

# Lie Groups and Lie Algebras

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**Part I**

**Finite Groups**

# Chapter 1

## finite groups

- a useful reference: <https://sites.ualberta.ca/~vbouchar/MAPH464/notes.html>.

- 
- def. of groups (Abelian groups, cyclic groups, symmetry groups, permutation groups).
  - order of  $G$  denoted by  $|G|$ , order of element  $g$ .
- 

- conjugated element  $ghg^{-1} = g'$ , conjugacy class.
- subgroup, (left/right) coset of a subgroup (2 theorems + Lagrange theorem).
- conjugacy subgroup  $hHh^{-1}$ .
- **normal subgroup** (i.e. invariant subgroup)  $N \triangleleft G, gNg^{-1} \subseteq N, \forall g$ .
  - **center**,  $Z(G) = \{z \in G | gzg^{-1} = z, \forall g\}$ .  
center is normal, but normal subgroup is not necessarily central.
  - the center of a Lie algebra is  $\mathfrak{h} = \{A \in \mathfrak{g} | [A, B] = 0, \forall B\} \equiv \{A \in \mathfrak{g} | \text{ad}_A = 0\}$ .  
center is an ideal, but ideal is not necessarily a center.
- groups without nontrivial normal subgroups are **simple**.
- **direct product group**  $G \times H$  (Cartesian product, direct product and direct sum).  
**def.:**  $G \times H = \{(g, h) | g \in G, h \in H\}$  with group product defined by  $(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_1 \circ h_2)$ .
- factor (quotient) group  $G/H_N$ .
- isomorphism vs. homomorphism.
  - kernel  $K \mapsto \{e\}$  of a homomorphism.

### 1.1 representation theory

- representation of a group  $D(g)$ .
- 用 basis of functions 来构建 rep. of  $G$ ,

$$\Omega_g \psi_i(\vec{x}) = \psi_i(g^{-1}\vec{x}) \quad (1.1.1)$$

- trivial rep. (1 dim.)  $D_{11}(\forall g) = 1$ .
- regular rep.  $D_{ij}(g) = \langle g_i | gg_j \rangle \equiv \delta_{g_i, gg_j}$ .

#### 1.1.1 reducibility

- reducible rep. vs. completely reducible (semisimple) rep..  
completely reducible rep.,

$$TD(g)T^{-1} = D^{(1)}(g) \oplus D^{(2)}(g) \oplus \dots \quad (1.1.2)$$

- completely reducible  $\iff$  invariant subspace is trivial.

## 1.2 unitarity theorem

- any finite-dim. rep. of a finite group are equivalent to a unitary rep..

**proof:**

for a finite-dim. rep.  $\Gamma = \{D(g), \dots\}$ , consider  $H = \sum_g D^\dagger(g)D(g)$ , we have,

$$D^\dagger(h)HD(h) = H \quad (1.2.1)$$

$H$  is a Hermitian matrix which can be diagonalized by a unitary matrix,

$$M \equiv \text{diag}(\lambda_1, \dots) = UHU^\dagger \quad (1.2.2)$$

then let,

$$B(g) = M^{1/2}UD(g)U^\dagger M^{-1/2} \quad (1.2.3)$$

where  $M^{1/2} = \text{diag}(\lambda_1^{1/2}, \dots)$ , we can see that,

$$\begin{aligned} B^\dagger(g)B(g) &= M^{-1/2}UD^\dagger(g)U^\dagger MUD(g)U^\dagger M^{-1/2} \\ &= M^{-1/2}MM^{-1/2} = I \end{aligned} \quad (1.2.4)$$

so  $\{B(g), \dots\}$  is a unitary rep..

- all the reducible unitary rep. are completely reducible.

**proof:**

unitary rep. 作用于  $V = W \oplus W^\perp$ , 其中  $V$  是 Hilbert 空间, 内积为  $\langle \cdot, \cdot \rangle$ ,  $W^\perp$  与  $W$  正交,  $W$  是表示的不变子空间, 下面证明  $W^\perp$  也是不变子空间,

$$\langle B(g)w^\perp | w \rangle = \langle w^\perp | B(-g)w \rangle = \langle w^\perp | w' \rangle = 0, \forall w^\perp \in W^\perp, w \in W \quad (1.2.5)$$

其中,  $w' \in W$ , 可见  $B(g)[W^\perp] \subseteq W^\perp$

其实不需要要求表示么正, 只需要  $B$  和  $B^\dagger$  拥有同一个不变子空间  $W$  就行.

这对 infinite group 也成立.

## 1.3 Schur's lemmas

- Schur's 1st lemma

for 2 irreducible real or complex rep.  $\Gamma_1 = \{D^{(1)}(g), \dots\}$  and  $\Gamma_2 = \{D^{(2)}(g), \dots\}$ ,  $\exists A$  s.t.  $\forall g$ ,

$$AD^{(1)}(g) = D^{(2)}(g)A \quad (1.3.1)$$

then, there are only 2 possibilities:

- $A = 0$ ,
- $A$  is reversible matrix and  $\Gamma_1, \Gamma_2$  are equivalent.

**proof:**

consider,

$$\begin{aligned} AD^{(1)}(g)[\ker A] &= D^{(2)}(g)A[\ker A] = 0 \\ \implies D^{(1)}(g)[\ker A] &\subseteq \ker A \end{aligned} \quad (1.3.2)$$

so,  $\ker A$  is a invariant subspace of rep.  $\Gamma_1$

but  $\Gamma_1$  is irreducible, so  $\ker A$  is trivial, i.e.  $\ker A$  is either 0 or  $V$ , which implies that...

对 infinite group 也成立.

- **Schur's 2nd lemma**

for a **irreducible complex rep.**  $\Gamma = \{D(g), \dots\}$ , if  $\forall g$ ,

$$AD(g) = D(g)A \quad (1.3.3)$$

then  $A = \lambda I$  for some  $\lambda \in \mathbb{C}$ .

**proof:**

$A$  must have (at least) one eigenvalue  $\lambda$ , then  $\det(A - \lambda I) = 0$  is irreversible matrix,

$$AD(g) = D(g)A \implies (A - \lambda I)D(g) = D(g)(A - \lambda I) \quad (1.3.4)$$

by Schur's 1st lemma, irreversible matrix  $A - \lambda I$  must be 0.

- **Schur's 3rd lemma**

for 2 **irreducible complex rep.**  $\Gamma_1 = \{D^{(1)}(g), \dots\}$  and  $\Gamma_2 = \{D^{(2)}(g), \dots\}$ , if  $\forall g$ ,

$$\begin{cases} AD^{(1)}(g) = D^{(2)}(g)A \\ BD^{(1)}(g) = D^{(2)}(g)B \end{cases} \quad (1.3.5)$$

then  $B = \lambda A$  for some  $\lambda \in \mathbb{C}$ .

**proof:**

$$(A - \lambda B)D^{(1)}(g) = D^{(2)}(g)(A - \lambda B) \quad (1.3.6)$$

choose  $\lambda$  s.t.  $\det(A - \lambda B) = 0$ , then, according to Schur's 1st lemma,  $A - \lambda B = 0$ .

## 1.4 the great orthogonal theorem

- **the great orthogonality theorem**

for 2 inequivalent irreducible rep.  $\Gamma^a = \{D^{(a)}(g), \dots\}$  where  $a = 1, 2$ ,

$$\frac{1}{|G|} \sum_g D_{ij}^{(a)}(g^{-1}) D_{j'i'}^{(b)}(g) = \frac{1}{d} \delta_{ii'} \delta_{jj'} \delta^{ab} \quad (1.4.1)$$

or for unitary rep.,

$$\frac{1}{|G|} \sum_g B_{ij}^{(a)*}(g) B_{i'j'}^{(b)}(g) = \frac{1}{d} \delta_{ii'} \delta_{jj'} \delta^{ab} \quad (1.4.2)$$

where  $d$  is the dim. of the rep..

**proof:**

for  $a = b$ :

consider  $A = \sum_g B^{(a)\dagger}(g) X B^{(a)}(g)$  where  $B^{(a)}(g) = T D^{(a)}(g) T^{-1}$  is the unitary rep. equivalent to  $\Gamma_a$ , then,

$$AB^{(a)}(h) = B^{(a)\dagger}(h^{-1})A \implies AB^{(a)}(h) = B^{(a)}(h)A \quad (1.4.3)$$

according to Schur's 1st lemma,  $A = \lambda I$ , then,

$$\begin{aligned} \lambda I &= \sum_g (T^{-1} B^{(a)\dagger}(g) T) (T^{-1} X T) (T^{-1} B^{(a)}(g) T) \\ &= \sum_g D^{(a)}(g^{-1}) X' D^{(a)}(g) \end{aligned} \quad (1.4.4)$$

choose  $X'_{,,} = \delta_{.,j} \delta_{j',.}$ , then we have  $\lambda I = \sum_g D_{.,j}^{(a)}(g^{-1}) D_{j',.}^{(a)}(g)$ , calculate the trace of the matrix,

$$\lambda d_a = \sum_g \delta_{jj'} = |G| \delta_{jj'} \quad (1.4.5)$$



so we can conclude that,

$$\frac{1}{|G|} \sum_g D_{ij}^{(a)}(g^{-1}) D_{j'i'}^{(a)}(g) = \frac{1}{d_a} \delta_{ii'} \delta_{jj'} \quad (1.4.6)$$

for  $a \neq b$ :

still consider  $A = \sum_g B^{(a)\dagger}(g) X B^{(b)}(g)$  then,

$$AB^{(b)}(h) = B^{(a)}(h)A \quad (1.4.7)$$

according to Schur's 1st lemma,  $A = 0$ , consequently,

$$\sum_g D_{ij}^{(a)}(g^{-1}) D_{j'i'}^{(b)}(g) = 0 \quad (1.4.8)$$

- characters of the rep.  $\Gamma_a$  of group  $G$  is the set  $\{\chi^{(a)}(g) = \text{tr} D^{(a)}(g) | g \in G\}$
- character table is the matrix  $X = \{X^a_i = \chi^{(a=1, \dots, \rho)}(g_{i=1, \dots, c})\}$ .  
where  $g_i$  is the rep. of the  $i$ th conjugacy class, and  $\rho$  is the number of the irreducible inequivalent rep. of  $G$ . ( $\rho = c$ , as to be proved later).

#### • 1st theorem of the orthogonality of the characters

the character of irreducible inequivalent rep. of  $G$  are orthogonal to each other, which can be derived easily from the great orthogonality theorem.

$$\frac{1}{|G|} \sum_g \chi^{(a)*}(g) \chi^{(b)}(g) = \delta^{ab} \quad (1.4.9)$$

#### • 2nd theorem of the orthogonality of the characters

$$\sum_{a=1}^{\rho} \chi^{(a)*}(g_i) \chi^{(a)}(g_j) = \frac{|G|}{n_i} \delta_{ij} \quad (1.4.10)$$

where  $g_i$  is the rep. of the  $i$ th conjugacy class,  $n_i$  is the number of elements in this conjugacy class, and  $\rho$  is the number of the irreducible inequivalent rep. of  $G$ .

**proof:**

by 1st theorem,

$$X \text{diag}\left(\frac{n_1}{|G|}, \dots, \frac{n_c}{|G|}\right) X^\dagger = I \quad (1.4.11)$$

then,

$$\Rightarrow \sum_j \left( X^\dagger X \text{diag}\left(\frac{n_1}{|G|}, \dots, \frac{n_c}{|G|}\right) \right)_{ij} X^\dagger_j{}^a = X^\dagger_i{}^a \quad (1.4.12)$$

since vectors  $(X^a_1, \dots, X^a_c)$  forms an orthogonal basis of the vector space, then we must have,

$$\left( X^\dagger X \text{diag}\left(\frac{n_1}{|G|}, \dots, \frac{n_c}{|G|}\right) \right)_{ij} = \delta_{ij} \quad (1.4.13)$$

then, finally, we have,

$$\sum_{a=1}^{\rho} \chi^{(a)*}(g_i) \chi^{(a)}(g_j) = \frac{|G|}{n_i} \delta_{ij} \quad (1.4.14)$$

- 群  $G$  的 irreducible inequivalent rep. 的数量等于其 conjugacy class 的数量  $c$ .

**proof:**

一个 irreducible inequivalent rep. 由其 characters 表示  $\{\chi^{(a)}(g), \dots\}$   
 (根据 theorem of the orthogonality of the characters) 不同的 irreducible inequivalent rep. 的 characters 一定不同.  
 且 conjugacy class 内的元素的 character 一定相等, 所以一个 rep. 实际上只有 conjugacy class 的数量  $c$  个不同的 characters, 所以可以将 characters 视为  $c$  维向量  $\frac{1}{\sqrt{|G|}}(\chi^{(a)}(g), \dots)$ , 那么  $c$  维向量空间中互相正交归一的向量最多只有  $c$  个.  
 利用 2nd theorem of... 可证... 最少有  $c$  个. 所以... 等于...

- characters of completely reducible rep..

suppose a completely reducible rep.  $\Gamma = \oplus_{a=1}^c m_a \Gamma_a$ , where  $m_a = 0, 1, 2, \dots$ , then,

$$\chi(g) = \sum_a m_a \chi^{(a)}(g) \quad (1.4.15)$$

(e.g. for  $D(g) = D^{(1)}(g) \oplus D^{(1)}(g)$ ,  $m_1 = 2$ ).

and,

$$\frac{1}{|G|} \sum_g \chi^*(g) \chi(g) = \sum_a m_a^2 > 1 \quad (1.4.16)$$

- Burnside theorem**

$$\sum_{a=1}^c d_a^2 = |G| \quad (1.4.17)$$

where  $d_a$  is the dim. of the  $a$ th inequivalent irreducible rep. of  $G$ .

**proof:**

by 2nd orthogonality theorem of characters,

$$\sum_{a=1}^c \chi^{(a)*}(e) (\chi^{(a)}(e) = d_a) = \frac{|G|}{(n_e = 1)} \implies \sum_{a=1}^c d_a^2 = |G| \quad (1.4.18)$$

- rep. of **direct product group**  $G = H \times F$  is derived from **irreducible rep.** of  $H$  and  $F$  by  $\Gamma = \Gamma_H \times \Gamma_F = \{D(hf) = D_H(h) \otimes D_F(f)\}$ , then  $\Gamma$  is also an irreducible rep..

**proof:**

利用 characters of completely reducible rep. 的性质.

- direct product of group rep.:  $\Gamma = \Gamma_a \times \Gamma_b$ , then  $\chi(g) = \chi^{(a)}(g) \chi^{(b)}(g)$

- projection operator is,

$$P_a = \frac{d_a}{|G|} \sum_g \chi^{(a)*}(g) T^{-1} \begin{pmatrix} \ddots & & \\ & D^{(b)}(g) & \\ & & \ddots \end{pmatrix} T = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} I & \\ & & \ddots \end{pmatrix} T \quad (1.4.19)$$

i.e.,

$$P_a = \frac{d_a}{|G|} \sum_g \chi^{(a)*}(g) D(g) = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} I & \\ & & \ddots \end{pmatrix} T \quad (1.4.20)$$

where  $TD(g)T^{-1} = \dots \oplus D^{(b)}(g) \oplus \dots$ .

notice that  $P_a$  is not necessarily a diagonal matrix, unless  $T$  consists of orthogonal column vectors.

- how to use a projection operator:

$$P_a D(g) = T^{-1} \begin{pmatrix} \ddots & & \\ & \delta^{ab} D^{(a)}(g) & \\ & & \ddots \end{pmatrix} T \quad (1.4.21)$$

and  $\text{tr}(P_a) = m_a d_a$ .

---

- about 1-dim. rep.  $\Gamma_1 = \{D^{(1)}(g), \dots\}$ :

1-dim. rep. must be **irreducible** and **unitary**, so,

$$\chi^{(1)}(g) = D^{(1)}(g) \quad \chi^{(1)}(g^{-1}) = \chi^{(1)*}(g) \quad (1.4.22)$$

so we can conclude that,

$$|\chi^{(1)}(g)| = |D^{(1)}(g)| = 1 \quad (1.4.23)$$

- $\Gamma_a$  is a n-dim. irreducible rep., then  $\Gamma_1 \times \Gamma_a$  is also an irreducible rep..

**proof:**

let  $\Gamma = \Gamma_1 \times \Gamma_a = \{D^{(1)}(g) \otimes D^{(a)}(g), \dots\}$ , then,

$$\frac{1}{|G|} \sum_g |\chi(g)|^2 = \frac{1}{|G|} \sum_g \underbrace{|\chi^{(1)}(g)|^2}_{=1} |\chi^{(a)}(g)|^2 = 1 \quad (1.4.24)$$

# **Part II**

## **General Theory**

# Chapter 2

## Lie groups

### 2.1 Lie groups

- **Lie group**  $G$  is a group and a manifold,
  - group multiplication,  $G \times G \rightarrow G$ , is  $C^\infty$ .
  - inverse,  $G \rightarrow G$ , is  $C^\infty$ .
- **left transformation**,  $L_g : G \rightarrow G, L_g(h) = gh$ .
  - $L_e = \text{id}$ .
  - $L_g L_h = L_{gh}$ .
  - $L_g^{-1} = L_{g^{-1}}$ .
  - $L_g$  is diffeomorphism, i.e. bijective +  $C^\infty$ .
- property of elements near  $e$ , if  $x^i(e) = 0$ , then,

$$x^i(gh) = x^i(g) + x^i(h) \quad (2.1.1)$$

**proof:**

$$\begin{aligned} gh &= \left( e + x^i(g) \frac{\partial g}{\partial x^i} \Big|_e + \cdots \right) \left( e + x^i(h) \frac{\partial g}{\partial x^i} \Big|_e + \cdots \right) \\ &= e + (x^i(g) + x^i(h)) \frac{\partial g}{\partial x^i} \Big|_e + \cdots \end{aligned} \quad (2.1.2)$$

consequently,  $x^i(g^{-1}) = -x^i(g)$ .

- for example, GL,

$$x_{ij}(I + \Delta) = \Delta_{ij} \quad (2.1.3)$$

### 2.2 topological properties

#### 2.2.1 compactness

- compactness is a property that seeks to generalize the notion of a **closed** and **bounded** subset of Euclidean space.

The idea is that a compact space has no "punctures" or "missing endpoints", i.e. it includes all **limiting** values of points.
- **def.:** compact Lie group:
  - 有限个  $\mathbb{R}^n$  中的闭集通过坐标映射到 Lie group 上可以覆盖整个 Lie group.
  - 注意,  $\mathbb{R}$  不是闭集,  $\mathbb{R} \cup \{\pm\infty\}$  才是闭集.

- **Heine-Borel theorem:**

a **matrix** Lie group is compact  $\iff$  it is topologically **closed** as a subset of  $\mathcal{M}_m(\mathbb{C})$  and **bounded**.

compact	noncompact
$O(m), SO(m), U(m), SU(m), Sp(m)$	$SL(m, \mathbb{R})$ (not bounded)

### 2.2.2 connectedness

- a topological space is connected if it is not the union of two **disjoint nonempty open sets**.
- matrix** Lie group is **connected**  $\iff$  it is **path-connected**.

- the **identity component** of  $G$ , denoted by  $G_0$ , is the biggest connected subset containing  $I$ .
  - $G_0$  is a **normal subgroup** of  $G$ .

**proof:**

- \*  $G_0$  is a subgroup.
  - $\forall A, B \in G_0$  there are paths  $A(t), B(t)$  connecting to  $I$ .
  - then  $A(t)B(t)$  is a continuous path connecting  $I$  and  $AB$ .
  - $(A(t))^{-1}$  is...  $I$  and  $A^{-1}$ .
- \*  $G_0$  is invariant.
  - $\forall A \in G_0, B \in G$  there are a path  $BA(t)B^{-1}$  connecting  $BAB^{-1}$  and  $I$ .

### 2.2.3 simple connectedness

- a topological space is **simply connected**  $\iff$  it is **path connected** and every **loop** can be **shrunk continuously into a point**.

**more precisely:**

- for every loop  $A(t), t \in [0, 1]$  in  $G$ ,  $A(0) = A(1)$ .  
 there exist a function  $A(s, t), s, t \in [0, 1]$  such that:
- $A(0, t) = A(t)$  is the original loop.
  - $A(1, t) = A(1, 0)$  is a point.
  - $A(s, 0) = A(s, 1)$  which means  $A(s, t)$  is a loop.

- summary:

matrix Lie groups	compactness	components	simple connectedness
$GL(m, \mathbb{C})$	no	1	no
$GL(m, \mathbb{R})$	no	2	no
$SL(m, \mathbb{C})$	no	1	yes
$SL(m, \mathbb{R})$	no	1	no
$O(m)$	yes	2	
$SO(m)$	yes	1	no
$U(m)$	yes	1	no
$SU(m)$	yes	1	yes
$O(m, 1)$	yes	4	
$SO(m, 1)$	yes	2	$m = 1$ , yes; $m \geq 2$ , no
$E(m)$ (Euclidean group)		2	
$P(m, 1)$ (Poincaré group)		4	

## 2.3 Lie subgroups

- def.:** a **Lie subgroup**  $H$  of a Lie group  $G$  is a subgroup which is also a submanifold.

- **closed subgroup theorem:**  $\{\text{closed subgroups}\} = \{\text{Lie subgroups}\}.$

**proof:**

first, let's prove that a closed subgroup  $H$  is a Lie subgroup.

– let,

$$\mathfrak{h} = \{A \in \mathfrak{g} \mid \exp(tA) \in H, \forall t \in \mathbb{R}\} \quad (2.3.1)$$

\*  $\mathfrak{h}$  is a subspace of  $\mathfrak{g}$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \exp\left(\frac{A}{n}\right) \exp\left(\frac{B}{n}\right) \right)^n &= \lim_{n \rightarrow \infty} \left( \exp\left(\frac{A}{n} + \frac{B}{n} + O\left(\frac{1}{n^2}\right)\right) \right)^n \\ &= \exp(A + B) \in H \end{aligned} \quad (2.3.2)$$

极限存在要求  $H$  是闭集.

- $W \subset \mathfrak{h}$  is a neighborhood of 0, which is small enough that  $\exp : W \rightarrow H$  is a one-to-one homomorphism (**local diffeomorphism**).
- $\exp^{-1} : \exp[V] \rightarrow V$  with  $V \cap \mathfrak{h} = W$  is a diffeomorphism, so  $(\exp^{-1}, \exp[V], V)$  is a chart on  $G$ , which can be extended by left translation. so,  $H$  is a submanifold.

---

second, let's prove that Lie subgroups are closed.

– 暂时不会证.

# Chapter 3

## Lie algebras

### 3.1 left-invariant vector fields

- vector field  $\bar{A}$  is invariant under push-forward,  $L_{g*} : V_h \rightarrow V_{gh}, \forall h$ ,

$$(L_{g*}\bar{A})|_{gh} = \bar{A}|_{gh} \quad (3.1.1)$$

i.e.,

$$\bar{A}(x^i)|_h = \bar{A}(y^i)|_{gh} \quad (3.1.2)$$

where  $L_g^* y^i = x^i \iff y^i(gh) = x^i(h)$ .

- see appendix B, maps between manifolds.
- the set of all left invariant vector field is denoted by  $\mathfrak{g}$ , and  $\mathfrak{g} \simeq V_e$ .

### 3.2 Lie algebras

- $A \equiv \bar{A}_e$  and  $\bar{A}_g = L_{g*}A, \forall g$ .
- a vector space,  $V$ , along with Lie bracket,  $[\cdot, \cdot] : V \times V \rightarrow V$ , is a **Lie algebra**,
  - $[A, B] = -[B, A]$ .
  - Jacob identity,  $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$ .
- for a Lie group  $G$ , its Lie bracket is the commutator,

$$[\bar{A}, \bar{B}]^a = \bar{A}^b \nabla_b \bar{B}^a - \bar{B}^b \nabla_b \bar{A}^a \quad (3.2.1)$$

$$- L_{g*}[\bar{A}, \bar{B}] = [L_{g*}\bar{A}, L_{g*}\bar{B}] = [\bar{A}, \bar{B}] \in \mathfrak{g}.$$

**proof:**

$$\begin{aligned} L_{g*}[\bar{A}, \bar{B}] &= L_{g*}\left(\frac{\partial}{\partial x^i}\bigg|_h\right)\left(A^j \frac{\partial}{\partial x^j} B^i - B^j \frac{\partial}{\partial x^j} A^i\right)\bigg|_{h,x} \\ &= \left(\frac{\partial}{\partial y^i}\bigg|_{gh}\right)\left(A^j \frac{\partial}{\partial x^j} B^i - B^j \frac{\partial}{\partial x^j} A^i\right)\bigg|_{h,x} \end{aligned} \quad (3.2.2)$$

notice that for left-invariant v. f. as a scalar field,  $(L_g^* A^i|_y)|_h = A^i|_{gh,y}$  and,

$$\begin{aligned} \left(\frac{\partial}{\partial x^j} A^i\right)\bigg|_{h,x} &\equiv \left(\frac{\partial}{\partial x^j}\right)\bigg|_h (L_g^* A^i|_y)\bigg|_h = L_{g*}\left(\frac{\partial}{\partial x^j}\bigg|_h\right)(A^i|_{gh,y}) \\ \implies \left(\frac{\partial}{\partial x^j} A^i\right)\bigg|_{h,x} &= \left(\frac{\partial}{\partial y^j} A^i\right)\bigg|_{gh,y} \end{aligned} \quad (3.2.3)$$

so  $L_{g*}[\bar{A}, \bar{B}] = [\bar{A}, \bar{B}]$ .

- satisfies the Jacob identity.



proof:

$$\begin{aligned}
& [A, [B, C]] + [C, [A, B]] + [B, [C, A]] \\
&= A^c \partial_c (B^b \partial_b C^a - C^b \partial_b B^a) - (B^c \partial_c C^b - C^c \partial_c B^b) \partial_b A^a + \dots \\
&= A^c \partial_c (B^b) \partial_b C^a + A^c B^b \partial_c \partial_b C^a - A^c \partial_c (C^b) \partial_b B^a + A^c C^b \partial_c \partial_b B^a \\
&\quad - B^c \partial_c (C^b) \partial_b A^a + C^c \partial_c (B^b) \partial_b A^a \\
&\quad + (B \partial C \partial A - B \partial A \partial C - C \partial A \partial B + A \partial C \partial B) \\
&\quad + (BC \partial \partial A - BA \partial \partial C) \\
&\quad + (C \partial A \partial B - C \partial B \partial A - A \partial B \partial C + B \partial A \partial C) \\
&\quad + (CA \partial \partial B - CB \partial \partial A) = 0
\end{aligned} \tag{3.2.4}$$

- **def.:** the **Lie algebra direct sum** of two Lie algebras,  $\mathfrak{g}_1, \mathfrak{g}_2$ , is the **vector space direct sum** (i.e.  $\mathfrak{g}_1, \mathfrak{g}_2$  are linearly independent  $\iff \mathfrak{g}_1 \cap \mathfrak{g}_2 = \{0\}$ ),  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , with the Lie bracket defined to be,

$$\begin{aligned}
& [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \\
& [A_1 + A_2, B_1 + B_2] = [A_1, B_1] + [A_2, B_2] \quad \forall A_1, B_1 \in \mathfrak{g}_1, A_2, B_2 \in \mathfrak{g}_2
\end{aligned} \tag{3.2.5}$$

i.e. we define the Lie bracket in the way that  $[\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$ .

### 3.2.1 subalgebras, ideals & simple, solvable, nilpotent Lie algebras

- **def.:** subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  is a subspace, satisfying that  $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ .
  - **def.:** Abelian subalgebra  $\mathfrak{h}$  is a subalgebra, satisfying that  $[A, B] = 0, \forall A, B \in \mathfrak{h}$ .
- **def.:** invariant subalgebra (i.e. **ideal**)  $\mathfrak{h}$  is a subalgebra, satisfying that  $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$ .
  - Abelian ideal.
  - proper invariant subalgebra (also called **proper ideal**) is an ideal that is not  $\mathfrak{g}, \{0\}$ .
  - trivial subalgebras are  $\mathfrak{g}, \{0\}$ .

- Lie algebra decomposes as the direct sum of its ideals,  $\mathfrak{h}_1, \mathfrak{h}_2, \dots$ , i.e.,

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \dots \tag{3.2.6}$$

then  $\oplus$  is called **Lie algebra direct sum**.

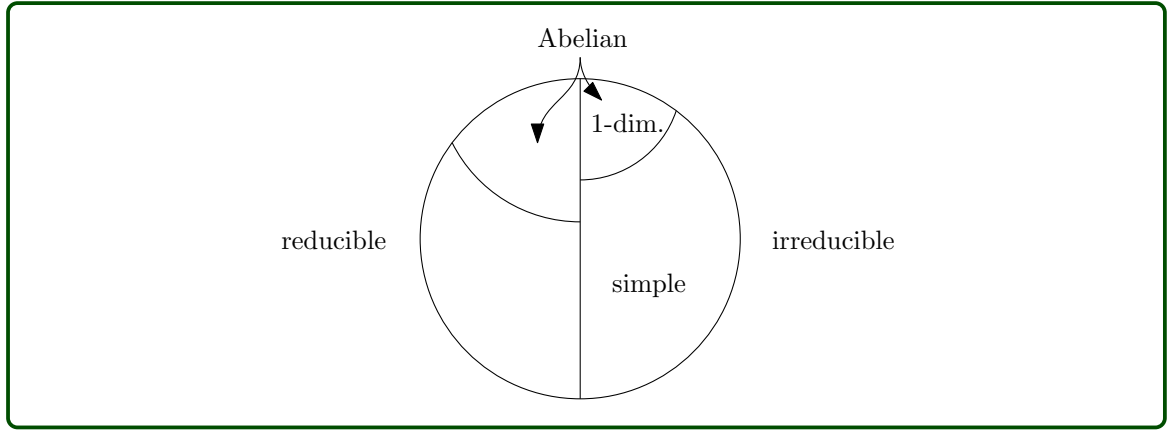
proof:

by def.,  $[\mathfrak{h}_i, \mathfrak{h}_j] \subseteq \mathfrak{h}_i \cap \mathfrak{h}_j = \{0\}$ , if  $i \neq j$ .

- **def.:** a Lie algebra **without nontrivial ideal** is **irreducible**.
  - all 1-dim. Lie algebras are irreducible.
- **def.:** a **irreducible** Lie algebra with  $\dim \mathfrak{g} \geq 2$  is **simple**.
  - equivalent **def.:** irreducible non-Abelian Lie algebras are simple.

proof:

all the subspaces of an Abelian Lie algebra is its ideal  $\implies$  Abelian Lie algebras aren't irreducible unless  $\dim = 1$ , so,



- **def.:** a Lie algebra  $\mathfrak{g}$  is **solvable** if  $\mathfrak{g}_i = \{0\}$  for some  $i$ , where,

$$\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}_i] \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{g} \quad (3.2.7)$$

- $\mathfrak{g}_i$  is an ideal in  $\mathfrak{g}_{i-1}$ , but not necessarily an ideal in  $\mathfrak{g}$ .

**proof:**

$$\forall A \in \mathfrak{g}_i \subseteq \mathfrak{g}_{i-1} \text{ and } \forall B \in \mathfrak{g}_{i-1}, [A, B] \in \mathfrak{g}_i, \text{ which means } [\mathfrak{g}_i, \mathfrak{g}_{i-1}] \subseteq \mathfrak{g}_i.$$

- **def.:** a Lie algebra  $\mathfrak{g}$  is **nilpotent** if  $\mathfrak{g}^i = \{0\}$  for some  $i$ , where,

$$\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i] \quad \text{and} \quad \mathfrak{g}^0 = \mathfrak{g} \quad (3.2.8)$$

- $\mathfrak{g}^{i+1} \subseteq \mathfrak{g}^i$ .
- $\mathfrak{g}^i$  is an ideal in  $\mathfrak{g}$ .
- nilpotent Lie algebra is solvable.

### 3.2.2 structure constants

- structure constants,

$$[X_i, X_j] = if_{ij}^k X_k \iff [X_i, X_j]^a = if_{bc}^a (X_i)^b (X_j)^c \quad (3.2.9)$$

$$[A_i, A_j] = -if_{ij}^k A_k \iff [A_i, A_j]^a = -f_{bc}^a (A_i)^b (A_j)^c \quad (3.2.10)$$

where  $X_i = -iA_i$  are called the generators.

- if the generators are Hermitian, then the structure constants are real,

$$[X_i, X_j]^\dagger = -if_{ij}^{*k} X_k = [X_j, X_i] = i \underbrace{f_{ji}^k}_{=-f_{ij}^k} X_k \implies f_{ij}^{*k} = f_{ij}^k \quad (3.2.11)$$

# Chapter 4

## exponential maps

### 4.1 one-parameter subgroups

- a  $C^\infty$  (Lie group) homomorphism  $\gamma : \mathbb{R} \rightarrow G$ , with  $\gamma(s)\gamma(t) = \gamma(s+t)$ .
- $\{\gamma(s) | s \in \mathbb{R}\}$  is an **integral curve** (passing through  $e$ ) of a **left-invariant vector field**.
  - the integral curve of a left-invariant vector field is complete, i.e. it's homomorphism to  $\mathbb{R}$ .

**proof:**

notation:  $\frac{d}{dt}\gamma(t) \equiv \frac{\partial}{\partial t} (\equiv \frac{dx^i(\mu(t))}{dt} \frac{\partial}{\partial x^i})$   
 let  $\mu : (-\epsilon, \epsilon) \rightarrow G$  be an integral curve of  $\bar{A}$ , with  $\mu(0) = e$ , then,

$$\frac{d}{dt}\Big|_s \mu(t) = A_{\mu(s)} = L_{\mu(s)*}(A_e) = L_{\mu(s)*} \frac{d}{dt}\Big|_0 \mu(t) = \frac{d}{dt}\Big|_{t=0} (\mu(s)\mu(t)) \quad (4.1.1)$$

**calculation:**

$$\frac{dx^i(\mu(t))}{dt}\Big|_s = \left( L_{\mu(s)*} \frac{d}{dt}\Big|_0 \mu(t) \right) x^i\Big|_{\mu(s)} = \left( \frac{d}{dt}\Big|_0 \mu(t) \right) y^i\Big|_e \quad (4.1.2)$$

where  $y^i\Big|_g \equiv L_{\mu(s)*} x^i\Big|_g = x^i\Big|_{\mu(s)g}$  so,

$$\left( \frac{d}{dt}\Big|_0 \mu(t) \right) y^i\Big|_e = \frac{dy^i(\mu(t))}{dt}\Big|_e = \frac{dx^i(\mu(s)\mu(t))}{dt}\Big|_{t=0} \quad (4.1.3)$$

so, as we can see,  $\nu : (-\epsilon + s, \epsilon + s) \rightarrow G, t \mapsto \mu(s)\mu(t-s)$  is also an integral curve of  $\bar{A}$ , with at least one intersection with  $\mu$ ,  $\nu(s) = \mu(s)$ .

since a vector field only has one integral curve through a fixed point,

**proof:**

for a vector field  $A$ , the integral curve  $\mu$  through point  $p$  must satisfy,

$$\frac{dx^i(\mu(t))}{dt}\Big|_s = A^i\Big|_{\mu(s)} \quad (4.1.4)$$

which is a linear differential equation of order one, consequently, the solution can be determined by  $x^i(\mu(t)) = \text{Const.}$

we can conclude that  $\mu$  and  $\nu$  is all part of one complete integral curve through  $e$ ,  $\gamma : \mathbb{R} \rightarrow G$ .

- the integral curve of  $\bar{A}$  through  $e$  is a one-parameter subgroup.

**proof:**

we have already proved that  $\nu(s+t) = \mu(s)\mu(t)$  and  $\mu = \nu = \gamma$ .  
so  $\gamma(s+t) = \gamma(s)\gamma(t)$ .

- the tangent vector of  $\gamma$  is left-invariant.

