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Unsupervised Accelerated MRI Reconstruction via Ground-Truth-Free Flow Matching

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Motivation

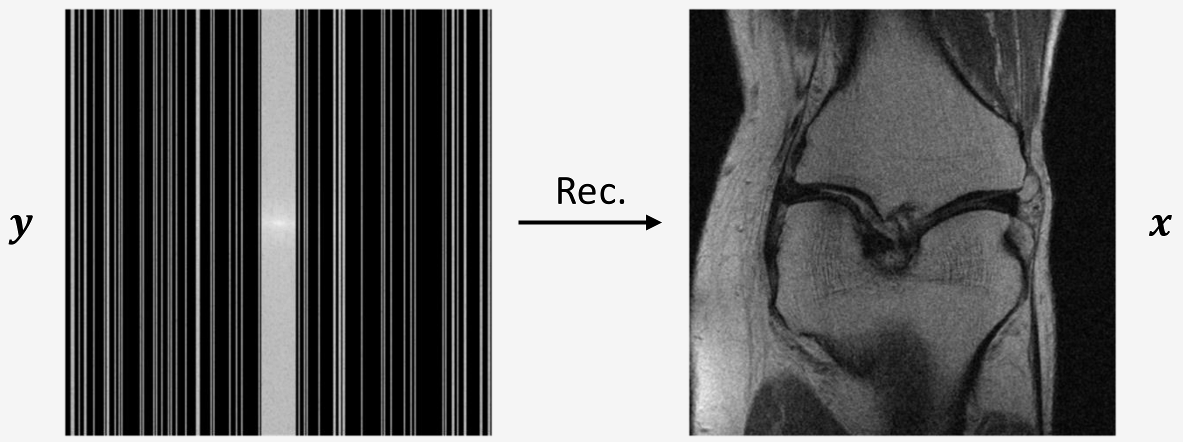
Motivation
MRI reconstruction

- Reconstruct the fully-sampled MR image $\boldsymbol{x} \in \mathbb{C}^D$ from under-sampled k-space measurements $\boldsymbol{y} \in \mathbb{C}^d$ through the forward model

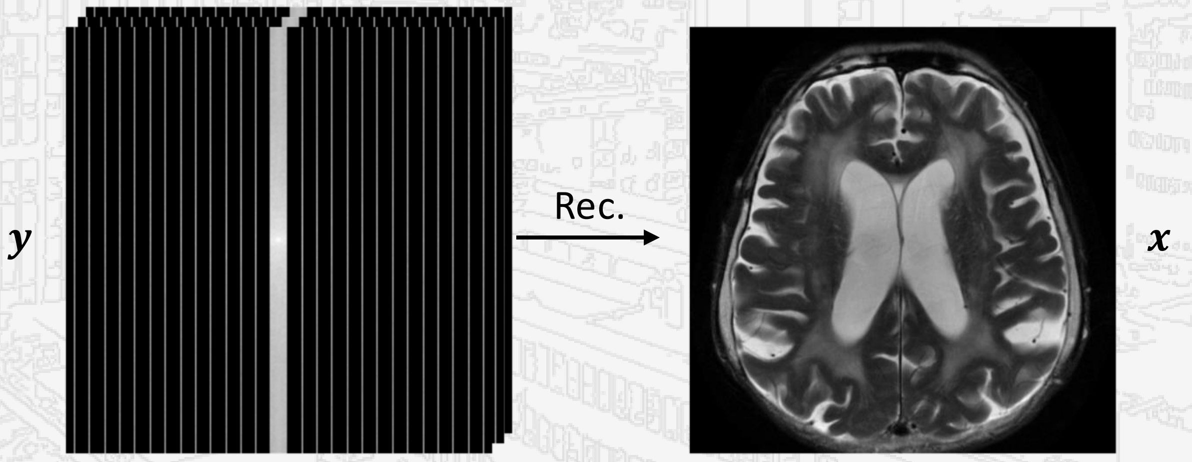
$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_C \end{bmatrix} \coloneqq \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e} \coloneqq \begin{bmatrix} \boldsymbol{MFS}_1 \\ \boldsymbol{MFS}_2 \\ \vdots \\ \boldsymbol{MFS}_C \end{bmatrix} \boldsymbol{x} + \boldsymbol{e},$$

where $\boldsymbol{M} \in \{0,1\}^{d \times D}$ denotes the under-sampling mask, $\boldsymbol{F} \in \mathbb{C}^{D \times D}$ the discrete Fourier transform, $\boldsymbol{S}_i \in \mathbb{C}^{D \times D}$ the sensitivity map of the i-th coil, and $\boldsymbol{e} \sim \mathcal{CN}(\boldsymbol{0}, 2\sigma^2 \boldsymbol{I}_{Cd})$ the complex Gaussian noise in k-space.

Emulated single coil



Multi-coil k-space



Motivation
Research gaps of MRI reconstruction

Previous work

- Optimisation-based:

$$x^* \in \arg\min_{x}\{-\log p(\, y \mid x\,) + \mathcal{R}(x)\},$$

where $\mathcal{R}(\cdot)$ is some regularization term.

- Supplied learning-based:
  - given paired training data $\{(x_i, y_i)\}_{i=1}^{N}$, train a neural network $f_{\theta}: y \mapsto x$ via loss minimization[1,2]
- Bayesian inference:
  - Learn the prior distribution of fully-sampled images by generative models
  - Reconstruct the observation through posterior sampling[3,4,5,6]

[1] *Aggarwal et al., TMI 2018*
[2] *Hammernik et al., MRM 2018*
[3] *Song et al., ICLR 2022*
[4] *Wang et al., ICLR 2023*
[5] *Chung et al., ICLR 2023*
[6] *Song et al., ICLR 2023*

Research gaps

- Both supervised and prior learning approaches require large datasets of fully-sampled MR images, which can be inaccessible.
- The high number of neural function evaluations (NFEs) of diffusion-model-based MRI reconstruction is computationally prohibitive in practice.

