Statistical Physics of Fields

Siyang Wan (万思扬)

May 29, 2025

Contents

convention, notation, and units

- 使用 Planck units, 此时 $G, \hbar \equiv \frac{h}{2\pi}, c, k_B = 1$, 因此:

names/dimensions	expressions/values
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

- 下面是 *Statistical Field Theory*, David Tong, 中引言的一部分.

> ... This phenomenon is known as *universality*.
> All of this makes phase transitions interesting. They involve violence, universal truths and competition between rival states. The story of phase transitions is, quite literally, the song of fire and ice.
> ...
> ... This leads us to a paradigm which now underlies huge swathes of physics, far removed from its humble origin of a pot on a stove. This paradigm revolves around two deep facts about the Universe we inhabit: **Nature is organised by symmetry. And Nature is organised by scale.**

其中 symmetry 指 Landau's approach to phase transitions, scale 是指 renormalization group.

Chapter 1

collective behavior, from particles to fields

1.1 phonons and elasticity

-

Appendices

Appendix A

probability

A.1 general definition

- 对于一个 random variable x, 用 $\mathcal{S} = \{x_1, x_2, \cdots\}$ 表示其 possible outcomes.
- to each event $E \subset \mathcal{S}$ is assigned a probability $p(E)$ that must satisfy:
  1. positivity. $p(E) \geq 0$.
  2. additivity. $p(A \text{ or } B) = p(A) + p(B), \forall A \cap B = \emptyset$.
  3. normalization. $p(\mathcal{S}) = 1$.

A.2 one random variable

- 考虑 $\mathcal{S} = \mathbb{R}$.
- cumulative probability function (CPF) 定义为
$$P(y) := \text{prob}(\{x | x < y\}), \tag{A.2.1}$$
  那么 $P(-\infty) = 0, P(+\infty) = 1$, 且 $P(x)$ 是 monotonically increasing.
- probability density function (PDF) 定义为
$$p(x) := \frac{dP}{dx}. \tag{A.2.2}$$
- the expectation value of any function $F(x)$ is
$$\langle F(x) \rangle := \int_{-\infty}^{\infty} dx\, p(x) F(x). \tag{A.2.3}$$
- 将 $F(x)$ 视为一个 random variable, 那么它的 probability density function 为
$$p_F(f) df = \sum_{F(x_i)=f} p(x_i) dx_i \Longrightarrow p_F(f) = \sum_{F(x_i)=f} \frac{p(x_i)}{\left|\frac{dF(x_i)}{dx}\right|}. \tag{A.2.4}$$

- the n^{th} moment of the PDF is
$$m_n := \langle x^n \rangle = \int_{-\infty}^{\infty} dx\, x^n p(x). \tag{A.2.5}$$
- the characteristic function is
$$\tilde{p}(k) := \langle e^{-ikx} \rangle = \int dx\, p(x) e^{-ikx}, \tag{A.2.6}$$
  并有 (Taylor expansion)
$$\begin{cases} \tilde{p}(x) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} m_n \\ e^{ikx_0} \tilde{p}(x) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \left\langle (x - x_0)^n \right\rangle \end{cases}, \tag{A.2.7}$$

因此 $\tilde{p}(x)$ is the generator of moments.

- -

- 类似 moments, 将 cumulants $\langle x^n \rangle_c$ 定义为

$$\ln \tilde{p}(k) = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle_c \,, \tag{A.2.8}$$

注意到 $\ln(1+\epsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\epsilon^n}{n}$.

- 前 4 个 cumulants 分别为

$$\begin{cases} \langle x \rangle_c = \langle x \rangle & \text{mean} \\ \langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2 & \text{variance} \\ \langle x^3 \rangle_c = \langle x^3 \rangle - 3 \langle x^2 \rangle \langle x \rangle + 2 \langle x \rangle^3 & \text{skewness} \\ \langle x^4 \rangle_c = \langle x^4 \rangle - 4 \langle x^3 \rangle \langle x \rangle - 3 \langle x^2 \rangle^2 + 12 \langle x^2 \rangle \langle x \rangle^2 - 6 \langle x \rangle^4 & \text{curtosis} \end{cases}. \tag{A.2.9}$$