**proof:**

$$\left( L_{\gamma(t_2)*} \frac{d}{dt} \Big|_{t_1} \gamma(t) \right) x^i \Big|_{\gamma(t_2+t_1)} = \frac{dx^i(\gamma(t_2+t))}{dt} \Big|_{t_1} = \left( \frac{d}{dt} \gamma(t) \right) x^i \Big|_{\gamma(t_2+t_1)} \quad (4.1.5)$$

- a useful lemma: for a curve  $\gamma$  on manifold  $M_1$ , and a map  $\psi : M_1 \rightarrow M_2$ , then,

$$\psi_* \left( \frac{d}{dt} \Big|_{p \in M_1} \gamma \right) = \frac{d}{dt} \Big|_{\psi(p) \in M_2} \psi \circ \gamma \quad (4.1.6)$$

the proof is in appendix [B.1.4](#).

## 4.2 exponential maps

- **def.:** exp. map on a **Riemann manifold**,  $\exp_p : V_p$  (or its subspace)  $\rightarrow M$ .
  - $\exp_p(v) = \gamma(1)$ , where  $\gamma$  is the geodesic determined by  $v$  and  $p$ .
- **def.:** exp. map on a **Lie group**,  $\exp : V_e \rightarrow G$ .
  - $\exp(A) = \gamma(1)$  where  $\gamma$  is the one-para. subgroup determined by  $\bar{A}$ .
  - def. for physicists:  $\exp : \mathfrak{g} \rightarrow G$ , with  $\exp(iX) = \exp(A) = \gamma(1)$ .
- **theorem:** for **compact** Lie group, the exponential map,  $\exp : V_e \rightarrow G$ , is **onto**.

### 4.2.1 matrix exponential and logarithm

- properties of exp. function of matrices (in general linear group):
  - $(e^A)^\dagger = e^{A^\dagger}$ .
  - if  $\det e^A \neq 0$ , then  $(e^A)^{-1} = e^{-A}$ .
  - $\det e^A = e^{\text{tr} A}$ .

**proof:**

\* if  $A$  is diagonalizable,

diagonalize  $A$  by  $T$ ,  $TAT^{-1} = D = \text{diag}(\lambda_1, \dots, \lambda_m)$ , then,

$$\det e^A = \det(Te^AT^{-1}) = \det e^D = e^{\lambda_1 + \dots + \lambda_m} = e^{\text{tr} A} \quad (4.2.1)$$

\* otherwise, it is still can be proved as follow,

$$\frac{d}{dt} \Big|_t \det(e^{tA}) = \frac{d}{ds} \Big|_{s=0} \det(e^{(s+t)A}) = \det(e^{tA}) \frac{d}{ds} \Big|_{s=0} \det(e^{sA}) \quad (4.2.2)$$

and,

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \det(e^{sA}) &= \frac{d}{ds} \Big|_{s=0} \det(I + sA) \\ &= \frac{d}{ds} \Big|_{s=0} \epsilon_{ij\dots k} (\delta_1^i + sA_1^i) \dots (\delta_m^k + sA_m^k) \\ &= \epsilon_{i2\dots m} A_1^i + \dots + \epsilon_{12\dots k} A_m^k = \text{tr} A \end{aligned} \quad (4.2.3)$$

so we have,

$$\begin{cases} \frac{1}{\det(e^{tA})} \frac{d}{dt} \Big|_t \det(e^{tA}) = \text{tr} A \\ \det(e^{tA}) \Big|_{t=0} = 1 \end{cases} \implies \det(e^{tA}) = e^{t \text{tr} A} \quad (4.2.4)$$

– Baker-Campbell-Hausdorff formula,

$$e^A e^B = \exp \left( A + B + \frac{1}{2} [A, B] + \frac{1}{12} ([A, [A, B]] + [B, [B, A]]) + \cdots \right) \quad (4.2.5)$$

- the Hilbert-Schmidt norm of  $A \in \mathcal{M}_m(\mathbb{C})$  is,

$$\|A\| = \left( \sum_{i,j=1}^m |A_{ij}|^2 \right)^{1/2} \quad (4.2.6)$$

- matrix logarithm is,

$$\ln M = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(M - I)^n}{n} \quad (4.2.7)$$

where  $M$  is a complex matrix with  $\|M - I\| < 1$ .

- $\forall M$  with  $\|M - I\| < 1$ ,  $e^{\ln M} = M$ .
- $\forall A$  with  $\|A\| < \ln 2$  then  $\|e^A - I\| < 1$  and  $\ln e^A = A$ .

- for a **connected** Lie group  $G$ , every element  $g \in G$  can be written in the form,

$$g = \exp(A_1) \exp(A_2) \cdots \exp(A_N) \quad (4.2.8)$$

for some  $A_1, A_2, \dots, A_N \in \mathfrak{g}$ .

**proof:**

曲线  $\gamma : [0, 1] \rightarrow G, \gamma(0) = I, \gamma(1) = g$ .

选取  $N$  足够大, 使得  $\gamma^{-1}(\frac{i-1}{N})\gamma(\frac{i}{N})$  在  $I$  的邻域, 那么, 存在  $A_i \in \mathfrak{g}$  使得,

$$\gamma^{-1}(\frac{i-1}{N})\gamma(\frac{i}{N}) = \exp(A_i) \quad (4.2.9)$$

所以,

$$g = \gamma^{-1}(0)\gamma(1) = \exp(A_1) \cdots \exp(A_N) \quad (4.2.10)$$

错误的推断:

combined with BCH formula,  $\exp : \mathfrak{g} \rightarrow G$  is onto for connected Lie groups, i.e.  $G \neq \exp[\mathfrak{g}]$ .

- onto 仅对 **compact connected** Lie groups 成立,
- 原因: BCH 公式中的级数展开可能不存在.

## 4.3 Baker-Campbell-Hausdorff formula

### 4.3.1 the Campbell's identity

- $\text{Ad}_{\exp(A)} = e^{\text{ad}_A} : V_e \rightarrow V_e$ .

**proof: (maybe not very rigorously)**

consider,

$$B(s) = \text{Ad}_{\exp(sA)}(B) = \frac{d}{dt} \Big|_0 \exp(sA) \exp(tB) \exp(-sA) \quad (4.3.1)$$

the derivative of  $B(s)$  is,

$$\frac{dB(s)}{ds} = \lim_{\Delta s \rightarrow 0} \frac{\text{numerator}}{\Delta s} = [A, \text{Ad}_{\exp(sA)}(B)] = \text{ad}_A B(s) \quad (4.3.2)$$

where the numerator is:

$$\begin{aligned} & \text{numerator} \\ &= \left. \frac{d}{dt} \right|_0 \exp(sA)(1 + \Delta sA) \exp(tB) \exp(-sA)(1 - \Delta sA) \\ & \quad - \left. \frac{d}{dt} \right|_0 \exp(sA) \exp(tB) \exp(-sA) \\ &= \Delta s [A, \text{Ad}_{\exp(sA)}(B)] \end{aligned} \quad (4.3.3)$$

so, the  $n$ th derivative is  $\frac{d^n}{ds^n} B(s) = (\text{ad}_A)^n B(s)$ , then naturally,

$$B(s) = e^{\text{ad}_A} B \quad (4.3.4)$$

### 4.3.2 BCH formula

- theorem 1 (Campbell's identity in the case of  $\mathfrak{gl}(m)$ ):

$$e^A B e^{-A} = e^{\text{ad}_A} B \quad (4.3.5)$$

**proof:**

consider  $F(t) = e^{tA} B e^{-tA}$ , so  $F(0) = B$ , and,

$$\frac{d}{dt} F(t) = [A, F(t)] = \text{ad}_A F(t) \implies \frac{d^n}{dt^n} F(t) = (\text{ad}_A)^n F(t) \quad (4.3.6)$$

so it is clear that  $F(t) = e^{\text{ad}_A} B$ .

- theorem 2:

$$e^{A(t)} \frac{d}{dt} e^{-A(t)} = -f(\text{ad}_A) \frac{dA(t)}{dt} \quad (4.3.7)$$

where  $f(z) = \frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$ .

**proof:**

consider  $F(s, t) = e^{sA(t)} \frac{d}{dt} e^{-sA(t)}$ , with  $F(0, t) = 0$ , and,

$$\begin{aligned} \frac{d}{ds} F(s, t) &= A(t) F(s, t) - e^{sA(t)} \frac{d}{dt} (A(t) e^{-sA(t)}) \\ &= -e^{sA(t)} \frac{dA(t)}{dt} e^{-sA(t)} \\ &= -e^{\text{ad}(sA(t))} \frac{dA(t)}{dt} \end{aligned} \quad (4.3.8)$$

and the  $n$ th derivative is,

$$\frac{d^n}{ds^n} F(s, t) = \text{ad}^{n-1}(A(t)) \frac{d}{ds} F(s, t) \quad (4.3.9)$$

when  $s = 0$ ,  $\left. \frac{d^n}{ds^n} \right|_{s=0} F(s, t) = -\text{ad}^{n-1}(A(t)) \frac{dA(t)}{dt}$ , so,

$$F(s = 1, t) = - \sum_{n=1}^{\infty} \frac{\text{ad}^{n-1}(A(t))}{n!} \frac{dA(t)}{dt} \quad (4.3.10)$$

(the 0th order term is 0)

- theorem 3:

$$\frac{d}{dt}e^{-A(t)} = - \int_0^1 e^{-sA(t)} \frac{dA(t)}{dt} e^{-(1-s)A(t)} ds \quad (4.3.11)$$

**proof:**

consider the following equation,

$$e^{-A} - e^{-B} = \int_0^1 e^{-sA} (B - A) e^{-(1-s)B} ds \quad (4.3.12)$$

**proof:**

consider the following equation,

$$e^{-sA} (B - A) e^{-(1-s)B} = \frac{d}{ds} \left( e^{-sA} e^{-(1-s)B} \right) \quad (4.3.13)$$

integrate both side of the equation,

$$\int_0^1 \dots ds = e^{-A} - e^{-B} \quad (4.3.14)$$

take  $A = A(t)$ ,  $B = A(t - \Delta t)$ , with  $\Delta t \rightarrow 0$ , then,

$$\frac{d}{dt}e^{-A(t)} = - \int_0^1 e^{-sA(t)} \frac{dA(t)}{dt} e^{-(1-s)A(t)} ds \quad (4.3.15)$$

- theorem 3 is equivalent to theorem 2.

**calculation:**

$$\begin{aligned} e^{A(t)} \frac{d}{dt} e^{-A(t)} &= - \int_0^1 e^{(1-s)A(t)} \frac{dA(t)}{dt} e^{-(1-s)A(t)} ds \\ &= - \int_0^1 \underbrace{e^{\text{ad}((1-s)A(t))}}_{=e^{(1-s)\text{ad}_{A(t)}}} \frac{dA(t)}{dt} ds \\ &= -f(\text{ad}_{A(t)}) \frac{dA(t)}{dt} \end{aligned} \quad (4.3.16)$$

where  $f(z)$  is defined in theorem 2.

- the Baker-Campbell-Hausdorff formula is,

$$\begin{aligned} e^A e^B &= \exp \left( B + \left( \int_0^1 g(e^{t\text{ad}_A} e^{\text{ad}_B}) dt \right) A \right) \\ &= \exp \left( A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] + \dots \right) \end{aligned} \quad (4.3.17)$$

where  $g(z) = \frac{\ln z}{z-1} = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n+1}$ , for  $|z-1| < 1$ .

**proof:**

consider  $e^{C(t)} = e^{tA} e^B$ , then,

$$e^{\text{ad}_{C(t)}} = e^{t\text{ad}_A} e^{\text{ad}_B} \quad (4.3.18)$$

**proof:**

consider the following equation,

$$e^{\text{ad}_{C(t)}} W = e^{C(t)} W e^{-C(t)}$$

$$\begin{aligned}
&= e^{tA} e^B W e^{-B} e^{-tA} \\
&= e^{tA} e^{\text{ad}_B} W e^{-tA} \\
&= e^{t \text{ad}_A} e^{\text{ad}_B} W
\end{aligned} \tag{4.3.19}$$

---

then, let's consider, (notice that  $\text{ad}_A A = 0$ ),

$$\begin{aligned}
e^{C(t)} \frac{d}{dt} e^{-C(t)} &= -f(\text{ad}_{C(t)}) \frac{dC(t)}{dt} \\
&= e^{tA} e^B \frac{d}{dt} e^{-B} e^{-tA} \\
&= e^{tA} \frac{d}{dt} e^{-tA} \\
&= -f(t \text{ad}_A) A = -A
\end{aligned} \tag{4.3.20}$$

$$\implies f(\text{ad}_{C(t)}) \frac{dC(t)}{dt} = A \tag{4.3.21}$$

notice that  $g(e^z) = 1/f(z)$ , so we have,

$$\frac{dC(t)}{dt} = g(e^{\text{ad}_{C(t)}}) A \implies C(1) - \underbrace{C(0)}_{=B} = \left( \int_0^1 g(e^{t \text{ad}_A} e^{\text{ad}_B}) dt \right) A \tag{4.3.22}$$



# Chapter 5

## basic representation theory

### 5.1 Lie group and Lie algebra homomorphisms

- $\Phi : G \rightarrow H$  is a **Lie group homomorphism**, then there exists a unique real-linear map  $\phi = \Phi_* : \mathfrak{g} \rightarrow \mathfrak{h}$  s.t.,

$$\Phi \circ \exp(A) = \exp(\phi A) \quad (5.1.1)$$

$\phi$  has the following properties:

1.  $\phi \text{Ad}_g(A) = \text{Ad}_{\Phi(g)}(A), \forall A, g,$
2.  $\phi$  is **Lie algebra homomorphism**,
3.  $\phi(A) = \left. \frac{d}{dt} \right|_0 \Phi \circ \exp(tA).$

**proof:**

let's prove the 3rd identity first,

$$\begin{aligned} \left( \Phi_* \left. \frac{d}{dt} \right|_s \gamma(t) \right) y^i &= \left( \left. \frac{d}{dt} \right|_s \gamma(t) \right) \Phi^* y^i = \left. \frac{d \Phi^* y^i(\gamma(t))}{dt} \right|_s = \left. \frac{dy^i(\Phi \gamma(t))}{dt} \right|_s \\ \implies \Phi_* \circ L_{\exp(sA)} A &= \left. \frac{d}{dt} \right|_s \Phi \exp(tA) \end{aligned} \quad (5.1.2)$$

and,

$$\begin{cases} L_{\Phi(g)*} \circ \Phi_* A = (L_{\Phi(g)} \circ \Phi)_* A \\ L_{\Phi(g)} \circ \Phi = \Phi \circ L_g \end{cases} \implies L_{\Phi(g)*} \circ \Phi_* A = \Phi_* \circ L_{g*} A \quad (5.1.3)$$

so,

$$\left. \frac{d}{dt} \right|_s \Phi \exp(tA) = L_{\Phi \exp(sA)*} \circ \Phi_* A \implies \exp(\Phi_* A) = \Phi \exp(A) \quad (5.1.4)$$

the 1st identity is easy to prove,

$$\text{Ad}_g \equiv I_{g*} \implies \begin{cases} \Phi_* \circ I_{g*} = (\Phi \circ I_g)_* \\ \Phi \circ I_g = I_{\Phi(g)} \circ \Phi \end{cases} \implies \dots \quad (5.1.5)$$

now let's prove the 2nd identity,

$$L_{\Phi(g)*} \circ \Phi_* A = \Phi_* \circ L_{g*} A \implies (\Phi_* A)_{\Phi(g)} = \Phi_* A_g \quad (5.1.6)$$

$$\implies ((\Phi_* A)_{\Phi(g)})^i \Phi_* \frac{\partial}{\partial x^i} = (A_g)^i \Phi_* \frac{\partial}{\partial x^i} \quad (5.1.7)$$

$$\implies A^i \Big|_g = \Phi^* ((\Phi_* A)^i \Big|_{\Phi(g)}) \quad (5.1.8)$$

where  $A^i$  and  $(\Phi_* A)^i$  are treated as functions on  $G$  and  $H$ .

so,

$$(\Phi_* [A, B]_g)^i \Phi_* \frac{\partial}{\partial x^i} = \left( (A_g)^j \frac{\partial}{\partial x^j} (B_g)^i - \dots \right) \Phi_* \frac{\partial}{\partial x^i} \quad (5.1.9)$$

and,

$$([\Phi_*A, \Phi_*B]_{\Phi(g)})^i \Phi_* \frac{\partial}{\partial x^i} = \left( (\Phi_*A)^a \nabla_a (\Phi_*B)^i - \dots \right) \Big|_{\Phi(g)} \Phi_* \frac{\partial}{\partial x^i} \quad (5.1.10)$$

where,

$$\begin{aligned} (\Phi_*A)^a \nabla_a (\Phi_*B)^i \Big|_{\Phi(g)} &= (A_g)^i \Phi_* \frac{\partial}{\partial x^i} (\Phi_*B)^i \\ &= (A_g)^i \frac{\partial}{\partial x^i} \Big|_{\Phi(g)} \Phi^* (\Phi_*B)^i \\ &= (A_g)^i \frac{\partial}{\partial x^i} \Big|_g B^i \end{aligned} \quad (5.1.11)$$

so, we proved that  $\Phi_*[A, B] = [\Phi_*A, \Phi_*B]$ .

- for a Lie group homomorphism  $\Phi : G \rightarrow H$  and  $\phi = \Phi_*$ ,

$$\text{Lie}(\ker \Phi) = \ker \phi \quad (5.1.12)$$

**proof:**

–  $\ker \Phi = \{g \in G \mid \Phi(g) = I\}$  is a **closed normal subgroup** of  $G$ .

\*  $G(\ker \Phi)G^{-1} \subseteq \ker \Phi$ .

\*  $\{I\}$  is a closed subgroup, and  $\Phi$  is continuous.

–  $\text{Lie}(\ker \Phi) \subseteq \ker \phi$ .

for all  $A \in \text{Lie}(\ker \Phi)$ ,

$$\Phi \exp(tA) \in \Phi(\ker \Phi) = \{I\} \implies \phi A = \frac{d}{dt} \Big|_0 \Phi \exp(tA) = 0 \quad (5.1.13)$$

so,  $A \in \ker \phi$ .

–  $\text{Lie}(\ker \Phi) \supseteq \ker \phi$ .

for all  $A \in \ker \phi$ ,

$$\exp(\phi A) = \Phi \exp(A) = I \implies \exp(A) \in \ker \Phi \quad (5.1.14)$$

so,  $A \in \text{Lie}(\ker \Phi)$ .

### 5.1.1 simply connected Lie groups

- Lie algebra homomorphism  $\implies$  Lie group homomorphism, when  $G$  is **simply connected**.

$\phi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a Lie algebra homomorphism, (if  $G$  is simply connected) then there **exist** a **unique** Lie group homomorphism  $\Phi : G \rightarrow H$  s.t.  $\Phi(\exp(A)) = \exp(\phi A)$  and  $\phi = \Phi_*$ .

**proof:**

$G$  is **connected**, so, for all  $g \in G$  there exists a path  $g(t)$  s.t.  $g(0) = I, g(1) = g$   
 $N$  is large enough that,

$$g^{-1} \left( \frac{i-1}{N} \right) g \left( \frac{i}{N} \right) \in U \quad (5.1.15)$$

where  $U \subset G$  is a neighborhood of  $I$  s.t. there exists an isomorphism,

$$\begin{aligned} \ln : U &\rightarrow \ln[U] \subset \mathfrak{g} \\ g = \exp(A) &\mapsto A, \forall g \in U \end{aligned} \quad (5.1.16)$$

which implies that there exists a unique local homomorphism,

$$\begin{aligned} f : U &\rightarrow H \\ g &\mapsto \exp(\phi \ln g), \forall g \in U \end{aligned} \quad (5.1.17)$$

where,

$$\begin{aligned}
f(g_1 g_2) &= \exp(\phi \ln(\exp(A_1) \exp(A_2))) \\
&= \exp\left(\phi \ln \exp\left(A + B + \frac{1}{2}[A, B] + \frac{1}{12} \cdots\right)\right) \\
&= \exp(\phi A) \exp(\phi B) \\
&= f(g_1) f(g_2)
\end{aligned} \tag{5.1.18}$$

so, there **exists** a homomorphism,

$$\begin{aligned}
\Phi : G &\rightarrow H \\
g &\mapsto f\left(g^{-1}(0)g\left(\frac{1}{N}\right)\right) \cdots f\left(g^{-1}\left(\frac{N-1}{N}\right)g(1)\right), \forall g \in G
\end{aligned} \tag{5.1.19}$$

finally, the **uniqueness**:

$\Phi$  is independent from the choice of path  $g(t)$  and the choice of partition  $0 = t_0 < t_1 < \cdots t_N = 1$ .

– independence of the partition:

for any good partition (partition that guarantees  $g^{-1}(t_{i-1})g(t_i) \in U$ ) insert  $s$  between  $t_{i-1}$  and  $t_i$ , since  $f$  is a local homomorphism,

$$f(g^{-1}(t_{i-1})g(s))f(g^{-1}(s)g(t_i)) = f(g^{-1}(t_{i-1})g(t_i)) \tag{5.1.20}$$

– independence of the path:

since  $G$  is **simply connected**, there exists a continuous map,

$$\begin{aligned}
g : [0, 1] \times [0, 1] &\rightarrow G \\
g(s, t) &= g_s(t) \\
g(s, 0) &= I, g(s, 1) = g
\end{aligned} \tag{5.1.21}$$

and choose a good partition that  $g_{s_{j-1}}^{-1}(t)g_{s_j}(t) \in U$ , so,

$$\begin{cases} \Phi_{s_{j-1}}(g) = \cdots f(g_{s_{j-1}}^{-1}(t_{i-1})g_{s_{j-1}}(t_i)) \cdots \\ \Phi_{s_j}(g) = \cdots f(g_{s_j}^{-1}(t_{i-1})g_{s_{j-1}}(t_{i-1})g_{s_{j-1}}^{-1}(t_{i-1})g_{s_{j-1}}(t_i)g_{s_{j-1}}^{-1}(t_i)g_{s_j}(t_i)) \cdots \end{cases} \tag{5.1.22}$$

the red terms will be canceled due to  $f$  is homomorphism.

so  $\Phi_{s_{j-1}} = \Phi_{s_j}$  which implies that  $\Phi_0 = \Phi_1$ .

显然, 根据上述选择,

$$\begin{cases} \Phi \circ \exp(A) = \exp(\phi A) \\ \Phi(g) = \exp(\phi A_1) \cdots \exp(\phi A_N) \end{cases} \tag{5.1.23}$$

now, let's prove  $\phi = \Phi_*$ .

consider,

$$\exp(\Phi_* A) = \exp(\phi A) \tag{5.1.24}$$

and if  $A$  is close to 0 enough,  $\exp$  is one-to-one, moreover,  $\Phi_*$  and  $\phi$  is linear, so  $\phi = \Phi_*$ .

- for 2 **simply connected** Lie groups  $G, H$ , there exists a Lie algebra **isomorphism**  $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ , then  $G, H$  are **isomorphic** to each other.

换句话说: simply connected Lie groups are determined by their Lie algebra.

– but, exponential maps,  $\exp : \mathfrak{g} \rightarrow G$ , are **not** one-to-one even for simply connected Lie groups.

e.g. in  $SU(2)$ ,  $\exp(4\pi i J_3) = I$ .

**proof:**

let  $\Phi, \Psi$  correspond to  $\phi, \phi^{-1}$  respectively, then,

$$\Phi \circ \Psi(\exp(A_1) \cdots \exp(A_N)) = \exp(\phi \circ \phi^{-1} A_1) \cdots \exp(\phi \circ \phi^{-1} A_N) \quad (5.1.25)$$

which means  $\Phi \circ \Psi = I$  similarly,  $\Psi \circ \Phi = I$ .

so  $\Phi$  is a reversible homomorphism, i.e. an isomorphism.

- for a simply connected Lie group  $G$ , its Lie algebra  $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ , then, there exist 2 **closed, simply connected** subgroups  $H_1, H_2$  corresponded to  $\mathfrak{h}_1, \mathfrak{h}_2$  and  $G \simeq H_1 \times H_2$ .

**proof:**

consider the projection map  $\phi_1 \in \text{End}(\mathfrak{g})$ , s.t.  $\phi_1(A + B) = A, \forall A \in \mathfrak{h}_1, B \in \mathfrak{h}_2$ .

- since  $G$  is simply connected,  $\Phi_1$  is the corresponding Lie group homomorphism.
- according to (5.1.12),  $\ker \phi_1 = \mathfrak{h}_2 = \text{Lie}(\ker \Phi_1)$ .
- let  $H_2$  be the identity component of  $\ker \Phi_1$ , thus  $H_2$  is a **closed connected** Lie subgroup.
- construct  $H_1$  in a similar way.

---

$\phi_1$  is the identity on  $\mathfrak{h}_1$ , so  $\Phi_1$  is the identity on  $H_1$ .

- consider a loop  $h(t)$  on  $H_1$ .
- there is a way to shrink  $h(t)$  into a point on  $G$ , say  $g(s, t)$  with  $g(0, t) = h(t)$  and  $g(1, t)$  is a point.
- define  $h(s, t) = \Phi_1(g(s, t))$ , then  $h(0, t) = h(t)$  and  $h(1, t)$  is a point.

so,  $H_1$  is **simply connected**.

---

finally, let's prove  $G \simeq H_1 \times H_2$ .

- since  $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ ,  $[\mathfrak{h}_1, \mathfrak{h}_2] = \{0\}$ , so  $h_1 h_2 = h_2 h_1, \forall h_1 \in H_1, h_2 \in H_2$ .
- $\Psi : H_1 \times H_2 \rightarrow G, (h_1, h_2) \mapsto h_1 h_2$  is a Lie group homomorphism.  
(we don't know  $H_1 \times H_2$  is simply connected yet)
- $\psi = \Psi_* : \mathfrak{h}_1 \oplus \mathfrak{h}_2 \rightarrow \mathfrak{g}$  is the original isomorphism.

$$\exp(\psi(A + B)) = \Psi \circ \exp(A + B) = \exp(A + B) \implies \psi(A + B) = A + B \quad (5.1.26)$$

- so the homomorphism  $\Psi' : G \rightarrow H_1 \times H_2$  associated with  $\psi^{-1}$  is an isomorphism.

### 5.1.2 universal covers

- $G$  is a **connected** Lie group,  $H$  is a **simply connected** Lie group with  $\mathfrak{g} \simeq \mathfrak{h}$ .  
then,  $H$  is the **universal cover** of  $G$  and the homomorphism  $\Phi : H \rightarrow G$  associated to the isomorphism  $\phi : \mathfrak{h} \rightarrow \mathfrak{g}$  is called the **covering map**.

- 
- the universal cover of  $\text{SO}(3)$  is  $\text{SU}(2)$ , and  $\ker \Phi = \{\pm I\}$ .
  - the universal cover of  $\text{SO}(n \geq 3)$  is  $\text{Spin}(n)$  and may be constructed as a certain group of invertible elements in the **Clifford algebra** over  $\mathbb{R}^n$ .
    - the covering map is two-to-one.
    - and  $\text{Spin}(4) \simeq \text{SU}(2) \times \text{SU}(2)$ .

## 5.2 basic representation theory

- **def.:** a **finite-dimensional representation** of a Lie group  $G$  (or a Lie algebra  $\mathfrak{g}$ ) is a **Lie group** (or a Lie algebra) **homomorphism**,

$$\begin{cases} \Pi : G \rightarrow \mathrm{GL}(V) \\ \pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \end{cases} \quad (5.2.1)$$

where  $\mathrm{GL}(V)$  is the group of invertible linear transformations of  $V$  and  $\mathfrak{gl}(V) = \mathrm{End}(V)$  is the space of all linear operators from  $V$  to itself with Lie bracket  $[A, B] = AB - BA$ .

- for a finite-dimensional representation of  $G$ ,

$$\pi(A) = \left. \frac{d}{dt} \right|_0 \Pi(e^{tA}) \quad (5.2.2)$$

then  $\Pi(\exp(A)) = e^{\pi(A)}$  and  $\pi$  is the representation of  $\mathfrak{g}$  on the same vector space.

- subspace  $W \subset V$  is **invariant** if  $\Pi(g)[W] \subseteq W, \forall g \in G$ .
- **def.:** a representation without nontrivial invariant subspaces ( $\{0\}, V$ ) is called **irreducible**.  
对 Lie algebra 的 irreducible rep. 的定义是一样的.
- $\Pi, \pi$  are associated representations of **connected** Lie group  $G$  and its Lie algebra  $\mathfrak{g}$ , then:
  - $\Pi$  is **irreducible**  $\iff \pi$  is **irreducible**.

**proof:**

\*  $\Pi$  is irreducible  $\implies \pi$  is irreducible.

设  $W \subseteq V$  是  $\pi$  的不变子空间, 那么  $\forall g$ ,

$$\Pi(g)[W] = e^{\pi(A_1)} \dots e^{\pi(A_N)}[W] \subseteq W \quad (5.2.3)$$

(其中用到了 (4.2.8) 式), 而  $\Pi$  是不可约表示, 所以  $W = \{0\}$  or  $V$

\*  $\Pi$  is irreducible  $\iff \pi$  is irreducible.

设  $W \subseteq V$  是  $\Pi$  的不变子空间, 那么  $\forall A$ ,

$$\pi(A)[W] = \left. \frac{d}{dt} \right|_0 \Pi(\exp(tA))[W] \subseteq W \quad (5.2.4)$$

所以...

- $\Pi_1, \Pi_2$  are **isomorphic**  $\iff \pi_1, \pi_2$  are **isomorphic**.
- $\pi$  is a **irreducible rep.** of  $\mathfrak{g}_{\mathbb{C}} \iff \pi$  is a (complex) **irreducible rep.** of  $\mathfrak{g}$ .  
where the rep. of  $\mathfrak{g}_{\mathbb{C}}$  is  $\pi(A + iB) = \pi(A) + i\pi(B)$  which is the unique extension of the rep. of  $\mathfrak{g}$ ,  $\pi$ .

### 5.2.1 new representations from old

- three ways to obtain new rep. from old:
  1. direct sums,
  2. tensor products,
  3. dual representations.

#### direct sums

- **def.:** the direct sum of  $\Pi_1, \dots, \Pi_m$  is a rep. of  $G$  on  $V_1 \oplus \dots \oplus V_m$ , defined by,

$$\Pi_1 \oplus \dots \oplus \Pi_m(g)(v_1, \dots, v_m) = (\Pi_1(g)v_1, \dots, \Pi_m(g)v_m) \quad (5.2.5)$$

对 Lie algebra rep.  $\pi_1, \dots, \pi_m$  的直和的定义是一样的.

## tensor products

- $\Pi_1, \Pi_2$  are rep. of  $G, H$  respectively. then, the tensor product rep.  $\Pi_1 \otimes \Pi_2$  of  $G \times H$  is defined to be,

$$(\Pi_1 \otimes \Pi_2)(g, h) = \Pi_1(g) \otimes \Pi_2(h) \quad (5.2.6)$$

- the tensor product rep.  $\pi_1 \otimes \pi_2$  of  $\mathfrak{g} \oplus \mathfrak{h}$  is,

$$(\pi_1 \otimes \pi_2)(A, B) = \pi_1(A) \otimes I + I \otimes \pi_2(B) \quad (5.2.7)$$

**proof:**

令  $\pi_1 : \mathfrak{g} \rightarrow \text{End}(U), \pi_2 : \mathfrak{h} \rightarrow \text{End}(V)$ , 那么,

$$\begin{aligned} (\pi_1 \otimes \pi_2)(A, B)(u \otimes v) &= \left( \frac{d}{dt} \Big|_0 (\Pi_1 \otimes \Pi_2)(\exp(tA), \exp(tB)) \right) (u \otimes v) \\ &= \frac{d}{dt} \Big|_0 \underbrace{\Pi_1(\exp(tA))u}_{=u(t)} \otimes \underbrace{\Pi_2(\exp(tB))v}_{=v(t)} \end{aligned} \quad (5.2.8)$$

其中,  $u(t), v(t)$  是  $U, V$  中的两条  $C^\infty$  的曲线,

$$(u + du) \otimes (v + dv) - u \otimes v = du \otimes v + u \otimes dv \quad (5.2.9)$$

代入, 所以,

$$(\pi_1 \otimes \pi_2)(A, B)(u \otimes v) = \pi_1(A)u \otimes v + u \otimes \pi_2(B)v \quad (5.2.10)$$

## dual representations

- 对于  $\Pi : G \rightarrow \text{End}(V)$ , dual rep. 就是  $\Pi^\dagger : G \rightarrow \text{End}(V^*)$ , 其中  $V^*$  是  $V$  的对偶空间.

### 5.2.2 complete reducibility

- 参见有限群中的定义 (group 和 Lie algebra 的定义都一样).
- a group or Lie algebra is said to have the **complete reducibility property** if every finite-dim. rep. of it is completely reducible.

- **unitary** rep. of  $G, \mathfrak{g}$  is **completely reducible**.

notice, the 'unitary' (skew self-adjoint) rep. of  $\mathfrak{g}$  is  $\pi^\dagger(A) = -\pi(A)$

证明参见有限群.

- **compact** Lie groups have the **complete reducibility property**.

**proof:**

for an  $n$ -dim. Lie group  $G$ ,

$$\epsilon = A^1 \wedge \cdots \wedge A^n \quad (5.2.11)$$

is a **right-invariant  $n$ -form** composed of the dual vectors of a basis of  $\mathfrak{g}$ .

if  $G$  is **compact**, we can integrate any smooth function over all  $G$ , denoted by,

$$\int_G f(g) \epsilon(g) \quad (5.2.12)$$

and, since  $\epsilon$  is right-invariant,

$$\int_G f(gh) \epsilon(g) = \int_G f(g) \epsilon(g) \quad (5.2.13)$$

for a rep. of  $G, \Pi : G \rightarrow \text{End}(V)$ , define an arbitrary inner product  $\langle \cdot, \cdot \rangle$  on  $V$ , then define another inner product on  $V$  by,

$$\langle \cdot, \cdot \rangle_G : V \times V \rightarrow \mathbb{C}$$

$$\langle u, v \rangle_G = \int_G \langle \Pi(g)u | \Pi(g)v \rangle \epsilon(g) \quad (5.2.14)$$

then,

$$\langle u, v \rangle_G = \langle \Pi(h)u, \Pi(h)v \rangle_G \quad (5.2.15)$$

and  $\langle v, v \rangle_G > 0$  for all  $v \neq 0$ .

so,  $\Pi(g)$  is **unitary** with respect to  $\langle \cdot, \cdot \rangle_G$ .

–  $\mathrm{SU}(m)$  are compact, hence have the complete reducibility property.

### 5.2.3 Schur's lemma

- **def.:** an **intertwining map** of rep.  $\Pi_1, \Pi_2$  (or  $\pi_1, \pi_2$ ) is a linear map  $\phi : V \rightarrow W$ , s.t.,

$$\begin{cases} \phi \Pi_1(g) = \Pi_2(g) \phi \\ \phi \pi_1(A) = \pi_2(A) \phi \end{cases} \in \mathrm{End}(W) \quad (5.2.16)$$

- **Schur's 1st lemma**

for 2 **irreducible real or complex rep.**  $\Pi_1, \Pi_2$  (or  $\pi_1, \pi_2$ ) on  $V, W$ , the intertwining map  $\phi$  is either 0 or an isomorphism.

证明参见有限群.

- **Schur's 2nd lemma**

for a **irreducible complex rep.**  $\Pi$  (or  $\pi$ ) on  $V$ , the intertwining map  $\phi : V \rightarrow V$  is  $\lambda I$  for some  $\lambda \in \mathbb{C}$ .

- **Schur's 3rd lemma**

for 2 **irreducible complex rep.**  $\Pi_1, \Pi_2$  (or  $\pi_1, \pi_2$ ) on  $V, W$ , and 2 intertwining map  $\phi_1, \phi_2 : V \rightarrow V$ , then  $\phi_1 = \lambda \phi_2$  for some  $\lambda \in \mathbb{C}$ .

## 5.3 Lie's third theorem

- **Lie's third theorem:** every **finite-dimensional** Lie algebra  $\mathfrak{g}$  over  $\mathbb{R}$  is associated to a Lie group  $G$ .
- every **finite-dimensional** Lie algebra is isomorphic to the Lie algebra of some **matrix** Lie group.

