

# Lie Groups and Lie Algebras

a study note based on Brian Hall's textbook

---

Siyang Wan (万思扬)

# Contents

<b>I</b>	<b>Finite Groups</b>	<b>4</b>
<b>1</b>	<b>finite groups</b>	<b>5</b>
1.1	representation theory . . . . .	5
1.1.1	reducibility . . . . .	5
1.2	unitarity theorem . . . . .	6
1.3	Schur's lemmas . . . . .	6
1.4	the great orthogonal theorem . . . . .	7
<b>II</b>	<b>General Theory</b>	<b>11</b>
<b>2</b>	<b>Lie groups</b>	<b>12</b>
2.1	Lie groups . . . . .	12
2.2	topological properties . . . . .	12
2.2.1	compactness . . . . .	12
2.2.2	connectedness . . . . .	13
2.2.3	simple connectedness . . . . .	13
2.3	Lie subgroups . . . . .	13
<b>3</b>	<b>Lie algebras</b>	<b>15</b>
3.1	left-invariant vector fields . . . . .	15
3.2	Lie algebras . . . . .	15
3.2.1	subalgebras, ideals & simple, solvable, nilpotent Lie algebras . . . . .	16
3.2.2	structure constants . . . . .	17
<b>4</b>	<b>exponential maps</b>	<b>18</b>
4.1	one-parameter subgroups . . . . .	18
4.2	exponential maps . . . . .	19
4.2.1	matrix exponential and logarithm . . . . .	19
4.3	Baker-Campbell-Hausdorff formula . . . . .	20
4.3.1	the Campbell's identity . . . . .	20
4.3.2	BCH formula . . . . .	21
<b>5</b>	<b>basic representation theory</b>	<b>24</b>
5.1	Lie group and Lie algebra homomorphisms . . . . .	24
5.1.1	simply connected Lie groups . . . . .	25
5.1.2	universal covers . . . . .	27
5.2	basic representation theory . . . . .	28
5.2.1	new representations from old . . . . .	28
5.2.2	complete reducibility . . . . .	29
5.2.3	Schur's lemma . . . . .	30
5.3	Lie's third theorem . . . . .	30
5.4	adjoint representations . . . . .	30
5.4.1	adjoint rep. of Lie groups . . . . .	30
5.4.2	adjoint rep. of Lie algebras . . . . .	31
5.5	Killing forms . . . . .	31

<b>III</b>	<b>Semisimple Lie Algebras</b>	<b>33</b>
<b>6</b>	<b>semisimple Lie algebras</b>	<b>34</b>
6.1	semisimple and reductive Lie algebras	34
6.1.1	some properties of reductive and semisimple Lie algebras	35
6.2	Cartan subalgebra	37
6.3	roots and root spaces	38
6.3.1	subalgebras isomorphic to $\mathfrak{su}(2)_{\mathbb{C}}$	39
6.3.2	root systems	41
6.4	Cartan's criterion	41
6.5	the Weyl group (from the Lie algebra approach)	42
6.6	simple Lie algebras	43
6.7	the root systems of the classical Lie algebras	46
6.7.1	the special linear algebras, $\mathfrak{sl}(n+1, \mathbb{C}) = \mathfrak{su}(n+1)_{\mathbb{C}}$ , and $A_n$	46
6.7.2	the orthogonal algebras, $\mathfrak{so}(2n, \mathbb{C})$ , and $D_n$	47
6.7.3	the orthogonal algebras, $\mathfrak{so}(2n+1, \mathbb{C})$ , and $B_n$	48
6.7.4	the symplectic algebras, $\mathfrak{sp}(2n, \mathbb{C})$ , and $C_n$	49
<b>7</b>	<b>root systems</b>	<b>51</b>
7.1	abstract root systems	51
7.2	rank-two systems	52
7.3	duality	52
7.4	bases and Weyl chambers	52
7.5	Weyl chambers and Weyl group	55
7.6	Dynkin diagrams	56
7.7	integral and dominant integral elements	57
7.8	the partial ordering	58
7.9	rank-three systems	60
7.10	the classical root systems	61
7.11	the classification	61
<b>8</b>	<b>representations of semisimple Lie algebras</b>	<b>63</b>
8.1	weights of representations	63
8.2	the highest weight cyclic representations & an introduction to Verma modules	65
8.3	universal enveloping algebras, $U(\mathfrak{g})$	65
8.4	Poincaré-Birkhoff-Witt theorem	67
8.5	construction of Verma modules, $W_{\mu}$	68
8.6	irreducible quotient modules, $V^{\mu} = W_{\mu}/U_{\mu}$	70
8.7	finite-dimensional quotient modules	71
<b>9</b>	<b>further properties of the representations</b>	<b>72</b>
9.1	the structure of weights	72
9.2	the Casimir element	72
<b>10</b>	<b><math>\mathfrak{su}(2)_{\mathbb{C}}</math> algebra</b>	<b>74</b>
10.1	representations of $\mathfrak{su}(2)_{\mathbb{C}}$ algebra	74
10.1.1	$\text{spin}-\frac{1}{2}$ representation	76
10.1.2	$\text{spin}-1$ representation	76
10.2	direct product representation	76
10.2.1	Clebsch-Gordan coefficients	77
<b>IV</b>	<b>Applications</b>	<b>79</b>
<b>11</b>	<b>some examples of Lie groups and Lie algebras</b>	<b>80</b>
11.1	general linear groups and algebras	80
11.2	special linear groups and algebras	81
11.3	the Lorentz group and the Lorentz algebra	81
11.3.1	indefinite orthogonal groups	81
11.3.2	the Lorentz group	82
11.3.3	the Lorentz algebra	82
11.4	unitary groups and algebras	82

11.5	special unitary groups and algebras . . . . .	82
11.6	symplectic groups . . . . .	82
11.7	the representations of $\mathfrak{sl}(3, \mathbb{C})$ . . . . .	82
<b>12</b>	<b>the spin groups, <math>\text{Spin}(n)</math></b>	<b>83</b>
<b>Appendices</b>		<b>85</b>
<b>A</b>	<b>linear algebra review</b>	<b>85</b>
A.1	eigenvalues and eigenspaces . . . . .	86
A.2	spectral theorem for normal matrices . . . . .	86
A.2.1	diagonalization . . . . .	86
A.2.2	geometric multiplicity & algebraic multiplicity . . . . .	86
A.2.3	Schur decomposition . . . . .	87
A.2.4	spectral theorem for normal matrices . . . . .	87
A.3	simultaneous diagonalization . . . . .	88
A.3.1	weights and weight spaces . . . . .	88
A.3.2	simultaneous diagonalization . . . . .	89
A.4	obtuse basis corresponds to acute dual basis . . . . .	90
<b>B</b>	<b>maps between manifolds</b>	<b>91</b>
B.1	pushforward & pullback . . . . .	91
B.1.1	pullback . . . . .	91
B.1.2	pushforward . . . . .	91
B.1.3	pullback . . . . .	91
B.1.4	曲綫像的切矢等於曲綫切矢的像 . . . . .	92
B.2	diffeomorphisms & Lie derivatives . . . . .	92
B.2.1	Lie derivatives . . . . .	92
B.3	consider two maps, $\psi \circ \phi$ . . . . .	93
B.3.1	pullback . . . . .	93
B.3.2	pushforward . . . . .	93
B.3.3	pullback . . . . .	93
B.4	Weyl transformations & conformal transformations . . . . .	93
B.4.1	Weyl transformations . . . . .	93
B.4.2	conformal isometries . . . . .	94
B.4.3	conformal Killing vector fields . . . . .	94

**Part I**

**Finite Groups**

# Chapter 1

## finite groups

- a useful reference: <https://sites.ualberta.ca/~vbouchar/MAPH464/notes.html>.

- 
- def. of groups (Abelian groups, cyclic groups, symmetry groups, permutation groups).
  - order of  $G$  denoted by  $|G|$ , order of element  $g$ .
- 

- conjugated element  $ghg^{-1} = g'$ , conjugacy class.
- subgroup, (left/right) coset of a subgroup (2 theorems + Lagrange theorem).
- conjugacy subgroup  $hHh^{-1}$ .
- **normal subgroup** (i.e. invariant subgroup)  $N \triangleleft G, gNg^{-1} \subseteq N, \forall g$ .
  - **center**,  $Z(G) = \{z \in G | gzg^{-1} = z, \forall g\}$ .  
center is normal, but normal subgroup is not necessarily central.
  - the center of a Lie algebra is  $\mathfrak{h} = \{A \in \mathfrak{g} | [A, B] = 0, \forall B\} \equiv \{A \in \mathfrak{g} | \text{ad}_A = 0\}$ .  
center is an ideal, but ideal is not necessarily a center.
- groups without nontrivial normal subgroups are **simple**.
- **direct product group**  $G \times H$  (Cartesian product, direct product and direct sum).  
**def.:**  $G \times H = \{(g, h) | g \in G, h \in H\}$  with group product defined by  $(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_1 \circ h_2)$ .
- factor (quotient) group  $G/H_N$ .
- isomorphism vs. homomorphism.
  - kernel  $K \mapsto \{e\}$  of a homomorphism.

### 1.1 representation theory

- representation of a group  $D(g)$ .
- 用 basis of functions 来构建 rep. of  $G$ ,

$$\Omega_g \psi_i(\vec{x}) = \psi_i(g^{-1}\vec{x}) \quad (1.1.1)$$

- trivial rep. (1 dim.)  $D_{11}(\forall g) = 1$ .
- regular rep.  $D_{ij}(g) = \langle g_i | gg_j \rangle \equiv \delta_{g_i, gg_j}$ .

#### 1.1.1 reducibility

- reducible rep. vs. completely reducible (semisimple) rep..  
completely reducible rep.,

$$TD(g)T^{-1} = D^{(1)}(g) \oplus D^{(2)}(g) \oplus \dots \quad (1.1.2)$$

- completely reducible  $\iff$  invariant subspace is trivial.

## 1.2 unitarity theorem

- any finite-dim. rep. of a finite group are equivalent to a unitary rep..

**proof:**

for a finite-dim. rep.  $\Gamma = \{D(g), \dots\}$ , consider  $H = \sum_g D^\dagger(g)D(g)$ , we have,

$$D^\dagger(h)HD(h) = H \quad (1.2.1)$$

$H$  is a Hermitian matrix which can be diagonalized by a unitary matrix,

$$M \equiv \text{diag}(\lambda_1, \dots) = UHU^\dagger \quad (1.2.2)$$

then let,

$$B(g) = M^{1/2}UD(g)U^\dagger M^{-1/2} \quad (1.2.3)$$

where  $M^{1/2} = \text{diag}(\lambda_1^{1/2}, \dots)$ , we can see that,

$$\begin{aligned} B^\dagger(g)B(g) &= M^{-1/2}UD^\dagger(g)U^\dagger MUD(g)U^\dagger M^{-1/2} \\ &= M^{-1/2}MM^{-1/2} = I \end{aligned} \quad (1.2.4)$$

so  $\{B(g), \dots\}$  is a unitary rep..

- all the reducible unitary rep. are completely reducible.

**proof:**

unitary rep. 作用于  $V = W \oplus W^\perp$ , 其中  $V$  是 Hilbert 空间, 内积为  $\langle \cdot, \cdot \rangle$ ,  $W^\perp$  与  $W$  正交,  $W$  是表示的不变子空间, 下面证明  $W^\perp$  也是不变子空间,

$$\langle B(g)w^\perp | w \rangle = \langle w^\perp | B(-g)w \rangle = \langle w^\perp | w' \rangle = 0, \forall w^\perp \in W^\perp, w \in W \quad (1.2.5)$$

其中,  $w' \in W$ , 可见  $B(g)[W^\perp] \subseteq W^\perp$

---

其实不需要要求表示么正, 只需要  $B$  和  $B^\dagger$  拥有同一个不变子空间  $W$  就行.

这对 infinite group 也成立.

## 1.3 Schur's lemmas

- Schur's 1st lemma

for 2 irreducible real or complex rep.  $\Gamma_1 = \{D^{(1)}(g), \dots\}$  and  $\Gamma_2 = \{D^{(2)}(g), \dots\}$ ,  $\exists A$  s.t.  $\forall g$ ,

$$AD^{(1)}(g) = D^{(2)}(g)A \quad (1.3.1)$$

then, there are only 2 possibilities:

- $A = 0$ ,
- $A$  is reversible matrix and  $\Gamma_1, \Gamma_2$  are equivalent.

**proof:**

consider,

$$\begin{aligned} AD^{(1)}(g)[\ker A] &= D^{(2)}(g)A[\ker A] = 0 \\ \implies D^{(1)}(g)[\ker A] &\subseteq \ker A \end{aligned} \quad (1.3.2)$$

so,  $\ker A$  is a invariant subspace of rep.  $\Gamma_1$

but  $\Gamma_1$  is irreducible, so  $\ker A$  is trivial, i.e.  $\ker A$  is either 0 or  $V$ , which implies that...

对 infinite group 也成立.

- **Schur's 2nd lemma**

for a **irreducible complex rep.**  $\Gamma = \{D(g), \dots\}$ , if  $\forall g$ ,

$$AD(g) = D(g)A \quad (1.3.3)$$

then  $A = \lambda I$  for some  $\lambda \in \mathbb{C}$ .

**proof:**

$A$  must have (at least) one eigenvalue  $\lambda$ , then  $\det(A - \lambda I) = 0$  is irreversible matrix,

$$AD(g) = D(g)A \implies (A - \lambda I)D(g) = D(g)(A - \lambda I) \quad (1.3.4)$$

by Schur's 1st lemma, irreversible matrix  $A - \lambda I$  must be 0.

- **Schur's 3rd lemma**

for 2 **irreducible complex rep.**  $\Gamma_1 = \{D^{(1)}(g), \dots\}$  and  $\Gamma_2 = \{D^{(2)}(g), \dots\}$ , if  $\forall g$ ,

$$\begin{cases} AD^{(1)}(g) = D^{(2)}(g)A \\ BD^{(1)}(g) = D^{(2)}(g)B \end{cases} \quad (1.3.5)$$

then  $B = \lambda A$  for some  $\lambda \in \mathbb{C}$ .

**proof:**

$$(A - \lambda B)D^{(1)}(g) = D^{(2)}(g)(A - \lambda B) \quad (1.3.6)$$

choose  $\lambda$  s.t.  $\det(A - \lambda B) = 0$ , then, according to Schur's 1st lemma,  $A - \lambda B = 0$ .

## 1.4 the great orthogonal theorem

- **the great orthogonality theorem**

for 2 inequivalent irreducible rep.  $\Gamma^a = \{D^{(a)}(g), \dots\}$  where  $a = 1, 2$ ,

$$\frac{1}{|G|} \sum_g D_{ij}^{(a)}(g^{-1}) D_{j'i'}^{(b)}(g) = \frac{1}{d} \delta_{ii'} \delta_{jj'} \delta^{ab} \quad (1.4.1)$$

or for unitary rep.,

$$\frac{1}{|G|} \sum_g B_{ij}^{(a)*}(g) B_{i'j'}^{(b)}(g) = \frac{1}{d} \delta_{ii'} \delta_{jj'} \delta^{ab} \quad (1.4.2)$$

where  $d$  is the dim. of the rep..

**proof:**

for  $a = b$ :

consider  $A = \sum_g B^{(a)\dagger}(g) X B^{(a)}(g)$  where  $B^{(a)}(g) = T D^{(a)}(g) T^{-1}$  is the unitary rep. equivalent to  $\Gamma_a$ , then,

$$AB^{(a)}(h) = B^{(a)\dagger}(h^{-1})A \implies AB^{(a)}(h) = B^{(a)}(h)A \quad (1.4.3)$$

according to Schur's 1st lemma,  $A = \lambda I$ , then,

$$\begin{aligned} \lambda I &= \sum_g (T^{-1} B^{(a)\dagger}(g) T) (T^{-1} X T) (T^{-1} B^{(a)}(g) T) \\ &= \sum_g D^{(a)}(g^{-1}) X' D^{(a)}(g) \end{aligned} \quad (1.4.4)$$

choose  $X'_{,,} = \delta_{.,j} \delta_{j',.}$ , then we have  $\lambda I = \sum_g D_{.,j}^{(a)}(g^{-1}) D_{j',.}^{(a)}(g)$ , calculate the trace of the matrix,

$$\lambda d_a = \sum_g \delta_{jj'} = |G| \delta_{jj'} \quad (1.4.5)$$



so we can conclude that,

$$\frac{1}{|G|} \sum_g D_{ij}^{(a)}(g^{-1}) D_{j'i'}^{(a)}(g) = \frac{1}{d_a} \delta_{ii'} \delta_{jj'} \quad (1.4.6)$$

for  $a \neq b$ :

still consider  $A = \sum_g B^{(a)\dagger}(g) X B^{(b)}(g)$  then,

$$AB^{(b)}(h) = B^{(a)}(h)A \quad (1.4.7)$$

according to Schur's 1st lemma,  $A = 0$ , consequently,

$$\sum_g D_{ij}^{(a)}(g^{-1}) D_{j'i'}^{(b)}(g) = 0 \quad (1.4.8)$$

- characters of the rep.  $\Gamma_a$  of group  $G$  is the set  $\{\chi^{(a)}(g) = \text{tr} D^{(a)}(g) | g \in G\}$
- character table is the matrix  $X = \{X^a_i = \chi^{(a=1, \dots, \rho)}(g_{i=1, \dots, c})\}$ .  
where  $g_i$  is the rep. of the  $i$ th conjugacy class, and  $\rho$  is the number of the irreducible inequivalent rep. of  $G$ . ( $\rho = c$ , as to be proved later).

#### • 1st theorem of the orthogonality of the characters

the character of irreducible inequivalent rep. of  $G$  are orthogonal to each other, which can be derived easily from the great orthogonality theorem.

$$\frac{1}{|G|} \sum_g \chi^{(a)*}(g) \chi^{(b)}(g) = \delta^{ab} \quad (1.4.9)$$

#### • 2nd theorem of the orthogonality of the characters

$$\sum_{a=1}^{\rho} \chi^{(a)*}(g_i) \chi^{(a)}(g_j) = \frac{|G|}{n_i} \delta_{ij} \quad (1.4.10)$$

where  $g_i$  is the rep. of the  $i$ th conjugacy class,  $n_i$  is the number of elements in this conjugacy class, and  $\rho$  is the number of the irreducible inequivalent rep. of  $G$ .

**proof:**

by 1st theorem,

$$X \text{diag}\left(\frac{n_1}{|G|}, \dots, \frac{n_c}{|G|}\right) X^\dagger = I \quad (1.4.11)$$

then,

$$\Rightarrow \sum_j \left( X^\dagger X \text{diag}\left(\frac{n_1}{|G|}, \dots, \frac{n_c}{|G|}\right) \right)_{ij} X^\dagger_j{}^a = X^\dagger_i{}^a \quad (1.4.12)$$

since vectors  $(X^a_1, \dots, X^a_c)$  forms an orthogonal basis of the vector space, then we must have,

$$\left( X^\dagger X \text{diag}\left(\frac{n_1}{|G|}, \dots, \frac{n_c}{|G|}\right) \right)_{ij} = \delta_{ij} \quad (1.4.13)$$

then, finally, we have,

$$\sum_{a=1}^{\rho} \chi^{(a)*}(g_i) \chi^{(a)}(g_j) = \frac{|G|}{n_i} \delta_{ij} \quad (1.4.14)$$

- 群  $G$  的 irreducible inequivalent rep. 的数量等于其 conjugacy class 的数量  $c$ .

**proof:**

一个 irreducible inequivalent rep. 由其 characters 表示  $\{\chi^{(a)}(g), \dots\}$   
 (根据 theorem of the orthogonality of the characters) 不同的 irreducible inequivalent rep. 的 characters 一定不同.  
 且 conjugacy class 内的元素的 character 一定相等, 所以一个 rep. 实际上只有 conjugacy class 的数量  $c$  个不同的 characters, 所以可以将 characters 视为  $c$  维向量  $\frac{1}{\sqrt{|G|}}(\chi^{(a)}(g), \dots)$ , 那么  $c$  维向量空间中互相正交归一的向量最多只有  $c$  个.  
 利用 2nd theorem of... 可证... 最少有  $c$  个. 所以... 等于...

- characters of completely reducible rep..

suppose a completely reducible rep.  $\Gamma = \oplus_{a=1}^c m_a \Gamma_a$ , where  $m_a = 0, 1, 2, \dots$ , then,

$$\chi(g) = \sum_a m_a \chi^{(a)}(g) \quad (1.4.15)$$

(e.g. for  $D(g) = D^{(1)}(g) \oplus D^{(1)}(g)$ ,  $m_1 = 2$ ).

and,

$$\frac{1}{|G|} \sum_g \chi^*(g) \chi(g) = \sum_a m_a^2 > 1 \quad (1.4.16)$$

- Burnside theorem**

$$\sum_{a=1}^c d_a^2 = |G| \quad (1.4.17)$$

where  $d_a$  is the dim. of the  $a$ th inequivalent irreducible rep. of  $G$ .

**proof:**

by 2nd orthogonality theorem of characters,

$$\sum_{a=1}^c \chi^{(a)*}(e) (\chi^{(a)}(e) = d_a) = \frac{|G|}{(n_e = 1)} \implies \sum_{a=1}^c d_a^2 = |G| \quad (1.4.18)$$

- rep. of **direct product group**  $G = H \times F$  is derived from **irreducible rep.** of  $H$  and  $F$  by  $\Gamma = \Gamma_H \times \Gamma_F = \{D(hf) = D_H(h) \otimes D_F(f)\}$ , then  $\Gamma$  is also an irreducible rep..

**proof:**

利用 characters of completely reducible rep. 的性质.

- direct product of group rep.:  $\Gamma = \Gamma_a \times \Gamma_b$ , then  $\chi(g) = \chi^{(a)}(g) \chi^{(b)}(g)$

- projection operator is,

$$P_a = \frac{d_a}{|G|} \sum_g \chi^{(a)*}(g) T^{-1} \begin{pmatrix} \ddots & & \\ & D^{(b)}(g) & \\ & & \ddots \end{pmatrix} T = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} I & \\ & & \ddots \end{pmatrix} T \quad (1.4.19)$$

i.e.,

$$P_a = \frac{d_a}{|G|} \sum_g \chi^{(a)*}(g) D(g) = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} I & \\ & & \ddots \end{pmatrix} T \quad (1.4.20)$$

where  $TD(g)T^{-1} = \dots \oplus D^{(b)}(g) \oplus \dots$ .

notice that  $P_a$  is not necessarily a diagonal matrix, unless  $T$  consists of orthogonal column vectors.

- how to use a projection operator:

$$P_a D(g) = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} D^{(a)}(g) & \\ & & \ddots \end{pmatrix} T \quad (1.4.21)$$

and  $\text{tr}(P_a) = m_a d_a$ .

---

- about 1-dim. rep.  $\Gamma_1 = \{D^{(1)}(g), \dots\}$ :

1-dim. rep. must be **irreducible** and **unitary**, so,

$$\chi^{(1)}(g) = D^{(1)}(g) \quad \chi^{(1)}(g^{-1}) = \chi^{(1)*}(g) \quad (1.4.22)$$

so we can conclude that,

$$|\chi^{(1)}(g)| = |D^{(1)}(g)| = 1 \quad (1.4.23)$$

- $\Gamma_a$  is a n-dim. irreducible rep., then  $\Gamma_1 \times \Gamma_a$  is also an irreducible rep..

**proof:**

let  $\Gamma = \Gamma_1 \times \Gamma_a = \{D^{(1)}(g) \otimes D^{(a)}(g), \dots\}$ , then,

$$\frac{1}{|G|} \sum_g |\chi(g)|^2 = \frac{1}{|G|} \sum_g \underbrace{|\chi^{(1)}(g)|^2}_{=1} |\chi^{(a)}(g)|^2 = 1 \quad (1.4.24)$$

# **Part II**

## **General Theory**

# Chapter 2

## Lie groups

### 2.1 Lie groups

- **Lie group**  $G$  is a group and a manifold,
  - group multiplication,  $G \times G \rightarrow G$ , is  $C^\infty$ .
  - inverse,  $G \rightarrow G$ , is  $C^\infty$ .
- **left transformation**,  $L_g : G \rightarrow G, L_g(h) = gh$ .
  - $L_e = \text{id}$ .
  - $L_g L_h = L_{gh}$ .
  - $L_g^{-1} = L_{g^{-1}}$ .
  - $L_g$  is diffeomorphism, i.e. bijective +  $C^\infty$ .
- property of elements near  $e$ , if  $x^i(e) = 0$ , then,

$$x^i(gh) = x^i(g) + x^i(h) \quad (2.1.1)$$

**proof:**

$$\begin{aligned} gh &= \left( e + x^i(g) \frac{\partial g}{\partial x^i} \Big|_e + \cdots \right) \left( e + x^i(h) \frac{\partial g}{\partial x^i} \Big|_e + \cdots \right) \\ &= e + (x^i(g) + x^i(h)) \frac{\partial g}{\partial x^i} \Big|_e + \cdots \end{aligned} \quad (2.1.2)$$

consequently,  $x^i(g^{-1}) = -x^i(g)$ .

- for example, GL,

$$x_{ij}(I + \Delta) = \Delta_{ij} \quad (2.1.3)$$

### 2.2 topological properties

#### 2.2.1 compactness

- compactness is a property that seeks to generalize the notion of a **closed** and **bounded** subset of Euclidean space.

The idea is that a compact space has no "punctures" or "missing endpoints", i.e. it includes all **limiting** values of points.
- **def.:** compact Lie group:
  - 有限个  $\mathbb{R}^n$  中的闭集通过坐标映射到 Lie group 上可以覆盖整个 Lie group.
  - 注意,  $\mathbb{R}$  不是闭集,  $\mathbb{R} \cup \{\pm\infty\}$  才是闭集.

- **Heine-Borel theorem:**

a **matrix** Lie group is compact  $\iff$  it is topologically **closed** as a subset of  $\mathcal{M}_m(\mathbb{C})$  and **bounded**.

compact	noncompact
$O(m), SO(m), U(m), SU(m), Sp(m)$	$SL(m, \mathbb{R})$ (not bounded)

### 2.2.2 connectedness

- a topological space is connected if it is not the union of two **disjoint nonempty open sets**.
- matrix** Lie group is **connected**  $\iff$  it is **path-connected**.

- the **identity component** of  $G$ , denoted by  $G_0$ , is the biggest connected subset containing  $I$ .
  - $G_0$  is a **normal subgroup** of  $G$ .

**proof:**

- \*  $G_0$  is a subgroup.
  - $\forall A, B \in G_0$  there are paths  $A(t), B(t)$  connecting to  $I$ .
  - then  $A(t)B(t)$  is a continuous path connecting  $I$  and  $AB$ .
  - $(A(t))^{-1}$  is...  $I$  and  $A^{-1}$ .
- \*  $G_0$  is invariant.
  - $\forall A \in G_0, B \in G$  there are a path  $BA(t)B^{-1}$  connecting  $BAB^{-1}$  and  $I$ .

### 2.2.3 simple connectedness

- a topological space is **simply connected**  $\iff$  it is **path connected** and every **loop** can be **shrunk continuously into a point**.

**more precisely:**

- for every loop  $A(t), t \in [0, 1]$  in  $G$ ,  $A(0) = A(1)$ .  
 there exist a function  $A(s, t), s, t \in [0, 1]$  such that:
- $A(0, t) = A(t)$  is the original loop.
  - $A(1, t) = A(1, 0)$  is a point.
  - $A(s, 0) = A(s, 1)$  which means  $A(s, t)$  is a loop.

- summary:

matrix Lie groups	compactness	components	simple connectedness
$GL(m, \mathbb{C})$	no	1	no
$GL(m, \mathbb{R})$	no	2	no
$SL(m, \mathbb{C})$	no	1	yes
$SL(m, \mathbb{R})$	no	1	no
$O(m)$	yes	2	
$SO(m)$	yes	1	no
$U(m)$	yes	1	no
$SU(m)$	yes	1	yes
$O(m, 1)$	yes	4	
$SO(m, 1)$	yes	2	$m = 1$ , yes; $m \geq 2$ , no
$E(m)$ (Euclidean group)		2	
$P(m, 1)$ (Poincaré group)		4	

## 2.3 Lie subgroups

- def.:** a **Lie subgroup**  $H$  of a Lie group  $G$  is a subgroup which is also a submanifold.

- **closed subgroup theorem:**  $\{\text{closed subgroups}\} = \{\text{Lie subgroups}\}$ .

**proof:**

first, let's prove that a closed subgroup  $H$  is a Lie subgroup.

– let,

$$\mathfrak{h} = \{A \in \mathfrak{g} \mid \exp(tA) \in H, \forall t \in \mathbb{R}\} \quad (2.3.1)$$

\*  $\mathfrak{h}$  is a subspace of  $\mathfrak{g}$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \exp\left(\frac{A}{n}\right) \exp\left(\frac{B}{n}\right) \right)^n &= \lim_{n \rightarrow \infty} \left( \exp\left(\frac{A}{n} + \frac{B}{n} + O\left(\frac{1}{n^2}\right)\right) \right)^n \\ &= \exp(A + B) \in H \end{aligned} \quad (2.3.2)$$

极限存在要求  $H$  是闭集.

- $W \subset \mathfrak{h}$  is a neighborhood of 0, which is small enough that  $\exp : W \rightarrow H$  is a one-to-one homomorphism (**local diffeomorphism**).
- $\exp^{-1} : \exp[V] \rightarrow V$  with  $V \cap \mathfrak{h} = W$  is a diffeomorphism, so  $(\exp^{-1}, \exp[V], V)$  is a chart on  $G$ , which can be extended by left translation. so,  $H$  is a submanifold.

---

second, let's prove that Lie subgroups are closed.

- 暂时不会证 (?).

# Chapter 3

## Lie algebras

### 3.1 left-invariant vector fields

- vector field  $\bar{A}$  is invariant under push-forward,  $L_{g*} : V_h \rightarrow V_{gh}, \forall h$ ,

$$(L_{g*}\bar{A})|_{gh} = \bar{A}|_{gh} \quad (3.1.1)$$

i.e.,

$$\bar{A}(x^i)|_h = \bar{A}(y^i)|_{gh} \quad (3.1.2)$$

where  $L_g^* y^i = x^i \iff y^i(gh) = x^i(h)$ .

- see appendix B, maps between manifolds.
- the set of all left invariant vector field is denoted by  $\mathfrak{g}$ , and  $\mathfrak{g} \simeq V_e$ .

### 3.2 Lie algebras

- $A \equiv \bar{A}_e$  and  $\bar{A}_g = L_{g*}A, \forall g$ .
- a vector space,  $V$ , along with Lie bracket,  $[\cdot, \cdot] : V \times V \rightarrow V$ , is a **Lie algebra**,
  - $[A, B] = -[B, A]$ .
  - Jacob identity,  $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$ .
- for a Lie group  $G$ , its Lie bracket is the commutator,

$$[\bar{A}, \bar{B}]^a = \bar{A}^b \nabla_b \bar{B}^a - \bar{B}^b \nabla_b \bar{A}^a \quad (3.2.1)$$

$$- L_{g*}[\bar{A}, \bar{B}] = [L_{g*}\bar{A}, L_{g*}\bar{B}] = [\bar{A}, \bar{B}] \in \mathfrak{g}.$$

**proof:**

$$\begin{aligned} L_{g*}[\bar{A}, \bar{B}] &= L_{g*}\left(\frac{\partial}{\partial x^i}\bigg|_h\right)\left(A^j \frac{\partial}{\partial x^j} B^i - B^j \frac{\partial}{\partial x^j} A^i\right)\bigg|_{h,x} \\ &= \left(\frac{\partial}{\partial y^i}\bigg|_{gh}\right)\left(A^j \frac{\partial}{\partial x^j} B^i - B^j \frac{\partial}{\partial x^j} A^i\right)\bigg|_{h,x} \end{aligned} \quad (3.2.2)$$

notice that for left-invariant v. f. as a scalar field,  $(L_g^* A^i|_y)|_h = A^i|_{gh,y}$  and,

$$\begin{aligned} \left(\frac{\partial}{\partial x^j} A^i\right)\bigg|_{h,x} &\equiv \left(\frac{\partial}{\partial x^j}\right)\bigg|_h (L_g^* A^i|_y)\bigg|_h = L_{g*}\left(\frac{\partial}{\partial x^j}\bigg|_h\right)(A^i|_{gh,y}) \\ \implies \left(\frac{\partial}{\partial x^j} A^i\right)\bigg|_{h,x} &= \left(\frac{\partial}{\partial y^j} A^i\right)\bigg|_{gh,y} \end{aligned} \quad (3.2.3)$$

so  $L_{g*}[\bar{A}, \bar{B}] = [\bar{A}, \bar{B}]$ .

- satisfies the Jacob identity.



proof:

$$\begin{aligned}
& [A, [B, C]] + [C, [A, B]] + [B, [C, A]] \\
&= A^c \partial_c (B^b \partial_b C^a - C^b \partial_b B^a) - (B^c \partial_c C^b - C^c \partial_c B^b) \partial_b A^a + \dots \\
&= A^c \partial_c (B^b) \partial_b C^a + A^c B^b \partial_c \partial_b C^a - A^c \partial_c (C^b) \partial_b B^a + A^c C^b \partial_c \partial_b B^a \\
&\quad - B^c \partial_c (C^b) \partial_b A^a + C^c \partial_c (B^b) \partial_b A^a \\
&\quad + (B \partial C \partial A - B \partial A \partial C - C \partial A \partial B + A \partial C \partial B) \\
&\quad + (BC \partial \partial A - BA \partial \partial C) \\
&\quad + (C \partial A \partial B - C \partial B \partial A - A \partial B \partial C + B \partial A \partial C) \\
&\quad + (CA \partial \partial B - CB \partial \partial A) = 0
\end{aligned} \tag{3.2.4}$$

- **def.:** the **Lie algebra direct sum** of two Lie algebras,  $\mathfrak{g}_1, \mathfrak{g}_2$ , is the **vector space direct sum** (i.e.  $\mathfrak{g}_1, \mathfrak{g}_2$  are linearly independent  $\iff \mathfrak{g}_1 \cap \mathfrak{g}_2 = \{0\}$ ),  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , with the Lie bracket defined to be,

$$\begin{aligned}
& [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \\
& [A_1 + A_2, B_1 + B_2] = [A_1, B_1] + [A_2, B_2] \quad \forall A_1, B_1 \in \mathfrak{g}_1, A_2, B_2 \in \mathfrak{g}_2
\end{aligned} \tag{3.2.5}$$

i.e. we define the Lie bracket in the way that  $[\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$ .

### 3.2.1 subalgebras, ideals & simple, solvable, nilpotent Lie algebras

- **def.:** subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  is a subspace, satisfying that  $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ .
  - **def.:** Abelian subalgebra  $\mathfrak{h}$  is a subalgebra, satisfying that  $[A, B] = 0, \forall A, B \in \mathfrak{h}$ .
- **def.:** invariant subalgebra (i.e. **ideal**)  $\mathfrak{h}$  is a subalgebra, satisfying that  $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$ .
  - Abelian ideal.
  - proper invariant subalgebra (also called **proper ideal**) is an ideal that is not  $\mathfrak{g}, \{0\}$ .
  - trivial subalgebras are  $\mathfrak{g}, \{0\}$ .

- Lie algebra decomposes as the direct sum of its ideals,  $\mathfrak{h}_1, \mathfrak{h}_2, \dots$ , i.e.,

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \dots \tag{3.2.6}$$

then  $\oplus$  is called **Lie algebra direct sum**.

proof:

by def.,  $[\mathfrak{h}_i, \mathfrak{h}_j] \subseteq \mathfrak{h}_i \cap \mathfrak{h}_j = \{0\}$ , if  $i \neq j$ .

- **def.:** a Lie algebra **without nontrivial ideal** is **irreducible**.
  - all 1-dim. Lie algebras are irreducible.
- **def.:** a **irreducible** Lie algebra with  $\dim \mathfrak{g} \geq 2$  is **simple**.
  - equivalent **def.:** irreducible non-Abelian Lie algebras are simple.

proof:

all the subspaces of an Abelian Lie algebra is its ideal  $\implies$  Abelian Lie algebras aren't irreducible unless  $\dim = 1$ , so,



- **def.:** a Lie algebra  $\mathfrak{g}$  is **solvable** if  $\mathfrak{g}_i = \{0\}$  for some  $i$ , where,

$$\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}_i] \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{g} \quad (3.2.7)$$

- $\mathfrak{g}_i$  is an ideal in  $\mathfrak{g}_{i-1}$ , but not necessarily an ideal in  $\mathfrak{g}$ .

**proof:**

$\forall A \in \mathfrak{g}_i \subseteq \mathfrak{g}_{i-1}$  and  $\forall B \in \mathfrak{g}_{i-1}$ ,  $[A, B] \in \mathfrak{g}_i$ , which means  $[\mathfrak{g}_i, \mathfrak{g}_{i-1}] \subseteq \mathfrak{g}_i$ .