- 有

$$\frac{\langle x^n \rangle}{n!} = \sum \Big(\prod_{\{1 \le m_1 \le n, 0 \le m_2 \le n | \sum m_1 m_2 = n\}} \frac{1}{(m_1!)^{m_2} m_2!} \langle x^{m_1} \rangle_c^{m_2} \Big). \tag{A.2.10}$$

calculation:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle &= \exp \Big(\sum_{m=1}^{\infty} \frac{(-ik)^m}{m!} \langle x^m \rangle_c \Big) = \prod_{m=1}^{\infty} \exp \Big(\frac{(-ik)^m}{m!} \langle x^m \rangle_c \Big) \\ &= \prod_{m_1=1}^{\infty} \Big(\sum_{m_2=0}^{\infty} \frac{(-ik)^{m_1 m_2}}{(m_1!)^{m_2} m_2!} \langle x^{m_1} \rangle_c^{m_2} \Big). \end{aligned} \tag{A.2.11}$$

- 用 cumulants 表示 moments 的系数 $\frac{n!}{(m_1!)^{m_2} m_2!}$ 可以用一些 graphs 来计算 (cluster expansion), 见 page 39, *Statistical Physics of Particles*, Kardar.

A.3 some important probability distributions

- the normal (Gaussian) distribution 的 PDF 为

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\lambda)^2}{2\sigma^2}}, \tag{A.3.1}$$

相应地, characteristic function 为

$$\tilde{p}(k) = e^{-ik\langle x \rangle_c - \frac{k^2}{2} \langle x^2 \rangle_c + \cdots} = e^{-ik\lambda - \frac{k^2\sigma^2}{2}}, \tag{A.3.2}$$

那么

$$\langle x \rangle_c = \lambda, \quad \langle x^2 \rangle_c = \sigma^2, \quad \langle x^3 \rangle_c = \langle x^4 \rangle_c = \cdots = 0. \tag{A.3.3}$$

  – higher cumulants 消失意味着 cluster expansion 中只含有 one-point and two-point (propagators) clusters.

- the binomial distribution, $\mathcal{S} = \{A, B\}$, 有

$$p_N(N_A) = \binom{N}{N_A} p_A^{N_A} (1-p_A)^{N-N_A}, \quad \binom{N}{N_A} = \frac{N!}{N_A!(N-N_A)!}, \tag{A.3.4}$$

其 characteristic function 为

$$\tilde{p}_N(k) = \sum_{N_A=0}^{N} p_N(N_A) e^{-ikN_A} = (p_A e^{-ik} + p_B)^N, \tag{A.3.5}$$

那么

$$\langle N_A \rangle_c = N p_A, \quad \langle N_A^2 \rangle_c = N p_A p_B, \quad \langle N_A^3 \rangle_c = N p_A p_B (1-2p_A), \quad \langle N_A^4 \rangle_c = p_A p_B (1-6p_A+6p_A^2). \tag{A.3.6}$$

calculation:

$$\begin{aligned}\ln\tilde{p}_N(k) =& N\ln(1+p_A(e^{-ik}-1))\\ =& N\Big((-ik)p_A+\frac{(-ik)^2}{2}(p_A-p_A^2)+\frac{(-ik)^3}{3!}(p_A-3p_A^2+2p_A^3)\\ &+\frac{(-ik)^4}{4!}(p_A-7p_A^2+12p_A^3-6p_A^4)+O(k^5)\Big)\end{aligned} \tag{A.3.7}$$

– 可以推广为 multinomial distribution,

$$p_N(N_A,N_B,\cdots,N_M)=\frac{N!}{N_A!N_B!\cdots N_M!}p_A^{N_A}p_B^{N_B}\cdots p_M^{N_M}. \tag{A.3.8}$$

- the Poisson distribution 考虑时间长度 T 范围内观察到 M 次衰变的概率 (dt 内发生一次衰变的概率为 αdt), 那么

$$\begin{cases} p_\lambda(M)=\dfrac{\lambda^M e^{-\lambda}}{M!} \\ \tilde{p}_\lambda=e^{\lambda(e^{-ik}-1)}\end{cases}, \quad \lambda=\alpha T, \quad \text{and} \quad \langle M^n\rangle_c=\lambda. \tag{A.3.9}$$