## 5.4 adjoint representations

### 5.4.1 adjoint rep. of Lie groups

- consider the adjoint diffeomorphism on  $G$ ,

$$I_g : G \rightarrow G, h \mapsto ghg^{-1} \quad (5.4.1)$$

- $\mathrm{Ad}_g = I_{g*} : V_e \rightarrow V_e$  is the pushforward,

$$\mathrm{Ad}_g \left( \frac{d}{dt} \Big|_0 \gamma(t) \right) x^i \Big|_e = \frac{dy^i(\gamma(t))}{dt} \Big|_0 \quad (5.4.2)$$

where  $y^i(h) = x^i(ghg^{-1})$ , so we have,

$$\mathrm{Ad}_g \left( \frac{d}{dt} \Big|_0 \gamma(t) \right) = \frac{d}{dt} \Big|_0 g \gamma(t) g^{-1} \quad (5.4.3)$$

i.e.  $\exp(\mathrm{Ad}_g(A)) = I_g \exp(A)$ .

– as we can see,  $\mathrm{Ad}_g \in \mathrm{Aut}(V_e)$  is a linear and reversible automorphism on  $V_e$ , since  $\mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}} = I$ .

- $\mathrm{Ad} : G \rightarrow \mathrm{Aut}(V_e) \simeq \mathrm{GL}(m, \mathbb{R})$  is the **adjoint representation of the Lie group**,  $G$ .

–  $\mathrm{Ad}$  is a homomorphism.

proof:

$$\text{Ad}_g \circ \text{Ad}_h = I_{g*} \circ I_{h*} = (I_g \circ I_h)_* = \text{Ad}_{gh} \quad (5.4.4)$$

### 5.4.2 adjoint rep. of Lie algebras

- The **structure constants** themselves generate a **representation of the Lie algebra**, called the **adjoint representation**.
- the Jacob identity written in the structure constants is,

$$f_{il}^m f_{jk}^l + f_{kl}^m f_{ij}^l + f_{jl}^m f_{ki}^l = 0 \quad (5.4.5)$$

consider the structure constants as the components of matrices,  $-if_{ij}^k = (T_i)_j^k$ , since  $f_{ij}^k = -f_{ji}^k$ , the matrices have the property that  $(T_i)_j^k = -(T_j)_i^k$ , then,

$$\begin{aligned} if_{jk}^l (T_l)_i^m + \underbrace{(T_i T_k)_j^m}_{=-(T_j T_k)_i^m} + (T_k T_j)_i^m &= 0 \\ \implies [T_j, T_k]_i^m = if_{jk}^l (T_l)_i^m \end{aligned} \quad (5.4.6)$$

or, more compactly,  $[T_i, T_j] = if_{ij}^k T_k$ .

- $\{(T_i)_j^k = -if_{ij}^k\}$  is called the adjoint representation of the Lie algebra  $\{X_i\}$ .

- more formally, adjoint representation is a map,  $\text{ad}_A : \mathfrak{g} \rightarrow \mathfrak{g}$ , where  $\mathfrak{g}$  is the Lie algebra of the group  $G$ ,

$$\text{ad}_A(B) = [A, B] \quad (5.4.7)$$

as one can see,  $(\text{ad}_A)^a_b = -f_{cb}^a A^c \in \mathcal{L}(\mathfrak{g})$ , or written in components,

$$(\text{ad}_{A_i})_j^k = -f_{ij}^k \implies \text{ad}_{A_i} = (iT_i)^T \quad (5.4.8)$$

and  $[\text{ad}_{A_i}, \text{ad}_{A_j}] = \text{ad}_{[A_i, A_j]} = -f_{ij}^k \text{ad}_{A_k}$ .

- $\text{ad} : \mathfrak{g} \rightarrow \mathcal{L}(\mathfrak{g})$  is a homomorphism, i.e.,

$$\text{ad}_{[A, B]} = [\text{ad}_A, \text{ad}_B] \quad (5.4.9)$$

proof:

$$\begin{aligned} (\text{ad}_A \text{ad}_B - \text{ad}_B \text{ad}_A)C &= [A, [B, C]] - [B, [A, C]] \\ &= [[A, B], C] = \text{ad}_{[A, B]}C \end{aligned} \quad (5.4.10)$$

## 5.5 Killing forms

- $\forall A, B \in \mathfrak{g}$ , the Killing form is,

$$B(A, B) = \text{tr}(\text{ad}_A \circ \text{ad}_B) \quad (5.5.1)$$

which can be written in terms of structure constants,

$$B_{ij} = f_{ik}^l f_{jl}^k \quad (5.5.2)$$

proof:

$$B(A_i, A_j) = \text{tr}(\text{ad}_{A_i} \text{ad}_{A_j}) = (-f_{ik}^l)(-f_{jl}^k) \quad (5.5.3)$$

$$- B([A, B], C) = B(A, [B, C]).$$



**proof:**

recall that,

$$\text{ad}_{[A,B]} = [\text{ad}_A, \text{ad}_B] \quad (5.5.4)$$

so,

$$\begin{aligned} B([A, B], C) &= \text{tr}([\text{ad}_A, \text{ad}_B] \text{ad}_C) \\ &= \text{tr}(\text{ad}_A \text{ad}_B \text{ad}_C) - \text{tr}(\text{ad}_A \text{ad}_C \text{ad}_B) \\ &= B(A, [B, C]) \end{aligned} \quad (5.5.5)$$

- two basis-independent properties of the Killing form:
  - the **number** of zero eigenvalues.
  - the **sign** of the non-zero eigenvalues.
- the structure constants with lowered indices are **completely antisymmetric**,

$$f_{ij}{}^l B_{lk} = -f_{ijk} = -f_{[ijk]} \quad (5.5.6)$$

**proof:**

$$f_{ij}{}^l B_{lk} = f_{ij}{}^l f_{lm}{}^n f_{kn}{}^m \quad (5.5.7)$$

notice that, according to Jacob identity,  $f_{ij}{}^l f_{lm}{}^n = 2f_{[i|l}{}^n f_{|j]m}{}^l$ , then,

$$f_{ijk} = -2f_{[i|l}{}^n f_{|j]m}{}^l f_{kn}{}^m \quad (5.5.8)$$

we can see that the equation holds under index permutation like  $(i, j, k) \rightarrow (k, i, j) \rightarrow (j, k, i)$ , and consequently, all three indices of  $f_{ijk}$  are antisymmetric.

**Part III**

**Semisimple Lie Algebras**

# Chapter 6

## semisimple Lie algebras

### 6.1 semisimple and reductive Lie algebras

- **def.:** a complex Lie algebra is **reductive** if there exists a **compact** Lie group  $K$  s.t.,

$$\mathfrak{g} \simeq \mathfrak{k}_{\mathbb{C}} \quad (6.1.1)$$

- an alternate def. from Wikipedia: a Lie algebra is reductive if its adjoint rep. is completely reducible.

#### proof of equivalence:

$\implies$ , complexification of a compact Lie group is reductive:

- the adjoint rep. of a compact Lie group is completely reducible, so is its complexification (they have the same invariant subspaces,  $W, W^{\perp}$ , only complexified).

$\impliedby$ , reductive is isomorphic to the complexification of some compact Lie group:

- the invariant subspaces of the adjoint representation are the ideals of  $\mathfrak{g}$ , especially, the kernel of the adjoint rep. is the center,  $\mathfrak{z}$ .
- $\mathfrak{g}$  decomposes as  $\mathfrak{z} \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \dots$ , where  $\mathfrak{h}_1, \dots$  are the smallest ideals of  $\mathfrak{g}$ , i.e. they don't have nontrivial ideals themselves  $\implies$  irreducible.
- moreover, if  $\dim \mathfrak{h}_i = 1$ , then,

$$[\mathfrak{h}_i, \mathfrak{z} \oplus \bigoplus_{j \neq i} \mathfrak{h}_j] = [\mathfrak{h}_i, \bigoplus_{j \neq i} \mathfrak{h}_j] \subseteq \mathfrak{h}_i \cap \bigoplus_{j \neq i} \mathfrak{h}_j = \{0\} \quad (6.1.2)$$

then  $\mathfrak{h}_i$  is just part of the center.

- so,  $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}_1 \oplus \dots$ , where  $\mathfrak{h}_1, \dots$  are simple Lie subalgebras.
- according to the converse of (6.1.13) (?),  $\mathfrak{h}_1 \oplus \dots$  is a semisimple Lie algebra.
- according to the converse of (6.1.6) (?), a Lie algebra decomposes as its center and a semisimple Lie algebra is compact.

- **def.:** a complex Lie algebra is **semisimple** if it is reductive and the center of  $\mathfrak{g}$  is trivial, i.e.  $\mathfrak{z} = \{A \in \mathfrak{g} | \text{ad}_A = 0\} = \{0\}$ .
- **def.:**  $\mathfrak{k}$  in (6.1.1) is the **compact real form** of the semisimple Lie algebra.
- some semisimple Lie algebras:

Lie algebras	reductive	semisimple	compact real forms
$\mathfrak{sl}(m \geq 2, \mathbb{C})$	yes	yes	$\mathfrak{su}(m)$
$\mathfrak{so}(m \geq 3, \mathbb{C})$	yes	yes	$\mathfrak{so}(m)$
$\mathfrak{so}(2, \mathbb{C})$	yes	no	$\mathfrak{so}(2)$
$\mathfrak{sp}(m \geq 1, \mathbb{C})$	yes	yes	$\mathfrak{sp}(m, \mathbb{R})$
$\mathfrak{gl}(m, \mathbb{C})$	yes	no	$\mathfrak{u}(m)$

### 6.1.1 some properties of reductive and semisimple Lie algebras

- let  $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  be a **reductive** Lie algebra, then there exists an inner product s.t.,

$$\langle \text{ad}_X A, B \rangle = -\langle A, \text{ad}_X B \rangle \quad (6.1.3)$$

for all  $A, B \in \mathfrak{g}, X \in \mathfrak{k}$ .

**proof:**

$\text{Ad} : K \rightarrow \text{End}(\mathfrak{k})$  is a unitary representation under the inner product chosen in (5.2.14) (which requires **compactness**),

$$\langle A, B \rangle = \int_K (\text{Ad}_g A, \text{Ad}_g B) \epsilon(g) \quad (6.1.4)$$

where  $(A, B)$  is some real positive definite inner product on  $\mathfrak{k}$ , and  $\epsilon$  is the volume form composed by right invariant dual vector fields.

so, the associated Lie algebra rep.  $\text{ad} : \mathfrak{k} \rightarrow \text{End}(\mathfrak{k})$  satisfies  $\text{ad}_X^\dagger = -\text{ad}_X$  (skew self-adjoint).

- for a **reductive** Lie algebra  $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ ,  $\mathfrak{h}$  is one of its ideals, then,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \quad (6.1.5)$$

where  $\mathfrak{h}^\perp$  is orthogonal to  $\mathfrak{h}$  with respect to the inner product in (6.1.3), and it is also an **ideal**.

**proof:**

- if  $\mathfrak{h} (\text{ad}_A[\mathfrak{h}] \subseteq \mathfrak{h}, \forall A)$  is an ideal of  $\mathfrak{g}$ , then it is also an ideal of  $\mathfrak{k}$  (obviously).
- unitary rep. is completely reducible, so both  $\mathfrak{h}$  and  $\mathfrak{h}^\perp$  are its invariant subspace, i.e. ideals.
- $[\mathfrak{h}, \mathfrak{h}^\perp] \subseteq \mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$ .
- so,  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ .

- every **complex reductive** Lie algebra,  $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ , decomposes as,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{z} \quad (6.1.6)$$

where  $\mathfrak{g}_1$  is **semisimple** and  $\mathfrak{z}$  is its **center**.

moreover,

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}' \quad (6.1.7)$$

where  $\mathfrak{z}'$  is the center of  $\mathfrak{k}$  and  $\mathfrak{k}_1$  is the compact real form of  $\mathfrak{g}_1$ .

**proof:**

center is an ideal, so,

$$\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z} \quad (6.1.8)$$

now we have to prove  $\mathfrak{g}_1 = \mathfrak{z}^\perp$  is semisimple,

- first, the **center** of  $\mathfrak{z}^\perp$  is **trivial**, for obvious reasons.

- $A \in \mathfrak{z} \iff \text{ad}_A[\mathfrak{k}] = \{0\}$ , so, for all  $A = X + iY \in \mathfrak{z}, X, Y \in \mathfrak{k}$ ,

$$A^* := X - iY \in \mathfrak{z} \quad (6.1.9)$$

i.e.  $\mathfrak{z}$  is closed under conjugation  $*$  :  $X + iY \mapsto X - iY$

so,  $\mathfrak{g}_1$  is also closed under conjugation.

\* 注意, 这里的定义和 Hall 书上的不一样, Hall 的定义是  $A^* = -X + iY, \bar{A} = X - iY$ .

- so, for  $\mathfrak{z}' := \mathfrak{z} \cap \mathfrak{k}, \mathfrak{k}_1 := \mathfrak{g}_1 \cap \mathfrak{k}$ ,

$$\mathfrak{z} = \mathfrak{z}'_{\mathbb{C}} \quad \mathfrak{g}_1 = \mathfrak{k}_1_{\mathbb{C}} \quad (6.1.10)$$

- consider the adjoint representation of  $K$  and  $\mathfrak{k}$ ,

$$\text{Lie}(\text{Ad}[K]) = \text{ad}[\mathfrak{k}] \simeq \mathfrak{k} / \ker(\text{ad}) = \mathfrak{k} / \mathfrak{z}' = \mathfrak{k}_1 \quad (6.1.11)$$

$\text{Ad}$  is a continuous map, so  $\text{Ad}[K]$  is a **compact** Lie group as  $K$ .

- so,  $\mathfrak{k}_1$  is the **compact real form** of  $\mathfrak{g}_1$ .

- if  $K$  is a **simply connected compact** Lie group, then  $\mathfrak{g} = \mathfrak{k}_\mathbb{C}$  is **semisimple**.

**proof:**

since  $K$  is simply connected and  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}'$ , so  $K$  decomposes as,

$$K = K_1 \times Z' \quad (6.1.12)$$

where  $K_1, Z'$  are closed simply connected subgroup associated with  $\mathfrak{k}_1, \mathfrak{z}'$ . simply connected Lie group  $Z'$  is isomorphic to  $\mathbb{R}^n$  for some  $n$ , but  $Z'$  is closed subgroup of a compact group, it is also compact, which means  $n = 0$ , i.e.  $\mathfrak{z}' = \{0\} = \mathfrak{z}$ , the center is trivial.

- an important **theorem**:

**semisimple** Lie algebra  $\mathfrak{g}$  decomposes as,

$$\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i \quad (6.1.13)$$

where  $\mathfrak{g}_i$  are **simple** (see 3.2.1) and **unique** up to order (the converse of the theorem is also true (?)).

**proof:**

first, let's prove  $\mathfrak{g}_i$  are simple,

- according to (6.1.5), semisimple Lie algebra with ideal  $\mathfrak{h}$  decomposes as,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \quad (6.1.14)$$

suppose  $\mathfrak{h}'$  is an ideal of  $\mathfrak{h}$ , notice that  $[\mathfrak{h}, \mathfrak{h}^\perp] = \{0\}$ , so  $\mathfrak{h}'$  is also an ideal of  $\mathfrak{g}$ .

- let  $\mathfrak{h}'' = \mathfrak{h}'^\perp \cap \mathfrak{h}$ , and  $[\mathfrak{h}'', \mathfrak{h}' \oplus \mathfrak{h}^\perp] = \{0\}$ , so it is also an ideal, then,

$$\mathfrak{g} = \mathfrak{h}'' \oplus \mathfrak{h}' \oplus \mathfrak{h}^\perp \quad (6.1.15)$$

- proceeding on the same way,

$$\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i \quad (6.1.16)$$

where  $\mathfrak{g}_i$  are ideals without nontrivial ideals, i.e. **irreducible**.

- if  $\dim \mathfrak{g}_i = 1$ , then  $\mathfrak{g}_i$  is Abelian, moreover,

$$[\mathfrak{g}_i, \bigoplus_{j \neq i} \mathfrak{g}_j] = \{0\} \quad (6.1.17)$$

$\mathfrak{g}_i \subseteq \mathfrak{z}$  which contradicts to semisimpleness (without nontrivial center). so,  $\dim \mathfrak{g}_i \geq 2$ .

now, let's prove uniqueness,

- $\pi_i := \text{ad}|_{\mathfrak{g}_i} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}_i)$  is an **irreducible rep.**, since the nontrivial invariant subspace of  $\pi_i$  is  $\{\text{an ideal of } \mathfrak{g}\} \cap \mathfrak{g}_i$ , and consider (6.1.17), it is also an ideal of  $\mathfrak{g}_i$ , which doesn't exist.
- since  $\pi_i[\mathfrak{g}_{j \neq i}] = \{0\}$  while  $\pi_i[\mathfrak{g}_i] \neq \{0\}$  (simple Lie algebras are non-Abelian)  $\implies$  these rep. are **not isomorphic** to each other.

- for a simple ideal  $\mathfrak{h}$  of  $\mathfrak{g}$ ,  $\pi_{\mathfrak{h}} := \text{ad}|_{\mathfrak{h}} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{h})$  is an irreducible rep..
- the projection map  $p_i : \mathfrak{g} \rightarrow \mathfrak{g}_i$  is an intertwining map,

$$p_i|_{\mathfrak{g}_j} \pi_j(A) = \pi_i(A) p_i|_{\mathfrak{g}_j} \begin{cases} = 0 & i \neq j \text{ or } A \notin \mathfrak{g}_{i=j} \\ \neq 0 & i = j, A \in \mathfrak{g}_{i=j} \end{cases} \quad (6.1.18)$$

and,

$$p_i|_{\mathfrak{h}} \pi_{\mathfrak{h}}(A) = \pi_i(A) p_i|_{\mathfrak{h}} \quad (6.1.19)$$

according to Schur's lemma,  $p_i|_{\mathfrak{h}} = 0$  or isomorphism.

- $p_i|_{\mathfrak{h}}$  is a projection map, so there must be some  $i$  so that  $p_i|_{\mathfrak{h}} \neq 0$ , so  $\mathfrak{h} = \mathfrak{g}_i$  for some  $i$ .

## 6.2 Cartan subalgebra

- **def.:**  $\mathfrak{g}$  is a complex semisimple Lie algebra, its subalgebra  $\mathfrak{h}$  is called **Cartan subalgebra** if:

1. it is Abelian,
2. if for some  $A \in \mathfrak{g}$  and  $[A, H] = 0, \forall H \in \mathfrak{h}$ , then  $A \in \mathfrak{h}$ , (make sure it is maximal),
3.  $\forall H \in \mathfrak{h}, \text{ad}_H$  is diagonalizable.

some remark:

- condition 1 and 2 say that  $\mathfrak{h}$  is a **maximal Abelian subalgebra** (not contained in a larger Abelian subalgebra) of  $\mathfrak{g}$  (there may be more than one maximal Abelian subalgebra).
- $[\text{ad}_{H_1}, \text{ad}_{H_2}] = \text{ad}_{[H_1, H_2]} = 0$ , so they are **simultaneously diagonalizable**.
- the def. makes sense in any Lie algebra, but if  $\mathfrak{g}$  is not semisimple, it may not have any Cartan subalgebra.

- now, let's prove **Cartan subalgebra exists in semisimple Lie algebras**.
- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is a complex semisimple Lie algebra,  $\mathfrak{t}$  is a **maximal Abelian subalgebra of  $\mathfrak{k}$** , then, the **Cartan subalgebra** of  $\mathfrak{g}$  is,

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \quad (6.2.1)$$

**proof:**

first, let's prove  $\mathfrak{h}$  is maximal Abelian,

- $\mathfrak{h}$  is obviously Abelian.
- if  $[A, \mathfrak{h}] = \{0\}$ , for some  $A = X + iY \in \mathfrak{g}$ , then  $[X, \mathfrak{h}] = [Y, \mathfrak{h}] = \{0\}$ , which means  $\mathfrak{t}$  is not maximal.

now, let's show that  $\text{ad}_H, \forall H \in \mathfrak{h}$  are diagonalizable,

- choose inner product shown in (5.2.14), so  $\text{ad}_X$  is skew self-adjoint for all  $X \in \mathfrak{k}$ , which means it is diagonalizable.
- $\text{ad}_T, \forall T \in \mathfrak{t}$  is diagonalizable, and  $[\text{ad}_T, \text{ad}_H] = 0, \forall H \in \mathfrak{h}$ , so  $\text{ad}_H, \forall H \in \mathfrak{h}$  are simultaneously diagonalizable.

- **def.:** the **rank**,  $r = \dim \mathfrak{h}$ , of a semisimple Lie algebra is the dimension of any of its Cartan subalgebras.
  - any two Cartan subalgebra  $\mathfrak{h}_1, \mathfrak{h}_2$  of a semisimple Lie algebra are isomorphic to each other (?).

## 6.3 roots and root spaces

- from now on, we only consider the Cartan subalgebra in (6.2.1),  $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ .
- def.:** a **nonzero** element  $\alpha \in \mathfrak{h}$  (because  $\langle \alpha | \in \mathfrak{h}^*$ ) is called a **root** if there exists a nonzero  $A \in \mathfrak{g}$  s.t.,

$$[H, A] = \langle \alpha, H \rangle A \quad (6.3.1)$$

for all  $H \in \mathfrak{h}$ .

- the inner product (on  $\mathfrak{h}$ ) is arbitrarily chosen.
- the set of all root is denoted as  $R = \{\alpha\}$ .
- if we choose the inner product in (5.2.14), then, for all root  $\alpha \in i\mathfrak{t}$ .

**proof:**

- choose  $H \in \mathfrak{t}$ ,  $\text{ad}_H$  is skew self-adjoint under the chosen inner product.
- the eigenvalue  $\langle \alpha, H \rangle$  is pure imaginary (and nonzero).
- the inner product is real on  $\mathfrak{k}$ .
- so,  $\alpha \in i\mathfrak{k} \cap \mathfrak{h} = i\mathfrak{t}$ .

- def.:** for a root  $\alpha$ , the **root space** is,

$$\mathfrak{g}_{\alpha} = \{A \in \mathfrak{g} | [H, A] = \langle \alpha, H \rangle A, \forall H \in \mathfrak{h}\} \quad (6.3.2)$$

a nonzero element of  $\mathfrak{g}_{\alpha}$  is called a **root vector**.

- more generally, for any element  $\alpha \in \mathfrak{h}$ , we can define  $\mathfrak{g}_{\alpha}$  as in (6.3.2), but we don't call it a root space unless  $\alpha$  is a root.
  - \* if  $\alpha$  is not a root, then,  $\mathfrak{g}_{\alpha}$  is either  $\{0\}$  ( $\alpha \neq 0$ ) or  $\mathfrak{h}$  ( $\alpha = 0$ ).
  - \* by def.  $[\mathfrak{h}, \mathfrak{g}_{\alpha}] = \mathfrak{g}_{\alpha}$ .
- the complex semisimple Lie algebra decomposes as,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha} \quad (6.3.3)$$

and  $\mathfrak{h} \cap \mathfrak{g}_{\alpha} = \mathfrak{g}_{\alpha} \cap \mathfrak{g}_{\beta} = \{0\}$ , furthermore,  $\mathfrak{h}$  and  $\mathfrak{g}_{\alpha}, \forall \alpha \in R$  are linearly independent.

note that  $\oplus$  is **not Lie algebra direct sum**, as that  $\mathfrak{h}, \mathfrak{g}_{\alpha}$  are not ideals.

**proof:**

$\text{ad}_H, H \in \mathfrak{h}$  can be simultaneously diagonalized, so, according to (A.3.9) in appendix A.3,

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}} \mathfrak{g}_{\alpha} \quad (6.3.4)$$

and  $\mathfrak{g}_{\alpha} \cap \mathfrak{g}_{\beta} = \{0\}, \forall \alpha \neq \beta \in \mathfrak{h}$ .

but if  $\alpha = 0$ ,  $\mathfrak{g}_0 = \mathfrak{h}$  and if  $\alpha \neq 0$  and not a root,  $\mathfrak{g}_{\alpha} = \{0\}$ , so...

- for any  $\alpha, \beta \in \mathfrak{h}$ , we have,

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta} \quad (6.3.5)$$

**proof:**

for all  $A \in \mathfrak{g}_{\alpha}, B \in \mathfrak{g}_{\beta}$ ,

$$[H, [A, B]] = -[B, [H, A]] - [A, [B, H]] = \langle \alpha + \beta, H \rangle [A, B] \quad (6.3.6)$$

- two useful propositions:

- if  $\alpha$  is a root, so does  $-\alpha$ , and for all  $A = X + iY \in \mathfrak{g}_{\alpha}, A^* = X - iY \in \mathfrak{g}_{-\alpha}$  (where  $X, Y \in \mathfrak{k}$ ).

**proof:**

for any  $H \in \mathfrak{t}$ ,

$$[H, A^*] = ([H, A])^* = (\langle \alpha, H \rangle)^* A^* \quad (6.3.7)$$

and because  $\alpha \in i\mathfrak{t}$ , so  $(\langle \alpha, H \rangle)^* = -\langle \alpha, H \rangle$ .

–  $\text{span}(R) = \mathfrak{h}$ .

**proof:**

if the root doesn't span  $\mathfrak{h}$ , then there nonzero exists  $H \in \mathfrak{h}$  s.t.,

$$\langle \alpha, H \rangle = 0, \forall \alpha \in R \implies [H, A] = 0, \forall A \in \mathfrak{g} \quad (6.3.8)$$

i.e.  $H$  is in the center of  $\mathfrak{g}$ , which contradicts to semisimplicity of  $\mathfrak{g}$  (without nontrivial center).

### 6.3.1 subalgebras isomorphic to $\mathfrak{su}(2)_{\mathbb{C}}$

- for each root  $\alpha \in R$ , we have the **coroot**,

$$H_{\alpha} = 2 \frac{\alpha}{\langle \alpha, \alpha \rangle} \in \mathfrak{h} \quad (6.3.9)$$

associated to it, and  $\forall A_{\alpha} \in \mathfrak{g}_{\alpha}, B_{\alpha} \in \mathfrak{g}_{-\alpha}$  there is,

$$\begin{cases} [H_{\alpha}, A_{\alpha}] = 2A_{\alpha} \\ [H_{\alpha}, B_{\alpha}] = -2B_{\alpha} \\ [A_{\alpha}, B_{\alpha}] = H_{\alpha} \end{cases} \quad (\text{with } \mathbf{normalization}) \quad (6.3.10)$$

and  $B_{\alpha} = -A_{\alpha}^*$  (as part of the normalization).

**proof:**

for all  $A \in \mathfrak{g}_{\alpha}, B \in \mathfrak{g}_{-\alpha}, H \in \mathfrak{h}$ , then  $[A, B] \in \mathfrak{h}$  and,

$$[A, B] = \langle -A^*, B \rangle \alpha \quad (6.3.11)$$

**proof:**

–  $[A, B] \in \mathfrak{h}$  because  $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$  and  $\mathfrak{g}_0 = \mathfrak{h}$ .

– and,

$$\begin{aligned} \langle H, [A, B] \rangle &= \langle \text{ad}_A^{\dagger} H, B \rangle = \langle \text{ad}_{-A^*} H, B \rangle \\ &= \langle [H, A^*], B \rangle = \langle \langle -\alpha, H \rangle A^*, B \rangle \\ &= \langle H, \alpha \rangle \langle -A^*, B \rangle \end{aligned} \quad (6.3.12)$$

for all  $H \in \mathfrak{h}$ , so,

$$[A, B] = \langle -A^*, B \rangle \alpha \quad (6.3.13)$$

choose the **normalization**,

$$\begin{cases} B_{\alpha} = -A_{\alpha}^* \\ \langle A_{\alpha}, A_{\alpha} \rangle^* \langle \alpha, \alpha \rangle = 2 \end{cases} \iff \begin{cases} H = [A, -A^*] = \langle A, A \rangle^* \alpha \\ H_{\alpha} = \frac{2}{\langle \alpha, H \rangle} H \\ A_{\alpha} = \sqrt{\frac{2}{\langle \alpha, H \rangle}} A \\ B_{\alpha} = -A_{\alpha}^* \end{cases} \quad (6.3.14) \quad \text{notice } \langle \alpha, H \rangle \in \mathbb{R}$$

$\forall A \in \mathfrak{g}_{\alpha}$  (notice that  $\langle \alpha, \alpha \rangle \in \mathbb{R}^+$  and  $\langle A, A \rangle = \langle X, X \rangle + \langle Y, Y \rangle - 2\text{Im} \langle X, Y \rangle \in \mathbb{R}, \forall A \in \mathfrak{g}$ ).



- compare  $\text{span}(H_\alpha, A_\alpha, B_\alpha)_\mathbb{C}$  with  $\mathfrak{su}(2)_\mathbb{C}$ , we have,

$$H_\alpha \mapsto 2J_3 \quad A_\alpha \mapsto \sqrt{2}J_+ \quad B_\alpha \mapsto \sqrt{2}J_- \quad (6.3.15)$$

- from the complex subalgebra  $\mathfrak{s}^\alpha = \text{span}(H_\alpha, A_\alpha, B_\alpha)$ , we can conclude that,

1. if  $\alpha$  and  $c\alpha$  are both roots, then  $c = \pm 1$ ,
2.  $\dim \mathfrak{g}_\alpha = 1$  for all root spaces.

**proof:**

consider  $A_{c\alpha} \in \mathfrak{g}_{c\alpha}$ ,

$$[H_\alpha, A_{c\alpha}] = \underbrace{\langle c\alpha, H_\alpha \rangle}_{=2c^*} A \quad (6.3.16)$$

$2c^*$  is an eigenvalue of  $\text{ad}_{H_\alpha} \in \text{End}(\mathfrak{g})$ , which is a finite-dim. rep. of  $\mathfrak{su}(2)_\mathbb{C}$ , so the eigenvalue must be an integer, i.e.,

$$2c^*, 2\frac{1}{c^*} \in \mathbb{Z} \implies c = \pm 1, \pm 2, \pm \frac{1}{2} \quad (6.3.17)$$

let  $\pm\alpha, \pm 2\alpha$  (notice  $\pm 4\alpha$  are not roots) be all the roots  $\propto \alpha$ , then let,

$$V^\alpha = \text{span}(H_\alpha) \oplus \bigoplus_{\beta=\pm\alpha, \pm 2\alpha} \mathfrak{g}_\beta \quad (6.3.18)$$

where  $\oplus$  is not Lie algebra direct sum.

$V^\alpha \supseteq \mathfrak{s}^\alpha$  is a subalgebra of  $\mathfrak{g}$ .

**proof:**

for all  $\beta, \beta' = \pm\alpha, \pm 2\alpha$ , we have,

- according to (6.3.11),  $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \propto \alpha \propto H_\alpha$ .
- $[H_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_\beta$ .
- $[\mathfrak{g}_\beta, \mathfrak{g}_{\beta'}] \subseteq \mathfrak{g}_{\beta+\beta'} = \mathfrak{g}_{\pm 2\alpha}$  or  $\{0\}$  (where  $\beta + \beta' \neq 0$ ).

now, let's prove  $V^\alpha = \mathfrak{s}^\alpha$ ,

- consider the 'unitary' (skew self-adjoint) rep.  $(\text{ad}, V^\alpha)$  of  $\text{span}(H_\alpha, A_\alpha, B_\alpha) \simeq \mathfrak{su}(2)_\mathbb{C}$ ,  $\mathfrak{s}^\alpha$  is the invariant subspace of the rep., and the rep. is completely reducible, so  $\mathfrak{s}^{\alpha\perp}$  is also an invariant subspace.
- the eigenvalues of  $\text{ad}_{H_\alpha}$  in  $V^\alpha$  are 0 and  $\langle \beta, H_\alpha \rangle = \pm 2, \pm 4$ .
- recall the property of the eigenvalues of  $\pi(H)$ , 0 must be one of the eigenvalues of  $\text{ad}_{H_\alpha}$  in the rep.  $(\text{ad}, \mathfrak{s}^{\alpha\perp})$ , which is **impossible** since  $H_\alpha \in \mathfrak{s}^\alpha$  is the only vector with eigenvalue 0.
- so,  $\mathfrak{s}^{\alpha\perp} = \{0\}$ , i.e. the only roots  $\propto \alpha$  are  $\pm\alpha$ , and,

$$\text{span}(H_\alpha) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{s}^\alpha \equiv \text{span}(H_\alpha, A_\alpha, B_\alpha) \quad (6.3.19)$$

i.e.  $\mathfrak{g}_\alpha = \text{span}(A_\alpha)$  or  $\dim \mathfrak{g}_\alpha = 1$ .

- a rephrase of (6.3.3): for all  $A \in \mathfrak{g}$ ,  $A$  is either a root or in a root space, and,

$$\begin{cases} \mathfrak{s}^\alpha \cap \mathfrak{s}^\beta = \{0\} & \alpha \neq \pm\beta \\ \mathfrak{s}^\alpha = \mathfrak{s}^{-\alpha} & H_\alpha = -H_{-\alpha} \quad A_\alpha = B_{-\alpha} \quad B_\alpha = A_{-\alpha} \end{cases} \quad (6.3.20)$$

- $\mathfrak{s}^\alpha, \mathfrak{h}, \mathfrak{g}_\alpha, \forall \alpha \in R$  are not ideals.
- the set of roots,  $R$ , may not be linearly independent.
  - the maximal set of linearly independent roots is called the **simple root**.
  - but  $\mathfrak{g}_\alpha, \forall \alpha \in R$  are linearly independent, as stated in (6.3.3).

### 6.3.2 root systems

- for all roots  $\alpha, \beta \in R \subset \mathfrak{it}$ , we have,

$$\langle \alpha, H_\beta \rangle = 2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \quad (6.3.21)$$

**proof:**

consider  $\mathfrak{s}^\beta = \text{span}(H_\beta, A_\beta, B_\beta)$ , and its adjoint representation  $\text{ad} : \mathfrak{s}^\beta \rightarrow \text{End}(\mathfrak{g}/\mathfrak{h})$  (which is finite dimensional),

$$[H_\beta, A_\alpha] = \langle \alpha, H_\beta \rangle A_\alpha \quad (6.3.22)$$

the eigenvalue of  $\text{ad}_{H_\beta}$  must be an integer, according to (9.1.6), so,

$$\langle \alpha, H_\beta \rangle \in \mathbb{Z} \quad (6.3.23)$$

- the **projection** of  $\alpha$  to  $\beta$  ( $\alpha \cdot \hat{e}_\beta$ ) is a (half-)integer multiple of  $|\beta|$ ,

$$\frac{\langle \alpha, \beta \rangle}{\sqrt{\langle \beta, \beta \rangle}} = (0, \pm \frac{1}{2}, \pm 1, \dots) |\beta| \quad (6.3.24)$$

- summary:

- the roots span  $\mathfrak{it}$ .
- if  $\alpha \in R$ , the only multiples of  $\alpha$  in  $R$  is  $-\alpha$ .
- $\alpha \in R$ , then  $s_\beta \alpha \in R$ , where  $s_\beta = I - 2 \frac{|\beta\rangle\langle\beta|}{\langle\beta,\beta\rangle}$  (see (6.5.2)).
- for all  $\alpha, \beta \in R$ , their inner product  $2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$ .

any such collection of vectors is called a **root system**.

## 6.4 Cartan's criterion

- **Cartan's criterion for simplicity:**

complex Lie algebra  $\mathfrak{g}$  is semisimple  $\iff$  its Killing form is non-degenerate.

**proof:**

first, let's prove  $\implies$ ,

- consider,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad (6.4.1)$$

(where  $\oplus$  is the vector space direct sum) and the adjoint representation is  $\text{ad} : \mathfrak{s}^\alpha \rightarrow \text{End}(\mathfrak{s}^\alpha)$ .  
and notice  $\mathfrak{h} = \text{span}(R)$ .

- so, for any  $\alpha \in R$ , we have,

$$\begin{cases} H_\alpha & B(H_\alpha, H_\alpha) = 8 \\ A_\alpha \text{ or } B_\alpha & B(A_\alpha, B_\alpha) = 4 \end{cases} \quad (6.4.2)$$

- so, for all  $A \neq 0 \in \mathfrak{g}$ , there exists some  $B \in \mathfrak{g}$  s.t.  $B(A, B) \neq 0$ , i.e. the Killing form is non-degenerate.

