- there is an orthogonal basis consisting of eigenvectors, i.e. $\gamma_H(\lambda_i) = \mu_H(\lambda_i)$.

for unitary matrices

- for a unitary matrix U, $|\lambda_i| = 1$.
- if $\lambda_i \neq \lambda_j$ then their eigenvectors are orthogonal.

proof:

$$\underbrace{\langle v_i| U^\dagger U |j\rangle}_{\langle v_i|v_j\rangle} = \lambda_i^* \lambda_j \langle v_i|v_j\rangle \Longrightarrow \begin{cases} i = j & |\lambda_i| = 1 \\ \lambda_i \neq \lambda_j & \langle v_i|v_j\rangle = 0 \end{cases} \tag{A.2.15}$$

- there is an orthogonal basis consisting of eigenvectors, i.e. $\gamma_U(\lambda_i) = \mu_U(\lambda_i)$.

for skew self-adjoint matrices

- for a skew self-adjoint matrix A ($A^\dagger = -A$), $\lambda_i \in i\mathbb{R}$.
- if $\lambda_i \neq \lambda_j$ then their eigenvectors are orthogonal.

proof:

$$\langle v_i|A|v_j\rangle = \lambda_j \langle v_i|v_j\rangle = (-\langle v_j|A|v_i\rangle)^* = -\lambda_i^* \langle v_i|v_j\rangle \Longrightarrow \begin{cases} i = j & \lambda_i \in i\mathbb{R} \\ i \neq j & \langle v_i|v_j\rangle = 0 \end{cases} \tag{A.2.16}$$

- there is an orthogonal basis...

A.3 simultaneous diagonalization

A.3.1 weights and weight spaces

- V is a vector space, $\mathcal{A}$ is a vector space of linear operators on V, and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{A}$.
- **def.:** a **weight** for $\mathcal{A}$ is an element $\mu \in \mathcal{A}$ s.t. there exists a nonzero $v \in V$,
$$Av = \langle \mu, A\rangle v \tag{A.3.1}$$
for all $A \in \mathcal{A}$.
- **def.:** $V_\mu = \{v \in V | A|v\rangle = |v\rangle \langle \mu, A\rangle, \forall A \in \mathcal{A}\}$ is called the **weight space** of μ.
- if $\mathcal{A}$ is **Abelian**, then there **exists** (at least) one weight for $\mathcal{A}$.

proof:

- assume W is the **minimal nonzero invariant subspace** of $\mathcal{A}$, meaning that,

$$A[W] \subseteq W, \forall A \in \mathcal{A} \tag{A.3.2}$$

and every subspace of U, except $\{0\}$, is not nonzero invariant under some operator in $\mathcal{A}$. (V is invariant but may not be minimal, so W **exists**)

- there exists $u \in W$ s.t. u is an eigenvector of $A \in \mathcal{A}$, with eigenvalue λ.

---

proof:
let $\{w_1, \cdots, w_m\}$ be the basis of W, then,

$$Aw_i = \sum_{j=1}^{m} \alpha_{ij} w_j \tag{A.3.3}$$

the eigenvector of $\{\alpha_{ij}\}$ is ξ with $\sum_i \xi^i \alpha_{ij} = \lambda_\alpha \xi^j$, then,

$$A\xi^i w_i = \lambda_\alpha \xi^j w_j \tag{A.3.4}$$

so, $u = \xi^i w_i$ is an eigenvector of A.

- the eigenspace $E_{A,\lambda}$ is an invariant subspace of $\mathcal{A}$,

$$ABv = BAv = \lambda Bv \Longrightarrow B[E_{A,\lambda}] \subseteq E_{A,\lambda}, \forall B \tag{A.3.5}$$

- for $u \in W \cap E_{A,\lambda}$,

$$Bu \in W \text{ and } E_{A,\lambda} \tag{A.3.6}$$

so, $W \cap E_{A,\lambda} \subseteq W$ is an invariant subspace of $\mathcal{A}$, which contradicts to the def. of W.

- so all the elements in W are eigenvectors of A, i.e. it is the **simultaneous eigenspace** of $\mathcal{A}$.