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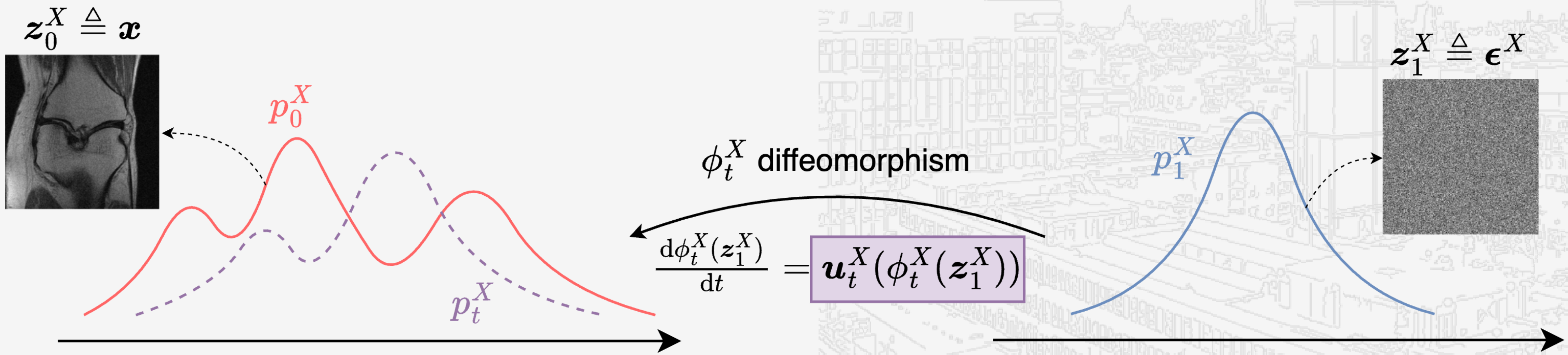
Methodology

Methodology
Flow matching

- Goal: learn a continuous normalising flow (CNF) from $p_1^X = \mathcal{N}(\mathbf{0}, \boldsymbol{I}_D)$ to p_0^X of the fully-sampled MR images.
  - Flow matching offers a simulation-free approach to learning the CNF.

- Formulation: define the CNF in the image space (X) by a diffeomorphism $\phi_t^X: [0,1] \times \mathbb{R}^D \to \mathbb{R}^D$
  - ϕ_t^X parameterised by a time-dependent vector field $\boldsymbol{u}_t^X: [0,1] \times \mathbb{R}^D \to \mathbb{R}^D$,

$$\boldsymbol{z}_t^X \triangleq \phi_t^X(\boldsymbol{z}_1^X), \qquad \forall\, \boldsymbol{z}_1^X \sim p_1^X, \qquad \mathrm{d}\boldsymbol{z}_t^X = \boldsymbol{u}_t^X(\boldsymbol{z}_t^X)\mathrm{d}t, \qquad \phi_1^X = \mathrm{id}.$$



Methodology
Flow matching

Question: How to construct the vector field $\boldsymbol{u}_t^X$ such that $p_0^X = [\phi_{1\to 0}^X]_\# p_1^X$ (push-forward of p_1^X by $\phi_{1\to 0}^X$)?

Conditions

1. conditional variable $\boldsymbol{\omega}^X$ independent of t
2. conditional probability path s.t. $\forall \boldsymbol{z}$,
   $$\mathbb{E}_{\boldsymbol{\omega}^X \sim p(\boldsymbol{\omega}^X)}\left[p_0^X(\boldsymbol{z} \mid \boldsymbol{\omega}^X)\right] = p_0^X(\boldsymbol{z})$$
   $$\mathbb{E}_{\boldsymbol{\omega}^X \sim p(\boldsymbol{\omega}^X)}\left[p_1^X(\boldsymbol{z} \mid \boldsymbol{\omega}^X)\right] = p_1^X(\boldsymbol{z})$$
3. continuity equation
   $$\partial_t p_t + \nabla \cdot (\boldsymbol{u}_t p_t) = 0$$

conditioning variable

$$\forall\, \boldsymbol{\omega}^X \sim p(\boldsymbol{\omega}^X)$$

conditional probability path

conditional vector field

$$\forall\, \boldsymbol{z}_{t|\boldsymbol{\omega}}^X \sim p_t^X(\boldsymbol{z}_t \mid \boldsymbol{\omega}^X) \xleftrightarrow[\text{continuity eq.}]{\text{determines explicitly by}} \boldsymbol{u}_{t|\boldsymbol{\omega}}^X(\boldsymbol{z}_t)$$

expectation over $\boldsymbol{\omega}^X \sim p(\boldsymbol{\omega}^X)$

expectation over $\boldsymbol{\omega}^X \sim p_t^X(\boldsymbol{\omega}^X \mid \boldsymbol{z}_t)$

$$\boldsymbol{z}_t^X \sim p_t^X(\boldsymbol{z}_t) \xleftrightarrow[\text{continuity eq.}]{\text{determines implicitly by}} \boxed{\boldsymbol{u}_t^X(\boldsymbol{z}_t) \triangleq \mathbb{E}_{\boldsymbol{\omega}^X \sim p(\boldsymbol{\omega}^X)}\left[\boldsymbol{u}_{t|\boldsymbol{\omega}}^X(\boldsymbol{z}_t)\frac{p_t^X(\boldsymbol{z}_t|\boldsymbol{\omega}^X)}{p_t^X(\boldsymbol{z}_t)}\right]}$$

Methodology
Flow matching

- Specification of the flow:
  - Choose the conditioning variable $\boldsymbol{\omega}^X \triangleq (\boldsymbol{x}, \boldsymbol{\epsilon}^X) \sim p_0^X \times p_1^X$.
  - Choose the conditional probability path $p_t^X(\boldsymbol{z}_t \mid \boldsymbol{\omega}^X) \triangleq \delta_{a_t\boldsymbol{x}+b_t\boldsymbol{\epsilon}^X}(\boldsymbol{z}_t)$ (linear interpolation).
  - Then, the conditional vector field is $\boldsymbol{u}_{t|\boldsymbol{\omega}}^X(\boldsymbol{z}_{t|\boldsymbol{\omega}}^X) = a_t'\boldsymbol{x} + b_t'\boldsymbol{\epsilon}^X$ satisfying the continuity equation.

- Training of the flow:
  - Flow matching (FM) objective:

$$\mathcal{L}_{\mathrm{FM}}(\boldsymbol{\theta}) \coloneqq \mathbb{E}_{t\sim p_T,\, \boldsymbol{z}_t^X \sim p_t^X(\boldsymbol{z}_t)} \left\| \boldsymbol{v}_{\boldsymbol{\theta}}^X(\boldsymbol{z}_t^X, t) - \boldsymbol{u}_t^X(\boldsymbol{z}_t^X) \right\|_2^2$$

  - Problem: no closed-form expression for the marginal vector field $\boldsymbol{u}_t^X(\boldsymbol{z}_t^X)$

  - Conditional flow matching (CFM) objective:

$$\mathcal{L}_{\mathrm{CFM}}(\boldsymbol{\theta}) \coloneqq \mathbb{E}_{t\sim p_T,\, \boldsymbol{\omega}^X \sim p(\boldsymbol{\omega}^X),\, \boldsymbol{z}_{t|\boldsymbol{\omega}}^X \sim p_t^X(\boldsymbol{z}_t|\boldsymbol{\omega}^X)} \left\| \boldsymbol{v}_{\boldsymbol{\theta}}^X(\boldsymbol{z}_{t|\boldsymbol{\omega}}^X, t) - \boldsymbol{u}_{t|\boldsymbol{\omega}}^X(\boldsymbol{z}_{t|\boldsymbol{\omega}}^X) \right\|_2^2$$

  - The gradient of the CFM is equivalent to that of the FM objective.
  - Problem: training requires large number of fully-sampled images.