---

now, let's prove  $\impliedby$ ,

- first, the center  $\mathfrak{z} = \{0\}$ , otherwise, there exists some  $A \in \mathfrak{g}$  s.t.  $\text{ad}_A = 0$ , which contradicts to the non-degeneracy.
- second, the adjoint rep. of  $\mathfrak{g}$  is completely reducible, otherwise, **the Killing form is degenerate (?)**.

## 6.5 the Weyl group (from the Lie algebra approach)

- **def.:** for each root  $\alpha \in R$ , define a linear map,

$$s_\alpha = I - \overbrace{|\alpha\rangle\langle H_\alpha|}^{=2\frac{|\alpha\rangle\langle\alpha|}{\langle\alpha,\alpha\rangle}} : \mathfrak{h} \rightarrow \mathfrak{h} \text{ or } i\mathfrak{t} \rightarrow i\mathfrak{t} \\ H \mapsto H - \alpha \langle H_\alpha, H \rangle \quad (6.5.1)$$

notice  $s_\alpha$  is the reflection about the hyperplane orthogonal to  $\alpha$ , i.e.,

- $s_\alpha |H\rangle = |H\rangle$  for all  $|H\rangle$  orthogonal to  $\alpha$ .
- $s_\alpha |\alpha\rangle = -|\alpha\rangle$ .

also notice  $s_\alpha = s_{-\alpha}$  and  $s_\alpha^2 = I$ .

- **def.:** the **Weyl group** is  $W = \langle \{s_\alpha, \alpha \in R\} \rangle$ , i.e. every element in  $W$  can be expressed as a combination of finite  $s_\alpha, \alpha \in R$ .
  - $W$  is a subgroup of the orthogonal group  $O(i\mathfrak{t})$ .

- for all  $\alpha \in R, w \in W$ ,

$$w|\alpha\rangle \in R \quad (6.5.2)$$

**proof:**

equivalently, we need to prove for all  $\alpha, \beta \in R$ ,

$$s_\alpha |\beta\rangle \in R \quad (6.5.3)$$

notice that for all  $H \in \mathfrak{h}$ ,

$$\begin{cases} \text{Ad}_{S_\alpha} H = s_\alpha |H\rangle \implies \text{Ad}_{S_\alpha} \text{ad}_H \text{Ad}_{S_\alpha}^{-1} = \text{ad}_{s_\alpha |H\rangle} \\ \text{Ad}_{S_\alpha}^{-1} H = s_\alpha |H\rangle \implies \text{Ad}_{S_\alpha}^{-1} \text{ad}_H \text{Ad}_{S_\alpha} = \text{ad}_{s_\alpha |H\rangle} \end{cases} \quad (6.5.4)$$

where  $\text{Ad}_{S_\alpha} = e^{\text{ad}_{A_\alpha}} e^{-\text{ad}_{B_\alpha}} e^{\text{ad}_{A_\alpha}} \in \text{End}(\mathfrak{g})$ .

**proof:**

notice that if  $\langle \alpha, H \rangle = 0$ , then  $[H, A_\alpha] = [H, B_\alpha] = 0$ , which implies  $[\text{ad}_H, \text{ad}_{A_\alpha} \text{ or } B_\alpha] = 0$ , so,

$$\begin{cases} \text{Ad}_{S_\alpha}^{-1} H = e^{-\text{ad}_{A_\alpha}} e^{\text{ad}_{B_\alpha}} e^{-\text{ad}_{A_\alpha}} H = H & \langle \alpha, H \rangle = 0 \\ \text{Ad}_{S_\alpha}^{-1} H = -H & H \propto \alpha \end{cases} \quad (6.5.5)$$

consider any  $H \in \mathfrak{h}$  and  $A_\beta \in \mathfrak{g}_\beta$  with  $\beta \in R$ ,

$$\text{Ad}_{S_\alpha} A_\beta \in \mathfrak{g} \quad (6.5.6)$$

and,

$$\begin{aligned} [H, \text{Ad}_{S_\alpha} A_\beta] &= \text{ad}_H \text{Ad}_{S_\alpha} A_\beta \\ &= \text{Ad}_{S_\alpha} (\text{Ad}_{S_\alpha}^{-1} \text{ad}_H \text{Ad}_{S_\alpha}) A_\beta \\ &= \text{Ad}_{S_\alpha} [s_\alpha H, A_\beta] = \langle \beta, s_\alpha H \rangle \text{Ad}_{S_\alpha} A_\beta \end{aligned} \quad (6.5.7)$$

and notice that  $\alpha \in i\mathfrak{t} \implies s_\alpha^\dagger = s_\alpha$ , so,

$$[H, \text{Ad}_{S_\alpha} A_\beta] = \langle s_\alpha \beta, H \rangle \text{Ad}_{S_\alpha} A_\beta \quad (6.5.8)$$

which means  $s_\alpha \beta \in R$  and  $\text{Ad}_{S_\alpha} A_\beta \in \mathfrak{g}_{s_\alpha \beta}$ .

- the Weyl group is **finite**.

**proof:**

since there are only finite roots,  $s_\alpha$  (which is reversible) is nothing but a **permutation** of the roots, so is every element in the Weyl group.

## 6.6 simple Lie algebras

- recall the def. of simple Lie algebra in section 3.2.1.
- see (6.1.13),  $\mathfrak{g}$  is simple  $\implies \mathfrak{g}$  is semisimple (不会证).

- $\mathfrak{g}_\mathbb{C}$  is simple  $\implies \mathfrak{g}$  is also simple.  
but,  $\mathfrak{g}$  is simple  $\not\implies \mathfrak{g}_\mathbb{C}$  is not necessarily simple.

**proof:**

- $\dim \mathfrak{g} = \dim \mathfrak{g}_\mathbb{C} \geq 2$ .
- if  $\mathfrak{g}$  has a nontrivial ideal,  $\mathfrak{h}$ , then  $\mathfrak{h}_\mathbb{C}$  is a nontrivial ideal of  $\mathfrak{g}_\mathbb{C}$ .

- **def.:** a real Lie algebra,  $\mathfrak{g}$ , is said to **admit a complex structure** if it is isomorphic to a complex Lie algebra,  $\mathfrak{h}$ ,

$$\begin{aligned} \phi : \mathfrak{g} &\rightarrow \mathfrak{h} \\ A &\mapsto \phi_1(A) + i\phi_2(A) \end{aligned} \quad (6.6.1)$$

and,

$$\phi([A, B]) = [\phi(A), \phi(B)] \implies \begin{cases} \phi_1([A, B]) = [\phi_1(A), \phi_1(B)] - [\phi_2(A), \phi_2(B)] \\ \phi_2([A, B]) = [\phi_1(A), \phi_2(B)] + [\phi_2(A), \phi_1(B)] \end{cases} \quad (6.6.2)$$

and  $\phi_1, \phi_2$  are not one-to-one.

- equivalently, there exists a "multiplication by  $i$ " map on  $\mathfrak{g}$ ,  $J : \mathfrak{g} \rightarrow \mathfrak{g}$ , s.t.,

$$J^2 = -I \quad \text{and} \quad [A, B + JC] = [A, B] + J[A, C] \quad (6.6.3)$$

**proof:**

let's prove def. 1.  $\implies$  there exists a  $J$  on  $\mathfrak{g}$ ,

- let  $J = (\phi^{-1} \circ iI \circ \phi) \in \text{End}(\mathfrak{g})$ .
- for all  $X \in \mathfrak{h}$ , there exists some  $A = \phi^{-1}X$ , so,

$$\begin{aligned} (\phi \circ J)A &= (\phi \circ J \circ \phi^{-1})X = iX = i\phi(A) \\ \implies \phi([A, JB]) &= [\phi(A), i\phi(B)] = i\phi([A, B]) = \phi(J[A, B]) \end{aligned} \quad (6.6.4)$$

- a non-Abelian compact Lie algebra,  $\mathfrak{k}$ , doesn't admit a complex structure.

**proof:**

- if  $\mathfrak{k}$  admits a complex structure, it has a "multiplication by  $i$ " map,  $J \in \text{End}(\mathfrak{k})$ .
- choose the inner product on  $\mathfrak{k}$ , so that  $\text{ad}_X, \forall X \in \mathfrak{k}$  are skew self-adjoint, hence diagonalizable in  $\mathbb{C}$ , with pure-imaginary (not all-zero) eigenvalues.

- \*  $\mathfrak{k} \simeq \mathfrak{h}$  where  $\mathfrak{h}$  is a complex Lie algebra.
- \* there exists  $H = \phi(X) \in \mathfrak{h}$  and  $A = \phi(Y) \in \mathfrak{h}$ , s.t.,

$$\phi([X, Y]) = i\phi(Y) \implies [X, Y] = JaY \quad (6.6.5)$$

where  $a \in \mathbb{R}$  since  $\text{ad}_X$  has pure imaginary eigenvalues.

\* which is **impossible**, because  $\text{ad}_{JX}$  has real eigenvalue,

$$[JX, Y] = -aY \quad (6.6.6)$$

- $\mathfrak{k}$  is the Lie algebra of a compact Lie group, then,  $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is simple  $\iff \mathfrak{k}$  is simple.

**proof:**

we only need to prove  $\Leftarrow$ ,

- $\mathfrak{k}$  is simple  $\implies$  without a nontrivial center  $\implies \mathfrak{g}$  is semisimple  $\implies$  is a direct sum of simple Lie algebras (and the decomposition is unique up to ordering, see (6.1.13)),

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{g} = \bigoplus_i \mathfrak{g}_i \quad (6.6.7)$$

- if  $\mathfrak{g}_i$  is a simple ideal of  $\mathfrak{g}$ , so is  $\mathfrak{g}_i^* = \{A^* | A \in \mathfrak{g}_i\}$ , which (together with the uniqueness of decomposition) implies  $\mathfrak{g}_i^* = \mathfrak{g}_j$  for some  $j$

\* if  $\mathfrak{g}_i^* = \mathfrak{g}_i$ , then  $\mathfrak{g}_i \cap \mathfrak{k}$  is a nontrivial ideal of  $\mathfrak{k}$ , contradicts to simplicity.

\* if  $\mathfrak{g}_i^* = \mathfrak{g}_j$  with  $i \neq j$ , then let  $\mathfrak{g}' = \mathfrak{g}_i \cup \mathfrak{g}_i^*$ , we have  $\mathfrak{g}'^* = \mathfrak{g}'$ , thus  $\mathfrak{g}' \cap \mathfrak{k}$  is a nontrivial ideal of  $\mathfrak{k}$ , unless  $\mathfrak{g}' = \mathfrak{g}$ .

now, let's discuss what happens if  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^*$ , where  $\mathfrak{g}_1, \mathfrak{g}_1^*$  are both simple ideals of  $\mathfrak{g}$ .

- define a linear map (projection),

$$\begin{aligned} \phi : \mathfrak{g}_1 &\rightarrow \mathfrak{k} \\ A &\mapsto \frac{1}{2}(A + A^*) \end{aligned} \quad (6.6.8)$$

notice that for all  $A \in \mathfrak{g}_1$ , we have  $A^* \in \mathfrak{g}_1^*$ , thus  $[A, A^*] = 0$ , so,

$$\phi([A, B]) = \frac{1}{2}([A, B] + [A^*, B^*]) = \frac{1}{2}([A + A^*, B + B^*]) = [\phi(A), \phi(B)] \quad (6.6.9)$$

\* furthermore,  $\phi$  is **one-to-one**, because,

$$A + A^* = B + B^* \implies A - B = B^* - A^* \in \mathfrak{g}_1 \cap \mathfrak{g}_1^* = \{0\} \implies A = B \quad (6.6.10)$$

\*  $\phi$  is also **on-to**, because as a complex Lie algebra,  $\mathfrak{g}_1$  has the same dimension of the real Lie algebra,  $\mathfrak{k}$ , thus for every  $X \in \mathfrak{k}$ , there exists some  $A \in \mathfrak{g}_1$ , s.t.  $X = \phi(A)$ .

- so,  $\mathfrak{k}$  is isomorphic to a complex Lie algebra  $\mathfrak{g}_1$ , i.e. it **admits a complex structure**, which contradicts to compactness.

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is simple.

- $\mathfrak{g}$  is not simple  $\iff \mathfrak{h}$  decomposes into  $\mathfrak{h}_1 \oplus \mathfrak{h}_2$  and  $\mathfrak{h}_1 \perp \mathfrak{h}_2$  (orthogonal direct sum), and every root is either in  $\mathfrak{h}_1$  or  $\mathfrak{h}_2$ .

where,

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is a complex semisimple Lie algebra.
- $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$  is the complexification of the maximal Abelian subalgebra of  $\mathfrak{k}$ ,  $\mathfrak{t}$ , i.e. the Cartan subalgebra.

**proof:**

first, let's prove  $\implies$ ,

- $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$  is not simple  $\implies \mathfrak{k}$  is not simple (from the theorem above)  $\implies \mathfrak{k}_1$  is the nontrivial ideal of  $\mathfrak{k}$ , i.e. an invariant subspace of  $\text{ad} : \mathfrak{k} \rightarrow \text{End}(\mathfrak{k})$ .

- notice the adjoint representation on  $\mathfrak{k}$  is completely reducible, there is another ideal  $\mathfrak{k}_2$  s.t.  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ .
- \* if we choose the inner product so that the adjoint rep. on  $\mathfrak{k}$  is unitary, then  $\mathfrak{h}_1 \perp \mathfrak{h}_2$  (see section 1.2).
- now, we have  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_1^\perp$ , which implies  $\mathfrak{k}_\mathbb{C} = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , where  $\mathfrak{g}_i = \mathfrak{k}_{i\mathbb{C}}$ , and, of course,  $\mathfrak{g}_1 \perp \mathfrak{g}_2$ .
- the maximal Abelian subalgebra,  $\mathfrak{t}$ , decomposes as  $\mathfrak{t}_1 \oplus \mathfrak{t}_2$ , where  $\mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{k}_i$ .

**proof:**

- \* consider  $T = X + Y \in \mathfrak{t}$  with  $X \in \mathfrak{k}_1$  and  $Y \in \mathfrak{k}_2$ , then,

$$[T_1, T_2] = \underbrace{[X_1, X_2]}_{\in \mathfrak{k}_1} + \underbrace{[Y_1, Y_2]}_{\in \mathfrak{k}_2} = 0 \quad (6.6.11)$$

notice that  $\mathfrak{k}_1, \mathfrak{k}_2$  are linearly independent, so,  $[X_1, X_2] = [Y_1, Y_2] = 0$ .

- \* which means  $[X, \mathfrak{t}] = \{0\}$ , but  $\mathfrak{t}$  is maximal, so  $X \in \mathfrak{t} \cap \mathfrak{k}_1$ , similarly,  $Y \in \mathfrak{t} \cap \mathfrak{k}_2$ .
- \* so,  $\mathfrak{t}_1 = \mathfrak{t} \cap \mathfrak{k}_1$  and  $\mathfrak{t}_2 = \mathfrak{t} \cap \mathfrak{k}_2$ , then, we have the Lie algebra direct sum,  $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ .

- consequently, the Cartan subalgebra decomposes as  $\mathfrak{t}_\mathbb{C} = \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ , with  $\mathfrak{h}_i = \mathfrak{t}_{i\mathbb{C}}$ , and, of course,  $\mathfrak{h}_1 \perp \mathfrak{h}_2$ .
- every root is either in  $\mathfrak{h}_1$  or  $\mathfrak{h}_2$ .

**proof:**

- \* let  $R_i$  be the roots for  $\mathfrak{g}_i$  in  $\mathfrak{h}_i$ .  
(i.e., excuse the sloppy notation, there exists a nonzero  $A \in \mathfrak{g}_i$  s.t.  $[\mathfrak{h}_i, A] = \langle R_i, \mathfrak{h}_i \rangle A$ ).
- \* now, we claim  $R_{i=1,2} \subset R$ , because for all  $\alpha \in R_1$ ,

$$[H_1 + H_2, A] = \langle \alpha, H_1 \rangle A + 0 = \langle \alpha, H_1 + H_2 \rangle A \quad (6.6.12)$$

where we noticed that the root vector  $A \in \mathfrak{g}_1 = \mathfrak{k}_{1\mathbb{C}}$  and  $H_2 \in \mathfrak{t}_{2\mathbb{C}}$  commutes with  $A$ , and  $\alpha \in \mathfrak{h}_1 \perp \mathfrak{h}_2$ .

- \* notice that  $R - (R_1 \cup R_2)$  are the roots associated to root vectors neither in  $\mathfrak{g}_1$  nor  $\mathfrak{g}_2$ .  
· consider  $A = A_1 + A_2$ , with  $A_i \in \mathfrak{g}_i$ , is a root vector of  $\alpha \in R$ , then, consider,

$$\begin{aligned} [H_1, A_1 + A_2] &= [H_1, A_1] = \langle \alpha, H_1 \rangle A_1 \propto A_1 + A_2 \\ \implies \text{either } A_2 &= 0 \text{ or } \langle \alpha, H_1 \rangle = 0 \end{aligned} \quad (6.6.13)$$

so, if  $A_2 = 0$ , then  $\alpha \in R_1$ , else,  $\alpha \in \mathfrak{h}_2$ , which means  $\alpha \in R_2$ .

- \* so, either  $\alpha$  is in  $R_1$  or in  $R_2$ .

- $\implies$  is proved.

now, let's prove  $\Leftarrow$ ,

- $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$  with  $\mathfrak{h}_1 \perp \mathfrak{h}_2$ , and  $R_i = R \cap \mathfrak{h}_i$ .
- then,  $\mathfrak{g}$  decomposes as,

$$\mathfrak{g} = \overbrace{\left( \mathfrak{h}_1 \oplus \bigoplus_{\alpha \in R_1} \mathfrak{g}_\alpha \right)}^{=\mathfrak{g}_1} \oplus \overbrace{\left( \mathfrak{h}_2 \oplus \bigoplus_{\beta \in R_2} \mathfrak{g}_\beta \right)}^{=\mathfrak{g}_2} \quad (6.6.14)$$

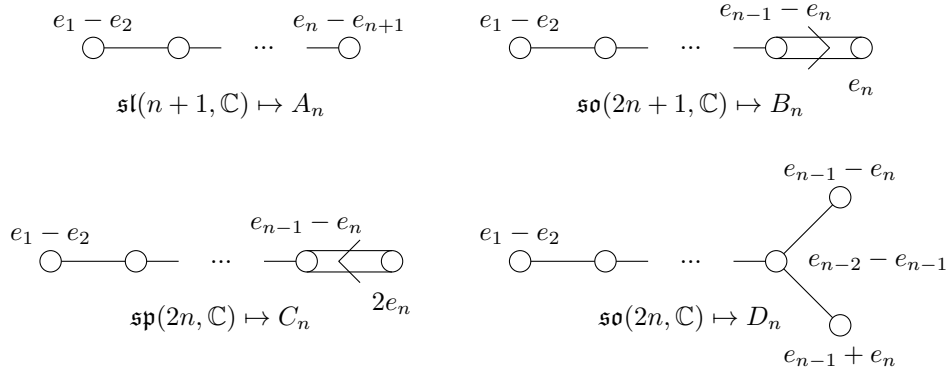
where  $\mathfrak{g}_\alpha, \forall \alpha \in R$  are linearly independent (see (6.3.3)).

- \* and it is easy to see that  $[\mathfrak{g}_\alpha, \mathfrak{h}_2] = \{0\}, \alpha \in R_1$  since  $\alpha \perp \mathfrak{h}_2$ , and,  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} = \{0\}$  if  $\alpha \in R_1, \beta \in R_2$  ( $\alpha + \beta \notin R$ ).

- so,  $\mathfrak{g}$  decomposes as the Lie algebra direct sum,  $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ , i.e. it is not simple.

## 6.7 the root systems of the classical Lie algebras

- 四个 root systems 的 Dynkin diagrams (见 section 7.6) 如下,



- $B_2$  and  $C_2$ ,  $A_3$  and  $D_3$  are isomorphic to each other.
- $D_2$  的 Dynkin diagram is not connected  $\implies D_2$  is reducible  $\implies \mathfrak{so}(4, \mathbb{C})$  is not simple,

$$\mathfrak{so}(4, \mathbb{C}) = \left( \text{span}(e_1 - e_2) \oplus \mathfrak{g}_{\pm(e_1 - e_2)} \right) \oplus \left( \text{span}(e_1 + e_2) \oplus \mathfrak{g}_{\pm(e_1 + e_2)} \right) \quad (6.7.1)$$

中间粗体的  $\oplus$  是 Lie algebra direct sum, (两个  $\mathfrak{su}(2)_{\mathbb{C}}$ ).

- $A_n, B_n, C_n, n \geq 1$  和  $D_n, n \geq 3$  都对应 simple Lie algebra,

$$\begin{array}{cccc} \mathfrak{sl}(n+1, \mathbb{C}) \mapsto A_n & \mathfrak{so}(2n+1, \mathbb{C}) \mapsto B_n & \mathfrak{sp}(2n, \mathbb{C}) \mapsto C_n & \mathfrak{so}(2n, \mathbb{C}) \mapsto D_n \\ n \geq 1 & n \geq 1 & n \geq 1 & n \geq 3 \end{array} \quad (6.7.2)$$

### 6.7.1 the special linear algebras, $\mathfrak{sl}(n+1, \mathbb{C}) = \mathfrak{su}(n+1)_{\mathbb{C}}$ , and $A_n$

- $\mathfrak{su}(n+1) = \{A \in \mathcal{M}_{n+1}(\mathbb{C}) | A^\dagger = -A \text{ and } \text{tr} A = 0\}$ , 它的 maximal commutative subalgebra 是,

$$\mathfrak{t} = \{\text{diag}(ia_1, \dots, ia_{n+1}) | a_i \in \mathbb{R} \text{ and } a_1 + \dots + a_{n+1} = 0\} \quad (6.7.3)$$

从而得到 Cartan subalgebra,  $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \{\text{diag}(\lambda_1, \dots, \lambda_{n+1}) | \lambda_i \in \mathbb{C} \text{ and } \lambda_1 + \dots + \lambda_{n+1} = 0\}$ .

- 令  $E_{ij}, i \neq j \in \{1, \dots, n+1\}$  是第  $i$  行第  $j$  列的分量为 1, 其余位置为零的矩阵,  $H = \text{diag}(\lambda_1, \dots) \in \mathfrak{h}$ , 那么,

$$[H, E_{ij}] = (\lambda_i - \lambda_j) E_{ij} \quad (6.7.4)$$

- 选择一个内积, 使得  $\text{ad}_X, \forall X \in \mathfrak{su}(n+1)$  是 skew self-adjoint,

$$\langle A, B \rangle = \text{tr}(A^\dagger B), \forall A, B \in \mathfrak{su}(n+1)_{\mathbb{C}} \quad (6.7.5)$$

**proof:**

注意这个内积在任何李代数中都保证  $\text{ad}_X, \forall X \in \mathfrak{t}$  是 skew self-adjoint, 但是根据 Cartan's criterion, 只有 semisimple 才能保证它 non-degenerate.

$$\text{tr}(A^\dagger \text{ad}_X B) = \text{tr}(A^\dagger X B - A^\dagger B X) = \text{tr}(A^\dagger X B - X A^\dagger B) = \text{tr}(-\text{ad}_X A B) \quad (6.7.6)$$

注意, 对于  $H, H' \in \mathfrak{h}$ , 有  $\langle H, H' \rangle = \sum_i \lambda_i^* \lambda'_i$ .

- 可见  $E_{ij}$  对应的 root 为,

$$[H, E_{ij}] = \underbrace{\langle e_i - e_j, H \rangle}_{=\alpha_{ij}} E_{ij}, i \neq j \quad (6.7.7)$$

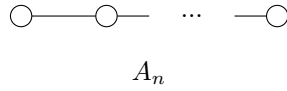
- $\mathfrak{sl}(n+1, \mathbb{C})$  对应的 root system 用  $A_n$  表示,

$E = \{v \in \mathbb{R}^{n+1} | v_1 + \dots + v_n = 0\}$ , 所以  $\dim E = n$ .

- $R = \{\alpha_{ij} = e_i - e_j | i \neq j \in \{1, \dots, n+1\}\}$ , 共有  $n(n+1)$  个根. ( $\dim \mathfrak{sl}(n+1, \mathbb{C}) = (n+1)^2 - 1$ )
- $\Delta = \{e_1 - e_2, \dots, e_n - e_{n+1}\}$  is a base, and  $R^+ = \{e_i - e_j | i < j\}$ , with,

$$e_i - e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \dots + (e_{j-1} - e_j) \quad (6.7.8)$$

- 所有根的长度为  $\sqrt{2}$ , 因此  $\langle \alpha, \beta \rangle = \langle \alpha, H_\beta \rangle$ .
- $\langle \alpha, \beta \rangle = 0, \pm 1$  (when  $\alpha \neq \pm \beta$ ).
- 两个 roots ( $\alpha \neq \pm \beta$ ) 之间的夹角可能是  $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}$ .
- 对于 base 中的根, 相邻 (consecutive) 的根夹角为  $\frac{2\pi}{3}$ , 不相邻的互相垂直, 所以其 Dynkin 图如下,



- $s_{\alpha_{ij}}$  作用到向量  $|v\rangle$  使其  $i, j$  分量的位置交换, 因此  $A_n$  的 Weyl 群是  $n+1$  个元素的 permutation group.

## 6.7.2 the orthogonal algebras, $\mathfrak{so}(2n, \mathbb{C})$ , and $D_n$

- $\mathfrak{so}(2n, \mathbb{R}) = \mathfrak{o}(2n, \mathbb{R}) = \{A \in \mathcal{M}_{2n}(\mathbb{R}) | A^T = -A\}$ , 它的 maximal commutative subalgebra 是,

$$\mathfrak{t} = \{H_a = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} | a = \text{diag}(a_1, \dots, a_n) \text{ with } a_i \in \mathbb{R}\} \quad (6.7.9)$$

**proof:**

任何  $\mathfrak{so}(2n, \mathbb{C})$  中的元素都可以展开成  $\mathfrak{h} = \mathfrak{t}_\mathbb{C}$  和  $D_{ij}^\alpha$  (见下文) 的叠加, 那么, 与  $\mathfrak{h}$  对易的元素一定不含有  $D_{ij}^\alpha$  分量, 所以... 是 maximal. (总共有  $2n^2 - 2n$  个根, 且 rank 为  $n$ , 所以总维数为  $2n^2 - n = \frac{2n(2n-1)}{2}$ )  
另外, 注意如果  $n = 2$ ,  $D_{11}^1 = D_{11}^2 = 0$  而,

$$D_{11}^3 = -D_{11}^4 = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \in \mathfrak{h} \quad (6.7.10)$$

也即  $\mathfrak{so}(2, \mathbb{C}) = \mathfrak{h}$ , 与不存在 nontrivial center 的对应不符, 不是 semisimple.

- the root vectors are  $D_{ij}^\alpha = C_{ij}^\alpha - (C_{ij}^\alpha)^T$ , where  $\alpha = 1, 2, 3, 4$  and,

$$\begin{aligned} C_{ij}^1 &= \begin{pmatrix} E_{ij} & iE_{ij} \\ iE_{ij} & -E_{ij} \end{pmatrix} & C_{ij}^2 &= \begin{pmatrix} E_{ij} & -iE_{ij} \\ -iE_{ij} & -E_{ij} \end{pmatrix} \\ C_{ij}^3 &= \begin{pmatrix} E_{ij} & -iE_{ij} \\ iE_{ij} & E_{ij} \end{pmatrix} & C_{ij}^4 &= \begin{pmatrix} E_{ij} & iE_{ij} \\ -iE_{ij} & E_{ij} \end{pmatrix} \end{aligned} \quad (6.7.11)$$

where  $i \neq j \in \{1, \dots, n\}$  (如果  $i = j$ , 那么  $D_{ii}^{1,2} = 0, D_{ii}^{3,4} \in \mathfrak{h}$ ), and we have,

$$\begin{aligned} [H_a, D_{ij}^1] &= i(a_i + a_j)D_{ij}^1 & [H_a, D_{ij}^2] &= -i(a_i + a_j)D_{ij}^2 \\ [H_a, D_{ij}^3] &= i(a_i - a_j)D_{ij}^3 & [H_a, D_{ij}^4] &= -i(a_i - a_j)D_{ij}^4 \end{aligned} \quad (6.7.12)$$

**calculation:**

we have  $D_{ij}^1 = C_{ij}^1 - C_{ji}^1, D_{ij}^2 = C_{ij}^2 - C_{ji}^2, D_{ij}^3 = C_{ij}^3 - C_{ji}^4, D_{ij}^4 = C_{ij}^4 - C_{ji}^3$ , and,

$$\begin{aligned} [H_a, C_{ij}^1] &= i(a_i + a_j)C_{ij}^1 & [H_a, C_{ij}^2] &= -i(a_i + a_j)C_{ij}^2 \\ [H_a, C_{ij}^3] &= i(a_i - a_j)C_{ij}^3 & [H_a, C_{ij}^4] &= -i(a_i - a_j)C_{ij}^4 \end{aligned} \quad (6.7.13)$$



- 内积定义为  $\langle A, B \rangle = \frac{1}{2} \text{tr}(A^\dagger B)$ , 那么,

$$\langle H_a, H_b \rangle = - \sum_{i=1}^n a_i^* b_i \quad (6.7.14)$$

所以, 可以将  $H_a$  视作  $i(a_1, \dots, a_n)$ .

- 可见 root vectors 和 roots 的对应关系为  $(i \neq j \in \{1, \dots, n\})$ ,

$$D_{ij}^1 \mapsto \alpha_{ij} = e_i + e_j \quad D_{ij}^2 \mapsto -\alpha_{ij} \quad D_{ij}^3 \mapsto \beta_{ij} = e_i - e_j \quad D_{ij}^4 \mapsto -\beta_{ij} \quad (6.7.15)$$

- $\mathfrak{so}(2n, \mathbb{C})$  对应的 root system 用  $D_n$  表示,

–  $E = \mathbb{R}^n$ .

–  $R = \{\pm e_i \pm e_j | i \neq j \in \{1, \dots, n\}\}$ , 共有  $\frac{n(n-1)}{2} \times 4 = 2n^2 - 2n$  个根. ( $\dim \mathfrak{so}(2n, \mathbb{C}) = \frac{2n(2n-1)}{2}$ )

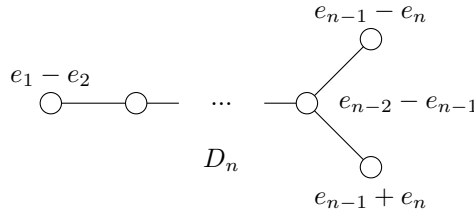
–  $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\} \cup \{e_{n-1} + e_n\}$  is a base, and  $R^+ = \{e_i - e_j | i < j\} \cup \{e_i + e_j\}$ , with,

$$e_i + e_j = \underbrace{(e_i - e_{i+1}) + \dots + (e_{n-1} - e_n)}_{=e_i + e_n} + \underbrace{(e_j - e_{j+1}) + \dots + (e_{n-1} - e_n)}_{=e_j - e_n} \quad (6.7.16)$$

– 所有根的长度为  $\sqrt{2}$ , 因此也有  $\langle \alpha, \beta \rangle = \langle \alpha, H_\beta \rangle$ .

–  $\langle \alpha, \beta \rangle = 0, \pm 1$  (when  $\alpha \neq \pm \beta$ ), 所以两个根之间的夹角可能是  $\frac{\pi}{2}$  或  $\frac{\pi}{3}, \frac{2\pi}{3}$ .

–  $D_n$  的 Dynkin 图如下,



–  $s_\alpha = s_{-\alpha}, \alpha \in R$  分别为,

$$\begin{cases} s_{\alpha_{ij}} : (\dots, v_i, \dots, v_j, \dots) \mapsto (\dots, -v_j, \dots, -v_i, \dots) \\ s_{\beta_{ij}} : (\dots, v_i, \dots, v_j, \dots) \mapsto (\dots, v_j, \dots, v_i, \dots) \end{cases} \quad (6.7.17)$$

### 6.7.3 the orthogonal algebras, $\mathfrak{so}(2n+1, \mathbb{C})$ , and $B_n$

- its maximal commutative subalgebra is,

$$\mathfrak{t} = \left\{ \left( \begin{array}{cc|c} 0 & a & \\ -a & 0 & \\ \hline & & 0 \end{array} \right) \mid a = \text{diag}(a_1, \dots, a_n) \text{ with } a_i \in \mathbb{R} \right\} \quad (6.7.18)$$

both  $\mathfrak{so}(2n+1, \mathbb{C})$  and  $\mathfrak{so}(2n, \mathbb{C})$  have rank  $n$ .

- every root in  $\mathfrak{so}(2n, \mathbb{C})$  is a root in  $\mathfrak{so}(2n+1, \mathbb{C})$ , but there are  $2n$  additional roots in  $\mathfrak{so}(2n+1, \mathbb{C})$ .
- the additional root vectors are,

$$B_k^1 = \left( \begin{array}{ccc|ccc|c} & & & & & & \vdots \\ & & & & & & 1 \\ & & & & & & \vdots \\ \hline & & & & & & \vdots \\ & & & & & & i \\ & & & & & & \vdots \\ \hline \dots & -1 & \dots & \dots & -i & \dots & 0 \end{array} \right) \quad B_k^2 = \left( \begin{array}{ccc|ccc|c} & & & & & & \vdots \\ & & & & & & 1 \\ & & & & & & \vdots \\ \hline & & & & & & \vdots \\ & & & & & & -i \\ & & & & & & \vdots \\ \hline \dots & -1 & \dots & \dots & i & \dots & 0 \end{array} \right) \quad (6.7.19)$$

其中  $B_k^{1,2}$  的非零元素位于  $(k, 2n+1), (n+k, 2n+1)$  和通过转置相对应的位置, 有对易关系,

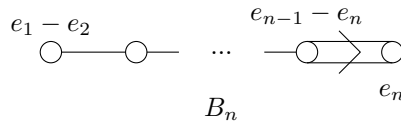
$$[H_a, B_k^1] = ia_k B_k^1 \quad [H_a, B_k^2] = -ia_k B_k^2 \quad (6.7.20)$$

- 选取与上一 subsection 一样的内积, 那么 root vectors 和 roots 的对应关系为,

$$B_k^1 \mapsto e_k \quad B_k^2 \mapsto -e_k \quad (6.7.21)$$

- $\mathfrak{so}(2n+1, \mathbb{C})$  对应的 root system 用  $B_n$  表示,

- $E = \mathbb{R}^n$ .
- $R = \{\pm e_i \pm e_j \text{ and } \pm e_k | i \neq j, k \in \{1, \dots, n\}\}$ , 共有  $2n^2$  个根. ( $\dim \mathfrak{so}(2n+1, \mathbb{C}) = \frac{(2n+1)2n}{2}$ )
- $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\} \cup \{e_n\}$  is a base, and  $R^+ = \{e_i - e_j | i < j\} \cup \{e_i + e_j\} \cup \{e_k\}$ .
- $\langle \alpha, \beta \rangle = 0, \pm 1$  (when  $\alpha \neq \pm \beta$ ), 所以两个根之间的夹角可能为  $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}$ .
- $B_n$  的 Dynkin 图如下,



#### 6.7.4 the symplectic algebras, $\mathfrak{sp}(2n, \mathbb{C})$ , and $C_n$

- $\mathfrak{sp}(2n, \mathbb{C}) = \{A \in \mathcal{M}_{2n}(\mathbb{C}) | \Omega A^T \Omega = A\}$ , where,

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (6.7.22)$$

$\mathfrak{sp}(2n, \mathbb{C})$  中的矩阵可以写成如下形式,

$$A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \quad (6.7.23)$$

where  $a, b, c \in \mathcal{M}_n(\mathbb{C})$ , and  $b, c$  are symmetric.