- **def.:** a Lie algebra  $\mathfrak{g}$  is **nilpotent** if  $\mathfrak{g}^i = \{0\}$  for some  $i$ , where,

$$\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i] \quad \text{and} \quad \mathfrak{g}^0 = \mathfrak{g} \quad (3.2.8)$$

- $\mathfrak{g}^{i+1} \subseteq \mathfrak{g}^i$ .
- $\mathfrak{g}^i$  is an ideal in  $\mathfrak{g}$ .
- nilpotent Lie algebra is solvable.

### 3.2.2 structure constants

- structure constants,

$$[X_i, X_j] = if_{ij}^k X_k \iff [X_i, X_j]^a = if_{bc}^a (X_i)^b (X_j)^c \quad (3.2.9)$$

$$[A_i, A_j] = -if_{ij}^k A_k \iff [A_i, A_j]^a = -f_{bc}^a (A_i)^b (A_j)^c \quad (3.2.10)$$

where  $X_i = -iA_i$  are called the generators.

- if the generators are Hermitian, then the structure constants are real,

$$[X_i, X_j]^\dagger = -if_{ij}^{*k} X_k = [X_j, X_i] = i \underbrace{f_{ji}^k}_{=-f_{ij}^k} X_k \implies f_{ij}^{*k} = f_{ij}^k \quad (3.2.11)$$

# Chapter 4

## exponential maps

### 4.1 one-parameter subgroups

- a  $C^\infty$  (Lie group) homomorphism  $\gamma : \mathbb{R} \rightarrow G$ , with  $\gamma(s)\gamma(t) = \gamma(s+t)$ .
- $\{\gamma(s) | s \in \mathbb{R}\}$  is an **integral curve** (passing through  $e$ ) of a **left-invariant vector field**.
  - the integral curve of a left-invariant vector field is complete, i.e. it's homomorphism to  $\mathbb{R}$ .

**proof:**

notation:  $\frac{d}{dt}\gamma(t) \equiv \frac{\partial}{\partial t} (\equiv \frac{dx^i(\mu(t))}{dt} \frac{\partial}{\partial x^i})$   
 let  $\mu : (-\epsilon, \epsilon) \rightarrow G$  be an integral curve of  $\bar{A}$ , with  $\mu(0) = e$ , then,

$$\left. \frac{d}{dt} \right|_s \mu(t) = A_{\mu(s)} = L_{\mu(s)*}(A_e) = L_{\mu(s)*} \left. \frac{d}{dt} \right|_0 \mu(t) = \left. \frac{d}{dt} \right|_{t=0} (\mu(s)\mu(t)) \quad (4.1.1)$$

**calculation:**

$$\left. \frac{dx^i(\mu(t))}{dt} \right|_s = \left( L_{\mu(s)*} \left. \frac{d}{dt} \right|_0 \mu(t) \right) x^i \Big|_{\mu(s)} = \left( \left. \frac{d}{dt} \right|_0 \mu(t) \right) y^i \Big|_e \quad (4.1.2)$$

where  $y^i \Big|_g \equiv L_{\mu(s)*} x^i \Big|_g = x^i \Big|_{\mu(s)g}$  so,

$$\left( \left. \frac{d}{dt} \right|_0 \mu(t) \right) y^i \Big|_e = \left. \frac{dy^i(\mu(t))}{dt} \right|_e = \left. \frac{dx^i(\mu(s)\mu(t))}{dt} \right|_{t=0} \quad (4.1.3)$$

so, as we can see,  $\nu : (-\epsilon + s, \epsilon + s) \rightarrow G, t \mapsto \mu(s)\mu(t-s)$  is also an integral curve of  $\bar{A}$ , with at least one intersection with  $\mu$ ,  $\nu(s) = \mu(s)$ .

since a vector field only has one integral curve through a fixed point,

**proof:**

for a vector field  $A$ , the integral curve  $\mu$  through point  $p$  must satisfy,

$$\left. \frac{dx^i(\mu(t))}{dt} \right|_s = A^i \Big|_{\mu(s)} \quad (4.1.4)$$

which is a linear differential equation of order one, consequently, the solution can be determined by  $x^i(\mu(t)) = \text{Const.}$

we can conclude that  $\mu$  and  $\nu$  is all part of one complete integral curve through  $e$ ,  $\gamma : \mathbb{R} \rightarrow G$ .

- the integral curve of  $\bar{A}$  through  $e$  is a one-parameter subgroup.

**proof:**

we have already proved that  $\nu(s+t) = \mu(s)\mu(t)$  and  $\mu = \nu = \gamma$ .  
so  $\gamma(s+t) = \gamma(s)\gamma(t)$ .

- the tangent vector of  $\gamma$  is left-invariant.

**proof:**

$$\left( L_{\gamma(t_2)*} \frac{d}{dt} \Big|_{t_1} \gamma(t) \right) x^i \Big|_{\gamma(t_2+t_1)} = \frac{dx^i(\gamma(t_2+t))}{dt} \Big|_{t_1} = \left( \frac{d}{dt} \gamma(t) \right) x^i \Big|_{\gamma(t_2+t_1)} \quad (4.1.5)$$

- a useful lemma: for a curve  $\gamma$  on manifold  $M_1$ , and a map  $\psi : M_1 \rightarrow M_2$ , then,

$$\psi_* \left( \frac{d}{dt} \Big|_{p \in M_1} \gamma \right) = \frac{d}{dt} \Big|_{\psi(p) \in M_2} \psi \circ \gamma \quad (4.1.6)$$

the proof is in appendix [B.1.4](#).

## 4.2 exponential maps

- **def.:** exp. map on a **Riemann manifold**,  $\exp_p : V_p$  (or its subspace)  $\rightarrow M$ .
  - $\exp_p(v) = \gamma(1)$ , where  $\gamma$  is the geodesic determined by  $v$  and  $p$ .
- **def.:** exp. map on a **Lie group**,  $\exp : V_e \rightarrow G$ .
  - $\exp(A) = \gamma(1)$  where  $\gamma$  is the one-para. subgroup determined by  $\bar{A}$ .
  - def. for physicists:  $\exp : \mathfrak{g} \rightarrow G$ , with  $\exp(iX) = \exp(A) = \gamma(1)$ .
- **theorem:** for **compact** Lie group, the exponential map,  $\exp : V_e \rightarrow G$ , is **onto**.

### 4.2.1 matrix exponential and logarithm

- properties of exp. function of matrices (in general linear group):
  - $(e^A)^\dagger = e^{A^\dagger}$ .
  - if  $\det e^A \neq 0$ , then  $(e^A)^{-1} = e^{-A}$ .
  - $\det e^A = e^{\text{tr} A}$ .

**proof:**

\* if  $A$  is diagonalizable,

diagonalize  $A$  by  $T$ ,  $TAT^{-1} = D = \text{diag}(\lambda_1, \dots, \lambda_m)$ , then,

$$\det e^A = \det(Te^AT^{-1}) = \det e^D = e^{\lambda_1 + \dots + \lambda_m} = e^{\text{tr} A} \quad (4.2.1)$$

\* otherwise, it is still can be proved as follow,

$$\frac{d}{dt} \Big|_t \det(e^{tA}) = \frac{d}{ds} \Big|_{s=0} \det(e^{(s+t)A}) = \det(e^{tA}) \frac{d}{ds} \Big|_{s=0} \det(e^{sA}) \quad (4.2.2)$$

and,

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \det(e^{sA}) &= \frac{d}{ds} \Big|_{s=0} \det(I + sA) \\ &= \frac{d}{ds} \Big|_{s=0} \epsilon_{ij\dots k} (\delta_1^i + sA_1^i) \dots (\delta_m^k + sA_m^k) \\ &= \epsilon_{i2\dots m} A_1^i + \dots + \epsilon_{12\dots k} A_m^k = \text{tr} A \end{aligned} \quad (4.2.3)$$

so we have,

$$\begin{cases} \frac{1}{\det(e^{tA})} \frac{d}{dt} \Big|_t \det(e^{tA}) = \text{tr} A \\ \det(e^{tA}) \Big|_{t=0} = 1 \end{cases} \implies \det(e^{tA}) = e^{t \text{tr} A} \quad (4.2.4)$$

– Baker-Campbell-Hausdorff formula,

$$e^A e^B = \exp \left( A + B + \frac{1}{2} [A, B] + \frac{1}{12} ([A, [A, B]] + [B, [B, A]]) + \cdots \right) \quad (4.2.5)$$

- the Hilbert-Schmidt norm of  $A \in \mathcal{M}_m(\mathbb{C})$  is,

$$\|A\| = \left( \sum_{i,j=1}^m |A_{ij}|^2 \right)^{1/2} \quad (4.2.6)$$

- matrix logarithm is,

$$\ln M = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(M - I)^n}{n} \quad (4.2.7)$$

where  $M$  is a complex matrix with  $\|M - I\| < 1$ .

- $\forall M$  with  $\|M - I\| < 1$ ,  $e^{\ln M} = M$ .
- $\forall A$  with  $\|A\| < \ln 2$  then  $\|e^A - I\| < 1$  and  $\ln e^A = A$ .

- for a **connected** Lie group  $G$ , every element  $g \in G$  can be written in the form,

$$g = \exp(A_1) \exp(A_2) \cdots \exp(A_N) \quad (4.2.8)$$

for some  $A_1, A_2, \dots, A_N \in \mathfrak{g}$ .

**proof:**

曲线  $\gamma : [0, 1] \rightarrow G, \gamma(0) = I, \gamma(1) = g$ .  
选取  $N$  足够大, 使得  $\gamma^{-1}(\frac{i-1}{N})\gamma(\frac{i}{N})$  在  $I$  的邻域, 那么, 存在  $A_i \in \mathfrak{g}$  使得,

$$\gamma^{-1}(\frac{i-1}{N})\gamma(\frac{i}{N}) = \exp(A_i) \quad (4.2.9)$$

所以,

$$g = \gamma^{-1}(0)\gamma(1) = \exp(A_1) \cdots \exp(A_N) \quad (4.2.10)$$

错误的推断:

combined with BCH formula,  $\exp : \mathfrak{g} \rightarrow G$  is onto for connected Lie groups, i.e.  $G \neq \exp[\mathfrak{g}]$ .

- onto 仅对 **compact connected** Lie groups 成立,
- 原因: BCH 公式中的级数展开可能不存在.

## 4.3 Baker-Campbell-Hausdorff formula

### 4.3.1 the Campbell's identity

- $\text{Ad}_{\exp(A)} = e^{\text{ad} A} : V_e \rightarrow V_e$ .

**proof: (maybe not very rigorously)**

consider,

$$B(s) = \text{Ad}_{\exp(sA)}(B) = \frac{d}{dt} \Big|_0 \exp(sA) \exp(tB) \exp(-sA) \quad (4.3.1)$$

the derivative of  $B(s)$  is,

$$\frac{dB(s)}{ds} = \lim_{\Delta s \rightarrow 0} \frac{\text{numerator}}{\Delta s} = [A, \text{Ad}_{\exp(sA)}(B)] = \text{ad}_A B(s) \quad (4.3.2)$$

where the numerator is:

$$\begin{aligned} & \text{numerator} \\ &= \left. \frac{d}{dt} \right|_0 \exp(sA)(1 + \Delta sA) \exp(tB) \exp(-sA)(1 - \Delta sA) \\ & \quad - \left. \frac{d}{dt} \right|_0 \exp(sA) \exp(tB) \exp(-sA) \\ &= \Delta s [A, \text{Ad}_{\exp(sA)}(B)] \end{aligned} \quad (4.3.3)$$

so, the  $n$ th derivative is  $\frac{d^n}{ds^n} B(s) = (\text{ad}_A)^n B(s)$ , then naturally,

$$B(s) = e^{\text{ad}_A} B \quad (4.3.4)$$

### 4.3.2 BCH formula

- theorem 1 (Campbell's identity in the case of  $\mathfrak{gl}(m)$ ):

$$e^A B e^{-A} = e^{\text{ad}_A} B \quad (4.3.5)$$

**proof:**

consider  $F(t) = e^{tA} B e^{-tA}$ , so  $F(0) = B$ , and,

$$\frac{d}{dt} F(t) = [A, F(t)] = \text{ad}_A F(t) \implies \frac{d^n}{dt^n} F(t) = (\text{ad}_A)^n F(t) \quad (4.3.6)$$

so it is clear that  $F(t) = e^{\text{ad}_A} B$ .

- theorem 2:

$$e^{A(t)} \frac{d}{dt} e^{-A(t)} = -f(\text{ad}_A) \frac{dA(t)}{dt} \quad (4.3.7)$$

where  $f(z) = \frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$ .

**proof:**

consider  $F(s, t) = e^{sA(t)} \frac{d}{dt} e^{-sA(t)}$ , with  $F(0, t) = 0$ , and,

$$\begin{aligned} \frac{d}{ds} F(s, t) &= A(t) F(s, t) - e^{sA(t)} \frac{d}{dt} (A(t) e^{-sA(t)}) \\ &= -e^{sA(t)} \frac{dA(t)}{dt} e^{-sA(t)} \\ &= -e^{\text{ad}(sA(t))} \frac{dA(t)}{dt} \end{aligned} \quad (4.3.8)$$

and the  $n$ th derivative is,

$$\frac{d^n}{ds^n} F(s, t) = \text{ad}^{n-1}(A(t)) \frac{d}{ds} F(s, t) \quad (4.3.9)$$

when  $s = 0$ ,  $\left. \frac{d^n}{ds^n} \right|_{s=0} F(s, t) = -\text{ad}^{n-1}(A(t)) \frac{dA(t)}{dt}$ , so,

$$F(s = 1, t) = - \sum_{n=1}^{\infty} \frac{\text{ad}^{n-1}(A(t))}{n!} \frac{dA(t)}{dt} \quad (4.3.10)$$

(the 0th order term is 0)

- theorem 3:

$$\frac{d}{dt}e^{-A(t)} = - \int_0^1 e^{-sA(t)} \frac{dA(t)}{dt} e^{-(1-s)A(t)} ds \quad (4.3.11)$$

**proof:**

consider the following equation,

$$e^{-A} - e^{-B} = \int_0^1 e^{-sA} (B - A) e^{-(1-s)B} ds \quad (4.3.12)$$

**proof:**

consider the following equation,

$$e^{-sA} (B - A) e^{-(1-s)B} = \frac{d}{ds} \left( e^{-sA} e^{-(1-s)B} \right) \quad (4.3.13)$$

integrate both side of the equation,

$$\int_0^1 \dots ds = e^{-A} - e^{-B} \quad (4.3.14)$$

take  $A = A(t)$ ,  $B = A(t - \Delta t)$ , with  $\Delta t \rightarrow 0$ , then,

$$\frac{d}{dt}e^{-A(t)} = - \int_0^1 e^{-sA(t)} \frac{dA(t)}{dt} e^{-(1-s)A(t)} ds \quad (4.3.15)$$

- theorem 3 is equivalent to theorem 2.

**calculation:**

$$\begin{aligned} e^{A(t)} \frac{d}{dt} e^{-A(t)} &= - \int_0^1 e^{(1-s)A(t)} \frac{dA(t)}{dt} e^{-(1-s)A(t)} ds \\ &= - \int_0^1 \underbrace{e^{\text{ad}((1-s)A(t))}}_{=e^{(1-s)\text{ad}_{A(t)}}} \frac{dA(t)}{dt} ds \\ &= -f(\text{ad}_{A(t)}) \frac{dA(t)}{dt} \end{aligned} \quad (4.3.16)$$

where  $f(z)$  is defined in theorem 2.

- the Baker-Campbell-Hausdorff formula is,

$$\begin{aligned} e^A e^B &= \exp \left( B + \left( \int_0^1 g(e^{t\text{ad}_A} e^{\text{ad}_B}) dt \right) A \right) \\ &= \exp \left( A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] + \dots \right) \end{aligned} \quad (4.3.17)$$

where  $g(z) = \frac{\ln z}{z-1} = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n+1}$ , for  $|z-1| < 1$ .

**proof:**

consider  $e^{C(t)} = e^{tA} e^B$ , then,

$$e^{\text{ad}_{C(t)}} = e^{t\text{ad}_A} e^{\text{ad}_B} \quad (4.3.18)$$

**proof:**

consider the following equation,

$$e^{\text{ad}_{C(t)}} W = e^{C(t)} W e^{-C(t)}$$

$$\begin{aligned}
&= e^{tA} e^B W e^{-B} e^{-tA} \\
&= e^{tA} e^{\text{ad}_B} W e^{-tA} \\
&= e^{t \text{ad}_A} e^{\text{ad}_B} W
\end{aligned} \tag{4.3.19}$$

---

then, let's consider, (notice that  $\text{ad}_A A = 0$ ),

$$\begin{aligned}
e^{C(t)} \frac{d}{dt} e^{-C(t)} &= -f(\text{ad}_{C(t)}) \frac{dC(t)}{dt} \\
&= e^{tA} e^B \frac{d}{dt} e^{-B} e^{-tA} \\
&= e^{tA} \frac{d}{dt} e^{-tA} \\
&= -f(t \text{ad}_A) A = -A
\end{aligned} \tag{4.3.20}$$

$$\implies f(\text{ad}_{C(t)}) \frac{dC(t)}{dt} = A \tag{4.3.21}$$

notice that  $g(e^z) = 1/f(z)$ , so we have,

$$\frac{dC(t)}{dt} = g(e^{\text{ad}_{C(t)}}) A \implies C(1) - \underbrace{C(0)}_{=B} = \left( \int_0^1 g(e^{t \text{ad}_A} e^{\text{ad}_B}) dt \right) A \tag{4.3.22}$$



# Chapter 5

## basic representation theory

### 5.1 Lie group and Lie algebra homomorphisms

- $\Phi : G \rightarrow H$  is a **Lie group homomorphism**, then there exists a unique real-linear map  $\phi = \Phi_* : \mathfrak{g} \rightarrow \mathfrak{h}$  s.t.,

$$\Phi \circ \exp(A) = \exp(\phi A) \quad (5.1.1)$$

$\phi$  has the following properties:

1.  $\phi \text{Ad}_g(A) = \text{Ad}_{\Phi(g)}(A), \forall A, g,$
2.  $\phi$  is **Lie algebra homomorphism**,
3.  $\phi(A) = \left. \frac{d}{dt} \right|_0 \Phi \circ \exp(tA).$

**proof:**

let's prove the 3rd identity first,

$$\begin{aligned} \left( \Phi_* \left. \frac{d}{dt} \right|_s \gamma(t) \right) y^i &= \left( \left. \frac{d}{dt} \right|_s \gamma(t) \right) \Phi^* y^i = \left. \frac{d \Phi^* y^i(\gamma(t))}{dt} \right|_s = \left. \frac{dy^i(\Phi \gamma(t))}{dt} \right|_s \\ \implies \Phi_* \circ L_{\exp(sA)*} A &= \left. \frac{d}{dt} \right|_s \Phi \exp(tA) \end{aligned} \quad (5.1.2)$$

and,

$$\begin{cases} L_{\Phi(g)*} \circ \Phi_* A = (L_{\Phi(g)} \circ \Phi)_* A \\ L_{\Phi(g)} \circ \Phi = \Phi \circ L_g \end{cases} \implies L_{\Phi(g)*} \circ \Phi_* A = \Phi_* \circ L_{g*} A \quad (5.1.3)$$

so,

$$\left. \frac{d}{dt} \right|_s \Phi \exp(tA) = L_{\Phi \exp(sA)*} \circ \Phi_* A \implies \exp(\Phi_* A) = \Phi \exp(A) \quad (5.1.4)$$

the 1st identity is easy to prove,

$$\text{Ad}_g \equiv I_{g*} \implies \begin{cases} \Phi_* \circ I_{g*} = (\Phi \circ I_g)_* \\ \Phi \circ I_g = I_{\Phi(g)} \circ \Phi \end{cases} \implies \dots \quad (5.1.5)$$

now let's prove the 2nd identity,

$$L_{\Phi(g)*} \circ \Phi_* A = \Phi_* \circ L_{g*} A \implies (\Phi_* A)_{\Phi(g)} = \Phi_* A_g \quad (5.1.6)$$

$$\implies ((\Phi_* A)_{\Phi(g)})^i \Phi_* \frac{\partial}{\partial x^i} = (A_g)^i \Phi_* \frac{\partial}{\partial x^i} \quad (5.1.7)$$

$$\implies A^i \Big|_g = \Phi^* ((\Phi_* A)^i \Big|_{\Phi(g)}) \quad (5.1.8)$$

where  $A^i$  and  $(\Phi_* A)^i$  are treated as functions on  $G$  and  $H$ .

so,

$$(\Phi_* [A, B]_g)^i \Phi_* \frac{\partial}{\partial x^i} = \left( (A_g)^j \frac{\partial}{\partial x^j} (B_g)^i - \dots \right) \Phi_* \frac{\partial}{\partial x^i} \quad (5.1.9)$$

and,

$$([\Phi_*A, \Phi_*B]_{\Phi(g)})^i \Phi_* \frac{\partial}{\partial x^i} = \left( (\Phi_*A)^a \nabla_a (\Phi_*B)^i - \dots \right) \Big|_{\Phi(g)} \Phi_* \frac{\partial}{\partial x^i} \quad (5.1.10)$$

where,

$$\begin{aligned} (\Phi_*A)^a \nabla_a (\Phi_*B)^i \Big|_{\Phi(g)} &= (A_g)^i \Phi_* \frac{\partial}{\partial x^i} (\Phi_*B)^i \\ &= (A_g)^i \frac{\partial}{\partial x^i} \Big|_{\Phi(g)} \Phi^* (\Phi_*B)^i \\ &= (A_g)^i \frac{\partial}{\partial x^i} \Big|_g B^i \end{aligned} \quad (5.1.11)$$

so, we proved that  $\Phi_*[A, B] = [\Phi_*A, \Phi_*B]$ .

- for a Lie group homomorphism  $\Phi : G \rightarrow H$  and  $\phi = \Phi_*$ ,

$$\text{Lie}(\ker \Phi) = \ker \phi \quad (5.1.12)$$

**proof:**

–  $\ker \Phi = \{g \in G \mid \Phi(g) = I\}$  is a **closed normal subgroup** of  $G$ .

\*  $G(\ker \Phi)G^{-1} \subseteq \ker \Phi$ .

\*  $\{I\}$  is a closed subgroup, and  $\Phi$  is continuous.

–  $\text{Lie}(\ker \Phi) \subseteq \ker \phi$ .

for all  $A \in \text{Lie}(\ker \Phi)$ ,

$$\Phi \exp(tA) \in \Phi(\ker \Phi) = \{I\} \implies \phi A = \frac{d}{dt} \Big|_0 \Phi \exp(tA) = 0 \quad (5.1.13)$$

so,  $A \in \ker \phi$ .

–  $\text{Lie}(\ker \Phi) \supseteq \ker \phi$ .

for all  $A \in \ker \phi$ ,

$$\exp(\phi A) = \Phi \exp(A) = I \implies \exp(A) \in \ker \Phi \quad (5.1.14)$$

so,  $A \in \text{Lie}(\ker \Phi)$ .

### 5.1.1 simply connected Lie groups

- Lie algebra homomorphism  $\implies$  Lie group homomorphism, when  $G$  is **simply connected**.

$\phi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a Lie algebra homomorphism, (if  $G$  is simply connected) then there **exist** a **unique** Lie group homomorphism  $\Phi : G \rightarrow H$  s.t.  $\Phi(\exp(A)) = \exp(\phi A)$  and  $\phi = \Phi_*$ .

**proof:**

$G$  is **connected**, so, for all  $g \in G$  there exists a path  $g(t)$  s.t.  $g(0) = I, g(1) = g$   
 $N$  is large enough that,

$$g^{-1} \left( \frac{i-1}{N} \right) g \left( \frac{i}{N} \right) \in U \quad (5.1.15)$$

where  $U \subset G$  is a neighborhood of  $I$  s.t. there exists an isomorphism,

$$\begin{aligned} \ln : U &\rightarrow \ln[U] \subset \mathfrak{g} \\ g = \exp(A) &\mapsto A, \forall g \in U \end{aligned} \quad (5.1.16)$$

which implies that there exists a unique local homomorphism,

$$\begin{aligned} f : U &\rightarrow H \\ g &\mapsto \exp(\phi \ln g), \forall g \in U \end{aligned} \quad (5.1.17)$$

where,

$$\begin{aligned}
f(g_1 g_2) &= \exp(\phi \ln(\exp(A_1) \exp(A_2))) \\
&= \exp\left(\phi \ln \exp\left(A + B + \frac{1}{2}[A, B] + \frac{1}{12} \cdots\right)\right) \\
&= \exp(\phi A) \exp(\phi B) \\
&= f(g_1) f(g_2)
\end{aligned} \tag{5.1.18}$$

so, there **exists** a homomorphism,

$$\begin{aligned}
\Phi : G &\rightarrow H \\
g &\mapsto f\left(g^{-1}(0)g\left(\frac{1}{N}\right)\right) \cdots f\left(g^{-1}\left(\frac{N-1}{N}\right)g(1)\right), \forall g \in G
\end{aligned} \tag{5.1.19}$$

finally, the **uniqueness**:

$\Phi$  is independent from the choice of path  $g(t)$  and the choice of partition  $0 = t_0 < t_1 < \cdots t_N = 1$ .

– independence of the partition:

for any good partition (partition that guarantees  $g^{-1}(t_{i-1})g(t_i) \in U$ ) insert  $s$  between  $t_{i-1}$  and  $t_i$ , since  $f$  is a local homomorphism,

$$f(g^{-1}(t_{i-1})g(s))f(g^{-1}(s)g(t_i)) = f(g^{-1}(t_{i-1})g(t_i)) \tag{5.1.20}$$

– independence of the path:

since  $G$  is **simply connected**, there exists a continuous map,

$$\begin{aligned}
g : [0, 1] \times [0, 1] &\rightarrow G \\
g(s, t) &= g_s(t) \\
g(s, 0) &= I, g(s, 1) = g
\end{aligned} \tag{5.1.21}$$

and choose a good partition that  $g_{s_{j-1}}^{-1}(t)g_{s_j}(t) \in U$ , so,

$$\begin{cases} \Phi_{s_{j-1}}(g) = \cdots f(g_{s_{j-1}}^{-1}(t_{i-1})g_{s_{j-1}}(t_i)) \cdots \\ \Phi_{s_j}(g) = \cdots f(g_{s_j}^{-1}(t_{i-1})g_{s_{j-1}}(t_{i-1})g_{s_{j-1}}^{-1}(t_{i-1})g_{s_{j-1}}(t_i)g_{s_{j-1}}^{-1}(t_i)g_{s_j}(t_i)) \cdots \end{cases} \tag{5.1.22}$$

the red terms will be canceled due to  $f$  is homomorphism.

so  $\Phi_{s_{j-1}} = \Phi_{s_j}$  which implies that  $\Phi_0 = \Phi_1$ .

显然, 根据上述选择,

$$\begin{cases} \Phi \circ \exp(A) = \exp(\phi A) \\ \Phi(g) = \exp(\phi A_1) \cdots \exp(\phi A_N) \end{cases} \tag{5.1.23}$$

now, let's prove  $\phi = \Phi_*$ .

consider,

$$\exp(\Phi_* A) = \exp(\phi A) \tag{5.1.24}$$

and if  $A$  is close to 0 enough,  $\exp$  is one-to-one, moreover,  $\Phi_*$  and  $\phi$  is linear, so  $\phi = \Phi_*$ .

- for 2 **simply connected** Lie groups  $G, H$ , there exists a Lie algebra **isomorphism**  $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ , then  $G, H$  are **isomorphic** to each other.

换句话说: simply connected Lie groups are determined by their Lie algebra.

– but, exponential maps,  $\exp : \mathfrak{g} \rightarrow G$ , are **not** one-to-one even for simply connected Lie groups.

e.g. in  $SU(2)$ ,  $\exp(4\pi i J_3) = I$ .

**proof:**

let  $\Phi, \Psi$  correspond to  $\phi, \phi^{-1}$  respectively, then,

$$\Phi \circ \Psi(\exp(A_1) \cdots \exp(A_N)) = \exp(\phi \circ \phi^{-1} A_1) \cdots \exp(\phi \circ \phi^{-1} A_N) \quad (5.1.25)$$

which means  $\Phi \circ \Psi = I$  similarly,  $\Psi \circ \Phi = I$ .

so  $\Phi$  is a reversible homomorphism, i.e. an isomorphism.

- for a simply connected Lie group  $G$ , its Lie algebra  $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ , then, there exist 2 **closed, simply connected** subgroups  $H_1, H_2$  corresponded to  $\mathfrak{h}_1, \mathfrak{h}_2$  and  $G \simeq H_1 \times H_2$ .

**proof:**

consider the projection map  $\phi_1 \in \text{End}(\mathfrak{g})$ , s.t.  $\phi_1(A + B) = A, \forall A \in \mathfrak{h}_1, B \in \mathfrak{h}_2$ .

- since  $G$  is simply connected,  $\Phi_1$  is the corresponding Lie group homomorphism.
- according to (5.1.12),  $\ker \phi_1 = \mathfrak{h}_2 = \text{Lie}(\ker \Phi_1)$ .
- let  $H_2$  be the identity component of  $\ker \Phi_1$ , thus  $H_2$  is a **closed connected** Lie subgroup.
- construct  $H_1$  in a similar way.

---

$\phi_1$  is the identity on  $\mathfrak{h}_1$ , so  $\Phi_1$  is the identity on  $H_1$ .

- consider a loop  $h(t)$  on  $H_1$ .
- there is a way to shrink  $h(t)$  into a point on  $G$ , say  $g(s, t)$  with  $g(0, t) = h(t)$  and  $g(1, t)$  is a point.
- define  $h(s, t) = \Phi_1(g(s, t))$ , then  $h(0, t) = h(t)$  and  $h(1, t)$  is a point.

so,  $H_1$  is **simply connected**.

---

finally, let's prove  $G \simeq H_1 \times H_2$ .

- since  $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ ,  $[\mathfrak{h}_1, \mathfrak{h}_2] = \{0\}$ , so  $h_1 h_2 = h_2 h_1, \forall h_1 \in H_1, h_2 \in H_2$ .
- $\Psi : H_1 \times H_2 \rightarrow G, (h_1, h_2) \mapsto h_1 h_2$  is a Lie group homomorphism.  
(we don't know  $H_1 \times H_2$  is simply connected yet)
- $\psi = \Psi_* : \mathfrak{h}_1 \oplus \mathfrak{h}_2 \rightarrow \mathfrak{g}$  is the original isomorphism.

$$\exp(\psi(A + B)) = \Psi \circ \exp(A + B) = \exp(A + B) \implies \psi(A + B) = A + B \quad (5.1.26)$$

- so the homomorphism  $\Psi' : G \rightarrow H_1 \times H_2$  associated with  $\psi^{-1}$  is an isomorphism.

### 5.1.2 universal covers

- $G$  is a **connected** Lie group,  $H$  is a **simply connected** Lie group with  $\mathfrak{g} \simeq \mathfrak{h}$ .

then,  $H$  is the **universal cover** of  $G$  and the homomorphism  $\Phi : H \rightarrow G$  associated to the isomorphism  $\phi : \mathfrak{h} \rightarrow \mathfrak{g}$  is called the **covering map**.

- 
- the universal cover of  $\text{SO}(3)$  is  $\text{SU}(2)$ , and  $\ker \Phi = \{\pm I\}$ .
  - the universal cover of  $\text{SO}(n \geq 3)$  is  $\text{Spin}(n)$  and may be constructed as a certain group of invertible elements in the **Clifford algebra** over  $\mathbb{R}^n$ .
    - the covering map is two-to-one.
    - and  $\text{Spin}(4) \simeq \text{SU}(2) \times \text{SU}(2)$ .

## 5.2 basic representation theory

- **def.:** a **finite-dimensional representation** of a Lie group  $G$  (or a Lie algebra  $\mathfrak{g}$ ) is a **Lie group** (or a Lie algebra) **homomorphism**,

$$\begin{cases} \Pi : G \rightarrow \mathrm{GL}(V) \\ \pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \end{cases} \quad (5.2.1)$$

where  $\mathrm{GL}(V)$  is the group of invertible linear transformations of  $V$  and  $\mathfrak{gl}(V) = \mathrm{End}(V)$  is the space of all linear operators from  $V$  to itself with Lie bracket  $[A, B] = AB - BA$ .

- for a finite-dimensional representation of  $G$ ,

$$\pi(A) = \left. \frac{d}{dt} \right|_0 \Pi(e^{tA}) \quad (5.2.2)$$

then  $\Pi(\exp(A)) = e^{\pi(A)}$  and  $\pi$  is the representation of  $\mathfrak{g}$  on the same vector space.

- subspace  $W \subset V$  is **invariant** if  $\Pi(g)[W] \subseteq W, \forall g \in G$ .
- **def.:** a representation without nontrivial invariant subspaces ( $\{0\}, V$ ) is called **irreducible**.  
对 Lie algebra 的 irreducible rep. 的定义是一样的.
- $\Pi, \pi$  are associated representations of **connected** Lie group  $G$  and its Lie algebra  $\mathfrak{g}$ , then:
  - $\Pi$  is **irreducible**  $\iff \pi$  is **irreducible**.

**proof:**

\*  $\Pi$  is irreducible  $\implies \pi$  is irreducible.

设  $W \subseteq V$  是  $\pi$  的不变子空间, 那么  $\forall g$ ,

$$\Pi(g)[W] = e^{\pi(A_1)} \dots e^{\pi(A_N)}[W] \subseteq W \quad (5.2.3)$$

(其中用到了 (4.2.8) 式), 而  $\Pi$  是不可约表示, 所以  $W = \{0\}$  or  $V$

\*  $\Pi$  is irreducible  $\iff \pi$  is irreducible.

设  $W \subseteq V$  是  $\Pi$  的不变子空间, 那么  $\forall A$ ,

$$\pi(A)[W] = \left. \frac{d}{dt} \right|_0 \Pi(\exp(tA))[W] \subseteq W \quad (5.2.4)$$

所以...

- $\Pi_1, \Pi_2$  are **isomorphic**  $\iff \pi_1, \pi_2$  are **isomorphic**.
- $\pi$  is a **irreducible** rep. of  $\mathfrak{g}_{\mathbb{C}} \iff \pi$  is a (complex) **irreducible** rep. of  $\mathfrak{g}$ .  
where the rep. of  $\mathfrak{g}_{\mathbb{C}}$  is  $\pi(A + iB) = \pi(A) + i\pi(B)$  which is the unique extension of the rep. of  $\mathfrak{g}$ ,  $\pi$ .

### 5.2.1 new representations from old

- three ways to obtain new rep. from old:
  1. direct sums,
  2. tensor products,
  3. dual representations.

#### direct sums

- **def.:** the direct sum of  $\Pi_1, \dots, \Pi_m$  is a rep. of  $G$  on  $V_1 \oplus \dots \oplus V_m$ , defined by,

$$\Pi_1 \oplus \dots \oplus \Pi_m(g)(v_1, \dots, v_m) = (\Pi_1(g)v_1, \dots, \Pi_m(g)v_m) \quad (5.2.5)$$

对 Lie algebra rep.  $\pi_1, \dots, \pi_m$  的直和的定义是一样的.

## tensor products

- $\Pi_1, \Pi_2$  are rep. of  $G, H$  respectively. then, the tensor product rep.  $\Pi_1 \otimes \Pi_2$  of  $G \times H$  is defined to be,

$$(\Pi_1 \otimes \Pi_2)(g, h) = \Pi_1(g) \otimes \Pi_2(h) \quad (5.2.6)$$

- the tensor product rep.  $\pi_1 \otimes \pi_2$  of  $\mathfrak{g} \oplus \mathfrak{h}$  is,

$$(\pi_1 \otimes \pi_2)(A, B) = \pi_1(A) \otimes I + I \otimes \pi_2(B) \quad (5.2.7)$$

**proof:**

令  $\pi_1 : \mathfrak{g} \rightarrow \text{End}(U), \pi_2 : \mathfrak{h} \rightarrow \text{End}(V)$ , 那么,

$$\begin{aligned} (\pi_1 \otimes \pi_2)(A, B)(u \otimes v) &= \left( \frac{d}{dt} \Big|_0 (\Pi_1 \otimes \Pi_2)(\exp(tA), \exp(tB)) \right) (u \otimes v) \\ &= \frac{d}{dt} \Big|_0 \underbrace{\Pi_1(\exp(tA))u}_{=u(t)} \otimes \underbrace{\Pi_2(\exp(tB))v}_{=v(t)} \end{aligned} \quad (5.2.8)$$

其中,  $u(t), v(t)$  是  $U, V$  中的两条  $C^\infty$  的曲线,

$$(u + du) \otimes (v + dv) - u \otimes v = du \otimes v + u \otimes dv \quad (5.2.9)$$

代入, 所以,

$$(\pi_1 \otimes \pi_2)(A, B)(u \otimes v) = \pi_1(A)u \otimes v + u \otimes \pi_2(B)v \quad (5.2.10)$$

## dual representations

- 对于  $\Pi : G \rightarrow \text{End}(V)$ , dual rep. 就是  $\Pi^\dagger : G \rightarrow \text{End}(V^*)$ , 其中  $V^*$  是  $V$  的对偶空间.

