MA6201: Shimura varieties

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Abstract

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Contents

1 Modular curves

Let

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}.$$

There is a left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ given as follows: for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \quad \text{and} \quad z \in \mathbb{H},$$

define

$$g \cdot z := \frac{az+b}{cz+d}.$$

Note that this yields a surjection

$$\mathrm{SL}_2(\mathbb{R}) \twoheadrightarrow \mathrm{Aut}_{\mathrm{hol}}(\mathbb{H}),$$

where $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{H})$ is the holomorphic automorphism group of $\mathbb{H}$. We are interested in certain discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ whose actions produce interesting quotients of $\mathbb{H}$.

Definition 1.1. A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a *congruence subgroup* if it is a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma(N) \subset \Gamma$ with finite index for some $N \geq 1$, where

$$\Gamma(N) = \left\{ g \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

We will also usually assume that Γ is *small enough*, i.e., that $\Gamma \subset \Gamma(N)$ for some $N \geq 3$.

The group Γ (if small enough) acts freely and properly discontinuously on $\mathbb{H}$, and this implies that $\Gamma\backslash\mathbb{H}$ has a canonical complex manifold structure given by the complex structure on $\mathbb{H}$. Furthermore, the quotient map $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering.

Definition 1.2. The complex manifold $\Gamma\backslash\mathbb{H}$ is called a *modular curve.*

Proposition 1.3. A modular curve $\Gamma\backslash\mathbb{H}$ has the following properties:

(a) $\Gamma\backslash\mathbb{H}$ has the canonical structure of an algebraic variety over $\mathbb{C}$, which is compatible with the complex manifold structure.

(b) $\Gamma\backslash\mathbb{H}$ is the moduli space of elliptic curves over $\mathbb{C}$ with "Γ-level structure".

(c) The moduli interpretation in (b) also makes sense over some number field E (depending on Γ; e.g., $E = \mathbb{Q}(\zeta_N)$ if $\Gamma = \Gamma(N)$). This moduli problem over E is represented by a quasi-projective E-scheme whose base change to $\mathbb{C}$ recovers $\Gamma\backslash\mathbb{H}$ as a $\mathbb{C}$-scheme. We say that $\Gamma\backslash\mathbb{H}$ has a *model* over E.

(d) This moduli interpretation even extends integrally over $\mathbb{Z}[\zeta_N, 1/N]$, to produce a smooth scheme.

(e) The $\mathbb{C}$-scheme $\Gamma\backslash\mathbb{H}$ has a canonical compactification obtained by adding certain "special points" (cusps), giving a proper algebraic curve.

We will now explain how to see (a)-(d).

Remark 1.4. To show (a), even after giving a structure of a complex manifold to $\Gamma\backslash\mathbb{H}$ one cannot appeal to the usual GAGA equivalence between smooth projective curves over $\mathbb{C}$ and compact Riemann surfaces, because $\Gamma\backslash\mathbb{H}$ is not compact. However, there is a canonical compactification of $\Gamma\backslash\mathbb{H}$ which is a compact Riemann surface (the Baily–Borel compactification).

1.1 Elliptic curves over $\mathbb{C}$

Definition 1.5. An elliptic curve E over $\mathbb{C}$ is a smooth projective algebraic group of dimension 1.

Over $\mathbb{C}$, every elliptic curve arises as a quotient $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. More precisely, given an elliptic curve E, there is a holomorphic group homomorphism $\exp : \mathrm{Lie}E \to E$ coming from Lie group theory. Here $\mathrm{Lie}E$ is the tangent space at the identity O, and is a 1-dimensional complex vector space. (Note that exp is not algebraic.) Then $\ker(\exp)$ is a lattice in $\mathrm{Lie}E$ and $(\mathrm{Lie}E)/\ker(\exp) \xrightarrow{\sim} E$ is an isomorphism of elliptic curves. Notice also that (non-canonically) the left hand side is isomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$.

Remark 1.6. The map $\exp : \mathrm{Lie}E \to E$ is a universal covering. Hence we have the following canonical isomorphisms: $\ker(\exp) = \pi_1(E, O) = H^1(E, \mathbb{Z})$.

Suppose E and E' are elliptic curves. We have

$$\mathrm{Hom}(E, E') \cong \big\{ f : \mathrm{Lie}E \to \mathrm{Lie}E' \;\big|\; f \text{ is } \mathbb{C}\text{-linear and } f\big(H^1(E,\mathbb{Z})\big) \subset H^1(E',\mathbb{Z}) \big\}$$

where the assignment is given by $F \longmapsto dF\big|_{\mathrm{Lie}E}$. Combining the above facts, we have an equivalence of categories

$$\big((V,\Lambda),\ V \text{ a 1-dimensional } \mathbb{C}\text{-vector space and } \Lambda \subset V \text{ a } \mathbb{Z}\text{-lattice}\big) \xrightarrow{\sim} \big(\text{Elliptic curves}/\mathbb{C}\big)$$

given by

$$(V,\Lambda) \longmapsto V/\Lambda.$$

and the reverse map is given by

$$E \mapsto (\mathrm{Lie}E, H^1(E,\mathbb{Z})).$$

We first observe that two elliptic curves $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic if and only if there is a $\mathbb{C}$-linear holomorphic automorphism of $\mathbb{C}$ that takes Λ to Λ'. Every such holomorphic automorphism of $\mathbb{C}$ is given by multiplication by an element $\alpha \in \mathbb{C}^\times$. Indeed, any continuous group automorphism of a real vector space is necessarily an $\mathbb{R}$-linear map, and one checks that $\varphi(a+bi) = a\varphi(1) + b\varphi(i)$ is holomorphic if and only if $\varphi(i) = i\varphi(1)$. So $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ if and only if $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^\times$. This gives us:

$$\{\text{Lattices inside } \mathbb{C}\}/\text{homothety} \xrightarrow{\sim} \{\text{Elliptic curves over} \mathbb{C}\}/\text{isomorphism}.$$

Now we orient $\mathbb{C}$, as real vector space, so that $(1, i)$ is a positive orientation. We let

$$\mathcal{Z} = \{\text{pairs } (\omega, \omega') \text{ of positively oriented } \mathbb{R}\text{-bases of } \mathbb{C}\}.$$

The group $\mathrm{SL}(2, \mathbb{Z})$ acts on $\mathcal{Z}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\omega, \omega') = (a\omega + b\omega', c\omega + d\omega').$$

This action fixes the lattice $\mathbb{Z}\omega + \mathbb{Z}\omega' \subset \mathbb{C}$, and the quotient $\mathrm{SL}(2, \mathbb{Z})\backslash\mathcal{Z}$ is thus identified with the set of all lattices in $\mathbb{C}$. Thus we have a bijection

$$\mathrm{SL}(2, \mathbb{Z})\backslash\mathcal{Z}/\mathbb{C}^\times \xrightarrow{\sim} \{\text{Elliptic curves}\}/\text{isomorphism}.$$

where $\alpha \in \mathbb{C}^\times$ takes (ω, ω') to $(\alpha\omega, \alpha\omega')$, or equivalently takes the lattice Λ to $\alpha\Lambda$.

We further observe that

$$(\omega, \omega') \longmapsto \frac{\omega'}{\omega}$$

identifies $\mathcal{Z}/\mathbb{C}^\times$ with the upper half plane $\mathbb{H} \subset \mathbb{C}$:

$$\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}.$$

Thus, we have

$$\mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \{\text{Elliptic curves}\}/\text{isomorphism}.$$

Let $\mathbb{H}^\pm = \mathbb{C} \setminus \mathbb{R}$, the union of the upper and lower half planes. The group $\mathrm{GL}(2, \mathbb{Z})$ similarly acts by fractional linear transformations on $\mathbb{H}^\pm$ as above. Note that we further have an isomorphism

$$\mathrm{GL}(2, \mathbb{Z})\backslash\mathbb{H}^\pm \simeq \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \{\text{Elliptic curves}\}/\text{isomorphism}.$$

Let us now give these sets some complex structures. Recall that we have an $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic morphism $j : \mathbb{H}^+ \to \mathbb{C}$ inducing a bijection

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+ \xrightarrow{\sim} \mathbb{C}.$$

Here j corresponds to evaluating the classical j-invariant of an elliptic curve, which we recall is defined as follows:

Definition 1.7. Take an elliptic curve $E/\mathbb{C}$ and write it in Weierstrass form

$$y^2 = x^3 + ax + b.$$

The *j-invariant* is given by

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

We may use this map to identify the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+$ with $\mathbb{C}$ in order to give the former a complex manifold structure.

Note that $\mathbb{H}^+ \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+$ is a holomorphic map, but not a local isomorphism. In other words, this is not a covering map; there is ramification over the images of i and $e^{\frac{2\pi i}{3}}$ with branching of order 2 and 3 respectively. This is related to the fact that the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}^+$ is problematic in the following sense:

- $-I \in \mathrm{SL}_2(\mathbb{Z})$ acts trivially on $\mathbb{H}^+$. In particular, the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}^+$ is not free.
- The naive solution is to now consider the action of $\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ on $\mathbb{H}^+$. For this action, most points in $\mathbb{H}^+$ have trivial stabilizer, but points in the orbit of i and the orbit of $e^{\frac{2\pi i}{3}}$ have nontrivial stabilizers. So this is also not a solution.

This phenomenon exactly corresponds to the fact that for any elliptic curve E over $\mathbb{C}$ (or in fact any algebraically closed field of characteristic away from 2 or 3), the automorphism group of E is either:

1. $\mathbb{Z}/2\mathbb{Z}$, where the nontrivial automorphism is negation. This corresponds to the inclusion of $\{\pm I\}$ in all stabilizers.
2. $\mathbb{Z}/4\mathbb{Z}$. This automorphism group applies to a unique isomorphism class of elliptic curves.
3. $\mathbb{Z}/6\mathbb{Z}$. This automorphism group applies to a unique isomorphism class of elliptic curves.

Remark 1.8. The complex manifold structure we put on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+$ (using j) is the unique one such that the projection $\mathbb{H}^+ \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+$ is holomorphic.

Remark 1.9. It is "more correct", in some sense, to define the orbifold (or Deligne-Mumford stack) quotient of $\mathbb{H}^+$ by $\mathrm{SL}_2(\mathbb{Z})$. This allows us to still form something like a fine moduli space of elliptic curves that remembers the automorphisms (including the generic $\mathbb{Z}/2\mathbb{Z}$-automorphisms). We will discuss this in more detail when we talk about moduli spaces.

1.2 Alternative point of view: Hodge structures

In the previous subsection, we classified elliptic curves $\mathbb{C}/\Lambda$ up to isomorphism by morally fixing the complex vector space $\mathbb{C}$ and varying Λ. We now introduce a different way to think about the upper half plane with its complex structure which is more amenable to generalization to higher dimensions. We may fix an abstract $\mathbb{Z}$-module Λ, finite free of rank 2, and ask how we could vary the $\mathbb{C}$-structure.

As before, an elliptic curve is given by $E = (\mathrm{Lie}E)/H^1(E,\mathbb{Z})$. Also, we have a canonical isomorphism of 2-dimensional $\mathbb{R}$-vector spaces: $\mathrm{Lie}E \cong H^1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Notice of course that $\mathrm{Lie}E$ also has a complex structure. Thus in order to reconstruct E, we need the abstract $\mathbb{Z}$-module $H^1(E,\mathbb{Z})$ together with a complex structure on the $\mathbb{R}$-vector space $H^1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. In general, to define a complex structure on an $\mathbb{R}$-vector space V, it suffices to define multiplication by i such that $i^2 = -1$. In other words, a complex structure on V is exactly an element $J \in \mathrm{End}_{\mathbb{R}}(V)$ such that $J^2 = -1$, and this element corresponds to scalar multiplication by i. Then, we extend this to all $\mathbb{C}$ by setting:

$$(x + iy) \cdot v \;=\; x \cdot v \;+\; y \cdot J(v)$$

for all real numbers x, y. However, we will give a slightly different definition here:

Definition 1.10. A complex structure on $\mathbb{R}^2$ is a homomorphism

$$h : \mathbb{C}^\times \to \mathrm{GL}(2, \mathbb{R}) = \mathrm{Aut}(\mathbb{R}^2)$$

such that the eigenvalues of $h(z) \in \mathbb{C}^\times$ on $\mathbb{R}^2$ are z and $\bar{z}$.

The equivalence between the two definitions can be seen by taking $J = h(i)$, and observing that the condition on the eigenvalues forces the characteristic polynomial of J to be $X^2 + 1$.

Choosing the base point $e_0 = (1, 0) \in \mathbb{R}^2$, we see that any complex structure h defines an isomorphism $i_h : \mathbb{R}^2 \to \mathbb{C}$ of complex vector spaces, via $i_h^{-1}(z) = h(z) \cdot e_0$.

Now, denote $V = \mathbb{R}^2$, and let $h : \mathbb{C}^\times \to \mathrm{Aut}(\mathbb{R}^2)$ be a complex structure. Then for any $z \in \mathbb{C}$, $z \notin \mathbb{R}$, $h(z)$ is diagonalizable and by definition has two eigenvalues on $V \otimes \mathbb{C}$, namely z and $\bar{z}$. For some $z \in \mathbb{C}^\times$, let $V^{-1,0} = V_h^{-1,0}$, resp. $V^{0,-1} = V_h^{0,-1}$, denote the z-eigenspace, resp. the $\bar{z}$-eigenspace, for $h(z)$ on $V_{\mathbb{C}}$. Observe that since h is a homomorphism, the subspaces $V^{-1,0}, V^{0,-1}$ are independent of the choice of $z \in \mathbb{C}^\times \backslash \mathbb{R}$.

Example 1.11. We can define a complex structure by the homomorphism

$$h_0 : \mathbb{C}^\times \to \mathrm{GL}(2, \mathbb{R}) \quad \text{such that} \quad h_0(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix};$$

this obviously satisfies the hypothesis, and if we look at the eigenspaces for $z = i$, we get

$$V^{-1,0} = \mathbb{C} \cdot v_0, \qquad V^{0,-1} = \mathbb{C} \cdot v_0'$$

where

$$v_0 = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \qquad v_0' = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Proposition 1.12. Let $V^{-1,0}, V^{0,-1}$ as above. Then under the complex conjugation on $V \otimes \mathbb{R}$, we have

$$V^{-1,0} = \overline{V^{0,-1}}.$$

Proof. Let v, v' be the basis of $V \otimes \mathbb{C}$ such that $h(i)v = iv$, $h(i)v' = -iv'$. Thus $V^{-1,0} = \mathbb{C} \cdot v$, $V^{0,-1} = \mathbb{C} \cdot v'$. On the other hand, $h(i) \in \mathrm{Aut}(\mathbb{R}^2) = \mathrm{GL}(2, \mathbb{R})$ is a real matrix with eigenvalues i, $-i$, hence there is a real matrix γ such that

$$\gamma^{-1} h(i) \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = h_0(i).$$

Then we see that, with v_0 and v_0' as above, we have $\gamma\mathbb{C} \cdot v_0 = V_h^{-1,0}$, resp. $\gamma\mathbb{C} \cdot v_0' = V_h^{0,-1}$. Now, we see that the claim is true for the complex structure h_0, since $\bar{v_0} = v_0'$. Thus, since γ is real, the claim follows. □

Now, we can relate the space of complex structures to $\mathbb{H}$ in the following way. Observe that $\mathrm{GL}(2, \mathbb{R})$ acts by fractional linear transformations on $\mathbb{H}^\pm$. The complex number $\tau_h = \gamma(i)$ then belongs to $\mathbb{H}^\pm$. Moreover we can define a map

$$\pi : \{\text{complex structures}\} \longrightarrow \mathbb{H}^\pm$$

by $\pi(h) = \tau_h$. This map may appear to depend on the choice of the matrix γ such that $h(i) = \gamma h_0(i)\gamma^{-1}$, but we have the following:

Proposition 1.13. The map π is well-defined.

Proof. We write $\tau_h(\gamma)$ to take provisional account of this dependence. Note first of all that h_0 and h both extend to algebra homomorphisms $\mathbb{C} \to M(2, \mathbb{R})$, and since i generates $\mathbb{C}$ as $\mathbb{R}$-algebra it follows that $\gamma h_0 \gamma^{-1} = h$. If γ' is another choice, then $k = \gamma'^{-1}\gamma$ belongs to the centralizer in $\mathrm{GL}(2, \mathbb{R})$ of h_0, i.e. to the centralizer of $h_0(\mathbb{C})$, which is just $h_0(\mathbb{C})$. Thus $k \in h_0(\mathbb{C})$, and if

$$k = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

we have $\tau_h(\gamma') = \gamma'(i) = \gamma(k(i))$. Since $k(i) = \dfrac{xi + y}{-yi + x} = i$, there is no dependence. □

In other words, letting $K_\infty = h_0(\mathbb{C}^\times) \subset \mathrm{GL}(2, \mathbb{R})$, there is a sequence of identifications

$$\{\text{complex structures}\} \simeq \mathrm{GL}(2, \mathbb{R})/K_\infty \simeq \mathbb{H}^\pm.$$

The significance of this is that the final term has an obvious complex structure, hence so do the first two terms. Moreover, this complex structure is $\mathrm{GL}(2, \mathbb{R})$-invariant.

We can expland on this a little bit more, to define the Borel embedding. The function associating the normalized vector $v' = v_h' \in V_h^{0,-1}$ to h is compatible with the complex structure. Now $V_h^{0,-1} \subset V_\mathbb{C}$ is a variable line in $V_\mathbb{C}$, hence defines a variable point $p_h \in \mathbb{P}(V_\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$. If $\binom{\alpha}{\beta}$ is the homogeneous coordinate of a point in $\mathbb{P}^1$, we use the standard inhomogeneous coordinate α/β. Then the inhomogeneous coordinate of $V_h^{0,-1}$ is just τ_h. We thus have a holomorphic embedding

$$\{\text{complex structures}\} \simeq \mathrm{GL}(2, \mathbb{R})/K_\infty \hookrightarrow \mathbb{P}(V_\mathbb{C})$$

obtained by associating the subspace $V_h^{0,-1}$ to h.

Now, we define a family of elliptic curves $\mathcal{E}$ over $\mathbb{H}$ which was given, for each complex structure h, some elliptic curve E_h given as $\mathbb{C}/i_h(\mathbb{Z}^2)$. Recall the formula for $i_h : \mathbb{R}^2 \simeq \mathbb{C}$:

$$i_h\big(h(z)e_0\big) = z \cdot i_h(e_0).$$

The map i_h extends by linearity to a surjective homomorphism

$$\mathbb{R}^2 \otimes \mathbb{C} = V_{\mathbb{C}} \longrightarrow \mathbb{C}.$$

The left hand side is $V^{-1,0} \oplus V^{0,-1}$, and since the formula shows that i_h commutes with the action of $\mathbb{C}^\times$ on both sides, it follows that the map $V_{\mathbb{C}} \to \mathbb{C}$ is the projection

$$V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}/V^{0,-1}.$$

In other words, the $\mathbb{C}$ in the numerator is identified with $V^{-1,0}$, and we have the formula

$$E_h = (V_{\mathbb{C}}/V_h^{0,-1})/i_h(\mathbb{Z}^2).$$

We can further check that $i_h(\mathbb{Z}^2)$ is given by $\mathbb{Z} \oplus \mathbb{Z} \cdot \tau_h$.

1.3 Moduli interpretation

We have constructed a family of elliptic curves $\mathcal{E}$ over $\mathbb{H}$, but as we saw above, elliptic curves over $\mathbb{C}$ are parametrized by $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. We want to say that the family $\mathcal{E}/\mathbb{H}$ descends to this quotient. This would imply the existence of a universal family over $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, and hence this would imply that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ a fine moduli space. (We will discuss this notion rigorously later.)

However, we see that $\mathcal{E}$ does not admit a quotient by $\mathrm{GL}(2,\mathbb{Z})$. More precisely, there is an action of $\mathrm{GL}(2,\mathbb{Z})$ on the family $\mathcal{E}$ preserving the subgroup $i_h(\mathbb{Z}^2)$; we simply let $g \in \mathrm{GL}(2,\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^2)$ act naturally on $i_h(\mathbb{Z}^2) \subset V_{\mathbb{C}}$. We run into the same issue as before: the element $-I_2 \in \mathrm{GL}(2,\mathbb{Z})$ acts as -1 on each $\mathcal{E}_h$ and the quotient is no longer a family of elliptic curves; and there are other elliptic fixed points in $\mathbb{H}$ (namely i and $e^{\frac{2\pi i}{3}}$) whose stabilizers define automorphisms of the corresponding elliptic curves.

Now, if we instead consider the group

$$\Gamma(N) = \{g \in \mathrm{GL}(2,\mathbb{Z}) \mid g \equiv I_2 \pmod{N}, \}$$

then there are no fixed points in $\mathbb{H}$ for any integer $N \geq 3$.

Proposition 1.14. For $N \geq 3$, $\Gamma(N)$ acts freely and properly discontinuously on $\mathbb{H}^+$.