A.3.2 simultaneous diagonalization

- **def.:** $\mathcal{A}$ is **simultaneously diagonalizable** if there exists a basis $\{v_1, \cdots, v_n\}$ s.t. each v_i is a simultaneous eigenvector of $\mathcal{A}$.
- if $\mathcal{A}$ is **Abelian** and each of $A \in \mathcal{A}$ is **diagonalizable**, then $\mathcal{A}$ is simultaneously diagonalizable.

proof:

if A, B commute and are diagonal, then, the vector space decomposes as,

$$V = \bigoplus_{i=1}^{r} E_{A,\lambda_i} \tag{A.3.7}$$

choose the eigenvectors of A as basis, then,

$$B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{pmatrix} \quad A = \begin{pmatrix} \lambda_1 I_1 & & \\ & \ddots & \\ & & \lambda_r I_r \end{pmatrix} \tag{A.3.8}$$

because $E_{A,\lambda_{i=1,\cdots,r}}$ are invariant subspaces of B.
each $B_{i=1,\cdots,r}$ is diagonalizable by $P_i \in \mathrm{End}(E_{A,\lambda_i})$ (or B won't be diagonalizable), and $\lambda_i I_i$ remains diagonal.

---

repeat this process, all matrices in $\mathcal{A}$ can be diagonalized.

- if $\mathcal{A}$ is **simultaneously diagonalizable**, then,

$$V = \bigoplus_{\mu} V_{\mu} \tag{A.3.9}$$

where weight spaces are **linearly independent**, i.e.,

- $\mu_1 \neq \mu_2 \neq \cdots \neq \mu_m$ are distinct weights, then, $\{v_i \neq 0 | v_i \in V_{\mu_i}\}$ is linearly independent.

proof:

first, $V_{\mu_1} \cap V_{\mu_2} = \{0\}$ for distinct weights $\mu_1 \neq \mu_2$, and $\bigcup_{\mu} V_{\mu} = V$.

---

then, let's prove linear independence,

- consider,

$$(A - \langle \mu_j, A \rangle I) \sum_{i=1}^{m} |v_i\rangle = \sum_{i=1}^{m} (\langle \mu_i, A \rangle - \langle \mu_j, A \rangle) |v_i\rangle \tag{A.3.10}$$

- so, if $v_1 + \cdots + v_m = 0$, then we must have,

$$v_1 + \cdots + v_{j-1} + v_{j+1} + \cdots + v_m = 0 \tag{A.3.11}$$

- repeat the process, every element in $\{v_i\}$ is zero.
- i.e. $\{v_i \neq 0 | v_i \in V_{\mu_i}\}$ is linearly independent.

A.4 obtuse basis corresponds to acute dual basis

- $\{v_1, \cdots, v_n\}$ is an obtuse (钝角) basis (i.e. $\langle v_i, v_j \rangle \leq 0, \forall i \neq j$), then its dual basis is acute (锐角) (i.e. $\langle v_i^*, v_j^* \rangle \geq 0, \forall i, j$).

proof:

用指标写出来就是,

$$g_{ab} (v_i)^a (v_j)^b \leq 0, i \neq j \iff g^{ab} (v_i^*)_a (v_j^*)_b \geq 0 \tag{A.4.1}$$

或者 $g_{ij} \leq 0, i \neq j \iff g^{ij} \geq 0$.

用数学归纳法证明. 首先, 在 $n = 1, 2$ 的情况下, 定理成立.
在 $n > 2$ 的情况下, 考虑投影算符,

$$P_i = 1 - \frac{|v_i\rangle \langle v_i|}{\langle v_i, v_i \rangle} \tag{A.4.2}$$

那么 $P_i |v_1\rangle, \cdots, P_i |v_{i-1}\rangle, P_i |v_{i+1}\rangle, \cdots, P_i |v_n\rangle$ 构成 $\text{span}(v_i)^{\perp} = \{u \in V | u \perp v_i\}$ 的钝角基底 (显然构成基底),

$$\langle P_i v_j, P_i v_k \rangle = \langle v_j, P_i v_k \rangle = \underbrace{\langle v_j, v_k \rangle}_{\leq 0} - \frac{\langle v_j, v_i \rangle \langle v_i, v_k \rangle}{\langle v_i, v_i \rangle} \leq 0 \tag{A.4.3}$$

其中 $j, k \neq i$ (注意 $\langle v_i, v_i \rangle > 0$). 并且,

$$(P_i v_j)^* = v_j^* \in \text{span}(v_i)^{\perp}, j \neq i \tag{A.4.4}$$

不断重复以上过程直至维数降低到 2, 从而证明 $\langle v_i^*, v_j^* \rangle \geq 0$.