### 5.2.2 complete reducibility

- 参见有限群中的定义 (group 和 Lie algebra 的定义都一样).
- a group or Lie algebra is said to have the **complete reducibility property** if every finite-dim. rep. of it is completely reducible.

- **unitary** rep. of  $G, \mathfrak{g}$  is **completely reducible**.

notice, the 'unitary' (skew self-adjoint) rep. of  $\mathfrak{g}$  is  $\pi^\dagger(A) = -\pi(A)$

证明参见有限群.

- **compact** Lie groups have the **complete reducibility property**.

**proof:**

for an  $n$ -dim. Lie group  $G$ ,

$$\epsilon = A^1 \wedge \cdots \wedge A^n \quad (5.2.11)$$

is a **right-invariant  $n$ -form** composed of the dual vectors of a basis of  $\mathfrak{g}$ .

if  $G$  is **compact**, we can integrate any smooth function over all  $G$ , denoted by,

$$\int_G f(g) \epsilon(g) \quad (5.2.12)$$

and, since  $\epsilon$  is right-invariant,

$$\int_G f(gh) \epsilon(g) = \int_G f(g) \epsilon(g) \quad (5.2.13)$$

for a rep. of  $G, \Pi : G \rightarrow \text{End}(V)$ , define an arbitrary inner product  $\langle \cdot, \cdot \rangle$  on  $V$ , then define another inner product on  $V$  by,

$$\langle \cdot, \cdot \rangle_G : V \times V \rightarrow \mathbb{C}$$

$$\langle u, v \rangle_G = \int_G \langle \Pi(g)u | \Pi(g)v \rangle \epsilon(g) \quad (5.2.14)$$

then,

$$\langle u, v \rangle_G = \langle \Pi(h)u, \Pi(h)v \rangle_G \quad (5.2.15)$$

and  $\langle v, v \rangle_G > 0$  for all  $v \neq 0$ .

so,  $\Pi(g)$  is **unitary** with respect to  $\langle \cdot, \cdot \rangle_G$ .

–  $\mathrm{SU}(m)$  are compact, hence have the complete reducibility property.

### 5.2.3 Schur's lemma

- **def.:** an **intertwining map** of rep.  $\Pi_1, \Pi_2$  (or  $\pi_1, \pi_2$ ) is a linear map  $\phi : V \rightarrow W$ , s.t.,

$$\begin{cases} \phi \Pi_1(g) = \Pi_2(g) \phi \\ \phi \pi_1(A) = \pi_2(A) \phi \end{cases} \in \mathrm{End}(W) \quad (5.2.16)$$

- **Schur's 1st lemma**

for 2 **irreducible real or complex rep.**  $\Pi_1, \Pi_2$  (or  $\pi_1, \pi_2$ ) on  $V, W$ , the intertwining map  $\phi$  is either 0 or an isomorphism.

证明参见有限群.

- **Schur's 2nd lemma**

for a **irreducible complex rep.**  $\Pi$  (or  $\pi$ ) on  $V$ , the intertwining map  $\phi : V \rightarrow V$  is  $\lambda I$  for some  $\lambda \in \mathbb{C}$ .

- **Schur's 3rd lemma**

for 2 **irreducible complex rep.**  $\Pi_1, \Pi_2$  (or  $\pi_1, \pi_2$ ) on  $V, W$ , and 2 intertwining map  $\phi_1, \phi_2 : V \rightarrow V$ , then  $\phi_1 = \lambda \phi_2$  for some  $\lambda \in \mathbb{C}$ .

## 5.3 Lie's third theorem

- **Lie's third theorem:** every **finite-dimensional** Lie algebra  $\mathfrak{g}$  over  $\mathbb{R}$  is associated to a Lie group  $G$ .
- every **finite-dimensional** Lie algebra is isomorphic to the Lie algebra of some **matrix** Lie group.

## 5.4 adjoint representations

### 5.4.1 adjoint rep. of Lie groups

- consider the adjoint diffeomorphism on  $G$ ,

$$I_g : G \rightarrow G, h \mapsto ghg^{-1} \quad (5.4.1)$$

- $\mathrm{Ad}_g = I_{g*} : V_e \rightarrow V_e$  is the pushforward,

$$\mathrm{Ad}_g \left( \frac{d}{dt} \Big|_0 \gamma(t) \right) x^i \Big|_e = \frac{dy^i(\gamma(t))}{dt} \Big|_0 \quad (5.4.2)$$

where  $y^i(h) = x^i(ghg^{-1})$ , so we have,

$$\mathrm{Ad}_g \left( \frac{d}{dt} \Big|_0 \gamma(t) \right) = \frac{d}{dt} \Big|_0 g \gamma(t) g^{-1} \quad (5.4.3)$$

i.e.  $\exp(\mathrm{Ad}_g(A)) = I_g \exp(A)$ .

– as we can see,  $\mathrm{Ad}_g \in \mathrm{Aut}(V_e)$  is a linear and reversible automorphism on  $V_e$ , since  $\mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}} = I$ .

- $\mathrm{Ad} : G \rightarrow \mathrm{Aut}(V_e) \simeq \mathrm{GL}(m, \mathbb{R})$  is the **adjoint representation of the Lie group**,  $G$ .

–  $\mathrm{Ad}$  is a homomorphism.

proof:

$$\text{Ad}_g \circ \text{Ad}_h = I_{g*} \circ I_{h*} = (I_g \circ I_h)_* = \text{Ad}_{gh} \quad (5.4.4)$$

### 5.4.2 adjoint rep. of Lie algebras

- The **structure constants** themselves generate a **representation of the Lie algebra**, called the **adjoint representation**.
- the Jacob identity written in the structure constants is,

$$f_{il}^m f_{jk}^l + f_{kl}^m f_{ij}^l + f_{jl}^m f_{ki}^l = 0 \quad (5.4.5)$$

consider the structure constants as the components of matrices,  $-if_{ij}^k = (T_i)_j^k$ , since  $f_{ij}^k = -f_{ji}^k$ , the matrices have the property that  $(T_i)_j^k = -(T_j)_i^k$ , then,

$$\begin{aligned} if_{jk}^l (T_l)_i^m + \underbrace{(T_i T_k)_j^m}_{=-(T_j T_k)_i^m} + (T_k T_j)_i^m &= 0 \\ \implies [T_j, T_k]_i^m = if_{jk}^l (T_l)_i^m \end{aligned} \quad (5.4.6)$$

or, more compactly,  $[T_i, T_j] = if_{ij}^k T_k$ .

- $\{(T_i)_j^k = -if_{ij}^k\}$  is called the adjoint representation of the Lie algebra  $\{X_i\}$ .

- more formally, adjoint representation is a map,  $\text{ad}_A : \mathfrak{g} \rightarrow \mathfrak{g}$ , where  $\mathfrak{g}$  is the Lie algebra of the group  $G$ ,

$$\text{ad}_A(B) = [A, B] \quad (5.4.7)$$

as one can see,  $(\text{ad}_A)^a_b = -f_{cb}^a A^c \in \mathcal{L}(\mathfrak{g})$ , or written in components,

$$(\text{ad}_{A_i})_j^k = -f_{ij}^k \implies \text{ad}_{A_i} = (iT_i)^T \quad (5.4.8)$$

and  $[\text{ad}_{A_i}, \text{ad}_{A_j}] = \text{ad}_{[A_i, A_j]} = -f_{ij}^k \text{ad}_{A_k}$ .

- $\text{ad} : \mathfrak{g} \rightarrow \mathcal{L}(\mathfrak{g})$  is a homomorphism, i.e.,

$$\text{ad}_{[A, B]} = [\text{ad}_A, \text{ad}_B] \quad (5.4.9)$$

proof:

$$\begin{aligned} (\text{ad}_A \text{ad}_B - \text{ad}_B \text{ad}_A)C &= [A, [B, C]] - [B, [A, C]] \\ &= [[A, B], C] = \text{ad}_{[A, B]}C \end{aligned} \quad (5.4.10)$$

## 5.5 Killing forms

- $\forall A, B \in \mathfrak{g}$ , the Killing form is,

$$B(A, B) = \text{tr}(\text{ad}_A \circ \text{ad}_B) \quad (5.5.1)$$

which can be written in terms of structure constants,

$$B_{ij} = f_{ik}^l f_{jl}^k \quad (5.5.2)$$

proof:

$$B(A_i, A_j) = \text{tr}(\text{ad}_{A_i} \text{ad}_{A_j}) = (-f_{ik}^l)(-f_{jl}^k) \quad (5.5.3)$$

$$- B([A, B], C) = B(A, [B, C]).$$



**proof:**

recall that,

$$\text{ad}_{[A,B]} = [\text{ad}_A, \text{ad}_B] \quad (5.5.4)$$

so,

$$\begin{aligned} B([A, B], C) &= \text{tr}([\text{ad}_A, \text{ad}_B] \text{ad}_C) \\ &= \text{tr}(\text{ad}_A \text{ad}_B \text{ad}_C) - \text{tr}(\text{ad}_A \text{ad}_C \text{ad}_B) \\ &= B(A, [B, C]) \end{aligned} \quad (5.5.5)$$

- two basis-independent properties of the Killing form:
  - the **number** of zero eigenvalues.
  - the **sign** of the non-zero eigenvalues.
- the structure constants with lowered indices are **completely antisymmetric**,

$$f_{ij}{}^l B_{lk} = -f_{ijk} = -f_{[ijk]} \quad (5.5.6)$$

**proof:**

$$f_{ij}{}^l B_{lk} = f_{ij}{}^l f_{lm}{}^n f_{kn}{}^m \quad (5.5.7)$$

notice that, according to Jacob identity,  $f_{ij}{}^l f_{lm}{}^n = 2f_{[i|l}{}^n f_{|j]m}{}^l$ , then,

$$f_{ijk} = -2f_{[i|l}{}^n f_{|j]m}{}^l f_{kn}{}^m \quad (5.5.8)$$

we can see that the equation holds under index permutation like  $(i, j, k) \rightarrow (k, i, j) \rightarrow (j, k, i)$ , and consequently, all three indices of  $f_{ijk}$  are antisymmetric.

**Part III**

**Semisimple Lie Algebras**

# Chapter 6

## semisimple Lie algebras

### 6.1 semisimple and reductive Lie algebras

- **def.:** a complex Lie algebra is **reductive** if there exists a **compact** Lie group  $K$  s.t.,

$$\mathfrak{g} \simeq \mathfrak{k}_{\mathbb{C}} \quad (6.1.1)$$

- an alternate def. from Wikipedia: a Lie algebra is reductive if its adjoint rep. is completely reducible.

#### proof of equivalence:

$\implies$ , complexification of a compact Lie group is reductive:

- the adjoint rep. of a compact Lie group is completely reducible, so is its complexification (they have the same invariant subspaces,  $W, W^{\perp}$ , only complexified).

$\impliedby$ , reductive is isomorphic to the complexification of some compact Lie group:

- the invariant subspaces of the adjoint representation are the ideals of  $\mathfrak{g}$ , especially, the kernel of the adjoint rep. is the center,  $\mathfrak{z}$ .
- $\mathfrak{g}$  decomposes as  $\mathfrak{z} \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \dots$ , where  $\mathfrak{h}_1, \dots$  are the smallest ideals of  $\mathfrak{g}$ , i.e. they don't have nontrivial ideals themselves  $\implies$  irreducible.
- moreover, if  $\dim \mathfrak{h}_i = 1$ , then,

$$[\mathfrak{h}_i, \mathfrak{z} \oplus \bigoplus_{j \neq i} \mathfrak{h}_j] = [\mathfrak{h}_i, \bigoplus_{j \neq i} \mathfrak{h}_j] \subseteq \mathfrak{h}_i \cap \bigoplus_{j \neq i} \mathfrak{h}_j = \{0\} \quad (6.1.2)$$

then  $\mathfrak{h}_i$  is just part of the center.

- so,  $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}_1 \oplus \dots$ , where  $\mathfrak{h}_1, \dots$  are simple Lie subalgebras.
- according to the converse of (6.1.13) (?),  $\mathfrak{h}_1 \oplus \dots$  is a semisimple Lie algebra.
- according to the converse of (6.1.6) (?), a Lie algebra decomposes as its center and a semisimple Lie algebra is compact.

- **def.:** a complex Lie algebra is **semisimple** if it is reductive and the center of  $\mathfrak{g}$  is trivial, i.e.  $\mathfrak{z} = \{A \in \mathfrak{g} | \text{ad}_A = 0\} = \{0\}$ .
- **def.:**  $\mathfrak{k}$  in (6.1.1) is the **compact real form** of the semisimple Lie algebra.
- some semisimple Lie algebras:

Lie algebras	reductive	semisimple	compact real forms
$\mathfrak{sl}(m \geq 2, \mathbb{C})$	yes	yes	$\mathfrak{su}(m)$
$\mathfrak{so}(m \geq 3, \mathbb{C})$	yes	yes	$\mathfrak{so}(m)$
$\mathfrak{so}(2, \mathbb{C})$	yes	no	$\mathfrak{so}(2)$
$\mathfrak{sp}(m \geq 1, \mathbb{C})$	yes	yes	$\mathfrak{sp}(m, \mathbb{R})$
$\mathfrak{gl}(m, \mathbb{C})$	yes	no	$\mathfrak{u}(m)$

### 6.1.1 some properties of reductive and semisimple Lie algebras

- let  $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  be a **reductive** Lie algebra, then there exists an inner product s.t.,

$$\langle \text{ad}_X A, B \rangle = -\langle A, \text{ad}_X B \rangle \quad (6.1.3)$$

for all  $A, B \in \mathfrak{g}, X \in \mathfrak{k}$ .

**proof:**

$\text{Ad} : K \rightarrow \text{End}(\mathfrak{k})$  is a unitary representation under the inner product chosen in (5.2.14) (which requires **compactness**),

$$\langle A, B \rangle = \int_K (\text{Ad}_g A, \text{Ad}_g B) \epsilon(g) \quad (6.1.4)$$

where  $(A, B)$  is some real positive definite inner product on  $\mathfrak{k}$ , and  $\epsilon$  is the volume form composed by right invariant dual vector fields.

so, the associated Lie algebra rep.  $\text{ad} : \mathfrak{k} \rightarrow \text{End}(\mathfrak{k})$  satisfies  $\text{ad}_X^\dagger = -\text{ad}_X$  (skew self-adjoint).

- for a **reductive** Lie algebra  $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ ,  $\mathfrak{h}$  is one of its ideals, then,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \quad (6.1.5)$$

where  $\mathfrak{h}^\perp$  is orthogonal to  $\mathfrak{h}$  with respect to the inner product in (6.1.3), and it is also an **ideal**.

**proof:**

- if  $\mathfrak{h} (\text{ad}_A[\mathfrak{h}] \subseteq \mathfrak{h}, \forall A)$  is an ideal of  $\mathfrak{g}$ , then it is also an ideal of  $\mathfrak{k}$  (obviously).
- unitary rep. is completely reducible, so both  $\mathfrak{h}$  and  $\mathfrak{h}^\perp$  are its invariant subspace, i.e. ideals.
- $[\mathfrak{h}, \mathfrak{h}^\perp] \subseteq \mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$ .
- so,  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ .

- every **complex reductive** Lie algebra,  $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ , decomposes as,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{z} \quad (6.1.6)$$

where  $\mathfrak{g}_1$  is **semisimple** and  $\mathfrak{z}$  is its **center**.

moreover,

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}' \quad (6.1.7)$$

where  $\mathfrak{z}'$  is the center of  $\mathfrak{k}$  and  $\mathfrak{k}_1$  is the compact real form of  $\mathfrak{g}_1$ .

**proof:**

center is an ideal, so,

$$\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z} \quad (6.1.8)$$

now we have to prove  $\mathfrak{g}_1 = \mathfrak{z}^\perp$  is semisimple,

- first, the **center** of  $\mathfrak{z}^\perp$  is **trivial**, for obvious reasons.

- $A \in \mathfrak{z} \iff \text{ad}_A[\mathfrak{k}] = \{0\}$ , so, for all  $A = X + iY \in \mathfrak{z}, X, Y \in \mathfrak{k}$ ,

$$A^* := X - iY \in \mathfrak{z} \quad (6.1.9)$$

i.e.  $\mathfrak{z}$  is closed under conjugation  $*$  :  $X + iY \mapsto X - iY$

so,  $\mathfrak{g}_1$  is also closed under conjugation.

\* 注意, 这里的定义和 Hall 书上的不一样, Hall 的定义是  $A^* = -X + iY, \bar{A} = X - iY$ .

- so, for  $\mathfrak{z}' := \mathfrak{z} \cap \mathfrak{k}, \mathfrak{k}_1 := \mathfrak{g}_1 \cap \mathfrak{k}$ ,

$$\mathfrak{z} = \mathfrak{z}'_{\mathbb{C}} \quad \mathfrak{g}_1 = \mathfrak{k}_1_{\mathbb{C}} \quad (6.1.10)$$

- consider the adjoint representation of  $K$  and  $\mathfrak{k}$ ,

$$\text{Lie}(\text{Ad}[K]) = \text{ad}[\mathfrak{k}] \simeq \mathfrak{k} / \ker(\text{ad}) = \mathfrak{k} / \mathfrak{z}' = \mathfrak{k}_1 \quad (6.1.11)$$

$\text{Ad}$  is a continuous map, so  $\text{Ad}[K]$  is a **compact** Lie group as  $K$ .

- so,  $\mathfrak{k}_1$  is the **compact real form** of  $\mathfrak{g}_1$ .

- if  $K$  is a **simply connected compact** Lie group, then  $\mathfrak{g} = \mathfrak{k}_\mathbb{C}$  is **semisimple**.

**proof:**

since  $K$  is simply connected and  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}'$ , so  $K$  decomposes as,

$$K = K_1 \times Z' \quad (6.1.12)$$

where  $K_1, Z'$  are closed simply connected subgroup associated with  $\mathfrak{k}_1, \mathfrak{z}'$ .  
simply connected Lie group  $Z'$  is isomorphic to  $\mathbb{R}^n$  for some  $n$ , but  $Z'$  is closed subgroup of a compact group, it is also compact, which means  $n = 0$ , i.e.  $\mathfrak{z}' = \{0\} = \mathfrak{z}$ , the center is trivial.

- an important **theorem**:

**semisimple** Lie algebra  $\mathfrak{g}$  decomposes as,

$$\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i \quad (6.1.13)$$

where  $\mathfrak{g}_i$  are **simple** (see 3.2.1) and **unique** up to order (the converse of the theorem is also true (?)).

**proof:**

first, let's prove  $\mathfrak{g}_i$  are simple,

- according to (6.1.5), semisimple Lie algebra with ideal  $\mathfrak{h}$  decomposes as,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \quad (6.1.14)$$

suppose  $\mathfrak{h}'$  is an ideal of  $\mathfrak{h}$ , notice that  $[\mathfrak{h}, \mathfrak{h}^\perp] = \{0\}$ , so  $\mathfrak{h}'$  is also an ideal of  $\mathfrak{g}$ .

- let  $\mathfrak{h}'' = \mathfrak{h}'^\perp \cap \mathfrak{h}$ , and  $[\mathfrak{h}'', \mathfrak{h}' \oplus \mathfrak{h}^\perp] = \{0\}$ , so it is also an ideal, then,

$$\mathfrak{g} = \mathfrak{h}'' \oplus \mathfrak{h}' \oplus \mathfrak{h}^\perp \quad (6.1.15)$$

- proceeding on the same way,

$$\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i \quad (6.1.16)$$

where  $\mathfrak{g}_i$  are ideals without nontrivial ideals, i.e. **irreducible**.

- if  $\dim \mathfrak{g}_i = 1$ , then  $\mathfrak{g}_i$  is Abelian, moreover,

$$[\mathfrak{g}_i, \bigoplus_{j \neq i} \mathfrak{g}_j] = \{0\} \quad (6.1.17)$$

$\mathfrak{g}_i \subseteq \mathfrak{z}$  which contradicts to semisimpleness (without nontrivial center). so,  $\dim \mathfrak{g}_i \geq 2$ .

now, let's prove uniqueness,

- $\pi_i := \text{ad}|_{\mathfrak{g}_i} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}_i)$  is an **irreducible rep.**, since the nontrivial invariant subspace of  $\pi_i$  is  $\{\text{an ideal of } \mathfrak{g}\} \cap \mathfrak{g}_i$ , and consider (6.1.17), it is also an ideal of  $\mathfrak{g}_i$ , which doesn't exist.
- since  $\pi_i[\mathfrak{g}_{j \neq i}] = \{0\}$  while  $\pi_i[\mathfrak{g}_i] \neq \{0\}$  (simple Lie algebras are non-Abelian)  $\implies$  these rep. are **not isomorphic** to each other.

- for a simple ideal  $\mathfrak{h}$  of  $\mathfrak{g}$ ,  $\pi_{\mathfrak{h}} := \text{ad}|_{\mathfrak{h}} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{h})$  is an irreducible rep..
- the projection map  $p_i : \mathfrak{g} \rightarrow \mathfrak{g}_i$  is an intertwining map,

$$p_i|_{\mathfrak{g}_j} \pi_j(A) = \pi_i(A) p_i|_{\mathfrak{g}_j} \begin{cases} = 0 & i \neq j \text{ or } A \notin \mathfrak{g}_{i=j} \\ \neq 0 & i = j, A \in \mathfrak{g}_{i=j} \end{cases} \quad (6.1.18)$$

and,

$$p_i|_{\mathfrak{h}} \pi_{\mathfrak{h}}(A) = \pi_i(A) p_i|_{\mathfrak{h}} \quad (6.1.19)$$

according to Schur's lemma,  $p_i|_{\mathfrak{h}} = 0$  or isomorphism.

- $p_i|_{\mathfrak{h}}$  is a projection map, so there must be some  $i$  so that  $p_i|_{\mathfrak{h}} \neq 0$ , so  $\mathfrak{h} = \mathfrak{g}_i$  for some  $i$ .

## 6.2 Cartan subalgebra

- **def.:**  $\mathfrak{g}$  is a complex semisimple Lie algebra, its subalgebra  $\mathfrak{h}$  is called **Cartan subalgebra** if:

1. it is Abelian,
2. if for some  $A \in \mathfrak{g}$  and  $[A, H] = 0, \forall H \in \mathfrak{h}$ , then  $A \in \mathfrak{h}$ , (make sure it is maximal),
3.  $\forall H \in \mathfrak{h}, \text{ad}_H$  is diagonalizable.

some remark:

- condition 1 and 2 say that  $\mathfrak{h}$  is a **maximal Abelian subalgebra** (not contained in a larger Abelian subalgebra) of  $\mathfrak{g}$  (there may be more than one maximal Abelian subalgebra).
- $[\text{ad}_{H_1}, \text{ad}_{H_2}] = \text{ad}_{[H_1, H_2]} = 0$ , so they are **simultaneously diagonalizable**.
- the def. makes sense in any Lie algebra, but if  $\mathfrak{g}$  is not semisimple, it may not have any Cartan subalgebra.

- now, let's prove **Cartan subalgebra exists in semisimple Lie algebras**.
- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is a complex semisimple Lie algebra,  $\mathfrak{t}$  is a **maximal Abelian subalgebra of  $\mathfrak{k}$** , then, the **Cartan subalgebra** of  $\mathfrak{g}$  is,

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \quad (6.2.1)$$

**proof:**

first, let's prove  $\mathfrak{h}$  is maximal Abelian,

- $\mathfrak{h}$  is obviously Abelian.
- if  $[A, \mathfrak{h}] = \{0\}$ , for some  $A = X + iY \in \mathfrak{g}$ , then  $[X, \mathfrak{h}] = [Y, \mathfrak{h}] = \{0\}$ , which means  $\mathfrak{t}$  is not maximal.

now, let's show that  $\text{ad}_H, \forall H \in \mathfrak{h}$  are diagonalizable,

- choose inner product shown in (5.2.14), so  $\text{ad}_X$  is skew self-adjoint for all  $X \in \mathfrak{k}$ , which means it is diagonalizable.
- $\text{ad}_T, \forall T \in \mathfrak{t}$  is diagonalizable, and  $[\text{ad}_T, \text{ad}_H] = 0, \forall H \in \mathfrak{h}$ , so  $\text{ad}_H, \forall H \in \mathfrak{h}$  are simultaneously diagonalizable.

- **def.:** the **rank**,  $r = \dim \mathfrak{h}$ , of a semisimple Lie algebra is the dimension of any of its Cartan subalgebras.
  - any two Cartan subalgebra  $\mathfrak{h}_1, \mathfrak{h}_2$  of a semisimple Lie algebra are isomorphic to each other (?).

## 6.3 roots and root spaces

- from now on, we only consider the Cartan subalgebra in (6.2.1),  $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ .
- def.:** a **nonzero** element  $\alpha \in \mathfrak{h}$  (because  $\langle \alpha | \in \mathfrak{h}^*$ ) is called a **root** if there exists a nonzero  $A \in \mathfrak{g}$  s.t.,

$$[H, A] = \langle \alpha, H \rangle A \quad (6.3.1)$$

for all  $H \in \mathfrak{h}$ .

- the inner product (on  $\mathfrak{h}$ ) is arbitrarily chosen.
- the set of all root is denoted as  $R = \{\alpha\}$ .
- if we choose the inner product in (5.2.14), then, for all root  $\alpha \in i\mathfrak{t}$ .

**proof:**

- choose  $H \in \mathfrak{t}$ ,  $\text{ad}_H$  is skew self-adjoint under the chosen inner product.
- the eigenvalue  $\langle \alpha, H \rangle$  is pure imaginary (and nonzero).
- the inner product is real on  $\mathfrak{k}$ .
- so,  $\alpha \in i\mathfrak{k} \cap \mathfrak{h} = i\mathfrak{t}$ .

- def.:** for a root  $\alpha$ , the **root space** is,

$$\mathfrak{g}_{\alpha} = \{A \in \mathfrak{g} | [H, A] = \langle \alpha, H \rangle A, \forall H \in \mathfrak{h}\} \quad (6.3.2)$$

a nonzero element of  $\mathfrak{g}_{\alpha}$  is called a **root vector**.

- more generally, for any element  $\alpha \in \mathfrak{h}$ , we can define  $\mathfrak{g}_{\alpha}$  as in (6.3.2), but we don't call it a root space unless  $\alpha$  is a root.
  - \* if  $\alpha$  is not a root, then,  $\mathfrak{g}_{\alpha}$  is either  $\{0\}$  ( $\alpha \neq 0$ ) or  $\mathfrak{h}$  ( $\alpha = 0$ ).
  - \* by def.  $[\mathfrak{h}, \mathfrak{g}_{\alpha}] = \mathfrak{g}_{\alpha}$ .
- the complex semisimple Lie algebra decomposes as,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha} \quad (6.3.3)$$

and  $\mathfrak{h} \cap \mathfrak{g}_{\alpha} = \mathfrak{g}_{\alpha} \cap \mathfrak{g}_{\beta} = \{0\}$ , furthermore,  $\mathfrak{h}$  and  $\mathfrak{g}_{\alpha}, \forall \alpha \in R$  are linearly independent.

note that  $\oplus$  is **not Lie algebra direct sum**, as that  $\mathfrak{h}, \mathfrak{g}_{\alpha}$  are not ideals.

**proof:**

$\text{ad}_H, H \in \mathfrak{h}$  can be simultaneously diagonalized, so, according to (A.3.9) in appendix A.3,

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}} \mathfrak{g}_{\alpha} \quad (6.3.4)$$

and  $\mathfrak{g}_{\alpha} \cap \mathfrak{g}_{\beta} = \{0\}, \forall \alpha \neq \beta \in \mathfrak{h}$ .

but if  $\alpha = 0$ ,  $\mathfrak{g}_0 = \mathfrak{h}$  and if  $\alpha \neq 0$  and not a root,  $\mathfrak{g}_{\alpha} = \{0\}$ , so...

- for any  $\alpha, \beta \in \mathfrak{h}$ , we have,

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta} \quad (6.3.5)$$

**proof:**

for all  $A \in \mathfrak{g}_{\alpha}, B \in \mathfrak{g}_{\beta}$ ,

$$[H, [A, B]] = -[B, [H, A]] - [A, [B, H]] = \langle \alpha + \beta, H \rangle [A, B] \quad (6.3.6)$$

- two useful propositions:

- if  $\alpha$  is a root, so does  $-\alpha$ , and for all  $A = X + iY \in \mathfrak{g}_{\alpha}, A^* = X - iY \in \mathfrak{g}_{-\alpha}$  (where  $X, Y \in \mathfrak{k}$ ).

**proof:**

for any  $H \in \mathfrak{t}$ ,

$$[H, A^*] = ([H, A])^* = (\langle \alpha, H \rangle)^* A^* \quad (6.3.7)$$

and because  $\alpha \in i\mathfrak{t}$ , so  $(\langle \alpha, H \rangle)^* = -\langle \alpha, H \rangle$ .

–  $\text{span}(R) = \mathfrak{h}$ .

**proof:**

if the root doesn't span  $\mathfrak{h}$ , then there nonzero exists  $H \in \mathfrak{h}$  s.t.,

$$\langle \alpha, H \rangle = 0, \forall \alpha \in R \implies [H, A] = 0, \forall A \in \mathfrak{g} \quad (6.3.8)$$

i.e.  $H$  is in the center of  $\mathfrak{g}$ , which contradicts to semisimpleness of  $\mathfrak{g}$  (without nontrivial center).

### 6.3.1 subalgebras isomorphic to $\mathfrak{su}(2)_{\mathbb{C}}$

- for each root  $\alpha \in R$ , we have the **coroot**,

$$H_{\alpha} = 2 \frac{\alpha}{\langle \alpha, \alpha \rangle} \in \mathfrak{h} \quad (6.3.9)$$

associated to it, and  $\forall A_{\alpha} \in \mathfrak{g}_{\alpha}, B_{\alpha} \in \mathfrak{g}_{-\alpha}$  there is,

$$\begin{cases} [H_{\alpha}, A_{\alpha}] = 2A_{\alpha} \\ [H_{\alpha}, B_{\alpha}] = -2B_{\alpha} \\ [A_{\alpha}, B_{\alpha}] = H_{\alpha} \end{cases} \quad (\text{with } \mathbf{normalization}) \quad (6.3.10)$$

and  $B_{\alpha} = -A_{\alpha}^*$  (as part of the normalization).

**proof:**

for all  $A \in \mathfrak{g}_{\alpha}, B \in \mathfrak{g}_{-\alpha}, H \in \mathfrak{h}$ , then  $[A, B] \in \mathfrak{h}$  and,

$$[A, B] = \langle -A^*, B \rangle \alpha \quad (6.3.11)$$

**proof:**

–  $[A, B] \in \mathfrak{h}$  because  $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$  and  $\mathfrak{g}_0 = \mathfrak{h}$ .

– and,

$$\begin{aligned} \langle H, [A, B] \rangle &= \langle \text{ad}_A^{\dagger} H, B \rangle = \langle \text{ad}_{-A^*} H, B \rangle \\ &= \langle [H, A^*], B \rangle = \langle \langle -\alpha, H \rangle A^*, B \rangle \\ &= \langle H, \alpha \rangle \langle -A^*, B \rangle \end{aligned} \quad (6.3.12)$$

for all  $H \in \mathfrak{h}$ , so,

$$[A, B] = \langle -A^*, B \rangle \alpha \quad (6.3.13)$$

choose the **normalization**,

$$\begin{cases} B_{\alpha} = -A_{\alpha}^* \\ \langle A_{\alpha}, A_{\alpha} \rangle^* \langle \alpha, \alpha \rangle = 2 \end{cases} \iff \begin{cases} H = [A, -A^*] = \langle A, A \rangle^* \alpha \\ H_{\alpha} = \frac{2}{\langle \alpha, H \rangle} H \\ A_{\alpha} = \sqrt{\frac{2}{\langle \alpha, H \rangle}} A \\ B_{\alpha} = -A_{\alpha}^* \end{cases} \quad (6.3.14)$$

notice  $\langle \alpha, H \rangle \in \mathbb{R}$

$\forall A \in \mathfrak{g}_{\alpha}$  (notice that  $\langle \alpha, \alpha \rangle \in \mathbb{R}^+$  and  $\langle A, A \rangle = \langle X, X \rangle + \langle Y, Y \rangle - 2\text{Im} \langle X, Y \rangle \in \mathbb{R}, \forall A \in \mathfrak{g}$ ).



- compare  $\text{span}(H_\alpha, A_\alpha, B_\alpha)_\mathbb{C}$  with  $\mathfrak{su}(2)_\mathbb{C}$ , we have,

$$H_\alpha \mapsto 2J_3 \quad A_\alpha \mapsto \sqrt{2}J_+ \quad B_\alpha \mapsto \sqrt{2}J_- \quad (6.3.15)$$

- from the complex subalgebra  $\mathfrak{s}^\alpha = \text{span}(H_\alpha, A_\alpha, B_\alpha)$ , we can conclude that,

1. if  $\alpha$  and  $c\alpha$  are both roots, then  $c = \pm 1$ ,
2.  $\dim \mathfrak{g}_\alpha = 1$  for all root spaces.

**proof:**

consider  $A_{c\alpha} \in \mathfrak{g}_{c\alpha}$ ,

$$[H_\alpha, A_{c\alpha}] = \underbrace{\langle c\alpha, H_\alpha \rangle}_{=2c^*} A \quad (6.3.16)$$

$2c^*$  is an eigenvalue of  $\text{ad}_{H_\alpha} \in \text{End}(\mathfrak{g})$ , which is a finite-dim. rep. of  $\mathfrak{su}(2)_\mathbb{C}$ , so the eigenvalue must be an integer, i.e.,

$$2c^*, 2\frac{1}{c^*} \in \mathbb{Z} \implies c = \pm 1, \pm 2, \pm \frac{1}{2} \quad (6.3.17)$$

let  $\pm\alpha, \pm 2\alpha$  (notice  $\pm 4\alpha$  are not roots) be all the roots  $\propto \alpha$ , then let,

$$V^\alpha = \text{span}(H_\alpha) \oplus \bigoplus_{\beta=\pm\alpha, \pm 2\alpha} \mathfrak{g}_\beta \quad (6.3.18)$$

where  $\oplus$  is not Lie algebra direct sum.

$V^\alpha \supseteq \mathfrak{s}^\alpha$  is a subalgebra of  $\mathfrak{g}$ .

**proof:**

for all  $\beta, \beta' = \pm\alpha, \pm 2\alpha$ , we have,

- according to (6.3.11),  $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \propto \alpha \propto H_\alpha$ .
- $[H_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_\beta$ .
- $[\mathfrak{g}_\beta, \mathfrak{g}_{\beta'}] \subseteq \mathfrak{g}_{\beta+\beta'} = \mathfrak{g}_{\pm 2\alpha}$  or  $\{0\}$  (where  $\beta + \beta' \neq 0$ ).

now, let's prove  $V^\alpha = \mathfrak{s}^\alpha$ ,

- consider the 'unitary' (skew self-adjoint) rep.  $(\text{ad}, V^\alpha)$  of  $\text{span}(H_\alpha, A_\alpha, B_\alpha) \simeq \mathfrak{su}(2)_\mathbb{C}$ ,  $\mathfrak{s}^\alpha$  is the invariant subspace of the rep., and the rep. is completely reducible, so  $\mathfrak{s}^{\alpha\perp}$  is also an invariant subspace.
- the eigenvalues of  $\text{ad}_{H_\alpha}$  in  $V^\alpha$  are 0 and  $\langle \beta, H_\alpha \rangle = \pm 2, \pm 4$ .
- recall the property of the eigenvalues of  $\pi(H)$ , 0 must be one of the eigenvalues of  $\text{ad}_{H_\alpha}$  in the rep.  $(\text{ad}, \mathfrak{s}^{\alpha\perp})$ , which is **impossible** since  $H_\alpha \in \mathfrak{s}^\alpha$  is the only vector with eigenvalue 0.
- so,  $\mathfrak{s}^{\alpha\perp} = \{0\}$ , i.e. the only roots  $\propto \alpha$  are  $\pm\alpha$ , and,

$$\text{span}(H_\alpha) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{s}^\alpha \equiv \text{span}(H_\alpha, A_\alpha, B_\alpha) \quad (6.3.19)$$

i.e.  $\mathfrak{g}_\alpha = \text{span}(A_\alpha)$  or  $\dim \mathfrak{g}_\alpha = 1$ .

- a rephrase of (6.3.3): for all  $A \in \mathfrak{g}$ ,  $A$  is either a root or in a root space, and,

$$\begin{cases} \mathfrak{s}^\alpha \cap \mathfrak{s}^\beta = \{0\} & \alpha \neq \pm\beta \\ \mathfrak{s}^\alpha = \mathfrak{s}^{-\alpha} & H_\alpha = -H_{-\alpha} \quad A_\alpha = B_{-\alpha} \quad B_\alpha = A_{-\alpha} \end{cases} \quad (6.3.20)$$

- $\mathfrak{s}^\alpha, \mathfrak{h}, \mathfrak{g}_\alpha, \forall \alpha \in R$  are not ideals.
- the set of roots,  $R$ , may not be linearly independent.
  - the maximal set of linearly independent roots is called the **simple root**.
  - but  $\mathfrak{g}_\alpha, \forall \alpha \in R$  are linearly independent, as stated in (6.3.3).