Proof. We sketch the proof that the action is free. Suppose $\gamma \in \Gamma(N)$ has a fixed point in $\mathbb{H}^+$. Since the stabilizer of $i \in \mathbb{H}^+$ in $\mathrm{SL}_2(\mathbb{R})$ is $\mathrm{SO}_2(\mathbb{R})$ and since $\mathbb{H}^+$ is transitive under $\mathrm{SL}_2(\mathbb{R})$, we see that γ must lie in a $\mathrm{SL}_2(\mathbb{R})$-conjugate of $\mathrm{SO}_2(\mathbb{R})$. In particular γ must be semi-simple and its eigenvalues in $\mathbb{C}$ have absolute value 1. On the other hand, the characteristic polynomial of γ is monic with integer coefficients, so the eigenvalues of γ are algebraic integers. Combined with the previous fact, we see that the eigenvalues of γ must be roots of unity. In particular, we see that $\langle\gamma\rangle$ is a torsion subgroup of $\Gamma(N)$, but we can check that $\Gamma(N)$ is torsion free.

We omit the proof that $\Gamma(N)$ acts properly discontinuously. See [DS06, §2.1]. □

In particular, this implies that $\Gamma(N)\backslash\mathbb{H}^+$ has the natural structure of a Riemann surface and $\mathbb{H}^+ \to \Gamma(N)\backslash\mathbb{H}^+$ is a covering. Further, this is obviously the universal covering, since $\mathbb{H}^+$ is simply connected.

Definition 1.15. The *modular curve* $Y(N)$ is the complex manifold

$$Y(N) := \bigsqcup_{j\in(\mathbb{Z}/N\mathbb{Z})^*} \Gamma(N)\backslash\mathbb{H}.$$

It also follows that the quotient $\Gamma(N)\backslash\mathcal{E}$ is a family of elliptic curves over $\Gamma(N)\backslash\mathbb{H}$.

We can ask what this space classifies. Observe that $\Gamma(N)$ fixes the group $N^{-1}\mathbb{Z}^2/\mathbb{Z}^2$, the basis of points of order N in $\mathcal{E}_h$ as defined by the generators

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

modulo N is fixed for all $h \in \mathbb{H}$. Thus, we see that points of $\Gamma(N)\backslash\mathbb{H}$ carry more than the data of the elliptic curve E_h: we also have an isomorphism

$$\alpha_N : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N],$$

where

$$E[N] := \{z \in E \mid z + \cdots + z \text{ (N times) } = 0\}.$$

Recall that $E[N]$ is non-canonically isomorphic to $(\mathbb{Z}/\mathbb{Z})^2 = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$ as $\mathbb{Z}/N\mathbb{Z}$-modules.

Definition 1.16. A choice of an isomorphism $\gamma : E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$ is called a *level-N structure* on E. Equivalently, this is a choice of an ordered basis (P, Q) of $E[N]$ as a free $\mathbb{Z}/N\mathbb{Z}$-module.

Now, we consider the following: For each $j \in (\mathbb{Z}/N\mathbb{Z})^\times$, fix once and for all an element $g_j \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ such that $\det(g_j) = j$. For instance, we may take $g_j = \begin{pmatrix} j & 0 \\ 0 & 1\end{pmatrix}$.

For each $j \in (\mathbb{Z}/N\mathbb{Z})^\times$ and each $h \in \mathbb{H}^+$, we can define an elliptic curve together with a level-N structure: $(E_h, \gamma_h = g_j \circ \alpha_N : E_h[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2)$. Moreover, we say that two tuples (E, γ) and (E', γ') are isomorphic if we have an isomorphism $f : E \mapsto E'$ such that we have a commutative square

$$\begin{tikzcd} E[N] \arrow[r, "f"] \arrow[d, "\gamma"] & E'[N] \arrow[d, "\gamma'"] \\ (\mathbb{Z}/N\mathbb{Z})^2 \arrow[r, equal] & (\mathbb{Z}/N\mathbb{Z})^2. \end{tikzcd}$$

Thus, we have a map

$$Y(N) \to \{\text{elliptic curves with level } N\text{-structure}\}/\text{isomorphism} \tag{1.3.1}$$

Proposition 1.17. The map (1.3.1) is a bijection.

Proof. We sketch here surjectivity: Let (E, γ) be an elliptic curve with level N structure. As before, we can identify E with a complex structure on given by some $h \in \mathbb{H}$ on $\Lambda = \mathbb{Z}^2$. Fix an isomorphism

$$u : \Lambda/N\Lambda \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2.$$

Observe that a level N structure on E induces an isomorphism (also denoted γ)

$$\gamma : \Lambda/N\Lambda \simeq (\mathbb{Z}/N\mathbb{Z})^2,$$

we compose u with some element of $\mathrm{GL}_2(\mathbb{Z})$ of determinant -1, and as a result we can always assume that $h \in \mathbb{H}^+$. Now let γ' be the composition

$$(\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{u^{-1}} \Lambda/N\Lambda \xrightarrow{\gamma} (\mathbb{Z}/N\mathbb{Z})^2.$$

Then $\gamma' \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. We note the following fact:

Fact. (Strong approximation for SL_2.) The natural map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.

For a proof, see [DS06, Exercise 1.2.2]. Note that the statement is not true if we replace SL_2 by GL_2, since elements of $\mathrm{GL}_2(\mathbb{Z})$ all have determinants ± 1.

Let $j = \det(\gamma')$, so $\gamma' g_j^{-1} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. By the above fact, we can compose u with a suitable element of $\mathrm{SL}_2(\mathbb{Z})$ to arrange that $\gamma' = g_j$. When we do this the element h we found in the above will be moved by the element of $\mathrm{SL}_2(\mathbb{Z})$. The image of such h under the above map will hence be the isomorphism (E, γ). $\square$

Remark 1.18. Note that to make a distinguished choice of one connected component $\Gamma(N)\backslash\mathbb{H}$ in $Y_(N)$ amounts to choosing a primitive N-th root of unity. In particular, we see that $Y(N)$ is a more natural space to consider, since we are not required to make this choice. This difference will become important later when we want to define the canonical model of the modular curve.

It is natural to simply 'extend the moduli problem' to classify elliptic curves over arbitrary bases (at least away from primes dividing N), where we have $E[N]$ is (at least étale locally) isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$. We will make this precise later.

2 Shimura data

For the modular curve the complex upper half plane $\mathbb{H}$ played a significant role; our first step is to understand its generalization.

2.1 Hodge structures

Definition 2.1. Let V be a finite-dimensional $\mathbb{R}$-vector space. A (real) *Hodge structure* on V is a $\mathbb{C}$-linear direct sum decomposition $V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ such that the natural conjugate-linear action $c \otimes v \mapsto \overline{c} \otimes v$ of complex conjugation on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ swaps $V^{p,q}$ and $V^{q,p}$. The decreasing chain of subspaces

$$F^p = \bigoplus_{p' \geq p} V^{p',q'}$$

is called the *Hodge filtration* of $V_{\mathbb{C}}$, and if the (p,q) for which $V^{p,q} \neq 0$ all satisfy $p+q=n$ then we say that the Hodge structure is *pure of weight* n.

Note that in general $F^p \cap \overline{F}^q = \bigoplus_{p' \geq p, q' \geq q} V^{p',q'}$, so if the Hodge structure is pure of weight n then $F^p \cap \overline{F}^q = V^{p,q}$ whenever $p+q=n$ (so in the pure case the Hodge structure can be recovered from the Hodge filtration by means of the complex conjugation action on $V_{\mathbb{C}}$). Also, it is generally harmless to restrict attention to pure Hodge structures, since if V is equipped with a Hodge structure then for any $n \in \mathbb{Z}$ the $\mathbb{C}$-subspaces $\bigoplus_{p+q=n} V^{p,q}$ in $V_{\mathbb{C}}$ are stable under the complex conjugation action and hence descend to $\mathbb{R}$-subspaces $V_n \subset V$ such that $\bigoplus V_n \simeq V$ (as scalar extension to $\mathbb{C}$ recovers the Hodge decomposition of $V_{\mathbb{C}}$, with the $V^{p,q}$ collected according to the value of $p+q$). Each V_n is equipped with a pure Hodge structure of weight n, so we can study these separately.

Remark 2.2. The terminology (and motivation) for a Hodge structure that is pure of weight n occurs for $V_n = H^n(Y(\mathbb{C}), \mathbb{R})$ with a smooth proper variety Y over $\mathbb{C}$. Indeed, we have that

$$H^n(Y(\mathbb{C}), \mathbb{R}) \otimes \mathbb{C} \simeq H^n(Y(\mathbb{C}), \mathbb{C}),$$

and by the Poincaré Lemma, we can use the complex-analytic (Kahler manifold) structure on $Y(\mathbb{C})$ to define a decreasing *Hodge filtration* F^p (consisting of de Rham cohomology classes represented by closed $\mathbb{C}$-valued C^∞ differential forms whose local expression in terms of dz_j's and $d\overline{z}_k$'s only involves wedge products with at least p of the dz_j's; e.g., $F^0 = (V_n)_{\mathbb{C}}, F^n$ contains the space $H^0(Y(\mathbb{C}), \Omega^n_Y)$ of global holomorphic n-forms, and $F^{n+1} = 0$). It is a deep theorem in Hodge theory that for V_n as above, $V^{p,q} := F^p \cap \overline{F}^q$ defines a Hodge structure on V_n that is pure of weight n (e.g., $V^{p,q} = \emptyset$ whenever $p+q \neq n$). Traditionally this is proved by exhibiting another construction of $V^{p,q}$ in terms of "harmonic forms" (relative to a choice of Kähler metric, say when Y is projective), but the final output of the construction turns out to be metric-independent. More explanations can be found in [Voi02]. There also exists a purely algebraic proof by Deligne–Illusie.

The cohomology algebra $\bigoplus_{n \geq 0} V_n = H^*(Y(\mathbb{C}), \mathbb{R})$ naturally inherits a Hodge structure for which V_n is the associated subspace that is pure of weight n.

For instance, if $Y = E$ is an elliptic curve over $\mathbb{C}$ and we take $n = 1$ then $F^1 = H^0(E, \Omega^1_E)$ is the space of global 1-forms on E and it is the kernel of the natural map $F^0 = (V_n)_{\mathbb{C}} = H^1(E(\mathbb{C}), \mathbb{C}) \to H^1(E(\mathbb{C}), \mathcal{O}_E(\mathbb{C})) = H^1(E, \mathcal{O}_E)$ which turns out to be surjective.

Proposition 2.3. There is an equivalence of categories between pairs $(V, h : \mathbb{C}^\times \to \mathrm{Aut}(V_{\mathbb{R}}))$, where V is a rational vector space and h is a homomorphism of real algebraic groups, and Hodge structures on V.

Proof. Let V be a rational vector space with a Hodge structure. Define $h : \mathbb{C}^\times \to \mathrm{Aut}(V_{\mathbb{R}})$ by letting $h(z)$ act as $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$ (this is the convention). One then has to check that h actually gives us a homomorphism of real algebraic groups. □

Remark 2.4. Another way to say this is to let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Then we want an algebraic homomorphism of $\mathbb{R}$-groups $h : \mathbb{S}_{\mathbb{R}} \to \mathrm{Aut}(V_{\mathbb{R}})$.

Definition 2.5. A polarization for a Hodge structure $(V, V^{p,q})$ of weight n is a nondegenerate bilinear form

$$\Psi : V \times V \to \mathbb{R}$$

which extends to $V_{\mathbb{C}}$ by linearity and is symmetric if n is even and alternating if n is odd. We also require that Ψ be subject to the following relations:

$$\Psi\left(V^{p,q}, V^{p',q'}\right) = 0 \quad \text{if } (p', q') \neq (q, p)$$

$$i^{p-q}\,\Psi(x, \overline{x}) > 0 \quad \text{for nonzero } x \in V^{p,q}.$$

These conditions are called the Hodge-Riemann bilinear relations.

Remark 2.6. Another equivalent definition is as follows. We denote by $\mathbb{R}(-n)$ the Hodge structure defined by $z \mapsto |z|^{-n}$. Then a polarization is a morphism of Hodge structures

$$\Psi : V \times V \to \mathbb{R}(-n)$$

such that $\Psi(v, h(i)w)$ is symmetric and positive definite.

Example 2.7. Fix a compact Kähler manifold X with Kähler form ω. Fix an integer $k \geq 0$. Let $(V, V^{p,q})$ be the pure Hodge structure obtained from $H^k(X(\mathbb{C}), \mathbb{R})$. We use ω to define a nondegenerate bilinear form $Q : V \times V \to \mathbb{R}$ by

$$Q(\alpha, \beta) := (-1)^{k(k-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{\dim(X)-k}.$$

This form Q is a polarization for $(V, V^{p,q})$, so we get the polarized Hodge structure $(V, V^{p,q}, Q)$.

Now, let us consider Hodge structures in families.

Definition 2.8. A variation of Hodge structures of weight n on a $\mathbb{C}$-scheme S is a $\mathbb{R}$-local system V on S, together with a decreasing filtration $F^pV_S \subset V_S := V \otimes \mathcal{O}_S$ by holomorphic vector bundles, such that:

1. The F^p define a Hodge structure on each fiber of V_S, that is, $V_S = F^p V_S \oplus \overline{F^{n-p+1} V_S}$.

2. (Griffiths Transversality) Let $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$. Then by the Riemann-Hilbert correspondence, $V_{\mathbb{C}}$ defines an integrable holomorphic connection

$$\nabla : V_S \to V_S \otimes \Omega^1_S.$$

We require that the F^p satisfy Griffiths transversality with respect to ∇, that is,

$$\nabla(F^p V_S) \subset F^{p-1} V_S \otimes \Omega^1_S.$$

Definition 2.9. A polarized variation of Hodge structure of weight n on S is a variation of Hodge structure $(V, F^p V_S)$ on S along with a bilinear form $Q : V \otimes V \to \mathbb{R}(-n)$, so that the restriction of $(V, F^p V_S, Q)$ to each fiber is a polarized Hodge structure.

Example 2.10. The key (and motivating) example of a variation of Hodge structures is the (relative) de Rham cohomology of the fibers of a morphism $f : X \to S$ that is the analytification of a smooth proper morphism between smooth complex algebraic varieties. More specifically, one defines"relative de Rham cohomology" sheaves $\mathcal{H}^n_{\mathrm{dR}}(X/S) = R^n f_*(f^{-1}\mathcal{O}_S) \simeq R^n f_*(\mathbb{C}) \otimes \mathcal{O}_S$, and homological methods provide a relative Hodge to de Rham spectral sequence that defines a natural decreasing filtration on $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ by $\mathcal{O}_S$-submodules $\mathcal{F}$. One can show (eg. [Sta25, 0FK4], and Ehresmann's theorem) that these subsheaves are actually locally free with formation commuting with any base change, inducing on s-fibers exactly the traditional Hodge filtration on $H^n(X_s, \mathbb{C})$ determined by the complex structure on X_s. The connection ∇ here is the Gauss-Manin connection.

A variation of Hodge structures always admits a period morphism, which looks at how the stages of the Hodge filtration move within this fixed space. We saw this in the the case of the Weierstrass family of elliptic curves over $\mathbb{C} - \mathbb{R}$, for which the line $H^0(\mathcal{E}_\tau, \Omega^1) = \mathbb{C}dz$ moves within $H^1(\mathcal{E}_\tau, \mathbb{C}) = \mathbb{C} \otimes \Lambda = \mathbb{C}^2$ via the expression $dz = \tau e_1^* + e_2^*$ with varying τ, where we had the Borel embedding $\mathbb{H} \hookrightarrow \mathbb{P}^1(\mathbb{C})$. In general, given a variation of Hodge structures on S, for each p we can define a holomorphic map

$$S \to \mathrm{Gr}_d(n)$$

which is the Grassmannian of dimension d subspaces of a dimension n vector space, where d is the dimension of $\mathcal{F}^p$, and n is the dimension of V.

We now want to understand moduli spaces of polarized variation of Hodge structures, and relate them to Hermitian symmetric domains.

2.2 Hermitian symmetric domains

Let M be a complex manifold, and let $J_p : T_p M \to T_p M$ denote the action of i on the tangent space at a point p of M.

Definition 2.11. A *Hermitian metric* on M is a Riemannian metric g on the underlying smooth manifold of M such that J_p is an isometry for all p. Recall that a *Riemannian metric* on M is a positive-definite smooth contravariant 2-tensor field g on M. This means: g consists of positive-definite bilinear forms $g_p : T_p(M) \times T_p(M) \to \mathbb{R}$ for each point $p \in M$ such that for smooth vector fields X, Y on M, the function $g(X,Y) : p \mapsto g_p(X_p, Y_p)$ is smooth. In local coordinates $(x^i)_{1\leq i\leq n}$ a Riemannian metric has the form

$$g_p = \sum_{i,j} g_{ij}(p) dx^i \otimes dx^j$$

for some symmetric positive-definite matrix $g_{ij}(p)$ that depends smoothly on p.

A *Hermitian manifold* is a complex manifold equipped with a hermitian metric g, and a *Hermitian symmetric space* is a connected homogeneous Hermitian manifold M that admits a symmetry at each point p, i.e., the group of holomorphic isometries acts transitively, and there exists an involution s_p having p as an isolated fixed point.

We denote by $\mathrm{Hol}(M)$ the group of holomorphic automorphisms of a Hermitian symmetric space M, and $\mathrm{Is}(M,g)$ the group of homomorphic isometries. These are real Lie groups and we denote by $\mathrm{Hol}(M)^+$, $\mathrm{Is}(M,g)^+$ the connected component of the identity.

Theorem 2.12. Every hermitian symmetric space M is a product of hermitian symmetric spaces of the following types:

- Noncompact type — the sectional curvature is negative
- Compact type — the sectional curvature is positive
- Euclidean type — the sectional curvature is zero.

In the first two cases, the space is simply connected. A hermitian symmetric space is *indecomposable* if it is not a product of two hermitian symmetric spaces of lower dimension.

A proof can be found in [Hel79, Chapter VIII].

Definition 2.13. A Hermitian symmetric space of non-compact type is called a Hermitian symmetric domain.

Example 2.14.
1. Let Λ be a discrete subgroup of $\mathbb{C}$. Then $\mathbb{C}/\Lambda$ is a Hermitian symmetric space. After identifying $\mathbb{C}$ with $\mathbb{R}^2$, the metric is given by the standard form $dx^2 + dy^2$. The translations act transitively on $\mathbb{C}/\Lambda$ and $x \mapsto -x$ is a symmetry at 0. Geodesics are images of straight lines in $\mathbb{C}$, and this has zero curvature.

2. The Riemann sphere $\mathbb{P}^1(\mathbb{C}) \cong S^2$ is a Hermitian symmetric space. The metric is induced from the standard metric $dx^2 + dy^2 + dz^2$ of ambient space $\mathbb{R}^3$ of S^2. Rotations act transitively on S^2 and symmetries are given by rotations by π around a diameter. Geodesics are great circles, and this has positive curvature.

3. The upper half-plane $\mathbb{H}_1$ is a Hermitian symmetric domain. After identifying $\mathbb{C}$ with $\mathbb{R}^2$, the metric is given by $\frac{dx^2+dy^2}{y^2}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}_1$ by Möbius transforms. This space has negative curvature.

We observe the following:

Proposition 2.15. Let M be a Hermitian symmetric space, $p \in M$, and denote by K_p the stabilizer subgroup of $\mathrm{Isom}(M,g)^+$. Then K_p is compact, and we have an isomorphism of smooth manifolds

$$\mathrm{Isom}(M,g)^+/K_p \simeq M.$$

We are interested in the situation where $\mathrm{Isom}(M,g)^+ \simeq G(\mathbb{R})^+$ for some connected reductive group G. One way to arrange for this is to mimic the GL_2 case and suppose that M is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$. (It will turn out that *every* Hermitian symmetric domain has such a form.)

2.3 Shimura data

We now suppose that G is a connected reductive group over $\mathbb{Q}$, and we let X be a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$.

We first explain how to get a variation of Hodge structures over X.

Firstly, from the previous section, we see that the points h serve as abstract encodings of G-valued Hodge structures, using a Tannakian perspective. More precisely, what we see is that given any algebraic representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{R}$, we may look at the compositions

$$\rho_{\mathbb{R}} \circ h : \mathbb{C}^\times \to \mathrm{Aut}(V)$$

(where $\rho_{\mathbb{R}}$ denotes ρ restricted to $\mathbb{R}$-points) and hence over X we get a "family" of Hodge structures on V. Now, there is a particularly special choice for (ρ, V), namely the adjoint representation

$$\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g}) \qquad \text{for } \mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}\big(G(\mathbb{R})\big).$$

One of the key observations we will see is that Hodge-theoretic conditions on the Hodge structures arising from every (ρ, V) should be expressed entirely in terms of the pair (G, X) and the Hodge structure on $\mathfrak{g}$ arising from $\rho = \mathrm{Ad}_G$.