Methodology

Ground-Truth-Free Flow Matching (GTF^{2}M): Preliminaries

- Dual-space conditional vector fields
  - Measurement (Y)-space conditioning variable: $\boldsymbol{\omega}^Y \triangleq (\boldsymbol{y}, \boldsymbol{\epsilon}^Y)$, $\boldsymbol{\epsilon}^Y \triangleq \boldsymbol{A}\boldsymbol{\epsilon}^X$.
  - Y-space conditional probability path: $p_t^Y(\boldsymbol{z}_t \mid \boldsymbol{\omega}^Y) \triangleq \delta_{a_t\boldsymbol{y}+b_t\boldsymbol{\epsilon}^Y}(\boldsymbol{z}_t)$.
  - Y-space conditional vector field: $\boldsymbol{u}_{t|\boldsymbol{\omega}}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\right) = a_t'\boldsymbol{y} + b_t'\boldsymbol{\epsilon}^Y$.

- Forward model of the dual-space conditional paths and vector fields
  - Using the conditions $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e}$ and $\boldsymbol{\epsilon}^Y \triangleq \boldsymbol{A}\boldsymbol{\epsilon}^X$, we can derive

$$\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y = \boldsymbol{A}\boldsymbol{z}_{t|\boldsymbol{\omega}}^X + a_t\boldsymbol{e},$$

where $\boldsymbol{z}_{t|\boldsymbol{\omega}}^X \triangleq a_t\boldsymbol{x} + b_t\boldsymbol{\epsilon}^X$, $\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y \triangleq a_t\boldsymbol{y} + b_t\boldsymbol{\epsilon}^Y$ and

$$\boldsymbol{u}_{t|\boldsymbol{\omega}}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\right) = \boldsymbol{A}\boldsymbol{u}_{t|\boldsymbol{\omega}}^X\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^X\right) + a_t'\boldsymbol{e}.$$

Methodology
Ground-Truth-Free Flow Matching (GTF²M)

- Goal: to learn the X-space marginal vector field $\boldsymbol{u}_t^X(\boldsymbol{z}_t^X)$ in a ground-truth-free manner.
  - Denote $\boldsymbol{h}_{\boldsymbol{\theta}}^X(\cdot)$: the predictor network for $\boldsymbol{u}_t^X(\boldsymbol{z}_t^X)$.

- GTF²M objective:

$$\mathcal{L}_{\mathrm{GTF^2M}}(\boldsymbol{\theta}) \triangleq \mathbb{E}_{t\sim p_T,\,\boldsymbol{\omega}^X\sim p(\boldsymbol{\omega}^X),\,\boldsymbol{z}_{t|\boldsymbol{\omega}}^X\sim p_t^X(\boldsymbol{z}_t|\boldsymbol{\omega}^X),\,\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\sim p_t^Z(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y|\boldsymbol{z}_{t|\boldsymbol{\omega}}^X)}\left\|\boldsymbol{h}_{\boldsymbol{\theta}}^X(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y, t) - \boldsymbol{u}_{t|\boldsymbol{\omega}}^X(\boldsymbol{z}_{t|\boldsymbol{\omega}}^X)\right\|_2^2$$

where $p_t^Z(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y \mid \boldsymbol{z}_{t|\boldsymbol{\omega}}^X)$ is induced from $\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y = \boldsymbol{A}\boldsymbol{z}_{t|\boldsymbol{\omega}}^X + a_t\boldsymbol{e}$.

  - It turns out that $\mathcal{L}_{\mathrm{GTF^2M}}(\boldsymbol{\theta})$ can be written as

$$\mathcal{L}_{\mathrm{GTF^2M}}(\boldsymbol{\theta}) = \mathbb{E}_{t\sim p_T,\,\boldsymbol{z}_t^X\sim p_t^X(\boldsymbol{z}_t),\,\boldsymbol{z}_t^Y\sim p_t^Z(\boldsymbol{z}_t^Y|\boldsymbol{z}_t^X)}\left\|\boldsymbol{h}_{\boldsymbol{\theta}}^X(\boldsymbol{z}_t^Y, t) - \boldsymbol{u}_t^X(\boldsymbol{z}_t^X)\right\|_2^2 + \mathrm{const.}$$

- $\Longrightarrow$ The GTF²M objective drives $\boldsymbol{h}_{\boldsymbol{\theta}}^X(\cdot)$ to predict $\boldsymbol{u}_t^X(\boldsymbol{z}_t^X)$.

Methodology
Ground-Truth-Free Flow Matching (GTF^{2}M)

- Goal: to learn the X-space marginal vector field $\boldsymbol{u}_t^X(\boldsymbol{z}_t^X)$ in a ground-truth-free manner.

- ENsemble Stein's Unbiased Risk Estimator (ENSURE) *(Aggarwal et al., TMI 2022)*
  - Assume the forward operator $\boldsymbol{A}_s$ is random and parameterised by a random variable s;
  - Assume the forward model $\boldsymbol{y}_s = \boldsymbol{A}_s\boldsymbol{x} + \boldsymbol{e}$, $\boldsymbol{e} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{C})$;
  - Denote $\boldsymbol{\rho}_s \triangleq \boldsymbol{A}_s^*\boldsymbol{C}^{-1}\boldsymbol{y}_s$ the sufficient statistic, and $\widehat{\boldsymbol{x}} \triangleq f_{\boldsymbol{\theta}}(\boldsymbol{\rho}_s)$ the reconstruction network.