### 6.3.2 root systems

- for all roots  $\alpha, \beta \in R \subset \mathfrak{it}$ , we have,

$$\langle \alpha, H_\beta \rangle = 2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \quad (6.3.21)$$

**proof:**

consider  $\mathfrak{s}^\beta = \text{span}(H_\beta, A_\beta, B_\beta)$ , and its adjoint representation  $\text{ad} : \mathfrak{s}^\beta \rightarrow \text{End}(\mathfrak{g}/\mathfrak{h})$  (which is finite dimensional),

$$[H_\beta, A_\alpha] = \langle \alpha, H_\beta \rangle A_\alpha \quad (6.3.22)$$

the eigenvalue of  $\text{ad}_{H_\beta}$  must be an integer, according to (10.1.6), so,

$$\langle \alpha, H_\beta \rangle \in \mathbb{Z} \quad (6.3.23)$$

- the **projection** of  $\alpha$  to  $\beta$  ( $\alpha \cdot \hat{e}_\beta$ ) is a (half-)integer multiple of  $|\beta|$ ,

$$\frac{\langle \alpha, \beta \rangle}{\sqrt{\langle \beta, \beta \rangle}} = (0, \pm \frac{1}{2}, \pm 1, \dots) |\beta| \quad (6.3.24)$$

- summary:

- the roots span  $\mathfrak{it}$ .
- if  $\alpha \in R$ , the only multiples of  $\alpha$  in  $R$  is  $-\alpha$ .
- $\alpha \in R$ , then  $s_\beta \alpha \in R$ , where  $s_\beta = I - 2 \frac{|\beta\rangle\langle\beta|}{\langle\beta,\beta\rangle}$  (see (6.5.2)).
- for all  $\alpha, \beta \in R$ , their inner product  $2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$ .

any such collection of vectors is called a **root system**.

## 6.4 Cartan's criterion

- **Cartan's criterion for simplicity:**

complex Lie algebra  $\mathfrak{g}$  is semisimple  $\iff$  its Killing form is non-degenerate.

**proof:**

first, let's prove  $\implies$ ,

- consider,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad (6.4.1)$$

(where  $\oplus$  is the vector space direct sum) and the adjoint representation is  $\text{ad} : \mathfrak{s}^\alpha \rightarrow \text{End}(\mathfrak{s}^\alpha)$ .  
and notice  $\mathfrak{h} = \text{span}(R)$ .

- so, for any  $\alpha \in R$ , we have,

$$\begin{cases} H_\alpha & B(H_\alpha, H_\alpha) = 8 \\ A_\alpha \text{ or } B_\alpha & B(A_\alpha, B_\alpha) = 4 \end{cases} \quad (6.4.2)$$

- so, for all  $A \neq 0 \in \mathfrak{g}$ , there exists some  $B \in \mathfrak{g}$  s.t.  $B(A, B) \neq 0$ , i.e. the Killing form is non-degenerate.

---

now, let's prove  $\impliedby$ ,

- first, the center  $\mathfrak{z} = \{0\}$ , otherwise, there exists some  $A \in \mathfrak{g}$  s.t.  $\text{ad}_A = 0$ , which contradicts to the non-degeneracy.
- second, the adjoint rep. of  $\mathfrak{g}$  is completely reducible, otherwise, **the Killing form is degenerate (?)**.

## 6.5 the Weyl group (from the Lie algebra approach)

- **def.:** for each root  $\alpha \in R$ , define a linear map,

$$s_\alpha = I - \overbrace{|\alpha\rangle\langle H_\alpha|}^{=2\frac{|\alpha\rangle\langle\alpha|}{\langle\alpha,\alpha\rangle}} : \mathfrak{h} \rightarrow \mathfrak{h} \text{ or } i\mathfrak{t} \rightarrow i\mathfrak{t} \\ H \mapsto H - \alpha \langle H_\alpha, H \rangle \quad (6.5.1)$$

notice  $s_\alpha$  is the reflection about the hyperplane orthogonal to  $\alpha$ , i.e.,

- $s_\alpha |H\rangle = |H\rangle$  for all  $|H\rangle$  orthogonal to  $\alpha$ .
- $s_\alpha |\alpha\rangle = -|\alpha\rangle$ .

also notice  $s_\alpha = s_{-\alpha}$  and  $s_\alpha^2 = I$ .

- **def.:** the **Weyl group** is  $W = \langle \{s_\alpha, \alpha \in R\} \rangle$ , i.e. every element in  $W$  can be expressed as a combination of finite  $s_\alpha, \alpha \in R$ .
  - $W$  is a subgroup of the orthogonal group  $O(i\mathfrak{t})$ .

- for all  $\alpha \in R, w \in W$ ,

$$w|\alpha\rangle \in R \quad (6.5.2)$$

**proof:**

equivalently, we need to prove for all  $\alpha, \beta \in R$ ,

$$s_\alpha |\beta\rangle \in R \quad (6.5.3)$$

notice that for all  $H \in \mathfrak{h}$ ,

$$\begin{cases} \text{Ad}_{S_\alpha} H = s_\alpha |H\rangle \implies \text{Ad}_{S_\alpha} \text{ad}_H \text{Ad}_{S_\alpha}^{-1} = \text{ad}_{s_\alpha |H\rangle} \\ \text{Ad}_{S_\alpha}^{-1} H = s_\alpha |H\rangle \implies \text{Ad}_{S_\alpha}^{-1} \text{ad}_H \text{Ad}_{S_\alpha} = \text{ad}_{s_\alpha |H\rangle} \end{cases} \quad (6.5.4)$$

where  $\text{Ad}_{S_\alpha} = e^{\text{ad}_{A_\alpha}} e^{-\text{ad}_{B_\alpha}} e^{\text{ad}_{A_\alpha}} \in \text{End}(\mathfrak{g})$ .

**proof:**

notice that if  $\langle \alpha, H \rangle = 0$ , then  $[H, A_\alpha] = [H, B_\alpha] = 0$ , which implies  $[\text{ad}_H, \text{ad}_{A_\alpha} \text{ or } B_\alpha] = 0$ , so,

$$\begin{cases} \text{Ad}_{S_\alpha}^{-1} H = e^{-\text{ad}_{A_\alpha}} e^{\text{ad}_{B_\alpha}} e^{-\text{ad}_{A_\alpha}} H = H & \langle \alpha, H \rangle = 0 \\ \text{Ad}_{S_\alpha}^{-1} H = -H & H \propto \alpha \end{cases} \quad (6.5.5)$$

consider any  $H \in \mathfrak{h}$  and  $A_\beta \in \mathfrak{g}_\beta$  with  $\beta \in R$ ,

$$\text{Ad}_{S_\alpha} A_\beta \in \mathfrak{g} \quad (6.5.6)$$

and,

$$\begin{aligned} [H, \text{Ad}_{S_\alpha} A_\beta] &= \text{ad}_H \text{Ad}_{S_\alpha} A_\beta \\ &= \text{Ad}_{S_\alpha} (\text{Ad}_{S_\alpha}^{-1} \text{ad}_H \text{Ad}_{S_\alpha}) A_\beta \\ &= \text{Ad}_{S_\alpha} [s_\alpha H, A_\beta] = \langle \beta, s_\alpha H \rangle \text{Ad}_{S_\alpha} A_\beta \end{aligned} \quad (6.5.7)$$

and notice that  $\alpha \in i\mathfrak{t} \implies s_\alpha^\dagger = s_\alpha$ , so,

$$[H, \text{Ad}_{S_\alpha} A_\beta] = \langle s_\alpha \beta, H \rangle \text{Ad}_{S_\alpha} A_\beta \quad (6.5.8)$$

which means  $s_\alpha \beta \in R$  and  $\text{Ad}_{S_\alpha} A_\beta \in \mathfrak{g}_{s_\alpha \beta}$ .

- the Weyl group is **finite**.

**proof:**

since there are only finite roots,  $s_\alpha$  (which is reversible) is nothing but a **permutation** of the roots, so is every element in the Weyl group.

## 6.6 simple Lie algebras

- recall the def. of simple Lie algebra in section 3.2.1.
- see (6.1.13),  $\mathfrak{g}$  is simple  $\implies \mathfrak{g}$  is semisimple (不会证).

- $\mathfrak{g}_\mathbb{C}$  is simple  $\implies \mathfrak{g}$  is also simple.  
but,  $\mathfrak{g}$  is simple  $\not\implies \mathfrak{g}_\mathbb{C}$  is not necessarily simple.

**proof:**

- $\dim \mathfrak{g} = \dim \mathfrak{g}_\mathbb{C} \geq 2$ .
- if  $\mathfrak{g}$  has a nontrivial ideal,  $\mathfrak{h}$ , then  $\mathfrak{h}_\mathbb{C}$  is a nontrivial ideal of  $\mathfrak{g}_\mathbb{C}$ .

- **def.:** a real Lie algebra,  $\mathfrak{g}$ , is said to **admit a complex structure** if it is isomorphic to a complex Lie algebra,  $\mathfrak{h}$ ,

$$\begin{aligned} \phi : \mathfrak{g} &\rightarrow \mathfrak{h} \\ A &\mapsto \phi_1(A) + i\phi_2(A) \end{aligned} \quad (6.6.1)$$

and,

$$\phi([A, B]) = [\phi(A), \phi(B)] \implies \begin{cases} \phi_1([A, B]) = [\phi_1(A), \phi_1(B)] - [\phi_2(A), \phi_2(B)] \\ \phi_2([A, B]) = [\phi_1(A), \phi_2(B)] + [\phi_2(A), \phi_1(B)] \end{cases} \quad (6.6.2)$$

and  $\phi_1, \phi_2$  are not one-to-one.

- equivalently, there exists a "multiplication by  $i$ " map on  $\mathfrak{g}$ ,  $J : \mathfrak{g} \rightarrow \mathfrak{g}$ , s.t.,

$$J^2 = -I \quad \text{and} \quad [A, B + JC] = [A, B] + J[A, C] \quad (6.6.3)$$

**proof:**

let's prove def. 1.  $\implies$  there exists a  $J$  on  $\mathfrak{g}$ ,

- let  $J = (\phi^{-1} \circ iI \circ \phi) \in \text{End}(\mathfrak{g})$ .
- for all  $X \in \mathfrak{h}$ , there exists some  $A = \phi^{-1}X$ , so,

$$\begin{aligned} (\phi \circ J)A &= (\phi \circ J \circ \phi^{-1})X = iX = i\phi(A) \\ \implies \phi([A, JB]) &= [\phi(A), i\phi(B)] = i\phi([A, B]) = \phi(J[A, B]) \end{aligned} \quad (6.6.4)$$

- a non-Abelian compact Lie algebra,  $\mathfrak{k}$ , doesn't admit a complex structure.

**proof:**

- if  $\mathfrak{k}$  admits a complex structure, it has a "multiplication by  $i$ " map,  $J \in \text{End}(\mathfrak{k})$ .
- choose the inner product on  $\mathfrak{k}$ , so that  $\text{ad}_X, \forall X \in \mathfrak{k}$  are skew self-adjoint, hence diagonalizable in  $\mathbb{C}$ , with pure-imaginary (not all-zero) eigenvalues.

- \*  $\mathfrak{k} \simeq \mathfrak{h}$  where  $\mathfrak{h}$  is a complex Lie algebra.
- \* there exists  $H = \phi(X) \in \mathfrak{h}$  and  $A = \phi(Y) \in \mathfrak{h}$ , s.t.,

$$\phi([X, Y]) = i\phi(Y) \implies [X, Y] = JaY \quad (6.6.5)$$

where  $a \in \mathbb{R}$  since  $\text{ad}_X$  has pure imaginary eigenvalues.

\* which is **impossible**, because  $\text{ad}_{JX}$  has real eigenvalue,

$$[JX, Y] = -aY \quad (6.6.6)$$

- $\mathfrak{k}$  is the Lie algebra of a compact Lie group, then,  $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is simple  $\iff \mathfrak{k}$  is simple.

**proof:**

we only need to prove  $\Leftarrow$ ,

- $\mathfrak{k}$  is simple  $\implies$  without a nontrivial center  $\implies \mathfrak{g}$  is semisimple  $\implies$  is a direct sum of simple Lie algebras (and the decomposition is unique up to ordering, see (6.1.13)),

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{g} = \bigoplus_i \mathfrak{g}_i \quad (6.6.7)$$

- if  $\mathfrak{g}_i$  is a simple ideal of  $\mathfrak{g}$ , so is  $\mathfrak{g}_i^* = \{A^* | A \in \mathfrak{g}_i\}$ , which (together with the uniqueness of decomposition) implies  $\mathfrak{g}_i^* = \mathfrak{g}_j$  for some  $j$

\* if  $\mathfrak{g}_i^* = \mathfrak{g}_i$ , then  $\mathfrak{g}_i \cap \mathfrak{k}$  is a nontrivial ideal of  $\mathfrak{k}$ , contradicts to simplicity.

\* if  $\mathfrak{g}_i^* = \mathfrak{g}_j$  with  $i \neq j$ , then let  $\mathfrak{g}' = \mathfrak{g}_i \cup \mathfrak{g}_i^*$ , we have  $\mathfrak{g}'^* = \mathfrak{g}'$ , thus  $\mathfrak{g}' \cap \mathfrak{k}$  is a nontrivial ideal of  $\mathfrak{k}$ , unless  $\mathfrak{g}' = \mathfrak{g}$ .

now, let's discuss what happens if  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^*$ , where  $\mathfrak{g}_1, \mathfrak{g}_1^*$  are both simple ideals of  $\mathfrak{g}$ .

- define a linear map (projection),

$$\begin{aligned} \phi : \mathfrak{g}_1 &\rightarrow \mathfrak{k} \\ A &\mapsto \frac{1}{2}(A + A^*) \end{aligned} \quad (6.6.8)$$

notice that for all  $A \in \mathfrak{g}_1$ , we have  $A^* \in \mathfrak{g}_1^*$ , thus  $[A, A^*] = 0$ , so,

$$\phi([A, B]) = \frac{1}{2}([A, B] + [A^*, B^*]) = \frac{1}{2}([A + A^*, B + B^*]) = [\phi(A), \phi(B)] \quad (6.6.9)$$

\* furthermore,  $\phi$  is **one-to-one**, because,

$$A + A^* = B + B^* \implies A - B = B^* - A^* \in \mathfrak{g}_1 \cap \mathfrak{g}_1^* = \{0\} \implies A = B \quad (6.6.10)$$

\*  $\phi$  is also **on-to**, because as a complex Lie algebra,  $\mathfrak{g}_1$  has the same dimension of the real Lie algebra,  $\mathfrak{k}$ , thus for every  $X \in \mathfrak{k}$ , there exists some  $A \in \mathfrak{g}_1$ , s.t.  $X = \phi(A)$ .

- so,  $\mathfrak{k}$  is isomorphic to a complex Lie algebra  $\mathfrak{g}_1$ , i.e. it **admits a complex structure**, which contradicts to compactness.

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is simple.

- $\mathfrak{g}$  is not simple  $\iff \mathfrak{h}$  decomposes into  $\mathfrak{h}_1 \oplus \mathfrak{h}_2$  and  $\mathfrak{h}_1 \perp \mathfrak{h}_2$  (orthogonal direct sum), and every root is either in  $\mathfrak{h}_1$  or  $\mathfrak{h}_2$ .

where,

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is a complex semisimple Lie algebra.
- $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$  is the complexification of the maximal Abelian subalgebra of  $\mathfrak{k}$ ,  $\mathfrak{t}$ , i.e. the Cartan subalgebra.

**proof:**

first, let's prove  $\implies$ ,

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is not simple  $\implies \mathfrak{k}$  is not simple (from the theorem above)  $\implies \mathfrak{k}_1$  is the nontrivial ideal of  $\mathfrak{k}$ , i.e. an invariant subspace of  $\text{ad} : \mathfrak{k} \rightarrow \text{End}(\mathfrak{k})$ .

- notice the adjoint representation on  $\mathfrak{k}$  is completely reducible, there is another ideal  $\mathfrak{k}_2$  s.t.  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ .
- \* if we choose the inner product so that the adjoint rep. on  $\mathfrak{k}$  is unitary, then  $\mathfrak{h}_1 \perp \mathfrak{h}_2$  (see section 1.2).
- now, we have  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_1^\perp$ , which implies  $\mathfrak{k}_\mathbb{C} = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , where  $\mathfrak{g}_i = \mathfrak{k}_{i\mathbb{C}}$ , and, of course,  $\mathfrak{g}_1 \perp \mathfrak{g}_2$ .
- the maximal Abelian subalgebra,  $\mathfrak{t}$ , decomposes as  $\mathfrak{t}_1 \oplus \mathfrak{t}_2$ , where  $\mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{k}_i$ .

-----  
**proof:**

- \* consider  $T = X + Y \in \mathfrak{t}$  with  $X \in \mathfrak{k}_1$  and  $Y \in \mathfrak{k}_2$ , then,

$$[T_1, T_2] = \underbrace{[X_1, X_2]}_{\in \mathfrak{k}_1} + \underbrace{[Y_1, Y_2]}_{\in \mathfrak{k}_2} = 0 \quad (6.6.11)$$

notice that  $\mathfrak{k}_1, \mathfrak{k}_2$  are linearly independent, so,  $[X_1, X_2] = [Y_1, Y_2] = 0$ .

- \* which means  $[X, \mathfrak{t}] = \{0\}$ , but  $\mathfrak{t}$  is maximal, so  $X \in \mathfrak{t} \cap \mathfrak{k}_1$ , similarly,  $Y \in \mathfrak{t} \cap \mathfrak{k}_2$ .
- \* so,  $\mathfrak{t}_1 = \mathfrak{t} \cap \mathfrak{k}_1$  and  $\mathfrak{t}_2 = \mathfrak{t} \cap \mathfrak{k}_2$ , then, we have the Lie algebra direct sum,  $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ .

- 
- consequently, the Cartan subalgebra decomposes as  $\mathfrak{t}_\mathbb{C} = \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ , with  $\mathfrak{h}_i = \mathfrak{t}_{i\mathbb{C}}$ , and, of course,  $\mathfrak{h}_1 \perp \mathfrak{h}_2$ .
  - every root is either in  $\mathfrak{h}_1$  or  $\mathfrak{h}_2$ .

-----  
**proof:**

- \* let  $R_i$  be the roots for  $\mathfrak{g}_i$  in  $\mathfrak{h}_i$ .  
(i.e., excuse the sloppy notation, there exists a nonzero  $A \in \mathfrak{g}_i$  s.t.  $[\mathfrak{h}_i, A] = \langle R_i, \mathfrak{h}_i \rangle A$ ).
- \* now, we claim  $R_{i=1,2} \subset R$ , because for all  $\alpha \in R_1$ ,

$$[H_1 + H_2, A] = \langle \alpha, H_1 \rangle A + 0 = \langle \alpha, H_1 + H_2 \rangle A \quad (6.6.12)$$

where we noticed that the root vector  $A \in \mathfrak{g}_1 = \mathfrak{k}_{1\mathbb{C}}$  and  $H_2 \in \mathfrak{t}_{2\mathbb{C}}$  commutes with  $A$ , and  $\alpha \in \mathfrak{h}_1 \perp \mathfrak{h}_2$ .

- \* notice that  $R - (R_1 \cup R_2)$  are the roots associated to root vectors neither in  $\mathfrak{g}_1$  nor  $\mathfrak{g}_2$ .  
· consider  $A = A_1 + A_2$ , with  $A_i \in \mathfrak{g}_i$ , is a root vector of  $\alpha \in R$ , then, consider,

$$\begin{aligned} [H_1, A_1 + A_2] &= [H_1, A_1] = \langle \alpha, H_1 \rangle A_1 \propto A_1 + A_2 \\ \implies \text{either } A_2 &= 0 \text{ or } \langle \alpha, H_1 \rangle = 0 \end{aligned} \quad (6.6.13)$$

so, if  $A_2 = 0$ , then  $\alpha \in R_1$ , else,  $\alpha \in \mathfrak{h}_2$ , which means  $\alpha \in R_2$ .

- \* so, either  $\alpha$  is in  $R_1$  or in  $R_2$ .

-----  
–  $\implies$  is proved.

now, let's prove  $\Leftarrow$ ,

- $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$  with  $\mathfrak{h}_1 \perp \mathfrak{h}_2$ , and  $R_i = R \cap \mathfrak{h}_i$ .
- then,  $\mathfrak{g}$  decomposes as,

$$\mathfrak{g} = \overbrace{\left( \mathfrak{h}_1 \oplus \bigoplus_{\alpha \in R_1} \mathfrak{g}_\alpha \right)}^{=\mathfrak{g}_1} \oplus \overbrace{\left( \mathfrak{h}_2 \oplus \bigoplus_{\beta \in R_2} \mathfrak{g}_\beta \right)}^{=\mathfrak{g}_2} \quad (6.6.14)$$

where  $\mathfrak{g}_\alpha, \forall \alpha \in R$  are linearly independent (see (6.3.3)).

- \* and it is easy to see that  $[\mathfrak{g}_\alpha, \mathfrak{h}_2] = \{0\}, \alpha \in R_1$  since  $\alpha \perp \mathfrak{h}_2$ , and,  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} = \{0\}$  if  $\alpha \in R_1, \beta \in R_2$  ( $\alpha + \beta \notin R$ ).

- so,  $\mathfrak{g}$  decomposes as the Lie algebra direct sum,  $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ , i.e. it is not simple.

## 6.7 the root systems of the classical Lie algebras

- 四个 root systems 的 Dynkin diagrams (见 section 7.6) 如下,

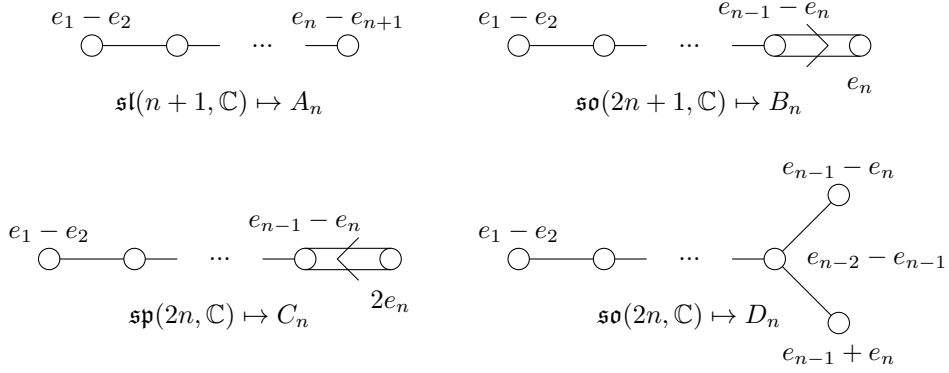


Figure 6.1: classical Dynkin diagrams

- $B_2$  and  $C_2$ ,  $A_3$  and  $D_3$  are isomorphic to each other.
- $D_2$  的 Dynkin diagram is not connected  $\implies D_2$  is reducible  $\implies \mathfrak{so}(4, \mathbb{C})$  is not simple,

$$\mathfrak{so}(4, \mathbb{C}) = \left( \text{span}(e_1 - e_2) \oplus \mathfrak{g}_{\pm(e_1 - e_2)} \right) \oplus \left( \text{span}(e_1 + e_2) \oplus \mathfrak{g}_{\pm(e_1 + e_2)} \right) \quad (6.7.1)$$

中间粗体的  $\oplus$  是 Lie algebra direct sum, (两个  $\mathfrak{su}(2)_{\mathbb{C}}$ ).

- $A_n, B_n, C_n, n \geq 1$  和  $D_n, n \geq 3$  都对应 simple Lie algebra,

$$\begin{array}{ccccccc} \mathfrak{sl}(n+1, \mathbb{C}) \mapsto A_n & \mathfrak{so}(2n+1, \mathbb{C}) \mapsto B_n & \mathfrak{sp}(2n, \mathbb{C}) \mapsto C_n & \mathfrak{so}(2n, \mathbb{C}) \mapsto D_n \\ n \geq 1 & n \geq 1 & n \geq 1 & n \geq 3 \end{array} \quad (6.7.2)$$

### 6.7.1 the special linear algebras, $\mathfrak{sl}(n+1, \mathbb{C}) = \mathfrak{su}(n+1)_{\mathbb{C}}$ , and $A_n$

- $\mathfrak{su}(n+1) = \{A \in \mathcal{M}_{n+1}(\mathbb{C}) | A^\dagger = -A \text{ and } \text{tr} A = 0\}$ , 它的 maximal commutative subalgebra 是,

$$\mathfrak{t} = \{\text{diag}(ia_1, \dots, ia_{n+1}) | a_i \in \mathbb{R} \text{ and } a_1 + \dots + a_{n+1} = 0\} \quad (6.7.3)$$

从而得到 Cartan subalgebra,  $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \{\text{diag}(\lambda_1, \dots, \lambda_{n+1}) | \lambda_i \in \mathbb{C} \text{ and } \lambda_1 + \dots + \lambda_{n+1} = 0\}$ .

- 令  $E_{ij}, i \neq j \in \{1, \dots, n+1\}$  是第  $i$  行第  $j$  列的分量为 1, 其余位置为零的矩阵,  $H = \text{diag}(\lambda_1, \dots) \in \mathfrak{h}$ , 那么,

$$[H, E_{ij}] = (\lambda_i - \lambda_j) E_{ij} \quad (6.7.4)$$

- 选择一个内积, 使得  $\text{ad}_X, \forall X \in \mathfrak{su}(n+1)$  是 skew self-adjoint,

$$\langle A, B \rangle = \text{tr}(A^\dagger B), \forall A, B \in \mathfrak{su}(n+1)_{\mathbb{C}} \quad (6.7.5)$$

**proof:**

注意这个内积在任何李代数中都保证  $\text{ad}_X, \forall X \in \mathfrak{t}$  是 skew self-adjoint, 但是根据 Cartan's criterion, 只有 semisimple 才能保证它 non-degenerate.

$$\text{tr}(A^\dagger \text{ad}_X B) = \text{tr}(A^\dagger XB - A^\dagger BX) = \text{tr}(A^\dagger XB - XA^\dagger B) = \text{tr}(-\text{ad}_X AB) \quad (6.7.6)$$

注意, 对于  $H, H' \in \mathfrak{h}$ , 有  $\langle H, H' \rangle = \sum_i \lambda_i^* \lambda'_i$ .

- 可见  $E_{ij}$  对应的 root 为,

$$[H, E_{ij}] = \underbrace{\langle e_i - e_j, H \rangle}_{=\alpha_{ij}} E_{ij}, i \neq j \quad (6.7.7)$$

- $\mathfrak{sl}(n+1, \mathbb{C})$  对应的 root system 用  $A_n$  表示,

- $E = \{v \in \mathbb{R}^{n+1} | v_1 + \cdots + v_n = 0\}$ , 所以  $\dim E = n$ .
- $R = \{\alpha_{ij} = e_i - e_j | i \neq j \in \{1, \cdots, n+1\}\}$ , 共有  $n(n+1)$  个根. ( $\dim \mathfrak{sl}(n+1, \mathbb{C}) = (n+1)^2 - 1$ )
- $\Delta = \{e_1 - e_2, \cdots, e_n - e_{n+1}\}$  is a base, and  $R^+ = \{e_i - e_j | i < j\}$ , with,

$$e_i - e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{j-1} - e_j) \quad (6.7.8)$$

- 所有根的长度为  $\sqrt{2}$ , 因此  $\langle \alpha, \beta \rangle = \langle \alpha, H_\beta \rangle$ .
- $\langle \alpha, \beta \rangle = 0, \pm 1$  (when  $\alpha \neq \pm \beta$ ).
- 两个 roots ( $\alpha \neq \pm \beta$ ) 之间的夹角可能是  $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}$ .
- 对于 base 中的根, 相邻 (consecutive) 的根夹角为  $\frac{2\pi}{3}$ , 不相邻的互相垂直, 所以其 Dynkin 图如下,

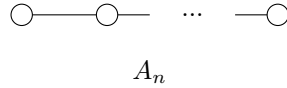


Figure 6.2: Dynkin diagram for  $A_n$

- $s_{\alpha_{ij}}$  作用到向量  $|v\rangle$  使其  $i, j$  分量的位置交换, 因此  $A_n$  的 Weyl 群是  $n+1$  个元素的 permutation group.

### 6.7.2 the orthogonal algebras, $\mathfrak{so}(2n, \mathbb{C})$ , and $D_n$

- $\mathfrak{so}(2n, \mathbb{R}) = \mathfrak{o}(2n, \mathbb{R}) = \{A \in \mathcal{M}_{2n}(\mathbb{R}) | A^T = -A\}$ , 它的 maximal commutative subalgebra 是,

$$\mathfrak{t} = \{H_a = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} | a = \text{diag}(a_1, \cdots, a_n) \text{ with } a_i \in \mathbb{R}\} \quad (6.7.9)$$

#### proof:

任何  $\mathfrak{so}(2n, \mathbb{C})$  中的元素都可以展开成  $\mathfrak{h} = \mathfrak{t}_\mathbb{C}$  和  $D_{ij}^\alpha$  (见下文) 的叠加, 那么, 与  $\mathfrak{h}$  对易的元素一定不含有  $D_{ij}^\alpha$  分量, 所以... 是 maximal. (总共有  $2n^2 - 2n$  个根, 且 rank 为  $n$ , 所以总维数为  $2n^2 - n = \frac{2n(2n-1)}{2}$ )  
另外, 注意如果  $n = 2$ ,  $D_{11}^1 = D_{11}^2 = 0$  而,

$$D_{11}^3 = -D_{11}^4 = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \in \mathfrak{h} \quad (6.7.10)$$

也即  $\mathfrak{so}(2, \mathbb{C}) = \mathfrak{h}$ , 与不存在 nontrivial center 的对应不符, 不是 semisimple.

- the root vectors are  $D_{ij}^\alpha = C_{ij}^\alpha - (C_{ij}^\alpha)^T$ , where  $\alpha = 1, 2, 3, 4$  and,

$$C_{ij}^1 = \begin{pmatrix} E_{ij} & iE_{ij} \\ iE_{ij} & -E_{ij} \end{pmatrix} \quad C_{ij}^2 = \begin{pmatrix} E_{ij} & -iE_{ij} \\ -iE_{ij} & -E_{ij} \end{pmatrix} \quad C_{ij}^3 = \begin{pmatrix} E_{ij} & -iE_{ij} \\ iE_{ij} & E_{ij} \end{pmatrix} \quad C_{ij}^4 = \begin{pmatrix} E_{ij} & iE_{ij} \\ -iE_{ij} & E_{ij} \end{pmatrix} \quad (6.7.11)$$

where  $i \neq j \in \{1, \cdots, n\}$  (如果  $i = j$ , 那么  $D_{ii}^{1,2} = 0, D_{ii}^{3,4} \in \mathfrak{h}$ ), and we have,

$$\begin{aligned} [H_a, D_{ij}^1] &= i(a_i + a_j)D_{ij}^1 & [H_a, D_{ij}^2] &= -i(a_i + a_j)D_{ij}^2 \\ [H_a, D_{ij}^3] &= i(a_i - a_j)D_{ij}^3 & [H_a, D_{ij}^4] &= -i(a_i - a_j)D_{ij}^4 \end{aligned} \quad (6.7.12)$$

#### calculation:

we have  $D_{ij}^1 = C_{ij}^1 - C_{ji}^1, D_{ij}^2 = C_{ij}^2 - C_{ji}^2, D_{ij}^3 = C_{ij}^3 - C_{ji}^4, D_{ij}^4 = C_{ij}^4 - C_{ji}^3$ , and,

$$\begin{aligned} [H_a, C_{ij}^1] &= i(a_i + a_j)C_{ij}^1 & [H_a, C_{ij}^2] &= -i(a_i + a_j)C_{ij}^2 \\ [H_a, C_{ij}^3] &= i(a_i - a_j)C_{ij}^3 & [H_a, C_{ij}^4] &= -i(a_i - a_j)C_{ij}^4 \end{aligned} \quad (6.7.13)$$



- 内积定义为  $\langle A, B \rangle = \frac{1}{2} \text{tr}(A^\dagger B)$ , 那么,

$$\langle H_a, H_b \rangle = - \sum_{i=1}^n a_i^* b_i \quad (6.7.14)$$

所以, 可以将  $H_a$  视作  $i(a_1, \dots, a_n)$ .

- 可见 root vectors 和 roots 的对应关系为  $(i \neq j \in \{1, \dots, n\})$ ,

$$D_{ij}^1 \mapsto \alpha_{ij} = e_i + e_j \quad D_{ij}^2 \mapsto -\alpha_{ij} \quad D_{ij}^3 \mapsto \beta_{ij} = e_i - e_j \quad D_{ij}^4 \mapsto -\beta_{ij} \quad (6.7.15)$$

- $\mathfrak{so}(2n, \mathbb{C})$  对应的 root system 用  $D_n$  表示,

–  $E = \mathbb{R}^n$ .

–  $R = \{\pm e_i \pm e_j | i \neq j \in \{1, \dots, n\}\}$ , 共有  $\frac{n(n-1)}{2} \times 4 = 2n^2 - 2n$  个根. ( $\dim \mathfrak{so}(2n, \mathbb{C}) = \frac{2n(2n-1)}{2}$ )

–  $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\} \cup \{e_{n-1} + e_n\}$  is a base, and  $R^+ = \{e_i - e_j | i < j\} \cup \{e_i + e_j\}$ , with,

$$e_i + e_j = \underbrace{(e_i - e_{i+1}) + \dots + (e_{n-1} + e_n)}_{=e_i + e_n} + \underbrace{(e_j - e_{j+1}) + \dots + (e_{n-1} - e_n)}_{=e_j - e_n} \quad (6.7.16)$$

– 所有根的长度为  $\sqrt{2}$ , 因此也有  $\langle \alpha, \beta \rangle = \langle \alpha, H_\beta \rangle$ .

–  $\langle \alpha, \beta \rangle = 0, \pm 1$  (when  $\alpha \neq \pm \beta$ ), 所以两个根之间的夹角可能是  $\frac{\pi}{2}$  或  $\frac{\pi}{3}, \frac{2\pi}{3}$ .

–  $D_n$  的 Dynkin 图如下,

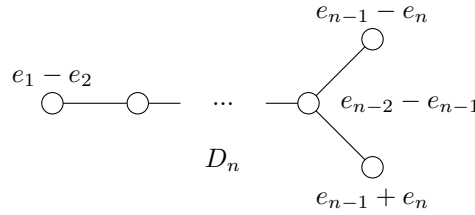


Figure 6.3: Dynkin diagram for  $D_n$

–  $s_\alpha = s_{-\alpha}, \alpha \in R$  分别为,

$$\begin{cases} s_{\alpha_{ij}} : (\dots, v_i, \dots, v_j, \dots) \mapsto (\dots, -v_j, \dots, -v_i, \dots) \\ s_{\beta_{ij}} : (\dots, v_i, \dots, v_j, \dots) \mapsto (\dots, v_j, \dots, v_i, \dots) \end{cases} \quad (6.7.17)$$

### 6.7.3 the orthogonal algebras, $\mathfrak{so}(2n+1, \mathbb{C})$ , and $B_n$

- its maximal commutative subalgebra is,

$$\mathfrak{t} = \left\{ \left( \begin{array}{cc|c} 0 & a & \\ -a & 0 & \\ \hline & & 0 \end{array} \right) \mid a = \text{diag}(a_1, \dots, a_n) \text{ with } a_i \in \mathbb{R} \right\} \quad (6.7.18)$$

both  $\mathfrak{so}(2n+1, \mathbb{C})$  and  $\mathfrak{so}(2n, \mathbb{C})$  have rank  $n$ .

- every root in  $\mathfrak{so}(2n, \mathbb{C})$  is a root in  $\mathfrak{so}(2n+1, \mathbb{C})$ , but there are  $2n$  additional roots in  $\mathfrak{so}(2n+1, \mathbb{C})$ .