Now, we fix a pair (ρ, V), and for each h we consider the $\mathbb{R}$-linear weight decomposition $\bigoplus_{n \in \mathbb{Z}} V_{n,h}$ on V arising from each $h \in X$ is (by definition) the decomposition arising from the $\mathbb{G}_m$-action $\rho_{\mathbb{R}} \circ h|_{\mathbb{R}^\times}$ on V over $\mathbb{R}$. Concretely, $V_{n,h}$ consists of those $v \in V$ such that $\rho(h(x))(v) = x^n v$ for all $x \in \mathbb{R}^\times$. We first wish to formulate a condition on (G, X) that is equivalent to the property that for any (ρ, V) the subspaces $V_{n,h} \subseteq V$ are the same for all $h \in X$.

Definition 2.16. Let $\mathbb{G}_m \to \mathbb{S}$ be the algebraic map which on $\mathbb{R}$-points sends $z \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$ to $z^{-1} \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. For any $h \in X$, we let $w_h : \mathbb{G}_m \to G$ be the composition

$$\mathbb{G}_m \to \mathbb{S} \xrightarrow{h} G,$$

and we call this the weight homomorphism.

Lemma 2.17. The subspaces $V_{n,h} \subseteq V$ are independent of $h \in X$ for every $\rho : G \to \mathrm{GL}(V)$ if and only if w_h has image in the center of G for some (equivalently, all) $h \in X$.

Proof. For any $g \in G(\mathbb{R})$, $\rho\big(w_{g,h}(x)\big) = \rho(g)\,\rho\big(w_h(x)\big)\,\rho(g)^{-1}$ for all $x \in \mathbb{R}^\times$. If we look at a faithful representation of G, then we see that the image of $\rho \circ w_h$ must lie in the center of $\rho(G)$, so the image of w_h is in Z_G. The reverse implication is clear. $\square$

Observe that we see that the composition $\mathrm{Ad}_G \circ h$ must be trivial when restricted to $\mathbb{R}^\times$, and thus it has weight 0. Now, we have not given any holomorphic structure on X, but it turns out we have the following:

Proposition 2.18. X has a unique complex structure for which each h defines a holomorphic family of Hodge structures.

Proof. (Sketch) This is [Mil05, 2.14(a)], and the rough idea is that after composing with a faithful representation of G, we can use the Borel embedding to embed X into a flag variety by looking at the Hodge filtration F^p for each p. We then check that the tangent spaces at each h define a complex subspace of the tangent space of the corresponding point in the Grassmannian. $\square$

We want this family of Hodge structures to be a polarizable variation of Hodge structures. We first look at the condition to be a variation of Hodge structures:

Proposition 2.19. Fix $\rho : G \to \mathrm{GL}(V)$ be a faithful representation. Let V_n be the holomorphic bundles on X corresponding to the weight filtration on the Hodge structure. Denote by the Hodge filtration by $\{F^p_{n,h}\}$ on V_n as h varies in X, and $\mathcal{H}_n = (V_n)_\mathbb{C} \otimes \mathcal{O}_X$. Define the connection

$$\nabla = 1 \otimes d : \mathcal{H}_n \otimes \mathcal{O}_X \to \mathcal{H}_n \otimes \Omega^1_X$$

Then V_n defines a variation of Hodge structures, i.e.

$$\nabla_\rho\big(\mathcal{F}^p_n\big) \subseteq \mathcal{F}^{p-1}_n \otimes_{\mathcal{O}_X} \Omega^1_X \qquad \text{for all } p \in \mathbb{Z}$$

if and only if for every $h \in X$ the weight-0 Hodge structure $\{V_h^{p,-p}\}_{p \in \mathbf{Z}}$ on $\mathfrak{g}$ associated to $\mathrm{Ad}_G \circ h$ is of type

$$\{(-1,1),(0,0),(1,-1)\}.$$

Proof. (Sketch) We use the construction as in the previous proof of the Borel embedding, which we denote by ϕ. Note that on tangent spaces, we have a map

$$d\phi : \mathfrak{g}/\mathfrak{g}^{00} \to \mathrm{End}(V)/\mathrm{End}(V)^{00} \simeq \mathrm{End}(V_\mathbb{C})/F^0\mathrm{End}(V_\mathbb{C})$$

and Griffiths transversality corresponds to the condition that the image of $d\phi$ lies in the coset classes of those $T \in \mathrm{End}(V_\mathbb{C})$ such that $T(F^p) \subseteq F^{p-1}$ for all $p \in \mathbb{Z}$, which is exactly $F^{-1}(\mathrm{End}(V_\mathbb{C}))$. Now, since $d\phi$ is injective, this is true if and only if we have $\mathfrak{g}_\mathbf{C} = F^{-1}(\mathfrak{g}_\mathbf{C})$, which is to say that the Hodge structure on $\mathfrak{g}$ defined by h makes $\mathfrak{g}^{p,q}_\mathbb{C}$ vanish whenever $p < -1$. Since complex conjugation on $\mathfrak{g}_\mathbb{C}$ swaps $\mathfrak{g}^{q,p}_\mathbb{C}$ and $\mathfrak{g}^{p,q}_\mathbb{C}$, we conclude that the transversality condition on fibers at $h \in X$ is equivalent to the vanishing of $\mathfrak{g}^{p,-p}_\mathbb{C}$ whenever $p \neq 0, \pm 1$, which is to say that the Hodge structure on $\mathfrak{g}$ defined by h has type $\{(1,-1),(0,0),(-1,1)\}$. $\square$

We now turn our attention to getting a polarization. The following result is [Del79, 1.1.14(iii)].

Theorem 2.20. In order that the variation of Hodge structures above admits a polarization, it is necessary and sufficient that the following two conditions hold.

1. Let G_1 be the minimal connected closed $\mathbb{R}$-subgroup in G such that all maps $h : \mathbb{C}^\times \to G(\mathbb{R})$ factor through $G_1(\mathbb{R})$. Then G_1 is reductive.

2. Let G_1' be the connected semisimple derived group of G_1 (e.g., $G_1' = \mathrm{SL}_n$ when $G_1 = \mathrm{GL}_n$). Then the intersection of the fixed-point locus of ι_h in $G(\mathbb{R})$ with $G_1'(\mathbb{R})^0$ in a maximal compact subgroup for some (equivalently, any) $h \in X$.

Here, we note that since for any $h \in X$, the conjugation action by $h(i)$ on G is independent of the choice of $i = \sqrt{-1} \in \mathbb{C}$ and is an involution since $h(-1)$ is central in G (by the weight-0 hypothesis), we obtain an involution denoted by ι_h.

Note that if we assumed that G was reductive, then the first condition clearly holds. Moreover, for the second condition, we can equivalently look at the image in the adjoint group $G^{\mathrm{ad}}_{\mathbb{R}}$, which we require to be maximal compact.

Definition 2.21. A involution θ of G is called a Cartan involution if the real form G^θ is compact. Here G^θ is a real algebraic group whose $\mathbb{R}$-points are

$$G^\theta(\mathbb{R}) = \{\, g \in G(\mathbb{C}) \mid \theta(g) = \overline{g} \,\}.$$

Remark 2.22. Observe that condition (2) implies that $\theta = i_h$ is a Cartan involution of G, and thus G^θ is hence an inner form of G.

To summarize, to get a Hermitian symmetric domain, we require the pair (G, X) to satisfy:

SV1: For all $h \in X$, the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{Ad}_G \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\};$$

SV2: For all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$,

Definition 2.23. A *Shimura datum* is a pair (G, X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

SV1: For all $h \in X$, the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{Ad}_G \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\};$$

SV2: For all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$,

SV3: G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Remark 2.24. SV3 is a mild extra condition that ensures some 'minimality' of the group G. There are situations (eg. quaternionic Shimura sets) where we ignore this condition.

Remark 2.25. Observe that for $n \geq 3$, GL_n cannot admit a Shimura datum because the quotient $\mathrm{GL}_n(\mathbb{R})/K_\infty$ where $K_\infty = \mathbb{R}^\times K_\infty^\circ$, and K_∞° is a maximal compact in $\mathrm{GL}_n(\mathbb{R})$, is not a Hermitian symmetric domain. The easiest way to see this is that if this were the case $\mathrm{GL}_{n,\mathbb{R}}$ must admit an inner form which is compact, and $\mathrm{GL}_{n,\mathbb{R}}$ does not for $n \geq 3$

Remark 2.26. One can ask about quotients $G(\mathbb{R})/K_\infty$ which are not Hermitian symmetric domains, merely Riemannian symmetric domains. An example of this is the arithmetic hyperbolic 3-manifold, for the group $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, for F an imaginary quadratic field. It turns out that some of these locally symmetric spaces can be found in the boundary of the Borel-Serre compactification of some Shimura variety, and this observation has been used quite successfully to prove results in this case, c.f. [?Scholzetorsion].

2.4 List of Hermitian symmetric domains

We can give an exhaustive list of the possible Hermitian symmetric domains, using the above characterization. Indeed, from [Lan17, §3]we have the following list of possible Hermitian symmetric domain $\mathcal{D} = G(\mathbb{R})/K$:

1. $G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}), K \cong U_n, \mathcal{D} = \mathcal{H}_n$, where
$$\mathcal{H}_n := \{\, Z \in \mathrm{Sym}_n(\mathbb{C}) : \mathrm{Im}(Z) > 0 \,\}$$
known as the Siegel upper half space, and
$$K := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}) \right\} \simeq U_n := \{\, g = A + iB : {}^t\overline{g}\, g = 1_n \,\}.$$

2. For all $a, b \geq 1$,
$$G(\mathbb{R}) = U_{a,b} := \{\, g \in \mathrm{GL}_{a+b}(\mathbb{C}) : {}^t\overline{g}\, 1_{a,b}\, g = 1_{a,b} \,\},$$
where $1_{a,b} := \begin{pmatrix} 1_a & 0 \\ 0 & -1_b \end{pmatrix}$,
$$\mathcal{D}_{a,b} := \left\{ U \in M_{a,b}(\mathbb{C}) : {}^t\begin{pmatrix} \overline{U} \\ 1 \end{pmatrix} \begin{pmatrix} 1_a & 0 \\ 0 & -1_b \end{pmatrix} \begin{pmatrix} U \\ 1 \end{pmatrix} = {}^t\overline{U}\, U - 1_b < 0 \right\},$$
$$K := \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in U_{a,b} \right\} \simeq U_a \times U_b$$

3. $G(\mathbb{R}) = \mathrm{SO}^*_{2n}, K \cong U_n, \mathcal{D} = \mathcal{H}_{\mathrm{SO}^*_{2n}},$ with $n \geq 2$:

4. $G(\mathbb{R})^+ = \mathrm{SO}_{a,2}(\mathbb{R})^+, K \cong \mathrm{SO}_a(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R}), \mathcal{D}^+ = \mathcal{H}^+_{\mathrm{SO}_{a,2}},$ with $a \geq 1$ but $a \neq 2$:

5. Lie $G \cong \mathfrak{e}_{6(-14)}$, Lie $K \cong \mathfrak{so}_{10} \oplus \mathbb{R}, \mathcal{D} = \mathcal{H}_{E_6}$:

6. Lie $G \cong \mathfrak{e}_{7(-25)}$, Lie $K \cong \mathfrak{e}_6 \oplus \mathbb{R}, \mathcal{D} = \mathcal{H}_{E_7}$:

Example 2.27. Let B be a quaternion algebra over a totally real field F, and let G be the algebraic group over $\mathbb{Q}$ with $G(\mathbb{Q}) = B^\times$. Then $B \otimes_{\mathbb{Q}} F = \prod_\nu B \otimes_{F,\nu} \mathbb{R}$, where ν runs over the embeddings of F into $\mathbb{R}$. We have

$$B \otimes_{\mathbb{Q}} \mathbb{R} \approx \mathbb{H} \times \cdots \times \mathbb{H} \times M_2(\mathbb{R}) \times \cdots \times M_2(\mathbb{R})$$

$$G(\mathbb{R}) \approx \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times \times \mathrm{GL}_2(\mathbb{R}) \times \cdots \times \mathrm{GL}_2(\mathbb{R})$$

$$h(a+ib) = 1 \quad \cdots \quad 1 \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \cdots \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Let X denote the $G(\mathbb{R})$-conjugacy class of h. Then (G, X) satisfies SV1 and SV2, and so it is a Shimura datum provided B splits (i.e., becomes isomorphic to $M_2(\mathbb{R})$) at at least one real prime of F. It then satisfies SV3 because G^{ad} is simple (as an algebraic group over $\mathbb{Q}$). Let $I = \mathrm{Hom}(F, \mathbb{Q}^a) = \mathrm{Hom}(F, \mathbb{R})$, and let I_{nc} be the set of ν such that $B \otimes_{F,\nu} \mathbb{R}$ is split. If B is non-split at all real places, then this forms a quaternionic Shimura set.

Example 2.28. Let $G = \mathrm{GSp}_{2n}/\mathbb{Q}$, and $X = \mathcal{H}_n^\pm$. Then this is a Shimura datum, corresponding to the *Siegel modular variety.*

Finally, we can ask how to realize the original SL_2 formulation of the modular curve in this picture; these form connected Shimura varieties.

Definition 2.29. A *connected Shimura datum* is a pair (G, X^+) consisting of a semisimple algebraic group over $\mathbb{Q}$ and a $G^{ad}(\mathbb{R})^+$-conjugacy class of homomorphisms $h : \mathbb{S} \to G^{ad}_{\mathbb{R}}$ satisfying

1. For all $h \in X^+$, the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{Ad}_G \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\};$$

2. For all $h \in X^+$, $\mathrm{ad}(h(i))$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$.

3. G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

2.5 Adelic description

Finally, we want to explain how the adelic description of $\mathrm{Sh}_K(G, X)$ has the structure of a locally symmetric space. We define

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

for an open compact subgroup. We want to show that this is a locally symmetric space, that is, we can write it in the form $\Gamma\backslash X$ for a discrete subgroup Γ of the automorphisms of X. For simplicity, we will only cover the case of the modular curve, as the proof in general is quite similar.

Proposition 2.30. We have a bijection:

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{A}_f)/K(N) \cong \Gamma\backslash\mathbb{H}.$$

where $K(N) = \ker(\mathrm{SL}_2(\hat{\mathbb{Z}}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$.

Proof. We first consider $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_f)/K(N)$. Observe that the natural inclusion

$$\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \subset \mathrm{SL}_2(\mathbb{A}_f)/K(N).$$

induces an inclusion

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \subset \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_f)/K(N),$$

since $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Q}) \cap \mathrm{SL}_2(\hat{\mathbb{Z}})$.

Now, observe that we have an equality $\mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(\mathbb{Q})\mathrm{SL}_2(\hat{\mathbb{Z}})$, so this inclusion is surjective hence a bijection. But by definition

$$\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \cong \mathrm{SL}_2(\mathbb{Z}/NZ),$$

so we conclude using strong approximation

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{Z}/NZ) \cong \{1\}.$$

It follows that

$$\mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(\mathbb{Q})K(N).$$

Consequently, we see that

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{A}_f)/K(N) \cong (\mathrm{SL}_2(\mathbb{Q}) \cap K(N))\backslash\mathbb{H} \cong \Gamma\backslash\mathbb{H},$$

where the first isomorphism follows because every representative (h, g) with $h \in \mathbb{H}$ and $g \in \mathrm{SL}_2(\mathbb{A}_f)$ is equivalent to a representative of the form (h', k), for $h' \in \mathbb{H}$ and $k \in K(N)$, and the first map is induced by $(h, g) \mapsto h'$. Moreover, $(h_1, k_1) \sim (h_2, k_2)$ if and only if there exists $\gamma \in \mathrm{SL}_2(\mathbb{Q}) \cap K(N)$ such that $\gamma h_1 = h_2$. □

As explained above, we would like to get the whole $Y(N)$, rather than just one connected component. To do this, it turns out that we have to work with the group $G = \mathrm{GL}_2$ instead of SL_2. Then denote by $\mathrm{GL}_2(\mathbb{R})_+ \subset \mathrm{GL}_2(\mathbb{R})$ the subset which stabilizes $\mathbb{H}^+ \subset \mathbb{H}^\pm$ (namely those with positive determinant).

Proposition 2.31.

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K'(N) \cong Y(N).$$

where $K'(N) = \ker(\mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$.

Proof. Let $\mathrm{GL}_2(\mathbb{Q})_+ = \mathrm{GL}_2(\mathbb{R})_+ \cap \mathrm{GL}_2(\mathbb{Q})$. Then we can argue as above to see that we have an isomorphism

$$\mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{A}_f)/K'(N) \cong \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

Moreover, we have an isomorphism

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K'(N) \simeq \mathrm{GL}_2(\mathbb{Q})_+\backslash \mathbb{H} \times \mathrm{GL}_2(\mathbb{A}_f)/K'(N) = \bigsqcup_{(\mathbb{Z}/NZ)^\times} \Gamma\backslash\mathbb{H}.$$

The first isomorphism follows from the fact that since $\mathrm{GL}_2(\mathbb{Q})$ is dense in $\mathrm{GL}_2(\mathbb{R})$ (from Real approximation, see [Mil05, Appendix A] for a proof) and $\mathrm{GL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$, every $h \in \mathbb{H}^\pm$ is of the form qh^+ with $q \in \mathrm{GL}_2(\mathbb{Q})$ and $h^+ \in \mathbb{H}^+$. This shows that the map is surjective. We leave injectivity as an exercise. □

2.6 Hodge type Shimura varieties

Definition 2.32. We define a morphism of Shimura data $(G, X) \to (G', X')$ to be a algebraic group homomorphism $f : G \to G'$ which maps the conjugacy class X to X'.

Given any morphism of Shimura data, and a level $K \subset G(\mathbb{A}_f)$ whose image lies in some $K' \subset G'(\mathbb{A}_f)$, we get a map of Shimura varieties

$$\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G', X').$$

If the map f is a closed embedding, we can even have the following:

Proposition 2.33 ([Del71, Proposition 1.15])**.** Let $(G, X) \hookrightarrow (G', X')$. For any compact open $K \subset G(\mathbb{A}_f)$, there exists a compact open $K' \subset G'(\mathbb{A}_f)$ such that the map

$$\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G', X')$$

is a closed immersion of schemes.

Definition 2.34. We say a Shimura datum (G, X) is of Hodge type of there is a closed embedding $(G, X) \hookrightarrow (\mathrm{GSp_{2n}}, \mathcal{H}_n^\pm)$ for some n.

We will focus on Hodge type Shimura varieties in this class, since we are able to deduce results about them from results about Siegel modular varieties.

For groups of type A,B and C, it is always possible, up to central isogeny of G, to assume that the Shimura datum is of Hodge type.

3 Abelian schemes

We would like to explain how to get a canonical model and integral model of Shimura varieties. In order to understand what structures we need to pay attention to, we first analyze what happens for generalizations of the modular curves to the higher-dimensional *Siegel modular varieties*. These are moduli spaces of polarized abelian schemes with full level N structure. Our next goal is to prove that such a moduli functor is indeed representable over a base like $\mathbb{Z}[1/N]$. Our main references here are [Mum08, MFK94]. Much of the exposition here is also from [Zhu22]

From now on, all schemes are assumed to be locally noetherian.

Definition 3.1. Let S be a scheme. An *abelian scheme* A over S is a smooth proper group scheme over S all of whose geometric fibers are connected.

In particular, for any geometric point s of S, A_s is an abelian variety over $k(s)$.

3.1 Abelian varieties over $\mathbb{C}$

We first want to understand abelian varieties over $\mathbb{C}$. We first see that by Serre's GAGA, the analytification is a complex torus:

Definition 3.2. A complex torus is a connected compact Lie group over $\mathbb{C}$.

Example 3.3. If $V \simeq \mathbb{C}^g$ and $\Lambda \subset V$ is a lattice (a discrete, co-compact subgroup) then V/Λ is a complex torus.

Theorem 3.4. Every complex torus A is commutative and the holomorphic exponential map $\exp\colon T_0(A) \to A$ is a surjective homomorphism with kernel $\Lambda \subset T_0(A)$ a lattice. Hence $A \simeq T_0(A)/\Lambda$.

Proof. See [Mum08, pg. 1-2]. To see commutativity, the key is to study the adjoint representation of A acting on $T_0(A)$. This map $a \mapsto dc_a(e)$ (with $c_a(x) = axa^{-1}$) is a holomorphic map $A \to \mathrm{GL}(T_0(A))$ from a connected compact complex manifold into an open submanifold of a Euclidean space, so it must be constant (by the maximum principle in several complex variables), and by setting $x = a$ we see that this must be $\mathrm{id}_{T_0(A)}$.

Now, the surjectivity of exp follows because the image is a subgroup of A which contains an open neighbourhood of 0, and by connectedness must necessarily be A. Finally, let Λ be the kernel of exp. It must be discrete because the exp is a local isomorphism at 0. Finally, we see that the induced homomorphism $T_0(A)/\Lambda \to A$ is holomorphic by the definition of the structure of complex manifold on $T_0(A)/\Lambda \to A$, and is thus an isomorphism. Since lattices are the only discrete subgroups of vector spaces with compact quotient, Λ must be a lattice. $\square$

Remark 3.5. Note that analytification does *not* induce an equivalence of categories between abelian varieties of dimension g over $\mathbb{C}$ and complex tori of dimension g, for $g \geq 2$.