  - Then, the MSE has an unbiased estimator as

$$\begin{aligned}\mathcal{L}_{\mathrm{MSE}}(\boldsymbol{\theta}) &= \mathbb{E}_{\boldsymbol{\rho}}\|f_{\boldsymbol{\theta}}(\boldsymbol{\rho}) - \boldsymbol{x}\|_2^2 \\ &= \mathbb{E}_s\mathbb{E}_{\boldsymbol{\rho}_s}\|\boldsymbol{R}_s(f_{\boldsymbol{\theta}}(\boldsymbol{\rho}_s) - \boldsymbol{x})\|_2^2 \\ &= \mathbb{E}_s\mathbb{E}_{\boldsymbol{\rho}_s}\left[\left\|\boldsymbol{R}_s\big(f_{\boldsymbol{\theta}}(\boldsymbol{\rho}_s) - \boldsymbol{\rho}_{s,\mathrm{ML}}\big)\right\|_2^2 + 2\nabla_{\boldsymbol{\rho}_s}\cdot\boldsymbol{R}_s^*\boldsymbol{R}_s f_{\boldsymbol{\theta}}(\boldsymbol{\rho}_s)\right] + \mathrm{const.}\end{aligned}$$

  where $\boldsymbol{R}_s \triangleq \boldsymbol{W}\boldsymbol{P}_s$ with $\boldsymbol{P}_s \triangleq \boldsymbol{A}_s^\dagger\boldsymbol{A}_s$ and $\boldsymbol{W} \triangleq \mathbb{E}_s[\boldsymbol{P}_s]^{-1/2}$, and $\boldsymbol{\rho}_{s,\mathrm{ML}} \triangleq (\boldsymbol{A}_s^*\boldsymbol{C}^{-1}\boldsymbol{A}_s)^\dagger\boldsymbol{A}_s^*\boldsymbol{C}^{-1}\boldsymbol{y}_s$ is the MLE solution for $\boldsymbol{y}_s = \boldsymbol{A}_s\boldsymbol{x} + \boldsymbol{e}$.

Methodology

Ground-Truth-Free Flow Matching (GTF^{2}M)

Goal: to learn the X-space marginal vector field $\boldsymbol{u}_t^X(\boldsymbol{z}_t^X)$ in a ground-truth-free manner.

Recall the following facts:

- Induced forward model over dual-space conditional vector fields:

$$\boldsymbol{u}_{t|\boldsymbol{\omega}}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\right) = \boldsymbol{A}\boldsymbol{u}_{t|\boldsymbol{\omega}}^X\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^X\right) + a_t'\boldsymbol{e}.$$

- Relationship between the measurement-space conditional path and vector field:

$$\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y = a_t\boldsymbol{y} + b_t\boldsymbol{\epsilon}^Y = \frac{a_t}{a_t'}\boldsymbol{u}_{t|\boldsymbol{\omega}}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\right) - b_t'\left(\frac{a_t}{a_t'} - \frac{b_t}{b_t'}\right)\boldsymbol{\epsilon}^Y,$$

  which implies that we can make prediction based on $\boldsymbol{u}_{t|\boldsymbol{\omega}}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\right)$ instead of $\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y$:

$$\boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y, t\right) = \boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{u}_{t|\boldsymbol{\omega}}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\right), t\right).$$

- The GTF^{2}M objective takes the form as an MSE:

$$\mathcal{L}_{\mathrm{GTF^2M}}(\boldsymbol{\theta}) = \mathbb{E}_{t\sim p_T,\, \boldsymbol{\omega}^X\sim p(\boldsymbol{\omega}^X),\, \boldsymbol{z}_{t|\boldsymbol{\omega}}^X\sim p_t^X\left(\boldsymbol{z}_t|\boldsymbol{\omega}^X\right),\, \boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\sim p_t^Z\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y|\boldsymbol{z}_{t|\boldsymbol{\omega}}^X\right)}\left\|\boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{u}_{t|\boldsymbol{\omega}}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\right), t\right) - \boldsymbol{u}_{t|\boldsymbol{\omega}}^X\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^X\right)\right\|_2^2.$$

Methodology

Ground-Truth-Free Flow Matching (GTF²M)

Goal: to learn the X-space marginal vector field $\boldsymbol{u}_t^X(\boldsymbol{z}_t^X)$ in a ground-truth-free manner.

ENSURE for GTF²M:

$$\begin{aligned}\mathcal{L}_{\mathrm{GTF^2M}}(\boldsymbol{\theta}) &= \mathbb{E}_{t,\,\boldsymbol{\omega}^X,\,\boldsymbol{z}_{t|\boldsymbol{\omega}}^X,\,\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y|\boldsymbol{z}_{t|\boldsymbol{\omega}}^X}\left\|\boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{u}_{t|\boldsymbol{\omega}}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^Y\right),t\right)-\boldsymbol{u}_{t|\boldsymbol{\omega}}^X\left(\boldsymbol{z}_{t|\boldsymbol{\omega}}^X\right)\right\|_2^2\\ &= \mathbb{E}_{s,\,t,\,\boldsymbol{\omega}^X,\,\boldsymbol{z}_{t|\boldsymbol{\omega},s}^X,\,\boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y|\boldsymbol{z}_{t|\boldsymbol{\omega},s}^X}\left\|\boldsymbol{R}_s\left[\boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{u}_{t|\boldsymbol{\omega},s}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y\right),t\right)-\boldsymbol{u}_{t|\boldsymbol{\omega},s}^X\left(\boldsymbol{z}_{t|\boldsymbol{\omega},s}^X\right)\right]\right\|_2^2\\ &= \mathbb{E}_{s,\,t,\,\boldsymbol{\omega}^Y,\,\boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y}\left[\left\|\boldsymbol{R}_s\left[\boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{\mu}_{t|\boldsymbol{\omega},s}^X,t\right)-\hat{\boldsymbol{u}}_{t|\boldsymbol{\omega},s,\mathrm{ML}}^X\right]\right\|_2^2+2\nabla_{\boldsymbol{\mu}_{t|\boldsymbol{\omega},s}^X}\cdot\boldsymbol{R}_s^*\boldsymbol{R}_s\boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{\mu}_{t|\boldsymbol{\omega},s}^X,t\right)\right]+\mathrm{const.}\end{aligned}$$

where $\boldsymbol{\mu}_{t|\boldsymbol{\omega},s}^X \triangleq \boldsymbol{A}_s^*\boldsymbol{C}_t^{-1}\boldsymbol{u}_{t|\boldsymbol{\omega},s}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y\right)$ is a sufficient statistic for $\boldsymbol{u}_{t|\boldsymbol{\omega},s}^X\left(\boldsymbol{z}_{t|\boldsymbol{\omega},s}^X\right)$ with $\boldsymbol{C}_t = (a_t'\sigma)^2\boldsymbol{I}_d$, and $\hat{\boldsymbol{u}}_{t|\boldsymbol{\omega},s,\mathrm{ML}}^X \triangleq (\boldsymbol{A}_s^*\boldsymbol{C}_t^{-1}\boldsymbol{A}_s)^\dagger\boldsymbol{A}_s^*\boldsymbol{C}_t^{-1}\boldsymbol{u}_{t|\boldsymbol{\omega},s}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y\right)$ the MLE solution for $\boldsymbol{u}_{t|\boldsymbol{\omega},s}^Y\left(\boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y\right) = \boldsymbol{A}\boldsymbol{u}_{t|\boldsymbol{\omega},s}^X\left(\boldsymbol{z}_{t|\boldsymbol{\omega},s}^X\right) + a_t'\boldsymbol{e}$.