- 可以认为  $\mathfrak{k} = \mathfrak{sp}(2n, \mathbb{C}) \cap \mathfrak{u}(2n)$  是其 compact real form,

$$\mathfrak{sp}(2n, \mathbb{C}) \cap \mathfrak{u}(2n) = \left\{ \begin{pmatrix} a & b \\ -b^\dagger & -a^T \end{pmatrix} \mid a^\dagger = -a, b^T = b \right\} \quad (6.7.24)$$

- the maximal commutative subalgebra of  $\mathfrak{k}$  is,

$$\mathfrak{t} = \{H_a = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a = \text{diag}(a_1, \dots, a_n), ia_i \in \mathbb{R}\} \quad (6.7.25)$$

- the root vectors are ( $i \neq j$ ),

$$\begin{aligned} A_{ij} &= \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix} & B_{ij} &= \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix} & C_{ij} &= \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \\ F_k &= \begin{pmatrix} 0 & E_{kk} \\ 0 & 0 \end{pmatrix} & G_k &= \begin{pmatrix} 0 & 0 \\ E_{kk} & 0 \end{pmatrix} \end{aligned} \quad (6.7.26)$$

对易关系为,

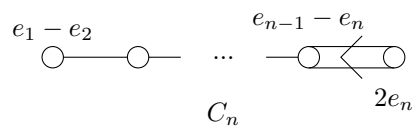
$$\begin{aligned} [H_a, A_{ij}] &= (a_i + a_j)A_{ij} & [H_a, B_{ij}] &= -(a_i + a_j)B_{ij} & [H_a, C_{ij}] &= (a_i - a_j)C_{ij} \\ [H_a, F_k] &= 2a_k F_k & [H_a, G_k] &= -2a_k G_k \end{aligned} \quad (6.7.27)$$

- 选取内积为  $\langle A, B \rangle = \frac{1}{2} \text{tr}(A^\dagger B)$ , 所以  $H_a$  可以视为  $(a_1, \dots, a_n)$ , 那么 root vectors 和 roots 的对应关系为,

$$A_{ij} \mapsto e_i + e_j \quad B_{ij} \mapsto -e_i - e_j \quad C_{ij} \mapsto e_i - e_j \quad F_k \mapsto 2e_k \quad G_k \mapsto -2e_k \quad (6.7.28)$$

- $\mathfrak{sp}(2n, \mathbb{C})$  对应的 root system 用  $C_n$  表示,

- $E = \mathbb{R}^n$ .
- $R = \{\pm e_i \pm e_j \text{ and } \pm 2e_k | i \neq j, k \in \{1, \dots, n\}\}$ , 与  $B_n$  相似 (区别是  $\pm e_k$  前的系数 2), 共有  $2n^2$  个根. ( $\dim \mathfrak{sp}(2n, \mathbb{C}) = n(2n+1)$ )
- $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\} \cup \{2e_n\}$  and  $R = \{e_i - e_j | i < j\} \cup \{e_i + e_j\} \cup \{2e_k\}$ .
- $\langle \alpha, \beta \rangle = 0, \pm 1, \pm 2$  (when  $\alpha \neq \pm \beta$ ), 所以两个根之间夹角可能为  $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}$ .
- $C_n$  的 Dynkin 图如下,



# Chapter 7

## root systems

### 7.1 abstract root systems

- **def.:** a **root system**  $(E, R)$  is a finite-dimensional vector space  $E = \text{span}(R)$ , with a finite collection of non-zero vectors  $R$ , and an inner product  $\langle \cdot, \cdot \rangle$ , and,

1.  $E = \text{span}(R)$ ,
2. if  $\alpha \in R$ , then  $c\alpha \in R \iff c = \pm 1$ ,
3. if  $\alpha, \beta \in R$ , then  $s_\alpha |\beta\rangle \in R$ , where  $s_\alpha = 1 - 2 \frac{\langle \alpha, \alpha \rangle}{\langle \alpha, \beta \rangle}$ ,
4. for all  $\alpha, \beta \in R$ ,  $2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ .

$\dim E$  is called the **rank** of the system, elements in  $R$  are called **roots**.

- **def.:** the **Weyl group**,  $W$ , of  $R$  is the finite subgroup of the orthogonal group of  $E$  generated by  $s_\alpha, \forall \alpha \in R$ .

- **def.:**  $(E, R)$  and  $(F, S)$  are two root systems, then  $(E \oplus F, R \cup S)$  is a root system, and  $R \cup S$  is called the **direct sum** of  $R$  and  $S$ .

(it is easy to see the direct sum root system satisfies the def. of root systems)

- **def.:** a root system is called **reducible** if there exists an orthogonal decomposition  $E = E_1 \oplus E_2$  with  $E_1 \perp E_2$  and  $\dim E_i > 0$ , and every root is either in  $E_1$  or  $E_2$ .

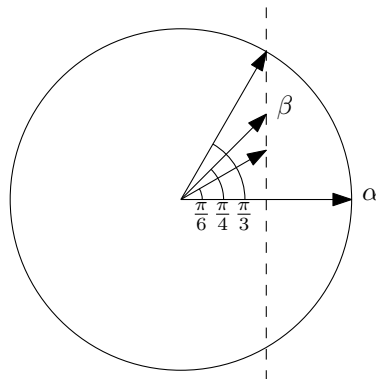
- the root system of a semisimple Lie algebra is irreducible  $\iff$  the semisimple Lie algebra is simple (见 section 6.6 最后一个定理).

- **def.:** an **isomorphism** is a linear map that **preserves the reflection**, not the inner product,

$$A : E \rightarrow F \quad \text{s.t.} \quad As_\alpha |\beta\rangle = s_{A\alpha} |A\beta\rangle \quad (7.1.1)$$

- 对于  $\langle \beta, \beta \rangle \leq \langle \alpha, \alpha \rangle$ , 且  $\beta \not\propto \alpha$ , 根  $\alpha, \beta$  之间可能的关系如下,

- $\beta \perp \alpha$ .
- or,  $\langle \alpha, \alpha \rangle = 1, 2, 3 \langle \beta, \beta \rangle$  (图中没有画出  $\beta \mapsto -\beta$  的情况, 那时夹角是图中夹角的补角).



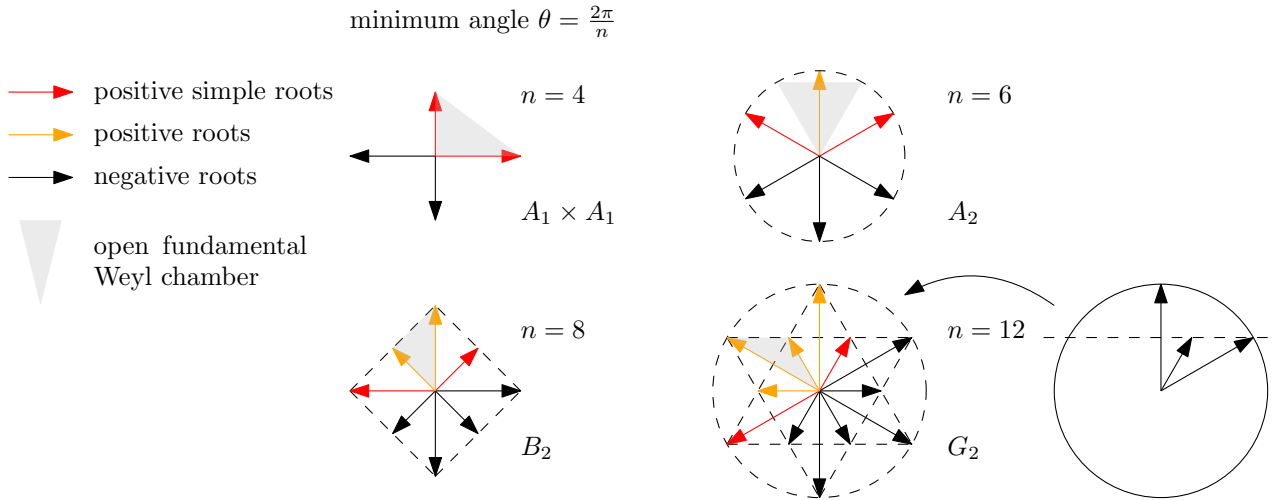
- 如果根  $\alpha, \beta$  之间夹锐角, 那么  $\pm(\alpha - \beta)$  也是根; 如果夹钝角, 那么  $\pm(\alpha + \beta)$  也是根.

**proof:**

假设  $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$ , 考虑夹锐角的情况, 此时,  $\beta - \alpha = s_\alpha |\beta\rangle$ ; 对于夹钝角的情况, 令  $\beta' = -\beta$  即可.

## 7.2 rank-two systems

- if rank is one, the roots are  $R = \{-\alpha, \alpha\}$ .
- **every rank-two system** is isomorphic to one of the systems below,



分别考虑两个根之间最小夹角为  $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$  的情况, 然后使用  $s_\alpha |\beta\rangle$  生成整个  $R$ .

for positive simple roots, positive roots, negative roots and Weyl chambers, see section 7.4.

- the **Weyl group** of a rank-two root system,  $R$ , with minimum angle  $\theta = \frac{2\pi}{n}$  is the symmetry group of a regular  $\frac{n}{2}$ -gon (正  $\frac{n}{2}$  边形).
  - 群元素包括  $\frac{n}{2}$  个镜面反射和  $2\theta$  转动.

## 7.3 duality

- **def.:** for a root  $\alpha \in R$  in a root system  $(E, R)$ , its **coroot** is,

$$H_\alpha = 2 \frac{\alpha}{\langle \alpha, \alpha \rangle} \quad \text{with} \quad \begin{cases} s_{H_\alpha} = s_\alpha \\ \frac{\langle H_\alpha, H_\beta \rangle}{\langle H_\alpha, H_\alpha \rangle} = \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \end{cases} \quad (7.3.1)$$

and the **dual root system** to  $R$  is  $R^\vee = \{H_\alpha | \alpha \in R\}$ .

- $R^\vee$  is also a root system, with the same Weyl group as  $R$  (because  $s_{H_\alpha} = s_\alpha$ ).
- $H_{H_\alpha} = \alpha$  and  $(R^\vee)^\vee = R$ .
- note that although  $H_{s_\alpha |\beta\rangle} = s_{H_\alpha} |H_\beta\rangle$ , the map  $H$  is not linear, so  $R^\vee$  and  $R$  are not necessarily isomorphic to each other.

## 7.4 bases and Weyl chambers

- **def.:** for a root system  $(E, R)$ , a subset  $\Delta \subset R$  is called a **base** if,
  1.  $\Delta$  is a basis of  $E$ ,
  2. each root  $\alpha \in R$  can be expressed as a linear combination of basis vectors in  $\Delta$  with non-negative (positive roots,  $R^+$ ) or non-positive (negative roots,  $R^-$ ) integer coefficients,  $R = R^+ \cup R^-$ .

elements in  $\Delta$  are called **positive simple roots**.

- $\alpha \neq \beta \in \Delta$ , then  $\langle \alpha, \beta \rangle \leq 0$ .

**proof:**

如果  $\alpha, \beta$  之间夹锐角, 那么  $\pm(\alpha - \beta)$  也是根, 不满足系数同时非负 (或非正) 的要求.

- for a root system  $(E, R)$ , there exists a hyperplane  $V$  through the origin in  $E$ , s.t.  $V$  does not contain any root.

**proof:**

考虑一个向量  $H \in E$ , 它不在任何一个垂直于某个根向量的超平面 (这样的超平面有限多, 所以  $H$  存在) 上, 那么  $V \perp H$  就是我们要找的超平面.

- **def.:** choose one side of  $V$  to be  $R^+$ , the other side to be  $R^-$ , an element  $\alpha \in R^+$  is **decomposable** if  $\alpha = \beta + \gamma$  for some  $\beta, \gamma \in R^+$ , otherwise,  $\alpha$  is **indecomposable**.
- the indecomposable roots in  $R^+$  form the base  $\Delta$ , and  $\Delta$  exists.

**proof:**

let  $\Delta$  denote the set of indecomposable elements in  $R^+$ , now we will prove  $\Delta$  is the base:

- every  $\alpha \in R^+$  can be expressed as a linear combination of elements in  $\Delta$  with non-negative integer coefficients.

**proof:**

- \* 考虑  $H \perp V$ , 且  $\langle \alpha, H \rangle > 0, \forall \alpha \in R^+$ .
- \* 考虑  $\Delta'$  是不能表示成  $\Delta$  的元素的非负整数系数的线性叠加的  $R^+$  元素的集合, 那么一定有  $\Delta' \cap \Delta = \emptyset$ .
- \* 考虑  $\alpha \in \Delta'$  且  $\langle \alpha, H \rangle$  是  $\Delta'$  中元素里最小的, 而且  $\alpha = \beta_1 + \beta_2$  (且  $\beta_1, \beta_2 \in R^+$ ), 那么  $\beta_1, \beta_2$  至少有一个是  $\Delta'$  的元素, 但是  $\langle \alpha, H \rangle = \langle \beta_1, H \rangle + \langle \beta_2, H \rangle$  这与  $\langle \alpha, H \rangle$  最小矛盾.
- \* 可见  $\beta_1, \beta_2 \notin \Delta'$ ,  $\alpha$  一定可以表示为  $\Delta$  的元素的... 的线性叠加.

- elements in  $\Delta$  are linearly independent.

**proof:**

如果,

$$\sum_{\alpha \in \Delta} c'_\alpha \alpha = 0 \implies \sum_{\alpha} c_\alpha \alpha = \sum_{\beta} d_\beta \beta = u \in R^+ \quad (7.4.1)$$

其中  $c_\alpha \geq 0, -d_\beta < 0$  分别是  $\{c'_\alpha\}$  中非负和负的系数, 等号两边对  $\Delta$  的两个无交集的子集求和.

考虑,

$$\langle u, u \rangle = \left\langle \sum_{\alpha} c_\alpha \alpha, \sum_{\beta} d_\beta \beta \right\rangle = \sum_{\alpha, \beta} c_\alpha d_\beta \langle \alpha, \beta \rangle \quad (7.4.2)$$

但是, 对于  $\alpha \neq \beta \in \Delta$ , 一定有  $\langle \alpha, \beta \rangle \leq 0$ , 所以  $\langle u, u \rangle = 0$ , 即  $u = 0$ , 这与  $u \in R^+$  矛盾.

**proof of  $\langle \alpha, \beta \rangle \leq 0, \forall \alpha \neq \beta \in \Delta$ :**

如果  $\alpha, \beta$  呈锐角, 那么  $\pm(\alpha - \beta)$  也是根, 且其中一个属于  $R^+$ , 比如  $\alpha - \beta \in R^+$ , 那么  $\alpha = (\alpha - \beta) + \beta$ , 与 indecomposable 矛盾.

最后, 注意到 indecomposable root 一定存在. 只需考虑  $\langle \alpha, H \rangle$  值最小的  $\alpha \in R^+$  即可证明存在.

- for any base  $\Delta$  for  $R$ , there exists a hyperplane  $V$ , s.t.  $\Delta$  arises as in the theorem above.

**proof:**

$\Delta$  是一组基底, 张成向量空间中的一个锥形, 存在一个区域, 这个区域中的每个向量都与基底夹角 (这个区域就是 fundamental Weyl chamber), 那么  $V$  就是垂直于这个区域中的某个矢量的超平面.

由于基向量线性无关, 所以任何基向量都不可分解 (indecomposable).

- $\alpha \in \Delta$  cannot be expressed as a linear combination of  $R^+ - \Delta$  with non-negative real (not integer) coefficients.

**proof:**

let  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ , suppose,

$$\alpha_1 = \sum_{\beta \in R^+ - \Delta} c_\beta \beta = \sum_{\beta, i} c_\beta d_{\beta, i} \alpha_i \quad (7.4.3)$$

where  $d_{\beta, i}$  are non-negative integers.

if  $c_\beta$  are non-negative, it will contradict to the linear independence.

- $\{H_\alpha | \alpha \in \Delta\}$  is the base of  $R^\vee$ .

**proof:**

– 首先, 选取  $\Delta$  对应的  $V$ , 并以这个平面推出  $\Delta^\vee$  (这个 base 存在), 那么  $H_\alpha \in R^{\vee+} \iff \alpha \in R^+$ .

– 考虑  $\alpha \in R^+ - \Delta$ , 那么  $\alpha$  是  $\alpha_1, \dots, \alpha_r$  的非负整数的线性叠加, 那么  $H_\alpha$  是  $H_{\alpha_1}, \dots, H_{\alpha_r}$  的非负实数的线性叠加.

– 根据上一个 theorem 可知  $H_\alpha \notin \Delta^\vee$  且  $H_{\alpha_1}, \dots, H_{\alpha_r}$  是  $E$  的基底, 所以一定有  $\Delta^\vee = \{H_{\alpha_1}, \dots, H_{\alpha_r}\}$ .

- **def.:** the open Weyl chambers for a root system  $(E, R)$  are connected components of,

$$E - \bigcup_{\alpha \in R} V_\alpha \quad (7.4.4)$$

where  $V_\alpha \perp \alpha$  is a hyperplane through the origin.

- **def.:** the open fundamental Weyl chamber (relative to  $\Delta$ ) is  $\{H | \langle \alpha, H \rangle > 0, \forall \alpha \in \Delta\}$ .
  - open fundamental Weyl chamber is connected (consider  $\langle H, \beta \rangle > \langle H, \alpha \rangle, \alpha \in \Delta, \beta \in R^+ - \Delta, H \perp V$ ).
  - every elements in the open fundamental Weyl chamber has a positive inner product with root in  $R^+$ , and negative inner product with root in  $R^-$ , so open fundamental Weyl chamber is an open Weyl chamber.
- for each open Weyl chamber  $C$ , there exists a unique base  $\Delta_C$ , s.t.  $C$  is the open fundamental Weyl chamber relative to  $\Delta_C$ .
  - there is a one-to-one correspondence between bases and Weyl chambers.

**proof:**

考虑  $H \in C$ , 以  $V \perp H$  建立起的 base 就是  $\Delta_C$ .

考虑  $\Delta, \Delta'$  都对应同一个  $C$ , 它们的  $R^+ = R'^+$ , 且可以选取  $V = V'$ , 那么一定有  $\Delta = \Delta'$  (都是不可分解的根).

- every root is an element of some base.

**proof:**

任何一个根  $\alpha$  对应的  $V_\alpha \perp \alpha$  都包含某个 open Weyl chamber  $C$  的边界。  
考虑  $H \in V_\alpha$  且  $H + \epsilon\alpha \in C$ , 选取  $V \perp H' = H + \epsilon\alpha$ , 显然  $\langle \alpha, H' \rangle$  是  $R^+$  中最小的, 所以一定有  $\alpha \in \Delta_C$ .

## 7.5 Weyl chambers and Weyl group

- the Weyl group act **transitively** on the set of Weyl chambers, i.e. for every open Weyl chamber  $C$ , we have,

$$\{w(C) | w \in W\} = E - \bigcup_{\alpha \in R} V_\alpha \quad (7.5.1)$$

**proof:**

consider chamber  $C$  with its base  $\Delta_C$ , we want to prove that  $wH' \in C$  for all  $H' \in E - \bigcup_{\alpha \in R} V_\alpha$  ( $H' \in C$  case is trivial) and  $w \in W'$  where  $W'$  is generated by  $s_\alpha, \alpha \in \Delta_C$ .

- in the case when  $H' \notin C$ , there exists some  $\alpha \in \Delta_C$  that  $\langle \alpha, H' \rangle < 0$  (夹钝角).
- since  $W'$  is a finite group, there exists a  $w \in W'$  that bring  $H'$  closest to some  $H \in C$ .
- if  $wH' \notin C$ , then there exists  $\alpha \in \Delta_C$  that  $\langle \alpha, wH' \rangle < 0$ , then,

$$\begin{aligned} |wH' - H|^2 - |s_\alpha wH' - H|^2 &= 2 \langle wH' | s_\alpha - 1 | H \rangle \\ &= -4 \frac{\langle wH' | \alpha \rangle \langle \alpha | H \rangle}{\langle \alpha, \alpha \rangle} > 0 \end{aligned} \quad (7.5.2)$$

which contradicts to the closest-ness.

- so, we must have  $wH' \in C$ .

- $W$  is generated by  $s_\alpha, \alpha \in \Delta$ .

**proof:**

we want to prove that for all  $\alpha$ , there exists some  $w \in W'$  (generated by  $s_\beta, \beta \in \Delta_C$ ) s.t.,

$$s_{w|\alpha} = ws_\alpha w^{-1} \in W' \quad (7.5.3)$$

- let  $\alpha \in \Delta_D$  where  $D$  is some chamber.
- we already proved that there is some  $w \in W'$  that  $w[D] = C$ , since  $w$  preserves inner product,  $w[\Delta_D] = \Delta_C$ .
- so,  $w|\alpha \rangle \in \Delta_C$ , i.e.  $s_{w|\alpha} \in W'$ .

- def.:** the **minimal expression** of  $w \in W$  is the expression of  $w$  in terms of  $s_\alpha, \alpha \in \Delta$  with the minimal number of  $s_\alpha$  (the minimal expression need not be unique).
- $\bar{C}$  is the closure of a Weyl chamber  $C$ , if  $H, H' \in \bar{C}$  and  $w|H \rangle = H'$ , then  $H = H'$ .  
i.e. two distinct elements of  $\bar{C}$  cannot be in the same orbit of  $W$ .

**proof:**

we proceed by induction on the number of the minimal expression of  $w$  in terms of  $s_\alpha, \alpha \in \Delta_C$ .

- if the minimal number is zero, i.e.  $w = I$ , the result holds.
- if the result holds when the minimal number is  $k - 1$ , then, consider  $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ .
- $C$  and  $w[C]$  lie on opposite sides of hyperplane  $V_{\alpha_1}$ , i.e.  $\bar{C} \cap w[\bar{C}] \subset V_{\alpha_1}$ .



**proof:**

let's prove by induction. for  $w = s_{\alpha_1}$ , the result holds, consider  $w = us_{\alpha_k}$ , where  $u = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{k-1}}$ ,

- \*  $C$  and  $u[C]$  lie on opposite sides of  $V_{\alpha_1}$  (by induction).
- \* if  $C$  and  $w[C]$  lie on the same side, then  $w[C] = u \circ s_{\alpha_k}[C]$  lies on the opposite side of  $u[C]$ , i.e.  $C$  and  $s_{\alpha_k}[C]$  lie on opposite sides of  $V_{u^{-1}|\alpha_1\rangle}$ .
- \* notice that  $\alpha_k \in \Delta_C$ , consider  $H \in V_{\alpha_k}$  which also lies on the boundary of  $C$ , then,  $s_{\alpha_k}H = H$  also lies on the boundary of  $s_{\alpha_k}[C]$ , which implies  $V_{u^{-1}|\alpha_1\rangle} = V_{\alpha_k}$ , so,

$$u^{-1}s_{\alpha_1}u = s_{u^{-1}|\alpha_1\rangle} = s_{\alpha_k} \implies w = s_{\alpha_1}u = s_{\alpha_2}\cdots s_{\alpha_k} \quad (7.5.4)$$

which contradicts to the minimal expression assumption.

- 
- since  $w|H\rangle = H' \in w[\bar{C}] \cap \bar{C} \subset V_{\alpha_1}$ , which implies,

$$s_{\alpha_1}H' = H' = s_{\alpha_2}\cdots s_{\alpha_k}H \quad (7.5.5)$$

by induction,  $H = H'$ .

- if  $H \in C$  for some chamber  $C$ , and  $w|H\rangle = H$ , then,  $w = I$  ( $W$  acts **freely**).

**proof:**

since  $w|H\rangle \in C$ , and  $w$  is a continuous map, so we must have  $w[C] = C$ , i.e. for all  $H' \in C$ , we have  $w|H'\rangle \in C \implies w|H'\rangle = H'$  (according to the theorem above), then  $w = I$ .

- $W$  acts **freely** and **transitively** on Weyl chambers, the same is true for bases, i.e. for two bases  $\Delta_1, \Delta_2$ , there exists (transitiveness) a unique (free-ness)  $w$ , s.t.  $w[\Delta_1] = \Delta_2$ .
- $C$  is a Weyl chamber,  $H \in E$ , then there is exactly one point in the  $W$ -orbit of  $H$  that lies in  $\bar{C}$  (but the  $w$  that  $w|H\rangle \in C$  is not necessarily unique).

**proof:**

- $H$  is in the closure of some chamber  $D$ , and there exists a  $w$  that  $w[\bar{D}] = \bar{C}$ , so  $w|H\rangle \in \bar{C}$ .
- if  $H', H'' \in \bar{C}$  are point in the  $W$ -orbit of  $H$ , then  $H' = H''$ .

- for all  $\alpha \in \Delta, \beta \in R^+$ , and  $\beta \neq \alpha$ , we have  $s_\alpha|\beta\rangle \in R^+$ .

**proof:**

- write  $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$  with  $c_\gamma \in \mathbb{Z}^+$ .
- notice that  $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ , so,  $s_\alpha|\beta\rangle = \beta - n\alpha$  for some integer  $n$ .
- in the expansion,

$$s_\alpha|\beta\rangle = \sum_{\gamma \in \Delta - \{\alpha\}} c_\gamma \gamma + (c_\alpha - n)\alpha \quad (7.5.6)$$

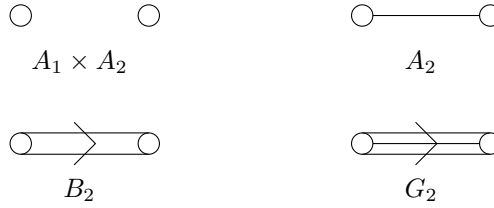
only the coefficient  $c_\alpha$  changes.

- if one coefficient is positive in the expansion, all other coefficients must be positive, so  $s_\alpha|\beta\rangle \in R^+$ .

## 7.6 Dynkin diagrams

- **def.:**  $\Delta = \{\alpha_1, \dots, \alpha_r\}$  is the base of  $R$ , the **Dynkin diagram** for  $R$  is:

1. 图中有  $r$  个**结点**,
2. 节点  $v_i, v_j$  之间根据  $\alpha_i, \alpha_j$  之间的夹角决定连线的**条数**,  $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$  分别对于 0, 1, 2, 3 条连线,
3. 如果  $\alpha_i, \alpha_j$  长度不同, 连线上画出一条**指向更短的根的箭头** (可以将箭头视作大于符号).



- 注意, 夹角为  $\frac{2\pi}{3}, \frac{3\pi}{4}$  的根长度一定不相等, 即, 2, 3 条线上一定有箭头; 相反, 一条线上一定没有箭头.
- 同一个 root system 的两个  $\Delta_1, \Delta_2$  的 Dynkin 图一定完全相同 (isomorphic).

**proof:**

there exists  $w \in W$  s.t.  $w[\Delta_1] = \Delta_2$ , and  $w$  preserves angles and lengths.

- a root system is irreducible (see section 7.1)  $\iff$  its Dynkin diagram is connected.
  - semisimple Lie algebra  $\mathfrak{g}$  is **simple**  $\iff$  the Dynkin diagram of  $R \subset \mathfrak{g}$  is **connected**.

**proof:**

如果  $R$  是 reducible, 那么  $\Delta = \Delta_1 \cup \Delta_2$  且  $\Delta_1 \perp \Delta_2$ , 则 Dynkin 图一定 not connected.

反之, Dynkin 图 not connected  $\implies \Delta = \Delta_1 \cup \Delta_2$  且  $\Delta_1 \perp \Delta_2$ , 那么  $E = E_1 \oplus E_2$  with  $E_i = \text{span}(\Delta_i)$ .

Weyl 群由  $s_\alpha, \alpha \in \Delta$  生成, 而  $s_\alpha, \alpha \in \Delta_1$  在  $E_2$  上是单位映射, 可见  $W = W_1 \times W_2$ , 因此,  $R = W[\Delta] = W_1[\Delta_1] \cup W_2[\Delta_2] = R_1 \cup R_2$ , 即根要么属于  $E_1$  要么属于  $E_2$ .

- Dynkin diagrams are isomorphic  $\iff$  root systems are isomorphic.

## 7.7 integral and dominant integral elements

- **def.:** an element  $\mu \in E$  is an **integral element** if for all  $\alpha \in R$ ,

$$\langle \mu, H_\alpha \rangle = 2 \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad (7.7.1)$$

$\mu$  is **dominant** (relative to  $\Delta$ ) if  $\langle \mu, \alpha \rangle \geq 0, \forall \alpha \in \Delta$ , and **strictly dominant** if  $\langle \mu, \alpha \rangle > 0, \forall \alpha \in \Delta$ .

- $\mu$  is (strictly) dominant (relative to  $\Delta_C$ )  $\iff \mu \in \bar{C}$  (or  $C$ ).
- for all  $\mu$ , there exists  $w \in W$  s.t.  $w|\mu \in \bar{C}$ .
- every integer linear combination of roots (e.g.  $2\alpha + 3\beta + 5\gamma$ ) is an integral element. 但一般不是所有 integral elements 都是根的整数线性组合.
- 注意  $\{H_\alpha | \alpha \in \Delta\}$  是  $R^\vee$  的 base (见 section 7.4), 所以  $\langle \mu, H_\alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta \implies \mu$  是 integral element.
- **def.:** the **fundamental weights** (relative to  $\Delta = \{\alpha_1, \dots, \alpha_l\}$ ) are  $\mu_1, \dots, \mu_r$  s.t.,

$$\langle \mu_i, H_{\alpha_j} \rangle = \delta_{ij} \quad (7.7.2)$$

i.e. the dual basis of  $\Delta^\vee$ .

- $\Delta^{\vee*}$  的非负 (正) 整数的线性组合是 (strictly) dominant integral element.
- $\Delta^{\vee*}$  的整数线性组合的集合 = integral elements 的集合.

- **def.:** half the sum of the positive roots (relative to  $\Delta$ ) is,

$$\delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \quad (7.7.3)$$

- $\delta$  is a strictly dominant integral element, and,

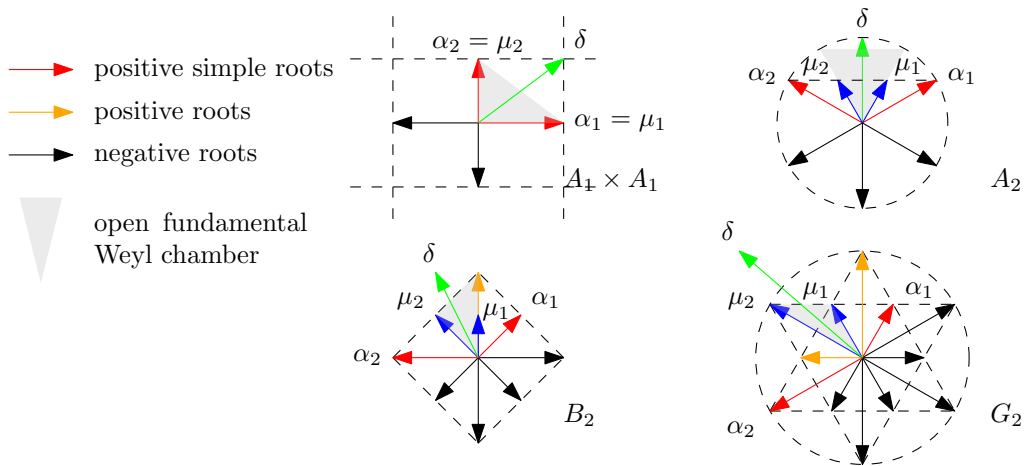
$$\langle \delta, H_\alpha \rangle = 1, \forall \alpha \in \Delta \iff \delta = \sum_{i=1}^r \mu_i \quad (7.7.4)$$

**proof:**

注意 section 7.5 最后一个定理,  $s_\alpha[R^+ - \{\alpha\}] = R^+ - \{\alpha\}$ , 所以  $R^+ - \{\alpha\} = \{\beta_1, s_\alpha\beta_1, \beta_2, s_\alpha\beta_2, \dots\}$ . 且有  $\langle \beta_1 + s_\alpha\beta_1, H_\alpha \rangle = 0$ , 所以,

$$\langle \delta, H_\alpha \rangle = \langle \frac{1}{2}\alpha, H_\alpha \rangle = 1 \quad (7.7.5)$$

- fundamental wights and half the sum of the positive roots in rank-two systems 见下图,



## 7.8 the partial ordering

- **def.:** relative to  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ ,  $\mu \succeq \nu$  ( $\mu$  is **higher** than  $\nu$ ) if,

$$\mu - \nu = c_1\alpha_1 + \dots + c_r\alpha_r \quad (7.8.1)$$

其中  $c_1, \dots, c_r \geq 0$ , 类似地, 可以定义  $\nu \preceq \mu$  (... **lower** than...).

–  $\succeq$  定义了一个 partial ordering on  $E$ , 但两个矢量之间可能既不存在  $\succeq$  也不存在  $\preceq$  的关系.

- $\mu \in E$  is dominant  $\implies \mu \succeq 0$ .

**proof:**

考虑  $\Delta$  的 dual basis  $\Delta^* = \{\alpha_1^*, \dots, \alpha_r^*\}$ , 有,

$$c_i = \langle \alpha_i^*, \mu \rangle = \sum_{j=1}^r \langle \alpha_i^*, \alpha_j^* \rangle \langle \alpha_j, \mu \rangle \quad (7.8.2)$$

$\Delta$  中的任何两个向量夹钝角 (见 section 7.4 定义后的第一条定理), 那么它的对偶基底中的任意两个向量夹锐角 (见 appendix A.4), 所以  $\langle \alpha_i^*, \alpha_j^* \rangle \geq 0, \langle \alpha_j, \mu \rangle \geq 0$ , 所以  $c_i \geq 0$ .

- if  $\mu$  is dominant (i.e.  $\mu \in \bar{C}$ ), then  $w|\mu \preceq \mu$  for all  $w \in W$ .

**proof:**

$O$  is the Weyl-group orbit of  $\mu$ . 考虑到  $O$  是有限集合, 令  $\nu \in O$  使得没有其它元素高于  $\nu$ , 那么一定有  $\nu \in \bar{C}$  (即 dominant), 否则, 如果  $\langle \nu, \alpha \rangle < 0, \exists \alpha \in \Delta_C$ , 那么,

$$s_\alpha |\nu\rangle = \nu - 2 \frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \succeq \nu \quad (7.8.3)$$

考虑到 section 7.5 的第四个结论, 可知  $\nu = \mu$ .

现在证明  $O$  中没有元素既不高于也不低于  $\mu$ .

考虑所有既不... 也不... 的元素的集合  $O'$ ,  $\xi \in O'$  且没有  $O'$  中的元素高于它, 那么,

– 如果  $o \in O - O'$ , 那么一定有  $\mu \succeq o$ , 且如果  $o \succeq \xi$ , 那么  $\mu \succeq o \succeq \xi$ , 与  $\xi \in O'$  矛盾.

所以  $O$  中没有元素高于  $\xi$ , 可知  $\xi \in \bar{C}$ , 矛盾.

- if  $\mu$  is a strictly dominant ( $\mu \in C$ ) integral element, then  $\mu \succeq \delta$  ( $\delta$  is half the sum of positive roots).

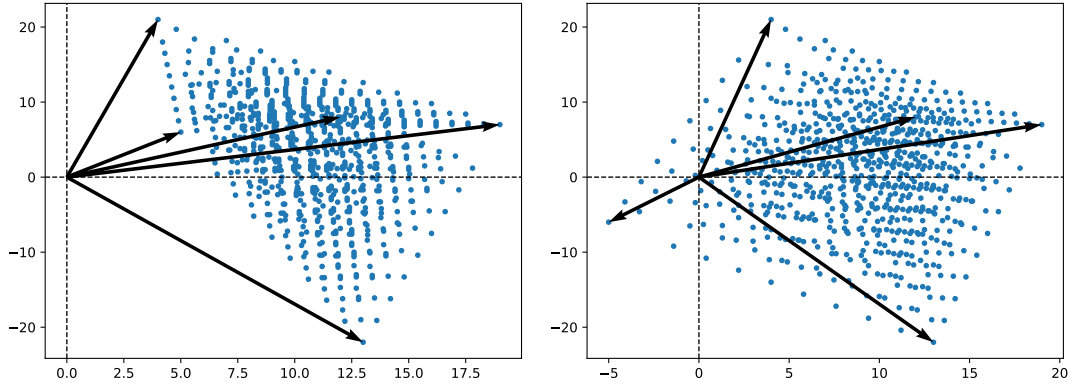
**proof:**

$\mu$  is a strictly dominant integral element  $\implies \langle \mu, \alpha \rangle \in \mathbb{Z}^+ - \{0\}, \forall \alpha \in \Delta_C; \langle \delta, \alpha \rangle = 1, \forall \alpha \in \Delta_C$ . 所以  $\mu - \delta \in \bar{C} \implies \mu \succeq \delta$ .