- the additional root vectors are,

$$B_k^1 = \left( \begin{array}{ccc|ccc|c} & & & & & & \vdots \\ & & & & & & 1 \\ & & & & & & \vdots \\ \hline & & & & & & \vdots \\ & & & & & & i \\ & & & & & & \vdots \\ \hline \dots & -1 & \dots & \dots & -i & \dots & 0 \end{array} \right) \quad B_k^2 = \left( \begin{array}{ccc|ccc|c} & & & & & & \vdots \\ & & & & & & 1 \\ & & & & & & \vdots \\ \hline & & & & & & \vdots \\ & & & & & & -i \\ & & & & & & \vdots \\ \hline \dots & -1 & \dots & \dots & i & \dots & 0 \end{array} \right) \quad (6.7.19)$$

其中  $B_k^{1,2}$  的非零元素位于  $(k, 2n+1), (n+k, 2n+1)$  和通过转置相对应的位置, 有对易关系,

$$[H_a, B_k^1] = ia_k B_k^1 \quad [H_a, B_k^2] = -ia_k B_k^2 \quad (6.7.20)$$

- 选取与上一 subsection 一样的内积, 那么 root vectors 和 roots 的对应关系为,

$$B_k^1 \mapsto e_k \quad B_k^2 \mapsto -e_k \quad (6.7.21)$$

- $\mathfrak{so}(2n+1, \mathbb{C})$  对应的 root system 用  $B_n$  表示,

- $E = \mathbb{R}^n$ .
- $R = \{\pm e_i \pm e_j \text{ and } \pm e_k | i \neq j, k \in \{1, \dots, n\}\}$ , 共有  $2n^2$  个根. ( $\dim \mathfrak{so}(2n+1, \mathbb{C}) = \frac{(2n+1)2n}{2}$ )
- $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\} \cup \{e_n\}$  is a base, and  $R^+ = \{e_i - e_j | i < j\} \cup \{e_i + e_j\} \cup \{e_k\}$ .
- $\langle \alpha, \beta \rangle = 0, \pm 1$  (when  $\alpha \neq \pm \beta$ ), 所以两个根之间的夹角可能为  $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}$ .
- $B_n$  的 Dynkin 图如下,

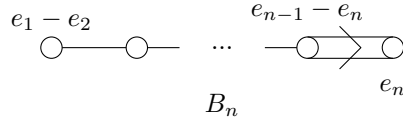


Figure 6.4: Dynkin diagram for  $B_n$

#### 6.7.4 the symplectic algebras, $\mathfrak{sp}(2n, \mathbb{C})$ , and $C_n$

- $\mathfrak{sp}(2n, \mathbb{C}) = \{A \in \mathcal{M}_{2n}(\mathbb{C}) | \Omega A^T \Omega = A\}$ , where,

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (6.7.22)$$

$\mathfrak{sp}(2n, \mathbb{C})$  中的矩阵可以写成如下形式,

$$A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \quad (6.7.23)$$

where  $a, b, c \in \mathcal{M}_n(\mathbb{C})$ , and  $b, c$  are symmetric.

- 可以认为  $\mathfrak{k} = \mathfrak{sp}(2n, \mathbb{C}) \cap \mathfrak{u}(2n)$  是其 compact real form,

$$\mathfrak{sp}(2n, \mathbb{C}) \cap \mathfrak{u}(2n) = \left\{ \begin{pmatrix} a & b \\ -b^\dagger & -a^T \end{pmatrix} \mid a^\dagger = -a, b^T = b \right\} \quad (6.7.24)$$

- the maximal commutative subalgebra of  $\mathfrak{k}$  is,

$$\mathfrak{t} = \{H_a = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a = \text{diag}(a_1, \dots, a_n), i a_i \in \mathbb{R}\} \quad (6.7.25)$$

- the root vectors are ( $i \neq j$ ),

$$\begin{aligned} A_{ij} &= \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix} & B_{ij} &= \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix} & C_{ij} &= \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \\ F_k &= \begin{pmatrix} 0 & E_{kk} \\ 0 & 0 \end{pmatrix} & G_k &= \begin{pmatrix} 0 & 0 \\ E_{kk} & 0 \end{pmatrix} \end{aligned} \quad (6.7.26)$$

对易关系为,

$$\begin{aligned} [H_a, A_{ij}] &= (a_i + a_j) A_{ij} & [H_a, B_{ij}] &= -(a_i + a_j) B_{ij} & [H_a, C_{ij}] &= (a_i - a_j) C_{ij} \\ [H_a, F_k] &= 2a_k F_k & [H_a, G_k] &= -2a_k G_k \end{aligned} \quad (6.7.27)$$

- 选取内积为  $\langle A, B \rangle = \frac{1}{2} \text{tr}(A^\dagger B)$ , 所以  $H_a$  可以视为  $(a_1, \dots, a_n)$ , 那么 root vectors 和 roots 的对应关系为,

$$A_{ij} \mapsto e_i + e_j \quad B_{ij} \mapsto -e_i - e_j \quad C_{ij} \mapsto e_i - e_j \quad F_k \mapsto 2e_k \quad G_k \mapsto -2e_k \quad (6.7.28)$$

- $\mathfrak{sp}(2n, \mathbb{C})$  对应的 root system 用  $C_n$  表示,

- $E = \mathbb{R}^n$ .
- $R = \{\pm e_i \pm e_j \text{ and } \pm 2e_k | i \neq j, k \in \{1, \dots, n\}\}$ , 与  $B_n$  相似 (区别是  $\pm e_k$  前的系数 2), 共有  $2n^2$  个根. ( $\dim \mathfrak{sp}(2n, \mathbb{C}) = n(2n+1)$ )
- $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\} \cup \{2e_n\}$  and  $R = \{e_i - e_j | i < j\} \cup \{e_i + e_j\} \cup \{2e_k\}$ .
- $\langle \alpha, \beta \rangle = 0, \pm 1, \pm 2$  (when  $\alpha \neq \pm \beta$ ), 所以两个根之间夹角可能为  $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}$ .
- $C_n$  的 Dynkin 图如下,

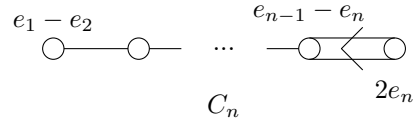


Figure 6.5: Dynkin diagram for  $C_n$

# Chapter 7

## root systems

### 7.1 abstract root systems

- **def.:** a **root system**  $(E, R)$  is a finite-dimensional vector space  $E = \text{span}(R)$ , with a finite collection of non-zero vectors  $R$ , and an inner product  $\langle \cdot, \cdot \rangle$ , and,

1.  $E = \text{span}(R)$ ,
2. if  $\alpha \in R$ , then  $c\alpha \in R \iff c = \pm 1$ ,
3. if  $\alpha, \beta \in R$ , then  $s_\alpha |\beta\rangle \in R$ , where  $s_\alpha = 1 - 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ ,
4. for all  $\alpha, \beta \in R$ ,  $2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ .

$\dim E$  is called the **rank** of the system, elements in  $R$  are called **roots**.

- **def.:** the **Weyl group**,  $W$ , of  $R$  is the finite subgroup of the orthogonal group of  $E$  generated by  $s_\alpha, \forall \alpha \in R$ .

- **def.:**  $(E, R)$  and  $(F, S)$  are two root systems, then  $(E \oplus F, R \cup S)$  is a root system, and  $R \cup S$  is called the **direct sum** of  $R$  and  $S$ .

(it is easy to see the direct sum root system satisfies the def. of root systems)

- **def.:** a root system is called **reducible** if there exists an orthogonal decomposition  $E = E_1 \oplus E_2$  with  $E_1 \perp E_2$  and  $\dim E_i > 0$ , and every root is either in  $E_1$  or  $E_2$ .

- the root system of a semisimple Lie algebra is irreducible  $\iff$  the semisimple Lie algebra is simple (见 section 6.6 最后一个定理).

- **def.:** an **isomorphism** is a linear map that **preserves the reflection**, not the inner product,

$$A : E \rightarrow F \quad \text{s.t.} \quad As_\alpha |\beta\rangle = s_{A\alpha} |A\beta\rangle \quad (7.1.1)$$

- 对于  $\langle \beta, \beta \rangle \leq \langle \alpha, \alpha \rangle$ , 且  $\beta \not\propto \alpha$ , 根  $\alpha, \beta$  之间可能的关系如下,

- $\beta \perp \alpha$ .
- or,  $\langle \alpha, \alpha \rangle = 1, 2, 3 \langle \beta, \beta \rangle$  (图中没有画出  $\beta \mapsto -\beta$  的情况, 那时夹角是图中夹角的补角).

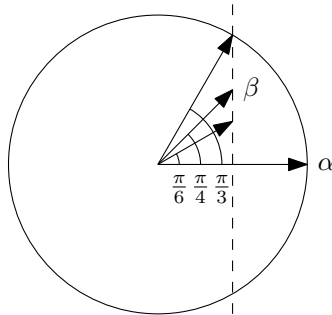


Figure 7.1: the basic acute angles and length ratios

- 如果根  $\alpha, \beta$  之间夹锐角, 那么  $\pm(\alpha - \beta)$  也是根; 如果夹钝角, 那么  $\pm(\alpha + \beta)$  也是根.

**proof:**

假设  $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$ , 考虑夹锐角的情况, 此时,  $\beta - \alpha = s_\alpha |\beta\rangle$ ; 对于夹钝角的情况, 令  $\beta' = -\beta$  即可.

## 7.2 rank-two systems

- if rank is one, the roots are  $R = \{-\alpha, \alpha\}$ .
- **every rank-two system** is isomorphic to one of the systems below,



Figure 7.2: the rank-two root systems

分别考虑两个根之间最小夹角为  $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$  的情况, 然后使用  $s_\alpha |\beta\rangle$  生成整个  $R$ .

for positive simple roots, positive roots, negative roots and Weyl chambers, see section 7.4.

- the **Weyl group** of a rank-two root system,  $R$ , with minimum angle  $\theta = \frac{2\pi}{n}$  is the symmetry group of a regular  $\frac{n}{2}$ -gon (正  $\frac{n}{2}$  边形).
  - 群元素包括  $\frac{n}{2}$  个镜面反射和  $2\theta$  转动.

## 7.3 duality

- **def.:** for a root  $\alpha \in R$  in a root system  $(E, R)$ , its **coroot** is,

$$H_\alpha = 2 \frac{\alpha}{\langle \alpha, \alpha \rangle} \quad \text{with} \quad \begin{cases} s_{H_\alpha} = s_\alpha \\ \frac{\langle H_\alpha, H_\beta \rangle}{\langle H_\alpha, H_\alpha \rangle} = \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \end{cases} \quad (7.3.1)$$

and the **dual root system** to  $R$  is  $R^\vee = \{H_\alpha | \alpha \in R\}$ .

- $R^\vee$  is also a root system, with the same Weyl group as  $R$  (because  $s_{H_\alpha} = s_\alpha$ ).
- $H_{H_\alpha} = \alpha$  and  $(R^\vee)^\vee = R$ .
- note that although  $H_{s_\alpha |\beta\rangle} = s_{H_\alpha} |H_\beta\rangle$ , the map  $H$  is not linear, so  $R^\vee$  and  $R$  are not necessarily isomorphic to each other.

## 7.4 bases and Weyl chambers

- **def.:** for a root system  $(E, R)$ , a subset  $\Delta \subset R$  is called a **base** if,
  1.  $\Delta$  is a basis of  $E$ ,

2. each root  $\alpha \in R$  can be expressed as a linear combination of basis vectors in  $\Delta$  with non-negative (positive roots,  $R^+$ ) or non-positive (negative roots,  $R^-$ ) integer coefficients,  $R = R^+ \cup R^-$ .

elements in  $\Delta$  are called **positive simple roots**.

- $\alpha \neq \beta \in \Delta$ , then  $\langle \alpha, \beta \rangle \leq 0$ .

**proof:**

如果  $\alpha, \beta$  之间夹锐角, 那么  $\pm(\alpha - \beta)$  也是根, 不满足系数同时非负 (或非正) 的要求.

- for a root system  $(E, R)$ , there exists a hyperplane  $V$  through the origin in  $E$ , s.t.  $V$  does not contain any root.

**proof:**

考虑一个向量  $H \in E$ , 它不在任何一个垂直于某个根向量的超平面 (这样的超平面有限多, 所以  $H$  存在) 上, 那么  $V \perp H$  就是我们要找的超平面.

- **def.:** choose one side of  $V$  to be  $R^+$ , the other side to be  $R^-$ , an element  $\alpha \in R^+$  is **decomposable** if  $\alpha = \beta + \gamma$  for some  $\beta, \gamma \in R^+$ , otherwise,  $\alpha$  is **indecomposable**.
- the indecomposable roots in  $R^+$  form the base  $\Delta$ , and  $\Delta$  exists.

**proof:**

let  $\Delta$  denote the set of indecomposable elements in  $R^+$ , now we will prove  $\Delta$  is the base:

- every  $\alpha \in R^+$  can be expressed as a linear combination of elements in  $\Delta$  with non-negative integer coefficients.

**proof:**

- \* 考虑  $H \perp V$ , 且  $\langle \alpha, H \rangle > 0, \forall \alpha \in R^+$ .
- \* 考虑  $\Delta'$  是不能表示成  $\Delta$  的元素的非负整数系数的线性叠加的  $R^+$  元素的集合, 那么一定有  $\Delta' \cap \Delta = \emptyset$ .
- \* 考虑  $\alpha \in \Delta'$  且  $\langle \alpha, H \rangle$  是  $\Delta'$  中元素里最小的, 而且  $\alpha = \beta_1 + \beta_2$  (且  $\beta_1, \beta_2 \in R^+$ ), 那么  $\beta_1, \beta_2$  至少有一个是  $\Delta'$  的元素, 但是  $\langle \alpha, H \rangle = \langle \beta_1, H \rangle + \langle \beta_2, H \rangle$  这与  $\langle \alpha, H \rangle$  最小矛盾.
- \* 可见  $\beta_1, \beta_2 \notin \Delta'$ ,  $\alpha$  一定可以表示为  $\Delta$  的元素的... 的线性叠加.

- elements in  $\Delta$  are linearly independent.

**proof:**

如果,

$$\sum_{\alpha \in \Delta} c'_\alpha \alpha = 0 \implies \sum_{\alpha} c_\alpha \alpha = \sum_{\beta} d_\beta \beta = u \in R^+ \quad (7.4.1)$$

其中  $c_\alpha \geq 0, -d_\beta < 0$  分别是  $\{c'_\alpha\}$  中非负和负的系数, 等号两边对  $\Delta$  的两个无交集的子集求和.

考虑,

$$\langle u, u \rangle = \langle \sum_{\alpha} c_\alpha \alpha, \sum_{\beta} d_\beta \beta \rangle = \sum_{\alpha, \beta} c_\alpha d_\beta \langle \alpha, \beta \rangle \quad (7.4.2)$$

但是, 对于  $\alpha \neq \beta \in \Delta$ , 一定有  $\langle \alpha, \beta \rangle \leq 0$ , 所以  $\langle u, u \rangle = 0$ , 即  $u = 0$ , 这与  $\alpha \in R^+$  矛盾.

**proof of  $\langle \alpha, \beta \rangle \leq 0, \forall \alpha \neq \beta \in \Delta$ :**

如果  $\alpha, \beta$  呈锐角, 那么  $\pm(\alpha - \beta)$  也是根, 且其中一个属于  $R^+$ , 比如  $\alpha - \beta \in R^+$ , 那么  $\alpha = (\alpha - \beta) + \beta$ , 与 indecomposable 矛盾.

最后, 注意到 indecomposable root 一定存在. 只需考虑  $\langle \alpha, H \rangle$  值最小的  $\alpha \in R^+$  即可证明存在.

- for any base  $\Delta$  for  $R$ , there exists a hyperplane  $V$ , s.t.  $\Delta$  arises as in the theorem above.

**proof:**

$\Delta$  是一组基底, 张成向量空间中的一个锥形, 存在一个区域, 这个区域中的每个向量都与基底夹角 (这个区域就是 fundamental Weyl chamber), 那么  $V$  就是垂直于这个区域中的某个矢量的超平面.

由于基向量线性无关, 所以任何基向量都不可分解 (indecomposable).

- $\alpha \in \Delta$  cannot be expressed as a linear combination of  $R^+ - \Delta$  with non-negative real (not integer) coefficients.

**proof:**

let  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ , suppose,

$$\alpha_1 = \sum_{\beta \in R^+ - \Delta} c_\beta \beta = \sum_{\beta, i} c_\beta d_{\beta, i} \alpha_i \quad (7.4.3)$$

where  $d_{\beta, i}$  are non-negative integers.

if  $c_\beta$  are non-negative, it will contradict to the linear independence.

- $\{H_\alpha | \alpha \in \Delta\}$  is the base of  $R^\vee$ .

**proof:**

– 首先, 选取  $\Delta$  对应的  $V$ , 并以这个平面推出  $\Delta^\vee$  (这个 base 存在), 那么  $H_\alpha \in R^{\vee+} \iff \alpha \in R^+$ .

– 考虑  $\alpha \in R^+ - \Delta$ , 那么  $\alpha$  是  $\alpha_1, \dots, \alpha_r$  的非负整数的线性叠加, 那么  $H_\alpha$  是  $H_{\alpha_1}, \dots, H_{\alpha_r}$  的非负实数的线性叠加.

– 根据上一个 theorem 可知  $H_\alpha \notin \Delta^\vee$  且  $H_{\alpha_1}, \dots, H_{\alpha_r}$  是  $E$  的基底, 所以一定有  $\Delta^\vee = \{H_{\alpha_1}, \dots, H_{\alpha_r}\}$ .

- **def.:** the open Weyl chambers for a root system  $(E, R)$  are connected components of,

$$E - \bigcup_{\alpha \in R} V_\alpha \quad (7.4.4)$$

where  $V_\alpha \perp \alpha$  is a hyperplane through the origin.

- **def.:** the open fundamental Weyl chamber (relative to  $\Delta$ ) is  $\{H | \langle \alpha, H \rangle > 0, \forall \alpha \in \Delta\}$ .
  - open fundamental Weyl chamber is connected (consider  $\langle H, \beta \rangle > \langle H, \alpha \rangle, \alpha \in \Delta, \beta \in R^+ - \Delta, H \perp V$ ).
  - every elements in the open fundamental Weyl chamber has a positive inner product with root in  $R^+$ , and negative inner product with root in  $R^-$ , so open fundamental Weyl chamber is an open Weyl chamber.
- for each open Weyl chamber  $C$ , there exists a unique base  $\Delta_C$ , s.t.  $C$  is the open fundamental Weyl chamber relative to  $\Delta_C$ .
  - there is a one-to-one correspondence between bases and Weyl chambers.

**proof:**

考虑  $H \in C$ , 以  $V \perp H$  建立起的 base 就是  $\Delta_C$ .

考虑  $\Delta, \Delta'$  都对应同一个  $C$ , 它们的  $R^+ = R'^+$ , 且可以选取  $V = V'$ , 那么一定有  $\Delta = \Delta'$  (都是不可分解的根).

- every root is an element of some base.

**proof:**

任何一个根  $\alpha$  对应的  $V_\alpha \perp \alpha$  都包含某个 open Weyl chamber  $C$  的边界。  
考虑  $H \in V_\alpha$  且  $H + \epsilon\alpha \in C$ , 选取  $V \perp H' = H + \epsilon\alpha$ , 显然  $\langle \alpha, H' \rangle$  是  $R^+$  中最小的, 所以一定有  $\alpha \in \Delta_C$ .

## 7.5 Weyl chambers and Weyl group

- the Weyl group act **transitively** on the set of Weyl chambers, i.e. for every open Weyl chamber  $C$ , we have,

$$\{w(C) | w \in W\} = E - \bigcup_{\alpha \in R} V_\alpha \quad (7.5.1)$$

**proof:**

consider chamber  $C$  with its base  $\Delta_C$ , we want to prove that  $wH' \in C$  for all  $H' \in E - \bigcup_{\alpha \in R} V_\alpha$  ( $H' \in C$  case is trivial) and  $w \in W'$  where  $W'$  is generated by  $s_\alpha, \alpha \in \Delta_C$ .

- in the case when  $H' \notin C$ , there exists some  $\alpha \in \Delta_C$  that  $\langle \alpha, H' \rangle < 0$  (夹钝角).
- since  $W'$  is a finite group, there exists a  $w \in W'$  that bring  $H'$  closest to some  $H \in C$ .
- if  $wH' \notin C$ , then there exists  $\alpha \in \Delta_C$  that  $\langle \alpha, wH' \rangle < 0$ , then,

$$\begin{aligned} |wH' - H|^2 - |s_\alpha wH' - H|^2 &= 2 \langle wH' | s_\alpha - 1 | H \rangle \\ &= -4 \frac{\langle wH' | \alpha \rangle \langle \alpha | H \rangle}{\langle \alpha, \alpha \rangle} > 0 \end{aligned} \quad (7.5.2)$$

which contradicts to the closest-ness.

- so, we must have  $wH' \in C$ .

- $W$  is generated by  $s_\alpha, \alpha \in \Delta$ .

**proof:**

we want to prove that for all  $\alpha$ , there exists some  $w \in W'$  (generated by  $s_\beta, \beta \in \Delta_C$ ) s.t.,

$$s_{w|\alpha} = ws_\alpha w^{-1} \in W' \quad (7.5.3)$$

- let  $\alpha \in \Delta_D$  where  $D$  is some chamber.
- we already proved that there is some  $w \in W'$  that  $w[D] = C$ , since  $w$  preserves inner product,  $w[\Delta_D] = \Delta_C$ .
- so,  $w|\alpha \rangle \in \Delta_C$ , i.e.  $s_{w|\alpha} \in W'$ .

- def.:** the **minimal expression** of  $w \in W$  is the expression of  $w$  in terms of  $s_\alpha, \alpha \in \Delta$  with the minimal number of  $s_\alpha$  (the minimal expression need not be unique).
- $\bar{C}$  is the closure of a Weyl chamber  $C$ , if  $H, H' \in \bar{C}$  and  $w|H \rangle = H'$ , then  $H = H'$ .  
i.e. two distinct elements of  $\bar{C}$  cannot be in the same orbit of  $W$ .

**proof:**

we proceed by induction on the number of the minimal expression of  $w$  in terms of  $s_\alpha, \alpha \in \Delta_C$ .

- if the minimal number is zero, i.e.  $w = I$ , the result holds.
  - if the result holds when the minimal number is  $k - 1$ , then, consider  $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ .
  - $C$  and  $w[C]$  lie on opposite sides of hyperplane  $V_{\alpha_1}$ , i.e.  $\bar{C} \cap w[\bar{C}] \subset V_{\alpha_1}$ .
-



**proof:**

let's prove by induction. for  $w = s_{\alpha_1}$ , the result holds, consider  $w = us_{\alpha_k}$ , where  $u = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{k-1}}$ ,

- \*  $C$  and  $u[C]$  lie on opposite sides of  $V_{\alpha_1}$  (by induction).
- \* if  $C$  and  $w[C]$  lie on the same side, then  $w[C] = u \circ s_{\alpha_k}[C]$  lies on the opposite side of  $u[C]$ , i.e.  $C$  and  $s_{\alpha_k}[C]$  lie on opposite sides of  $V_{u^{-1}|\alpha_1\rangle}$ .
- \* notice that  $\alpha_k \in \Delta_C$ , consider  $H \in V_{\alpha_k}$  which also lies on the boundary of  $C$ , then,  $s_{\alpha_k}H = H$  also lies on the boundary of  $s_{\alpha_k}[C]$ , which implies  $V_{u^{-1}|\alpha_1\rangle} = V_{\alpha_k}$ , so,

$$u^{-1}s_{\alpha_1}u = s_{u^{-1}|\alpha_1\rangle} = s_{\alpha_k} \implies w = s_{\alpha_1}u = s_{\alpha_2}\cdots s_{\alpha_k} \quad (7.5.4)$$

which contradicts to the minimal expression assumption.

- 
- since  $w|H\rangle = H' \in w[\bar{C}] \cap \bar{C} \subset V_{\alpha_1}$ , which implies,

$$s_{\alpha_1}H' = H' = s_{\alpha_2}\cdots s_{\alpha_k}H \quad (7.5.5)$$

by induction,  $H = H'$ .

- if  $H \in C$  for some chamber  $C$ , and  $w|H\rangle = H$ , then,  $w = I$  ( $W$  acts **freely**).

**proof:**

since  $w|H\rangle \in C$ , and  $w$  is a continuous map, so we must have  $w[C] = C$ , i.e. for all  $H' \in C$ , we have  $w|H'\rangle \in C \implies w|H'\rangle = H'$  (according to the theorem above), then  $w = I$ .

- $W$  acts **freely** and **transitively** on Weyl chambers, the same is true for bases, i.e. for two bases  $\Delta_1, \Delta_2$ , there exists (transitiveness) a unique (free-ness)  $w$ , s.t.  $w[\Delta_1] = \Delta_2$ .
- $C$  is a Weyl chamber,  $H \in E$ , then there is exactly one point in the  $W$ -orbit of  $H$  that lies in  $\bar{C}$  (but the  $w$  that  $w|H\rangle \in C$  is not necessarily unique).

**proof:**

- $H$  is in the closure of some chamber  $D$ , and there exists a  $w$  that  $w[\bar{D}] = \bar{C}$ , so  $w|H\rangle \in \bar{C}$ .
- if  $H', H'' \in \bar{C}$  are point in the  $W$ -orbit of  $H$ , then  $H' = H''$ .

- for all  $\alpha \in \Delta, \beta \in R^+$ , and  $\beta \neq \alpha$ , we have  $s_\alpha|\beta\rangle \in R^+$ .

**proof:**

- write  $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$  with  $c_\gamma \in \mathbb{Z}^+$ .
- notice that  $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ , so,  $s_\alpha|\beta\rangle = \beta - n\alpha$  for some integer  $n$ .
- in the expansion,

$$s_\alpha|\beta\rangle = \sum_{\gamma \in \Delta - \{\alpha\}} c_\gamma \gamma + (c_\alpha - n)\alpha \quad (7.5.6)$$

only the coefficient  $c_\alpha$  changes.

- if one coefficient is positive in the expansion, all other coefficients must be positive, so  $s_\alpha|\beta\rangle \in R^+$ .

## 7.6 Dynkin diagrams

- **def.:**  $\Delta = \{\alpha_1, \dots, \alpha_r\}$  is the base of  $R$ , the **Dynkin diagram** for  $R$  is:

1. 图中有  $r$  个结点,
2. 节点  $v_i, v_j$  之间根据  $\alpha_i, \alpha_j$  之间的夹角决定连线的条数,  $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$  分别对于 0, 1, 2, 3 条连线,
3. 如果  $\alpha_i, \alpha_j$  长度不同, 连线上画出一条指向更短的根的箭头 (可以将箭头视作大于符号).

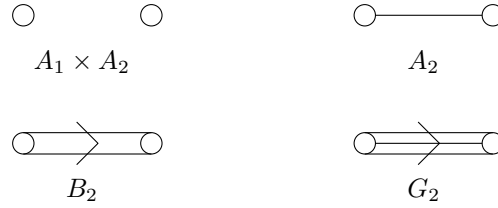


Figure 7.3: Dynkin diagrams for the rank-two root systems

- 注意, 夹角为  $\frac{2\pi}{3}, \frac{3\pi}{4}$  的根长度一定不相等, 即, 2, 3 条线上一定有箭头; 相反, 一条线上一定没有箭头.
- 同一个 root system 的两个  $\Delta_1, \Delta_2$  的 Dynkin 图一定完全相同 (isomorphic).

**proof:**

there exists  $w \in W$  s.t.  $w[\Delta_1] = \Delta_2$ , and  $w$  preserves angles and lengths.

- a root system is irreducible (see section 7.1)  $\iff$  its Dynkin diagram is connected.
  - semisimple Lie algebra  $\mathfrak{g}$  is **simple**  $\iff$  the Dynkin diagram of  $R \subset \mathfrak{g}$  is **connected**.

**proof:**

如果  $R$  是 reducible, 那么  $\Delta = \Delta_1 \cup \Delta_2$  且  $\Delta_1 \perp \Delta_2$ , 则 Dynkin 图一定 not connected.

反之, Dynkin 图 not connected  $\implies \Delta = \Delta_1 \cup \Delta_2$  且  $\Delta_1 \perp \Delta_2$ , 那么  $E = E_1 \oplus E_2$  with  $E_i = \text{span}(\Delta_i)$ .

Weyl 群由  $s_\alpha, \alpha \in \Delta$  生成, 而  $s_\alpha, \alpha \in \Delta_1$  在  $E_2$  上是单位映射, 可见  $W = W_1 \times W_2$ , 因此,  $R = W[\Delta] = W_1[\Delta_1] \cup W_2[\Delta_2] = R_1 \cup R_2$ , 即根要么属于  $E_1$  要么属于  $E_2$ .

- Dynkin diagrams are isomorphic  $\iff$  root systems are isomorphic.

## 7.7 integral and dominant integral elements

- **def.:** an element  $\mu \in E$  is an **integral element** if for all  $\alpha \in R$ ,

$$\langle \mu, H_\alpha \rangle = 2 \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad (7.7.1)$$

$\mu$  is **dominant** (relative to  $\Delta$ ) if  $\langle \mu, \alpha \rangle \geq 0, \forall \alpha \in \Delta$ , and **strictly dominant** if  $\langle \mu, \alpha \rangle > 0, \forall \alpha \in \Delta$ .

- $\mu$  is (strictly) dominant (relative to  $\Delta_C$ )  $\iff \mu \in \bar{C}$  (or  $C$ ).
- for all  $\mu$ , there exists  $w \in W$  s.t.  $w|\mu \in \bar{C}$ .
- every integer linear combination of roots (e.g.  $2\alpha + 3\beta + 5\gamma$ ) is an integral element. 但一般不是所有 integral elements 都是根的整数线性组合.
- 注意  $\{H_\alpha | \alpha \in \Delta\}$  是  $R^\vee$  的 base (见 section 7.4), 所以  $\langle \mu, H_\alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta \implies \mu$  是 integral element.
- **def.:** the **fundamental weights** (relative to  $\Delta = \{\alpha_1, \dots, \alpha_l\}$ ) are  $\mu_1, \dots, \mu_r$  s.t.,

$$\langle \mu_i, H_{\alpha_j} \rangle = \delta_{ij} \quad (7.7.2)$$

i.e. the dual basis of  $\Delta^\vee$ .

- $\Delta^{\vee*}$  的非负 (正) 整数的线性组合是 (strictly) dominant integral element.

–  $\Delta^{\vee*}$  的整数线性组合的集合 = integral elements 的集合.

- **def.:** half the sum of the positive roots (relative to  $\Delta$ ) is,

$$\delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \quad (7.7.3)$$

- $\delta$  is a strictly dominant integral element, and,

$$\langle \delta, H_\alpha \rangle = 1, \forall \alpha \in \Delta \iff \delta = \sum_{i=1}^r \mu_i \quad (7.7.4)$$

**proof:**

注意 section 7.5 最后一个定理,  $s_\alpha[R^+ - \{\alpha\}] = R^+ - \{\alpha\}$ , 所以  $R^+ - \{\alpha\} = \{\beta_1, s_\alpha\beta_1, \beta_2, s_\alpha\beta_2, \dots\}$ . 且有  $\langle \beta_1 + s_\alpha\beta_1, H_\alpha \rangle = 0$ , 所以,

$$\langle \delta, H_\alpha \rangle = \langle \frac{1}{2}\alpha, H_\alpha \rangle = 1 \quad (7.7.5)$$

- fundamental wights and half the sum of the positive roots in rank-two systems 见下图,

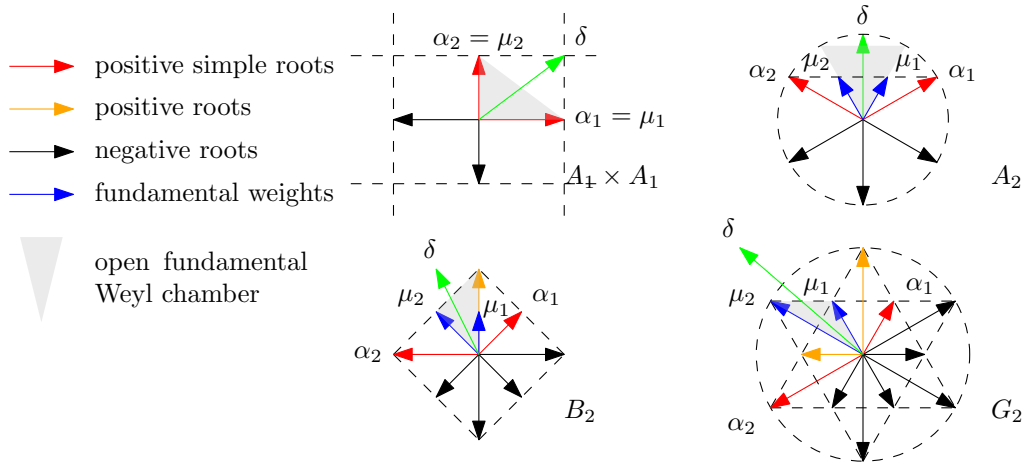


Figure 7.4: fundamental wights and half the sum of the positive roots in rank-two systems

## 7.8 the partial ordering

- **def.:** relative to  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ ,  $\mu \succeq \nu$  ( $\mu$  is **higher** than  $\nu$ ) if,

$$\mu - \nu = c_1\alpha_1 + \dots + c_r\alpha_r \quad (7.8.1)$$

其中  $c_1, \dots, c_r \geq 0$ , 类似地, 可以定义  $\nu \preceq \mu$  (... **lower** than...).

–  $\succeq$  定义了一个 partial ordering on  $E$ , 但两个矢量之间可能既不存在  $\succeq$  也不存在  $\preceq$  的关系.

- $\mu \in E$  is dominant  $\implies \mu \succeq 0$ .

**proof:**

考虑  $\Delta$  的 dual basis  $\Delta^* = \{\alpha_1^*, \dots, \alpha_r^*\}$ , 有,

$$c_i = \langle \alpha_i^*, \mu \rangle = \sum_{j=1}^r \langle \alpha_i^*, \alpha_j^* \rangle \langle \alpha_j, \mu \rangle \quad (7.8.2)$$

$\Delta$  中的任何两个向量夹钝角 (见 section 7.4 定义后的第一条定理), 那么它的对偶基底中的任意两个向量夹锐角 (见 appendix A.4), 所以  $\langle \alpha_i^*, \alpha_j^* \rangle \geq 0, \langle \alpha_j, \mu \rangle \geq 0$ , 所以  $c_i \geq 0$ .

- if  $\mu$  is dominant (i.e.  $\mu \in \bar{C}$ ), then  $w|\mu \preceq \mu$  for all  $w \in W$ .

**proof:**

$O$  is the Weyl-group orbit of  $\mu$ . 考虑到  $O$  是有限集合, 令  $\nu \in O$  使得没有其它元素高于  $\nu$ , 那么一定有  $\nu \in \bar{C}$  (即 dominant), 否则, 如果  $\langle \nu, \alpha \rangle < 0, \exists \alpha \in \Delta_C$ , 那么,

$$s_\alpha \nu = \nu - 2 \frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \succeq \nu \quad (7.8.3)$$

考虑到 section 7.5 的第四个结论, 可知  $\nu = \mu$ .

现在证明  $O$  中没有元素既不高于也不低于  $\mu$ .

考虑所有既不... 也不... 的元素的集合  $O'$ ,  $\xi \in O'$  且没有  $O'$  中的元素高于它, 那么,

– 如果  $o \in O - O'$ , 那么一定有  $\mu \succeq o$ , 且如果  $o \succeq \xi$ , 那么  $\mu \succeq o \succeq \xi$ , 与  $\xi \in O'$  矛盾.

所以  $O$  中没有元素高于  $\xi$ , 可知  $\xi \in \bar{C}$ , 矛盾.

- if  $\mu$  is a strictly dominant ( $\mu \in C$ ) integral element, then  $\mu \succeq \delta$  ( $\delta$  is half the sum of positive roots).

**proof:**

$\mu$  is a strictly dominant integral element  $\implies \langle \mu, \alpha \rangle \in \mathbb{Z}^+ - \{0\}, \forall \alpha \in \Delta_C; \langle \delta, \alpha \rangle = 1, \forall \alpha \in \Delta_C$ . 所以  $\mu - \delta \in \bar{C} \implies \mu \succeq \delta$ .