Appendix B

maps between manifolds

B.1 pushforward & pullback

- 對於一個 m- dim 李群 G 和 n- dim 流形 M, 它們之間存在映射 $\sigma : G \times M \to M$, 滿足,

$$\begin{cases} \sigma_g : M \to M \text{ is diffeomorphism} \\ \sigma_g \circ \sigma_h = \sigma_{gh} \end{cases} \tag{B.1.1}$$

- 可見, $\{\sigma_g : M \to M | g \in G\}$ is homomorphic to G, 且 $\sigma_p : G \to M$ is C^∞ and preserves the topology.
- 我們用 $\{x^\mu | \mu = 1, \cdots m\}$ 表示李群 G 上的坐標, 用 $\{y^\nu | \nu = 1, \cdots n\}$ 表示流形 M 上的坐標.

B.1.1 pullback

- 流形 M 上有坐標 $\{y^\mu | \mu = 1, \cdots n\}$, 那麼通過 pullback 可以得到李群 G 上的 n 個標量場,

$$\sigma_p^* : \mathcal{F}_M \to \mathcal{F}_G \quad (\sigma_p^* y^\mu)(g) = y^\mu(\sigma_p(g)) \tag{B.1.2}$$

- 不能 pushforward 的原因:

$$\sigma_{p*} x^\mu(\underline{\underline{\sigma_p(g)}}) = x^\mu(g) \tag{B.1.3}$$

  $\sigma_p(g)$ 這個 M 上的點可能對應不同的 g, 那麼標量場 $\sigma_{p*} x^\mu$ 在此處的取值也就無法確定.

- 注意: $\{\sigma_p^* y^\mu\}$ 是 G 上的一組 n 個標量場, 但是 $(\sigma_p^* y) : G \to n'$- dim Surface $\subset \mathbb{R}^n$, 其中,

$$\begin{cases} n' \leq m & \text{one-to-one 時取等 } (\dim \sigma_p[G] = \dim G) \\ n' \leq n & \text{onto 時取等 } (\dim \sigma_p[G] = n) \end{cases} \tag{B.1.4}$$

B.1.2 pushforward

- 將李群 G 上的矢量場 pushforward 到流形 M 上,

$$\sigma_{p*} : \mathcal{T}_G(1,0) \to \mathcal{T}_M(1,0) \quad \Big(\sigma_{p*} \frac{\partial}{\partial x^\mu}\Big)(\underline{\underline{y^\nu}})\Big|_{\sigma_p(g)} = \Big(\frac{\partial}{\partial x^\mu}\Big)(\sigma_p^* y^\nu)\Big|_g \tag{B.1.5}$$

  我們可以得到 pushforward 后的矢量場的全部 n 個分量.

- 但是由於 $\sigma_p^* y^\nu$ 只有 n' 個獨立變量 $(\dim \sigma_p^* y[G] = n')$, 所以 pushforward 后得到的 m 個矢量場中, 也只有 n' 個是綫性獨立的.
- 不能 pullback 的原因: 顯然無法確定 pullback 后的矢量場的 m 個分量, 最多 n' 個.

B.1.3 pullback

- 將流形 M 上的對偶矢量場 pullback 到李群 G 上,

$$(\sigma_p^* dy^\mu)_a \Big(\frac{\partial}{\partial x^\nu}\Big)^a\Big|_g = (dy^\mu)_a \Big(\sigma_{p*} \frac{\partial}{\partial x^\nu}\Big)^a\Big|_{\sigma_p(g)} \tag{B.1.6}$$

  同樣, pullback 得到的 n 個矢量場中, 綫性獨立的有 n' 個.