– the Poisson distribution 的图像如下所示:

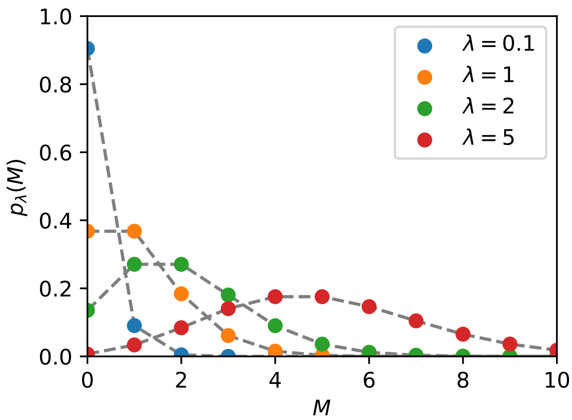

Figure A.1: the Poisson distribution.

calculation:

参考 binomial distribution, 将 dt 时间内发生一次衰变记为 A, 未衰变记为 B, 那么

$$p_A=\alpha dt, \quad p_B=1-p_A, \quad N=\frac{T}{dt}, \quad N_A=M, \tag{A.3.10}$$

因此 ($N\gg 1$)

$$\begin{cases}\ln\dbinom{N}{M}\simeq N\ln N-(N-M)\ln(N-M)-M-\ln M!\\ \ln((\alpha dt)^M(1-\alpha dt)^{N-M})=M\ln\dfrac{\lambda}{N}+(N-M)\ln\Big(1-\dfrac{\lambda}{N}\Big)\end{cases}$$

$$\Longrightarrow \quad \begin{aligned}\ln p_\lambda(M)\simeq&\Big(N\ln N-(N-M)\ln(N-M)-M-\ln M!\Big)\\ &+\Big(M\ln\lambda-M\ln N-\lambda\frac{N-M}{N}\Big)\\ =&N(\ln N-\ln(N-M))+M(\ln(N-M)-\ln N)\\ &-M-\ln M!+M\ln\lambda-\lambda\frac{N-M}{N}\\ \simeq&M+\frac{M^2}{2N}-\frac{M^2}{N}-M-\ln M!+M\ln\lambda-\lambda\frac{N-M}{N}\simeq-\ln M!+M\ln\lambda-\lambda\end{aligned}$$

$$\Longrightarrow p_\lambda(M) = \frac{\lambda^M e^{-\lambda}}{M!}. \tag{A.3.11}$$

更简单的方法是计算 $\tilde{p}_\lambda(M)$,

$$\tilde{p}_\lambda(M) = \lim_{dt\to 0}(1+\alpha dt(e^{-ik}-1))^N = \lim_{N\to\infty}\left(1+\frac{\lambda}{N}(e^{-ik}-1)\right)^N = e^{\lambda(e^{-ik}-1)}. \tag{A.3.12}$$

A.4 many random variables

- 考虑 N 个随机变量, 且 $\mathcal{S} = \mathbb{R}^N$.
- $d^N x$ 体元中的概率用 joint PDF $p(\vec{x})$ 描述, 如果 the N variables are independent, 那么

$$p(\vec{x}) = \prod_{i=1}^N p_i(x_i). \tag{A.4.1}$$

- unconditional PDF 描述 $(x_1, \cdots, x_m)$ 的概率密度, 而剩余的 $(x_{m+1}, \cdots, x_N)$ 取值任意, 那么

$$p(x_1, \cdots, x_m) = \int dx_{m+1} \cdots dx_N \, p(\vec{x}). \tag{A.4.2}$$

- conditional PDF 描述 $(x_{m+1}, \cdots, x_N)$ 条件下 $(x_1, \cdots, x_N)$ 的概率密度, 有

$$p(x_1, \cdots, x_m | x_{m+1}, \cdots, x_N) = \frac{p(\vec{x})}{p(x_{m+1}, \cdots, x_N)}. \tag{A.4.3}$$