Definition 3.6. Let $X = V/L$ be a complex torus of dimension g, and let E be a skew-symmetric form $E\colon L \times L \to \mathbb{Z}$. Since $L \otimes_{\mathbb{Z}} \mathbb{R} = V$, we can extend E to a skew-symmetric $\mathbb{R}$-bilinear form $E_{\mathbb{R}}\colon V \times V \to \mathbb{R}$. We call E a *Riemann form* if

(a) $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$;

(b) the associated Hermitian form $H(x, y) = E(ix, y) + iE(x, y)$ is positive definite.

Note that there is a natural bijection between Hermitian forms H on V and skew-symmetric $\mathbb{R}$ bilinear forms E on V satisfying $E(ix, iy) = E(x, y)$. The correspondence is given by

$$H \mapsto E = \operatorname{Im}(H) \quad \text{and} \quad E \mapsto H(x, y) = E(ix, y) + i\,E(x, y).$$

Observe that H is positive definite if and only if $E_{\mathbb{R}}(iv, v) > 0$ for all $v \neq 0$.

Remark 3.7. If X has dimension 1, then $\bigwedge^2 L \simeq \mathbb{Z}$, and so there is a skew-symmetric form $E\colon L \times L \to \mathbb{Z}$ such that every other such form is an integral multiple of it. The form E is uniquely determined up to sign, and exactly one of $\pm E$ is a Riemann form.

We say that X is *polarizable* if it admits a Riemann form.

Remark 3.8. Note that the complex analytic structure on a complex torus X induces a Hodge structure of weight $(-1, 0), (0, -1)$, since we have $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{C}^g$, exactly as in the case of elliptic curves. Note that E is a Riemann form exactly when $-E$ defines a polarization on this Hodge structure.

Theorem 3.9. A complex torus X is of the form $A(\mathbb{C})$ if and only if it is polarizable.

Proof. (Rough Sketch) The main idea here is that X is the analytification of an abelian variety if and only if it is algebraizable, and to show that it is algebraizable we need an embedding into projective space. This embedding is induced by sections of an ample line bundle.

More precisely, for a Hermitian form H, and $\alpha : \Lambda \to \{z \in \mathbb{C}^\times : |z| = 1\}$ be a map with

$$\alpha(\lambda_1 + \lambda_2) = e^{i\pi E}(\lambda_1, \lambda_2)\alpha(\lambda_1)\alpha(\lambda_2)$$

we let $\mathcal{L}(H, \alpha)$ be the line bundle given by the quotient of $\mathbb{C} \times V$ by the action of Λ defined by for all $\lambda \in \Lambda$,

$$\lambda \cdot (z, w) = \big(\alpha(\lambda)\, e^{\pi H(w,\lambda) + \frac{1}{2}\pi H(\lambda,\lambda)} \cdot z,\ w + \lambda\big).$$

Then we have the following theorem of Appell and Humbert that classifies all line bundles on X:

Theorem 3.10 (Appell-Humbert)**.** Let $\mathcal{L}$ be a line bundle on the complex torus X. Then $\mathcal{L}$ is isomorphic to a $\mathcal{L}(H, \alpha)$ for a uniquely determined (H, α) as defined above.

For each such $\mathcal{L}$, we can look at sections of $\mathcal{L}$, which are certain θ-functions on $\mathbb{C}^g$: ie. they satisfy

$$\theta(z + \lambda) = \alpha(\lambda) e^{\pi H(z,\lambda) + \frac{1}{2}\pi H(\lambda,\lambda)} \theta(z).$$

Theorem 3.11 (Lefschetz)**.** Let X be a complex torus V/Λ with $\mathcal{L} = \mathcal{L}(H, \alpha)$ the associated line bundle as above. The Riemann form H is positive-definite if and only if $\mathcal{L}$ is ample, and in this case the space of holomorphic sections of $\mathcal{L}^{\otimes n}$ gives an embedding of X as a closed complex submanifold in a projective space for each $n \geq 3$; i.e., a holomorphic map $\Theta : X \to \mathbb{CP}^d$ that is injective and induces an injective map on tangent spaces.

Assuming the two results above, we see that if X is the analytification of an abelian variety over $\mathbb{C}$, then by properness we get an embedding $A \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$. Therefore analytification gives us $\iota : X \hookrightarrow \mathbb{CP}^n$. There is an ample line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$, which pulls back via ι to an ample line bundle $\iota^*\mathcal{O}(1)$ on X. Applying Appell–Humbert Theorem then provides the positive-definite Riemann form H that we seek.

Conversely, suppose we are given a positive-definite Riemann form H on X. We thus have an ample line bundle on X. Thus, we have an embedding $X \hookrightarrow \mathbb{CP}^n$. Chow's Theorem gives that $X = A^{\mathrm{an}}$ for a smooth A such that $A \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ a closed embedding. GAGA gives that the group laws on X are algebraizable and come from the group laws on A. Therefore, we obtain that A is an abelian variety. ☐

In the case of abelian schemes over a general base, we first will need a rigidity result.

Theorem 3.12 (Rigidity Lemma)**.** Let S be a scheme, and G be a group scheme over S and separated over S. Let $f : X \to S$ be a scheme morphism such that

1. f is flat.
2. either f is proper, or f is closed and admits a section.
3. For each $s \in S$, the $k(s)$-vector space $H^0(X_s, \mathcal{O}_{X_s})$ is 1-dimensional.

Then for any two S-morphisms $\phi, \phi' : X \to G$, if ϕ and ϕ' agree on one geometric fiber (or equivalently, on one fiber) for each connected component of S, then ϕ and ϕ' differ by multiplication by a section in $G(S)$.

Proof. This is [MFK94, Proposition 6.1]. ☐

Corollary 3.13. Let X and G over S be as in Theorem 3.12. Assume either that $X \to S$ is proper, or that it is universally closed and admits a section. Let Y be a connected scheme over S and assume that $Y \to S$ admits a section ϵ. Then for any S-scheme morphism $\varphi : X \times_S Y \to G$, there are S-scheme morphisms $g : X \to G$ and $h : Y \to G$ such that φ is given by $(x, y) \mapsto g(x) \cdot h(y)$.

Proof. Let $f : X \to S$ be the structure map. We consider the following commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[rr, shift left, "\Phi"] \arrow[rr, shift right, "\Phi'"'] \arrow[dr] & & G \times_S Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where we define

$$\Phi(x,y) := \big(\varphi(x,y),\, y\big), \qquad \Phi'(x,y) := \big(\varphi\big(x,\epsilon(f(x))\big),\, y\big).$$

The Y-scheme $X \times_S Y$ and the Y-group scheme $G \times_S Y$ satisfy the hypotheses of Theorem 3.12. Now Φ, Φ' are Y-morphisms and for any $y_0 \in \mathrm{im}(\epsilon)$, the morphisms Φ and Φ' agree on the fiber of $X \times_S Y$ over y_0. Hence by Theorem 3.12 we know that Φ and Φ' differ by multiplication by a section of $G \times_S Y \to Y$, which is of the form $y \mapsto (h(y), y)$ for some S-map $h : Y \to G$. Then we have $\varphi(x,y) \;=\; \varphi\big(x, \epsilon(f(x))\big) \cdot h(y)$. Setting $g(x) := \varphi\big(x, \epsilon(f(x))\big)$ we can conclude the proof. $\square$

Corollary 3.14. Suppose X/S is an abelian scheme and G/S is a separated group scheme. Then any S-map $\varphi : X \to G$ preserving the identity section is a group homomorphism. In particular, the group structure on X is determined by the identity section.

Remark 3.15. The corollary shows that the group structure on an abelian variety is uniquely determined by the choice of a zero element (as in the case of an elliptic curve).

Proof. We may assume that S is connected. Then X is connected, since $X \to S$ is closed, surjective, and has connected fibers. Consider the composition $\Phi : X \times_S X \xrightarrow{\mu} X \xrightarrow{\varphi} G$ where μ is the multiplication map, i.e., $\Phi(x,y) = \varphi(x \cdot y)$. Then by Corollary 3.13, we have $\varphi(x \cdot y) \;=\; g(x) \cdot h(y)$ for some $g : X \to G$ and $h : X \to G$. Now observe that $e \;=\; \varphi(e \cdot e) \;=\; g(e)\, h(e)$. This implies that $h(e) = g(e)^{-1}$. Then we have

$$\varphi(x) \;=\; \varphi(x \cdot e) \;=\; g(x)\, h(e) \;=\; g(x)\, g(e)^{-1}.$$

Also,

$$\varphi(x) \;=\; \varphi(e \cdot x) \;=\; g(e)\, h(x).$$

So we have $g(x) \;=\; \varphi(x)\, g(e), \qquad h(x) \;=\; g(e)^{-1}\, \varphi(x)$. Hence we have

$$\varphi(x \cdot y) \;=\; g(x)\, h(y) \;=\; \varphi(x)\, g(e)\, g(e)^{-1}\, \varphi(y) \;=\; \varphi(x)\, \varphi(y).$$

$\square$

Corollary 3.16. Suppose X/S is an abelian scheme. Then the group structure is commutative.

Proof. Apply the previous result to the inversion $X \to X$, $x \mapsto x^{-1}$. $\square$

Corollary 3.17. Every map $A \to B$ of abelian varieties is the composite of a homomorphism with a translation.

Proof. The map will send the k-rational point 0 of A to a k-rational point b of B. After composing with translation by $-b$, we get a map which preserves the identity. Applying the corollary above, we can conclude that this map is a homomorphism. $\square$

3.2 Polarization

3.2.1 Picard schemes

We first recall the following from algebraic geometry. Given a scheme X, we can consider the contravariant functor

$$h_X : (\mathrm{Sch})^{\mathrm{op}} \longrightarrow (\mathrm{Set}),$$
$$Y \mapsto \mathrm{Mor}(Y, X).$$

A morphism $Y \to Z$ gives a map of sets $\mathrm{Mor}(Z, X) \to \mathrm{Mor}(Y, X)$, and hence h_X is really a functor. This is called the functor of points.

Lemma 3.18 (Yoneda's Lemma)**.** Let X, X' be two schemes, and h_X and $h_{X'}$ the associated functors. If h_X and $h_{X'}$ are naturally isomorphic, then X and X' are canonically isomorphic.

Definition 3.19. A functor h is representable (in the category of schemes) if there is a scheme X and a natural isomorphism of functors $h \to h_X$.

For a scheme X, we define $\mathrm{Pic}(X)$ to be the abelian group of isomorphism classes of invertible $\mathcal{O}_X$-modules. A morphism $f : X \to Y$ of schemes gives a group homomorphism

$$f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X), \qquad \mathcal{L} \longmapsto f^*\mathcal{L}.$$

Thus, we may consider the *absolute Picard functor*

$$\mathrm{Pic} : (\mathrm{Sch})^{\mathrm{op}} \longrightarrow (\mathrm{Ab}), \qquad X \mapsto \mathrm{Pic}(X).$$

Suppose $f : X \to S$ is a scheme morphism. It will be easier to work with the *relative Picard functor*, defined as

$$\mathrm{Pic}_{X/S} : (S\text{-schemes})^{\mathrm{op}} \longrightarrow (\mathrm{Ab}), \qquad T \longmapsto \mathrm{Pic}(X_T)/f_T^*\mathrm{Pic}(T),$$

where $X_T := X \times_S T$ and $f_T : X_T \to T$ is the base change of f.

Theorem 3.20 (Grothendieck)**.** Suppose $X \to S$ is a flat projective morphism with all geometric fibers integral (irreducible and reduced). Also assume that X/S has a section. Then $\mathrm{Pic}_{X/S}$ is representable by a commutative group scheme over S which is locally of finite type and separated over S.

Remark 3.21. If $e \in X(S)$ is a section, then $\mathrm{Pic}_{X/S} \cong \mathrm{Pic}_{X/S,e}$, where $\mathrm{Pic}_{X/S,e}$ is the *rigidified Picard functor* sending each S-scheme T to the group of isomorphism classes of pairs $(\mathcal{L}, \rho)$, where $\mathcal{L}$ is a line bundle on X_T and ρ is an isomorphism $e_T^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_T$ (called a *rigidification* of $\mathcal{L}$ along e_T). Here, e_T denotes the section of $X_T \to T$ induced by the section e of $X \to S$. More explicitly, we have inverse bijections $\mathrm{Pic}(X_T)/f_T^*\,\mathrm{Pic}(T) \longleftrightarrow \mathrm{Pic}_{X/S,e}(T)$

$$\mathcal{L} \longleftarrow (\mathcal{L}, \rho)$$
$$\mathcal{L} \longmapsto \big(\mathcal{L} \otimes f_T^* e_T^* \mathcal{L}^{-1},\ \text{canonical } \rho\big).$$

Here the canonical ρ is defined by noting that

$$e_T^*\big(\mathcal{L} \otimes f_T^* e_T^* \mathcal{L}^{-1}\big) \cong e_T^*\mathcal{L} \otimes e_T^*\mathcal{L}^{-1} \cong \mathcal{O}_T.$$

Definition 3.22. Recall that a morphism $f : X \to S$ is called *projective*, if the S-scheme X is isomorphic to a closed subscheme of the projective bundle $\mathbb{P}(\mathcal{E})$ over S attached to some coherent $\mathcal{O}_S$-module $\mathcal{E}$ on S.

Remark 3.23. Note that in general this is not the same as requiring that X is isomorphic to a closed subscheme of $\mathbb{P}^n_S$ for some n. However, when S admits an ample invertible sheaf (e.g. when S is affine), the two definitions are the same; see for instance [Sta25, Tag 0B45].

If $f : X \to S$ is projective, then it is proper, and there is an open covering (U_i) of S such that $X|_{U_i}$ is U_i-isomorphic to a closed subscheme of $\mathbb{P}^{n_i}_{U_i}$ for each i (see [Sta25, Tag 01WB]). The converse is not true. Thus a locally projective morphism (i.e., one that becomes projective after passing to an open covering of the target) need not be projective.

Theorem 3.24 (Grothendieck)**.** Let $f : X \to S$ be a flat projective morphism whose geometric fibers are integral. Assume that f admits a section, and that f is smooth. Assume that S is noetherian. Then there is a closed and open subgroup scheme $\mathrm{Pic}^\tau_{X/S}$ of $\mathrm{Pic}_{X/S}$ (over S), called the *torsion component*, satisfying the following conditions:

1. For each $s \in S$, the fiber of $\mathrm{Pic}^\tau_{X/S}$ over s consists of the torsion connected components of $(\mathrm{Pic}_{X/S})_s$. Here we say that a connected component is torsion if its image under the multiplication-by-n map $[n] : (\mathrm{Pic}_{X/S}) \to (\mathrm{Pic}_{X/S})$ lies in the identity connected component for some $n \geq 1$.

2. $\mathrm{Pic}^\tau_{X/S}$ is projective over S.

Remark 3.25. Another way $\mathrm{Pic}^\tau_{X/S}$ is usually defined when S is an algebraically closed field, is as the component containing line bundles which are algebraically equivalent to the trivial line bundle.

Definition 3.26. Let M, M' be two line bundles on A. We say that M is algebraically equivalent to 0, if there exists a connected k-scheme T and a line bundle on $A \times_k T$ specializing to M and $\mathcal{O}_A$ at two k-points of T. We way M, M' are algebraically equivalent if $M^{-1} \otimes M'$ is algebraically equivalent to 0.

3.2.2 Dual abelian schemes

Now if X/S is an abelian scheme, then all the assumptions in Theorem 3.20 are satisfied. If we assume that X/S is projective and that S is noetherian, then the assumptions in Theorem 3.24 are satisfied as well. Furthermore we have the following result:

Theorem 3.27. Let X/S be a projective abelian scheme, and assume that S is noetherian. Then $\mathrm{Pic}^\tau_{X/S}$ is smooth and has connected geometric fibers. Hence in view of Theorem 3.24 we know that $\mathrm{Pic}^\tau_{X/S}$ is a projective abelian scheme.

Remark 3.28. To show for the above theorem that $\mathrm{Pic}^\tau_{X/S}$ has connected geometric fibers, we reduce to the case where S is the spectrum of an algebraically closed field (since the formation of $\mathrm{Pic}^\tau_{X/S}$ commutes with base change). Then this is a fundamental result in the theory of abelian varieties over a field; see [Mum08, §13]. The proof that $\mathrm{Pic}^\tau_{X/S}$ is smooth is found in [MFK94, Prop. 6.7].

Definition 3.29. In the setting of Theorem 3.27, we call $\mathrm{Pic}^{\tau}_{X/S}$ the *dual abelian scheme* of X, and denote it by $X^{\vee}$.

Remark 3.30. One can ask about the case of a general abelian scheme, not just one which is projective. In this case, the dual abelian scheme also exists and is smooth and geometrically connected. This is an unpublished result of Raynaud, as detailed in [FC90, Theorem 1.9].

Definition 3.31. Let A, B be two abelian schemes over an arbitrary (locally noetherian) S. By an *isogeny*, we mean an S-group scheme homomorphism $A \to B$ that is surjective and quasi-finite.

Lemma 3.32. Any isogeny $\phi : A \to B$ is finite and flat.

Proof. Since both A and B are proper over S, we know that ϕ is proper. But a proper and quasi-finite map is finite ([Sta25, Tag 02LS]), so ϕ is finite.

Since both A and B are flat and finite-type over S, we can use the fiberwise criterion for flatness to reduce to the case where S is the spectrum of a field k. We may also assume that k is algebraically closed since flatness satisfies fpqc descent. Now ϕ is a surjective map between two finite-type schemes over a field, and the target is integral. Hence we can make use of *generic flatness* : There exists a non-empty open subscheme $U \subset B$ over which ϕ is flat.

Since ϕ is a group homomorphism, we can use the group structure on B to translate U, in order to obtain an open covering of B such that ϕ is flat over each member of the covering. (For this step we need to use that k is algebraic closed.) It follows that ϕ is flat, as desired. ◻

3.2.3 The Mumford Λ-construction

Let S be a noetherian scheme, and $f : A \to S$ an abelian scheme over S. For any line bundle L on A, we define the Mumford line bundle $\mathcal{M}(L)$ on $A \times_S A$ by

$$\mathcal{M}(L) := \mu^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1},$$

where p_1, p_2 are the two projections $A \times_S A \to A$, and μ is the group law $A \times_S A \to A$.

Recall that for any S-scheme T to give an S-map $T \to \mathrm{Pic}_{A/S}$ is the same as to specify an element of $\mathrm{Pic}(A_T)/f_T^*\mathrm{Pic}(T)$, where $A_T = A \times_S T$. Thus for $T = A$, the Mumford line bundle $\mathcal{M}(L)$ on $A \times_S A = A_A$ gives rise to an S-map $\Lambda(L) : A \longrightarrow \mathrm{Pic}_{A/S}$.

Lemma 3.33. The S-map $\Lambda(L)$ takes the identity section of A to the identity section of $\mathrm{Pic}_{A/S}$.

Proof. Let $e \in A(S)$ be the identity section. To compute $\Lambda(L) \circ e$, we need to compute the pullback of $\mathcal{M}(L)$ under

$$A = A \times_S S \xrightarrow{(\mathrm{id},e)} A \times_S A, \qquad x \longmapsto (x, e(f(x))).$$

Note that the compositions of the above map followed by $\mu, p_1, p_2 : A \times_S A \to A$ respectively are $\mathrm{id}, \mathrm{id}, e \circ f$. Hence the pullback of $\mathcal{M}(L)$ under the above map is isomorphic to $f^* e^* L^{-1}$. This line bundle on A represents the zero element of $\mathrm{Pic}_{A/S}(S) = \mathrm{Pic}(A)/f^*\mathrm{Pic}(S)$. Hence $\Lambda(L) \circ e$ is the identity section of $\mathrm{Pic}_{A/S}$. ◻

As a consequence, we know that $\Lambda(L) : A \to \mathrm{Pic}_{A/S}$ is a group homomorphism by the Rigidity Lemma (see Corollary 3.14). Moreover, by the fiberwise connectedness of A, we know that the image of $\Lambda(L)$ lies in $A^\vee$, so we get a homomorphism $A \to A^\vee$. Observe also that for two line bundles L, M on A we have

$$\Lambda(L \otimes M^{\pm 1}) = \Lambda(L) \pm \Lambda(M).$$

We now assume that A is an abelian variety over an algebraically closed field k. In this case A is automatically projective. Note that $\mathrm{Pic}_{A/k}(k) = \mathrm{Pic}(A)$, since $\mathrm{Pic}(k)$ is trivial.

Let L be a line bundle on A. Similar to the proof above, we see that at the level of k-points the map $\Lambda(L)$ is given by

$$A(k) \longrightarrow A^\vee \subset \mathrm{Pic}_{A/k}(k) = \mathrm{Pic}(A), \qquad x \longmapsto t_x^* L \otimes L^{-1},$$

where $t_x : A \to A$ is translation by x. The fact that $\Lambda(L)$ is a group homomorphism thus implies the following:

Theorem 3.34 (Theorem of the Square)**.** For any line bundle L on A and any $x, y \in A(k)$, we have an isomorphism of line bundles

$$t_{x+y}^* L \otimes L \;\cong\; t_x^* L \otimes t_y^* L.$$

We make two further observations.