B.1.4 曲綫像的切矢等於曲綫切矢的像

- 對於一個曲綫 $\gamma : \mathbb{R} \to M_1$, 流形間的映射 $\psi : M_1 \to M_2$ 將其映射為 $\psi \circ \gamma : \mathbb{R} \to M_2$.
- 曲綫 γ 的切矢為 $\frac{\partial}{\partial t} = \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu}$, 那麼,

$$\psi_*\Big(\frac{\partial}{\partial t}\Big) = \frac{dx^\mu(\gamma(t))}{dt} \psi_*\Big(\frac{\partial}{\partial x^\mu}\Big) \tag{B.1.7}$$

是曲綫 $\psi \circ \gamma$ 的切矢.

- 證明的方法是將 (B.1.7) 式兩邊作用于 M_2 上的坐標 y^ν,

$$\begin{aligned} &\psi_*\Big(\frac{\partial}{\partial t}\Big)(y^\nu) = \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu}(\psi^* y^\nu) \\ \Longrightarrow &\psi_*\Big(\frac{\partial}{\partial t}\Big) = \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu}(\psi^* y^\nu)\frac{\partial}{\partial y^\nu} = \frac{d\psi^* y^\nu(\gamma(t))}{dt}\frac{\partial}{\partial y^\nu} = \frac{dy^\nu(\psi\circ\gamma(t))}{dt}\frac{\partial}{\partial y^\nu} \end{aligned} \tag{B.1.8}$$

B.2 diffeomorphisms & Lie derivatives

- 在流形 M 上有個 one-parameter group of diffeomorphism, 即,

$$\begin{cases} \phi_t : M \to M \text{ is diffeomorphism} \\ \phi_s \circ \phi_t = \phi_{s+t} \end{cases} \tag{B.2.1}$$

且對應矢量場 $\xi^a\Big|_p = \frac{d}{dt}\Big|_{t=0} \phi_t(p)$.

B.2.1 Lie derivatives

- 對於流形 M 上的任意 (k, l) 型張量場,

$$\mathcal{L}_\xi T^{a\cdots}{}_{b\cdots}\Big|_p = \lim_{t\to 0}\frac{1}{t}\Big(T^{a\cdots}{}_{b\cdots}\Big|_{\phi_t(p)} - \phi_{t*}\big(T^{a\cdots}{}_{b\cdots}\Big|_p\big)\Big) \tag{B.2.2}$$

$$= \lim_{t\to 0}\frac{1}{t}\Big(\phi_t^*\big(T^{a\cdots}{}_{b\cdots}\Big|_{\phi_t(p)}\big) - T^{a\cdots}{}_{b\cdots}\Big|_p\Big) \tag{B.2.3}$$

$$= \xi^c\nabla_c T^{a\cdots}{}_{b\cdots} - (\nabla_c\xi^a)T^{c\cdots}{}_{b\cdots} - \cdots + (\nabla_b\xi^c)T^{a\cdots}{}_{c\cdots} + \cdots \tag{B.2.4}$$

proof:

– 選取滿足如下要求的坐標,

$$\{x^\mu | \mu = 0, \cdots n\} \quad \xi = \frac{\partial}{\partial x^0} \tag{B.2.5}$$

也就是說,

$$\phi_t^* x^\mu(p) = x^\mu(\phi_t(p)) = \begin{cases} x^0(p) + t & \mu = 0 \\ x^\mu(p) & \mu \neq 0 \end{cases} \tag{B.2.6}$$

– 那麼, 對矢量場和對偶矢量場的 pullback 和 pushforward 分別如下,

$$\begin{cases} \phi_t^*\Big(dx^\mu\Big|_{\phi_t(p)}\Big) = dx^\mu\Big|_p & \text{and} \quad \phi_t^*\Big(\frac{\partial}{\partial x^\mu}\Big|_{\phi_t(p)}\Big) = \frac{\partial}{\partial x^\mu}\Big|_p \\ \phi_{t*}\Big(dx^\mu\Big|_p\Big) = dx^\mu\Big|_{\phi_t(p)} & \text{and} \quad \phi_{t*}\Big(\frac{\partial}{\partial x^\mu}\Big|_p\Big) = \frac{\partial}{\partial x^\mu}\Big|_{\phi_t(p)} \end{cases} \tag{B.2.7}$$