- 如果变量独立, 那么 unconditional PDF 等于 conditional PDF.
- $F(\vec{x})$ 的 expectation value 为

$$\langle F(\vec{x})\rangle = \int d^N x \, p(\vec{x}) F(\vec{x}). \tag{A.4.4}$$

- the joint characteristic function is

$$\tilde{p}(\vec{k}) = \langle e^{-i\vec{k}\cdot\vec{x}}\rangle. \tag{A.4.5}$$

- the joint moments and joint cumulants are generated by $\tilde{p}(\vec{k})$ and $\ln \tilde{p}(\vec{k})$ respectively, as

$$\begin{cases} \langle x_1^{n_1} \cdots x_N^{n_N}\rangle = \dfrac{\partial^{n_1}}{\partial(-ik_1)^{n_1}} \cdots \dfrac{\partial^{n_N}}{\partial(-ik_N)^{n_N}} \tilde{p}(\vec{k}=0) \\ \langle x_1^{n_1} \cdots x_N^{n_N}\rangle_c = \dfrac{\partial^{n_1}}{\partial(-ik_1)^{n_1}} \cdots \dfrac{\partial^{n_N}}{\partial(-ik_N)^{n_N}} \ln \tilde{p}(\vec{k}=0) \end{cases}, \tag{A.4.6}$$

joint moments 和 joint cumulants 之间依然可以用 cluster expansion 建立关系,

$$\frac{\langle x_1^{n_1} \cdots x_N^{n_N}\rangle}{n_1! \cdots n_N!} = \sum \left(\prod_{\{1\le m_i \le n_i, 0\le l \le n_i | \sum l m_i = n_i\}} \frac{1}{(m_1!\cdots m_N!)^l l!} \langle x_1^{m_1} \cdots x_N^{m_N}\rangle_c^l \right). \tag{A.4.7}$$

calculation:

$$\begin{aligned} & \sum_{\vec{n}} \frac{(-ik_1)^{n_1} \cdots (-ik_N)^{n_N}}{n_1! \cdots n_N!} \langle x_1^{n_1} \cdots x_N^{n_N}\rangle \\ =& \prod_{\vec{m}\neq 0} \exp\left(\frac{(-ik_1)^{m_1} \cdots (-ik_N)^{m_N}}{m_1! \cdots m_N!} \langle x_1^{m_1} \cdots x_N^{m_N}\rangle_c \right) \\ =& \prod_{\vec{m}\neq 0} \left(\sum_{l=0}^\infty \frac{(-ik_1)^{l m_1} \cdots (-ik_N)^{l m_N}}{(m_1!\cdots m_N!)^l l!} \langle x_1^{m_1} \cdots x_N^{m_N}\rangle_c^l \right). \end{aligned} \tag{A.4.8}$$

- 如果 x_i, x_j 互相独立, 那么 $\langle x_i x_j\rangle_c = \langle x_i x_j\rangle - \langle x_i\rangle\langle x_j\rangle = 0$, (实际上, $\langle \cdots x_i \cdots x_j \cdots\rangle_c = 0$).

calculation:

注意到

$$\ln \tilde{p}_N(\vec{k}) = \ln \tilde{p}'(\cdots, k_i, \cdots) + \ln \tilde{p}''(\cdots, k_j, \cdots). \tag{A.4.9}$$

- the joint Gaussian distribution 是 (A.3.1) 的推广, 为

$$\begin{cases} p(\vec{x}) = \dfrac{1}{\sqrt{(2\pi)^N \det(C)}} \exp\Big(-\dfrac{1}{2} \sum_{ij} (C^{-1})_{ij} (x_i - \lambda_i)(x_j - \lambda_j) \Big) \\ \tilde{p}(\vec{k}) = \exp\Big(-i\vec{k}\cdot\vec{\lambda} - \dfrac{1}{2}\vec{k}\cdot C \cdot \vec{k} \Big) \end{cases}, \tag{A.4.10}$$

其中 $C = C^T$, 那么

$$\langle x_i \rangle = \lambda_i, \quad \langle x_i x_j \rangle_c = C_{ij}. \tag{A.4.11}$$

  - 更多结果 (Wick theorem) 参考笔记 A. Zee QFT 或 *Quantum Field Theory and Critical Phenomena*, Zinn-Justin.