- **def.:** the **convex hull** of vectors  $v_1, \dots, v_N$  is the set,

$$\text{Conv}(v_1, \dots, v_N) = \{c_1 v_1 + \dots + c_N v_N \mid c_1 + \dots + c_N = 1 \text{ and } c_i \in \mathbb{R}^+\} \quad (7.8.4)$$

两个例子如下图,

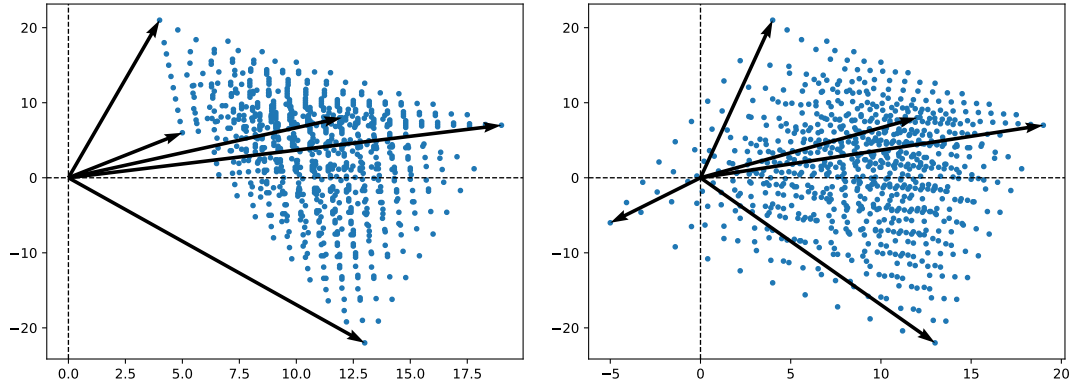


Figure 7.5: convex hulls

- $K$  is a compact, convex subset of  $E$ , and  $\lambda \in E - K$ , then there is an element  $\gamma \in E$  s.t.,

$$\langle \gamma, \lambda \rangle > \langle \gamma, \kappa \rangle, \forall \kappa \in K \quad (7.8.5)$$

**proof:**

由于  $K$  是紧致的, 存在  $\kappa_0 \in K$  使得  $|\lambda - \kappa_0|$  最小, 令  $\gamma = \lambda - \kappa_0$ , 那么,

$$\langle \gamma, \lambda - \kappa_0 \rangle > 0 \implies \langle \gamma, \lambda \rangle > \langle \gamma, \kappa_0 \rangle \quad (7.8.6)$$

对于  $K$  中的任意元素  $\kappa$ ,  $\kappa(s) = s\kappa + (1-s)\kappa_0, s \in [0, 1]$  属于  $K$ , 那么,

$$|\lambda - \kappa(s)|^2 \geq |\lambda - \kappa_0|^2 \implies s^2 |\kappa - \kappa_0|^2 - 2s \langle \lambda - \kappa_0, \kappa - \kappa_0 \rangle \geq 0 \quad (7.8.7)$$

考虑  $s \ll 1$  的情况, 可见,

$$\underbrace{\langle \lambda - \kappa_0, \kappa - \kappa_0 \rangle}_{=\gamma} \leq 0 \implies \langle \gamma, \lambda \rangle > \langle \gamma, \kappa_0 \rangle \geq \langle \gamma, \kappa \rangle \quad (7.8.8)$$

–  $\mu, \nu$  are dominant ( $\in \bar{C}$ ) and  $\nu \notin \text{Conv}(W|\mu)$ , then there exists a dominant element  $\gamma \in \bar{C}$  s.t.,

$$\langle \gamma, \nu \rangle > \langle \gamma, w\mu \rangle, \forall w \in W \quad (7.8.9)$$

meaning that  $\nu \not\leq w\mu, \forall w \in W$ .

**proof:**

根据上一个定理, 存在  $\gamma' \in E$  使得  $\langle \gamma', \nu \rangle > \langle \gamma', \kappa \rangle, \forall \kappa \in \text{Conv}(W|\mu)$ , 特别地,  $\langle \gamma', \nu \rangle > \langle \gamma', w\mu \rangle, \forall w \in W$ .

考虑  $\{\gamma\} = W|\gamma' \cap \bar{C}$ , 这个  $\gamma = w_0\gamma'$  是唯一的, 且  $\gamma \succeq \gamma'$ . 所以,

$$\gamma - \gamma' \in \bar{C} \implies \langle \gamma - \gamma', \nu \rangle \geq 0 \implies \langle \gamma, \nu \rangle > \langle w_0\gamma, w\mu \rangle, \forall w \in W \implies \dots \quad (7.8.10)$$

( $\gamma - \gamma'$  与 positive simple root 的内积为正, 且  $\nu$  可以展开成 positive simple root 的正系数叠加)

• 两个定理:

- if  $\mu, \nu$  are dominant, then  $\nu \in \text{Conv}(W|\mu) \iff \nu \preceq \mu$ .
- $\mu$  is dominant and  $\nu \in E$ , then  $\nu \in \text{Conv}(W|\mu) \iff w|\nu \preceq \mu, \forall w \in W$ .

**proof:**

上一个定理已经证明了  $\Leftarrow$ , 我们现在来证明  $\Rightarrow$ .  $\mu$  是 dominant, 那么  $w\mu \preceq \mu, \forall w \in W$ , 所以,

$$\left( \sum_{i=1}^{|W|} c_i w_i |\mu \rangle \right) - \mu = \sum_{i=1}^{|W|} c_i \underbrace{(w_i |\mu \rangle - \mu)}_{\preceq 0} \preceq 0 \quad (7.8.11)$$

所以  $\text{Conv}(W|\mu) \preceq \mu$ .

首先, 显然有  $\nu \in \text{Conv}(W|\mu) \iff w|\nu \in \text{Conv}(W|\mu), \forall w \in W$ . 那么考虑  $\nu' = w_0\nu \in \bar{C}$ , 有,

$$\nu \in \text{Conv}(W|\mu) \iff \nu' \in \text{Conv}(W|\mu) \iff \nu' \preceq \mu \quad (7.8.12)$$

而  $w|\nu \preceq \nu' \preceq \mu, \forall w \in W$ , 得证.

## 7.9 rank-three systems

- 本 section 只考虑 irreducible rank-three systems, 总共有三种, 分别是  $A_3, B_3, C_3$ , 它们分别来自  $\mathfrak{sl}(4, \mathbb{C})$ ,  $\mathfrak{so}(7, \mathbb{C})$  和  $\mathfrak{sp}(3, \mathbb{C})$ .
- $A_3$  root system 见下图, 其中, base 由红色向量组成, Weyl 群是右图中绿色正四面体的对称群,



Figure 7.6: the  $A_3$  root system and its Weyl group

- $B_3, C_3$  root systems 分别见下图,

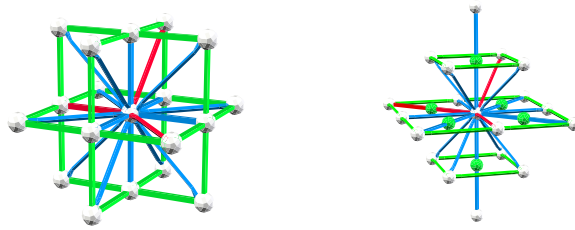


Figure 7.7: the  $B_3$  and  $C_3$  root systems

它们的 Weyl 群显然相同, 是下图中黄色立方体的对称群,

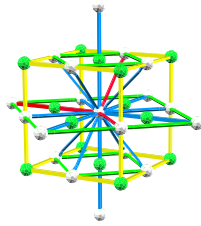


Figure 7.8: the Weyl group of  $C_3$

## 7.10 the classical root systems

- 见 section 6.7.

## 7.11 the classification

- every irreducible root system is either the root system of a classical Lie algebra (types  $A_n, B_n, C_n, n \geq 1$  and  $D_n, n \geq 3$ , with  $B_2 \simeq C_2, A_3 \simeq D_3$ ) or one of five **exceptional root systems**.
- the **exceptional root systems** are  $G_2, F_4, E_6, E_7, E_8$ , 它们的 Dynkin 图如下,

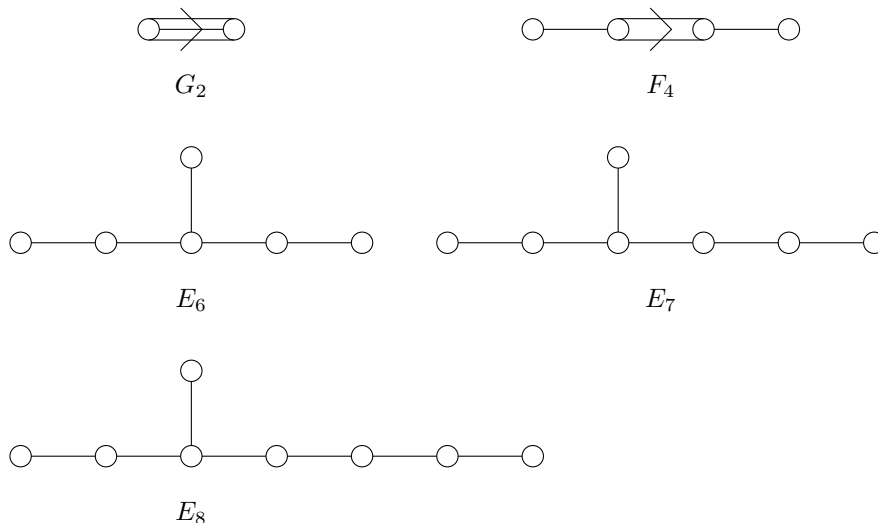


Figure 7.9: exceptional Dynkin diagrams

- 三个有用的定理:
  - $\mathfrak{h}_1, \mathfrak{h}_2$  are Cartan subalgebras of the semisimple Lie algebra  $\mathfrak{g}$ , then there exists a automorphism (自同构)  $\phi: \mathfrak{g} \rightarrow \mathfrak{g}$  s.t.  $\phi[\mathfrak{h}_1] = \mathfrak{h}_2$ . (见 section 6.2 末尾)

- the root systems associated to  $(\mathfrak{g}_1, \mathfrak{h}_1)$  and  $(\mathfrak{g}_2, \mathfrak{h}_2)$  are isomorphic  $\implies \mathfrak{g}_1, \mathfrak{g}_2$  are isomorphic.
- for every root system  $R$ , there exists a root system associated to  $(\mathfrak{g}, \mathfrak{h})$  isomorphic to  $R$ .

因此, 所有 simple Lie algebra 都与下表中的某个 classical Lie algebra,

$\mathfrak{sl}(n+1, \mathbb{C}) \mapsto A_n$	$\mathfrak{so}(2n+1, \mathbb{C}) \mapsto B_n$	$\mathfrak{sp}(2n, \mathbb{C}) \mapsto C_n$	$\mathfrak{so}(2n, \mathbb{C}) \mapsto D_n$
$n \geq 1$	$n \geq 2$	$n \geq 3$	$n \geq 4$
$n = 1$	$B_1 \simeq A_1$	$C_1 \simeq A_1$	$n \neq 1$
$n = 2$		$C_2 \simeq B_2$	$n \neq 2$
$n = 3$			$D_3 \simeq A_3$

或  $G_2, F_4, E_{6,7,8}$  中的某个 exceptional Lie algebra 相 isomorphic.

## Chapter 8

# representations of semisimple Lie algebras

### 8.1 weights of representations

- **def.:**  $(\pi, V)$  is a (possibly infinite dimensional) rep. of semisimple Lie algebra  $\mathfrak{g}$ , then  $\lambda \in \mathfrak{h}$  is the **weight** of  $\pi$  if there **exists** a  $v \neq 0 \in V$  s.t.,

$$\pi(H)v = \langle \lambda, H \rangle v, \forall H \in \mathfrak{h} \iff \det(\pi(H) - \langle \lambda, H \rangle I) = 0, \forall H \in \mathfrak{h} \quad (8.1.1)$$

the **weight space** of  $\lambda$  (denoted by  $V_\lambda$ ) is the set of all  $v \in V$  satisfying (8.1.1), and the dimension of the weight space is called the (geometric) **multiplicity**. (more about weights, see appendix A.3)

- $(\pi, V)$  is finite-dimensional  $\implies$  every weight of  $\pi$  is an **integral element**.

**proof:**

$\pi|_{\mathfrak{s}^\alpha}$  可以视为  $\mathfrak{s}^\alpha = \text{span}(H_\alpha, A_\alpha, B_\alpha) \simeq \mathfrak{su}(2)_\mathbb{C}$  的表示, 那么根据 (10.1.6),  $\pi(H_\alpha) \equiv \pi(2J_3)$  的 eigenvalue 是整数, 所以,

$$\langle \lambda, H_\alpha \rangle \in \mathbb{Z} \quad (8.1.2)$$

- for finite-dimensional rep., for a weight  $\lambda$  of  $\pi$ ,  $w|\lambda\rangle, \forall w \in W$  is still a weight and  $V_{w|\lambda} \simeq V_\lambda$ .

**proof:**

注意, 令  $S_\alpha = e^{A_\alpha} e^{-B_\alpha} e^{A_\alpha}$ , 那么,

$$\text{Ad}_{S_\alpha} H_\alpha = -H_\alpha \implies \text{Ad}_{S_\alpha} = s_\alpha \quad (8.1.3)$$

证明见 (10.1.7). 所以, 考虑  $s_\alpha|\lambda\rangle$  (注意到  $s_\alpha^{-1} = s_\alpha$ ),

$$\begin{aligned} & \begin{cases} \pi(s_\alpha^{-1}H)v = \langle \lambda, s_\alpha^{-1}H \rangle v \quad \forall v \in V_\lambda \\ \pi(s_\alpha^{-1}H) = \pi(\text{Ad}_{S_\alpha}H) = \Pi(S_\alpha)\pi(H)\Pi^{-1}(S_\alpha) \end{cases} \\ \implies & \pi(H)(\Pi^{-1}(S_\alpha)v) = \langle s_\alpha\lambda, H \rangle (\Pi^{-1}(S_\alpha)v) \\ \implies & \Pi^{-1}(S_\alpha)[V_\lambda] = V_{s_\alpha|\lambda} \end{aligned} \quad (8.1.4)$$

( $\Pi(S_\alpha)$  一定是可逆矩阵, 否则不存在逆元,  $\Pi$  就根本不是一个表示)

- 考虑半单李代数的正根为  $R^+ = \{\alpha_1, \dots, \alpha_N\}$ , 李代数的基底是  $\Delta \cup \{A_1, \dots, A_N\} \cup \{B_1, \dots, B_N\}$ , 其中  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ , 且  $A_i \in \mathfrak{g}_{\alpha_i}, B_i \in \mathfrak{g}_{-\alpha_i}$ .

– 那么,  $\forall \alpha \in R$ ,

$$\begin{cases} \pi(H)\pi(A_\alpha)v = \langle \lambda + \alpha, H \rangle \pi(A_\alpha)v \\ \pi(H)\pi(B_\alpha)v = \langle \lambda - \alpha, H \rangle \pi(B_\alpha)v \end{cases} \implies \begin{cases} \pi(A_\alpha)[V_\lambda] \subseteq V_{\lambda+\alpha} \\ \pi(B_\alpha)[V_\lambda] \subseteq V_{\lambda-\alpha} \end{cases} \quad (8.1.5)$$



- 对于所有的不可约表示,  $\pi(H), \forall H \in \mathfrak{h}$  都可以被对角化, 因此也可以被同时对角化.

**proof:**

$U$  是  $V$  的子空间, 由  $\mathfrak{h}$  的 simultaneous eigenvectors 构成, 根据 (8.1.5),  $\pi(A_\alpha)[U] \subseteq U$ , 所以  $U$  是不变子空间 (且不为零, 因为  $\mathfrak{h}$  是 Abelian, 至少存在一个权, 见 appendix A.3). 又因为  $(\pi, V)$  不可约, 所以  $V = U = \bigoplus_\lambda V_\lambda$ .

- 三个关于 **highest weight** 的定理:

- every irreducible, finite-dim. rep. of  $\mathfrak{g}$  has a highest weight. (最高权存在)
- two irreducible, finite-dim. rep. with the same highest weight are isomorphic. (一一对应)
- the **highest weight**  $\mu$  of a irreducible, finite-dim. rep. is a **dominant integral element**.

**proof:**

**reordering lemma:** 考虑李代数  $\mathfrak{g}$  及其表示  $\pi$ ,  $\{A_1, \dots, A_n\}$  是李代数的一组基底, 那么下式,

$$\pi(A_{i_1}) \cdots \pi(A_{i_N}) \quad (8.1.6)$$

可以表示成,

$$\pi(A_n)^{j_n} \cdots \pi(A_1)^{j_1} \quad (8.1.7)$$

的线性组合, 其中  $j_1 + \cdots + j_n \leq N$ .

**proof:**

用数学归纳法证明,  $N = 1$  时显然成立, 假设  $N - 1$  时成立, 那么  $N$  时,

$$\pi(A_{i_1}) \cdots \pi(A_{i_N}) = \pi(A_{i_1}) \left( \sum_{j_1 + \cdots + j_N \leq N-1} C_{j_1, \dots, j_N} \pi(A_n)^{j_n} \cdots \pi(A_1)^{j_1} \right) \quad (8.1.8)$$

用对易关系改变  $\pi(A_{i_1})$  的位置,

$$\pi(A_{i_1})\pi(A_k) = \pi(A_k)\pi(A_{i_1}) + \underbrace{\pi([A_{i_1}, A_k])}_{=\sum_l -f_{i_1 k}^l A_l} \quad (8.1.9)$$

右边的一项最多含  $N - 1$  个基矢, 所以命题得证.

- 令 (dominant) integral element  $\mu$  为  $(\pi, V)$  的 **highest weight**, 那么 (根据 (8.1.5)) 一定有  $\pi(A_{\alpha_i})[V_\mu] = \{0\}, \forall \alpha_i \in R^+$ .
- 选取  $\{B_1, \dots, B_N\} \cup \Delta \cup \{A_1, \dots, A_N\}$  为  $\mathfrak{g}$  的基底 (其中  $N$  是正根的个数), 那么考虑 some  $v \in V_\mu$ ,

$$\pi(B_{i_1}) \cdots \pi(B_{i_M})v = \text{linear combination of } \pi(B_N)^{j_N} \cdots \pi(B_1)^{j_1}v \quad (8.1.10)$$

(注意到  $v$  是  $\pi(H_i)$  的本征向量, 而  $\pi(A_i)v = 0$ )

另外, 一定有  $\mu - j_1\alpha_1 - \cdots - j_N\alpha_N \in \text{Conv}(W|\mu)$ , 否则  $\pi(B_N)^{j_N} \cdots \pi(B_1)^{j_1}v = 0$ .

- 考虑,

$$\text{linear combinations of } \pi(B_{i_1}) \cdots \pi(B_{i_M})v \text{ with } M \geq 0, \text{ for some } v \in V_\mu \quad (8.1.11)$$

这是  $V$  的不变子空间, 考虑到 irreducibility, (8.1.11) 等于  $V$ . 同时也证明了  $\dim V_\mu = 1$ , 且  $\mu$  是唯一的最高权, 因此它一定是 dominant.

- **theorem:** if  $\mu$  is a **dominant integral element**, there exists an irreducible, finite-dim. rep. of  $\mathfrak{g}$  with **highest weight**  $\mu$ .

本 chapter 的剩余部分将用来证明这个定理.

## 8.2 the highest weight cyclic representations & an introduction to Verma modules

- **def.:** for a (maybe infinite-dim.) rep.  $(\pi, V)$  of  $\mathfrak{g}$  with highest weight  $\mu \in \mathfrak{h}$  (不一定是 integral), if there exists  $v \neq 0 \in V$  s.t.,

1.  $\pi(H)v = \langle \mu, H \rangle v, \forall H \in \mathfrak{h}$  (simultaneously diagonalizable, 见 appendix A.3.2),
2.  $\pi(A)v = 0, \forall A \in \mathfrak{g}_\alpha$ , with  $\alpha \in R^+$ ,
3. the smallest invariant subspace (见 section 5.2 第三点,  $\pi(A)[W] \subseteq W, \forall A \in \mathfrak{g}$ ) containing  $v$  is  $V$ ,

then it is said to be **highest weight cyclic**.

- 有限维情况下, highest weight cyclic rep. 是 irreducible, 且最高权相同  $\mu$  的... 互相 isomorphic.

- 下面初步介绍构造 Verma module  $(\pi_\mu, V^\mu)$  的思路 ( $V^\mu$  选择上标, 以区分 weight space  $V_\mu$ ).
- 依旧是选取,

$$\{B_1, \dots, B_N\} \cup \Delta \cup \{A_1, \dots, A_N\} \quad \text{with} \quad \begin{cases} R^+ = \{\underbrace{\alpha_1, \dots, \alpha_r}_{=\Delta}, \alpha_{r+1}, \dots, \alpha_N\} \\ A_i \in \mathfrak{g}_{\alpha_i} \quad i = 1, \dots, N \\ B_i \in \mathfrak{g}_{-\alpha_i} \quad i = 1, \dots, N \end{cases} \quad (8.2.1)$$

作为  $\mathfrak{g}$  的基底.

- 由于对于  $(\pi_\mu, V^\mu)$ ,  $\mu$  是最高权, 所以一定存在,

$$v_0 \in V^\mu, \text{ s.t. } \pi_\mu(A)v_0 = 0, \forall A \in \mathfrak{g}_\alpha, \text{ with } \alpha \in R^+ \quad (8.2.2)$$

- 根据 (8.1.11), 考虑具有以下形式的向量,

$$\pi_\mu(B_1)^{n_1} \cdots \pi_\mu(B_N)^{n_N} v_0 \in V_{\mu - \sum_{i=1}^N n_i \alpha_i} \subset V^\mu, \text{ with } n_i \in \mathbb{Z}^+ \quad (8.2.3)$$

它们的线性组合张成  $V^\mu$ .

- Verma module 中的 weights 仅具有如下形式,

$$\mu - \sum_{i=1}^N n_i \alpha_i \quad (8.2.4)$$

其中  $n_i$  是非负整数.

- 这样定义后, 我们就能 (通过对易关系) 计算  $\mathfrak{g}$  中每个元素的表示如何作用于任何一个  $V^\mu$  中的向量.

## 8.3 universal enveloping algebras, $U(\mathfrak{g})$

- **def.:** 李代数  $\mathfrak{g}$  嵌入的 associative algebra (对 algebra 的一般定义见 appendix A 开头),  $\mathcal{A}$ , 是:

- 存在乘法单位元  $e$ , 且满足结合律 (unital, associative algebra).

- $\mathfrak{g}$  嵌入于  $\mathcal{A}$  ( $\hat{j}: \mathfrak{g} \rightarrow \mathcal{A}$ ).

(例如: 对于矩阵李群  $G \subseteq \text{GL}(n, \mathbb{C})$ , 那么  $\mathfrak{g}$  就是  $\mathcal{M}_n(\mathbb{C})$  的子空间)

- 李括号简化为,

$$\hat{j}([A, B]) = \hat{j}(A) \cdot \hat{j}(B) - \hat{j}(B) \cdot \hat{j}(A) \quad (8.3.1)$$

- $\mathcal{A}$  由单位元  $e$  和如下元素张成,

$$\hat{j}(A_1) \cdots \hat{j}(A_k) \quad (8.3.2)$$

其中  $k \geq 1$ .

另外, 对于  $\mathfrak{g}$  一般来说  $\mathcal{A}$  不唯一.

- **def.:** a pair  $(U(\mathfrak{g}), \hat{i})$  (需要满足结合律) with the following properties is called a **universal enveloping algebra**,

1.  $\hat{i}([A, B]) = \hat{i}(A) \cdot \hat{i}(B) - \hat{i}(B) \cdot \hat{i}(A), \forall A, B,$
2. the **smallest subalgebra** with **identity**  $e \in U(\mathfrak{g})$  **containing**  $\{\hat{i}(A), A \in \mathfrak{g}\}$  is  $U(\mathfrak{g})$ ,  
(这个条件称为  $U(\mathfrak{g})$  由  $\hat{i}(A), A \in \mathfrak{g}$  生成)
3. 考虑  $\mathfrak{g}$  嵌入的某个 associative algebra  $\mathcal{A}$  with identity, 那么  $U(\mathfrak{g})$  和  $\mathcal{A}$  之间存在 a **unique** algebra homomorphism  $\phi : U(\mathfrak{g}) \rightarrow \mathcal{A}$ , s.t.,

$$\begin{cases} \phi(e) = e' \in \mathcal{A} \\ \phi \circ \hat{i} = \hat{j} : \mathfrak{g} \rightarrow \mathcal{A} \end{cases} \quad (8.3.3)$$

即  $\mathcal{A} \simeq U(\mathfrak{g}) / \ker(\phi)$ , (只需要说明这个  $\ker(\phi)$  是唯一的就行).

- $\mathfrak{g}$  的任意两个 universal enveloping algebras 互相同构.  
(由于  $U(\mathfrak{g})$  本身也是 associated algebra, 再利用性质 3)
- **theorem:** 任何李代数都存在一个 universal enveloping algebra.

**proof:**

– **def.:** the **tensor algebra**  $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$ , (notation  $\mathfrak{g}^{\otimes k} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ ).

\*  $T(\mathfrak{g})$  是对于  $B(\cdot, \cdot) = \otimes$  满足结合律的代数.

\* 且存在单位元  $1 \in \mathbb{C} \equiv \mathfrak{g}^{\otimes 0}$ .

$(T(\mathfrak{g}), \otimes)$  满足  $U(\mathfrak{g})$  的第两个条件, 但是, 对于第三个条件, 考虑,

$$\begin{cases} \psi(1) = e \in \mathcal{A} \\ \psi : T(\mathfrak{g}) \rightarrow \mathcal{A}, A \mapsto \hat{j}(A) \end{cases} \quad (8.3.4)$$

显然, 这样的 homomorphism  $\psi$  不唯一, 实际上  $U(\mathfrak{g})$  是  $T(\mathfrak{g})$  的一个商空间 (见下文).

现在, 我们来构造  $U(\mathfrak{g})$ . 考虑双向不变子空间 (two-sided ideal)  $J$ ,

$$J = \left\{ \sum_i \alpha_i \otimes (A_i \otimes B_i - B_i \otimes A_i - [A_i, B_i]) \otimes \beta_i \mid A_i, B_i \in \mathfrak{g}, \alpha_i, \beta_i \in T(\mathfrak{g}) \right\} \quad (8.3.5)$$

那么  $U(\mathfrak{g}) = T(\mathfrak{g})/J$ .

– 注意,  $J$  是一个 **two-sided ideal**, 即  $\forall \alpha \in T(\mathfrak{g}), \beta \in J$ , 有  $\alpha \otimes \beta, \beta \otimes \alpha \in J$ .

– 且  $J$  是包含形如  $A \otimes B - B \otimes A - [A, B]$  的元素的最小的 two-sided ideal.

– 注意, the kernel of an algebra homomorphism is always a two-sided ideal. 考虑  $\phi : U \rightarrow \mathcal{A}$ , 那么,  $\forall \alpha \in \ker(\phi), \beta \in U$ ,

$$\phi(\beta \cdot \alpha) = \phi(\beta) \cdot 0 = 0 \quad (8.3.6)$$

**proof:**

– 第一条  $(T(\mathfrak{g})$  不满足, 但  $T(\mathfrak{g})/J$  满足),

$$[A, B] \sim A \otimes B - B \otimes A \quad (8.3.7)$$

– 第二条成立 ( $T(\mathfrak{g})$  和  $T(\mathfrak{g})/J$  都满足).

– 第三条 ( $T(\mathfrak{g})$  和  $T(\mathfrak{g})/J$  都满足), 考虑 algebra homomorphism  $\psi : T(\mathfrak{g}) \rightarrow \mathcal{A}$  s.t.,

$$\begin{cases} \psi(1) = e \in \mathcal{A} \\ \psi(A_1 \otimes \cdots \otimes A_k) = \hat{j}(A_1) \cdots \hat{j}(A_k) \end{cases} \quad (8.3.8)$$

那么, (考虑到 kernel 一定是 two-sided ideal), 必然有  $J \subset \ker(\psi)$ .

(令  $\phi = \psi|_{U(\mathfrak{g})}$ , 有  $\ker(\psi) = J \oplus \ker(\phi)$ , 即  $\mathcal{A} = T(\mathfrak{g})/\ker(\psi) = U(\mathfrak{g})/\ker(\phi)$ .)

注意,  $\mathcal{A}$  由  $e$  和 (8.3.2) 中的元素张成,  $\psi$  必须满足  $\psi(1) = e$ , 考虑第二个条件  $\phi \circ \hat{i} = \hat{j}$ , 考虑  $\forall A \in \mathfrak{g}$ ,

$$\phi(A) = \hat{j}(A) \quad (8.3.9)$$

且  $U(\mathfrak{g})$  由  $A_1 \oplus \cdots \oplus A_k, k \geq 0$  张成, 所以  $\phi$  的选取是唯一的.

- $(\pi, V)$  是李代数  $\mathfrak{g}$  的一个表示 (不一定是有限维), 那么存在一个 unique algebra homomorphism,

$$\tilde{\pi} : U(\mathfrak{g}) \rightarrow \text{End}(V) \quad \text{s.t.} \quad \begin{cases} \tilde{\pi}(1) = I \\ \tilde{\pi}(A) = \pi(A), \forall A \in \mathfrak{g} \subset U(\mathfrak{g}) \end{cases} \quad (8.3.10)$$

**proof:**

可以认为  $\mathcal{A} = \text{End}(V), \hat{j} = \pi$ , 那么, 存在 unique  $\tilde{\pi} = \phi : U(\mathfrak{g}) \rightarrow \mathcal{A}, \dots$

## 8.4 Poincaré-Birkhoff-Witt theorem

- **PBW theorem:** 对于有限维李代数  $\mathfrak{g}$  (不一定半单), 其基矢为  $\{A_1, \dots, A_k\}$ , 那么,

$$\hat{i}(A_1)^{n_1} \cdots \hat{i}(A_k)^{n_k} \quad (8.4.1)$$

其中  $n_i$  是非负整数, 构成  $U(\mathfrak{g})$  的基矢 (张成并线性独立).

– 同时意味着  $\hat{i} : \mathfrak{g} \rightarrow U(\mathfrak{g})$  是 injective (one-to-one).

**proof:**

证明方法类似于 reordering lemma (见 (8.1.7)).

首先 (8.4.1) 中的向量显然能张成  $U(\mathfrak{g})$ , 我需要证明它们线性独立, 方法如下:

考虑一个向量空间  $D$ , 其基底为  $\{v_{i_1, \dots, i_N}\}$ , 其中  $1 \leq i_1 \leq \cdots \leq i_N \leq k$ . 我们的目标是证明存在一个线性映射  $\gamma : U(\mathfrak{g}) \rightarrow D$ , (这个映射不必是同构), 使得,

$$\hat{i}(A_{i_1}) \cdots \hat{i}(A_{i_N}) \mapsto v_{i_1, \dots, i_N} \quad (8.4.2)$$

为此, 我们希望能构造一个线性映射  $\delta : T(\mathfrak{g}) \rightarrow D$ , s.t.,

1.  $\delta(A_{i_1} \otimes \cdots \otimes A_{i_N}) = v_{i_1, \dots, i_N}$  if  $1 \leq i_1 \leq \cdots \leq i_N \leq k$ ,
2.  $\delta[J] = \{0\}$ , 因此  $\delta$  自然能给出线性映射  $\gamma : U(\mathfrak{g}) \rightarrow D$ .

构造方法如下.

考虑  $n$  阶单项式  $A_{j_1} \otimes \cdots \otimes A_{j_n}$ , 令逆序的下标对数为其 index, (显然 0, 1 阶的单项式的 index 都是零),  $n \leq k$  阶单项式的 index 最高为  $\frac{n(n-1)}{2}$ . 下面用归纳法来确定  $\delta$ .

- 假设  $\delta$  的定义 (已经在 index 小于等于  $p$ , 或者阶数小于等于  $n-1$  下做出了定义) 使得, 下式在: 等号左边两相的 index 都不超过  $p \geq 1$  时, 且  $n \leq N$  时, 成立,

$$\delta(A_{i_1} \cdots (A_{i_j} A_{i_{j+1}} - A_{i_{j+1}} A_{i_j}) \cdots A_{i_n}) = \delta(A_{i_1} \cdots [A_{i_j}, A_{i_{j+1}}] \cdots A_{i_n}) \quad (8.4.3)$$

( $p=0$  一定成立, 因为  $i_j = i_{j+1}$ , 等号两边为零)

- 考虑等号左侧第一项的 index 为  $p+1$ , 且  $i_j > i_{j+1}$  是逆序, 那么, 定义  $\delta$  在 (8.4.3) 下依然成立. 这样我们就把  $\delta$  的定义拓展到了  $n$  阶, index 为  $p+1$  的情况,

$$\delta(A_{i_1} \cdots \underbrace{A_{i_j} A_{i_{j+1}}}_{\text{逆序}} \cdots A_{i_n}) = \delta(A_{i_1} \cdots A_{i_{j+1}} A_{i_j} \cdots A_{i_n}) + \delta(\cdots [A_{i_j}, A_{i_{j+1}}] \cdots) \quad (8.4.4)$$

- 由于 (8.4.4) 左侧至少有两处逆序 (假设另一个逆序对为  $i_l > i_{l+1}$  且  $j < l$ ), 那么还需要证明等式右侧与逆序对的选取无关, 我们通过分类讨论证明这一点.

分类讨论:

- 如果  $j+1 \leq l-1$ .

考虑,

$$\begin{aligned}
& \delta(\cdots A_{i_j} A_{i_{j+1}} \cdots A_{i_l} A_{i_{l+1}} \cdots) \\
&= \delta(\cdots A_{i_j} A_{i_{j+1}} \cdots A_{i_{l+1}} A_{i_l} \cdots) + \delta(\cdots A_{i_j} A_{i_{j+1}} \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) \\
&= \delta(\cdots A_{i_{j+1}} A_{i_j} \cdots A_{i_{l+1}} A_{i_l} \cdots) + \delta(\cdots [A_{i_j}, A_{i_{j+1}}] \cdots A_{i_{l+1}} A_{i_l} \cdots) \\
&\quad + \delta(\cdots A_{i_{j+1}} A_{i_j} \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) + \delta(\cdots [A_{i_j}, A_{i_{j+1}}] \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) \\
&= \cdots
\end{aligned} \tag{8.4.5}$$

最后一个等号右侧的第一, 三项和第二, 四项结合, 就得到 (8.4.4) 右侧.

(要注意, 证明过程中每一个单项式的 index 都小于等于  $p$ , 或者阶数小于等于  $n-1$ )

- 如果  $j+1 = l$ .

为了简洁, 用  $A = A_{i_j}, B = A_{i_{j+1}=l}, C = A_{i_{l+1}}$ , 那么,

$$\begin{aligned}
& \delta(\cdots BAC \cdots) + \delta(\cdots [A, B]C \cdots) \\
&= \delta(\cdots CBA \cdots) + \delta(\cdots [B, C]A \cdots) + \delta(\cdots B[A, C] \cdots) + \delta(\cdots [A, B]C \cdots)
\end{aligned} \tag{8.4.6}$$

同时,

$$\begin{aligned}
& \delta(\cdots ACB \cdots) + \delta(\cdots A[B, C] \cdots) \\
&= \delta(\cdots CBA \cdots) + \delta(\cdots [A, C]B \cdots) + \delta(\cdots C[A, B] \cdots) + \delta(\cdots A[B, C] \cdots)
\end{aligned} \tag{8.4.7}$$

那么, 只需要证明,

$$\begin{aligned}
& [[B, C], A] + \underbrace{[B, [A, C]]}_{=[C, A], B]} + [[A, B], C] = 0
\end{aligned} \tag{8.4.8}$$

而这就是 Jacobi identity.

## 8.5 construction of Verma modules, $W_\mu$

- **def.:** a left ideal of  $U(\mathfrak{g})$  generated by  $\{\alpha_i\}$  is,

$$I = \left\{ \sum_i \beta_i \alpha_i \mid \forall \beta_i \in U(\mathfrak{g}) \right\} \tag{8.5.1}$$

- 用  $I_\mu$  表示一个 left ideal generated by,

$$\{H - \langle \mu, H \rangle, \forall H \in \mathfrak{h}\} \cup \bigcup_{\alpha \in R^+} \mathfrak{g}_\alpha \tag{8.5.2}$$

(第一个集合中的元素是一个一阶向量减一个零阶向量)

- **def.:** the **Verma module** with highest weight  $\mu$  is,

$$W_\mu = U(\mathfrak{g})/I_\mu \tag{8.5.3}$$

用  $[\alpha]$  表示  $\alpha \in U(\mathfrak{g})$  在  $W_\mu$  中的像 (等价类).

- $(\pi_\mu, W_\mu)$  是 universal enveloping algebra 的一个表示,

$$\pi_\mu(\alpha)[\beta] = [\alpha\beta] \tag{8.5.4}$$

proof:

$$\pi_\mu(\alpha_1)\pi_\mu(\alpha_2)[\beta] = [\alpha_1\alpha_2\beta] = \pi_\mu(\alpha_1\alpha_2)[\beta] \quad (8.5.5)$$

且如果  $\beta \sim \beta'$ , 那么  $\alpha\beta \sim \alpha\beta'$ .

– 所以, (其中  $A \in \mathfrak{g}_{\alpha \in R^+}$ ),

$$\begin{cases} \pi_\mu(H)[1] = \langle \mu, H \rangle [1] \\ \pi_\mu(A)[1] = 0 \end{cases} \quad (8.5.6)$$

但要注意, 一般  $[A\alpha] \neq 0$ , 所以  $\pi_\mu(A) \neq [A] = 0$ , (不过  $[\alpha A] = 0$ ).

•  $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha$ , 由于  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ , 所以  $\mathfrak{n}^+, \mathfrak{n}^-$  都是  $\mathfrak{g}$  的子代数.

• theorem:

–  $(\pi_\mu, W_\mu)$  是一个 highest weight cyclic rep. (定义见 section 8.2 开头), 且最高权为  $\mu$  (不过, 由于  $W_\mu$  一定是无限维, 最高权不一定是 dominant), 最高权向量为  $v_0 = [1]$ .

–  $\{B_1, \dots, B_k\}$  是  $\mathfrak{n}^-$  的一组基底, 那么,

$$\pi_\mu(B_1)^{n_1} \dots \pi_\mu(B_k)^{n_k} v_0 \quad (8.5.7)$$

(其中  $n_i \in \mathbb{Z}^+$ ), 组成  $W_\mu$  的一组基底.

