Maps Between Manifolds

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https://github.com/siyang03/my-note---maps-between-manifolds

Contents

1 a map between two manifolds

- $\phi : M_1 \to M_2$ 是一个流形间的映射, 如下图所示,

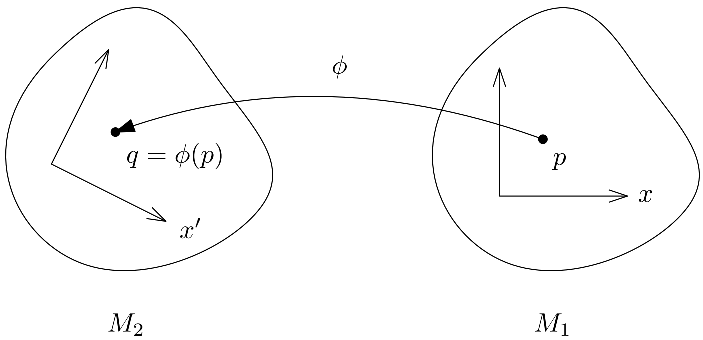

Figure 1: a map between two manifolds

1.1 pull-back

- 将 M_2 上的标量场拉回得到 M_1 上的标量场, 特别地,

$$x = \phi^* x' \quad \text{i.e.} \quad x'(q) = x'(\phi(p)) = (\phi^* x')(p) = x(p) \tag{1.1}$$

不过要注意, $\phi^* x'$ 不是 M_1 上的坐标, 因为 ϕ 不是 one-to-one, (一个有帮助的例子是 $\dim M_2 < \dim M_1$).

1.2 push-forward

- 将 M_1 上的矢量场推前得到 M_2 上的矢量场, 特别地,

$$\phi_* \frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \quad \text{i.e.} \quad \frac{\partial}{\partial x'^\mu}(x') = \left(\phi_* \frac{\partial}{\partial x^\mu}\right)(x') = \frac{\partial}{\partial x^\mu}(\phi^* x') = \frac{\partial}{\partial x^\mu}(x) \tag{1.2}$$

1.3 pull-back again

- 将 M_2 上的对偶矢量场拉回得到 M_1 上的对偶矢量场, 特别地,

$$dx = \phi^* dx' \quad \text{i.e.} \quad dx'^{\mu}\Big(\frac{\partial}{\partial x'}\Big) = dx'^{\mu}\Big(\phi_* \frac{\partial}{\partial x}\Big) = (\phi^* dx'^{\mu})\Big(\frac{\partial}{\partial x}\Big) = dx^{\mu}\Big(\frac{\partial}{\partial x}\Big) \tag{1.3}$$

2 some examples

2.1 a simple example

- 考虑 $\phi : \mathbb{R}^3 \to \mathbb{R}^2$, 两个流形上分别有直角坐标和极坐标, 映射具体形式为,

$$\phi(p) = q \quad \text{s.t.} \quad r(q) = x(p), \theta(q) = y(p) \tag{2.1}$$

如下图所示,

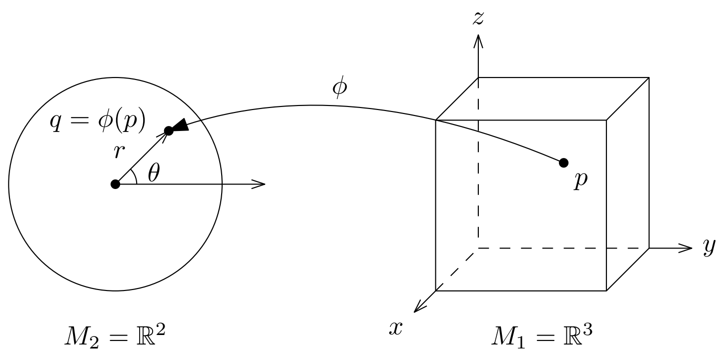

Figure 2: a simple example

- 此时 $x = \{x, y, z\}, x' = \{r, \theta\}$, 并且有,

$$\begin{cases} x = \phi^* r \quad y = \phi^* \theta \\ \phi_*\Big(\dfrac{\partial}{\partial x} + \alpha \dfrac{\partial}{\partial z}\Big) = \dfrac{\partial}{\partial r} \quad \phi_*\Big(\dfrac{\partial}{\partial y} + \beta \dfrac{\partial}{\partial z}\Big) = \dfrac{\partial}{\partial \theta} \quad \text{其中 } \alpha, \beta \text{ 是任意实数} \\ dx = \phi^* dr \quad dy = \phi^* d\theta \end{cases} \tag{2.2}$$

2.2 Lorentz transformation

- Lorentz 变换 $\phi : M \to M$ (其中 $M = \mathbb{R}^{3,1}$) 的具体形式为,

$$\phi : p \mapsto q \quad \text{s.t.} \quad x(q) = \Lambda^{-1} x(p) \tag{2.3}$$