Lemma 3.35. Let L be a line bundle on A. Then $\Lambda(L) = 0$ if and only if the Mumford line bundle $\mathcal{M}(L)$ is trivial.

Proof. The "if" direction is clear from definition. Suppose now that $\Lambda(L) = 0$. Then we see from definition of $\Lambda(L)$ that since the base change of f along $A \to \operatorname{Spec} k$ is p_2, this implies $\mathcal{M}(L)$ lies in $p_2^*\mathrm{Pic}(A)$. Thus suppose M is a line bundle on $A \times_k T$ which isomorphic to $p_2^*(N)$ for some line bundle N on T. Write e_T for the map $T \to A \times_k T$, $t \mapsto (e, t)$. Then $p_2^* e_T^* M \cong p_2^* e_T^* p_2^* N \cong p_2^* N \cong M$, where the second isomorphism is because $p_2 \circ e_T = \mathrm{id}_T$. In particular, $M \otimes p_2^* e_T^* M^{-1} \cong \mathcal{O}_{A \times T}$. Applying this to $M = \mathcal{M}(L)$, we know that $\mathcal{M}(L) \otimes p_2^*(e, \mathrm{id})^* \mathcal{M}(L)^{-1} \cong \mathcal{O}_{A \times A}$. One can check that the left hand side is isomorphic to $\mathcal{M}(L) \otimes e_2^* L$, where e_2 is the map $A \times_k A \to A, (x, y) \mapsto e$. But $e_2^* L$ is trivial since e_2 factors through $\operatorname{Spec} k$ and $\mathrm{Pic}(k) = 0$. Hence $\mathcal{M}(L)$ is trivial. $\square$

Since A/k is projective, it has ample line bundles.

Lemma 3.36. Let L be an ample line bundle on A. Then $\ker(\Lambda(L))$ is a finite subgroup scheme of A.

Proof. Suppose not. Then one can find a positive-dimensional abelian subvariety $B \subset A$ contained in $\ker(\Lambda(L))$. Note that $\Lambda(L)|_B = \Lambda(L|_B)$, and $L|_B$ is ample on B. Thus by considering $B = A$, we are in the case where $\Lambda(L) = 0$. By the previous lemma, $\mathcal{M}(L)$ is trivial. The pullback of $\mathcal{M}(L)^{-1}$ along the "anti-diagonal"

$$(\mathrm{id}, [-1]) : A \longrightarrow A \times_k A, \qquad x \longmapsto (x, -x)$$

is $L \otimes [-1]^*L$, and it must be trivial on A. Since L is ample and $[-1]$ is an automorphism of A, $L \otimes [-1]^*L$ is ample. Thus the trivial line bundle on A is ample, a contradiction with the fact that A is projective and positive-dimensional. $\square$

The following is the "main theorem" for line bundles on an abelian variety.

Theorem 3.37. Fix an ample line bundle L on A. For any line bundle M on A, we have $\Lambda(M) = 0$ if and only if $M \cong \Lambda(L)(x) = t_x^*L \otimes L^{-1}$ for some $x \in A(k)$.

Proof. (Sketch) This is explained in [Mum08, §8]. Suppose $M = t_x^*L \otimes L^{-1}$. Then for any $y \in A(k)$ we compute

$$\Lambda(M)(y) = t_y^*M \otimes M^{-1} \cong t_{x+y}^*L \otimes t_y^*L^{-1} \otimes t_x^*L^{-1} \otimes L,$$

which is trivial by the Theorem of Square 3.34. Thus $\Lambda(M) = 0$. Conversely, if $\Lambda(M) = 0$, then we consider the line bundle on $A \times_k A$ given by

$$K = \mu^*L \otimes p_1^*L^{-1} \otimes p_2^*\left(L^{-1} \otimes M^{-1}\right).$$

If $\Lambda(M) = 0$, M is non-trivial, and M is not isomorphic to $t_x^*L \otimes L^{-1}$ for some x, then we can look at the cohomology of $A \times_k A$ valued in K, and it turns out that using the Leray spectral sequence all these cohomology groups vanish in every degree. However, restricting to $A \times \{e\}$, we see that

$$K|_{A \times \{e\}} \simeq \mathcal{O}_A,$$

but the space of global sections of A does not vanish (it is exactly k). $\square$

Lemma 3.38. Two line bundles M and M' are algebraically equivalent if and only if the isomorphism class of $M \otimes M'^{-1}$ lies in $A^\vee(k) \subset \mathrm{Pic}_{A/k}(k) = \mathrm{Pic}(A)$.

Corollary 3.39. Let M be a line bundle on A. Then M is algebraically equivalent to zero if and only if $\Lambda(M) = 0$.

Proof. Suppose M is algebraically equivalent to zero. Then there is a connected k-scheme T and a line bundle $\widetilde{M}$ on $A \times_k T$ which specializes to M and to $\mathcal{O}_A$ at two points $t_1, t_2 \in T(k)$. Consider the T-group scheme homomorphism $\Lambda(\widetilde{M}) : A_T = A \times_k T \longrightarrow \mathrm{Pic}_{A_T/T}$. On the fiber of A_T over t_2, the map induced by $\Lambda(\widetilde{M})$ is $\Lambda(\mathcal{O}_A) = 0$. Thus $\Lambda(\widetilde{M})$ and the zero homomorphism agree on one fiber over T. Since they both preserve the identity section, they must be equal by the Rigidity Lemma . Thus $\Lambda(\widetilde{M}) = 0$. But on the fiber over t_1, the map induced by $\Lambda(\widetilde{M})$ is $\Lambda(M)$. Hence $\Lambda(M) = 0$.

Conversely, suppose that $\Lambda(M) = 0$. Then by Theorem 3.37, there exists an (ample) line bundle L on A and a point $x \in A(k)$ such that $M = \Lambda(L)(x)$. But $\Lambda(L)(A(k)) \subset A^\vee(k)$, so $M \in A^\vee(k)$. Thus M is algebraically equivalent to zero by Lemma 3.38. $\square$

Corollary 3.40. Let L be an ample line bundle on A. Then $\Lambda(L) : A \to A^\vee$ is an isogeny.

Proof. By Lemma 3.36, $\Lambda(L)$ is quasi-finite. To see that it is surjective, let $M \in A^\vee(k)$. Then $\Lambda(M) = 0$ by Corollary 3.39. Hence $M \in \mathrm{im}(\Lambda(L))$ by Theorem 3.37. $\square$

In particular, since $\Lambda(L)$ is finite flat, we can consider the degree $\deg \Lambda(L)$, which we shall see is always a square

4 Moduli spaces of abelian schemes

4.1 The Poincaré line bundle

Definition 4.1. If A/S is an abelian scheme, by a *polarization* on A, we mean an S-group homomorphism $\lambda : A \longrightarrow A^\vee$ such that for each geometric point $x : \operatorname{Spec} \bar{k} \to S$, the induced homomorphism $\lambda_x : A_x \longrightarrow (A^\vee)_x \cong (A_x)^\vee$ (from the abelian variety A_x over $\bar{k}$ to its dual) is of the form $\Lambda(L_x)$ for some ample line bundle L_x on A_x. Note that a polarization is necessarily an isogeny, since each $\lambda_x = \Lambda(L_x)$ is surjective and quasi-finite by the ampleness of L_x.

Definition 4.2. If $\Lambda(L_x)$ is of degree 1 at any point x, we say λ is principally polarized. Observe that this implies that $\Lambda(L_x)$ is an isomorphism.

Example 4.3. When A is an elliptic curve, we can consider the line bundle $\mathcal{O}([e])$, corresponding to the divisor of the identity $[e]$. Then $\mathcal{O}([e])$ is ample, and it turns out that the polarization so obtained if of degree 1.

We now want to make sense of this over S, rather than just fiberwise. Recall that $\operatorname{Pic}_{A/S}$ represents the rigidified Picard functor

$$\operatorname{Pic}_{A/S,e} : (\text{locally noetherian } S\text{-schemes}) \longrightarrow (\text{Abelian groups})$$

given by

$$T \longmapsto \{(L, \rho) \mid L \text{ a line bundle on } A_T = A \times_S T,\ \rho : e_T^* L \xrightarrow{\sim} \mathcal{O}_T\}/\cong .$$

Over $A_{\operatorname{Pic}_{A/S}} := A \times_S \operatorname{Pic}_{A/S}$ we have a universal pair (L, ρ) corresponding to the image on $1 \in \operatorname{Hom}(Pic_{A/S}, Pic_{A/S})$, where L is a line bundle on $A \operatorname{Pic}_{A/S}$ and $\rho : (e, \mathrm{id})^* L \xrightarrow{\sim} \mathcal{O}_{\operatorname{Pic}_{A/S}}$. This pair is unique up to isomorphism. We may restrict (L, ρ) to $A \times_S A^\vee$. In particular, we acquire the so-called Poincaré line bundle $\mathcal{P}$ on $A \times_S A^\vee$, which comes equipped with a trivialization along (e, id).

Remark 4.4.

1. Let $e^\vee$ be the identity section of $A^\vee \to S$. Then $(\mathrm{id}, e^\vee)^* \mathcal{P}$ on A is also equipped with a trivialization and this trivialization is compatible with the previous trivialization in the sense that these two induce the same isomorphism $(e, e^\vee)^* \mathcal{P} \xrightarrow{\sim} \mathcal{O}_S$.

2. We have the "flipping" identification $f : A^\vee \times A \xrightarrow{\sim} A \times A^\vee, \qquad (x, y) \longmapsto (y, x)$. Notice that $f^* \mathcal{P}$ is a line bundle on $A^\vee \times A$ again equipped with two compatible trivializations along $(e^\vee, \mathrm{id})$ and (id, e). Observe that this structure can be used to obtain a canonical map $A \to A^{\vee\vee}$, which turns out to be an isomorphism.

Suppose $\lambda : A \to A^\vee$ is an S-homomorphism. Consider the map $(\mathrm{id}_A, \lambda) : A \longrightarrow A \times_S A^\vee$. We set $L^\Delta(\lambda) := (\mathrm{id}_A, \lambda)^* \mathcal{P}$ where $\mathcal{P}$ is the Poincaré line bundle. Note that $L^\Delta(\lambda)$ is equipped with a canonical trivialization along e, i.e., an isomorphism $p_{\mathrm{can}} : e^* L^\Delta(\lambda) \xrightarrow{\sim} \mathcal{O}_S$.

Proposition 4.5. Let λ be a polarization on A. Then $\Lambda\big(L^\Delta(\lambda)\big) = 2\lambda$.

Proof. Since the construction $\lambda \mapsto \Lambda\big(L^{\Delta}(\lambda)\big)$ commutes with base change, by the Rigidity Lemma we reduce to the case where S is the spectrum of an algebraically closed field k. Since λ is a polarization and $S = \operatorname{Spec} k$, we have $\lambda = \Lambda(L)$ for some line bundle L on A, and we know that λ is an isogeny. We need to show that $\Lambda\big(L^{\Delta}(\lambda)\big) = 2\Lambda(L)$. Since the right hand side is $\Lambda(L^2)$, it suffices to show that $L^{\Delta}(\lambda)$ is algebraically equivalent to L^2 in view of Corollary 3.39. Since $\lambda = \Lambda(L)$, unraveling the definition one sees that $L^{\Delta}(\lambda)$ is the pullback of the Mumford line bundle $\mathcal{M}(L)$ on $A \times_k A$ along $\Delta : A \to A \times_k A$. This can be explicitly computed to be $[2]^*L \otimes L^{-2}$. Hence it suffices to note that $[2]^*L$ is algebraically equivalent to L^4, which follows from the Theorem of the Square, see [Mum08, §8]. ☐

Thus, we get a way to produce a sheaf from a polarization, which at least geometrically fiberwise is related to the ample line bundle L we want. We want to say that this line bundle determines the polarization.

Definition 4.6. We say that an isogeny $\lambda : A \to A^{\vee}$ is *symmetric* if it satisfies $\Lambda\big(L^{\Delta}(\lambda)\big) = 2\lambda$. Thus the previous proposition says that polarizations are special examples of symmetric isogenies. (Recall that a polarization is automatically an isogeny.)

Proposition 4.7. Let $\lambda : A \to A^{\vee}$ be a homomorphism. Then the following are equivalent. (In the literature one often finds (2) as the definition of a polarization.)

1. λ is a polarization.
2. λ is a symmetric isogeny, and $L^{\Delta}(\lambda)$ is relatively ample with respect to $A \to S$.

Proof. (Sketch) Everything can be checked geometrically fiberwise, so we assume that $S = \operatorname{Spec} k$ for k an algebraically closed field.

(1) $\Rightarrow$ (2) : We have that $\Lambda(L^{\Delta}(\lambda)) = \Lambda(L^2)$, and L^2 is ample. Observe that $L^{\Delta}(\lambda) \otimes L^{-2}$ is geometrically equivalent to 0, so there is some $x \in A(k)$ such that

$$L^{\Delta}(\lambda) \otimes L^{-2} \simeq \Lambda(L^2)(x) = t_x^* L^2 \otimes L^{-2}.$$

Then $L^{\Delta}(\lambda) \simeq t_x^* L^2$ is ample.

(2) $\Rightarrow$ (1): We use [Mum08, §23 Theorem 3] to find a line bundle L such that $L^2 \simeq L^{\Delta}(\lambda)$, then L is ample and $\lambda = \Lambda(L)$. ☐

The line bundle $L^{\Delta}(\lambda)$ also defines the polarization λ in the following sense:

Theorem 4.8 ([MFK94, Proposition 6.11])**.** Let $\pi \colon X \to S$ be a projective abelian scheme, and let L be an invertible sheaf on X, ample over S, such that $e^*(L) \cong \mathcal{O}_S$, where e is the identity section. If $f \colon T \to S$ is any base extension, let $X_T = X \times_S T$ and let $L_T = p_1^*(L)$ be the induced sheaf on X_T. Then for any k, there is at most one homomorphism $\lambda \colon X_T \to \widehat{X}_T$ such that $L_T = L^{\Delta}(k\lambda)$. Moreover, there is a closed subscheme $S_0 \subset S$ such that one such λ exists if and only if f factors through S_0.

We now introduce an equivalent way to understand polarizations of abelian varieties, which we shall see recovers the definition of polarization over $\mathbb{C}$.

Definition 4.9. Let X and Y be abelian varieties. A *divisorial correspondence* between X and Y is a line bundle Q on $X \times Y$ whose restrictions to $\{0\} \times Y$ and $X \times \{0\}$ are trivial.

Observe that $\mathcal{P}$ defines a divisorial correspondence between $X, X^\vee$.

Proposition 4.10. A polarization on an abelian variety A is a divisoral correspondence Q on $A \times A$ which is invariant under the involution on $A \times A$ such that the line bundle Δ^*Q on A is ample.

Proof. (Sketch) We check that Q defines a morphism $\phi : A \to A^\vee$, which satisfies $\Delta^*Q \simeq (\mathrm{id}, \phi)^*\mathcal{P}_A$ is ample. From [Mum08, p.60] we hence see that ϕ is an isogeny, hence this defines a polarization. $\square$

Proposition 4.11. Let X and Y be two abelian varieties of the same dimension, and Q a divisorial correspondence between X and Y. The following are equivalent:

1. If $Q|_{\{x\}\times Y}$ is trivial, then $x = 0$.
2. If $Q|_{X\times\{y\}}$ is trivial, then $y = 0$.

If these hold, then $X \simeq \widehat{Y}$ with Q isomorphic to the Poincaré bundle $\mathcal{P}_Y$ of Y, and $Y \simeq \widehat{X}$ with Q isomorphic to the Poincaré bundle $\mathcal{P}_X$ of X.

Proof. By symmetry, it suffices to deduce (2) from (1). If (1) holds, then by considering the map $x \mapsto Q|_{\{x\}\times Y}$ there is an injective morphism $\phi\colon X \to \widehat{Y}$ such that $Q \simeq (\phi \times \mathrm{id}_Y)^*\mathcal{P}_Y$, where $\mathcal{P}_Y$ is the Poincaré bundle on $\widehat{Y} \times Y$. Since $\dim X = \dim \widehat{Y}$ and ϕ is injective, ϕ is also surjective; since the characteristic is 0, this implies that ϕ is an isomorphism, i.e. $X \simeq \widehat{Y}$.

Now, let $\psi = \hat{\phi}\colon Y \to \widehat{X}$, so that if $\mathcal{P}_X$ is the Poincaré bundle on $X \times \widehat{X}$, $(1_X \times \psi)^*\mathcal{P}_X = Q$. To prove (2), we have to show that ψ is injective. If not, we can find a finite subgroup $K \subset \ker \psi$, $K \neq 0$, and ψ factorizes as

$$Y \xrightarrow{\eta} Y/K \xrightarrow{\tilde{\eta}} \widehat{X}$$

where η is the natural homomorphism. Thus, if L is the line bundle $(1_X \times \eta)^*\mathcal{P}_X$ on $X \times Y/K$, we have that $Q \simeq (1_X \times \tilde{\eta})^*(L)$. Now, L induces a homomorphism $\alpha\colon X \to \widehat{(Y/K)}$, and the isomorphism $Q \simeq (1_X \times \tilde{\eta})^*(L)$ means precisely that the composite $X \xrightarrow{\alpha} \widehat{(Y/K)} \xrightarrow{\widehat{\eta}} \widehat{Y}$ is the homomorphism defined by Q. Thus this composite is an isomorphism. Thus α is injective, and since $\dim X = \dim \widehat{(Y/K)}$, α and $\widehat{\eta}$ are both isomorphisms. But $\widehat{\eta}$ has a non-trivial kernel, (the dual abelian group of K). This is a contradiction, proving that ψ is injective. $\square$

We now return to the case of $k = \mathbb{C}$. Observe that by the Theorem of Appell-Humbert, any divisoral correspondence can be given by $\mathcal{L}(H, \alpha)$, for some H Hermitian form on $X \times Y$. More precisely, we have the following:

Let $X_i = V_i/U_i$, $i = 1, 2$, be two abelian varieties $\mathbb{C}$. Let $Q = L(H, \alpha)$ on $X_1 \times X_2$. The correspondence Q is divisorial if Q is trivial on $\{0\} \times X_2$ and on $X_1 \times \{0\}$. This means:

1. $H \equiv 0$ on $\{0\} \times V_2$ and on $V_1 \times \{0\}$,
2. $\alpha \equiv 1$ on $\{0\} \times U_2$ and on $U_1 \times \{0\}$.

Define
$$B(x_1, x_2) = H\big((x_1, 0), (0, x_2)\big).$$
Then B is an $\mathbb{R}$-bilinear form on $V_1 \times V_2$, complex-linear on V_1 and anti-linear on V_2. Let $\operatorname{Im}(B) = \beta$. Then β is integral on $U_1 \times U_2$, and we get
$$H\big((x_1, x_2), (y_1, y_2)\big) = B(x_1, y_2) + \overline{B(y_1, x_2)},$$
$$\alpha(u_1, u_2) = \alpha(u_1, 0)\, \alpha(0, u_2)\, e^{\pi i\, \beta(u_1, u_2)}.$$
Thus the divisorial correspondence Q is determined entirely by B.

Corollary 4.12. Via Q, $X_1 \simeq \widehat{X}_2$ and $X_2 \simeq \widehat{X}_1$ if and only if

1. B is non-degenerate;
2. under β, U_1 and U_2 are dual lattices, i.e.,
$$U_2 = \{\, x_2 \in V_2 \mid \beta(u_1, x_2) \in \mathbb{Z}\ \forall\, u_1 \in U_1 \,\}.$$
and vice versa.

Hence, if $X = V/U$, then the dual abelian variety $\widehat{X}$ can be explicitly constructed as follows. Let
$$\bar{V} = \operatorname{Hom}_{\text{conj. linear}}(V, \mathbb{C}),$$
$$U^\vee = \{\, \ell \in \bar{V} \mid \operatorname{Im} \ell(u) \in \mathbb{Z},\ \text{for all } u \in U \,\}.$$
Then
$$X^\vee \simeq \bar{V}/U^\vee.$$
The form $B\colon V \times \bar{V} \to \mathbb{C}$ is simply $B(x, \ell) = \overline{\ell(x)}$, and the Poincaré bundle $\mathcal{P}_X$ on $X \times \widehat{X}$ is simply $L(H, \alpha)$ with
$$H\big((x_1, \ell_1), (x_2, \ell_2)\big) = \overline{\ell_2(x_1)} + \ell_1(x_2),$$
$$\alpha\big((u, \ell)\big) = e^{-\pi i \operatorname{Im} \ell(u)}.$$

Proposition 4.13. A divisorial correspondence Q on $X \times X$ is a polarization exactly when the associated $\mathbb{R}$-bilinear form B on $V \times V$ satisfies $\beta(iv, iw) = \beta(v, w)$ and B is positive definite (so this B is exactly the H in the previous section).

4.2 Mumford's GIT Construction

We assume the following result:

Theorem 4.14 ([MFK94, Prop. 6.13])**.** Let $\pi : X \to S$ be a projective abelian scheme. Let L be a line bundle on X, relatively ample for π. Then the following statements hold:

1. $R^i\pi_*L = 0 \quad \text{for all } i > 0.$

2. π_*L is a vector bundle on S, and its formation commutes with arbitrary base change. Let r be its rank.