A.5 sums of random variables and the central limit theorem

- 考虑 $X = \sum_{i=1}^N x_i$, 那么 X 的 PDF 为

$$\begin{aligned} p_X(X) &= \int d^N x\, p(\vec{x}) \frac{d}{dX} \theta(X - \sum_{i=1}^N x_i) \\ &= \int d^N x\, p(\vec{x}) \delta(X - \sum_{i=1}^N x_i) = \int dx_1 \cdots dx_{N-1}\, p(x_1, \cdots, x_{N-1}, X - \textstyle\sum_{i=1}^{N-1} x_i), \end{aligned} \tag{A.5.1}$$

相应地, the characteristic function 为

$$\tilde{p}_X(K) = \langle e^{-iK\sum_{i=1}^N x_i} \rangle = \tilde{p}(k_1 = \cdots = k_N = K), \tag{A.5.2}$$

因此, cumulants 为

$$\begin{aligned} &\ln \tilde{p}_X(K) = (-iK) \sum_i \langle x_i \rangle_c + \frac{(-iK)^2}{2!} \sum_{i,j} \langle x_i x_j \rangle_c + \frac{(-iK)^3}{3!} \sum_{i,j,k} \langle x_i x_j x_k \rangle_c + \cdots \\ \Longrightarrow &\langle X \rangle_c = \sum_i \langle x_i \rangle_c, \quad \langle X^2 \rangle_c = \sum_{i,j} \langle x_i x_j \rangle_c, \quad \langle X^3 \rangle_c = \sum_{i,j,k} \langle x_i x_j x_k \rangle_c, \quad \cdots \end{aligned} \tag{A.5.3}$$

- the central limit theorem: 如果

$$p_N(\vec{x}) = p(x_1) \cdots p(x_N) \quad \text{and} \quad \langle x^n \rangle_c \ll N^{\frac{n}{2}-1}, \tag{A.5.4}$$

那么, 当 $N \to \infty$ 时, 有

$$\lim_{N\to\infty} p_X(X) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(X-\lambda_X)^2}{2\sigma_X^2}}, \quad \lambda_X = N\langle x \rangle_c, \quad \sigma_X^2 = N\langle x^2 \rangle_c. \tag{A.5.5}$$

proof:

令

$$Y = \frac{X - \langle X \rangle}{\sqrt{N}} \quad \text{with} \quad \langle X \rangle = N\langle x \rangle_c, \tag{A.5.6}$$

那么

$$\begin{aligned} p_Y(Y) &= \int d^N x\, p(x_1) \cdots p(x_N) \delta(Y + \tfrac{\langle X \rangle}{\sqrt{N}} - \tfrac{1}{\sqrt{N}} \textstyle\sum_{i=1}^N x_i) \\ &= \int \frac{dq}{2\pi} e^{iq\left(Y + \frac{\langle X \rangle}{\sqrt{N}}\right)} \int d^N x\, p(x_1) \cdots p(x_N) e^{-i\frac{q}{\sqrt{N}} \sum_{i=1}^N x_i} \end{aligned}$$

$$= \int \frac{dq}{2\pi} e^{iq\left(Y + \frac{\langle X \rangle}{\sqrt{N}}\right)} \tilde{p}^N\left(\frac{q}{\sqrt{N}}\right), \tag{A.5.7}$$

并注意到 (saddle point integration)

$$\tilde{p}(k) = e^{-ik\langle x \rangle_c + \frac{(-ik)^2}{2}\langle x^2 \rangle_c + \cdots}, \tag{A.5.8}$$

因此

$$p_Y(Y) = \int \frac{dq}{2\pi} \exp\left(iqY - \frac{q^2}{2}\langle x^2 \rangle_c + O\left(\frac{1}{\sqrt{N}}\right)\right) = \frac{1}{\sqrt{2\pi \langle x^2 \rangle_c}} e^{-\frac{Y^2}{2\langle x^2 \rangle_c}}. \tag{A.5.9}$$