结合 PBW theorem, 可见有向量空间同构  $W_\mu \simeq U(\mathfrak{n}^-)$ , 且  $\alpha \mapsto \pi_\mu(\alpha)v_0$ .

proof:

lemma: 令  $J_\mu$  是  $U(\mathfrak{n}^+ \oplus \mathfrak{h})$  上的, 由 (8.5.2) 中的元素生成的 left ideal, 那么  $v_0 = [1] \notin J_\mu$ .

proof:

考虑一维表示,

$$\sigma_\mu : \mathfrak{n}^+ \oplus \mathfrak{h} \rightarrow \underbrace{\text{End}(\mathbb{C})}_{=\mathbb{C}} \quad \text{s.t.} \quad \begin{cases} \sigma_\mu(A) = 0 & A \in \mathfrak{n}^+ \\ \sigma_\mu(H) = \langle \mu, H \rangle & H \in \mathfrak{h} \end{cases} \quad (8.5.8)$$

对比 (8.3.10), 可知存在一个唯一的  $\tilde{\sigma}_\mu : U(\mathfrak{n}^+ \oplus \mathfrak{h}) \rightarrow \mathbb{C}$ , s.t.,

$$\begin{cases} \tilde{\sigma}_\mu(1) = 1 \\ \tilde{\sigma}_\mu(A + H) = \langle \mu, H \rangle \end{cases} \quad \text{and} \quad \ker(\tilde{\sigma}_\mu) \supset \{0\} \cup \mathfrak{n}^+ \cup \{H \perp \mu\} \cup \{H - \langle \mu, H \rangle\} \quad (8.5.9)$$

且  $\ker(\tilde{\sigma}_\mu)$  是  $U(\mathfrak{n}^+ \oplus \mathfrak{h})$  上的一个 two-sided ideal, 所以  $J_\mu \subset \ker(\tilde{\sigma}_\mu)$ , 所以...

含有  $v_0$  的不变子空间  $U = W_\mu$ , 因为  $\pi_\mu(\alpha)v_0 = [\alpha]$ , 那么证明第一个 theorem 只需要再说明  $[1] \neq [0]$ , (highest weight cyclic rep. 的前两个性质见 (8.5.6)).

要说明  $[1] \neq [0]$ , 只需要证明  $1 \notin I_\mu$ .

考虑  $I_\mu$  中的元素按照 PBW theorem 展开,

$$\begin{aligned} I_\mu \ni \alpha &= \sum_{\beta_1 \in U(\mathfrak{g})} \beta_1 (H - \langle \mu, H \rangle) + \sum_{\beta_2 \in U(\mathfrak{g})} \beta_2 A_\alpha \\ &= \sum_{\alpha_1, \dots, \alpha_N \in U(\mathfrak{n}^+)} (B_{\alpha_1})^{n_1} \dots (B_{\alpha_N})^{n_N} \underbrace{\gamma_{n_1, \dots, n_N}}_{\in U(\mathfrak{n}^+)} (H - \langle \mu, H \rangle) + \dots \\ &= \sum_{\alpha_1, \dots, \alpha_N \in U(\mathfrak{n}^+)} (B_{\alpha_1})^{n_1} \dots (B_{\alpha_N})^{n_N} \underbrace{\delta_{n_1, \dots, n_N}}_{\in J_\mu} \end{aligned} \quad (8.5.10)$$

如果  $\alpha = 1 \in I_\mu$ , 那么  $n_1 = \dots = n_N = 0$ , 且  $\alpha = 1 = \delta_{0, \dots, 0} \in J_\mu$ , 与引理的结论矛盾, 所以  $1 \notin I_\mu$ .

现在来证明第二个 theorem. 已经说明了  $W_\mu$  是含  $v_0$  的最小的不变子空间, 所以 (8.5.7) 中的向量一定张成  $W_\mu$ , 我们还需要证明它们线性独立. 考虑, 如果它们线性相关,

$$\sum_{\substack{\in \mathbb{C} \\ C_{n_1, \dots, n_k}}} [(B_1)^{n_1} \cdots (B_k)^{n_k}] = 0 \\ \Rightarrow \alpha = \sum C_{n_1, \dots, n_k} (B_1)^{n_1} \cdots (B_k)^{n_k} \in I_\mu \quad (8.5.11)$$

但是, 对照 (8.5.10) (注意, 利用 PBW theorem 得到的展开式是唯一的), 可见  $C_{n_1, \dots, n_k} \in J_\mu$ , 而这不成立.

## 8.6 irreducible quotient modules, $V^\mu = W_\mu/U_\mu$

- 本节我们将证明 Verma module  $W_\mu$  有一个 largest nonzero invariant subspace  $U_\mu$ , 而商空间  $V^\mu = W_\mu/U_\mu$  是最高权为  $\mu$  的不可约表示. 且如果  $\mu$  是 dominant integral, 那么  $V^\mu$  是有限维.

- **def.:**  $U_\mu$  由如下向量  $v \in W_\mu$  组成 (注意, (8.5.7) 是  $W_\mu$  的一组基底):

1.  $v$  的  $v_0 = [1]$  分量为零,
  - 注意, 并不是所有由低于  $\mu$  的权对应的权向量组成的矢量都属于  $U_\mu$ , 例如  $[B_\alpha] \notin U_\mu, \alpha \in R^+$ , 因为  $\pi_\mu(A_\alpha)[B_\alpha] = \langle \mu, H_\alpha \rangle v_0$ , 见第二个条件.
2.  $\pi_\mu(A_1) \cdots \pi_\mu(A_k)v, k \geq 1$  的  $v_0$  分量也为零, 其中  $A_1, \dots, A_k \in \mathfrak{n}^+$ ,

也就是所有通过升算符无法达到  $v_0$  的向量.

- $U_\mu$  是一个不变子空间.

**proof:**

- 首先  $\pi_\mu(A)[U_\mu] \subseteq U_\mu, \forall A \in \mathfrak{n}^+$ .
- $\pi_\mu(A_1) \cdots \pi_\mu(A_k)v, k \geq 0$  是由低于  $\mu$  的权对应的权向量组成, 考虑,

$$\pi_\mu(A_1) \cdots \pi_\mu(A_k) \pi_\mu(C)v \quad (8.6.1)$$

其中  $C \in \mathfrak{h} \oplus \mathfrak{n}^-$ , reordering lemma 告诉我们 (8.6.1) 等于下列形式的向量的线性组合,

$$\pi_\mu^{n_1}(B_1) \cdots \pi_\mu^{n_N}(B_N) \pi_\mu^{n'_1}(H_1) \cdots \pi_\mu^{n'_r}(H_r) \pi_\mu^{n''_1}(A_1) \cdots \pi_\mu^{n''_N}(A_N)v \quad (8.6.2)$$

只能让这些权向量对应的权保持不变或降低, 所以...

- 商空间  $V^\mu = W_\mu/U_\mu$  构成  $\mathfrak{g}$  的一个不可约表示 (见 section 5.2).

**proof:**

显然, 对于  $V_\mu$  的不变子空间  $V'$ , 有  $V' \oplus U_\mu \subset W_\mu$  也是一个不变子空间 (因为已经证明了  $U_\mu$  是不变子空间).

那么, 现在只需要证明:  $W_\mu$  中, 包含子集  $U_\mu$  的不变子空间要么是  $U_\mu$ , 要么是  $W_\mu$ .

考虑不变子空间  $U'$  满足  $U_\mu \subset U' \subset W_\mu$ , 且  $U' \neq U_\mu$ , 那么,

- 有  $v \in U'$  且  $v \notin U_\mu$ .
- 由于  $v \notin U_\mu$ , 一定存在一些组合  $A_1, \dots, A_k$  使得  $u = \pi_\mu(A_1) \cdots \pi_\mu(A_k)v$  的  $v_0$  分量不为零.
- 由于  $U'$  是不变子空间,

$$\prod_{\lambda \neq \mu} (\pi_\mu(H) - \langle \lambda, H \rangle I) u \in U' \quad (8.6.3)$$

对于  $u$  在 (8.5.7) 中的其它 (非  $v_0$ ) 分量, 经过上式都被化为零 (注意  $\mathfrak{h}$  是 Abelian), 所剩的只有  $v_0$  分量, 因此  $v_0 \in U'$ .

–  $U'$  含有  $v_0$ , 因此必然有  $U' = W_\mu$ .

- $(\pi_\mu, V^\mu)$  是最高权为  $\mu$ , 对应权向量为  $v_0$  的 highest weight cyclic rep..

- 一些计算: 对于  $\alpha \in \Delta$  (这一点对 (8.6.6) 中的分析很重要, 因为  $\alpha$  无法表示为  $R^+$  中其它元素的线性组合) 有,

$$\pi_\mu(A_\alpha)\pi_\mu^i(B_\alpha)v_0 = i(\langle\mu, H_\alpha\rangle - (i-1))\pi_\mu^{i-1}(B_\alpha)v_0 \quad (8.6.4)$$

所以, 如果  $\langle\mu, H_\alpha\rangle \in \mathbb{Z}^+ \cup \{0\}$ , 那么,

$$\pi_\mu(A_\alpha)\underbrace{\pi_\mu^{\langle\mu, H_\alpha\rangle+1}(B_\alpha)v_0}_{\text{令其}=v} = 0 \quad (8.6.5)$$

且对于  $\forall \beta \in R^+, j \in \mathbb{Z}^+$ ,

$$\pi_\mu^j(A_\beta)v \in V_{\mu - \langle\mu, H_\alpha\rangle\alpha - \alpha + j\beta} \quad (8.6.6)$$

注意到  $\mu - \langle\mu, H_\alpha\rangle\alpha - \alpha + j\beta \not\leq \mu$ , 由于  $\mu$  是最高权, 所以  $\pi_\mu^j(A_\beta)v = 0$ , 所以  $v \in U_\mu$ , (但要注意, 对于 finite-dim. rep.,  $s_\alpha|\mu\rangle$  是一个 weight of the rep., 见 (8.1.4)).

## 8.7 finite-dimensional quotient modules

- 本 section 将表明, 对于 dominant integral element  $\mu$ , 不可约表示  $V^\mu = W_\mu/U_\mu$  是有限维的.
- 这里有一些关于 nilpotent 的讨论, 没太细看 (?).
- 现在证明 section 8.1 的最后一条 theorem: if  $\mu$  is a **dominant integral element**, there exists an irreducible, finite-dim. rep. of  $\mathfrak{g}$  with **highest weight**  $\mu$ .

**proof:**

$(\pi_\mu, V^\mu)$  是 highest weight 为  $\mu$  的 irreducible rep.. 它的所有 weight 满足  $\lambda \preceq \mu$ , 且  $w|\lambda\rangle, \forall w \in W$  也是 weight. 根据 section 7.8 的最后一条的第二个定理, 可知  $\lambda \in \text{Conv}(W|\mu\rangle)$ , 因此  $(\pi_\mu, V^\mu)$  只有有限多个 weights.

(8.5.7) 中的向量构成  $V^\mu$  的一组基, 且  $n_1, \dots, n_k$  不能太大, 因此  $V^\mu$  是有限维.



## Chapter 9

# further properties of the representations

### 9.1 the structure of weights

- **theorem:** 对于 semisimple Lie algebra  $\mathfrak{g}$  的一个 irreducible finite-dim. rep.  $(\pi_\mu, V^\mu)$ , 其 highest weight 为  $\mu$ , 那么, integral element  $\lambda$  是其 weight  $\iff \lambda$  满足以下两个条件,
  1.  $\lambda \in \text{Conv}(W|\mu)$ ,
  2.  $\mu - \lambda$  可以表示成 roots 的整数线性组合.

**proof:**

- **”no holes” lemma:** 对于一个 semisimple Lie algebra  $\mathfrak{g}$  的 finite-dim. rep.  $(\pi, V)$ ,  $\lambda$  是它的一个 weight, 那么, 对于一个 root  $\alpha$  满足  $\langle \lambda, \alpha \rangle > 0$ , 有,

$$\lambda - i\alpha, i \in \{0, 1, \dots, \langle \lambda, H_\alpha \rangle\} \quad (9.1.1)$$

都是 weights, (也就是  $\lambda, \lambda - \alpha, \dots, s_\alpha|\lambda\rangle$ ).

**proof:**

考虑如下 weight spaces 的直和,

$$V \supset U = \bigoplus_{i \in \mathbb{Z}} V_{\lambda - i\alpha} \quad (9.1.2)$$

(线性独立证明见 (A.3.9)), 那么  $U$  在  $\mathfrak{s}^\alpha = \text{span}(H_\alpha, A_\alpha, B_\alpha)$  的作用下保持不变. 并且注意到  $V_{\lambda - i\alpha}$  是以,

$$\langle \lambda, H_\alpha \rangle - 2i \quad (9.1.3)$$

为本征值的  $H_\alpha$  的本征空间, 根据 (10.1.6) (不需要 irreducibility) 可知  $\langle \lambda, H_\alpha \rangle, \dots, -\langle \lambda, H_\alpha \rangle$  都是本征值.

---

首先考虑  $\lambda$  是 dominant integral (结合条件 1 implies  $\lambda \preceq \mu$ ), 来证明  $\Leftarrow$  (去除条件 finite-dim.).

### 9.2 the Casimir element

- **def.:** the 2nd-order Casimir operator is,

$$C_2 = -B^{ij} A_i \otimes A_j \quad (9.2.1)$$

where  $B^{ij} = B_{ij}^{-1}$ .

- the 2nd-order Casimir operator commutes with all the generators.

proof:

$$\begin{aligned}[C_2, A_k] &= -B^{ij}[A_i A_j, A_k] \\ &= -B^{ij}(-f_{jk}{}^l A_i A_l - f_{ik}{}^l A_l A_j)\end{aligned}\tag{9.2.2}$$

notice that  $B^{ij}$  is symmetric, so,

$$\begin{aligned}[C_2, A_k] &= -B^{ij}(-f_{ik}{}^l A_j A_l - f_{ik}{}^l A_l A_j) \\ &= B^{ij} f_{ik}{}^l (A_j A_l + A_l A_j) \\ &= \underbrace{B^{ij} B^{lm} (A_j A_l + A_l A_j)}_{\text{symmetric about } (i,m)} f_{ikm} = 0\end{aligned}\tag{9.2.3}$$

# Chapter 10

## $\mathfrak{su}(2)_{\mathbb{C}}$ algebra

- $\mathfrak{su}(2) = \{A \in \mathcal{M}_2(\mathbb{C}) | A^\dagger = -A \text{ and } \text{tr} A = 0\}$ .
  - $\dim \mathfrak{su}(2) = 2^2 - 1 = 3$ .
  - $\mathfrak{su}(2) = \text{span}\{iJ_1, iJ_2, iJ_3\}$  is a real vector space.
- its structure is,

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (10.0.1)$$

where  $i, j, k = 1, 2, 3$ .

---

- ladder operators,

$$\begin{cases} J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \in \mathfrak{su}(2)_{\mathbb{C}} \\ [J_3, J_{\pm}] = \pm J_{\pm} \\ [J_+, J_-] = J_3 \\ J^2 = J_+J_- + J_-J_+ + J_3^2 \end{cases} \quad (10.0.2)$$

- another basis is  $H = 2J_3, A = \sqrt{2}J_+, B = \sqrt{2}J_-$ , and,

$$\begin{cases} [H, A] = 2A \\ [H, B] = -2B \\ [A, B] = H \end{cases} \quad \text{ad}_H = \begin{pmatrix} 0 & & \\ & 2 & \\ & & -2 \end{pmatrix} \quad \text{ad}_A = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{ad}_B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad (10.0.3)$$

so, the Killing form is,

$$B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \quad (10.0.4)$$


---

- its Killing form is  $B_{ij} = \epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$ .
- its 2nd order Casimir operator is,

$$C_2 = -B^{ij}A_iA_j = \frac{1}{2}\delta_{ij}J_iJ_j = \frac{1}{2}J^2 \quad (10.0.5)$$

### 10.1 representations of $\mathfrak{su}(2)_{\mathbb{C}}$ algebra

- for each (half-)integer  $j$ , there exists an  $2j + 1$  dimensional **irreducible** complex rep.,

$$\pi_j : \mathfrak{su}(2)_{\mathbb{C}} \rightarrow \text{span}(|j, m\rangle, m = -j, \dots, j) \quad (10.1.1)$$

and any two irreducible rep. with the same dimension are isomorphic.

**proof:**

let  $\pi$  be an irreducible rep. of  $\mathfrak{su}(2)_{\mathbb{C}}$  on a finite-dimensional complex vector space  $V$ , and  $|u\rangle$  is a eigenvector of  $\pi(J_3)$ ,

$$\begin{cases} \pi(J_3)|u\rangle = \alpha|u\rangle \\ \pi(J_3)\pi^k(J_{\pm})|u\rangle = (\alpha \pm k)\pi^k(J_{\pm})|u\rangle \end{cases} \quad (10.1.2)$$

since  $V$  is finite-dimensional, so there is some  $N_{\pm} \geq 0$ , s.t.,

$$\pi^{N_{\pm}}(J_{\pm})|u\rangle \neq 0 \quad \text{but} \quad \pi^{N_{\pm}+1}(J_{\pm})|u\rangle = 0 \quad (10.1.3)$$

let's set  $|u_0\rangle = \pi^{N_-}(J_-)|u\rangle$  and  $\lambda_0 = \alpha - N_-$ ,  $|u_k\rangle = \pi^k(J_+)|u_0\rangle$ , then,

$$\pi(J_3)|u_k\rangle = (\lambda_0 + k)|u_k\rangle, k = 0, \dots, 2j \quad (10.1.4)$$

where  $j = \frac{N_+ + N_-}{2}$ , and,

$$\begin{aligned} \pi(J_-)|u_k\rangle &= -k(\lambda_0 + \frac{k-1}{2})|u_{k-1}\rangle \\ \xrightarrow{k-1=2j} 0 &= -(2j+1)(\lambda_0 + j)|u_{2j-1}\rangle \implies \lambda_0 = -j \end{aligned} \quad (10.1.5)$$

so, for any **finite-dimensional** rep. of  $\mathfrak{su}(2)_{\mathbb{C}}$ ,  $\lambda_0 = -j$  must be a **(half-)integer**.

- according to appendix A.1,  $|u_0\rangle, \dots, |u_{2j}\rangle$  are **linearly independent**.
- $\text{span}(|u_0\rangle, \dots, |u_{2j}\rangle)$  is **invariant** under  $\pi(J_3), \pi(J_{\pm})$ , hence invariant under all  $\pi(A), A \in \mathfrak{su}(2)_{\mathbb{C}}$ .
- so every irreducible rep. is of the form as  $\text{span}(|u_0\rangle, \dots, |u_{2j}\rangle)$ .

- for any finite-dim. (not necessarily irreducible) rep.  $(\pi, V)$  of  $\mathfrak{su}(2)_{\mathbb{C}}$ ,

1. all eigenvalues of  $\pi(J_3)$  are **(half-)integer**,

$$-j, -j+1, \dots, j \quad (10.1.6)$$

2.  $\pi(J_{\pm})$  are nilpotent,

3. let  $S = e^A e^{-B} e^A \implies \Pi(S) = e^{\pi(A)} e^{-\pi(B)} e^{\pi(A)}$ , then,

$$\text{Ad}_S H = -H \implies \Pi(S)\pi(H)\Pi(S^{-1}) = -\pi(H) \quad (10.1.7)$$

**calculation:**

use the Campbell's identity,

$$\begin{aligned} \text{Ad}_{\Pi(S)}\pi(H) &= \pi(\text{Ad}_{e^A} \text{Ad}_{e^{-B}} \text{Ad}_{e^A} H) \\ &= \pi(e^{\text{ad}_A} e^{-\text{ad}_B} e^{\text{ad}_A} H) \end{aligned} \quad (10.1.8)$$

and,

$$\begin{aligned} e^{\text{ad}_A} H &= H - 2A \\ e^{-\text{ad}_B}(H - 2A) &= H - 2B - 2(A + H - B) = -H - 2A \\ e^{\text{ad}_A}(-H - 2A) &= -(H - 2A) - 2A = -H \end{aligned} \quad (10.1.9)$$

and,

$$\begin{aligned} \text{Ad}_S^{-1} H &= e^{-\text{ad}_A} e^{\text{ad}_B} e^{-\text{ad}_A} H \\ &= e^{-\text{ad}_A} e^{\text{ad}_B}(H + 2A) \\ &= e^{-\text{ad}_A} \underbrace{((H + 2B) + 2(A - H - B))}_{=-H+2A} = -H \end{aligned} \quad (10.1.10)$$

but,

$$\begin{aligned}
e^{\text{ad}_{J_+}} J_3 &= J_3 - J_+ \\
e^{-\text{ad}_{J_-}} (J_3 + J_+) &= (J_3 - J_-) - (J_+ + J_3 - \frac{1}{2} J_-) = -J_+ - \frac{1}{2} J_- \\
e^{\text{ad}_{J_+}} (-J_+ - \frac{1}{2} J_-) &= -J_+ - \frac{1}{2} (J_- + J_3 - \frac{1}{2} J_+) \quad (10.1.11)
\end{aligned}$$

- the eigenstates  $|j, m\rangle$  of the operators  $J_3, J^2$  are,

$$\begin{cases} J_3 |j, m\rangle = m |j, m\rangle \\ J^2 |j, m\rangle = j(j+1) |j, m\rangle \\ J_{\pm} |j, m\rangle = \frac{1}{\sqrt{2}} \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle \end{cases} \quad (10.1.12)$$

when  $J_1 = \frac{1}{\sqrt{2}}(J_+ + J_-)$  and  $J_2 = \frac{1}{i\sqrt{2}}(J_+ - J_-)$  act on  $|s, m\rangle$ ,

$$\begin{cases} J_1 |j, m\rangle = \lambda_+(j, m) |j, m+1\rangle + \lambda_-(j, m) |j, m-1\rangle \\ J_2 |j, m\rangle = -i\lambda_+(j, m) |j, m+1\rangle + i\lambda_-(j, m) |j, m-1\rangle \end{cases} \quad (10.1.13)$$

where  $\lambda_{\pm}(j, m) = \sqrt{\frac{j(j+1) - m(m \pm 1)}{2}}$ .

- spin- $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$  rep. are faithful, and spin-0, 1, 2,  $\dots$  rep. are not faithful.**

### 10.1.1 spin- $\frac{1}{2}$ representation

- choose  $s = 1/2$ , and  $|\frac{1}{2}, \frac{1}{2}\rangle = (1, 0)^T, |\frac{1}{2}, -\frac{1}{2}\rangle = (0, 1)^T$ , then  $J_i = \frac{1}{2}\sigma_i$ , where,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (10.1.14)$$

and the ladder operators are,

$$J_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (10.1.15)$$

### 10.1.2 spin-1 representation

- choose  $s = 1$ , and  $|1, 1\rangle = (1, 0, 0)^T, |1, 0\rangle = (0, 1, 0)^T, |1, -1\rangle = (0, 0, 1)^T$ , then,

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (10.1.16)$$

## 10.2 direct product representation

- the direct product representation of the SU(2) group is,

$$D_{ii'jj'}^{1 \otimes 2}(g) = D_{ij}^1(g) D_{i'j'}^2(g) \quad (10.2.1)$$

- consider a group element near the identity,

$$\begin{aligned}
(1 + i\alpha_i J_i^{1 \otimes 2})_{ii'jj'} &= (\delta_{ij}^1 + i\alpha_i (J_i^1)_{ij})(\delta_{i'j'}^2 + i\alpha_i (J_i^2)_{i'j'}) \\
&= \delta_{ij}^1 \delta_{i'j'}^2 + i\alpha_i (J_i^{1 \otimes 2})_{ii'jj'} \quad (10.2.2)
\end{aligned}$$

where  $(J_i^{1 \otimes 2})_{ii'jj'} = (J_i^1)_{ij} \delta_{i'j'}^2 + \delta_{ij}^1 (J_i^2)_{i'j'}$  or more compactly,

$$J_i^{1 \otimes 2} = J_i^1 \otimes I^2 + I^1 \otimes J_i^2 \quad (10.2.3)$$

- the eigenstates are,

$$J_3^{1\otimes 2} |j_1, m_1\rangle \otimes |j_2, m_2\rangle = (m_1 + m_2) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \quad (10.2.4)$$

- the  $(J^2)^{j_1 \otimes j_2}$  is,

$$\begin{aligned} (J^2)^{j_1 \otimes j_2} &= \sum_i (J_i^{j_1} \otimes I^{j_2} + I^{j_1} \otimes J_i^{j_2})^2 \\ &= (J^2)^{j_1} \otimes I^{j_2} + I^{j_1} \otimes (J^2)^{j_2} + 2 \sum_i J_i^{j_1} \otimes J_i^{j_2} \end{aligned} \quad (10.2.5)$$

**when  $(J^2)^{j_1 \otimes j_1}$  acts on  $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ :**

$$\begin{aligned} &(J^2)^{j_1 \otimes j_1} |j_1, m_1\rangle \otimes |j_2, m_2\rangle \\ &= (j_1(j_1 + 1) + j_2(j_2 + 1) + 2m_1m_2) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \\ &\quad + 2(J_1^{j_1} \otimes J_1^{j_2} + J_2^{j_1} \otimes J_2^{j_2}) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \end{aligned} \quad (10.2.6)$$

where,

$$\begin{aligned} &2(J_1^{j_1} \otimes J_1^{j_2} + J_2^{j_1} \otimes J_2^{j_2}) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \\ &= 4\lambda_+(j_1, m_1)\lambda_-(j_2, m_2) |j_1, m_1 + 1\rangle \otimes |j_2, m_2 - 1\rangle \\ &\quad + 4\lambda_-(j_1, m_1)\lambda_+(j_2, m_2) |j_1, m_1 - 1\rangle \otimes |j_2, m_2 + 1\rangle \end{aligned} \quad (10.2.7)$$

### 10.2.1 Clebsch-Gordan coefficients

- direct product representation and direct sum representation,

$$\{j_1\} \otimes \{j_2\} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \{j\} \quad (10.2.8)$$

where  $\{j\}$  means spin- $j$  representation.

**proof:**

the eigenvalue and corresponded eigenspace of  $J_3^{j_1 \otimes j_2}$  is (assuming  $j_1 \geq j_2$ ),

eigenvalue	basis of the eigenspace	dimension
$j_1 + j_2$	$ j_1, j_1, j_2, j_2\rangle$	1
$j_1 + j_2 - 1$	$ j_1, j_1 - 1, j_2, j_2\rangle,  j_1, j_1, j_2, j_2 - 1\rangle$	2
$\vdots$	$\vdots$	$\vdots$
$j_1 + j_2 - 2j_2$	$ j_1, j_1 - 2j_2, j_2, j_2\rangle, \dots,  j_1, j_1, j_2, -j_2\rangle$	$1 + 2j_2$
$j_1 - j_2 - 1$	$ j_1, j_1 - 2j_2 - 1, j_2, j_2\rangle, \dots,  j_1, j_1 - 1, j_2, -j_2\rangle$	$1 + 2j_2$
$\vdots$	$\vdots$	$\vdots$
$j_1 + j_2 - 2j_1$	$ j_1, -j_1, j_2, j_2\rangle, \dots,  j_1, -j_1 + 2j_2, j_2, -j_2\rangle$	$1 + 2j_2$
$-j_1 + j_2 - 1$	$ j_1, -j_1, j_2, j_2 - 1\rangle, \dots,  j_1, -j_1 + 2j_2 - 1, j_2, -j_2\rangle$	$2j_2$
$\vdots$	$\vdots$	$\vdots$
$-j_1 - j_2$	$ j_1, -j_1, j_2, -j_2\rangle$	1

so, it is clear that we can use  $|j_1, j_1, j_2, j_2\rangle$  and  $J_-^{j_1 \otimes j_2}$  to produce  $\{j_1 + j_2\}$ , and among the rest of the vectors, the highest eigenvalue of  $J_3^{j_1 \otimes j_2}$  is  $j_1 + j_2 - 1$  and there is only one vector with this eigenvalue is remained.

hence,

$$\{j_1\} \otimes \{j_2\} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \{j\} \quad (10.2.9)$$

– example:  $\{\frac{1}{2}\} \otimes \{\frac{1}{2}\} = \underbrace{\{1\}}_{\text{spin triplet}} \oplus \underbrace{\{0\}}_{\text{spin singlet}}$

---

- the Clebsch-Gordan coefficients are,

$$\langle j_1, m_1, j_2, m_2 | j_1, j_2, j, m \rangle \quad (10.2.10)$$

where  $|j_1, j_2, j, m\rangle$  (it is common to write  $|j, m\rangle$  for short) are the coupled eigenstates of  $J_3^{j_1 \otimes j_2}$  and  $(J^2)^{j_1 \otimes j_2}$ .

- the recursion relations are,

$$\begin{aligned} & \lambda_{\pm}(j_1, \mathbf{m_1} \mp 1) \langle j_1, \mathbf{m_1} \mp 1, j_2, m_2 | j, m \rangle \\ & + \lambda_{\pm}(j_2, \mathbf{m_2} \mp 1) \langle j_2, m_2, j_2, \mathbf{m_2} \mp 1 | j, m \rangle \\ & = \lambda_{\pm}(j, m) \langle j_1, m_1, j_2, m_2 | j, \mathbf{m} \mp 1 \rangle \end{aligned} \quad (10.2.11)$$

**proof:**

just consider the ladder operators  $J_{\pm}^{j_1 \otimes j_2} = J_{\pm}^{j_1} \otimes I^{j_2} + I^{j_1} \otimes J_{\pm}^{j_2}$ ,

$$\sum_{j_1, m_1, j_2, m_2} J_{\pm}^{j_1 \otimes j_2} |j_1, m_1, j_2, m_2\rangle \langle j_1, m_1, j_2, m_2 | j, m \rangle = \dots \quad (10.2.12)$$

taking  $m = j$  gives the initial recursion relation,

$$\begin{aligned} & \lambda_{+}(j_1, \mathbf{m_1} - 1) \langle j_1, \mathbf{m_1} - 1, j_2, m_2 | j, j \rangle \\ & + \lambda_{+}(j_2, \mathbf{m_2} - 1) \langle j_2, m_2, j_2, \mathbf{m_2} - 1 | j, j \rangle = 0 \end{aligned} \quad (10.2.13)$$

- use the phase convention that  $\langle j_1, m_1, j_2, m_2 | j, j \rangle \in \mathbb{R}$  and  $> 0$ , combined with the recursion relations, we can conclude that  $\langle j_1, m_1, j_2, m_2 | j, m \rangle \in \mathbb{R}$ .

# Part IV

## Applications



# Chapter 11

## some examples of Lie groups and Lie algebras

### 11.1 general linear groups and algebras

- $\mathrm{GL}(n, \mathbb{C}) = \{M \in \mathcal{M}_n(\mathbb{C}) \mid \det M \neq 0\}$ .
  - $\dim \mathrm{GL}(n, \mathbb{C}) = n^2$ .
  - $\mathrm{GL}(n, \mathbb{R})$  有两个连通分支,

$$\mathrm{GL}(n, \mathbb{R}) = \det^{-1}[(-\infty, 0)] \sqcup \det^{-1}[(0, \infty)] \quad (11.1.1)$$

- $\mathfrak{gl}(n, \mathbb{C}) = \mathcal{M}_n(\mathbb{C})$ .
- the left-invariant vector field at  $g$  is,

$$(A_g)^i_j = x^i_k(g)(A_e)^k_j \quad (11.1.2)$$

and the Lie bracket is,

$$[A, B] = AB - BA \quad (11.1.3)$$

**proof:**

for general linear group,  $x^i_j(gh) = x^i_k(g)x^k_j(h)$ .  
so, the pushforward of the left transformation is,

$$L_{g*}(A|_e)x^i_j|_g = A(y^i_j)|_e \quad (11.1.4)$$

where  $y^i_j(h) = (L_g^*x^i_j)(h) = x^i_k(g)x^k_j(h)$ , so we have,

$$A(y^i_j)|_e = A|_e(x^k_l) \underbrace{\frac{\partial y^i_j}{\partial x^k_l}|_e}_{=x^i_m(g)\delta^m_k\delta^l_j} = x^i_k(g)A|_e(x^k_j) \quad (11.1.5)$$

$$\begin{aligned} [A, B]^i_j &= (dx^i_j)_a(A^b\partial_b B^a - B^b\partial_b A^a) \\ &= A^k_l \frac{\partial}{\partial x^k_l} B^i_j - B^k_l \frac{\partial}{\partial x^k_l} A^i_j \end{aligned} \quad (11.1.6)$$

注意  $(A_g)^i_j = x^i_k(g)(A_e)^k_j$ , 所以,

$$\frac{\partial}{\partial x^k_l}(A^i_j)|_g = \underbrace{\frac{\partial}{\partial x^k_l}(x^i_m(g))}_{=\delta^i_k\delta^l_m}(A_e)^m_j = \delta^i_k(A_e)^l_j \quad (11.1.7)$$

代入得到,

$$[A, B]^i_j|_g = (A_g)^k_l \delta^i_k (B_e)^l_j - (B_g)^k_l \delta^i_k (A_e)^l_j$$

$$= x^i_k(g)(A^k_l B^l_j - B^k_l A^l_j) \quad (11.1.8)$$

## 11.2 special linear groups and algebras

- $\mathrm{SL}(n, \mathbb{C}) = \{M \in \mathrm{GL}(n, \mathbb{C}) | \det M = 1\}$ .
- $\mathfrak{sl}(n, \mathbb{C}) = \{A \in \mathcal{M}_n(\mathbb{C}) | \mathrm{tr} A = 0\}$ .

## 11.3 the Lorentz group and the Lorentz algebra

### 11.3.1 indefinite orthogonal groups

- $\mathrm{O}(p, q) = \{\Lambda \in \mathcal{M}_n(\mathbb{R}) | \Lambda^T \eta \Lambda = \eta\}$  is called the **indefinite orthogonal group**, where  $n = p + q$  and,

$$\eta = \mathrm{diag}(\underbrace{+1, \dots, +1}_p, \underbrace{-1, \dots, -1}_q) \quad (11.3.1)$$

– 将  $\lambda$  视作一组列向量  $(\lambda_1, \dots, \lambda_n)$ , 那么,

$$\eta(\lambda_\mu, \lambda_\nu) = \eta_{\mu\nu} \quad (11.3.2)$$

即  $n$  个互相正交的向量.

–  $\dim \mathrm{O}(p, q) = \frac{n(n-1)}{2}$ .

– 可以证明, 对于,

$$\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (11.3.3)$$

有  $\det \Lambda = \frac{\det A}{\det D}$ , 且  $|\det A|, |\det D| \geq 1$ .

**proof:**

分块矩阵满足,

$$\begin{cases} A^T B = C^T D \\ A^T A - C^T C = I_{p \times p} \\ D^T D - B^T B = I_{q \times q} \end{cases} \quad (11.3.4)$$

如果  $\det A \neq 0$ , 那么,

$$\det \Lambda = \det(A) \det(D - CA^{-1}B) \quad (11.3.5)$$

对 (11.3.4) 的第一行做变换, 得到,

$$A^{-1} = C^{-1}(D^T)^{-1}B^T \implies CA^{-1}B = (D^T)^{-1}B^T B \quad (11.3.6)$$

再代入 (11.3.4) 的第三行, 得到  $CA^{-1}B = D - (D^T)^{-1}$ , 所以...

由 (11.3.4) 的第二行,

$$\det^2 A = \det(I + C^T C) \stackrel{(?)}{\geq} 1 \quad (11.3.7)$$

- $\mathrm{O}(p, q)$  具有如下子群,

$$\begin{cases} \mathrm{SO}(p, q) = \{\Lambda \in \mathrm{O}(p, q) | \det \Lambda = 1\} \\ \mathrm{SO}_+(p, q) = \{\Lambda \in \mathrm{SO}(p, q) | \det A \geq 1\} \\ \mathrm{O}_+(p, q) = \{\Lambda \in \mathrm{O}(p, q) | \det A \geq 1\} \\ \mathrm{O}_-(p, q) = \{\Lambda \in \mathrm{O}(p, q) | \det D \geq 1\} \end{cases} \quad (11.3.8)$$

且有如下四个连通分支,

$$\mathrm{SO}_\pm(p, q) \quad \text{and} \quad \mathrm{O}'_\pm(p, q) = \{\det \Lambda = -1, \det A \geq 1 \text{ or } \det A \leq -1\} \quad (11.3.9)$$

### 11.3.2 the Lorentz group

- $L = O(3, 1)$  is called the Lorentz group.

### 11.3.3 the Lorentz algebra

- $\mathfrak{so}(3, 1) = \{A \in \mathcal{M}_4(\mathbb{R}) | A^T = -\eta A \eta\} \simeq \mathfrak{so}(4)$ .
- 考虑  $\mathfrak{so}(4, \mathbb{C})$  的 Dynkin diagram,  $D_2$ , (见 section 6.7), 可见  $\mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ .  
– 因此,  $\mathfrak{so}(3, 1)$  的 irreducible rep. 是  $\text{spin-}j_1 \oplus \text{spin-}j_2$ , 用  $(j_1, j_2)$  表示.