3. The S-homomorphism
$$\Lambda(L) : A \to A^\vee$$
is an isogeny, and its degree is r^2.

4. L^3 is relatively very ample, i.e., the canonical S-map
$$A \to \mathbb{P}_S\big(\pi_*L^3\big)$$
is a closed immersion.

Remark 4.15.
1. (1) and (2) can be checked fiber by fiber by standard cohomology and base change results, so it suffices to see that for an abelian variety $H^i(A_s, L|_{A_s})$ vanishes for $i \geq 1$, and r is equal to the dimension of $H^0(A_s, L|_{A_s})$. The former follows from vanishing of cohomology for ample line bundles, while the latter follows from the next remark:

2. (3) follows from a "Riemann–Roch" Theorem [Mum08, §16]: For L an ample line bundle on an abelian variety A over an algebraically closed field k, we have that $\mathrm{H}^0(A, L)$ has rank equal to $\sqrt{\deg \Lambda(L)}$ and $\mathrm{H}^i(A, L) = 0$ for $i > 0$.

3. (4) This can also be shown fiberwise, where the statement is shown in [Mum08, §17].

We need one last theorem of Grothendieck about deforming abelian schemes:

Theorem 4.16. Suppose S is a connected, locally noetherian scheme, $\pi : A \to S$ is a proper and smooth morphism, and $e : S \to A$ is a section of π. Suppose further that there is a geometric point $s : \operatorname{Spec} k \to S$ of S such that $\pi_s : A_s \to \operatorname{Spec} k$ is an abelian variety with zero section e_s (the base change of e). Then $\pi : A \to S$ is an abelian scheme with zero section e.

Remark 4.17. The proof is in [MFK94, §6.3]. Let us briefly explain the steps of the proof. In the following, we simply say that (π, e) is an abelian scheme when we mean that $\pi : A \to S$ is an abelian scheme with zero section e. Recall from the Rigidity Lemma that the group structure on an abelian scheme is uniquely determined by the zero section. Hence whether (π_s, e_s) or (π, e) is an abelian scheme is a question of the *existence* of a group scheme structure.

Now, we can use obstruction theory to first extend the group scheme structure over nilpotent thickenings: i.e. where R is a local Artin ring with maximal ideal $\mathfrak{m}$ and residue field k. Consider a quotient ring $R_0 = R/I$ where I satisfies $\mathfrak{m}I = 0$ (which in particular implies that $I^2 = 0$). The idea is that we can consider the difference map

$$\mu_0 : A_0 \times_{S_0} A_0 \to A_0, \quad (x, y) \mapsto x - y.$$

The goal is to deform μ_0 to a map $\mu : A \times_S A \to A$ satisfying the group axioms (that is, we define the group law $+ : A \times_S A \to A$ using the zero section e and the difference map μ by $x+y := x-(0-y)$, and require that $+$ satisfies the group axioms). The possibility of lifting μ_0 to μ (without worrying about the group axioms) is controlled by the vanishing of an element of a certain H^1 cohomology group, and if this obstruction vanishes then the space of all possible lifts is a torsor over some H^0 cohomology group.

Granted the above infinitesimal lifting result, we can show that the space of possible $s \in S$ for which an abelian scheme structure exists is open, and a result of Koizumi also implies that it is closed. Thus, since S is connected, we may conclude.

4.2.1 The moduli problem $\mathcal{A}_{g,d,N}$

We fix integers $g \geq 1$, $d \geq 1$, $N \geq 3$.

Definition 4.18. Let $\mathcal{A}_{g,d,N}$ be the functor from the category of noetherian schemes over $\mathbb{Z}[1/N]$ to the category of sets, sending a scheme S to the set of isomorphism classes of triples (A, λ, γ), where

- $\pi : A \to S$ is an abelian scheme of relative dimension g;
- $\lambda : A \to A^\vee$ is a polarization on A whose degree (as an isogeny) is d^2;
- γ is a level-N structure on A, namely an isomorphism of S-group schemes

$$\gamma : (\mathbb{Z}/N\mathbb{Z})^{2g}_S \xrightarrow{\sim} A[N].$$

On morphisms, the functor $\mathcal{A}_{g,d,N}$ is defined using the obvious notion of pullback.

Theorem 4.19. The functor $\mathcal{A}_{g,d,N}$ is representable.

Remark 4.20.
1. For $\pi : A \to S$ as above, $A[N]$ is a finite étale group scheme over S, and étale locally on S it is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^{2g}_S$. From this, similar to the modular curve case, it is easy to show that the functor from S-schemes to sets sending T to the set of level-N structures on $A \times_S T$ is representable by an S-scheme, and that this S-scheme is a finite étale $\mathrm{GL}_{2g}(\mathbb{Z}/N\mathbb{Z})$-torsor on S. This implies that $\mathcal{A}_{g,d,N}$ is relatively representable over $\mathcal{A}_{g,d,1}$.
2. For $N \geq 3$, if A is an abelian scheme over S and γ is a level-N structure on A, then there are no non-trivial S-automorphism of A preserving γ.
3. In view of (1) and (2), to prove the theorem for general $N \geq 3$ we only need to prove it for all sufficiently large N. Then, we may construct the scheme for the remaining $N' \leq N$ as a quotient by the group $\ker(\mathrm{GL}_{2g}(\mathbb{Z}/N\mathbb{Z}) \to \mathrm{GL}_{2g}(\mathbb{Z}/N'\mathbb{Z}))$.

4. The Siegel modular variety is a subfunctor of $\mathcal{A}_{g,d,N}$ where a certain compatibility condition between λ and γ is imposed. Once we know the representability of $\mathcal{A}_{g,d,N}$, we immediately deduce that the Siegel modular variety is represented by a union of some connected components of $\mathcal{A}_{g,d,N}$.

We first give a rough explanation of how to tackle the problem. Consider the easiest case of $g = 1$. Any elliptic curve over $\mathbb{C}$ is given as a double cover of $\mathbb{P}^1$ branched at 4 distinct points. Let

$$U \;=\; \big\{\{p_1, \ldots, p_4\} \subset \mathbb{P}^1 \;\big|\; p_i \neq p_j\big\}$$

be the set of distinct unordered 4-tuples. Observe that $\mathrm{PGL}_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by linear fractional transformations, and that $A_1 \;=\; U/\mathrm{PGL}_2(\mathbb{C})$, provided we understand how to make this into a variety. Mumford showed more generally how this can be done, which we do explicitly here. The set U can be identified with the subset of the projective space $\mathbb{P}^4$ of homogeneous quartic polynomials in x, y. We have $\mathrm{PGL}_2(\mathbb{C}) \;=\; \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$, and it is more convenient to work with $\mathrm{SL}_2(\mathbb{C})$. This acts on $\mathbb{P}^4$ by the substitutions $a_0x^4 + a_1x^3y + \cdots \;=\; f(x, y) \;\longmapsto\; f(ax + by,\; cx + dy)$.

We can first try to define $\mathbb{P}^4/\mathrm{SL}_2(\mathbb{C})$, and then pass to A_1. It is natural to try to identify $\mathbb{P}^4/\mathrm{SL}_2(\mathbb{C})$ with Proj of the graded ring of invariants

$$R \;=\; \mathbb{C}[a_0, \ldots, a_4]^{\mathrm{SL}_2(\mathbb{C})}.$$

The ring R is known to be generated by an explicit quadratic polynomial P and a cubic polynomial Q with no relations. It follows that R is a polynomial ring, although with a nonstandard grading, but in any case $\operatorname{Proj} R \;=\; \mathbb{P}^1$. However, we see that this is not a good notion, since there are no nonconstant maps from $\mathbb{P}^4$ to $\mathbb{P}^1$, so there is no quotient map. Instead, we consider the following: A point $p \in \mathbb{P}^4$ is called *semistable* if there exists a constant polynomial $f \in R$ such that $f(p) \neq 0$. A point is called *stable* if in addition, the orbit is closed and the stabilizer group is finite. There is a map from the locus $\mathbb{P}^4_{\mathrm{ss}}$ of semistable points to $\mathbb{P}^1$, and one usually writes $\mathbb{P}^1 \;=\; \mathbb{P}^4_{\mathrm{ss}} /\!\!/ \mathrm{SL}_2(\mathbb{C})$ to distinguish it from the orbit space $\mathbb{P}^4/\mathrm{SL}_2(\mathbb{C})$, which is different. However, on the stable locus, the quotients agree: $\mathbb{P}^4_{\mathrm{s}} /\!\!/ \mathrm{SL}_2(\mathbb{C}) \;=\; \mathbb{P}^4_{\mathrm{s}}/\mathrm{SL}_2(\mathbb{C})$. Now, we can check that the set U consists of points where the discriminant Δ, which is known to equal $\Delta \;=\; P^3 - 6Q^2$, is nonzero. Thus $U \subset \mathbb{P}^4_{\mathrm{ss}}$. In fact, U lies in $\mathbb{P}^4_{\mathrm{s}}$. Under the quotient map $\mathbb{P}^4_{\mathrm{ss}}/\mathrm{PGL}_2(\mathbb{C}) \;\longrightarrow\; \mathbb{P}^1$, the quotient $U/\mathrm{PGL}_2(\mathbb{C})$ is identified with $\mathbb{C} \subset \mathbb{P}^1$.

4.2.2 A framed moduli problem

Definition 4.21. A *linear rigidification* of (A, λ), is a choice of an S-isomorphism

$$\phi : \mathbf{P}\big(\pi_*\mathcal{L}^\Delta(3\lambda)\big) \xrightarrow{\;\sim\;} \mathbf{P}^m_S,$$

where $m = \sqrt{\deg \Lambda\big(\mathcal{L}^\Delta(3\lambda)\big)} - 1 = 6^g d - 1$.

Here m is $r - 1$, where r is the rank of the vector bundle $\pi_*\mathcal{L}^\Delta(3\lambda)$. Observe that a local trivialization of $\pi_*\mathcal{L}^\Delta(3\lambda)$ induces an isomorphism $\mathbf{P}\big(\pi_*\mathcal{L}^\Delta(3\lambda)\big) \cong \mathbf{P}^m_S$. We moreover note that a linear rigidification is compatible with base change.

Definition 4.22. Let $\mathcal{H}_{g,d,N}$ be the functor from the category of noetherian schemes over $\mathbf{Z}[1/N]$ to the category of sets, sending S to the set of isomorphism classes of quadruples $(A, \lambda, \gamma, \phi)$, where (A, λ, γ) is a triple over S as in the definition of $\mathcal{A}_{g,d,N}$, and ϕ is a linear rigidification of (A, λ).

The strategy to show that $\mathcal{H}_{g,d,N}$ is representable is to embed it into a larger functor which is closely related to Hilbert schemes.

Definition 4.23. Let $\widetilde{\mathcal{H}}$ be the functor from the category of noetherian schemes over $\mathbb{Z}[1/N]$ to the category of sets, sending S to the set of closed subschemes $Z \subset \mathbb{P}^m_S$ that are flat over S together with $2g+1$ sections $\sigma_0, \dots, \sigma_{2g} : S \to Z$.

We construct a natural transformation $\mathcal{H}_{g,d,N} \longrightarrow \widetilde{\mathcal{H}}$ as follows. Let S be a noetherian $\mathbb{Z}[1/N]$-scheme, and let $(A, \lambda, \gamma, \phi) \in \mathcal{H}_{g,d,N}(S)$. We define an element $(Z, \sigma_0, \dots, \sigma_{2g}) \in \widetilde{\mathcal{H}}(S)$ as follows. From (A, λ, ϕ), we obtain a closed immersion $i : A \hookrightarrow \mathbb{P}\big(\pi_* L^{\Delta}(3\lambda)\big) \xrightarrow{\sim} \mathbb{P}^m_S$; this is our definition of Z. Define σ_0 to be the neutral section e of A. The S-scheme $(\mathbb{Z}/N\mathbb{Z})^{2g}$ has $2g$ canonical sections, corresponding to the basis vectors $(0, \dots, 0, 1, 0, \dots, 0) \in (\mathbb{Z}/N\mathbb{Z})^{2g}$. Using γ, we view these $2g$ sections as sections $\sigma_1, \dots, \sigma_{2g}$ of A. This completes the definition of the map $\mathcal{H}_{g,d,N}(S) \to \widetilde{\mathcal{H}}(S)$.

Proposition 4.24. $\mathcal{H}_{g,d,N} \to \widetilde{\mathcal{H}}$ is a locally closed subfunctor (i.e., roughly, for any $\xi \in \widetilde{\mathcal{H}}(S)$, we have a locally closed locus in S over which ξ comes from $\mathcal{H}_{g,d,N}$).

Proof. (Rough Sketch) This is [MFK94, §7.2]. From $A \subset \mathbb{P}^m_S$ and $\sigma_0 = e$, we recover the group structure on A, while from $\sigma_1, ..., \sigma_{2g}$ we obtain the level structure. One can construct $L\Delta(3\lambda)$ from $\mathcal{O}(1)$ on $\mathbb{P}^m_S$, and hence the polarization. Thus, we indeed get a subfunctor.

Now, to show that this defines something localy closed, write $\xi = (i : A \hookrightarrow \mathbb{P}^m_S, e, \sigma_1, \dots, \sigma_{2g})$.

1. Note that there is a unique open subscheme $S' \subset S$ such that for all geometric points s of S, the base change A_s is smooth if and only if s belongs to S'. Hence we may replace S by S', and in particular we may assume that $A \to S$ is smooth.

2. Since $A \to S$ is smooth and proper, we can apply Theorem 4.16, which states that for each connected component S^+ of S, if (A, e) becomes an abelian scheme when base changed to one geometric point of S^+, then (A, e) is an abelian scheme on S^+. So we can and must throw away all connected components of S not satisfying this condition. Then for the new S, the pair (A, e) is an abelian scheme over S.

3. Asking that $\sigma_1, \dots, \sigma_{2g}$ are sections of $A[N]$ (instead of just sections of A) is a closed condition on S. For them to define a level structure γ is an open condition. Thus we may assume that $\sigma_1, \dots, \sigma_{2g}$ indeed come from a (unique) level structure γ.

4. To get the polarization, we use Theorem 4.8.

□

We will now see using the standard theory of Hilbert schemes that $\widetilde{\mathcal{H}}$ is representable. It then follows that the locally closed subfunctor $\mathcal{H}_{g,d,N}$ is also representable (by a locally closed subscheme of the scheme representing $\widetilde{\mathcal{H}}$).

4.2.3 Hilbert schemes

Fix an integer $n \geq 1$. We define a functor $\mathrm{Hilb}_{\mathbb{P}^n}$ from locally noetherian schemes to sets, sending each S to the set of all closed subschemes $Z \subset \mathbb{P}^n_S$ such that Z is flat over S.

Theorem 4.25 (Grothendieck)**.** The functor $\mathrm{Hilb}_{\mathbb{P}^n}$ is representable by a locally noetherian scheme, still denoted by $\mathrm{Hilb}_{\mathbb{P}^n}$.

Write H for $\mathrm{Hilb}_{\mathbb{P}^n}$. We have the universal closed subscheme $Z_{\mathrm{univ}} \subset \mathbb{P}^n_H$ which is flat over H. Using this, we can easily represent certain variations of the functor $\mathrm{Hilb}_{\mathbb{P}^n}$. For instance, for $r \geq 2$, we define a functor $\mathrm{Hilb}_{\mathbb{P}^n,r}$ by sending a locally noetherian scheme S to the set of all closed subschemes $Z \subset \mathbb{P}^n_S$ with Z flat over S, along with the data of r ordered sections $S \to Z$. The functor $\mathrm{Hilb}_{\mathbb{P}^n,r}$ is representable by the r-fold product $Z_{\mathrm{univ}} \times_H \cdots \times_H Z_{\mathrm{univ}}$.

In particular, we have that $\widetilde{\mathcal{H}} = \mathrm{Hilb}_{\mathbb{P}^m,\,2g+1} \times \operatorname{Spec} \mathbb{Z}[1/N]$ is representable. One issue is that $\mathrm{Hilb}_{\mathbb{P}^n}$ and $\mathrm{Hilb}_{\mathbb{P}^n,r}$ are too large. We want to shrink them to more manageable subschemes. To do this, we will need the notion of Hilbert polynomials.

Definition 4.26. For a closed subscheme $Z \subset \mathbb{P}^n_k$ where k is algebraically closed, we obtain a polynomial $P_Z(T) \in \mathbb{Z}[T]$ such that for all $n \in \mathbb{Z}$,

$$P_Z(n) = \chi\big(\mathcal{O}_Z(n)\big),$$

where $\mathcal{O}_Z(n)$ is the nth power of $\mathcal{O}_{\mathbb{P}^n}(1)|_Z$. A basic fact is that if S is locally noetherian and $Z \subset \mathbb{P}^n_S$ is a closed subscheme flat over S, then for geometric points s of S the polynomial $P_{Z_s}(T)$ is independent of s.

Thus we obtain a natural decomposition

$$\mathrm{Hilb}_{\mathbb{P}^n} \;=\; \coprod_{P(T)\in\mathbb{Z}[T]} \mathrm{Hilb}^{P(T)}_{\mathbb{P}^n},$$

where each $\mathrm{Hilb}^{P(T)}_{\mathbb{P}^n}$ is open and closed (i.e., a union of connected components) in $\mathrm{Hilb}_{\mathbb{P}^n}$, determined by the condition that over its geometric points the Hilbert polynomial of Z_{univ} is $P(T)$.

Theorem 4.27 (Grothendieck)**.** Each $\mathrm{Hilb}^{P(T)}_{\mathbb{P}^n}$ is a quasi-projective scheme over $\mathbb{Z}$.

Similarly, define $\mathrm{Hilb}^{P(T)}_{\mathbb{P}^n,r} := \mathrm{Hilb}_{\mathbb{P}^n,r} \times_{\mathrm{Hilb}_{\mathbb{P}^n}} \mathrm{Hilb}^{P(T)}_{\mathbb{P}^n}$; the moduli problem it represents is clear. This is again a quasi-projective scheme over $\mathbb{Z}$, since $\mathrm{Hilb}_{\mathbb{P}^n,r}$ is projective over $\mathrm{Hilb}_{\mathbb{P}^n}$.

Proposition 4.28. The subscheme $\mathcal{H}_{g,d,N} \subset \widetilde{\mathcal{H}} = \mathrm{Hilb}_{\mathbb{P}^m,2g+1}[1/N]$ is contained in $\mathrm{Hilb}^{P(T)}_{\mathbb{P}^m,2g+1}$ for an explicit $P(T)$. In particular, $\mathcal{H}_{g,d,N}$ is quasi-projective over $\mathbb{Z}[1/N]$.

Remark 4.29. One can in fact using the Riemann-Roch result to explicitly compute this Hilbert polynomial as $P(T) = 6^g d T^g$.

4.2.4 GIT quotient

As explained for $g = 1$, the idea is to roughly construct $\mathcal{A}_{g,d,N}$ as a quotient $\mathcal{H}_{g,d,N}/\mathrm{PGL}_{m+1}$, where PGL_{m+1} acts by modifying the linear rigidification $\phi : \mathbb{P}\big(\pi_*\mathcal{L}^{\Delta}(3\lambda)\big) \xrightarrow{\sim} \mathbb{P}^m_S$ via its natural action on $\mathbb{P}^m_S$.

We know that PGL_n is a smooth group scheme over $\mathbb{Z}$. For each scheme S, the group $\mathrm{PGL}_n(S)$ acts on $\mathbb{P}^n_S$ via linear transformations, which are S-scheme automorphisms of $\mathbb{P}^n_S$.

Remark 4.30. For each noetherian scheme S, the homomorphism $\mathrm{PGL}_n(S) \longrightarrow \mathrm{Aut}_{S\text{-sch}}\big(\mathbb{P}^n_S\big)$ is an isomorphism.

Definition 4.31. Let k be an algebraically closed field, and l a positive integer. We say a collection C of l points in $\mathbb{P}^n_k$ is *stable*, if for every proper linear subspace $H \subsetneq \mathbb{P}^n_k$, we have $\frac{|C \cap H|}{l} < \frac{\dim H + 1}{n+1}$.

For example, in $\mathbb{P}^2_k(k)$, a collection of 4 points is stable if and only if no 3 of the points are colinear. In general, by considering 0-dimensional H, we see that a necessary condition for any collection of l points in $\mathbb{P}^n_k(k)$ to be stable is that $l > n + 1$.

Proposition 4.32. Let $l \geq 1$. There is a unique maximal open subscheme $(\mathbb{P}^n)^l_{\text{stable}}$ of $(\mathbb{P}^n)^l$ (where $\mathbb{P}^n$ is over $\mathbb{Z}$) such that for every algebraically closed field k and every $\xi = (\xi_1, \cdots, \xi_l) \in (\mathbb{P}^n)^l_{\text{stable}}(k)$, the l components ξ_i of ξ form a stable collection of l points in $\mathbb{P}^n_k(k)$.