- **def.:** the **convex hull** of vectors  $v_1, \dots, v_N$  is the set,

$$\text{Conv}(v_1, \dots, v_N) = \{c_1 v_1 + \dots + c_N v_N | c_1 + \dots + c_N = 1 \text{ and } c_i \in \mathbb{R}^+\} \quad (7.8.4)$$

两个例子如下图,



- $K$  is a compact, convex subset of  $E$ , and  $\lambda \in E - K$ , then there is an element  $\gamma \in E$  s.t.,

$$\langle \gamma, \lambda \rangle > \langle \gamma, \kappa \rangle, \forall \kappa \in K \quad (7.8.5)$$

**proof:**

由于  $K$  是紧致的, 存在  $\kappa_0 \in K$  使得  $|\lambda - \kappa_0|$  最小, 令  $\gamma = \lambda - \kappa_0$ , 那么,

$$\langle \gamma, \lambda - \kappa_0 \rangle > 0 \implies \langle \gamma, \lambda \rangle > \langle \gamma, \kappa_0 \rangle \quad (7.8.6)$$

对于  $K$  中的任意元素  $\kappa$ ,  $\kappa(s) = s\kappa + (1-s)\kappa_0, s \in [0, 1]$  属于  $K$ , 那么,

$$|\lambda - \kappa(s)|^2 \geq |\lambda - \kappa_0|^2 \implies s^2 |\kappa - \kappa_0|^2 - 2s \langle \lambda - \kappa_0, \kappa - \kappa_0 \rangle \geq 0 \quad (7.8.7)$$

考虑  $s \ll 1$  的情况, 可见,

$$\underbrace{\langle \lambda - \kappa_0, \kappa - \kappa_0 \rangle}_{=\gamma} \leq 0 \implies \langle \gamma, \lambda \rangle > \langle \gamma, \kappa_0 \rangle \geq \langle \gamma, \kappa \rangle \quad (7.8.8)$$

- $\mu, \nu$  are dominant ( $\in \bar{C}$ ) and  $\nu \notin \text{Conv}(W|\mu)$ , then there exists a dominant element  $\gamma \in \bar{C}$  s.t.,

$$\langle \gamma, \nu \rangle > \langle \gamma, w\mu \rangle, \forall w \in W \quad (7.8.9)$$

meaning that  $\nu \not\preceq w\mu, \forall w \in W$ .

**proof:**

根据上一个定理, 存在  $\gamma' \in E$  使得  $\langle \gamma', \nu \rangle > \langle \gamma', \kappa \rangle, \forall \kappa \in \text{Conv}(W|\mu)$ , 特别地,  $\langle \gamma', \nu \rangle > \langle \gamma', w\mu \rangle, \forall w \in W$ .

考虑  $\{\gamma\} = W|\gamma' \cap \bar{C}$ , 这个  $\gamma = w_0\gamma'$  是唯一的, 且  $\gamma \succeq \gamma'$ . 所以,

$$\gamma - \gamma' \in \bar{C} \implies \langle \gamma - \gamma', \nu \rangle \geq 0 \implies \langle \gamma, \nu \rangle > \langle w_0\gamma, w\mu \rangle, \forall w \in W \implies \dots \quad (7.8.10)$$

( $\gamma - \gamma'$  与 positive simple root 的内积为正, 且  $\nu$  可以展开成 positive simple root 的正系数叠加)

#### • 两个定理:

- if  $\mu, \nu$  are dominant, then  $\nu \in \text{Conv}(W|\mu) \iff \nu \preceq \mu$ .
- $\mu$  is dominant and  $\nu \in E$ , then  $\nu \in \text{Conv}(W|\mu) \iff w|\nu \preceq \mu, \forall w \in W$ .

**proof:**

上一个定理已经证明了  $\Leftarrow$ , 我们现在来证明  $\Rightarrow$ .  $\mu$  是 dominant, 那么  $w\mu \preceq \mu, \forall w \in W$ , 所以,

$$\left( \sum_{i=1}^{|W|} c_i w_i |\mu \rangle \right) - \mu = \sum_{i=1}^{|W|} c_i \underbrace{(w_i |\mu \rangle - \mu)}_{\preceq 0} \preceq 0 \quad (7.8.11)$$

所以  $\text{Conv}(W|\mu) \preceq \mu$ .

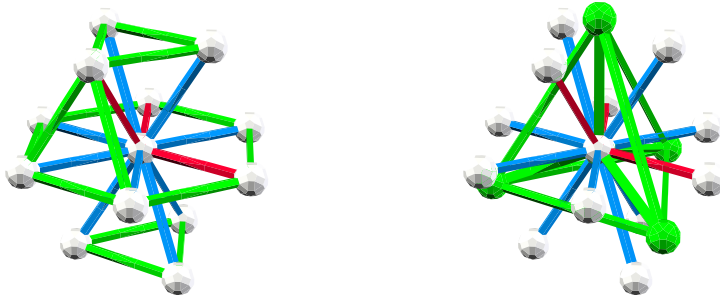
首先, 显然有  $\nu \in \text{Conv}(W|\mu) \iff w|\nu \in \text{Conv}(W|\mu), \forall w \in W$ . 那么考虑  $\nu' = w_0\nu \in \bar{C}$ , 有,

$$\nu \in \text{Conv}(W|\mu) \iff \nu' \in \text{Conv}(W|\mu) \iff \nu' \preceq \mu \quad (7.8.12)$$

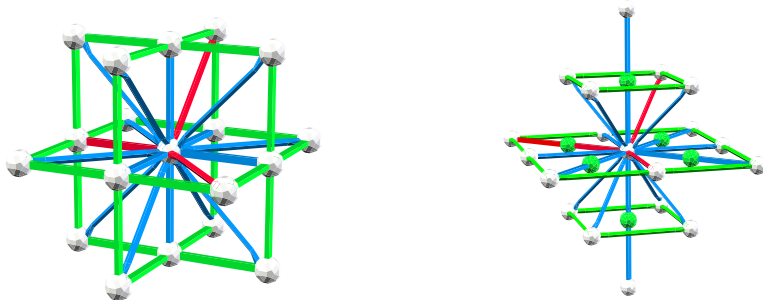
而  $w|\nu \preceq \nu' \preceq \mu, \forall w \in W$ , 得证.

## 7.9 rank-three systems

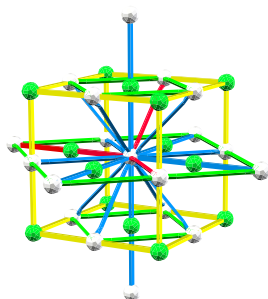
- 本 section 只考虑 irreducible rank-three systems, 总共有三种, 分别是  $A_3, B_3, C_3$ , 它们分别来自  $\mathfrak{sl}(4, \mathbb{C})$ ,  $\mathfrak{so}(7, \mathbb{C})$  和  $\mathfrak{sp}(3, \mathbb{C})$ .
- $A_3$  root system 见下图, 其中, base 由红色向量组成, Weyl 群是右图中绿色正四面体的对称群,



- $B_3, C_3$  root systems 分别见下图,



它们的 Weyl 群显然相同, 是下图中黄色立方体的对称群,

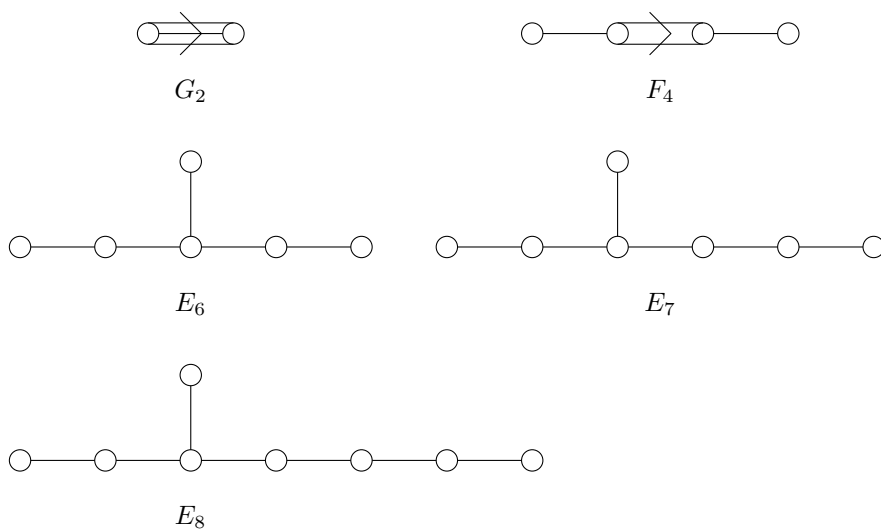


## 7.10 the classical root systems

- 见 section 6.7.

## 7.11 the classification

- every irreducible root system is either the root system of a classical Lie algebra (types  $A_n, B_n, C_n, n \geq 1$  and  $D_n, n \geq 3$ , with  $B_2 \simeq C_2, A_3 \simeq D_3$ ) or one of five **exceptional root systems**.
- the **exceptional root systems** are  $G_2, F_4, E_6, E_7, E_8$ , 它们的 Dynkin 图如下,



- 三个有用的定理:
  - $\mathfrak{h}_1, \mathfrak{h}_2$  are Cartan subalgebras of the semisimple Lie algebra  $\mathfrak{g}$ , then there exists a automorphism (自同构)  $\phi: \mathfrak{g} \rightarrow \mathfrak{g}$  s.t.  $\phi[\mathfrak{h}_1] = \mathfrak{h}_2$ . (见 section 6.2 末尾)
  - the root systems associated to  $(\mathfrak{g}_1, \mathfrak{h}_1)$  and  $(\mathfrak{g}_2, \mathfrak{h}_2)$  are isomorphic  $\implies \mathfrak{g}_1, \mathfrak{g}_2$  are isomorphic.
  - for every root system  $R$ , there exists a root system associated to  $(\mathfrak{g}, \mathfrak{h})$  isomorphic to  $R$ .

因此, 所有 simple Lie algebra 都与下表中的某个 classical Lie algebra,

$\mathfrak{sl}(n+1, \mathbb{C}) \mapsto A_n$	$\mathfrak{so}(2n+1, \mathbb{C}) \mapsto B_n$	$\mathfrak{sp}(2n, \mathbb{C}) \mapsto C_n$	$\mathfrak{so}(2n, \mathbb{C}) \mapsto D_n$
$n \geq 1$	$n \geq 2$	$n \geq 3$	$n \geq 4$
$n = 1$	$B_1 \simeq A_1$	$C_1 \simeq A_1$	$n \neq 1$
$n = 2$		$C_2 \simeq B_2$	$n \neq 2$
$n = 3$			$D_3 \simeq A_3$

或  $G_2, F_4, E_{6,7,8}$  中的某个 exceptional Lie algebra 相 isomorphic.

## Chapter 8

# representations of semisimple Lie algebras

### 8.1 weights of representations

- **def.:**  $(\pi, V)$  is a (possibly infinite dimensional) rep. of semisimple Lie algebra  $\mathfrak{g}$ , then  $\lambda \in \mathfrak{h}$  is the **weight** of  $\pi$  if there **exists** a  $v \neq 0 \in V$  s.t.,

$$\pi(H)v = \langle \lambda, H \rangle v, \forall H \in \mathfrak{h} \iff \det(\pi(H) - \langle \lambda, H \rangle I) = 0, \forall H \in \mathfrak{h} \quad (8.1.1)$$

the **weight space** of  $\lambda$  (denoted by  $V_\lambda$ ) is the set of all  $v \in V$  satisfying (8.1.1), and the dimension of the weight space is called the (geometric) **multiplicity**. (more about weights, see appendix A.3)

- $(\pi, V)$  is finite-dimensional  $\implies$  every weight of  $\pi$  is an **integral element**.

**proof:**

$\pi|_{\mathfrak{s}^\alpha}$  可以视为  $\mathfrak{s}^\alpha = \text{span}(H_\alpha, A_\alpha, B_\alpha) \simeq \mathfrak{su}(2)_\mathbb{C}$  的表示, 那么根据 (9.1.6),  $\pi(H_\alpha) \equiv \pi(2J_3)$  的 eigenvalue 是整数, 所以,

$$\langle \lambda, H_\alpha \rangle \in \mathbb{Z} \quad (8.1.2)$$

- for finite-dimensional rep., for a weight  $\lambda$  of  $\pi$ ,  $w|\lambda\rangle, \forall w \in W$  is still a weight and  $V_{w|\lambda} \simeq V_\lambda$ .

**proof:**

注意, 令  $S_\alpha = e^{A_\alpha} e^{-B_\alpha} e^{A_\alpha}$ , 那么,

$$\text{Ad}_{S_\alpha} H_\alpha = -H_\alpha \implies \text{Ad}_{S_\alpha} = s_\alpha \quad (8.1.3)$$

证明见 (9.1.7). 所以, 考虑  $s_\alpha|\lambda\rangle$  (注意到  $s_\alpha^{-1} = s_\alpha$ ),

$$\begin{aligned} & \begin{cases} \pi(s_\alpha^{-1}H)v = \langle \lambda, s_\alpha^{-1}H \rangle v \quad \forall v \in V_\lambda \\ \pi(s_\alpha^{-1}H) = \pi(\text{Ad}_{S_\alpha}H) = \Pi(S_\alpha)\pi(H)\Pi^{-1}(S_\alpha) \end{cases} \\ \implies & \pi(H)(\Pi^{-1}(S_\alpha)v) = \langle s_\alpha\lambda, H \rangle (\Pi^{-1}(S_\alpha)v) \\ \implies & \Pi^{-1}(S_\alpha)[V_\lambda] = V_{s_\alpha|\lambda} \end{aligned} \quad (8.1.4)$$

( $\Pi(S_\alpha)$  一定是可逆矩阵, 否则不存在逆元,  $\Pi$  就根本不是一个表示)

- 考虑半单李代数的正根为  $R^+ = \{\alpha_1, \dots, \alpha_N\}$ , 李代数的基底是  $\Delta \cup \{A_1, \dots, A_N\} \cup \{B_1, \dots, B_N\}$ , 其中  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ , 且  $A_i \in \mathfrak{g}_{\alpha_i}, B_i \in \mathfrak{g}_{-\alpha_i}$ .

– 那么,  $\forall \alpha \in R$ ,

$$\begin{cases} \pi(H)\pi(A_\alpha)v = \langle \lambda + \alpha, H \rangle \pi(A_\alpha)v \\ \pi(H)\pi(B_\alpha)v = \langle \lambda - \alpha, H \rangle \pi(B_\alpha)v \end{cases} \implies \begin{cases} \pi(A_\alpha)[V_\lambda] \subseteq V_{\lambda+\alpha} \\ \pi(B_\alpha)[V_\lambda] \subseteq V_{\lambda-\alpha} \end{cases} \quad (8.1.5)$$



- 对于所有的不可约表示,  $\pi(H), \forall H \in \mathfrak{h}$  都可以被对角化, 因此也可以被同时对角化.

**proof:**

$U$  是  $V$  的子空间, 由  $\mathfrak{h}$  的 simultaneous eigenvectors 构成, 根据 (8.1.5),  $\pi(A_\alpha)[U] \subseteq U$ , 所以  $U$  是不变子空间 (且不为零, 因为  $\mathfrak{h}$  是 Abelian, 至少存在一个权, 见 appendix A.3). 又因为  $(\pi, V)$  不可约, 所以  $V = U = \bigoplus_\lambda V_\lambda$ .

- 三个关于 **highest weight** 的定理:

- every irreducible, finite-dim. rep. of  $\mathfrak{g}$  has a highest weight. (最高权存在)
- two irreducible, finite-dim. rep. with the same highest weight are isomorphic. (一一对应)
- the **highest weight**  $\mu$  of a irreducible, finite-dim. rep. is a **dominant integral element**.

**proof:**

**reordering lemma:** 考虑李代数  $\mathfrak{g}$  及其表示  $\pi$ ,  $\{A_1, \dots, A_n\}$  是李代数的一组基底, 那么下式,

$$\pi(A_{i_1}) \cdots \pi(A_{i_N}) \quad (8.1.6)$$

可以表示成,

$$\pi(A_n)^{j_n} \cdots \pi(A_1)^{j_1} \quad (8.1.7)$$

的线性组合, 其中  $j_1 + \cdots + j_n \leq N$ .

**proof:**

用数学归纳法证明,  $N = 1$  时显然成立, 假设  $N - 1$  时成立, 那么  $N$  时,

$$\pi(A_{i_1}) \cdots \pi(A_{i_N}) = \pi(A_{i_1}) \left( \sum_{j_1 + \cdots + j_N \leq N-1} C_{j_1, \dots, j_N} \pi(A_n)^{j_n} \cdots \pi(A_1)^{j_1} \right) \quad (8.1.8)$$

用对易关系改变  $\pi(A_{i_1})$  的位置,

$$\pi(A_{i_1})\pi(A_k) = \pi(A_k)\pi(A_{i_1}) + \underbrace{\pi([A_{i_1}, A_k])}_{=\sum_l -f_{i_1 k}^l A_l} \quad (8.1.9)$$

右边的一项最多含  $N - 1$  个基矢, 所以命题得证.

- 令 (dominant) integral element  $\mu$  为  $(\pi, V)$  的 **highest weight**, 那么 (根据 (8.1.5)) 一定有  $\pi(A_{\alpha_i})[V_\mu] = \{0\}, \forall \alpha_i \in R^+$ .
- 选取  $\{B_1, \dots, B_N\} \cup \Delta \cup \{A_1, \dots, A_N\}$  为  $\mathfrak{g}$  的基底 (其中  $N$  是正根的个数), 那么考虑 some  $v \in V_\mu$ ,

$$\pi(B_{i_1}) \cdots \pi(B_{i_M})v = \text{linear combination of } \pi(B_N)^{j_N} \cdots \pi(B_1)^{j_1}v \quad (8.1.10)$$

(注意到  $v$  是  $\pi(H_i)$  的本征向量, 而  $\pi(A_i)v = 0$ )

另外, 一定有  $\mu - j_1\alpha_1 - \cdots - j_N\alpha_N \in \text{Conv}(W|\mu)$ , 否则  $\pi(B_N)^{j_N} \cdots \pi(B_1)^{j_1}v = 0$ .

- 考虑,

$$\text{linear combinations of } \pi(B_{i_1}) \cdots \pi(B_{i_M})v \text{ with } M \geq 0, \text{ for some } v \in V_\mu \quad (8.1.11)$$

这是  $V$  的不变子空间, 考虑到 irreducibility, (8.1.11) 等于  $V$ . 同时也证明了  $\dim V_\mu = 1$ , 且  $\mu$  是唯一的最高权, 因此它一定是 dominant.

- if  $\mu$  is a **dominant integral element**, there exists an irreducible, finite-dim. rep. of  $\mathfrak{g}$  with **highest weight**  $\mu$ .

本 chapter 的剩余部分将用来证明这个定理.

## 8.2 the highest weight cyclic representations & an introduction to Verma modules

- **def.:** for a (maybe infinite-dim.) rep.  $(\pi, V)$  of  $\mathfrak{g}$  with highest weight  $\mu \in \mathfrak{h}$  (不一定是 integral), if there exists  $v \neq 0 \in V$  s.t.,

1.  $\pi(H)v = \langle \mu, H \rangle v, \forall H \in \mathfrak{h}$  (simultaneously diagonalizable, 见 appendix A.3.2),
2.  $\pi(A)v = 0, \forall A \in \mathfrak{g}_\alpha$ , with  $\alpha \in R^+$ ,
3. the smallest invariant subspace (见 section 5.2 第三点,  $\pi(A)[W] \subseteq W, \forall A \in \mathfrak{g}$ ) containing  $v$  is  $V$ ,

then it is said to be **highest weight cyclic**.

- 有限维情况下, highest weight cyclic rep. 是 irreducible, 且最高权相同  $\mu$  的... 互相 isomorphic.

- 下面初步介绍构造 Verma module  $(\pi_\mu, V^\mu)$  的思路 ( $V^\mu$  选择上标, 以区分 weight space  $V_\mu$ ).
- 依旧是选取,

$$\{B_1, \dots, B_N\} \cup \Delta \cup \{A_1, \dots, A_N\} \quad \text{with} \quad \begin{cases} R^+ = \{\underbrace{\alpha_1, \dots, \alpha_r}_{=\Delta}, \alpha_{r+1}, \dots, \alpha_N\} \\ A_i \in \mathfrak{g}_{\alpha_i} & i = 1, \dots, N \\ B_i \in \mathfrak{g}_{-\alpha_i} & i = 1, \dots, N \end{cases} \quad (8.2.1)$$

作为  $\mathfrak{g}$  的基底.

- 由于对于  $(\pi_\mu, V^\mu)$ ,  $\mu$  是最高权, 所以一定存在,

$$v_0 \in V^\mu, \text{ s.t. } \pi_\mu(A)v_0 = 0, \forall A \in \mathfrak{g}_\alpha, \text{ with } \alpha \in R^+ \quad (8.2.2)$$

- 根据 (8.1.11), 考虑具有以下形式的向量,

$$\pi_\mu(B_1)^{n_1} \cdots \pi_\mu(B_N)^{n_N} v_0 \in V_{\mu - \sum_{i=1}^N n_i \alpha_i} \subset V^\mu, \text{ with } n_i \in \mathbb{Z}^+ \quad (8.2.3)$$

它们的线性组合张成  $V^\mu$ .

- Verma module 中的 weights 仅具有如下形式,

$$\mu - \sum_{i=1}^N n_i \alpha_i \quad (8.2.4)$$

其中  $n_i$  是非负整数.

- 这样定义后, 我们就能 (通过对易关系) 计算  $\mathfrak{g}$  中每个元素的表示如何作用于任何一个  $V^\mu$  中的向量.

## 8.3 universal enveloping algebras

- **def.:** 李代数  $\mathfrak{g}$  嵌入的 associative algebra (对 algebra 的一般定义见 appendix A 开头),  $\mathcal{A}$ , 是:

- 存在乘法单位元  $e$ , 且满足结合律 (unital, associative algebra).

- $\mathfrak{g}$  嵌入于  $\mathcal{A}$  ( $\hat{j}: \mathfrak{g} \rightarrow \mathcal{A}$ ).

(例如: 对于矩阵李群  $G \subseteq \text{GL}(n, \mathbb{C})$ , 那么  $\mathfrak{g}$  就是  $\mathcal{M}_n(\mathbb{C})$  的子空间)

- 李括号简化为,

$$\hat{j}([A, B]) = \hat{j}(A) \cdot \hat{j}(B) - \hat{j}(B) \cdot \hat{j}(A) \quad (8.3.1)$$

- $\mathcal{A}$  由单位元  $e$  和如下元素张成,

$$\hat{j}(A_1) \cdots \hat{j}(A_k) \quad (8.3.2)$$

其中  $k \geq 1$ .

另外, 对于  $\mathfrak{g}$  一般来说  $\mathcal{A}$  不唯一.

- **def.:** a pair  $(U(\mathfrak{g}), \hat{i})$  (需要满足结合律) with the following properties is called a **universal enveloping algebra**,

1.  $\hat{i}([A, B]) = \hat{i}(A) \cdot \hat{i}(B) - \hat{i}(B) \cdot \hat{i}(A), \forall A, B,$
2. the **smallest subalgebra** with **identity**  $e \in U(\mathfrak{g})$  **containing**  $\{\hat{i}(A), A \in \mathfrak{g}\}$  is  $U(\mathfrak{g})$ ,  
(这个条件称为  $U(\mathfrak{g})$  由  $\hat{i}(A), A \in \mathfrak{g}$  生成)
3. 考虑  $\mathfrak{g}$  嵌入的某个 associative algebra  $\mathcal{A}$  with identity, 那么  $U(\mathfrak{g})$  和  $\mathcal{A}$  之间存在 a **unique** algebra homomorphism  $\phi : U(\mathfrak{g}) \rightarrow \mathcal{A}$ , s.t.,

$$\begin{cases} \phi(e) = e' \in \mathcal{A} \\ \phi \circ \hat{i} = \hat{j} : \mathfrak{g} \rightarrow \mathcal{A} \end{cases} \quad (8.3.3)$$

即  $\mathcal{A} \simeq U(\mathfrak{g}) / \ker(\phi)$ , (只需要说明这个  $\ker(\phi)$  是唯一的就行).

- $\mathfrak{g}$  的任意两个 universal enveloping algebras 互相同构.  
(由于  $U(\mathfrak{g})$  本身也是 associated algebra, 再利用性质 3)
- **theorem:** 任何李代数都存在一个 universal enveloping algebra.

**proof:**

– **def.:** the **tensor algebra**  $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$ , (notation  $\mathfrak{g}^{\otimes k} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ ).

\*  $T(\mathfrak{g})$  是对于  $B(\cdot, \cdot) = \otimes$  满足结合律的代数.

\* 且存在单位元  $1 \in \mathbb{C} \equiv \mathfrak{g}^{\otimes 0}$ .

$(T(\mathfrak{g}), \otimes)$  满足  $U(\mathfrak{g})$  的第两个条件, 但是, 对于第三个条件, 考虑,

$$\begin{cases} \psi(1) = e \in \mathcal{A} \\ \psi : T(\mathfrak{g}) \rightarrow \mathcal{A}, A \mapsto \hat{j}(A) \end{cases} \quad (8.3.4)$$

显然, 这样的 homomorphism  $\psi$  不唯一, 实际上  $U(\mathfrak{g})$  是  $T(\mathfrak{g})$  的一个商空间 (见下文).

现在, 我们来构造  $U(\mathfrak{g})$ . 考虑双向不变子空间 (two-sided ideal)  $J$ ,

$$J = \left\{ \sum_i \alpha_i \otimes (A_i \otimes B_i - B_i \otimes A_i - [A_i, B_i]) \otimes \beta_i \mid A_i, B_i \in \mathfrak{g}, \alpha_i, \beta_i \in T(\mathfrak{g}) \right\} \quad (8.3.5)$$

那么  $U(\mathfrak{g}) = T(\mathfrak{g})/J$ .

– 注意,  $J$  是一个 **two-sided ideal**, 即  $\forall \alpha \in T(\mathfrak{g}), \beta \in J$ , 有  $\alpha \otimes \beta, \beta \otimes \alpha \in J$ .

– 且  $J$  是包含形如  $A \otimes B - B \otimes A - [A, B]$  的元素的最小的 two-sided ideal.

– 注意, the kernel of an algebra homomorphism is always a two-sided ideal. 考虑  $\phi : U \rightarrow \mathcal{A}$ , 那么,  $\forall \alpha \in \ker(\phi), \beta \in U$ ,

$$\phi(\beta \cdot \alpha) = \phi(\beta) \cdot 0 = 0 \quad (8.3.6)$$

**proof:**

– 第一条 ( $T(\mathfrak{g})$  不满足, 但  $T(\mathfrak{g})/J$  满足),

$$[A, B] \sim A \otimes B - B \otimes A \quad (8.3.7)$$

– 第二条成立 ( $T(\mathfrak{g})$  和  $T(\mathfrak{g})/J$  都满足).

– 第三条 ( $T(\mathfrak{g})$  和  $T(\mathfrak{g})/J$  都满足), 考虑 algebra homomorphism  $\psi : T(\mathfrak{g}) \rightarrow \mathcal{A}$  s.t.,

$$\begin{cases} \psi(1) = e \in \mathcal{A} \\ \psi(A_1 \otimes \cdots \otimes A_k) = \hat{j}(A_1) \cdots \hat{j}(A_k) \end{cases} \quad (8.3.8)$$

那么, (考虑到 kernel 一定是 two-sided ideal), 必然有  $J \subset \ker(\psi)$ .

(令  $\phi = \psi|_{U(\mathfrak{g})}$ , 有  $\ker(\psi) = J \oplus \ker(\phi)$ , 即  $\mathcal{A} = T(\mathfrak{g})/\ker(\psi) = U(\mathfrak{g})/\ker(\phi)$ .)

注意,  $\mathcal{A}$  由  $e$  和 (8.3.2) 中的元素张成,  $\psi$  必须满足  $\psi(1) = e$ , 考虑第二个条件  $\phi \circ \hat{i} = \hat{j}$ , 考虑  $\forall A \in \mathfrak{g}$ ,

$$\phi(A) = \hat{j}(A) \quad (8.3.9)$$

且  $U(\mathfrak{g})$  由  $A_1 \oplus \cdots \oplus A_k, k \geq 0$  张成, 所以  $\phi$  的选取是唯一的.

- $(\pi, V)$  是李代数  $\mathfrak{g}$  的一个表示 (不一定是有限维), 那么存在一个 unique algebra homomorphism,

$$\tilde{\pi} : U(\mathfrak{g}) \rightarrow \text{End}(V) \quad \text{s.t.} \quad \begin{cases} \tilde{\pi}(1) = I \\ \tilde{\pi}(A) = \pi(A), \forall A \in \mathfrak{g} \subset U(\mathfrak{g}) \end{cases} \quad (8.3.10)$$

**proof:**

可以认为  $\mathcal{A} = \text{End}(V), \hat{j} = \pi$ , 那么, 存在 unique  $\tilde{\pi} = \phi : U(\mathfrak{g}) \rightarrow \mathcal{A}, \dots$

## 8.4 Poincaré-Birkhoff-Witt theorem

- **PBW theorem:** 对于有限维李代数  $\mathfrak{g}$  (不一定半单), 其基矢为  $\{A_1, \dots, A_k\}$ , 那么,

$$\hat{i}(A_1)^{n_1} \cdots \hat{i}(A_k)^{n_k} \quad (8.4.1)$$

其中  $n_i$  是非负整数, 构成  $U(\mathfrak{g})$  的基矢 (张成并线性独立).

– 同时意味着  $\hat{i} : \mathfrak{g} \rightarrow U(\mathfrak{g})$  是 injective (one-to-one).

**proof:**

证明方法类似于 reordering lemma (见 (8.1.7)).

首先 (8.4.1) 中的向量显然能张成  $U(\mathfrak{g})$ , 我需要证明它们线性独立, 方法如下:

考虑一个向量空间  $D$ , 其基底为  $\{v_{i_1, \dots, i_N}\}$ , 其中  $1 \leq i_1 \leq \cdots \leq i_N \leq k$ . 我们的目标是证明存在一个线性映射  $\gamma : U(\mathfrak{g}) \rightarrow D$ , (这个映射不必是同构), 使得,

$$\hat{i}(A_{i_1}) \cdots \hat{i}(A_{i_N}) \mapsto v_{i_1, \dots, i_N} \quad (8.4.2)$$

为此, 我们希望能构造一个线性映射  $\delta : T(\mathfrak{g}) \rightarrow D$ , s.t.,

1.  $\delta(A_{i_1} \otimes \cdots \otimes A_{i_N}) = v_{i_1, \dots, i_N}$  if  $1 \leq i_1 \leq \cdots \leq i_N \leq k$ ,
2.  $\delta[J] = \{0\}$ , 因此  $\delta$  自然能给出线性映射  $\gamma : U(\mathfrak{g}) \rightarrow D$ .

构造方法如下.

考虑  $n$  阶单项式  $A_{j_1} \otimes \cdots \otimes A_{j_n}$ , 令逆序的下标对数为其 index, (显然 0, 1 阶的单项式的 index 都是零),  $n \leq k$  阶单项式的 index 最高为  $\frac{n(n-1)}{2}$ . 下面用归纳法来确定  $\delta$ .

- 假设  $\delta$  的定义 (已经在 index 小于等于  $p$ , 或者阶数小于等于  $n-1$  下做出了定义) 使得, 下式在: 等号左边两相的 index 都不超过  $p \geq 1$  时, 且  $n \leq N$  时, 成立,

$$\delta(A_{i_1} \cdots (A_{i_j} A_{i_{j+1}} - A_{i_{j+1}} A_{i_j}) \cdots A_{i_n}) = \delta(A_{i_1} \cdots [A_{i_j}, A_{i_{j+1}}] \cdots A_{i_n}) \quad (8.4.3)$$

( $p=0$  一定成立, 因为  $i_j = i_{j+1}$ , 等号两边为零)

- 考虑等号左侧第一项的 index 为  $p+1$ , 且  $i_j > i_{j+1}$  是逆序, 那么, 定义  $\delta$  在 (8.4.3) 下依然成立. 这样我们就把  $\delta$  的定义拓展到了  $n$  阶, index 为  $p+1$  的情况,

$$\delta(A_{i_1} \cdots \underbrace{A_{i_j} A_{i_{j+1}}}_{\text{逆序}} \cdots A_{i_n}) = \delta(A_{i_1} \cdots A_{i_{j+1}} A_{i_j} \cdots A_{i_n}) + \delta(\cdots [A_{i_j}, A_{i_{j+1}}] \cdots) \quad (8.4.4)$$

- 由于 (8.4.4) 左侧至少有两处逆序 (假设另一个逆序对为  $i_l > i_{l+1}$  且  $j < l$ ), 那么还需要证明等式右侧与逆序对的选取无关, 我们通过分类讨论证明这一点.

分类讨论:

- 如果  $j+1 \leq l-1$ .

考虑,

$$\begin{aligned}
& \delta(\cdots A_{i_j} A_{i_{j+1}} \cdots A_{i_l} A_{i_{l+1}} \cdots) \\
&= \delta(\cdots A_{i_j} A_{i_{j+1}} \cdots A_{i_{l+1}} A_{i_l} \cdots) + \delta(\cdots A_{i_j} A_{i_{j+1}} \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) \\
&= \delta(\cdots A_{i_{j+1}} A_{i_j} \cdots A_{i_{l+1}} A_{i_l} \cdots) + \delta(\cdots [A_{i_j}, A_{i_{j+1}}] \cdots A_{i_{l+1}} A_{i_l} \cdots) \\
&\quad + \delta(\cdots A_{i_{j+1}} A_{i_j} \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) + \delta(\cdots [A_{i_j}, A_{i_{j+1}}] \cdots [A_{i_l}, A_{i_{l+1}}] \cdots) \\
&= \cdots
\end{aligned} \tag{8.4.5}$$

最后一个等号右侧的第一, 三项和第二, 四项结合, 就得到 (8.4.4) 右侧.

(要注意, 证明过程中每一个单项式的 index 都小于等于  $p$ , 或者阶数小于等于  $n-1$ )

- 如果  $j+1 = l$ .

为了简洁, 用  $A = A_{i_j}, B = A_{i_{j+1}=l}, C = A_{i_{l+1}}$ , 那么,

$$\begin{aligned}
& \delta(\cdots BAC \cdots) + \delta(\cdots [A, B]C \cdots) \\
&= \delta(\cdots CBA \cdots) + \delta(\cdots [B, C]A \cdots) + \delta(\cdots B[A, C] \cdots) + \delta(\cdots [A, B]C \cdots)
\end{aligned} \tag{8.4.6}$$

同时,

$$\begin{aligned}
& \delta(\cdots ACB \cdots) + \delta(\cdots A[B, C] \cdots) \\
&= \delta(\cdots CBA \cdots) + \delta(\cdots [A, C]B \cdots) + \delta(\cdots C[A, B] \cdots) + \delta(\cdots A[B, C] \cdots)
\end{aligned} \tag{8.4.7}$$

那么, 只需要证明,

$$\begin{aligned}
& [[B, C], A] + \underbrace{[B, [A, C]]}_{=[C, A], B]} + [[A, B], C] = 0
\end{aligned} \tag{8.4.8}$$

而这就是 Jacobi identity.

## 8.5 construction of Verma modules

- **def.:** a left ideal of  $U(\mathfrak{g})$  generated by  $\{\alpha_i\}$  is,

$$I = \left\{ \sum_i \beta_i \alpha_i \mid \forall \beta_i \in U(\mathfrak{g}) \right\} \tag{8.5.1}$$

- 用  $I_\mu$  表示一个 left ideal generated by,

$$\{H - \langle \mu, H \rangle, \forall H \in \mathfrak{h}\} \cup \bigcup_{\alpha \in R^+} \mathfrak{g}_\alpha \tag{8.5.2}$$

(第一个集合中的元素是一个一阶向量减一个零阶向量)

- **def.:** the **Verma module** with highest weight  $\mu$  is,

$$W_\mu = U(\mathfrak{g})/I_\mu \tag{8.5.3}$$

用  $[\alpha]$  表示  $\alpha \in U(\mathfrak{g})$  在  $W_\mu$  中的像 (等价类).