所以,

$$\begin{aligned} \mathcal{L}_\xi T^{a\cdots}{}_{b\cdots}\Big|_p &= (\partial_0 T^{a\cdots}{}_{b\cdots})\Big|_p \\ &= \xi^c\Big(\nabla_c T^{a\cdots}{}_{b\cdots} - \Gamma^a_{dc}T^{d\cdots}{}_{b\cdots} - \cdots + \Gamma^d_{bc}T^{a\cdots}{}_{d\cdots} + \cdots\Big) \end{aligned} \tag{B.2.8}$$

由於,

$$(\nabla_d\xi^a)T^{d\cdots}{}_{b\cdots} = \partial_d\Big(\frac{\partial}{\partial x^0}\Big)^a + \Gamma^a_{cd}\Big(\frac{\partial}{\partial x^0}\Big)^c T^{d\cdots}{}_{b\cdots} \tag{B.2.9}$$

代入,

$$\mathcal{L}_\xi T^{a\cdots}{}_{b\cdots}\Big|_p = \xi^c\nabla_c T^{a\cdots}{}_{b\cdots} - (\nabla_c\xi^a)T^{c\cdots}{}_{b\cdots} - \cdots + (\nabla_b\xi^c)T^{a\cdots}{}_{c\cdots} + \cdots \tag{B.2.10}$$

B.3 consider two maps, $\psi \circ \phi$

- 三個流形 M_1, M_2, M_3, 維數分別爲 n_1, n_2, n_3, 其上分別有坐標 $\{x^\mu\}, \{y^\mu\}, \{z^\mu\}$.
- 它們之間存在兩個 C^∞ 的 homomorphism, $\phi : M_1 \to M_2$ 和 $\psi : M_2 \to M_3$.

B.3.1 pullback

- 考慮,

$$\begin{cases} \psi^* z^\mu(p_2) = z^\mu(\psi(p_2)) \\ \underbrace{\phi^* \circ \psi^*}_{(\psi\circ\phi)^*} z^\mu(p_1) = z^\mu(\psi\circ\phi(p_1)) \end{cases} \tag{B.3.1}$$

所以, $\phi^* \circ \psi^* = (\psi \circ \phi)^*$.

B.3.2 pushforward

- 考慮,

$$\frac{\partial}{\partial x^\mu}\left((\psi\circ\phi)^* z^\nu\right)\Big|_{p_1} = \left((\psi\circ\phi)_*\frac{\partial}{\partial x^\mu}\right)(z^\nu)\Big|_{\psi\circ\phi(p_1)} \tag{B.3.2}$$

并且,

$$\frac{\partial}{\partial x^\mu}\left(\phi^* y^\nu\right)\Big|_{p_1} = \phi_*\frac{\partial}{\partial x^\mu}(y^\nu)\Big|_{\phi(p_1)} \tag{B.3.3}$$

$$\frac{\partial}{\partial x^\mu}\left(\phi^*\circ\psi^* z^\nu\right)\Big|_{p_1} = \phi_*\frac{\partial}{\partial x^\mu}(\psi^* z^\nu)\Big|_{\phi(p_1)} = \psi_*\circ\phi_*\frac{\partial}{\partial x^\mu}(z^\nu)\Big|_{\psi\circ\phi(p_1)} \tag{B.3.4}$$

所以, $(\psi\circ\phi)_* = \psi_*\circ\phi_*$.

B.3.3 pullback

- 考慮,

$$\left((\psi\circ\phi)^* dz^\mu\right)_a\left(\frac{\partial}{\partial x^\nu}\right)^a\Big|_{p_1} = (dz^\mu)_a\left((\psi\circ\phi)_*\frac{\partial}{\partial x^\nu}\right)^a\Big|_{\psi\circ\phi(p_1)} \tag{B.3.5}$$

且,

$$(\phi^*\circ\psi^* dz^\mu)_a\left(\frac{\partial}{\partial x^\nu}\right)^a\Big|_{p_1} = (\psi^* dz^\mu)_a\left(\phi_*\frac{\partial}{\partial x^\nu}\right)^a\Big|_{\phi(p_1)} = (dz^\mu)_a\left(\psi_*\circ\phi_*\frac{\partial}{\partial x^\nu}\right)^a\Big|_{\psi\circ\phi(p_1)} \tag{B.3.6}$$

所以, 依舊有 $\phi^*\circ\psi^* = (\psi\circ\phi)^*$.