- For single-coil MRI $\boldsymbol{A}_s = \boldsymbol{M}_s\boldsymbol{F}$, $\boldsymbol{M}_s = [\,\boldsymbol{I}_d \mid \boldsymbol{0}\,]\boldsymbol{T}_s$ for some permutation matrix $\boldsymbol{T}_s \in \{0,1\}^{D\times D}$, the projection operator $\boldsymbol{R}_s \triangleq \boldsymbol{W}\boldsymbol{P}_s = \boldsymbol{F}^*\boldsymbol{T}_s^{\mathrm{T}}\mathrm{diag}\left(p_1^{-1/2},\dots,p_d^{-1/2},0,\dots,0\right)\boldsymbol{T}_s\boldsymbol{F}$, where p_i is the probability that the ith k-space measurement is acquired.

Methodology

Ground-Truth-Free Flow Matching (GTF²M)

Goal: to learn the X-space marginal vector field $\boldsymbol{u}_t^X(\boldsymbol{z}_t^X)$ in a ground-truth-free manner.

Computation of the GTF²M objective

- Note that $\boldsymbol{\mu}_{t|\boldsymbol{\omega},s}^X = \boldsymbol{A}_s^* \boldsymbol{C}_t^{-1} \boldsymbol{u}_{t|\boldsymbol{\omega},s}^Y(\boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y) = \frac{1}{a_t' a_t \sigma^2}\left[\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y + c\boldsymbol{\epsilon}^Y\right]$; we can write

$$\boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{\mu}_{t|\boldsymbol{\omega},s}^X, t\right) = \boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y, t\right).$$

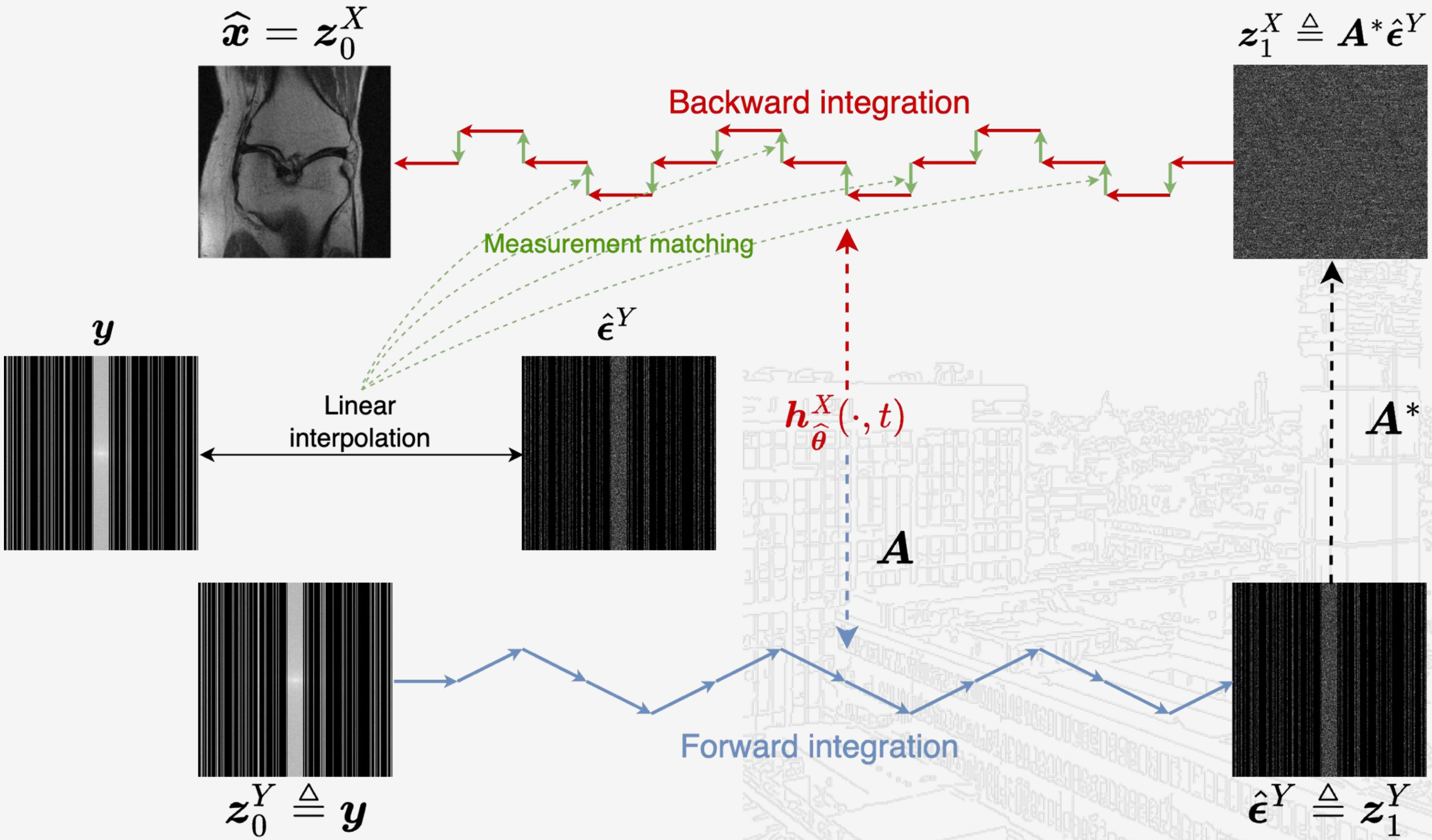
- By change of variables, the GTF²M objective can be written as

$$\mathcal{L}_{\mathrm{GTF^2M}}(\boldsymbol{\theta}) = \mathbb{E}_{s,\, t,\, \boldsymbol{\omega}^Y,\, \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y}\left[\left\|\boldsymbol{R}_s\left[\boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y, t\right) - \hat{\boldsymbol{u}}_{t|\boldsymbol{\omega},s,\mathrm{ML}}^X\right]\right\|_2^2 + 2a_t' a_t \sigma^2 \nabla_{\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y} \cdot \boldsymbol{R}_s^* \boldsymbol{R}_s \boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y, t\right)\right] + \text{const.}$$