- $(\pi_\mu, W_\mu)$  是 universal enveloping algebra 的一个表示,

$$\pi_\mu(\alpha)[\beta] = [\alpha\beta] \tag{8.5.4}$$

proof:

$$\pi_\mu(\alpha_1)\pi_\mu(\alpha_2)[\beta] = [\alpha_1\alpha_2\beta] = \pi_\mu(\alpha_1\alpha_2)[\beta] \quad (8.5.5)$$

且如果  $\beta \sim \beta'$ , 那么  $\alpha\beta \sim \alpha\beta'$ .

– 所以, (其中  $A \in \mathfrak{g}_{\alpha \in R^+}$ ),

$$\begin{cases} \pi_\mu(H)[1] = \langle \mu, H \rangle [1] \\ \pi_\mu(A)[1] = 0 \end{cases} \quad (8.5.6)$$

但要注意, 一般  $[A\alpha] \neq 0$ , 所以  $\pi_\mu(A) \neq [A] = 0$ , (不过  $[\alpha A] = 0$ ).

•  $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha$ , 由于  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ , 所以  $\mathfrak{n}^+, \mathfrak{n}^-$  都是  $\mathfrak{g}$  的子代数.

• theorem:

–  $(\pi_\mu, W_\mu)$  是一个 highest weight cyclic rep. (定义见 section 8.2 开头), 且最高权为  $\mu$  (不过, 由于  $W_\mu$  一定是无限维, 最高权不一定是 dominant), 最高权向量为  $v_0 = [1]$ .

–  $\{B_1, \dots, B_k\}$  是  $\mathfrak{n}^-$  的一组基底, 那么,

$$\pi_\mu(B_1)^{n_1} \dots \pi_\mu(B_k)^{n_k} v_0 \quad (8.5.7)$$

(其中  $n_i \in \mathbb{Z}^+$ ), 组成  $W_\mu$  的一组基底.

结合 PBW theorem, 可见有向量空间同构  $W_\mu \simeq U(\mathfrak{n}^-)$ , 且  $\alpha \mapsto \pi_\mu(\alpha)v_0$ .

proof:

lemma: 令  $J_\mu$  是  $U(\mathfrak{n}^+ \oplus \mathfrak{h})$  上的, 由 (8.5.2) 中的元素生成的 left ideal, 那么  $v_0 = [1] \notin J_\mu$ .

proof:

考虑一维表示,

$$\sigma_\mu : \mathfrak{n}^+ \oplus \mathfrak{h} \rightarrow \underbrace{\text{End}(\mathbb{C})}_{=\mathbb{C}} \quad \text{s.t.} \quad \begin{cases} \sigma_\mu(A) = 0 & A \in \mathfrak{n}^+ \\ \sigma_\mu(H) = \langle \mu, H \rangle & H \in \mathfrak{h} \end{cases} \quad (8.5.8)$$

对比 (8.3.10), 可知存在一个唯一的  $\tilde{\sigma}_\mu : U(\mathfrak{n}^+ \oplus \mathfrak{h}) \rightarrow \mathbb{C}$ , s.t.,

$$\begin{cases} \tilde{\sigma}_\mu(1) = 1 \\ \tilde{\sigma}_\mu(A + H) = \langle \mu, H \rangle \end{cases} \quad \text{and} \quad \ker(\tilde{\sigma}_\mu) \supset \{0\} \cup \mathfrak{n}^+ \cup \{H \perp \mu\} \cup \{H - \langle \mu, H \rangle\} \quad (8.5.9)$$

且  $\ker(\tilde{\sigma}_\mu)$  是  $U(\mathfrak{n}^+ \oplus \mathfrak{h})$  上的一个 two-sided ideal, 所以  $J_\mu \subset \ker(\tilde{\sigma}_\mu)$ , 所以...

含有  $v_0$  的不变子空间  $U = W_\mu$ , 因为  $\pi_\mu(\alpha)v_0 = [\alpha]$ , 那么证明第一个 theorem 只需要再说明  $[1] \neq [0]$ , (highest weight cyclic rep. 的前两个性质见 (8.5.6)).

要说明  $[1] \neq [0]$ , 只需要证明  $1 \notin I_\mu$ .

考虑  $I_\mu$  中的元素按照 PBW theorem 展开,

$$\begin{aligned} I_\mu \ni \alpha &= \sum_{\beta_1 \in U(\mathfrak{g})} \beta_1 (H - \langle \mu, H \rangle) + \sum_{\beta_2 \in U(\mathfrak{g})} \beta_2 A_\alpha \\ &= \sum_{\alpha_1, \dots, \alpha_N \in U(\mathfrak{n}^+)} (B_{\alpha_1})^{n_1} \dots (B_{\alpha_N})^{n_N} \underbrace{\gamma_{n_1, \dots, n_N}}_{\in U(\mathfrak{n}^+)} (H - \langle \mu, H \rangle) + \dots \\ &= \sum_{\alpha_1, \dots, \alpha_N \in U(\mathfrak{n}^+)} (B_{\alpha_1})^{n_1} \dots (B_{\alpha_N})^{n_N} \underbrace{\delta_{n_1, \dots, n_N}}_{\in J_\mu} \end{aligned} \quad (8.5.10)$$

如果  $\alpha = 1 \in I_\mu$ , 那么  $n_1 = \dots = n_N = 0$ , 且  $\alpha = 1 = \delta_{0, \dots, 0} \in J_\mu$ , 与引理的结论矛盾, 所以  $1 \notin I_\mu$ .

现在来证明第二个 theorem. 已经说明了  $W_\mu$  是含  $v_0$  的最小的不变子空间, 所以 (8.5.7) 中的向量一定张成  $W_\mu$ , 我们还需要证明它们线性独立. 考虑, 如果它们线性相关,

$$\sum_{\substack{\in \mathbb{C} \\ C_{n_1, \dots, n_k}}} [(B_1)^{n_1} \cdots (B_k)^{n_k}] = 0 \\ \Rightarrow \alpha = \sum C_{n_1, \dots, n_k} (B_1)^{n_1} \cdots (B_k)^{n_k} \in I_\mu \quad (8.5.11)$$

但是, 对照 (8.5.10) (注意, 利用 PBW theorem 得到的展开式是唯一的), 可见  $C_{n_1, \dots, n_k} \in J_\mu$ , 而这不成立.

## 8.6 irreducible quotient modules

- 本节我们将证明 Verma module  $W_\mu$  有一个 largest nonzero invariant subspace  $U_\mu$ , 而商空间  $V^\mu = W_\mu/U_\mu$  是最高权为  $\mu$  的不可约表示. 且如果  $\mu$  是 dominant integral, 那么  $V^\mu$  是有限维.

- def.:**  $U_\mu$  由如下向量  $v \in W_\mu$  组成 (注意, (8.5.7) 是  $W_\mu$  的一组基底):

- $v$  的  $v_0 = [1]$  分量为零,
  - 注意, 并不是所有由低于  $\mu$  的权对应的权向量组成的矢量都属于  $U_\mu$ , 例如  $[B_\alpha] \notin U_\mu, \alpha \in R^+$ , 因为  $\pi_\mu(A_\alpha)[B_\alpha] = \langle \mu, H_\alpha \rangle v_0$ , 见第二个条件.
- $\pi_\mu(A_1) \cdots \pi_\mu(A_k)v, k \geq 1$  的  $v_0$  分量也为零, 其中  $A_1, \dots, A_k \in \mathfrak{n}^+$ ,

也就是所有通过升算符无法达到  $v_0$  的向量.

- $U_\mu$  是一个不变子空间.

**proof:**

- 首先  $\pi_\mu(A)[U_\mu] \subseteq U_\mu, \forall A \in \mathfrak{n}^+$ .
- $\pi_\mu(A_1) \cdots \pi_\mu(A_k)v, k \geq 0$  是由低于  $\mu$  的权对应的权向量组成, 考虑,

$$\pi_\mu(A_1) \cdots \pi_\mu(A_k) \pi_\mu(C)v \quad (8.6.1)$$

其中  $C \in \mathfrak{h} \oplus \mathfrak{n}^-$ , reordering lemma 告诉我们 (8.6.1) 等于下列形式的向量的线性组合,

$$\pi_\mu^{n_1}(B_1) \cdots \pi_\mu^{n_N}(B_N) \pi_\mu^{n'_1}(H_1) \cdots \pi_\mu^{n'_r}(H_r) \pi_\mu^{n''_1}(A_1) \cdots \pi_\mu^{n''_N}(A_N)v \quad (8.6.2)$$

只能让这些权向量对应的权保持不变或降低, 所以...

- 商空间  $V^\mu = W_\mu/U_\mu$  构成  $\mathfrak{g}$  的一个不可约表示 (见 section 5.2).

**proof:**

显然, 对于  $V_\mu$  的不变子空间  $V'$ , 有  $V' \oplus U_\mu \subset W_\mu$  也是一个不变子空间 (因为已经证明了  $U_\mu$  是不变子空间).

那么, 现在只需要证明:  $W_\mu$  中, 包含子集  $U_\mu$  的不变子空间要么是  $U_\mu$ , 要么是  $W_\mu$ .

考虑不变子空间  $U'$  满足  $U_\mu \subset U' \subset W_\mu$ , 且  $U' \neq U_\mu$ , 那么,

- 有  $v \in U'$  且  $v \notin U_\mu$ .
- 由于  $v \notin U_\mu$ , 一定存在一些组合  $A_1, \dots, A_k$  使得  $u = \pi_\mu(A_1) \cdots \pi_\mu(A_k)v$  的  $v_0$  分量不为零.
- 由于  $U'$  是不变子空间,

$$\prod_{\lambda \neq \mu} (\pi_\mu(H) - \langle \lambda, H \rangle I) u \in U' \quad (8.6.3)$$

对于  $u$  在 (8.5.7) 中的其它 (非  $v_0$ ) 分量, 经过上式都被化为零 (注意  $\mathfrak{h}$  是 Abelian), 所剩的只有  $v_0$  分量, 因此  $v_0 \in U'$ .

–  $U'$  含有  $v_0$ , 因此必然有  $U' = W_\mu$ .

- $(\pi_\mu, V^\mu)$  是最高权为  $\mu$ , 对应权向量为  $v_0$  的 highest weight cyclic rep..

- 一些计算: 对于  $\alpha \in \Delta$  (这一点对 (8.6.6) 中的分析很重要, 因为  $\alpha$  无法表示为  $R^+$  中其它元素的线性组合) 有,

$$\pi_\mu(A_\alpha)\pi_\mu^i(B_\alpha)v_0 = i(\langle \mu, H_\alpha \rangle - (i-1))\pi_\mu^{i-1}(B_\alpha)v_0 \quad (8.6.4)$$

所以, 如果  $\langle \mu, H_\alpha \rangle \in \mathbb{Z}^+ \cup \{0\}$ , 那么,

$$\pi_\mu(A_\alpha) \underbrace{\pi_\mu^{\langle \mu, H_\alpha \rangle}(B_\alpha)v_0}_{\text{令其}=v} = 0 \quad (8.6.5)$$

且对于  $\forall \beta \in R^+, j \in \mathbb{Z}^+$ ,

$$\pi_\mu^j(A_\beta)v \in V_{\mu - \langle \mu, H_\alpha \rangle \alpha + j\beta} \quad (8.6.6)$$

注意到  $\mu - \langle \mu, H_\alpha \rangle \alpha + j\beta \not\leq \mu$ , 由于  $\mu$  是最高权, 所以  $\pi_\mu^j(A_\beta)v = 0$ , 所以  $v \in U_\mu$ .

## 8.7 finite-dimensional quotient modules

- 本 section 将表明, 对于 dominant integral element  $\mu$ , 不可约表示  $V^\mu = W_\mu/U_\mu$  是有限维的.

•

## 8.8 Casimir operators

- **def.:** the 2nd-order Casimir operator is,

$$C_2 = -B^{ij}A_i \otimes A_j \quad (8.8.1)$$

where  $B^{ij} = B_{ij}^{-1}$ .

- the 2nd-order Casimir operator commutes with all the generators.

**proof:**

$$\begin{aligned} [C_2, A_k] &= -B^{ij}[A_i A_j, A_k] \\ &= -B^{ij}(-f_{jk}^l A_i A_l - f_{ik}^l A_l A_j) \end{aligned} \quad (8.8.2)$$

notice that  $B^{ij}$  is symmetric, so,

$$\begin{aligned} [C_2, A_k] &= -B^{ij}(-f_{ik}^l A_j A_l - f_{ik}^l A_l A_j) \\ &= B^{ij}f_{ik}^l (A_j A_l + A_l A_j) \\ &= \underbrace{B^{ij}B^{lm}(A_j A_l + A_l A_j)}_{\text{symmetric about } (i,m)} f_{ikm} = 0 \end{aligned} \quad (8.8.3)$$



# Chapter 9

## $\mathfrak{su}(2)_{\mathbb{C}}$ algebra

- $\mathfrak{su}(2) = \{A \in \mathcal{M}_2(\mathbb{C}) | A^\dagger = -A \text{ and } \text{tr} A = 0\}$ .
  - $\dim \mathfrak{su}(2) = 2^2 - 1 = 3$ .
  - $\mathfrak{su}(2) = \text{span}\{iJ_1, iJ_2, iJ_3\}$  is a real vector space.
- its structure is,

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (9.0.1)$$

where  $i, j, k = 1, 2, 3$ .

- ladder operators,

$$\begin{cases} J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \in \mathfrak{su}(2)_{\mathbb{C}} \\ [J_3, J_{\pm}] = \pm J_{\pm} \\ [J_+, J_-] = J_3 \\ J^2 = J_+J_- + J_-J_+ + J_3^2 \end{cases} \quad (9.0.2)$$

- another basis is  $H = 2J_3, A = \sqrt{2}J_+, B = \sqrt{2}J_-$ , and,

$$\begin{cases} [H, A] = 2A \\ [H, B] = -2B \\ [A, B] = H \end{cases} \quad \text{ad}_H = \begin{pmatrix} 0 & & \\ & 2 & \\ & & -2 \end{pmatrix} \quad \text{ad}_A = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{ad}_B = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad (9.0.3)$$

so, the Killing form is,

$$B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \quad (9.0.4)$$

- its Killing form is  $B_{ij} = \epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$ .
- its 2nd order Casimir operator is,

$$C_2 = -B^{ij}A_iA_j = \frac{1}{2}\delta_{ij}J_iJ_j = \frac{1}{2}J^2 \quad (9.0.5)$$

### 9.1 representations of $\mathfrak{su}(2)_{\mathbb{C}}$ algebra

- for each (half-)integer  $j$ , there exists an  $2j + 1$  dimensional **irreducible** complex rep.,

$$\pi_j : \mathfrak{su}(2)_{\mathbb{C}} \rightarrow \text{span}(|j, m\rangle, m = -j, \dots, j) \quad (9.1.1)$$

and any two irreducible rep. with the same dimension are isomorphic.

**proof:**

let  $\pi$  be an irreducible rep. of  $\mathfrak{su}(2)_{\mathbb{C}}$  on a finite-dimensional complex vector space  $V$ , and  $|u\rangle$  is a eigenvector of  $\pi(J_3)$ ,

$$\begin{cases} \pi(J_3)|u\rangle = \alpha|u\rangle \\ \pi(J_3)\pi^k(J_{\pm})|u\rangle = (\alpha \pm k)\pi^k(J_{\pm})|u\rangle \end{cases} \quad (9.1.2)$$

since  $V$  is finite-dimensional, so there is some  $N_{\pm} \geq 0$ , s.t.,

$$\pi^{N_{\pm}}(J_{\pm})|u\rangle \neq 0 \quad \text{but} \quad \pi^{N_{\pm}+1}(J_{\pm})|u\rangle = 0 \quad (9.1.3)$$

let's set  $|u_0\rangle = \pi^{N_-}(J_-)|u\rangle$  and  $\lambda_0 = \alpha - N_-$ ,  $|u_k\rangle = \pi^k(J_+)|u_0\rangle$ , then,

$$\pi(J_3)|u_k\rangle = (\lambda_0 + k)|u_k\rangle, k = 0, \dots, 2j \quad (9.1.4)$$

where  $j = \frac{N_+ + N_-}{2}$ , and,

$$\begin{aligned} \pi(J_-)|u_k\rangle &= -k(\lambda_0 + \frac{k-1}{2})|u_{k-1}\rangle \\ \xrightarrow{k-1=2j} 0 &= -(2j+1)(\lambda_0 + j)|u_{2j-1}\rangle \implies \lambda_0 = -j \end{aligned} \quad (9.1.5)$$

so, for any **finite-dimensional** rep. of  $\mathfrak{su}(2)_{\mathbb{C}}$ ,  $\lambda_0 = -j$  must be a **(half-)integer**.

- according to appendix A.1,  $|u_0\rangle, \dots, |u_{2j}\rangle$  are **linearly independent**.
- $\text{span}(|u_0\rangle, \dots, |u_{2j}\rangle)$  is **invariant** under  $\pi(J_3), \pi(J_{\pm})$ , hence invariant under all  $\pi(A), A \in \mathfrak{su}(2)_{\mathbb{C}}$ .
- so every irreducible rep. is of the form as  $\text{span}(|u_0\rangle, \dots, |u_{2j}\rangle)$ .

- for any finite-dim. (not necessarily irreducible) rep.  $(\pi, V)$  of  $\mathfrak{su}(2)_{\mathbb{C}}$ ,

1. all eigenvalues of  $\pi(J_3)$  are **(half-)integer**,

$$-j, -j+1, \dots, j \quad (9.1.6)$$

2.  $\pi(J_{\pm})$  are nilpotent,

3. let  $S = e^A e^{-B} e^A \implies \Pi(S) = e^{\pi(A)} e^{-\pi(B)} e^{\pi(A)}$ , then,

$$\text{Ad}_S H = -H \implies \Pi(S)\pi(H)\Pi(S^{-1}) = -\pi(H) \quad (9.1.7)$$

**calculation:**

use the Campbell's identity,

$$\begin{aligned} \text{Ad}_{\Pi(S)}\pi(H) &= \pi(\text{Ad}_{e^A} \text{Ad}_{e^{-B}} \text{Ad}_{e^A} H) \\ &= \pi(e^{\text{ad}_A} e^{-\text{ad}_B} e^{\text{ad}_A} H) \end{aligned} \quad (9.1.8)$$

and,

$$\begin{aligned} e^{\text{ad}_A} H &= H - 2A \\ e^{-\text{ad}_B}(H - 2A) &= H - 2B - 2(A + H - B) = -H - 2A \\ e^{\text{ad}_A}(-H - 2A) &= -(H - 2A) - 2A = -H \end{aligned} \quad (9.1.9)$$

and,

$$\begin{aligned} \text{Ad}_S^{-1} H &= e^{-\text{ad}_A} e^{\text{ad}_B} e^{-\text{ad}_A} H \\ &= e^{-\text{ad}_A} e^{\text{ad}_B} (H + 2A) \\ &= e^{-\text{ad}_A} \underbrace{((H + 2B) + 2(A - H - B))}_{=-H+2A} = -H \end{aligned} \quad (9.1.10)$$

but,

$$\begin{aligned}
e^{\text{ad}_{J_+}} J_3 &= J_3 - J_+ \\
e^{-\text{ad}_{J_-}} (J_3 + J_+) &= (J_3 - J_-) - (J_+ + J_3 - \frac{1}{2} J_-) = -J_+ - \frac{1}{2} J_- \\
e^{\text{ad}_{J_+}} (-J_+ - \frac{1}{2} J_-) &= -J_+ - \frac{1}{2} (J_- + J_3 - \frac{1}{2} J_+) \tag{9.1.11}
\end{aligned}$$

- the eigenstates  $|j, m\rangle$  of the operators  $J_3, J^2$  are,

$$\begin{cases} J_3 |j, m\rangle = m |j, m\rangle \\ J^2 |j, m\rangle = j(j+1) |j, m\rangle \\ J_{\pm} |j, m\rangle = \frac{1}{\sqrt{2}} \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle \end{cases} \tag{9.1.12}$$

when  $J_1 = \frac{1}{\sqrt{2}}(J_+ + J_-)$  and  $J_2 = \frac{1}{i\sqrt{2}}(J_+ - J_-)$  act on  $|s, m\rangle$ ,

$$\begin{cases} J_1 |j, m\rangle = \lambda_+(j, m) |j, m+1\rangle + \lambda_-(j, m) |j, m-1\rangle \\ J_2 |j, m\rangle = -i\lambda_+(j, m) |j, m+1\rangle + i\lambda_-(j, m) |j, m-1\rangle \end{cases} \tag{9.1.13}$$

where  $\lambda_{\pm}(j, m) = \sqrt{\frac{j(j+1) - m(m \pm 1)}{2}}$ .

- spin- $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$  rep. are faithful, and spin-0, 1, 2,  $\dots$  rep. are not faithful.**

### 9.1.1 spin- $\frac{1}{2}$ representation

- choose  $s = 1/2$ , and  $|\frac{1}{2}, \frac{1}{2}\rangle = (1, 0)^T, |\frac{1}{2}, -\frac{1}{2}\rangle = (0, 1)^T$ , then  $J_i = \frac{1}{2}\sigma_i$ , where,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.1.14}$$

and the ladder operators are,

$$J_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{9.1.15}$$

### 9.1.2 spin-1 representation

- choose  $s = 1$ , and  $|1, 1\rangle = (1, 0, 0)^T, |1, 0\rangle = (0, 1, 0)^T, |1, -1\rangle = (0, 0, 1)^T$ , then,

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{9.1.16}$$

## 9.2 direct product representation

- the direct product representation of the SU(2) group is,

$$D_{ii'jj'}^{1 \otimes 2}(g) = D_{ij}^1(g) D_{i'j'}^2(g) \tag{9.2.1}$$

- consider a group element near the identity,

$$\begin{aligned}
(1 + i\alpha_i J_i^{1 \otimes 2})_{ii'jj'} &= (\delta_{ij}^1 + i\alpha_i (J_i^1)_{ij})(\delta_{i'j'}^2 + i\alpha_i (J_i^2)_{i'j'}) \\
&= \delta_{ij}^1 \delta_{i'j'}^2 + i\alpha_i (J_i^{1 \otimes 2})_{ii'jj'} \end{aligned} \tag{9.2.2}$$

where  $(J_i^{1 \otimes 2})_{ii'jj'} = (J_i^1)_{ij} \delta_{i'j'}^2 + \delta_{ij}^1 (J_i^2)_{i'j'}$  or more compactly,

$$J_i^{1 \otimes 2} = J_i^1 \otimes I^2 + I^1 \otimes J_i^2 \tag{9.2.3}$$

- the eigenstates are,

$$J_3^{1\otimes 2} |j_1, m_1\rangle \otimes |j_2, m_2\rangle = (m_1 + m_2) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \quad (9.2.4)$$

- the  $(J^2)^{j_1 \otimes j_2}$  is,

$$\begin{aligned} (J^2)^{j_1 \otimes j_2} &= \sum_i (J_i^{j_1} \otimes I^{j_2} + I^{j_1} \otimes J_i^{j_2})^2 \\ &= (J^2)^{j_1} \otimes I^{j_2} + I^{j_1} \otimes (J^2)^{j_2} + 2 \sum_i J_i^{j_1} \otimes J_i^{j_2} \end{aligned} \quad (9.2.5)$$

**when  $(J^2)^{j_1 \otimes j_1}$  acts on  $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ :**

$$\begin{aligned} &(J^2)^{j_1 \otimes j_1} |j_1, m_1\rangle \otimes |j_2, m_2\rangle \\ &= (j_1(j_1 + 1) + j_2(j_2 + 1) + 2m_1m_2) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \\ &\quad + 2(J_1^{j_1} \otimes J_1^{j_2} + J_2^{j_1} \otimes J_2^{j_2}) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \end{aligned} \quad (9.2.6)$$

where,

$$\begin{aligned} &2(J_1^{j_1} \otimes J_1^{j_2} + J_2^{j_1} \otimes J_2^{j_2}) |j_1, m_1\rangle \otimes |j_2, m_2\rangle \\ &= 4\lambda_+(j_1, m_1)\lambda_-(j_2, m_2) |j_1, m_1 + 1\rangle \otimes |j_2, m_2 - 1\rangle \\ &\quad + 4\lambda_-(j_1, m_1)\lambda_+(j_2, m_2) |j_1, m_1 - 1\rangle \otimes |j_2, m_2 + 1\rangle \end{aligned} \quad (9.2.7)$$

### 9.2.1 Clebsch-Gordan coefficients

- direct product representation and direct sum representation,

$$\{j_1\} \otimes \{j_2\} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \{j\} \quad (9.2.8)$$

where  $\{j\}$  means spin- $j$  representation.

**proof:**

the eigenvalue and corresponded eigenspace of  $J_3^{j_1 \otimes j_2}$  is (assuming  $j_1 \geq j_2$ ),

eigenvalue	basis of the eigenspace	dimension
$j_1 + j_2$	$ j_1, j_1, j_2, j_2\rangle$	1
$j_1 + j_2 - 1$	$ j_1, j_1 - 1, j_2, j_2\rangle,  j_1, j_1, j_2, j_2 - 1\rangle$	2
$\vdots$	$\vdots$	$\vdots$
$j_1 + j_2 - 2j_2$	$ j_1, j_1 - 2j_2, j_2, j_2\rangle, \dots,  j_1, j_1, j_2, -j_2\rangle$	$1 + 2j_2$
$j_1 - j_2 - 1$	$ j_1, j_1 - 2j_2 - 1, j_2, j_2\rangle, \dots,  j_1, j_1 - 1, j_2, -j_2\rangle$	$1 + 2j_2$
$\vdots$	$\vdots$	$\vdots$
$j_1 + j_2 - 2j_1$	$ j_1, -j_1, j_2, j_2\rangle, \dots,  j_1, -j_1 + 2j_2, j_2, -j_2\rangle$	$1 + 2j_2$
$-j_1 + j_2 - 1$	$ j_1, -j_1, j_2, j_2 - 1\rangle, \dots,  j_1, -j_1 + 2j_2 - 1, j_2, -j_2\rangle$	$2j_2$
$\vdots$	$\vdots$	$\vdots$
$-j_1 - j_2$	$ j_1, -j_1, j_2, -j_2\rangle$	1

so, it is clear that we can use  $|j_1, j_1, j_2, j_2\rangle$  and  $J_-^{j_1 \otimes j_2}$  to produce  $\{j_1 + j_2\}$ , and among the rest of the vectors, the highest eigenvalue of  $J_3^{j_1 \otimes j_2}$  is  $j_1 + j_2 - 1$  and there is only one vector with this eigenvalue is remained.

hence,

$$\{j_1\} \otimes \{j_2\} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \{j\} \quad (9.2.9)$$

– example:  $\{\frac{1}{2}\} \otimes \{\frac{1}{2}\} = \underbrace{\{1\}}_{\text{spin triplet}} \oplus \underbrace{\{0\}}_{\text{spin singlet}}$

---

- the Clebsch-Gordan coefficients are,

$$\langle j_1, m_1, j_2, m_2 | j_1, j_2, j, m \rangle \quad (9.2.10)$$

where  $|j_1, j_2, j, m\rangle$  (it is common to write  $|j, m\rangle$  for short) are the coupled eigenstates of  $J_3^{j_1 \otimes j_2}$  and  $(J^2)^{j_1 \otimes j_2}$ .

- the recursion relations are,

$$\begin{aligned} & \lambda_{\pm}(j_1, \mathbf{m_1} \mp 1) \langle j_1, \mathbf{m_1} \mp 1, j_2, m_2 | j, m \rangle \\ & + \lambda_{\pm}(j_2, \mathbf{m_2} \mp 1) \langle j_2, m_2, j_2, \mathbf{m_2} \mp 1 | j, m \rangle \\ & = \lambda_{\pm}(j, m) \langle j_1, m_1, j_2, m_2 | j, \mathbf{m} \mp 1 \rangle \end{aligned} \quad (9.2.11)$$

**proof:**

just consider the ladder operators  $J_{\pm}^{j_1 \otimes j_2} = J_{\pm}^{j_1} \otimes I^{j_2} + I^{j_1} \otimes J_{\pm}^{j_2}$ ,

$$\sum_{j_1, m_1, j_2, m_2} J_{\pm}^{j_1 \otimes j_2} |j_1, m_1, j_2, m_2\rangle \langle j_1, m_1, j_2, m_2 | j, m \rangle = \dots \quad (9.2.12)$$

taking  $m = j$  gives the initial recursion relation,

$$\begin{aligned} & \lambda_{+}(j_1, \mathbf{m_1} - 1) \langle j_1, \mathbf{m_1} - 1, j_2, m_2 | j, j \rangle \\ & + \lambda_{+}(j_2, \mathbf{m_2} - 1) \langle j_2, m_2, j_2, \mathbf{m_2} - 1 | j, j \rangle = 0 \end{aligned} \quad (9.2.13)$$

- use the phase convention that  $\langle j_1, m_1, j_2, m_2 | j, j \rangle \in \mathbb{R}$  and  $> 0$ , combined with the recursion relations, we can conclude that  $\langle j_1, m_1, j_2, m_2 | j, m \rangle \in \mathbb{R}$ .

# Part IV

## Applications

## Chapter 10

# some examples of Lie groups and Lie algebras

### 10.1 general linear groups and algebras

- $\mathrm{GL}(n, \mathbb{C}) = \{M \in \mathcal{M}_n(\mathbb{C}) \mid \det M \neq 0\}$ .
  - $\dim \mathrm{GL}(n, \mathbb{C}) = n^2$ .
  - $\mathrm{GL}(n, \mathbb{R})$  有两个连通分支,

$$\mathrm{GL}(n, \mathbb{R}) = \det^{-1}[(-\infty, 0)] \sqcup \det^{-1}[(0, \infty)] \quad (10.1.1)$$

- $\mathfrak{gl}(n, \mathbb{C}) = \mathcal{M}_n(\mathbb{C})$ .
- the left-invariant vector field at  $g$  is,

$$(A_g)^i_j = x^i_k(g)(A_e)^k_j \quad (10.1.2)$$

and the Lie bracket is,

$$[A, B] = AB - BA \quad (10.1.3)$$

**proof:**

for general linear group,  $x^i_j(gh) = x^i_k(g)x^k_j(h)$ .  
so, the pushforward of the left transformation is,

$$L_{g*}(A|_e)x^i_j|_g = A(y^i_j)|_e \quad (10.1.4)$$

where  $y^i_j(h) = (L_g^*x^i_j)(h) = x^i_k(g)x^k_j(h)$ , so we have,

$$A(y^i_j)|_e = A|_e(x^k_l) \underbrace{\frac{\partial y^i_j}{\partial x^k_l}|_e}_{=x^i_m(g)\delta_k^m\delta_j^l} = x^i_k(g)A|_e(x^k_j) \quad (10.1.5)$$

$$\begin{aligned} [A, B]^i_j &= (dx^i_j)_a(A^b\partial_b B^a - B^b\partial_b A^a) \\ &= A^k_l \frac{\partial}{\partial x^k_l} B^i_j - B^k_l \frac{\partial}{\partial x^k_l} A^i_j \end{aligned} \quad (10.1.6)$$

注意  $(A_g)^i_j = x^i_k(g)(A_e)^k_j$ , 所以,

$$\frac{\partial}{\partial x^k_l}(A^i_j)|_g = \underbrace{\frac{\partial}{\partial x^k_l}(x^i_m(g))}_{=\delta_k^i\delta_m^l}(A_e)^m_j = \delta_k^i(A_e)^l_j \quad (10.1.7)$$

代入得到,

$$[A, B]^i_j|_g = (A_g)^k_l \delta_k^i (B_e)^l_j - (B_g)^k_l \delta_k^i (A_e)^l_j$$

$$= x^i_k(g)(A^k_l B^l_j - B^k_l A^l_j) \quad (10.1.8)$$

## 10.2 special linear groups and algebras

- $\mathrm{SL}(n, \mathbb{C}) = \{M \in \mathrm{GL}(n, \mathbb{C}) | \det M = 1\}$ .
- $\mathfrak{sl}(n, \mathbb{C}) = \{A \in \mathcal{M}_n(\mathbb{C}) | \mathrm{tr} A = 0\}$ .

## 10.3 the Lorentz group and the Lorentz algebra

### 10.3.1 indefinite orthogonal groups

- $\mathrm{O}(p, q) = \{\Lambda \in \mathcal{M}_n(\mathbb{R}) | \Lambda^T \eta \Lambda = \eta\}$  is called the **indefinite orthogonal group**, where  $n = p + q$  and,

$$\eta = \mathrm{diag}(\underbrace{+1, \dots, +1}_p, \underbrace{-1, \dots, -1}_q) \quad (10.3.1)$$

– 将  $\lambda$  视作一组列向量  $(\lambda_1, \dots, \lambda_n)$ , 那么,

$$\eta(\lambda_\mu, \lambda_\nu) = \eta_{\mu\nu} \quad (10.3.2)$$

即  $n$  个互相正交的向量.

- $\dim \mathrm{O}(p, q) = \frac{n(n-1)}{2}$ .
- 可以证明, 对于,

$$\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (10.3.3)$$

有  $\det \Lambda = \frac{\det A}{\det D}$ , 且  $|\det A|, |\det D| \geq 1$ .

- $\mathrm{O}(p, q)$  具有如下子群,

$$\begin{cases} \mathrm{SO}(p, q) = \{\Lambda \in \mathrm{O}(p, q) | \det \Lambda = 1\} \\ \mathrm{SO}_+(p, q) = \{\Lambda \in \mathrm{SO}(p, q) | \det A \geq 1\} \\ \mathrm{O}_+(p, q) = \{\Lambda \in \mathrm{O}(p, q) | \det A \geq 1\} \\ \mathrm{O}_-(p, q) = \{\Lambda \in \mathrm{O}(p, q) | \det D \geq 1\} \end{cases} \quad (10.3.4)$$

且有如下四个连通分支,

$$\mathrm{SO}_\pm(p, q) \quad \text{and} \quad \mathrm{O}'_\pm(p, q) = \{\det \Lambda = -1, \det A \geq 1 \text{ or } \det A \leq -1\} \quad (10.3.5)$$

### 10.3.2 the Lorentz group

- $\mathrm{L} = \mathrm{O}(3, 1)$  is called the Lorentz group.

### 10.3.3 the Lorentz algebra

- $\mathfrak{so}(3, 1) = \{A \in \mathcal{M}_4(\mathbb{R}) | A^T = -\eta A \eta\} \simeq \mathfrak{so}(4)$ .
- 考虑  $\mathfrak{so}(4, \mathbb{C})$  的 Dynkin diagram,  $D_2$ , (见 section 6.7), 可见  $\mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ .  
– 因此,  $\mathfrak{so}(3, 1)$  的 irreducible rep. 是  $\mathrm{spin}\text{-}j_1 \oplus \mathrm{spin}\text{-}j_2$ , 用  $(j_1, j_2)$  表示.

## 10.4 unitary groups and algebras

- $\mathrm{U}(n) = \{U \in \mathrm{GL}(n, \mathbb{C}) | U^\dagger U = I\}$ .  
–  $\dim \mathrm{U}(n) = n^2$ .  
–  $\mathrm{U}(n)$  is connected.
- $\mathfrak{u}(n) = \{A \in \mathcal{M}_n(\mathbb{C}) | A^\dagger = -A\}$ .



## 10.5 special unitary groups and algebras

- $\mathrm{SU}(n) = \{U \in \mathrm{GL}(n, \mathbb{C}) | U^\dagger U = I, \det U = 1\}$ .  
–  $\dim \mathrm{SU}(n) = n^2 - 1$ .
- $\mathfrak{su}(n) = \{A \in \mathcal{M}_n(\mathbb{C}) | A^\dagger = -A, \mathrm{tr} A = 0\}$ .