It remains to see that $\mathcal{H}_{g,d,N}$ maps into the stable locus, and hence the quotient exists. We will leave this as a theorem, the proof of which is in [MFK94, §7.3].

Theorem 4.33. Consider the map $f : \mathcal{H}_{g,d,N} \to (\mathbb{P}^m)^{N^{2g}}$, by looking at the image of the elements of $(\mathbb{Z}/N\mathbb{Z})^{2g}$. Then this defines a PGL_{m+1}-equivariant map, and its image lies in the stable locus.

Hence, the quotient $\mathcal{H}_{g,d,N}/\mathrm{PGL}_{m+1}$ exists.

Finally, we discuss the precise relation between $\mathcal{A}_{g,d,N}$ and the Siegel modular variety:

4.3 Siegel modular variety

Let us first recall the construction of GSp_{2g}. Let (V, ψ) be a symplectic space of dimension $2g$ over k, i.e., V is a k-vector space of dimension $2g$ and ψ is a nondegenerate alternating form on V. A *symplectic basis* of V is a basis $(e_{\pm i})_{1 \leq i \leq g}$ such that

$$\psi(e_i, e_{-i}) = 1 \quad \text{for } 1 \leq i \leq g, \qquad \psi(e_i, e_j) = 0 \quad \text{for } j \neq \pm i.$$

This means that the matrix of ψ with respect to $(e_{\pm i})$ has ± 1 down the second diagonal, and zeros elsewhere:

$$\big(\psi(e_{\pm i}, e_{\pm j})\big)_{1 \leq i,j \leq g} = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix},$$

Let (V, ψ) be a nonzero symplectic space, and let $\mathrm{GSp}(\psi)$ be the group of symplectic similitudes of (V, ψ), i.e., the group of automorphisms of V preserving ψ up to a scalar. Thus

$$\mathrm{GSp}(\psi)(k) = \{\, g \in \mathrm{GL}(V) \mid \psi(gu, gv) = \nu(g)\, \psi(u, v) \text{ for some } \nu(g) \in k^\times \,\}.$$

Let (A, λ) be an abelian scheme of dimension $\frac{1}{2}\dim(V)$ over S with a principal polarization λ. From λ we get a perfect alternating pairing

$$e^N_\lambda \colon A[N] \times A^\vee[N] \longrightarrow \mu_N,$$

as follows. For any S-scheme T let $x \in A[N](T)$ and $\mathcal{L} \in A^\vee[N](T)$. By definition we have $Nx = 0$ and $\mathcal{L}^{\otimes N} \simeq \mathcal{O}$. We know (from Theorem of the cube) that we have an isomorphism $[N]^*\mathcal{L} \simeq \mathcal{L}^{\otimes N}$ so we may fix an isomorphism $u : \mathcal{O} \to [N]^*\mathcal{L}$. Pulling back the isomorphism via t_x we obtain

$$t_x^* u : t_x^*\mathcal{O} \to t_x^*[N]^*\mathcal{L} \simeq ([N] \circ tx)^*\mathcal{L} = [N]^*\mathcal{L}.$$

It follows in particular that $u \circ (t_x^* u)^{-1}$ is an S-isomorphism of $[N]^*\mathcal{L}$, and since $A \to S$ is proper this must be given by multiplication by an element $\zeta \in \mathbb{G}^\times_{m,S}(T)$, and we define $e_N(x, \mathcal{L}) = \zeta$. It is clear from the definition that $1 = e_N(Nt, \mathcal{L}) = e_N(x, \mathcal{L})^N = \zeta^N$, so ζ is in fact an N-th root of unity, i.e. it lies in $\mu_{N,S}(T)$.

Note that a choice of $i = \sqrt{-1} \in \mathbb{C}$ gives us an isomorphism $[n] \longmapsto e^{2\pi i n/N} \colon \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N$, and so e^N_λ becomes a pairing to $\mathbb{Z}/N\mathbb{Z}$. By a compatible *level-N structure* on an abelian scheme A, we mean an isomorphism

$$\eta \colon (\mathbb{Z}/N\mathbb{Z})^{2g} \longrightarrow A[N]$$

under which ψ_N (the mod N reduction of ψ) corresponds to a $(\mathbb{Z}/N\mathbb{Z})^\times$ multiple of e^N_λ.

Now, recall that we have the Siegel upper half plane

$$\mathcal{H}_g = \{\Omega \in \mathrm{Mat}_{g\times g}(\mathbb{C}) \mid \Omega = \Omega^T,\ \mathrm{Im}(\Omega) > 0\}.$$

Given $\Omega \in H_g$ we can construct a torus $X_\Omega = \mathbb{C}^g/\big(\Omega\,\mathbb{Z}^g + \mathbb{Z}^g\big)$. This carries a principal polarization H_Ω represented by the matrix Ω^{-1}. The associated symplectic form $E = \mathrm{Im}\, H_\Omega$ is the standard one $E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Thus (X_Ω, H_Ω) is an abelian variety with principal polarization. We have the following generalization of the results for the modular curve:

Proposition 4.34. There is a bijection of sets

$$\mathrm{GSp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g^{\pm} \simeq \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g \simeq \{\text{principally polarized abelian varieties over } \mathbb{C} \text{ of dimension } g\}$$

Let $K(N) = \ker(\mathrm{GSp}(\hat{\mathbb{Z}}) \to \mathrm{GSp}(\mathbb{Z}/N\mathbb{Z}))$.

Proposition 4.35. There is a bijection of sets

$$\mathrm{Sh}_{K(N)}(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})(\mathbb{C}) \simeq \left\{ \begin{array}{l} \text{principally polarized abelian varieties over } \mathbb{C} \text{ of dimension } g \\ \text{with compatible level } N\text{-structure.} \end{array} \right\}$$

5 Canonical Models

5.1 Descent

In this section, let C be an algebraically closed field of characteristic zero. Let V be a variety over C, and let k be a subfield of C.

Definition 5.1. A variety V_0 over k is a model of V if there is an isomorphism

$$\phi : V_0 \times_k C \simeq V.$$

For any $\sigma \in \mathrm{Aut}(C/k)$ we can consider the base change of V along σ, which we denote by σV. Observe that if V has a model over k, we have an isomorphism $f_\sigma : \sigma V \to V$, and it satisfies the following cocycle condition $f_{\sigma\tau} = f_\sigma \circ \sigma f_\tau$ for all $\sigma, \tau \in \mathrm{Aut}(C/k)$. This motivates the following definition:

Definition 5.2. Let $\Gamma = \mathrm{Aut}(C/k)$. A C/k-*descent system* on an algebraic scheme V over C is a family $(f_\sigma)_{\sigma\in\Gamma}$ of isomorphisms $f_\sigma : \sigma V \to V$ satisfying the cocycle condition,

$$f_\sigma \circ (\sigma f_\tau) \;=\; f_{\sigma\tau} \qquad \text{for all } \sigma, \tau \in \Gamma,$$

A model (V_0, ϕ) of V over a subfield K of C containing k *splits* the descent system if

$$f_\sigma \;=\; \phi \circ \sigma(\phi^{-1})$$

for all σ fixing K.

A descent system $(f_\sigma)_{\sigma\in\Gamma}$ is said to be *continuous* if it is split by some model over a subfield K of Ω that is finitely generated over k. (This is equivalent to asking for a continuous action of Γ, where the topology on Γ is the Krull topology.) A *descent datum* is a continuous descent system. A descent datum is *effective* if it is split by some model over k.

Let V_0 be an algebraic scheme over k, and let $V = V_{0,C}$. For $\sigma \in \Gamma$, we have a canonical isomorphism $f_\sigma : \sigma V \to V$ induced by the equality $\sigma \circ i = i$, where $i : k \hookrightarrow C$ is the inclusion. Then $(f_\sigma)_{\sigma\in\Gamma}$ is an effective C/k-descent datum, split by V_0.

Now, define a functor $\mathcal{F}$ from varieties over k to varieties over C with a C/k-descent datum.

Proposition 5.3. $\mathcal{F}$ defined above from varieties over k to varieties over C with a C/k-descent datum is fully faithful.

Remark 5.4. The functor is clearly not essentially surjective, since we only obtain effective descent datum. The difficulty is then in establishing when we have effective descent data.

The proof of Proposition 5.3 follows from the following:

Proposition 5.5. Let $(f_\sigma)_{\sigma\in\Gamma}$ and $(f'_\sigma)_{\sigma\in\Gamma}$ be C/k-descent data on algebraic varieties V and W over C. If (V_0, f) and (W_0, f') are models over k splitting $(f_\sigma)_{\sigma\in\Gamma}$ and $(f'_\sigma)_{\sigma\in\Gamma}$ respectively, then to give a regular map $\alpha_0 : V_0 \to W_0$ is the same as giving a regular map $\alpha : V \to W$ such that the diagrams

$$\begin{tikzcd} \sigma V \arrow[r, "f_\sigma"] \arrow[d, "\sigma\alpha"'] & V \arrow[d, "\alpha"] \\ \sigma W \arrow[r, "f'_\sigma"'] & W \end{tikzcd}$$

commute for all $\sigma \in \Gamma$.

Proof. (Sketch, see [Mil25, Proposition 6.1]) If α_0 exists, then we define α as the base change of α_0 via the inclusion $k \hookrightarrow C$, and it clearly satisfies the required condition.

Given α, we can consider the graph of the morphism $\Gamma_\alpha \subset V \times W$, and we need to show that this comes from a closed subvariety of $V_0 \times W_0$. We claim that this is true if and only if the C points $\Gamma_\alpha(C) \subset V(C) \times W(C)$ are stable under the action of Γ. Assuming this, we are done since by assumption the action of α commutes with the descent data.

To see the claim, we observe that this holds more generally for closed subvarieties over C, and comes from the fact that the ideal of definition descends (as a C-vector space to a k-vector space) if and only if it is stable under the action of Γ. (Here we used that $C^{\mathrm{Aut}(C/k)} = k$, which requires C to be characteristic 0 and Zorn's Lemma.) $\square$

The following is the key descent result we need:

Theorem 5.6. If V is quasi-projective over C, then every descent datum is effective.

We first see the following:

Proposition 5.7. A C/k-descent system $(f_\sigma)_{\sigma\in\Gamma}$ on a variety V over C is continuous if there exists a finite set S of points in $V(C)$ such that

(a) the only automorphism of V fixing all $P \in S$ is the identity, and

(b) there exists a subfield K of C finitely generated over k such that $\sigma P = P$ for all $\sigma \in \Gamma$ fixing K.

Proof. Let (V_0, ϕ) be a model of V over a subfield K of C finitely generated over k. Such a model exists because we see that on each affine open, we can get a model over a field finitely generated over k (namely, we can embed the this affine open as a closed subvariety of the spectrum of a polynomial algebra over C, and we adjoin the coefficients of the equations defining this subvariety) and since V is quasi-compact, after passing to a further finitely generated extension we can glue the models. After possibly replacing K by a larger finitely generated field, we may suppose

(i) $\sigma P = P$ for all $\sigma \in \Gamma$ fixing K and all $P \in S$ (because of (b)), and

(ii) $\phi(P) \in V_0(K)$ for all $P \in S$ (because S is finite).

Then, for $P \in S$ and every σ fixing K, $\phi_\sigma(\sigma P) \stackrel{\text{def}}{=} P$, and $(\sigma\phi)(\sigma P) = \sigma(\phi P) \stackrel{(ii)}{=} \phi P$. Hence both ϕ_σ and $\phi^{-1}\sigma\phi$ are isomorphisms $\sigma V \to V$ sending σP to P. Therefore, ϕ_σ and $\phi^{-1}\sigma\phi$ differ by an automorphism of V fixing the $P \in S$, and so are equal. This says that (V_0, ϕ) splits $(\phi_\sigma)_{\sigma\in\Gamma}$. □

Proposition 5.8. If every finite set of points of $V(C)$ is contained in an open affine algebraic subscheme of V_C, then every descent datum on V is effective.

Proof. Let $(\phi_\sigma)_{\sigma\in\Gamma}$ be a descent datum on V, and let U be an open subscheme of V. By continuity, (ϕ_σ) is split by a model (V_1, ϕ) of V over some finite extension k_1 of k. After possibly replacing k_1 with a larger finite extension, which we may suppose to be Galois over k, we have that there exists an open subscheme U_1 of V_1 such that $\phi(U) = U_{1,C}$. Now consider σU, observe that it depends only on the coset $\sigma\Delta$, where $\Delta = \mathrm{Gal}(C/k_1)$. In particular, $\{\sigma U \mid \sigma \in \Gamma\}$ is finite, and so the scheme $U' \stackrel{\text{def}}{=} \bigcap_{\sigma\in\Gamma} \sigma U$ is open in V. Note that we have

$$\tau U' \;=\; \tau\left(\bigcap_{\sigma\in\Gamma} \sigma U\right) \;=\; \bigcap_{\sigma\in\Gamma} \tau\sigma U \;=\; U'.$$

for all $\tau \in \Gamma$.

Let P be a closed point of V. Because $\{\sigma P \mid \sigma \in \Gamma\}$ is finite, it is contained in an open affine U of V. Now $U' \stackrel{\text{def}}{=} \bigcap_{\sigma\in\Gamma} \sigma U$ is an open affine in V containing P and such that $\sigma U' = U'$ for all $\sigma \in \Gamma$. Since V is quasi-compact, we can cover V using a finite number of such affine opens U which are fixed by σ for all $\sigma \in \Gamma$. Thus, we can reduce to the case where V is affine, and $\sigma V = V$ for all σ, since we can glue the affine opens.

If $V = \operatorname{Spec} R$, then consider the subring of Γ-invariants R^Γ, with σ acting via ϕ_σ. We have an isomorphism

$$C \otimes_k R^\Gamma \simeq R,$$

and thus we can take $V_0 = \operatorname{Spec} R^\Gamma$, with the isomorphism induced by the one above. This clearly splits the descent datum. □

Theorem 5.6 follows from the following lemma:

Lemma 5.9. Let V be a quasi-projective scheme over a field k. Every finite set S of closed points of V is contained in an open affine subset of V.

Proof. Embed V as a subscheme of $\mathbb{P}^n$, and let $\bar{V}$ be the closure of V in $\mathbb{P}^n$. For each $P \in S$, there exists an open subset U_P of $\bar{V}$ such that $U_P \cap S = \{P\}$, and a homogeneous polynomial $F_P \in I(\bar{V} \setminus U_P)$ such that $F_P(P) \neq 0$. We may suppose that the F_P have the same degree. As $F_P(Q) = 0$ for $Q \in S \setminus \{P\}$, the polynomial $F \stackrel{\text{def}}{=} \sum F_P$ has the property that, for all $P \in S$, $F(P) \neq 0$. Hence $\bar{V} \cap D(F)$ is an open affine subset of V containing S. □

Thus, we see that the key point in checking that a descent system is effective reduces to understanding the actions of $\mathrm{Aut}(C/k)$ on a finite set of points. In the context of Shimura varieties, these will be the *special points*.

5.2 Canonical models

5.2.1 Reflex field

Let (G, X) be a Shimura datum. Recall that X is a $G(\mathbb{R})$-conjugacy class of maps

$$h : \mathbb{S} \longrightarrow G_{\mathbb{R}}.$$

and for each such h we can define the cocharacter over $\mathbb{C}$

$$\mu_h : \ \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \underset{\sim}{\xrightarrow{c}} \mathbb{S}_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}}; \qquad z \longmapsto (z, 1).$$

Here c is the usual isomorphism whose inverse is induced by

$$R \otimes_{\mathbb{R}} \mathbb{C} \ \simeq \ R \times R, \qquad r \otimes z \ \longmapsto \ (rz,\ r\bar{z}),$$

for any $\mathbb{C}$-algebra R.

Of course, the conjugacy class $\mathcal{C}$ of μ_h is independent of h.

Let T be a maximal torus in G and $W := N_G(T)(\overline{\mathbb{Q}})/Z_G(T)(\overline{\mathbb{Q}})$ its absolute Weyl group, which is finite. Then the natural maps

$$\left\{ \begin{array}{l} W\text{-conjugacy classes} \\ \text{in } \mathrm{Hom}(\mathbb{G}_{m,\overline{\mathbb{Q}}}, T_{\overline{\mathbb{Q}}}) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} W\text{-conjugacy classes} \\ \text{in } \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, T_{\mathbb{C}}) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} G(\mathbb{C})\text{-conjugacy classes} \\ \text{in } \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}}) \end{array} \right\}$$

are all isomorphisms. Hence we may view $\mathcal{C}$ as an element of the leftmost set.

Definition 5.10. The *reflex field* $E(G, X)$ of (G, X) is the field of definition of $\mathcal{C}$, that is, the fixed field inside $\overline{\mathbb{Q}}$ of the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing $\mathcal{C}$. If there is a representative of $\mathcal{C}$ defined over $E \subset \overline{\mathbb{Q}}$ (in particular if E splits T) then $E \supset E(G, X)$. Hence $E(G, X)$ is a number field.

Definition 5.11. A point $h \in X$ is special if there exists a $\mathbb{Q}$-torus $T \subset G$ such that h factors through $T_{\mathbb{R}}$. For such h and T we call (T, h) a *special pair*.

Remark 5.12. Since the conjugacy class is just a point, we see that the Shimura variety $\mathrm{Sh}_{K'}(T, x)$ is just a finite disjoint union of points for all compact open $K' \subset T(\mathbb{A}_f)$.

Let (T, h) be a special pair and $E := E(\mu_h)$ the field of definition of μ_h, which as explained above is a number field containing $E(G, X)$.

Consider the composite map

$$\mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_m \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}}\, \mu_h} \mathrm{Res}_{E/\mathbb{Q}}\, T_E \xrightarrow{\mathrm{Nm}_{E/\mathbb{Q}}} T,$$

where on R-points the norm map $\mathrm{Nm}_{E/\mathbb{Q}}$ sends $t \in T(R \otimes_{\mathbb{Q}} E)$ to

$$\prod_{\varphi: E \hookrightarrow \overline{\mathbb{Q}}} \varphi_*(t) \in T(R \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}),$$

which is invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and hence lies in $T(R)$. Taking $\mathbb{A}_{\mathbb{Q}}$-points and further composing with the projection to $\mathbb{A}_{f,\mathbb{Q}}$-points, we get

$$r_h : \mathbb{A}_E^\times \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}} \mu_h} T(\mathbb{A}_E) \xrightarrow{\mathrm{Nm}_{E/\mathbb{Q}}} T(\mathbb{A}_{\mathbb{Q}}) \longrightarrow T(\mathbb{A}_{f,\mathbb{Q}}).$$

For $h \in X$ and $a \in G(\mathbb{A}_{f,\mathbb{Q}})$, let $[h,a]_K$ denote the class of (h,a) in

$$\mathrm{Sh}_K(G,X)(\mathbb{C}) \;=\; G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{f,\mathbb{Q}})/K.$$

Now, recall the main result in global class field theory: we have the global Artin map

$$\mathrm{art}_{E(\mu_h)} : \; \mathbb{A}_{E(\mu_h)}^\times \;\twoheadrightarrow \mathrm{Gal}\big(E(\mu_h)^{\mathrm{ab}}/E(\mu_h)\big),$$

such that, for every finite extension L of E contained in E^{ab}, art_E gives rise to a commutative diagram

$$\begin{tikzcd}
E^\times\backslash\mathbb{A}_E^\times \arrow[r, two heads, "\mathrm{art}_E"] \arrow[d] & \mathrm{Gal}(E^{\mathrm{ab}}/E) \arrow[d, "\sigma\mapsto\sigma|_L"] \\
E^\times\backslash\mathbb{A}_E^\times/\mathrm{Nm}_{L/E}(\mathbb{A}_L^\times) \arrow[r, "\mathrm{art}_{L/E}", "\simeq"'] & \mathrm{Gal}(L/E)
\end{tikzcd}$$

It has the following properties (which determine it):

(a) $\mathrm{art}_{L/E}(u) = 1$ for every $u = (u_v) \in \mathbb{A}_E^\times$ such that

 (i) if v is unramified in L, then u_v is a unit,

 (ii) if v is ramified in L, then u_v is sufficiently close to 1 (depending only on L/E), and

 (iii) if v is real but becomes complex in L, then $u_v > 0$.

(b) For every prime v of E unramified in L, the idèle

$$\alpha = (1, \ldots, 1, \pi^{-1}, 1, \ldots)$$

where π is a uniformizer of $\mathcal{O}_{E_v}$, maps to the Frobenius element in $\mathrm{Gal}(L/E)$ given by raising to the q-th power, where q is the size of the residue field of $L_{v'}$ for some $v'|v$.

Remark 5.13. The choice of taking π^{-1} rather than π is the correct one, see last note in [Mil05, §12].

Definition 5.14. A model $M_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ defined over $E(G,X)$ is *canonical* if for every special pair (T,h) in (G,X) and every $a \in G(\mathbb{A}_{f,\mathbb{Q}})$,

1. $[h, a]_K \in M_K(G, X)(\mathbb{C})$ is defined over $E(\mu_h)^{\mathrm{ab}}$; and

2. for all $s \in \mathbb{A}^{\times}_{E(\mu_h)}$, we have $\mathrm{art}_{E(\mu_h)}(s) \cdot [h, a]_K \ = \ [h,\, r_h(s)\, a]_K$.