B.4 Weyl transformations & conformal transformations

B.4.1 Weyl transformations

- Weyl 變換在保持流形不變的情況下, 改變流形上配備的度規, 此時, 流形的曲率等幾何性質也會發生改變.
- 背景流形上選取坐標 $\{x^\mu\}$, 那麼新度規與舊度規的關係為,

$$\tilde{g}_{\mu\nu} = e^{\Phi(x)} g_{\mu\nu} \tag{B.4.1}$$

其中, $\Phi(x)$ 是流形上的一個標量場.

- 在 Weyl 變換下, 仿射聯絡係數, 曲率張量都會發生變化, 但 Weyl 張量不會發生變換 (具体變換形式及计算过程见 GoodNotes 筆記: Weyl Transformation and Conformal Transformation).

B.4.2 conformal isometries

- 流行 M 上配備有兩套度規 g_{ab} 和 $\tilde{g}_{ab}$ (可見 Weyl 變換和共形變換都會改變流形的度規場).
- 映射 ϕ 是 conformal isometry, 其生成的拉回映射 ϕ^* 滿足,

$$(\phi^*(\tilde{g}\Big|_{\phi(p)}))_{ab} = \Omega^2 g_{ab}\Big|_p \tag{B.4.2}$$

  其中 Ω 是流形上的標量場.

- conformal transformations preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.
- 用坐標的拉回映射來表示這個變換, 那麼是, 對於流形上的坐標 $\{y^\mu\}$ 其拉回映射的像為 $\{x^\mu\}$, 即,

$$\begin{cases} (\phi^* y^\mu)(p) \equiv x^\mu(p) = y^\mu(\phi(p)) \\ \phi^* dy^\mu = dx^\mu \end{cases} \tag{B.4.3}$$

  那麼, conformal isometry ϕ 即滿足,

$$\tilde{g}_{\mu\nu}\Big|_{\phi(p)} \phi^*(dy^\mu \otimes dy^\nu) = \Omega^2 g_{\mu\nu}(dx^\mu \otimes dx^\nu) \tag{B.4.4}$$

$$\Longrightarrow \tilde{g}_{\mu\nu}\Big|_{\phi(p)} = (\Omega^2 g_{\mu\nu})\Big|_p \tag{B.4.5}$$

  其中 $\tilde{g}_{\mu\nu}$ 是度規 $\tilde{g}_{ab}$ 在 $\{y^\mu\}$ 坐標系下的分量.

B.4.3 conformal Killing vector fields

- 流形上的一個 one-parameter group of conformal isometry $\{\phi_t, t \in \mathbb{R}\}$, 其中每個 ϕ_t 都是 conformal isometry 且滿足如 (B.2.1) 式的群乘法, 且,

$$(\phi_t^* g)_{ab} = a(t) g_{ab} \tag{B.4.6}$$

  $a(t)$ 顯然要滿足某些性質, 目前可以確認 $a(0) = 1$.

- 矢量場 $\psi^a\big|_{\phi_s(p)} = \frac{d}{dt}\big|_s \phi_t(p)$ 稱爲 conformal Killing vector field, 相應的度規的李導數為,

$$(\mathcal{L}_\psi g)_{ab} = 2\nabla_{(a}\psi_{b)} = \alpha g_{ab} \tag{B.4.7}$$

  其中 $\alpha = \frac{d}{dt}\big|_{t=0} a(t)$, 對上式兩端求 trace, 得到,

$$2\nabla^a \psi_a = n\alpha \Longrightarrow \alpha = \frac{2}{n}\nabla^a\psi_a \tag{B.4.8}$$

  其中 n 是流形維數.

- 得到 conformal Killing vector field 滿足的方程,

$$\nabla_{(a}\psi_{b)} = \frac{1}{n}(\nabla^c\psi_c) g_{ab} \tag{B.4.9}$$