## 11.4 unitary groups and algebras

- $U(n) = \{U \in GL(n, \mathbb{C}) | U^\dagger U = I\}$ .  
–  $\dim U(n) = n^2$ .  
–  $U(n)$  is connected.
- $\mathfrak{u}(n) = \{A \in \mathcal{M}_n(\mathbb{C}) | A^\dagger = -A\}$ .

## 11.5 special unitary groups and algebras

- $SU(n) = \{U \in GL(n, \mathbb{C}) | U^\dagger U = I, \det U = 1\}$ .  
–  $\dim SU(n) = n^2 - 1$ .
- $\mathfrak{su}(n) = \{A \in \mathcal{M}_n(\mathbb{C}) | A^\dagger = -A, \text{tr} A = 0\}$ .

## 11.6 symplectic groups

- $Sp(2n, \mathbb{C}) = \{A \in \mathcal{M}_{2n}(\mathbb{C}) | -\Omega A^T \Omega = A^{-1}\}$ , where,

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (11.6.1)$$

- $\dim Sp(2n, \mathbb{C}) = 2n(2n + 1)$ .

## 11.7 the representations of $\mathfrak{sl}(3, \mathbb{C})$

- this section, we are going to discuss the classification of the irreducible rep. of  $SU(3)$  and  $\mathfrak{sl}(3, \mathbb{C})$ .
- $\mathfrak{sl}(3, \mathbb{C}) \simeq \mathfrak{su}(3)_{\mathbb{C}}$ .
- $SU(m)$  are **simply connected, compact** Lie groups.  
– according to section 5.1.1, 单连通李群 (的表示) 完全由其李代数 (的表示) 决定.  
rep. of  $\mathfrak{sl}(3, \mathbb{C}) \xrightarrow{\text{restrict to}} \text{rep. of } \mathfrak{su}(3) \xrightarrow{\text{simple connectedness}} \text{rep. of } SU(3)$ .  
– according to section 5.2,  $\Pi$  is irreducible  $\iff \pi$  is irreducible.  
and  $SU(3)$  is **compact**, so it has complete reducibility property  $\implies$  rep. of  $\mathfrak{sl}(3, \mathbb{C})$  is **completely reducible**. 可见, 半单李代数的表示都是 completely reducible.

## Chapter 12

# the spin groups, $\text{Spin}(n)$

- Wikipedia: [Spin group](#), [Indefinite orthogonal group](#),  $O(p, q)$ .
- 关于 universal cover &  $\text{Spin}(n)$  与  $\text{SO}(n \geq 3)$  和 Clifford algebra 的关系, 见 subsection [5.1.2](#).

# Appendices

# Appendix A

## linear algebra review

- **def.:** an **algebra** (over a field  $K$ ) is a vector space + bilinear product  $B : A \times A \rightarrow A$  (简写做  $\cdot$ ), 几个主要特征如下,

1. 双线性形式  $B(\cdot, \cdot)$  满足左, 右分配律和 (A.0.2),
2. 可能存在单位元 (不是零向量),

$$B(e, x) = x, \forall x \quad (\text{A.0.1})$$

存在单位元的代数称为 **unital algebra**.

- 注意区分 bilinear form 和 sesquilinear form,

$$\begin{cases} B(ax, by) = abB(x, y) & \text{双线性} \\ S(ax, by) = a^*bS(x, y) & \text{半双线性, 有复共轭} \end{cases} \quad (\text{A.0.2})$$

一般用  $(\cdot, \cdot)$  和  $\langle \cdot, \cdot \rangle$  区分.

- 李代数  $\mathfrak{g}$  一定不存在单位元, (因为一定有  $[E, E] = 0 \implies E = 0$  与单位元性质矛盾).

- 另外,

$$\begin{aligned} \text{injective} &\leftrightarrow \text{one-to-one function} \\ \text{surjective} &\leftrightarrow \text{onto} \\ \text{bijective} &\leftrightarrow \text{one-to-one correspondence} \end{aligned}$$

- a exact sequence (其中  $f_i$  都是 homomorphism),

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \cdots \quad (\text{A.0.3})$$

表示  $f_1[G_1] = \ker(f_2)$ . 例如,

- $G \rightarrow H \rightarrow 0$  表示  $f[G] = \ker(f_2) = H$ , 即  $f$  是 onto.
- $0 \rightarrow G \rightarrow H$  表示  $\{0\} = \ker(f)$ , 即  $f$  是 one-to-one.

- a short exact sequence,

$$0 \rightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow 0 \quad (\text{A.0.4})$$

表示  $f_1$  是 one-to-one,  $f_2$  是 onto, 且  $\ker(f_2) = f_1[G_1]$ , 所以,

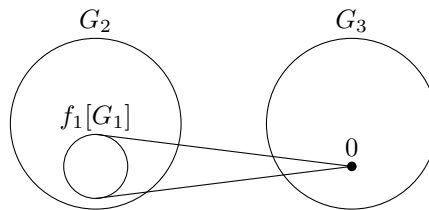


Figure A.1: short exact sequence

注意到  $f_1, f_2$  都是 homomorphism, 所以,

$$G_3 = G_2 / f_1[G_1] \quad (\text{A.0.5})$$

## A.1 eigenvalues and eigenspaces

- eigenvectors associated to different eigenvalues are linearly independent.

**proof:**

if  $v_1, \dots, v_k$  are linearly independent eigenvectors with different eigenvalues, and  $v_{k+1}$  is a linear combination of them and is also an eigenvector, then,

$$\begin{aligned} v_{k+1} &= \sum_{i=1}^k c^i v_i \implies \lambda_{k+1} v_{k+1} = \sum_i c^i \lambda_i v_i \\ \implies 0 &= \sum_i c^i (\lambda_i - \lambda_{k+1}) v_i \end{aligned} \quad (\text{A.1.1})$$

which contradicts to the linear independence.

## A.2 spectral theorem for normal matrices

### A.2.1 diagonalization

- we want to use an **reversible matrix** to **diagonalize** a diagonalizable matrix  $A \in \text{End}(\mathbb{C}^n)$ ,

$$P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n) \iff A = P \text{diag}(\lambda_1, \dots, \lambda_n) P^{-1} \quad (\text{A.2.1})$$

we can see that:

- $\det A = \prod_i \lambda_i$ .
- $\text{tr} A = \sum_i \lambda_i$ .

**method to find  $P$ :**

consider,

$$AP = P \text{diag}(\lambda_1, \dots, \lambda_n) \quad (\text{A.2.2})$$

let the column-vector be  $P_{ij} = \xi_i^{(j)}$ , then,

$$\sum_j A_{ij} \xi_j^{(k)} = \xi_i^{(k)} \lambda_k \quad \text{or} \quad A \xi^{(k)} = \lambda_k \xi^{(k)} \quad (\text{A.2.3})$$

it is clear that  $\{\xi^{(i)}\}$  are the eigenvectors of  $A$  with corresponding eigenvalues  $\{\lambda_i\}$ .

- $A$  is diagonalizable  $\iff$  the eigenspace of  $A$  is  $n$ -dimensional.

### A.2.2 geometric multiplicity & algebraic multiplicity

- the **dimension theorem**: let  $T : V \rightarrow W$ , then,

$$\dim V = \dim \ker T + \dim T(V) \quad (\text{A.2.4})$$

where  $T(\ker T) = 0 \in W$ .

**proof:**

let  $U \cap \ker T = \{0\}$  and  $V = U \oplus \ker T$ , so,

$$\dim V = \dim \ker T + \dim U \quad (\text{A.2.5})$$

$\forall |b_1\rangle, |b_2\rangle \in U$ , if  $|b_1\rangle \neq |b_2\rangle$  then  $T|b_1\rangle \neq T|b_2\rangle$ , so,

$$T(U) \simeq U \implies \dim U = \dim T(U) \quad (\text{A.2.6})$$

and notice that  $T(V) = T(U)$ , so we have  $\dim U = \dim T(V)$ .

- some times we use  $\dim T \equiv \dim T(V)$  for convenience.

- **def.:** the **geometric multiplicity** (of eigenvalue  $\lambda_i$ ),  $\gamma_A(\lambda_i)$ , is defined to be,

$$\gamma_A(\lambda_i) = \dim(\ker(A - \lambda_i I)) \equiv n - \dim(A - \lambda_i I) \quad (\text{A.2.7})$$

- **def.:** the **algebraic multiplicity**,  $\mu_A(\lambda_i)$ , is defined to be the multiplicity (重根数) of root  $\lambda_i$  in the polynomial  $\det(A - \lambda I) = 0$ .
- theorem of geometric multiplicity & algebraic multiplicity:

$$1 \leq \gamma_A(\lambda_i) \leq \mu_A(\lambda_i) \leq n \quad (\text{A.2.8})$$

**proof:**

let  $\{v_{i=1, \dots, \gamma_A(\lambda_i)}\}$  to be the orthogonal basis of the eigenspace of  $\lambda_i$ ,

$$A |v_j\rangle, j \in \{1, \dots, \gamma_A(\lambda_i)\} = \lambda_i |v_j\rangle \quad (\text{A.2.9})$$

and let  $\{v_1, \dots, v_{\gamma_A(\lambda_i)}, v_{\gamma_A(\lambda_i)+1}, \dots, v_n\}$  to be the orthogonal basis of the vector space  $V$ , (note that  $\{v_{\gamma_A(\lambda_i)+1}, \dots, v_n\}$  are not necessarily eigenvectors), then,

$$\langle v_j | A |v_k\rangle \equiv A'_{jk} = \begin{pmatrix} \lambda_i & & * & * & * \\ & \ddots & & & \\ & & \lambda_i & * & * & * \\ & & & \ddots & \\ & & & & \lambda_i & * & * & * \end{pmatrix} \quad (\text{A.2.10})$$

then we have,

$$\det(A - \lambda I) = \det(A' - \lambda I) = (\lambda - \lambda_i)^{\gamma_A(\lambda_i)} \mathcal{P}_{n-\gamma_A(\lambda_i)}^c(\lambda) \quad (\text{A.2.11})$$

so, it is clear that  $\mu_A(\lambda_i) \geq \gamma_A(\lambda_i)$ .

### A.2.3 Schur decomposition

- **Schur decomposition:** for any complex matrix  $M$ ,

$$M = U(\text{upper triangle matrix})U^\dagger \quad (\text{A.2.12})$$

**proof:**

let  $\lambda \in \mathbb{C}$  to be an eigenvalue of  $U$  with corresponding orthonormal eigenvectors  $\{v_1, \dots, v_{\gamma_M(\lambda)}\}$ , then use the eigenvectors to construct an orthonormal basis,

$$\langle v_i | M |v_j\rangle = \begin{pmatrix} \lambda I_{\gamma_M(\lambda) \times \gamma_M(\lambda)} & M_{12} \\ 0 & M_{22} \end{pmatrix} \quad (\text{A.2.13})$$

apply the exact procedure to  $M_{22}$  until  $M$  is completely trianglized.

### A.2.4 spectral theorem for normal matrices

- **def.:** matrix  $A$  is **normal** if and only if  $[A, A^\dagger] = 0$ .
- **spectral theorem** for normal matrices:

there is an orthogonal basis consisting of eigenvectors of  $A$ .

**proof:**

– **normal triangle** matrix must be **diagonal**.

-----  
**proof:**



assume  $A$  is an upper triangle normal matrix, then  $A^\dagger A$  is upper triangle and  $AA^\dagger$  is lower triangle, which implies both of them are diagonal.

$A^\dagger A$  is diagonal  $\implies$  matrix  $A$  is also diagonal (draw  $A$  and  $A^\dagger$  and it will become obvious).

- $A$  is similar to an upper triangle matrix which is also normal  $\implies$  similar to a diagonal matrix.

#### for Hermitian matrices

- for a Hermitian matrix  $H$ ,  $\lambda_i \in \mathbb{R}$ .
- if  $\lambda_i \neq \lambda_j$  then their eigenvectors are orthogonal.

**proof:**

$$\langle v_i | H | v_j \rangle = \lambda_j \langle v_i | v_j \rangle = (\langle v_j | H | v_i \rangle)^* = \lambda_i^* \langle v_i | v_j \rangle \implies \begin{cases} i = j & \lambda_i \in \mathbb{R} \\ i \neq j & \langle v_i | v_j \rangle = 0 \end{cases} \quad (\text{A.2.14})$$

- there is an orthogonal basis consisting of eigenvectors, i.e.  $\gamma_H(\lambda_i) = \mu_H(\lambda_i)$ .

#### for unitary matrices

- for a unitary matrix  $U$ ,  $|\lambda_i| = 1$ .
- if  $\lambda_i \neq \lambda_j$  then their eigenvectors are orthogonal.

**proof:**

$$\underbrace{\langle v_i | U^\dagger U | j \rangle}_{\langle v_i | v_j \rangle} = \lambda_i^* \lambda_j \langle v_i | v_j \rangle \implies \begin{cases} i = j & |\lambda_i| = 1 \\ \lambda_i \neq \lambda_j & \langle v_i | v_j \rangle = 0 \end{cases} \quad (\text{A.2.15})$$

- there is an orthogonal basis consisting of eigenvectors, i.e.  $\gamma_U(\lambda_i) = \mu_U(\lambda_i)$ .

#### for skew self-adjoint matrices

- for a skew self-adjoint matrix  $A$  ( $A^\dagger = -A$ ),  $\lambda_i \in i\mathbb{R}$ .
- if  $\lambda_i \neq \lambda_j$  then their eigenvectors are orthogonal.

**proof:**

$$\langle v_i | A | v_j \rangle = \lambda_j \langle v_i | v_j \rangle = (-\langle v_j | A | v_i \rangle)^* = -\lambda_i^* \langle v_i | v_j \rangle \implies \begin{cases} i = j & \lambda_i \in i\mathbb{R} \\ i \neq j & \langle v_i | v_j \rangle = 0 \end{cases} \quad (\text{A.2.16})$$

- there is an orthogonal basis...

## A.3 simultaneous diagonalization

### A.3.1 weights and weight spaces

- $V$  is a vector space,  $\mathcal{A}$  is a vector space of linear operators on  $V$ , and  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathcal{A}$ .
- **def.:** a **weight** for  $\mathcal{A}$  is an element  $\mu \in \mathcal{A}$  s.t. there exists a nonzero  $v \in V$ ,

$$Av = \langle \mu, A \rangle v \quad (\text{A.3.1})$$

for all  $A \in \mathcal{A}$ .

- **def.:**  $V_\mu = \{v \in V | A|v\rangle = |v\rangle \langle \mu, A \rangle, \forall A \in \mathcal{A}\}$  is called the **weight space** of  $\mu$ .
- if  $\mathcal{A}$  is **Abelian**, then there **exists** (at least) one weight for  $\mathcal{A}$ .

**proof:**

- assume  $W$  is the **minimal nonzero invariant subspace** of  $\mathcal{A}$ , meaning that,

$$A[W] \subseteq W, \forall A \in \mathcal{A} \quad (\text{A.3.2})$$

and every subspace of  $U$ , except  $\{0\}$ , is not nonzero invariant under some operator in  $\mathcal{A}$ .  
( $V$  is invariant but may not be minimal, so  $W$  **exists**)

- there exists  $u \in W$  s.t.  $u$  is an eigenvector of  $A \in \mathcal{A}$ , with eigenvalue  $\lambda$ .

**proof:**

let  $\{w_1, \dots, w_m\}$  be the basis of  $W$ , then,

$$Aw_i = \sum_{j=1}^m \alpha_{ij} w_j \quad (\text{A.3.3})$$

the eigenvector of  $\{\alpha_{ij}\}$  is  $\xi$  with  $\sum_i \xi^i \alpha_{ij} = \lambda_\alpha \xi^j$ , then,

$$A\xi^i w_i = \lambda_\alpha \xi^j w_j \quad (\text{A.3.4})$$

so,  $u = \xi^i w_i$  is an eigenvector of  $A$ .

- the eigenspace  $E_{A,\lambda}$  is an invariant subspace of  $\mathcal{A}$ ,

$$ABv = BAv = \lambda Bv \implies B[E_{A,\lambda}] \subseteq E_{A,\lambda}, \forall B \quad (\text{A.3.5})$$

- for  $u \in W \cap E_{A,\lambda}$ ,

$$Bu \in W \text{ and } E_{A,\lambda} \quad (\text{A.3.6})$$

so,  $W \cap E_{A,\lambda} \subseteq W$  is an invariant subspace of  $\mathcal{A}$ , which contradicts to the def. of  $W$ .

- so all the elements in  $W$  are eigenvectors of  $A$ , i.e. it is the **simultaneous eigenspace** of  $\mathcal{A}$ .

### A.3.2 simultaneous diagonalization

- **def.:**  $\mathcal{A}$  is **simultaneously diagonalizable** if there exists a basis  $\{v_1, \dots, v_n\}$  s.t. each  $v_i$  is a simultaneous eigenvector of  $\mathcal{A}$ .
- if  $\mathcal{A}$  is **Abelian** and each of  $A \in \mathcal{A}$  is **diagonalizable**, then  $\mathcal{A}$  is simultaneously diagonalizable.

**proof:**

if  $A, B$  commute and are diagonal, then, the vector space decomposes as,

$$V = \bigoplus_{i=1}^r E_{A,\lambda_i} \quad (\text{A.3.7})$$

choose the eigenvectors of  $A$  as basis, then,

$$B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{pmatrix} \quad A = \begin{pmatrix} \lambda_1 I_1 & & \\ & \ddots & \\ & & \lambda_r I_r \end{pmatrix} \quad (\text{A.3.8})$$

because  $E_{A,\lambda_{i=1,\dots,r}}$  are invariant subspaces of  $B$ .

each  $B_{i=1,\dots,r}$  is diagonalizable by  $P_i \in \text{End}(E_{A,\lambda_i})$  (or  $B$  won't be diagonalizable), and  $\lambda_i I_i$  remains diagonal.

repeat this process, all matrices in  $\mathcal{A}$  can be diagonalized.

- if  $\mathcal{A}$  is **simultaneously diagonalizable**, then,

$$V = \bigoplus_{\mu} V_{\mu} \quad (\text{A.3.9})$$

where weight spaces are **linearly independent**, i.e.,

–  $\mu_1 \neq \mu_2 \neq \cdots \neq \mu_m$  are distinct weights, then,  $\{v_i \neq 0 | v_i \in V_{\mu_i}\}$  is linearly independent.

**proof:**

first,  $V_{\mu_1} \cap V_{\mu_2} = \{0\}$  for distinct weights  $\mu_1 \neq \mu_2$ , and  $\bigcup_{\mu} V_{\mu} = V$ .

then, let's prove linear independence,

– consider,

$$(A - \langle \mu_j, A \rangle I) \sum_{i=1}^m |v_i\rangle = \sum_{i=1}^m (\langle \mu_i, A \rangle - \langle \mu_j, A \rangle) |v_i\rangle \quad (\text{A.3.10})$$

– so, if  $v_1 + \cdots + v_m = 0$ , then we must have,

$$v_1 + \cdots + v_{j-1} + v_{j+1} + \cdots + v_m = 0 \quad (\text{A.3.11})$$

– repeat the process, every element in  $\{v_i\}$  is zero.

– i.e.  $\{v_i \neq 0 | v_i \in V_{\mu_i}\}$  is linearly independent.

## A.4 obtuse basis corresponds to acute dual basis

- $\{v_1, \cdots, v_n\}$  is an obtuse (钝角) basis (i.e.  $\langle v_i, v_j \rangle \leq 0, \forall i \neq j$ ), then its dual basis is acute (锐角) (i.e.  $\langle v_i^*, v_j^* \rangle \geq 0, \forall i, j$ ).

**proof:**

用指标写出来就是,

$$g_{ab}(v_i)^a (v_j)^b \leq 0, i \neq j \iff g^{ab}(v_i^*)_a (v_j^*)_b \geq 0 \quad (\text{A.4.1})$$

或者  $g_{ij} \leq 0, i \neq j \iff g^{ij} \geq 0$ .

用数学归纳法证明. 首先, 在  $n = 1, 2$  的情况下, 定理成立.

在  $n > 2$  的情况下, 考虑投影算符,

$$P_i = 1 - \frac{|v_i\rangle \langle v_i|}{\langle v_i, v_i \rangle} \quad (\text{A.4.2})$$

那么  $P_i |v_1\rangle, \cdots, P_i |v_{i-1}\rangle, P_i |v_{i+1}\rangle, \cdots, P_i |v_n\rangle$  构成  $\text{span}(v_i)^\perp = \{u \in V | u \perp v_i\}$  的钝角基底 (显然构成基底),

$$\langle P_i v_j, P_i v_k \rangle = \langle v_j, P_i v_k \rangle = \underbrace{\langle v_j, v_k \rangle}_{\leq 0} - \frac{\langle v_j, v_i \rangle \langle v_i, v_k \rangle}{\langle v_i, v_i \rangle} \leq 0 \quad (\text{A.4.3})$$

其中  $j, k \neq i$  (注意  $\langle v_i, v_i \rangle > 0$ ). 并且,

$$(P_i v_j)^* = v_j^* \in \text{span}(v_i)^\perp, j \neq i \quad (\text{A.4.4})$$

不断重复以上过程直至维数降低到 2, 从而证明  $\langle v_i^*, v_j^* \rangle \geq 0$ .

# Appendix B

## maps between manifolds

### B.1 pushforward & pullback

- 對於一個  $m$ -dim 李群  $G$  和  $n$ -dim 流形  $M$ , 它們之間存在映射  $\sigma : G \times M \rightarrow M$ , 滿足,

$$\begin{cases} \sigma_g : M \rightarrow M \text{ is diffeomorphism} \\ \sigma_g \circ \sigma_h = \sigma_{gh} \end{cases} \quad (\text{B.1.1})$$

- 可見,  $\{\sigma_g : M \rightarrow M | g \in G\}$  is homomorphic to  $G$ , 且  $\sigma_p : G \rightarrow M$  is  $C^\infty$  and preserves the topology.
- 我們用  $\{x^\mu | \mu = 1, \dots, m\}$  表示李群  $G$  上的坐標, 用  $\{y^\nu | \nu = 1, \dots, n\}$  表示流形  $M$  上的坐標.

#### B.1.1 pullback

- 流形  $M$  上有坐標  $\{y^\mu | \mu = 1, \dots, n\}$ , 那麼通過 pullback 可以得到李群  $G$  上的  $n$  個標量場,

$$\sigma_p^* : \mathcal{F}_M \rightarrow \mathcal{F}_G \quad (\sigma_p^* y^\mu)(g) = y^\mu(\sigma_p(g)) \quad (\text{B.1.2})$$

- 不能 pushforward 的原因:

$$\sigma_{p*} x^\mu(\underline{\sigma_p(g)}) = x^\mu(g) \quad (\text{B.1.3})$$

$\sigma_p(g)$  這個  $M$  上的點可能對應不同的  $g$ , 那麼標量場  $\sigma_{p*} x^\mu$  在此處的取值也就無法確定.

- 注意:  $\{\sigma_p^* y^\mu\}$  是  $G$  上的一組  $n$  個標量場, 但是  $(\sigma_p^* y) : G \rightarrow n'\text{-dim Surface} \subset \mathbb{R}^n$ , 其中,

$$\begin{cases} n' \leq m & \text{one-to-one 時取等 } (\dim \sigma_p[G] = \dim G) \\ n' \leq n & \text{onto 時取等 } (\dim \sigma_p[G] = n) \end{cases} \quad (\text{B.1.4})$$

#### B.1.2 pushforward

- 將李群  $G$  上的矢量場 pushforward 到流形  $M$  上,

$$\sigma_{p*} : \mathcal{T}_G(1,0) \rightarrow \mathcal{T}_M(1,0) \quad \left( \sigma_{p*} \frac{\partial}{\partial x^\mu} \right) (\underline{y^\nu}) \Big|_{\sigma_p(g)} = \left( \frac{\partial}{\partial x^\mu} \right) (\sigma_p^* y^\nu) \Big|_g \quad (\text{B.1.5})$$

我們可以得到 pushforward 后的矢量場的全部  $n$  個分量.

- 但是由於  $\sigma_p^* y^\nu$  只有  $n'$  個獨立變量 ( $\dim \sigma_p^* y[G] = n'$ ), 所以 pushforward 后得到的  $m$  個矢量場中, 也只有  $n'$  個是綫性獨立的.
- 不能 pullback 的原因: 顯然無法確定 pullback 后的矢量場的  $m$  個分量, 最多  $n'$  個.

#### B.1.3 pullback

- 將流形  $M$  上的對偶矢量場 pullback 到李群  $G$  上,

$$(\sigma_p^* dy^\mu)_a \left( \frac{\partial}{\partial x^\nu} \right)^a \Big|_g = (dy^\mu)_a \left( \sigma_{p*} \frac{\partial}{\partial x^\nu} \right)^a \Big|_{\sigma_p(g)} \quad (\text{B.1.6})$$

同樣, pullback 得到的  $n$  個矢量場中, 綫性獨立的有  $n'$  個.

### B.1.4 曲綫像的切矢等於曲綫切矢的像

- 對於一個曲綫  $\gamma : \mathbb{R} \rightarrow M_1$ , 流形間的映射  $\psi : M_1 \rightarrow M_2$  將其映射為  $\psi \circ \gamma : \mathbb{R} \rightarrow M_2$ .
- 曲綫  $\gamma$  的切矢為  $\frac{\partial}{\partial t} = \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu}$ , 那麼,

$$\psi_* \left( \frac{\partial}{\partial t} \right) = \frac{dx^\mu(\gamma(t))}{dt} \psi_* \left( \frac{\partial}{\partial x^\mu} \right) \quad (\text{B.1.7})$$

是曲綫  $\psi \circ \gamma$  的切矢.

- 證明的方法是將 (B.1.7) 式兩邊作用于  $M_2$  上的坐標  $y^\nu$ ,

$$\begin{aligned} \psi_* \left( \frac{\partial}{\partial t} \right) (y^\nu) &= \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu} (\psi^* y^\nu) \\ \Rightarrow \psi_* \left( \frac{\partial}{\partial t} \right) &= \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu} (\psi^* y^\nu) \frac{\partial}{\partial y^\nu} = \frac{d\psi^* y^\nu(\gamma(t))}{dt} \frac{\partial}{\partial y^\nu} = \frac{dy^\nu(\psi \circ \gamma(t))}{dt} \frac{\partial}{\partial y^\nu} \end{aligned} \quad (\text{B.1.8})$$

## B.2 diffeomorphisms & Lie derivatives

- 在流形  $M$  上有個 one-parameter group of diffeomorphism, 即,

$$\begin{cases} \phi_t : M \rightarrow M \text{ is diffeomorphism} \\ \phi_s \circ \phi_t = \phi_{s+t} \end{cases} \quad (\text{B.2.1})$$

且對應矢量場  $\xi^a \Big|_p = \frac{d}{dt} \Big|_{t=0} \phi_t(p)$ .

### B.2.1 Lie derivatives

- 對於流形  $M$  上的任意  $(k, l)$  型張量場,

$$\mathcal{L}_\xi T^{a\dots}_{b\dots} \Big|_p = \lim_{t \rightarrow 0} \frac{1}{t} \left( T^{a\dots}_{b\dots} \Big|_{\phi_t(p)} - \phi_{t*} (T^{a\dots}_{b\dots} \Big|_p) \right) \quad (\text{B.2.2})$$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \left( \phi_t^* (T^{a\dots}_{b\dots} \Big|_{\phi_t(p)}) - T^{a\dots}_{b\dots} \Big|_p \right) \quad (\text{B.2.3})$$

$$= \xi^c \nabla_c T^{a\dots}_{b\dots} - (\nabla_c \xi^a) T^{c\dots}_{b\dots} - \dots + (\nabla_b \xi^c) T^{a\dots}_{c\dots} + \dots \quad (\text{B.2.4})$$

**proof:**

- 選取滿足如下要求的坐標,

$$\{x^\mu | \mu = 0, \dots, n\} \quad \xi = \frac{\partial}{\partial x^0} \quad (\text{B.2.5})$$

也就是說,

$$\phi_t^* x^\mu(p) = x^\mu(\phi_t(p)) = \begin{cases} x^0(p) + t & \mu = 0 \\ x^\mu(p) & \mu \neq 0 \end{cases} \quad (\text{B.2.6})$$

- 那麼, 對矢量場和對偶矢量場的 pullback 和 pushforward 分別如下,

$$\begin{cases} \phi_t^* (dx^\mu \Big|_{\phi_t(p)}) = dx^\mu \Big|_p & \text{and} & \phi_t^* \left( \frac{\partial}{\partial x^\mu} \Big|_{\phi_t(p)} \right) = \frac{\partial}{\partial x^\mu} \Big|_p \\ \phi_{t*} (dx^\mu \Big|_p) = dx^\mu \Big|_{\phi_t(p)} & \text{and} & \phi_{t*} \left( \frac{\partial}{\partial x^\mu} \Big|_p \right) = \frac{\partial}{\partial x^\mu} \Big|_{\phi_t(p)} \end{cases} \quad (\text{B.2.7})$$

所以,

$$\begin{aligned} \mathcal{L}_\xi T^{a\dots}_{b\dots} \Big|_p &= (\partial_0 T^{a\dots}_{b\dots}) \Big|_p \\ &= \xi^c \left( \nabla_c T^{a\dots}_{b\dots} - \Gamma_{dc}^a T^{d\dots}_{b\dots} - \dots + \Gamma_{bc}^d T^{a\dots}_{d\dots} + \dots \right) \end{aligned} \quad (\text{B.2.8})$$

由於,

$$(\nabla_d \xi^a) T^{d\dots}_{b\dots} = \partial_d \left( \frac{\partial}{\partial x^0} \right)^a + \Gamma_{cd}^a \left( \frac{\partial}{\partial x^0} \right)^c T^{d\dots}_{b\dots} \quad (\text{B.2.9})$$

代入,

$$\mathcal{L}_\xi T^{a\cdots}_{b\cdots} \Big|_p = \xi^c \nabla_c T^{a\cdots}_{b\cdots} - (\nabla_c \xi^a) T^{c\cdots}_{b\cdots} - \cdots + (\nabla_b \xi^c) T^{a\cdots}_{c\cdots} + \cdots \quad (\text{B.2.10})$$

### B.3 consider two maps, $\psi \circ \phi$

- 三個流形  $M_1, M_2, M_3$ , 維數分別為  $n_1, n_2, n_3$ , 其上分別有坐標  $\{x^\mu\}, \{y^\mu\}, \{z^\mu\}$ .
- 它們之間存在兩個  $C^\infty$  的 homomorphism,  $\phi: M_1 \rightarrow M_2$  和  $\psi: M_2 \rightarrow M_3$ .

#### B.3.1 pullback

- 考慮,

$$\begin{cases} \psi^* z^\mu(p_2) = z^\mu(\psi(p_2)) \\ \underbrace{\phi^* \circ \psi^*}_{(\psi \circ \phi)^*} z^\mu(p_1) = z^\mu(\psi \circ \phi(p_1)) \end{cases} \quad (\text{B.3.1})$$

所以,  $\phi^* \circ \psi^* = (\psi \circ \phi)^*$ .

#### B.3.2 pushforward

- 考慮,

$$\frac{\partial}{\partial x^\mu} ((\psi \circ \phi)^* z^\nu) \Big|_{p_1} = ((\psi \circ \phi)_* \frac{\partial}{\partial x^\mu}) (z^\nu) \Big|_{\psi \circ \phi(p_1)} \quad (\text{B.3.2})$$

并且,

$$\frac{\partial}{\partial x^\mu} (\phi^* y^\nu) \Big|_{p_1} = \phi_* \frac{\partial}{\partial x^\mu} (y^\nu) \Big|_{\phi(p_1)} \quad (\text{B.3.3})$$

$$\frac{\partial}{\partial x^\mu} (\phi^* \circ \psi^* z^\nu) \Big|_{p_1} = \phi_* \frac{\partial}{\partial x^\mu} (\psi^* z^\nu) \Big|_{\phi(p_1)} = \psi_* \circ \phi_* \frac{\partial}{\partial x^\mu} (z^\nu) \Big|_{\psi \circ \phi(p_1)} \quad (\text{B.3.4})$$

所以,  $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ .

#### B.3.3 pullback

- 考慮,

$$((\psi \circ \phi)^* dz^\mu)_a \left( \frac{\partial}{\partial x^\nu} \right)^a \Big|_{p_1} = (dz^\mu)_a \left( (\psi \circ \phi)_* \frac{\partial}{\partial x^\nu} \right)^a \Big|_{\psi \circ \phi(p_1)} \quad (\text{B.3.5})$$

且,

$$(\phi^* \circ \psi^* dz^\mu)_a \left( \frac{\partial}{\partial x^\nu} \right)^a \Big|_{p_1} = (\psi^* dz^\mu)_a \left( \phi_* \frac{\partial}{\partial x^\nu} \right)^a \Big|_{\phi(p_1)} = (dz^\mu)_a \left( \psi_* \circ \phi_* \frac{\partial}{\partial x^\nu} \right)^a \Big|_{\psi \circ \phi(p_1)} \quad (\text{B.3.6})$$

所以, 依舊有  $\phi^* \circ \psi^* = (\psi \circ \phi)^*$ .

## B.4 Weyl transformations & conformal transformations

### B.4.1 Weyl transformations

- Weyl 變換在保持流形不變的情況下, 改變流形上配備的度規, 此時, 流形的曲率等幾何性質也會發生改變.
- 背景流形上選取坐標  $\{x^\mu\}$ , 那麼新度規與舊度規的關係為,

$$\tilde{g}_{\mu\nu} = e^{\Phi(x)} g_{\mu\nu} \quad (\text{B.4.1})$$

其中,  $\Phi(x)$  是流形上的一個標量場.

- 在 Weyl 變換下, 仿射聯絡係數, 曲率張量都會發生變化, 但 Weyl 張量不會發生變換 (具體變換形式及計算過程見 GoodNotes 筆記: Weyl Transformation and Conformal Transformation).

### B.4.2 conformal isometries

- 流行  $M$  上配備有兩套度規  $g_{ab}$  和  $\tilde{g}_{ab}$  (可見 Weyl 變換和共形變換都會改變流形的度規場).
- 映射  $\phi$  是 conformal isometry, 其生成的拉回映射  $\phi^*$  滿足,

$$(\phi^*(\tilde{g})|_{\phi(p)})_{ab} = \Omega^2 g_{ab}|_p \quad (\text{B.4.2})$$

其中  $\Omega$  是流形上的標量場.

- conformal transformations preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.
- 用坐標的拉回映射來表示這個變換, 那麼是, 對於流形上的坐標  $\{y^\mu\}$  其拉回映射的像為  $\{x^\mu\}$ , 即,

$$\begin{cases} (\phi^* y^\mu)(p) \equiv x^\mu(p) = y^\mu(\phi(p)) \\ \phi^* dy^\mu = dx^\mu \end{cases} \quad (\text{B.4.3})$$

那麼, conformal isometry  $\phi$  即滿足,

$$\tilde{g}_{\mu\nu}|_{\phi(p)} \phi^*(dy^\mu \otimes dy^\nu) = \Omega^2 g_{\mu\nu}(dx^\mu \otimes dx^\nu) \quad (\text{B.4.4})$$

$$\implies \tilde{g}_{\mu\nu}|_{\phi(p)} = (\Omega^2 g_{\mu\nu})|_p \quad (\text{B.4.5})$$

其中  $\tilde{g}_{\mu\nu}$  是度規  $\tilde{g}_{ab}$  在  $\{y^\mu\}$  坐標系下的分量.

### B.4.3 conformal Killing vector fields

- 流形上的一個 one-parameter group of conformal isometry  $\{\phi_t, t \in \mathbb{R}\}$ , 其中每個  $\phi_t$  都是 conformal isometry 且滿足如 (B.2.1) 式的群乘法, 且,

$$(\phi_t^* g)_{ab} = a(t) g_{ab} \quad (\text{B.4.6})$$

$a(t)$  顯然要滿足某些性質, 目前可以確認  $a(0) = 1$ .

- 向量場  $\psi^a|_{\phi_s(p)} = \frac{d}{dt}|_s \phi_t(p)$  稱為 conformal Killing vector field, 相應的度規的李導數為,

$$(\mathcal{L}_\psi g)_{ab} = 2\nabla_{(a} \psi_{b)} = \alpha g_{ab} \quad (\text{B.4.7})$$

其中  $\alpha = \frac{d}{dt}|_{t=0} a(t)$ , 對上式兩端求 trace, 得到,

$$2\nabla^a \psi_a = n\alpha \implies \alpha = \frac{2}{n} \nabla^a \psi_a \quad (\text{B.4.8})$$

其中  $n$  是流形維數.

- 得到 conformal Killing vector field 滿足的方程,

$$\nabla_{(a} \psi_{b)} = \frac{1}{n} (\nabla^c \psi_c) g_{ab} \quad (\text{B.4.9})$$