- Using the Hutchinson trace estimator, the divergence term can be estimated as

$$\nabla_{\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y} \cdot \boldsymbol{R}_s^* \boldsymbol{R}_s \boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y, t\right) = \mathbb{E}_{\boldsymbol{b}\sim\mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_s^* \boldsymbol{R}_s)}\left[\boldsymbol{b}^{\mathrm{T}} \nabla_{\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y} \boldsymbol{h}_{\boldsymbol{\theta}}^X\left(\boldsymbol{A}_s^* \boldsymbol{z}_{t|\boldsymbol{\omega},s}^Y, t\right) \boldsymbol{b}\right].$$

Methodology
Reconstruction as decoupled continuous de-aliasing



Methodology

Reconstruction as decoupled continuous de-aliasing

Algorithm 1: Decoupled continuous de-aliasing via cyclic integration

- **Input**: k-space measurement $\boldsymbol{y} = (\boldsymbol{y}_c)_{c=1}^{C}$, pretrained flow predictor $\boldsymbol{h}_{\widehat{\boldsymbol{\theta}}}^{X}(\cdot, t)$, forward steps L, backward steps K, regularisation parameter ζ
- **Output**: reconstructed MR image $\widehat{\boldsymbol{x}}$ of $\boldsymbol{y}$
- **Steps**:

1. Set $\boldsymbol{z}_0^Y \coloneqq \boldsymbol{y}$;
2. For $t = 0, \dots, {}^{(L-1)}/_{L}$ do
   - $\boldsymbol{z}_{t+1/L}^{Y} \leftarrow \boldsymbol{z}_t^Y + \frac{1}{L}\boldsymbol{A}\boldsymbol{h}_{\widehat{\boldsymbol{\theta}}}^{X}(\boldsymbol{A}^*\boldsymbol{z}_t^Y, t)$; // forward integration
3. Set $\widehat{\boldsymbol{\epsilon}}^Y \coloneqq \boldsymbol{z}_1^Y$, $\boldsymbol{z}_1^X \coloneqq \boldsymbol{A}^*\widehat{\boldsymbol{\epsilon}}^Y$;
4. For $t \in \{1, \dots, {}^{1}/_{K}\}$ do
   - $\boldsymbol{z}_{t|\widehat{\boldsymbol{\omega}}}^{Y} = a_t\boldsymbol{y} + b_t\widehat{\boldsymbol{\epsilon}}^Y$;
   - $\boldsymbol{z}_{t-1/K}^{X} \leftarrow \boldsymbol{z}_t^X - \frac{1}{K}\boldsymbol{h}_{\widehat{\boldsymbol{\theta}}}^{X}\left(\boldsymbol{A}^*\boldsymbol{z}_{t|\widehat{\boldsymbol{\omega}}}^{Y}, t\right)$; // backward integration
   - $\boldsymbol{z}_t^X \leftarrow \boldsymbol{z}_t^X - \lambda_t\boldsymbol{A}^*\left(\boldsymbol{A}\boldsymbol{z}_t^X - \boldsymbol{z}_{t|\widehat{\boldsymbol{\omega}}}^{Y}\right)$; // measurement consistency update
5. Set $\widehat{\boldsymbol{x}} \coloneqq \boldsymbol{z}_0^X$ for single-coil or $\widehat{\boldsymbol{x}} \coloneqq \left(\sum_{c=1}^{C}\boldsymbol{S}_c^*\boldsymbol{S}_c\right)^{-1}\sum_{c=1}^{C}\boldsymbol{S}_c^*\boldsymbol{z}_{0,c}^X$ for multi-coil reconstruction;

- **Return** $\widehat{\boldsymbol{x}}$

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Experiments & Results

Experiments

Datasets and preprocessing

NYU fastMRI Initiative database:

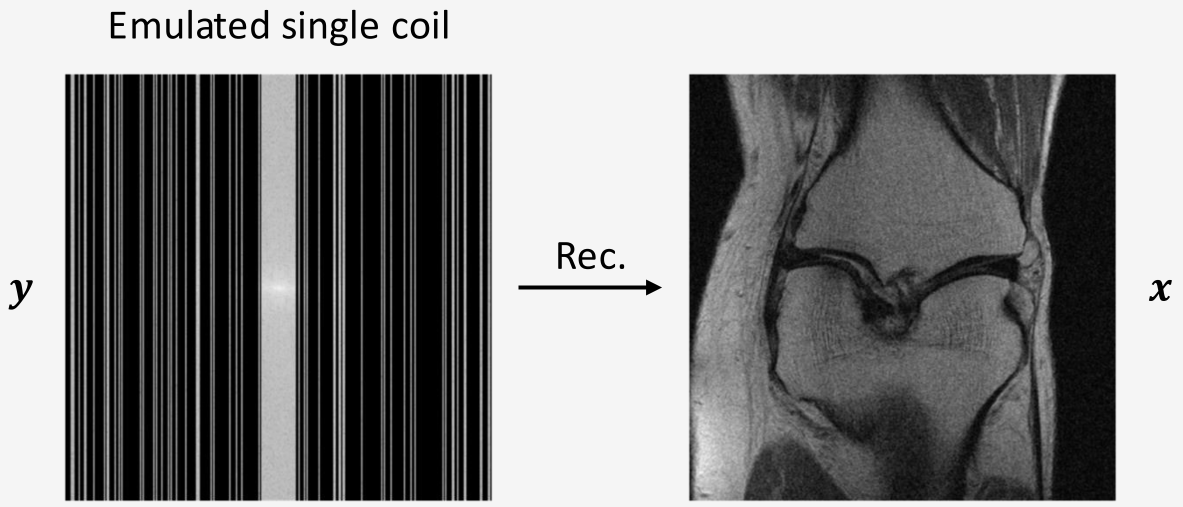
- Single-coil proton density (PD) weighted knee MRI without fat suppression.
- Multi-coil T2 weighted brain MRI.

