Statistical Field Theory

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Contents

convention, notation, and units

- 使用 Planck units, 此时 $G, \hbar \equiv \frac{h}{2\pi}, c, k_B = 1$, 因此:

names/dimensions	expressions/values
Planck length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$
Planck time (T)	$t_P = \frac{l_P}{c} = 5.391 \times 10^{-44}\,\mathrm{s}$
Planck mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} \simeq 10^{19}\,\mathrm{GeV}$
Planck temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.417 \times 10^{32}\,\mathrm{K}$

- 时空维度用 $d = D + 1$ 表示.

- 下面是 *Statistical Field Theory*, David Tong, 中引言的一部分.

> ... This phenomenon is known as *universality*.
> All of this makes phase transitions interesting. They involve violence, universal truths and competition between rival states. The story of phase transitions is, quite literally, the song of fire and ice.
> ...
> ... This leads us to a paradigm which now underlies huge swathes of physics, far removed from its humble origin of a pot on a stove. This paradigm revolves around two deep facts about the Universe we inhabit: **Nature is organised by symmetry. And Nature is organised by scale.**

其中 symmetry 指 Landau's approach to phase transitions, scale 是指 renormalization group.

Part I

from spins to fields

Chapter 1

the Ising model

- 考虑一个 D 维 lattice, 含有 N 个 sites, 每个 lattice site 可以处于两个状态 $s_i = \pm 1$.
- 系统的能量为
$$E\{s_i\} = -B\sum_i s_i - J\sum_{\langle ij\rangle} s_i s_j, \tag{1.0.1}$$
其中 $\langle ij\rangle$ 表示 nearest neighbor pairs.
  - $J > 0$, 系统称为 ferromagnet; $J < 0$, 系统称为 anti-ferromagnet.
- 系统的 partition function 为
$$Z_{\mathrm{C}}(T, B, N) = \sum_{\{s_i\}} e^{-\beta E\{s_i\}}. \tag{1.0.2}$$
- 那么, 系统的 magnetization 为
$$\langle m\rangle \equiv \frac{1}{N}\langle\sum_i s_i\rangle = \frac{1}{N\beta}\frac{\partial}{\partial B}\ln Z_{\mathrm{C}}(T, B, N) \equiv -\frac{1}{N}\Big(\frac{\partial F}{\partial B}\Big)_{T,N}, \tag{1.0.3}$$
并且 $m \in [-1, 1]$.

1.1 the effective free energy

- 将 partition function 写作
$$Z_{\mathrm{C}} = \sum_m \underbrace{\sum_{\{s_i\}|m} e^{-\beta E\{s_i\}}}_{:=e^{-\beta F_{\mathrm{eff}}(m)}} \simeq \frac{1}{2/N}\int_{-1}^{1} dm\, e^{-\beta F_{\mathrm{eff}}(m)}, \tag{1.1.1}$$
其中 $F_{\mathrm{eff}}(m)$ 称作 effective free energy.
  - m 对应的 number of microstates 为
$$\Omega(m) = \frac{N!}{N_\uparrow! N_\downarrow!} \simeq \exp\Big(N\Big(\ln 2 - \frac{1+m}{2}\ln(1+m) - \frac{1-m}{2}\ln(1-m)\Big)\Big) \le 2^N, \tag{1.1.2}$$
  其中 $N_{\uparrow,\downarrow} = \frac{1\pm m}{2}N$ 分别是处于 $s = \pm 1$ 的 lattice sites 数量.
- 注意到 $F_{\mathrm{eff}} \sim O(Nk_BT)$, 定义
$$f_{\mathrm{eff}} := \frac{F_{\mathrm{eff}}}{N} \sim O(k_BT), \tag{1.1.3}$$
那么
$$\ln Z_{\mathrm{C}} \simeq -\beta N f_{\mathrm{eff}}(m_0) + O(\ln N), \tag{1.1.4}$$
其中 $f_{\mathrm{eff}}(m_0)$ 是 $m \in (-1, 1)$ 上的 global minimum, 可见
$$F_{\mathrm{eff}}(m_0) \simeq F \quad \text{and} \quad m_0 \simeq \langle m\rangle\,. \tag{1.1.5}$$

proof:

使用 Laplace's method, 对于 $N \to \infty$ 有

$$I = \int_a^b e^{Nf(x)} dx \simeq e^{Nf(x_0)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}N|f''(x_0)|(x-x_0)^2} dx = \sqrt{\frac{2\pi}{N|f''(x_0)|}} e^{Nf(x_0)}, \tag{1.1.6}$$

其中 $f(x_0)$ 是区间 (a, b) 上的 global maximum. 因此

$$Z_C \simeq \frac{N}{2} \int_{-1}^{1} e^{-\beta N f_{\text{eff}}(m)} dm \simeq \sqrt{\frac{\pi N k_B T}{2|f''_{\text{eff}}(m_0)|}} e^{-\beta N f_{\text{eff}}(m_0)}. \tag{1.1.7}$$

- 接下来, 我们需要计算 $F_{\text{eff}}(m)$ 的具体形式.

1.2 mean field theory

- 作如下近似

$$s_i \approx m \Longrightarrow E\{s_i\} \approx N\Big(- Bm - \frac{q}{2} J m^2 \Big), \tag{1.2.1}$$

其中 q 表示每个 lattice site 拥有的 nearest neighbors 数量.

  - $D = 1$, 那么 $q = 2$; $D = 2$ 的 square lattice, 那么 $q = 4$; $D = 2$ 的 triangular lattice, 那么 $q = 6$.

- 那么

$$\begin{aligned} & e^{-\beta F_{\text{eff}}(m)} \approx \Omega(m) e^{-\beta E(m)} \\ \Longrightarrow & \begin{cases} f_{\text{eff}}(m) \approx -Bm - \frac{q}{2} J m^2 - k_B T \Big(\ln 2 - \frac{1+m}{2} \ln(1+m) - \frac{1-m}{2} \ln(1-m) \Big) \\ \frac{\partial f_{\text{eff}}}{\partial m} \approx -B - qJm + \frac{k_B T}{2} \ln \frac{1+m}{1-m} \end{cases}, \end{aligned} \tag{1.2.2}$$

得到

$$\langle m \rangle \simeq m_0 \approx \tanh(\beta B_{\text{eff}}(m_0)) \approx \begin{cases} 1 & T \to 0 \\ 0 & T \to \infty \end{cases}, \tag{1.2.3}$$

其中 $B_{\text{eff}}(m_0) = B + qJm_0$ 称为 mean field.

Appendices