## 10.6 symplectic groups

- $\mathrm{Sp}(2n, \mathbb{C}) = \{A \in \mathcal{M}_{2n}(\mathbb{C}) | -\Omega A^T \Omega = A^{-1}\}$ , where,

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (10.6.1)$$

- $\dim \mathrm{Sp}(2n, \mathbb{C}) = 2n(2n + 1)$ .

## 10.7 the representations of $\mathfrak{sl}(3, \mathbb{C})$

- this section, we are going to discuss the classification of the irreducible rep. of  $\mathrm{SU}(3)$  and  $\mathfrak{sl}(3, \mathbb{C})$ .
- $\mathfrak{sl}(3, \mathbb{C}) \simeq \mathfrak{su}(3)_{\mathbb{C}}$ .
- $\mathrm{SU}(m)$  are **simply connected**, **compact** Lie groups.
  - according to section 5.1.1, 单连通李群 (的表示) 完全由其李代数 (的表示) 决定.  
rep. of  $\mathfrak{sl}(3, \mathbb{C}) \xrightarrow{\text{restrict to}} \text{rep. of } \mathfrak{su}(3) \xrightarrow{\text{simple connectedness}} \text{rep. of } \mathrm{SU}(3)$ .
  - according to section 5.2,  $\Pi$  is irreducible  $\iff \pi$  is irreducible.  
and  $\mathrm{SU}(3)$  is **compact**, so it has complete reducibility property  $\implies$  rep. of  $\mathfrak{sl}(3, \mathbb{C})$  is **completely reducible**. 可见, 半单李代数的表示都是 completely reducible.

## Chapter 11

# the spin groups, $\text{Spin}(n)$

- Wikipedia: [https://en.wikipedia.org/wiki/Spin\\_group](https://en.wikipedia.org/wiki/Spin_group).
- 关于 universal cover &  $\text{Spin}(n)$  与  $\text{SO}(n \geq 3)$  和 Clifford algebra 的关系, 见 subsection 5.1.2.

# Appendices

# Appendix A

## linear algebra review

- **def.:** an **algebra** (over a field  $K$ ) is a vector space + bilinear product  $B : A \times A \rightarrow A$  (简写做  $\cdot$ ), 几个主要特征如下,

1. 双线性形式  $B(\cdot, \cdot)$  满足左, 右分配律和 (A.0.2),
2. 可能存在单位元 (不是零向量),

$$B(e, x) = x, \forall x \quad (\text{A.0.1})$$

存在单位元的代数称为 **unital algebra**.

- 注意区分 bilinear form 和 sesquilinear form,

$$\begin{cases} B(ax, by) = abB(x, y) & \text{双线性} \\ S(ax, by) = a^*bS(x, y) & \text{半双线性, 有复共轭} \end{cases} \quad (\text{A.0.2})$$

一般用  $(\cdot, \cdot)$  和  $\langle \cdot, \cdot \rangle$  区分.

- 李代数  $\mathfrak{g}$  一定不存在单位元, (因为一定有  $[E, E] = 0 \implies E = 0$  与单位元性质矛盾).

- 另外,

$$\begin{aligned} \text{injective} &\leftrightarrow \text{one-to-one function} \\ \text{surjective} &\leftrightarrow \text{onto} \\ \text{bijective} &\leftrightarrow \text{one-to-one correspondence} \end{aligned} \quad (\text{A.0.3})$$

- a exact sequence (其中  $f_i$  都是 homomorphism),

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \cdots \quad (\text{A.0.4})$$

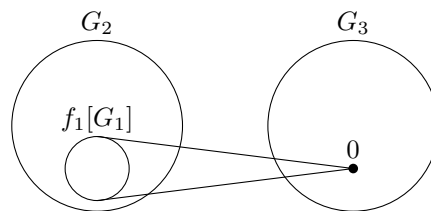
表示  $f_1[G_1] = \ker(f_2)$ . 例如,

- $G \rightarrow H \rightarrow 0$  表示  $f[G] = \ker(f_2) = H$ , 即  $f$  是 onto.
- $0 \rightarrow G \rightarrow H$  表示  $\{0\} = \ker(f)$ , 即  $f$  是 one-to-one.

- a short exact sequence,

$$0 \rightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow 0 \quad (\text{A.0.5})$$

表示  $f_1$  是 one-to-one,  $f_2$  是 onto, 且  $\ker(f_2) = f_1[G_1]$ , 所以,



注意到  $f_1, f_2$  都是 homomorphism, 所以,

$$G_3 = G_2 / f_1[G_1] \quad (\text{A.0.6})$$

## A.1 eigenvalues and eigenspaces

- eigenvectors associated to different eigenvalues are linearly independent.

**proof:**

if  $v_1, \dots, v_k$  are linearly independent eigenvectors with different eigenvalues, and  $v_{k+1}$  is a linear combination of them and is also an eigenvector, then,

$$\begin{aligned} v_{k+1} &= \sum_{i=1}^k c^i v_i \implies \lambda_{k+1} v_{k+1} = \sum_i c^i \lambda_i v_i \\ \implies 0 &= \sum_i c^i (\lambda_i - \lambda_{k+1}) v_i \end{aligned} \quad (\text{A.1.1})$$

which contradicts to the linear independence.

## A.2 spectral theorem for normal matrices

### A.2.1 diagonalization

- we want to use an **reversible matrix** to **diagonalize** a diagonalizable matrix  $A \in \text{End}(\mathbb{C}^n)$ ,

$$P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n) \iff A = P \text{diag}(\lambda_1, \dots, \lambda_n) P^{-1} \quad (\text{A.2.1})$$

we can see that:

- $\det A = \prod_i \lambda_i$ .
- $\text{tr} A = \sum_i \lambda_i$ .

**method to find  $P$ :**

consider,

$$AP = P \text{diag}(\lambda_1, \dots, \lambda_n) \quad (\text{A.2.2})$$

let the column-vector be  $P_{ij} = \xi_i^{(j)}$ , then,

$$\sum_j A_{ij} \xi_j^{(k)} = \xi_i^{(k)} \lambda_k \quad \text{or} \quad A \xi^{(k)} = \lambda_k \xi^{(k)} \quad (\text{A.2.3})$$

it is clear that  $\{\xi^{(i)}\}$  are the eigenvectors of  $A$  with corresponding eigenvalues  $\{\lambda_i\}$ .

- $A$  is diagonalizable  $\iff$  the eigenspace of  $A$  is  $n$ -dimensional.

### A.2.2 geometric multiplicity & algebraic multiplicity

- the **dimension theorem**: let  $T : V \rightarrow W$ , then,

$$\dim V = \dim \ker T + \dim T(V) \quad (\text{A.2.4})$$

where  $T(\ker T) = 0 \in W$ .

**proof:**

let  $U \cap \ker T = \{0\}$  and  $V = U \oplus \ker T$ , so,

$$\dim V = \dim \ker T + \dim U \quad (\text{A.2.5})$$

$\forall |b_1\rangle, |b_2\rangle \in U$ , if  $|b_1\rangle \neq |b_2\rangle$  then  $T|b_1\rangle \neq T|b_2\rangle$ , so,

$$T(U) \simeq U \implies \dim U = \dim T(U) \quad (\text{A.2.6})$$

and notice that  $T(V) = T(U)$ , so we have  $\dim U = \dim T(V)$ .

- some times we use  $\dim T \equiv \dim T(V)$  for convenience.

- **def.:** the **geometric multiplicity** (of eigenvalue  $\lambda_i$ ),  $\gamma_A(\lambda_i)$ , is defined to be,

$$\gamma_A(\lambda_i) = \dim(\ker(A - \lambda_i I)) \equiv n - \dim(A - \lambda_i I) \quad (\text{A.2.7})$$

- **def.:** the **algebraic multiplicity**,  $\mu_A(\lambda_i)$ , is defined to be the multiplicity (重根数) of root  $\lambda_i$  in the polynomial  $\det(A - \lambda I) = 0$ .
- theorem of geometric multiplicity & algebraic multiplicity:

$$1 \leq \gamma_A(\lambda_i) \leq \mu_A(\lambda_i) \leq n \quad (\text{A.2.8})$$

**proof:**

let  $\{v_{i=1, \dots, \gamma_A(\lambda_i)}\}$  to be the orthogonal basis of the eigenspace of  $\lambda_i$ ,

$$A |v_j\rangle, j \in \{1, \dots, \gamma_A(\lambda_i)\} = \lambda_i |v_j\rangle \quad (\text{A.2.9})$$

and let  $\{v_1, \dots, v_{\gamma_A(\lambda_i)}, v_{\gamma_A(\lambda_i)+1}, \dots, v_n\}$  to be the orthogonal basis of the vector space  $V$ , (note that  $\{v_{\gamma_A(\lambda_i)+1}, \dots, v_n\}$  are not necessarily eigenvectors), then,

$$\langle v_j | A |v_k\rangle \equiv A'_{jk} = \begin{pmatrix} \lambda_i & & * & * & * \\ & \ddots & & & \\ & & \lambda_i & * & * & * \\ & & & \ddots & \\ & & & & \lambda_i & * & * & * \end{pmatrix} \quad (\text{A.2.10})$$

then we have,

$$\det(A - \lambda I) = \det(A' - \lambda I) = (\lambda - \lambda_i)^{\gamma_A(\lambda_i)} \mathcal{P}_{n-\gamma_A(\lambda_i)}^c(\lambda) \quad (\text{A.2.11})$$

so, it is clear that  $\mu_A(\lambda_i) \geq \gamma_A(\lambda_i)$ .

### A.2.3 Schur decomposition

- **Schur decomposition:** for any complex matrix  $M$ ,

$$M = U(\text{upper triangle matrix})U^\dagger \quad (\text{A.2.12})$$

**proof:**

let  $\lambda \in \mathbb{C}$  to be an eigenvalue of  $U$  with corresponding orthonormal eigenvectors  $\{v_1, \dots, v_{\gamma_M(\lambda)}\}$ , then use the eigenvectors to construct an orthonormal basis,

$$\langle v_i | M |v_j\rangle = \begin{pmatrix} \lambda I_{\gamma_M(\lambda) \times \gamma_M(\lambda)} & M_{12} \\ 0 & M_{22} \end{pmatrix} \quad (\text{A.2.13})$$

apply the exact procedure to  $M_{22}$  until  $M$  is completely trianglized.

### A.2.4 spectral theorem for normal matrices

- **def.:** matrix  $A$  is **normal** if and only if  $[A, A^\dagger] = 0$ .
- **spectral theorem** for normal matrices:

there is an orthogonal basis consisting of eigenvectors of  $A$ .

**proof:**

– **normal triangle** matrix must be **diagonal**.

**proof:**

assume  $A$  is an upper triangle normal matrix, then  $A^\dagger A$  is upper triangle and  $AA^\dagger$  is lower triangle, which implies both of them are diagonal.

$A^\dagger A$  is diagonal  $\implies$  matrix  $A$  is also diagonal (draw  $A$  and  $A^\dagger$  and it will become obvious).

- $A$  is similar to an upper triangle matrix which is also normal  $\implies$  similar to a diagonal matrix.

### for Hermitian matrices

- for a Hermitian matrix  $H$ ,  $\lambda_i \in \mathbb{R}$ .
- if  $\lambda_i \neq \lambda_j$  then their eigenvectors are orthogonal.

**proof:**

$$\langle v_i | H | v_j \rangle = \lambda_j \langle v_i | v_j \rangle = (\langle v_j | H | v_i \rangle)^* = \lambda_i^* \langle v_i | v_j \rangle \implies \begin{cases} i = j & \lambda_i \in \mathbb{R} \\ i \neq j & \langle v_i | v_j \rangle = 0 \end{cases} \quad (\text{A.2.14})$$

- there is an orthogonal basis consisting of eigenvectors, i.e.  $\gamma_H(\lambda_i) = \mu_H(\lambda_i)$ .

### for unitary matrices

- for a unitary matrix  $U$ ,  $|\lambda_i| = 1$ .
- if  $\lambda_i \neq \lambda_j$  then their eigenvectors are orthogonal.

**proof:**

$$\underbrace{\langle v_i | U^\dagger U | j \rangle}_{\langle v_i | v_j \rangle} = \lambda_i^* \lambda_j \langle v_i | v_j \rangle \implies \begin{cases} i = j & |\lambda_i| = 1 \\ \lambda_i \neq \lambda_j & \langle v_i | v_j \rangle = 0 \end{cases} \quad (\text{A.2.15})$$

- there is an orthogonal basis consisting of eigenvectors, i.e.  $\gamma_U(\lambda_i) = \mu_U(\lambda_i)$ .

### for skew self-adjoint matrices

- for a skew self-adjoint matrix  $A$  ( $A^\dagger = -A$ ),  $\lambda_i \in i\mathbb{R}$ .
- if  $\lambda_i \neq \lambda_j$  then their eigenvectors are orthogonal.

**proof:**

$$\langle v_i | A | v_j \rangle = \lambda_j \langle v_i | v_j \rangle = (-\langle v_j | A | v_i \rangle)^* = -\lambda_i^* \langle v_i | v_j \rangle \implies \begin{cases} i = j & \lambda_i \in i\mathbb{R} \\ i \neq j & \langle v_i | v_j \rangle = 0 \end{cases} \quad (\text{A.2.16})$$

- there is an orthogonal basis...

## A.3 simultaneous diagonalization

### A.3.1 weights and weight spaces

- $V$  is a vector space,  $\mathcal{A}$  is a vector space of linear operators on  $V$ , and  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathcal{A}$ .
- **def.:** a **weight** for  $\mathcal{A}$  is an element  $\mu \in \mathcal{A}$  s.t. there exists a nonzero  $v \in V$ ,

$$Av = \langle \mu, A \rangle v \quad (\text{A.3.1})$$

for all  $A \in \mathcal{A}$ .

- **def.:**  $V_\mu = \{v \in V | A | v \rangle = | v \rangle \langle \mu, A \rangle, \forall A \in \mathcal{A}\}$  is called the **weight space** of  $\mu$ .
- if  $\mathcal{A}$  is **Abelian**, then there **exists** (at least) one weight for  $\mathcal{A}$ .

**proof:**

- assume  $W$  is the **minimal nonzero invariant subspace** of  $\mathcal{A}$ , meaning that,

$$A[W] \subseteq W, \forall A \in \mathcal{A} \quad (\text{A.3.2})$$

and every subspace of  $U$ , except  $\{0\}$ , is not nonzero invariant under some operator in  $\mathcal{A}$ .  
( $V$  is invariant but may not be minimal, so  $W$  **exists**)

- there exists  $u \in W$  s.t.  $u$  is an eigenvector of  $A \in \mathcal{A}$ , with eigenvalue  $\lambda$ .

**proof:**

let  $\{w_1, \dots, w_m\}$  be the basis of  $W$ , then,

$$Aw_i = \sum_{j=1}^m \alpha_{ij} w_j \quad (\text{A.3.3})$$

the eigenvector of  $\{\alpha_{ij}\}$  is  $\xi$  with  $\sum_i \xi^i \alpha_{ij} = \lambda_\alpha \xi^j$ , then,

$$A\xi^i w_i = \lambda_\alpha \xi^j w_j \quad (\text{A.3.4})$$

so,  $u = \xi^i w_i$  is an eigenvector of  $A$ .

- the eigenspace  $E_{A,\lambda}$  is an invariant subspace of  $\mathcal{A}$ ,

$$ABv = BAv = \lambda Bv \implies B[E_{A,\lambda}] \subseteq E_{A,\lambda}, \forall B \quad (\text{A.3.5})$$

- for  $u \in W \cap E_{A,\lambda}$ ,

$$Bu \in W \text{ and } E_{A,\lambda} \quad (\text{A.3.6})$$

so,  $W \cap E_{A,\lambda} \subseteq W$  is an invariant subspace of  $\mathcal{A}$ , which contradicts to the def. of  $W$ .

- so all the elements in  $W$  are eigenvectors of  $A$ , i.e. it is the **simultaneous eigenspace** of  $\mathcal{A}$ .

### A.3.2 simultaneous diagonalization

- **def.:**  $\mathcal{A}$  is **simultaneously diagonalizable** if there exists a basis  $\{v_1, \dots, v_n\}$  s.t. each  $v_i$  is a simultaneous eigenvector of  $\mathcal{A}$ .
- if  $\mathcal{A}$  is **Abelian** and each of  $A \in \mathcal{A}$  is **diagonalizable**, then  $\mathcal{A}$  is simultaneously diagonalizable.

**proof:**

if  $A, B$  commute and are diagonal, then, the vector space decomposes as,

$$V = \bigoplus_{i=1}^r E_{A,\lambda_i} \quad (\text{A.3.7})$$

choose the eigenvectors of  $A$  as basis, then,

$$B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{pmatrix} \quad A = \begin{pmatrix} \lambda_1 I_1 & & \\ & \ddots & \\ & & \lambda_r I_r \end{pmatrix} \quad (\text{A.3.8})$$

because  $E_{A,\lambda_{i=1,\dots,r}}$  are invariant subspaces of  $B$ .

each  $B_{i=1,\dots,r}$  is diagonalizable by  $P_i \in \text{End}(E_{A,\lambda_i})$  (or  $B$  won't be diagonalizable), and  $\lambda_i I_i$  remains diagonal.

repeat this process, all matrices in  $\mathcal{A}$  can be diagonalized.



- if  $\mathcal{A}$  is **simultaneously diagonalizable**, then,

$$V = \bigoplus_{\mu} V_{\mu} \quad (\text{A.3.9})$$

where weight spaces are **linearly independent**, i.e.,

–  $\mu_1 \neq \mu_2 \neq \cdots \neq \mu_m$  are distinct weights, then,  $\{v_i \neq 0 | v_i \in V_{\mu_i}\}$  is linearly independent.

**proof:**

first,  $V_{\mu_1} \cap V_{\mu_2} = \{0\}$  for distinct weights  $\mu_1 \neq \mu_2$ , and  $\bigcup_{\mu} V_{\mu} = V$ .

then, let's prove linear independence,

– consider,

$$(A - \langle \mu_j, A \rangle I) \sum_{i=1}^m |v_i\rangle = \sum_{i=1}^m (\langle \mu_i, A \rangle - \langle \mu_j, A \rangle) |v_i\rangle \quad (\text{A.3.10})$$

– so, if  $v_1 + \cdots + v_m = 0$ , then we must have,

$$v_1 + \cdots + v_{j-1} + v_{j+1} + \cdots + v_m = 0 \quad (\text{A.3.11})$$

– repeat the process, every element in  $\{v_i\}$  is zero.

– i.e.  $\{v_i \neq 0 | v_i \in V_{\mu_i}\}$  is linearly independent.

## A.4 obtuse basis corresponds to acute dual basis

- $\{v_1, \cdots, v_n\}$  is an obtuse (钝角) basis (i.e.  $\langle v_i, v_j \rangle \leq 0, \forall i \neq j$ ), then its dual basis is acute (锐角) (i.e.  $\langle v_i^*, v_j^* \rangle \geq 0, \forall i, j$ ).

**proof:**

用指标写出来就是,

$$g_{ab}(v_i)^a (v_j)^b \leq 0, i \neq j \iff g^{ab}(v_i^*)_a (v_j^*)_b \geq 0 \quad (\text{A.4.1})$$

或者  $g_{ij} \leq 0, i \neq j \iff g^{ij} \geq 0$ .

用数学归纳法证明. 首先, 在  $n = 1, 2$  的情况下, 定理成立.

在  $n > 2$  的情况下, 考虑投影算符,

$$P_i = 1 - \frac{|v_i\rangle \langle v_i|}{\langle v_i, v_i \rangle} \quad (\text{A.4.2})$$

那么  $P_i |v_1\rangle, \cdots, P_i |v_{i-1}\rangle, P_i |v_{i+1}\rangle, \cdots, P_i |v_n\rangle$  构成  $\text{span}(v_i)^\perp = \{u \in V | u \perp v_i\}$  的钝角基底 (显然构成基底),

$$\langle P_i v_j, P_i v_k \rangle = \langle v_j, P_i v_k \rangle = \underbrace{\langle v_j, v_k \rangle}_{\leq 0} - \frac{\langle v_j, v_i \rangle \langle v_i, v_k \rangle}{\langle v_i, v_i \rangle} \leq 0 \quad (\text{A.4.3})$$

其中  $j, k \neq i$  (注意  $\langle v_i, v_i \rangle > 0$ ). 并且,

$$(P_i v_j)^* = v_j^* \in \text{span}(v_i)^\perp, j \neq i \quad (\text{A.4.4})$$

不断重复以上过程直至维数降低到 2, 从而证明  $\langle v_i^*, v_j^* \rangle \geq 0$ .

# Appendix B

## maps between manifolds

### B.1 pushforward & pullback

- 對於一個  $m$ -dim 李群  $G$  和  $n$ -dim 流形  $M$ , 它們之間存在映射  $\sigma : G \times M \rightarrow M$ , 滿足,

$$\begin{cases} \sigma_g : M \rightarrow M \text{ is diffeomorphism} \\ \sigma_g \circ \sigma_h = \sigma_{gh} \end{cases} \quad (\text{B.1.1})$$

- 可見,  $\{\sigma_g : M \rightarrow M | g \in G\}$  is homomorphic to  $G$ , 且  $\sigma_p : G \rightarrow M$  is  $C^\infty$  and preserves the topology.
- 我們用  $\{x^\mu | \mu = 1, \dots, m\}$  表示李群  $G$  上的坐標, 用  $\{y^\nu | \nu = 1, \dots, n\}$  表示流形  $M$  上的坐標.

#### B.1.1 pullback

- 流形  $M$  上有坐標  $\{y^\mu | \mu = 1, \dots, n\}$ , 那麼通過 pullback 可以得到李群  $G$  上的  $n$  個標量場,

$$\sigma_p^* : \mathcal{F}_M \rightarrow \mathcal{F}_G \quad (\sigma_p^* y^\mu)(g) = y^\mu(\sigma_p(g)) \quad (\text{B.1.2})$$

- 不能 pushforward 的原因:

$$\sigma_{p*} x^\mu(\underline{\sigma_p(g)}) = x^\mu(g) \quad (\text{B.1.3})$$

$\sigma_p(g)$  這個  $M$  上的點可能對應不同的  $g$ , 那麼標量場  $\sigma_{p*} x^\mu$  在此處的取值也就無法確定.

- 注意:  $\{\sigma_p^* y^\mu\}$  是  $G$  上的一組  $n$  個標量場, 但是  $(\sigma_p^* y) : G \rightarrow n'$ -dim Surface  $\subset \mathbb{R}^n$ , 其中,

$$\begin{cases} n' \leq m & \text{one-to-one 時取等 } (\dim \sigma_p[G] = \dim G) \\ n' \leq n & \text{onto 時取等 } (\dim \sigma_p[G] = n) \end{cases} \quad (\text{B.1.4})$$

#### B.1.2 pushforward

- 將李群  $G$  上的矢量場 pushforward 到流形  $M$  上,

$$\sigma_{p*} : \mathcal{T}_G(1,0) \rightarrow \mathcal{T}_M(1,0) \quad \left( \sigma_{p*} \frac{\partial}{\partial x^\mu} \right) (\underline{y^\nu}) \Big|_{\sigma_p(g)} = \left( \frac{\partial}{\partial x^\mu} \right) (\sigma_p^* y^\nu) \Big|_g \quad (\text{B.1.5})$$

我們可以得到 pushforward 后的矢量場的全部  $n$  個分量.

- 但是由於  $\sigma_p^* y^\nu$  只有  $n'$  個獨立變量 ( $\dim \sigma_p^* y[G] = n'$ ), 所以 pushforward 后得到的  $m$  個矢量場中, 也只有  $n'$  個是綫性獨立的.
- 不能 pullback 的原因: 顯然無法確定 pullback 后的矢量場的  $m$  個分量, 最多  $n'$  個.

#### B.1.3 pullback

- 將流形  $M$  上的對偶矢量場 pullback 到李群  $G$  上,

$$(\sigma_p^* dy^\mu)_a \left( \frac{\partial}{\partial x^\nu} \right)^a \Big|_g = (dy^\mu)_a \left( \sigma_{p*} \frac{\partial}{\partial x^\nu} \right)^a \Big|_{\sigma_p(g)} \quad (\text{B.1.6})$$

同樣, pullback 得到的  $n$  個矢量場中, 綫性獨立的有  $n'$  個.

### B.1.4 曲綫像的切矢等於曲綫切矢的像

- 對於一個曲綫  $\gamma : \mathbb{R} \rightarrow M_1$ , 流形間的映射  $\psi : M_1 \rightarrow M_2$  將其映射為  $\psi \circ \gamma : \mathbb{R} \rightarrow M_2$ .
- 曲綫  $\gamma$  的切矢為  $\frac{\partial}{\partial t} = \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu}$ , 那麼,

$$\psi_* \left( \frac{\partial}{\partial t} \right) = \frac{dx^\mu(\gamma(t))}{dt} \psi_* \left( \frac{\partial}{\partial x^\mu} \right) \quad (\text{B.1.7})$$

是曲綫  $\psi \circ \gamma$  的切矢.

- 證明的方法是將 (B.1.7) 式兩邊作用於  $M_2$  上的坐標  $y^\nu$ ,

$$\begin{aligned} \psi_* \left( \frac{\partial}{\partial t} \right) (y^\nu) &= \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu} (\psi^* y^\nu) \\ \Rightarrow \psi_* \left( \frac{\partial}{\partial t} \right) &= \frac{dx^\mu(\gamma(t))}{dt} \frac{\partial}{\partial x^\mu} (\psi^* y^\nu) \frac{\partial}{\partial y^\nu} = \frac{d\psi^* y^\nu(\gamma(t))}{dt} \frac{\partial}{\partial y^\nu} = \frac{dy^\nu(\psi \circ \gamma(t))}{dt} \frac{\partial}{\partial y^\nu} \end{aligned} \quad (\text{B.1.8})$$

## B.2 diffeomorphisms & Lie derivatives

- 在流形  $M$  上有個 one-parameter group of diffeomorphism, 即,

$$\begin{cases} \phi_t : M \rightarrow M \text{ is diffeomorphism} \\ \phi_s \circ \phi_t = \phi_{s+t} \end{cases} \quad (\text{B.2.1})$$

且對應矢量場  $\xi^a \Big|_p = \frac{d}{dt} \Big|_{t=0} \phi_t(p)$ .

### B.2.1 Lie derivatives

- 對於流形  $M$  上的任意  $(k, l)$  型張量場,

$$\mathcal{L}_\xi T^{a\dots}_{b\dots} \Big|_p = \lim_{t \rightarrow 0} \frac{1}{t} \left( T^{a\dots}_{b\dots} \Big|_{\phi_t(p)} - \phi_{t*} (T^{a\dots}_{b\dots} \Big|_p) \right) \quad (\text{B.2.2})$$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \left( \phi_t^* (T^{a\dots}_{b\dots} \Big|_{\phi_t(p)}) - T^{a\dots}_{b\dots} \Big|_p \right) \quad (\text{B.2.3})$$

$$= \xi^c \nabla_c T^{a\dots}_{b\dots} - (\nabla_c \xi^a) T^{c\dots}_{b\dots} - \dots + (\nabla_b \xi^c) T^{a\dots}_{c\dots} + \dots \quad (\text{B.2.4})$$

**proof:**

- 選取滿足如下要求的坐標,

$$\{x^\mu \mid \mu = 0, \dots, n\} \quad \xi = \frac{\partial}{\partial x^0} \quad (\text{B.2.5})$$

也就是說,

$$\phi_t^* x^\mu(p) = x^\mu(\phi_t(p)) = \begin{cases} x^0(p) + t & \mu = 0 \\ x^\mu(p) & \mu \neq 0 \end{cases} \quad (\text{B.2.6})$$

- 那麼, 對矢量場和對偶矢量場的 pullback 和 pushforward 分別如下,

$$\begin{cases} \phi_t^* (dx^\mu \Big|_{\phi_t(p)}) = dx^\mu \Big|_p & \text{and} & \phi_t^* \left( \frac{\partial}{\partial x^\mu} \Big|_{\phi_t(p)} \right) = \frac{\partial}{\partial x^\mu} \Big|_p \\ \phi_{t*} (dx^\mu \Big|_p) = dx^\mu \Big|_{\phi_t(p)} & \text{and} & \phi_{t*} \left( \frac{\partial}{\partial x^\mu} \Big|_p \right) = \frac{\partial}{\partial x^\mu} \Big|_{\phi_t(p)} \end{cases} \quad (\text{B.2.7})$$

所以,

$$\begin{aligned} \mathcal{L}_\xi T^{a\dots}_{b\dots} \Big|_p &= (\partial_0 T^{a\dots}_{b\dots}) \Big|_p \\ &= \xi^c \left( \nabla_c T^{a\dots}_{b\dots} - \Gamma_{dc}^a T^{d\dots}_{b\dots} - \dots + \Gamma_{bc}^d T^{a\dots}_{d\dots} + \dots \right) \end{aligned} \quad (\text{B.2.8})$$

由於,

$$(\nabla_d \xi^a) T^{d\dots}_{b\dots} = \partial_d \left( \frac{\partial}{\partial x^0} \right)^a + \Gamma_{cd}^a \left( \frac{\partial}{\partial x^0} \right)^c T^{d\dots}_{b\dots} \quad (\text{B.2.9})$$

代入,

$$\mathcal{L}_\xi T^{a\cdots}_{b\cdots} \Big|_p = \xi^c \nabla_c T^{a\cdots}_{b\cdots} - (\nabla_c \xi^a) T^{c\cdots}_{b\cdots} - \cdots + (\nabla_b \xi^c) T^{a\cdots}_{c\cdots} + \cdots \quad (\text{B.2.10})$$

### B.3 consider two maps, $\psi \circ \phi$

- 三個流形  $M_1, M_2, M_3$ , 維數分別為  $n_1, n_2, n_3$ , 其上分別有坐標  $\{x^\mu\}, \{y^\mu\}, \{z^\mu\}$ .
- 它們之間存在兩個  $C^\infty$  的 homomorphism,  $\phi: M_1 \rightarrow M_2$  和  $\psi: M_2 \rightarrow M_3$ .

#### B.3.1 pullback

- 考慮,

$$\begin{cases} \psi^* z^\mu(p_2) = z^\mu(\psi(p_2)) \\ \underbrace{\phi^* \circ \psi^*}_{(\psi \circ \phi)^*} z^\mu(p_1) = z^\mu(\psi \circ \phi(p_1)) \end{cases} \quad (\text{B.3.1})$$

所以,  $\phi^* \circ \psi^* = (\psi \circ \phi)^*$ .

#### B.3.2 pushforward

- 考慮,

$$\frac{\partial}{\partial x^\mu} ((\psi \circ \phi)^* z^\nu) \Big|_{p_1} = ((\psi \circ \phi)_* \frac{\partial}{\partial x^\mu}) (z^\nu) \Big|_{\psi \circ \phi(p_1)} \quad (\text{B.3.2})$$

并且,

$$\frac{\partial}{\partial x^\mu} (\phi^* y^\nu) \Big|_{p_1} = \phi_* \frac{\partial}{\partial x^\mu} (y^\nu) \Big|_{\phi(p_1)} \quad (\text{B.3.3})$$

$$\frac{\partial}{\partial x^\mu} (\phi^* \circ \psi^* z^\nu) \Big|_{p_1} = \phi_* \frac{\partial}{\partial x^\mu} (\psi^* z^\nu) \Big|_{\phi(p_1)} = \psi_* \circ \phi_* \frac{\partial}{\partial x^\mu} (z^\nu) \Big|_{\psi \circ \phi(p_1)} \quad (\text{B.3.4})$$

所以,  $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ .

#### B.3.3 pullback

- 考慮,

$$((\psi \circ \phi)^* dz^\mu)_a \left( \frac{\partial}{\partial x^\nu} \right)^a \Big|_{p_1} = (dz^\mu)_a \left( (\psi \circ \phi)_* \frac{\partial}{\partial x^\nu} \right)^a \Big|_{\psi \circ \phi(p_1)} \quad (\text{B.3.5})$$

且,

$$(\phi^* \circ \psi^* dz^\mu)_a \left( \frac{\partial}{\partial x^\nu} \right)^a \Big|_{p_1} = (\psi^* dz^\mu)_a \left( \phi_* \frac{\partial}{\partial x^\nu} \right)^a \Big|_{\phi(p_1)} = (dz^\mu)_a \left( \psi_* \circ \phi_* \frac{\partial}{\partial x^\nu} \right)^a \Big|_{\psi \circ \phi(p_1)} \quad (\text{B.3.6})$$

所以, 依舊有  $\phi^* \circ \psi^* = (\psi \circ \phi)^*$ .

## B.4 Weyl transformations & conformal transformations

### B.4.1 Weyl transformations

- Weyl 變換在保持流形不變的情況下, 改變流形上配備的度規, 此時, 流形的曲率等幾何性質也會發生改變.
- 背景流形上選取坐標  $\{x^\mu\}$ , 那麼新度規與舊度規的關係為,

$$\tilde{g}_{\mu\nu} = e^{\Phi(x)} g_{\mu\nu} \quad (\text{B.4.1})$$

其中,  $\Phi(x)$  是流形上的一個標量場.

- 在 Weyl 變換下, 仿射聯絡係數, 曲率張量都會發生變化, 但 Weyl 張量不會發生變換 (具體變換形式及計算過程見 GoodNotes 筆記: Weyl Transformation and Conformal Transformation).

### B.4.2 conformal isometries

- 流行  $M$  上配備有兩套度規  $g_{ab}$  和  $\tilde{g}_{ab}$  (可見 Weyl 變換和共形變換都會改變流形的度規場).
- 映射  $\phi$  是 conformal isometry, 其生成的拉回映射  $\phi^*$  滿足,

$$(\phi^*(\tilde{g})|_{\phi(p)})_{ab} = \Omega^2 g_{ab}|_p \quad (\text{B.4.2})$$

其中  $\Omega$  是流形上的標量場.

- conformal transformations preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.
- 用坐標的拉回映射來表示這個變換, 那麼是, 對於流形上的坐標  $\{y^\mu\}$  其拉回映射的像為  $\{x^\mu\}$ , 即,

$$\begin{cases} (\phi^* y^\mu)(p) \equiv x^\mu(p) = y^\mu(\phi(p)) \\ \phi^* dy^\mu = dx^\mu \end{cases} \quad (\text{B.4.3})$$

那麼, conformal isometry  $\phi$  即滿足,

$$\tilde{g}_{\mu\nu}|_{\phi(p)} \phi^*(dy^\mu \otimes dy^\nu) = \Omega^2 g_{\mu\nu}(dx^\mu \otimes dx^\nu) \quad (\text{B.4.4})$$

$$\implies \tilde{g}_{\mu\nu}|_{\phi(p)} = (\Omega^2 g_{\mu\nu})|_p \quad (\text{B.4.5})$$

其中  $\tilde{g}_{\mu\nu}$  是度規  $\tilde{g}_{ab}$  在  $\{y^\mu\}$  坐標系下的分量.

### B.4.3 conformal Killing vector fields

- 流形上的一個 one-parameter group of conformal isometry  $\{\phi_t, t \in \mathbb{R}\}$ , 其中每個  $\phi_t$  都是 conformal isometry 且滿足如 (B.2.1) 式的群乘法, 且,

$$(\phi_t^* g)_{ab} = a(t) g_{ab} \quad (\text{B.4.6})$$

$a(t)$  顯然要滿足某些性質, 目前可以確認  $a(0) = 1$ .

- 矢量場  $\psi^a|_{\phi_s(p)} = \frac{d}{dt}|_s \phi_t(p)$  稱為 conformal Killing vector field, 相應的度規的李導數為,

$$(\mathcal{L}_\psi g)_{ab} = 2\nabla_{(a} \psi_{b)} = \alpha g_{ab} \quad (\text{B.4.7})$$

其中  $\alpha = \frac{d}{dt}|_{t=0} a(t)$ , 對上式兩端求 trace, 得到,

$$2\nabla^a \psi_a = n\alpha \implies \alpha = \frac{2}{n} \nabla^a \psi_a \quad (\text{B.4.8})$$

其中  $n$  是流形維數.

- 得到 conformal Killing vector field 滿足的方程,

$$\nabla_{(a} \psi_{b)} = \frac{1}{n} (\nabla^c \psi_c) g_{ab} \quad (\text{B.4.9})$$