Statistical Field Theory

Siyang Wan (万思扬)

April 19, 2025

Contents

convention, notation, and units

- 使用 Planck units, 此时 $G, \hbar, c, k_B = 1$, 因此:

name/dimension	expression/value
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

- 下面是 *Statistical Field Theory*, David Tong, 中引言的一部分.

> ... This phenomenon is known as *universality*.
> All of this makes phase transitions interesting. They involve violence, universal truths and competition between rival states. The story of phase transitions is, quite literally, the song of fire and ice.
> ...
> ... This leads us to a paradigm which now underlies huge swathes of physics, far removed from its humble origin of a pot on a stove. This paradigm revolves around two deep facts about the Universe we inhabit: **Nature is organised by symmetry. And Nature is organised by scale.**

其中 symmetry 指 Landau's approach to phase transitions, scale 是指 renormalization group.

Part I

thermodynamics and statistical physics

Chapter 1

fundamentals of statistical mechanics

1.1 the microscopic states

- microscopic state 就是系统中每个粒子的准确状态.
  - 量子力学中用 $|\psi\rangle$ 描述.
  - 经典力学中用 $p = (p_1, \cdots, p_N), q = (q_1, \cdots, q_N)$ 描述 (即相空间中的一个点).

1.2 the ensembles

- ensemble 是一定条件下, 系统的 microscopic state 的可能分布.
  - 量子力学用 density matrix 描述,

$$\rho = \begin{cases} |\psi\rangle \langle\psi| & \text{pure ensemble} \\ \sum_i p_i |\psi_i\rangle \langle\psi_i| & \text{mixed ensemble} \end{cases}. \tag{1.2.1}$$

  - 经典力学用 probability density 描述, 即系统处于 (p, q) 的概率密度

$$\rho(p, q, t). \tag{1.2.2}$$

- 微观态的概率分布随时间演化的方程为

$$\begin{cases} i\hbar \dfrac{\partial \rho}{\partial t} = [H, \rho] & \text{von Neumann's equation in QM} \\ \dfrac{\partial \rho}{\partial t} = \{H, \rho\}_{\mathrm{PB}} \equiv \sum_i \left(\dfrac{\partial H}{\partial q_i} \dfrac{\partial \rho}{\partial p_i} - \dfrac{\partial \rho}{\partial q_i} \dfrac{\partial H}{\partial p_i} \right) & \text{Liouville's equation in CM} \end{cases}. \tag{1.2.3}$$

1.3 the equilibrium ensembles

- 满足

$$\frac{\partial \rho}{\partial t} = 0 \tag{1.3.1}$$

  的 ensemble 称为 equilibrium ensemble.
  - 无论是量子理论还是经典理论, 如果 ρ 是 H 的函数, 那么就一定是 equilibrium ensemble.
- 下面是几种 equilibrium ensembles:

ensembles	macroscopic values	ρ
microcanonical	N, V, E	$\rho_{\mathrm{MC}} = \frac{1}{\Omega(E)} \delta(H - E)$
canonical	N, V, T	$\rho_{\mathrm{C}} = \frac{1}{Z_{\mathrm{C}}} e^{-\beta H}$
grand canonical	μ, V, T	$\rho_{\mathrm{GC}} = \frac{1}{Z_{\mathrm{GC}}} e^{-\beta(H - \mu N)}$

Part II

statistical field theory

Chapter 2

the Ising model

- Ising model 由 D 维空间中的 N 个格点 (lattice site) 组成, 每个格点可以处于两种态,
$$s_i = \pm 1. \tag{2.0.1}$$
- 一组 $\{s_i\}$ 的能量为
$$H[s_i] = -B\sum_i s_i - J\sum_{\langle ij\rangle} s_i s_j, \tag{2.0.2}$$
其中 $\langle ij\rangle$ 表示格点中所有"nearest neighbour" pairs, $\langle ij\rangle$ 的数量依赖于 D 和 the type of lattice.
  - 如果 $B > 0$, 格点倾向于 $s_i = +1$ 的态.
  - 如果 $J > 0$, 格点倾向于 $\uparrow\uparrow$ 或 $\downarrow\downarrow$, 这种系统称为 a ferromagnet, 反之则称为 an anti-ferromagnet.

- in canonical ensemble, 状态 $\{s_i\}$ 的概率为
$$p[s_i] = \frac{1}{Z}e^{-\beta H[s_i]}, \tag{2.0.3}$$
其中 $Z(\beta, B, J)$ 是 partition function.
- 定义 equilibrium magnetization 为
$$m = \frac{1}{N}\langle\sum_i s_i\rangle \in [-1, +1], \tag{2.0.4}$$
利用 partition function 可知,
$$m = \frac{1}{N\beta}\frac{\partial \ln Z}{\partial B}. \tag{2.0.5}$$

2.1 the effective free energy

- 用以下办法定义 $F(m)$,
$$Z = \sum_m \sum_{m|\{s_i\}} e^{-\beta H[s_i]} := \sum_m e^{-\beta F(m)} \overset{N \ll 1}{\approx} \frac{N}{2}\int dm\, e^{-\beta F(m)}, \tag{2.1.1}$$
其中 $\{s_i\}|m$ 表示所有满足 $\frac{1}{N}\sum_i s_i = m$ 的态 $\{s_i\}$.
  - 系数 $\frac{N}{2}$ 对物理没有影响, 可以忽略.