如下图所示,

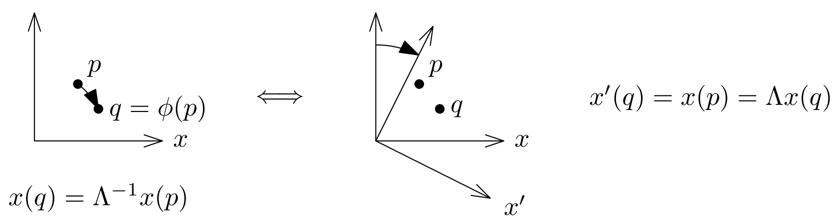

Figure 3: Lorentz transformation

- 有如下关系,

$$\begin{cases} x = \phi^* x' = \Lambda^{-1} x' \\ \phi_* \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x'} = \eta \Lambda \eta \dfrac{\partial}{\partial x} \\ dx = \phi^* dx' = \Lambda^{-1} dx' \end{cases} \tag{2.4}$$

calculation:

对于推前映射,

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} = \Lambda_{\mu}{}^{\nu} \frac{\partial}{\partial x^{\nu}} \tag{2.5}$$

其中,

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = (\Lambda^{-1})^{\nu}{}_{\mu} = \Lambda_{\mu}{}^{\nu} \tag{2.6}$$

2.3 Poincaré transformation

- Poincaré 变换 $\phi : M \to M$ (其中 $M = \mathbb{R}^{3,1}$) 的具体形式为,

$$\phi : p \mapsto q \quad \text{s.t.} \quad x(p) = \Lambda x(q) + a \tag{2.7}$$

如下图所示,

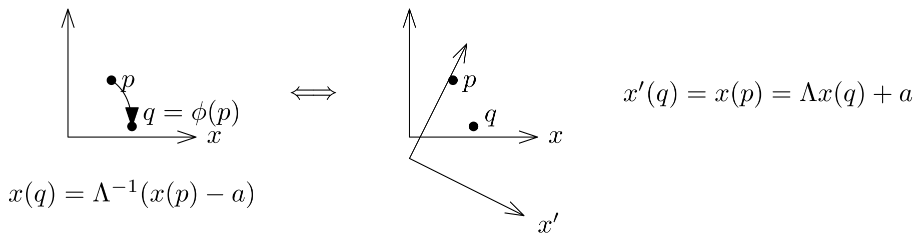

Figure 4: Poincaré transformation

- 有如下关系,

$$\begin{cases} x = \phi^* x' = \Lambda^{-1}(x' - a) \\ \phi_* \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x'} = \eta \Lambda \eta \dfrac{\partial}{\partial x} \\ dx = \phi^* dx' = \Lambda^{-1} dx' \end{cases} \tag{2.8}$$

2.4 left-invariant vector fields on the Poincaré group

- Poincaré 群的具体性质见笔记 Lie Groups and Lie Algebras.
- 在 Poincaré 群的单位元的邻域上建立坐标,

$$x : (\Lambda(\omega), a) \mapsto \{\omega, a\} \tag{2.9}$$

定义映射 $L_{(\Lambda(\omega_0), a_0)} : \mathrm{P} \to \mathrm{P}$, 使得,

$$(\Lambda(\omega), a) \mapsto (\Lambda(\omega_0), a_0)(\Lambda(\omega), a) \tag{2.10}$$

- 为了方便, 令 $\phi = L_{(\Lambda(\omega_0), a_0)}$, 以及 $g = (\Lambda(\omega), a), h = \phi(g)$.
- 定义新坐标 x', (有 $\phi^* x' = x$),

$$x' : h \mapsto x(g) \quad \text{i.e.} \quad \begin{cases} x(h) = \{\omega_h, a_h\} \\ x'(h) = \{\omega_g, a_g\} \end{cases} \tag{2.11}$$

其中,

$$(\Lambda(\omega_h), a_h) = (\Lambda(\omega_0), a_0)(\Lambda(\omega_g), a_g) \Longrightarrow \begin{cases} \Lambda(\omega_h) = \Lambda(\omega_0)\Lambda(\omega_g) \\ a_h = a_0 + \Lambda(\omega_0) a_g \end{cases} \tag{2.12}$$

- 因此, 对于推前映射,

$$\begin{cases} \phi_* \dfrac{\partial}{\partial a} = \dfrac{\partial}{\partial a'} = \Lambda(\omega_0)^{\nu}{}_{\mu} \dfrac{\partial}{\partial a^{\nu}} \\ \phi_* \dfrac{\partial}{\partial \omega_{\mu\nu}} = \dfrac{\partial}{\partial \omega'_{\mu\nu}} = (?) \end{cases} \tag{2.13}$$

calculation:

用 (2.12) 中给出的关系 (做替换 $x_g \mapsto x', x_h \mapsto x$), 所以,

$$\begin{cases} \Lambda(\omega) = \Lambda(\omega_0)\Lambda(\omega') \\ a = a_0 + \Lambda(\omega_0)a' \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial a^\nu}{\partial a'^\mu} = \Lambda(\omega_0)^\nu{}_\mu \\ \dfrac{\partial \omega_{\mu\nu}}{\partial \omega'_{\mu\nu}} = (?) \end{cases} \tag{2.14}$$