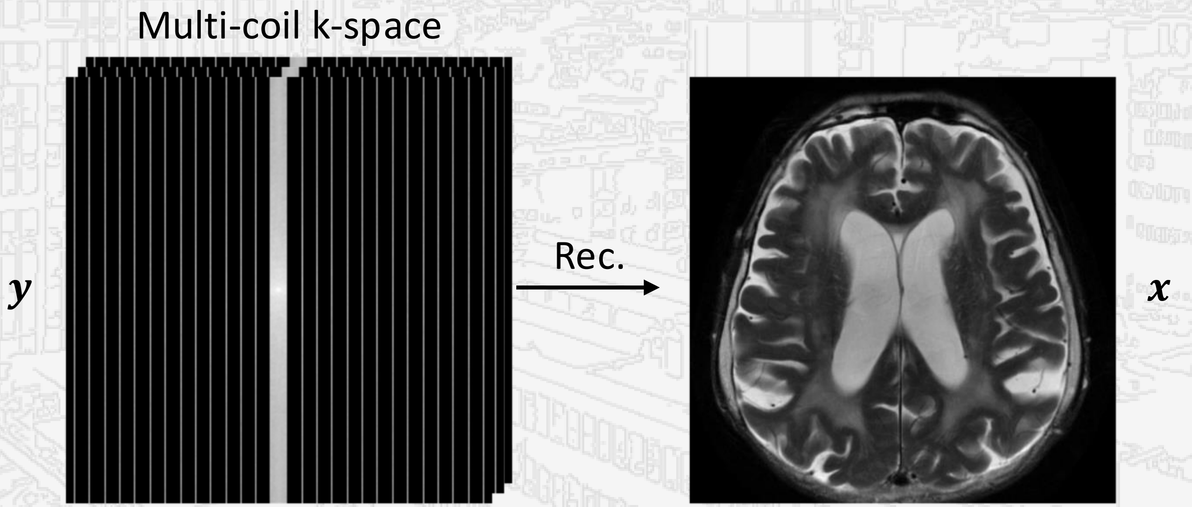
Cartesian sampling masks:

- 8% and 4% fully-sampled low-frequency k-space lines for 4x and 8x, respectively.
- The other lines are sampled random-uniformly and equidistantly for knee and brain MRI, respectively.

Ground truth:

- Knee: emulated single-coil image; Brain: SENSE reconstruction.





Experiments
Implementation details

Hyperparameters:

- Linear interpolation: $a_t = 1 - t$ and $b_t = t$;
- Noise level: $\sigma = 0.01$.

Network architecture: ADM (ablated diffusion model), *Dhariwal and Nichol, NeurIPS 2021*

- U-Net;
- Adaptive group normalisation (conditioned on linear projection of the positional embeddings of t);
- Self attention and dropout at the lowest three resolutions of the U-Net.

Training details:

- AdamW optimiser with learning rate 1×10^{-4}, weight decay coefficient 0.1;
- Exponential moving average of the network parameters.

Experiments
Compared baselines

Supervised end-to-end learning:

- MoDL: Model-based Deep Learning, *Aggarwal et al., TMI 2018;*

Diffusion model-based posterior sampling methods with prior learning:

- DDNM^{+}: Denoising Diffusion Null-space Models, *Wang et al., ICLR 2023*;
- ΠGDM: Pseudoinverse Guided Diffusion Models, *Song et al., ICLR 2023*;
- FlowPS: Flow-based Posterior Sampling, *Pokle et al., TMLR 2024*;

Unsupervised methods without prior learning:

- REI: Robust Equivariant Imaging, *Chen et al., CVPR 2022*;
- ENSURE: Ensemble Stein's Unbiased Risk Estimator + MoDL, *Aggarwal et al., TMI 2022*.

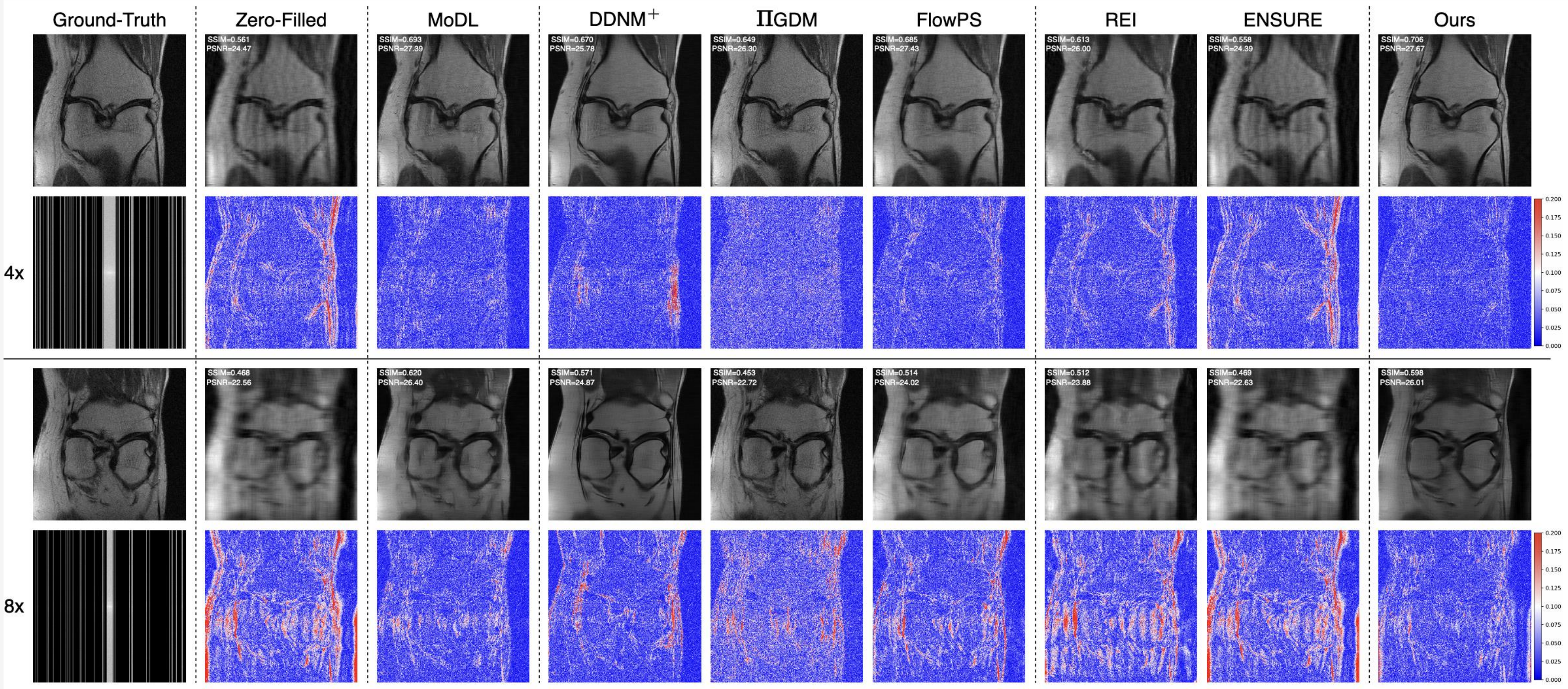
Results
Comparison study

Single-coil knee MRI

Method	SSIM ↑		PSNR ↑		NFEs ↓
	4×	8×	4×	8×	
Zero-filled	0.684 ± 0.086*	0.556 ± 0.106*	27.60 ± 2.78*	23.92 ± 2.75*	N/A
(a) Supervised methods using fully sampled images					
MoDL [1]	0.786 ± 0.069*	0.692 ± 0.107*	30.72 ± 3.07*	28.58 ± 2.96*	1
(b) Prior learning methods using fully sampled images					
DDNM$^+$ [30]	0.791 ± 0.076*	0.681 ± 0.108*	31.73 ± 3.29*	28.00 ± 3.20	100
ΠGDM [24]	0.728 ± 0.098*	0.581 ± 0.114*	30.27 ± 3.34*	25.83 ± 3.13*	100
FlowPS [20]	0.763 ± 0.077*	0.631 ± 0.101*	30.66 ± 2.73*	26.30 ± 2.55*	100
(c) Unsupervised methods w/o prior learning					
REI [4]	0.740 ± 0.087*	0.591 ± 0.110*	29.96 ± 2.87*	25.04 ± 2.97*	1
ENSURE [2]	0.684 ± 0.086*	0.556 ± 0.106*	27.65 ± 2.79*	23.91 ± 2.75*	1
(d) Unsupervised methods w/ prior learning					
Ours	0.801 ± 0.073	0.688 ± 0.095	31.64 ± 2.95	27.95 ± 2.64	20

Results
Comparison study

Single-coil knee MRI



Results

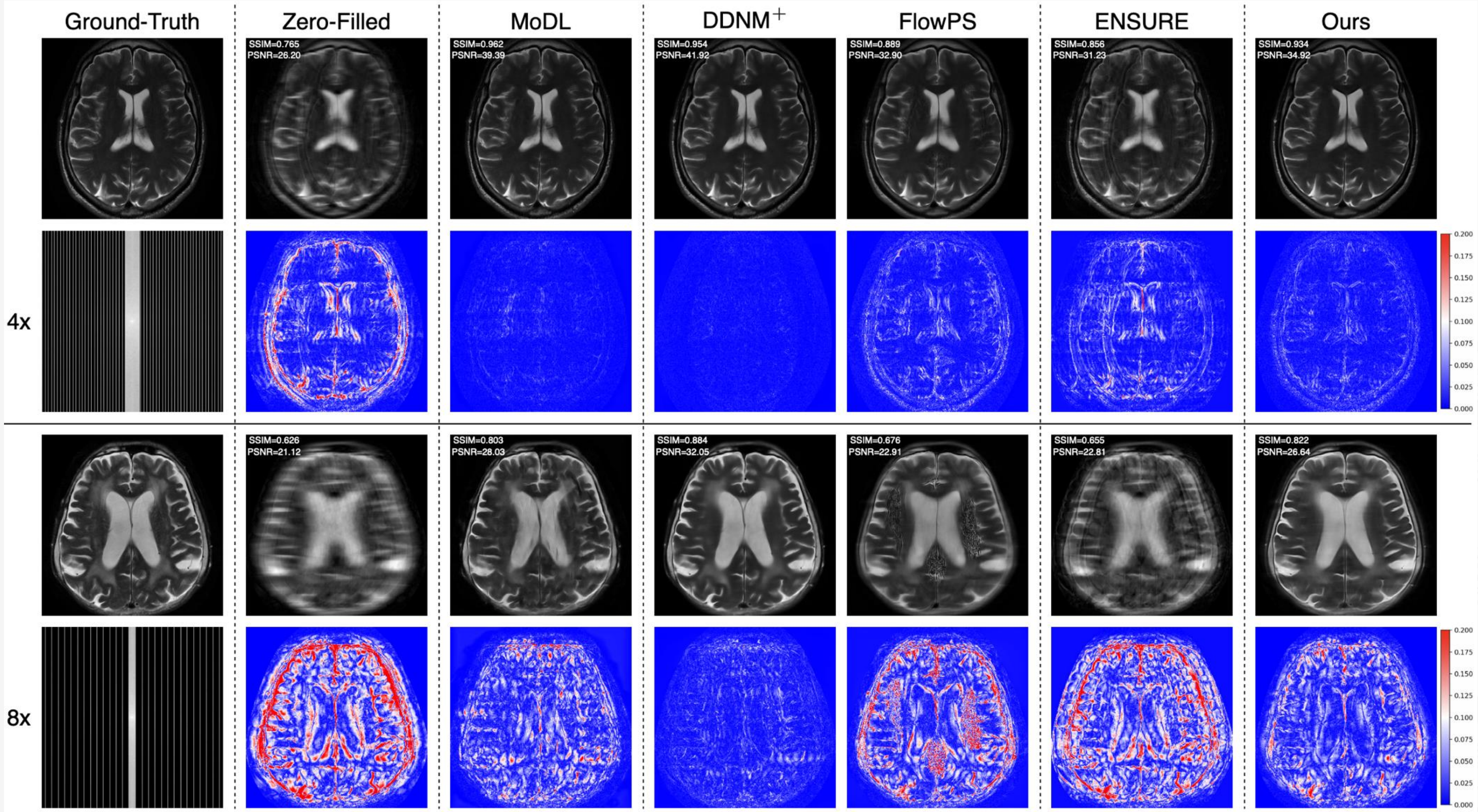
Comparison study

Multi-coil brain MRI

Method	SSIM ↑		PSNR ↑		NFEs ↓
	4×	8×	4×	8×	
Zero-filled	0.800 ± 0.089*	0.716 ± 0.117*	27.66 ± 3.78*	24.10 ± 3.97*	N/A
(a) Supervised methods using fully sampled images					
MoDL [1]	0.948 ± 0.044*	0.820 ± 0.051*	38.28 ± 3.37*	30.18 ± 3.04*	1
(b) Prior learning methods using fully sampled images					
DDNM^{+} [30]	0.929 ± 0.045*	0.887 ± 0.048*	40.61 ± 3.43*	34.13 ± 2.92*	100
FlowPS [20]	0.855 ± 0.060*	0.748 ± 0.069*	33.10 ± 2.73*	26.56 ± 3.50*	100
(c) Unsupervised methods w/o prior learning					
ENSURE [2]	0.825 ± 0.053*	0.739 ± 0.108*	31.75 ± 3.99*	25.56 ± 3.50*	1
(d) Unsupervised methods w/ prior learning					
Ours	0.920 ± 0.060	0.859 ± 0.054	34.65 ± 2.32	28.72 ± 2.92	20

Results
Comparison study

Multi-coil brain MRI



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Conclusion

Conclusion

Contributions

- An unsupervised prior learning framework for MRI reconstruction
  - No need for fully-sampled MRI during training;
  - An efficient cyclic integration algorithm as a decoupled continuous de-aliasing process.

Limitations and future work

- No closed-form expression of the GTF^{2}M objective for the combined multi-coil MR forward operator;
- Extension of the reconstruction algorithm to noisy data;
- Extension of the framework in the semi-supervised setup.

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Thank you!
Q&A

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