We first observe that in the case of a Shimura datum (T, h) for T a torus over $\mathbb{Q}$, this defines a canonical model. To see this, recall that to define a model for this zero-dimensional variety over $E := E(\mu_h)$, we need a descent datum, or equivalently a continuous action of $\mathrm{Gal}(\mathbb{C}/E)$, so we need the action to factor through $\mathrm{Gal}(L/E)$ for some finite Galois extension L of E.

Observe that part (2) of the definition of canonical model defines a continuous action of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ on $\mathrm{Sh}_K(T, h)$. This action defines a model of $\mathrm{Sh}_K(T, h)$ over E, which, by definition, is canonical.

5.2.2 Uniqueness

We now sketch how to see uniqueness of canonical models.

Lemma 5.15. There exists a special point in X.

Proof. (Sketch, see [Mil05, Lemma 13.3]) Let $x \in X$, and let T be a maximal torus in $G_{\mathbb{R}}$ containing $h_x(\mathbb{C}^{\times})$. We need to conjugate this to a maximal torus defined over $\mathbb{Q}$.

We consider $\mathfrak{g}$ the Lie algebra of $G_{\mathbb{C}}$. A subalgebra $\mathfrak{h}$ is Cartan if it is nilpotent and equal to its own normalizer. We know that the Cartan subalgebras are exactly the Lie algebras of maximal tori in G. Moreover, we also know that Cartan subalgebras are centralizers of regular elements in $\mathfrak{g}$, namely, elements x whose characteristic polynomial of $\mathrm{ad}(x)$ has non-vanishing coefficients in all degrees $\leq \dim(\mathfrak{g})$. The regular x form a connected dense open subset of $\mathfrak{g}$ for the Zariski topology

Hence, we see that T is the centralizer in $G_{\mathbb{R}}$ of a regular element λ of $\mathrm{Lie}(G_{\mathbb{R}})$. If $\lambda_0 \in \mathrm{Lie}(G)$ is chosen to be sufficiently close to λ in $\mathrm{Lie}(G_{\mathbb{R}})$, then it will also be regular, and so its centralizer T_0 in G is a maximal torus in G. Moreover, T_0 will become conjugate to T over $\mathbb{R}$: $(T_0)_{\mathbb{R}} = gTg^{-1}$ for some $g \in G(\mathbb{R})$. Now $h_{gx}(\mathbb{S}) \overset{\mathrm{def}}{=} ghg^{-1}(\mathbb{S}) \subset T_{0\mathbb{R}}$, and so gx is special. □

Lemma 5.16 ([Del71, §5.1])**.** For every finite extension L of $E(G, X)$ in $\mathbb{C}$, there exists a special point x_0 such that $E(x_0)$ is linearly disjoint from L.

Lemma 5.17. For any $x \in X$, $\{[x, a]_K \mid a \in G(\mathbb{A}_f)\}$ is dense in $\mathrm{Sh}_K(G, X)$ (in the Zariski topology).

Proof. Write $\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times (G(\mathbb{A}_f)/K)$ and note that the real approximation theorem (that $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$) implies that $G(\mathbb{Q})x$ is dense in X for the complex topology. Then $G(\mathbb{Q})x \times G(\mathbb{A}_f)$ is dense in $X \times G(\mathbb{A}_f)$, and its image in $\mathrm{Sh}_K(\mathbb{C})$ is dense for the complex topology and thus for the Zariski topology, becuase the Zariski topology is coarser than the complex topology. That image equals $\{[x, a]_K \mid a \in G(\mathbb{A}_f)\}$ because $[gx, b]_K = [x, g^{-1}b]_K$. □

Theorem 5.18. A canonical model for $\mathrm{Sh}_K(G, X)$, if it exists, is unique up to unique isomorphism.

Proof. Observe that it suffices to show that if we have two canonical models $M_K(G,X)$ and $M'_K(G,X)$ over $E(G,X)$, then the identity morphism on $\mathrm{Sh}_K(G,X)$ descends uniquely to a morphism (necessarily an isomorphism) between $M_K(G,X)$ and $M'_K(G,X)$. Observe that as in the proof of Proposition 5.5 such a morphism over $E(G,X)$ if it exists, is also necessarily unique, so we only need to show existence.

As in Proposition 5.5 it suffices to show that $\sigma\big(\mathrm{id}\big) = \mathrm{id}\sigma$ for all automorphisms σ of $\mathbb{C}$ fixing $E(G,X)$. Let $x_0 \in X$ be special. Then $E(x_0) \supset E(G,X)$ and we first show this for those σ fixing $E(x_0)$. But this follows from the property of canonical models as we can choose $\mathrm{art}(s) = \sigma|_{E(x_0)^{\mathrm{ab}}}$. and for such points $[x_0, a]_K$ we clearly have the desired commutativity. By density, we know that we have commutativity for all points on Sh_K. Using Lemma 5.16 shows that these σ's as we let the special point vary generate $\mathrm{Aut}(\mathbb{C}/E(G,X))$. ☐

Remark 5.19. Similar ideas as in the proof above allow us to see that

1. If, for all sufficiently small compact open subgroups K of $G(\mathbb{A}_f)$, (actually a cofinal collection of such) $\mathrm{Sh}_K(G,X)$ has a canonical model, then so also does

$$\mathrm{Sh}(G,X) := \varprojlim \mathrm{Sh}_K(G,X),$$

and it is unique up to a unique isomorphism.

2. Moreover, the action of $G(\mathbb{A})_f$, given by $[x,a] \mapsto [x,ag]$, also descends to $E(G,X)$. In particular, for all K such that $\mathrm{Sh}_K(G,X)$ is a complex algebraic variety, viewing $\mathrm{Sh}_K(G,X)$ as the geometric quotient $\mathrm{Sh}(G,X)/K$, we also have a model $M_K(G,X)$.

Proposition 5.20. Let $(G,X) \hookrightarrow (G',X')$ be an inclusion of Shimura data. If $\mathrm{Sh}(G',X')$ has a canonical model, then so does $\mathrm{Sh}(G,X)$.

Proof. We apply Proposition 2.33. Observe that from the definition of reflex field, we must have $E(G,X) \supset E(G',X')$. Moreover, it suffices to construct the model $M_K(G,X)$ as closed subvariety of $M_{K'}(G',X')_{E(G,X)}$, since such a model will clearly be canonical. Thus, the claim follows easily from the observation that for all $\sigma \in \mathrm{Aut}(\mathbb{C}/E(G,X))$, we have $\sigma\mathrm{Sh}(G,X)(\mathbb{C}) = \mathrm{Sh}(G,X)(\mathbb{C})$. But this is clear on special points (T,h) whose image factors through (G,X), ☐

5.3 CM abelian varieties

We now want to explain how the model for the Siegel modular variety that we have constructed is canonical. To see this, we need to understand what types of abelian varieties these special points correspond to.

Definition 5.21. A number field E is a *CM (complex multiplication) field* if it is a quadratic totally imaginary extension of a totally real field F.

Let $a \mapsto a^*$ denote the nontrivial automorphism of E fixing F. Then $\rho(a^*) = \overline{\rho(a)}$ for every embedding $\rho : E \hookrightarrow \mathbb{C}$.

Each embedding of F into $\mathbb{R}$ will extend to two conjugate embeddings of E into $\mathbb{C}$, interchanged by $*$. A *CM-type* Φ for E is a choice of one embedding from each conjugate pair. Thus, we have $\mathrm{Hom}(E,\mathbb{C}) \;=\; \Phi \sqcap \bar{\Phi}$

Definition 5.22. Let E be a CM field. An *abelian variety with CM by* E is a tuple (A, i), where A is an abelian variety of dimension $\frac{1}{2}[E : \mathbb{Q}]$ and $i : E \hookrightarrow \mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an embedding of E as a subfield of $\mathrm{End}^0(A)$.

Observe that the tuple if A is a complex abelian variety, (A, i) determines a CM type $\Phi \subseteq \mathrm{Hom}(E, \mathbb{C})$ on E associated to (A, i), via the following recipe.

Let (A, i) have CM by E. The choice of $i : E \hookrightarrow \mathrm{End}^0(A)$ determines a faithful action of E on $H_1(A(\mathbb{C}), \mathbb{Q})$, which gives $H_1\big(A(\mathbb{C}), \mathbb{Q}\big)$ the structure of a 1-dimensional E-vector space.

Recall that Hodge theory gave us a splitting

$$H^1\big(A(\mathbb{C}), \mathbb{C}\big) \;=\; H^{0,1} \oplus H^{1,0}$$

where $H^{1,0} = \overline{H^{0,1}}$. Now, we also have $H^{0,1} \;=\; H^0\big(A(\mathbb{C}), \Omega^1\big)$, and there is a natural isomorphism $H^0\big(A(\mathbb{C}), \Omega^1\big) = T_e(A)^\vee$, so dualizing gives a canonical decomposition

$$H_1\big(A(\mathbb{C}), \mathbb{C}\big) \;=\; \mathrm{Lie}(A) \oplus \overline{\mathrm{Lie}(A)}.$$

Since $H_1\big(A(\mathbb{C}), \mathbb{C}\big) \simeq H_1\big(A(\mathbb{C}), \mathbb{Q}\big) \otimes_{\mathbb{Q}} \mathbb{C}$, $H_1\big(A(\mathbb{C}), \mathbb{C}\big)$ inherits the structure of a $2\dim A$-dimensional E-module over $\mathbb{C}$. We thus have an isomorphism of E-modules

$$H_1\big(A(\mathbb{C}), \mathbb{C}\big) \;\simeq\; E \otimes_{\mathbb{Q}} \mathbb{C} \;\simeq\; \bigoplus_{\varphi \in \mathrm{Hom}(E,\mathbb{C})} \mathbb{C}_\varphi$$

where $\mathbb{C}_\varphi$ denotes the space $\mathbb{C}$ treated as an E-module via multiplication through φ. Thus, via the Hodge decomposition we have

$$\mathrm{Lie} A(C) \simeq \bigoplus_{\varphi \in \Phi} \mathbb{C}_\varphi =: \mathbb{C}^\Phi$$

where $\Phi \subseteq \mathrm{Hom}(E, \mathbb{C})$ is a CM type of E.

Proposition 5.23. Let (A, i) be an abelian variety of CM-type (E, Φ) over $\mathbb{C}$. Then (A, i) has a model over $\overline{\mathbb{Q}}$, uniquely determined up to isomorphism.

Proof. Let $k \subset \Omega$ be algebraically closed fields of characteristic zero. For an abelian variety A over k, the torsion points in $A(k)$ are Zariski dense, and the map on torsion points $A(k)_{\mathrm{tors}} \to A(\Omega)_{\mathrm{tors}}$ is bijective, and so every regular map $A_\Omega \to W_\Omega$ (with W a variety over k) is fixed by the automorphisms of Ω/k and is therefore defined over k. It follows that $A \mapsto A_\Omega : \mathrm{AV}(k) \to \mathrm{AV}(\Omega)$ is fully faithful.

It remains to show that every abelian variety (A, i) of CM-type over $\mathbb{C}$ arises from a pair over $\overline{\mathbb{Q}}$. The polynomials defining A and i have coefficients in some subring R of $\mathbb{C}$ that is finitely generated over $\mathbb{Q}$. There is a maximal ideal $\mathfrak{m}$ of R that will have residue field contained in $\overline{\mathbb{Q}}$, and the reduction of (A, i) mod $\mathfrak{m}$ is called a specialization of (A, i).

Observe that every specialization (A', i') of (A, i) to a pair over $\overline{\mathbb{Q}}$ with A' an abelian variety will still be of CM-type (E, Φ), because the CM-type is determined by the set of eigenvalues of a generator e of E over $\mathbb{Q}$ acting on the tangent space, and this set is unchanged by the change of ground ring from $\mathbb{C}$ to R to $\overline{\mathbb{Q}}$.

Now, we claim that (A, i) and (A', i') are isogenous. Assuming this, the kernel H of this isogeny is a finite subgroup of $A'(\mathbb{C})_{\mathrm{tors}} = A'(\overline{\mathbb{Q}})_{\mathrm{tors}}$, and $(A'/H, i)$ is a model of (A, i) over $\overline{\mathbb{Q}}$.

It remains to show the claim about the existence of an isogeny, which is the next proposition. □

Consider the image Λ_Φ of O_E under the map $\Phi : E \to \mathbb{C}^\Phi$, given by $a \mapsto (\varphi(a))_{\varphi \in \Phi}$.

Proposition 5.24. The quotient $A_\Phi = \mathbb{C}^\Phi / \Lambda_\Phi$ is an abelian variety of CM-type (E, Φ) for the natural homomorphism $i_\Phi : E \to \mathrm{End}^0(A_\Phi)$. Any other pair (A, i) of CM-type (E, Φ) is E-isogenous to (A_Φ, i_Φ).

Proof. (Sketch, see [Mil05, Proposition 10.2] for full proof.) We can check that such A_Φ is indeed an abelian variety, by checking that the form

$$\psi(u, v) := \mathrm{Tr}_{E/\mathbb{Q}}(\alpha u v^*) \qquad u, v \in O_E$$

for an $\alpha \in O_E$ which satisfies $\mathrm{Im}(\varphi(\alpha)) > 0$ for $\varphi \in \Phi$.

Let (A, i) be of CM-type (E, Φ). This means that there exists an isomorphism $\mathbb{C}^\Phi \xrightarrow{\sim} \mathrm{Lie}(A)$ of $E \otimes_{\mathbb{Q}} \mathbb{C}$-modules, and so $A(\mathbb{C})$ is a quotient of $\mathbb{C}^\Phi$ by a lattice Λ such that $\Lambda \otimes \mathbb{Q}$ is stable under the action of E on $\mathbb{C}^\Phi$ given by Φ. Therefore $\Lambda \otimes \mathbb{Q} = \Phi(E) \cdot \lambda$ for some $\lambda \in (E \otimes_{\mathbb{Q}} \mathbb{R})^\times$. After replacing the isomorphism $\mathbb{C}^\Phi \to \mathrm{Lie}(A)$ by its composite with $\mathbb{C}^\Phi \xrightarrow{\lambda} \mathbb{C}^\Phi$, we may suppose that $\Lambda \otimes \mathbb{Q} = \Phi(E)$, and so $\Lambda = \Phi(\Lambda')$, where Λ' is a lattice in E. Now, $N\Lambda' \subset \mathcal{O}_E$ for some N, and hence we can define $\mathcal{O}_E$-isogenies

$$A \simeq \mathbb{C}^\Phi / \Lambda \xrightarrow{N} \mathbb{C}^\Phi / N\Lambda \leftarrow \mathbb{C}^\Phi / \Phi(\mathcal{O}_E) \simeq A_\Phi.$$

□

Now, let E^* be the fixed field of Φ, that is, the subfield of $\overline{\mathbb{Q}}$ such that $\sigma \in \mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$ fixes E^* if and only if $\sigma\Phi = \Phi$. Let σA_Φ be the base change of A_Φ by σ, and observe that σA_Φ has CM by (E, Φ) as well, hence is isogenous to A_Φ. We want to understand this isogeny $\alpha : A_\Phi \to \sigma A_\Phi$.

To make our comparison, recall the (rational) adelic Tate module: write

$$\hat{T}(A) := \prod_\ell T_\ell(A) \quad \text{and} \quad \hat{V}(A) := \hat{T}(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Observe that each $V_\ell(A) \simeq H^1(A(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell$ is a 1-dimensional $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-vector space, and from this it follows that $\hat{V}(A)$ and $\hat{V}(\sigma A_\Phi)$ are rank 1 free $\mathbb{A}_{E,f}$-modules. We denote also by $\alpha : \hat{V}(A) \to$

$\hat{V}(\sigma A_\Phi)$. Note that this α is well defined up to an element of $E^\times$. Finally, observe that σ induces an isomorphism (also denoted σ)

$$\sigma : \hat{V}(A) \to \hat{V}(\sigma A_\Phi).$$

Thus, we see that there exists some $\eta(\sigma) \in \mathbb{A}_{E,f}^\times / E^\times$ such that

$$\alpha(\eta(\sigma)\, x) \;=\; \sigma(x) \qquad \text{for all } x \in \hat{V}(A),$$

yielding a well-defined group homomorphism $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/E^*) \longrightarrow \mathbb{A}_{E,f}^\times/E^\times$, necessarily factoring through $\mathrm{Gal}(E^{*,\mathrm{ab}}/E^*)$ since the image is abelian. We can thus compare this map with the one from global class field theory:

Theorem 5.25 (Shimura–Taniyama Main Theorem of Complex Multiplication)**.** For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E^*)$, and any $s \in \mathbb{A}_{f,E^*}^\times$ with $\mathrm{Art}(s) = \sigma\big|_{E^{*,\mathrm{ab}}}$, there exists a unique E-isogeny $\alpha : A \to {}^\sigma A$ such that $\alpha(N_\Phi(s) \cdot x) = \sigma(x)$ for all $x \in \hat{V}(A)$.

Another way to see this is that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/E^*) \arrow[r, "\eta"] \arrow[d] & \mathbb{A}_{E,f}^\times/E^\times \\
\mathrm{Gal}(E^{*,\mathrm{ab}}/E^*) & \mathbb{A}_{E^*,f}^\times/(E^*)^\times \arrow[l, "\mathrm{art}"'] \arrow[u, "N_\Phi"']
\end{tikzcd}$$

We define the reflex norm N_Φ as follows. By Galois descent for vector spaces, there exists a unique $E \otimes_{\mathbb{Q}} E^*$-module V_Φ such that $V_\Phi \otimes_{E^*} \mathbb{C} \simeq \prod_{\varphi\in\Phi} \mathbb{C}_\varphi$. The reflex norm $N_\Phi : (E^*)^\times \to E^\times$ is defined by

$$a \longmapsto \det\bigl(a|_{V_\Phi}\bigr),$$

viewing V_Φ as a free E-module and $a|_{V_\Phi}$ as a linear transformation on this space.

We are now interested in determining what abelian varieties correspond to special points of the Siegel modular variety.

Definition 5.26 (CM-algebras and CM-type)**.** A *CM-algebra* is a finite product of CM-fields. An abelian variety A over $\mathbb{C}$ is *CM* (or *of CM-type*) if there exists a CM-algebra E and a homomorphism $E \longrightarrow \mathrm{End}^0(A)$ such that $H_1(A(\mathbb{C}), \mathbb{Q})$ is a free E-module of rank 1.

Let $E \to \mathrm{End}^0(A)$ be as in the definition, and write E as a product of CM-fields, $E = E_1 \times \cdots \times E_m$. Then A is isogenous to a product of abelian varieties $A_1 \times \cdots \times A_m$ with A_i of CM-type (E_i, Φ_i) for some Φ_i.

Recall that, for an abelian variety A over $\mathbb{C}$, there is a homomorphism $h_A : \mathbb{C}^\times \longrightarrow \mathrm{GL}\bigl(H_1(A(\mathbb{C}), \mathbb{R})\bigr)$ describing the natural complex structure on $H_1(A(\mathbb{C}), \mathbb{R})$

Proposition 5.27. An abelian variety A over $\mathbb{C}$ is CM if and only if there exists a torus $T \subset \mathrm{GL}\bigl(H_1(A(\mathbb{C}), \mathbb{Q})\bigr)$ such that $h_A(\mathbb{C}^\times) \subset T(\mathbb{R})$.

Proof. (Sketch, see [Mil05, Proposition 14.10] for full proof.) The statements depend only on A up to isogeny, and every abelian variety is isogenous to a product of simple abelian varieties. It follows that we may assume that A is simple.

Let A be an abelian variety such that $\mathrm{End}^0(A)$ contains a field E for which $H_1(A(\mathbb{C}), \mathbb{Q})$ has dimension 1 as an E-vector space. The action of $E \otimes \mathbb{R}$ on $H_1(A(\mathbb{C}), \mathbb{Q})$ preserves the Hodge structure, and so $h_A(\mathbb{C}^\times)$ commutes with $E \otimes \mathbb{R}$ in $\mathrm{End}\big(H_1(A(\mathbb{C}), \mathbb{R})\big)$. Therefore $h_A(\mathbb{C}^\times) \subset (E \otimes \mathbb{R})^\times = \big(\mathbb{G}_m\big)_{E/\mathbb{Q}}(\mathbb{R})$.

Conversely, if such a torus T exists, then recall that for any abelian variety, $\mathrm{End}^0(A)$ is the subalgebra of $\mathrm{End}\big(H_1(A(\mathbb{C}), \mathbb{Q})\big)$ of elements preserving the Hodge structure or, equivalently, that commute with $\mu_A(\mathbb{G}_m)$ in $\mathrm{GL}\big(H_1(A(\mathbb{C}), \mathbb{C})\big)$. By assumption, there is a torus $T \subset \mathrm{GL}\big(H_1(A(\mathbb{C}), \mathbb{Q})\big)$ such that $\mu_A(\mathbb{C}^\times) \subset T(\mathbb{C})$. Therefore $\mathrm{End}^0(A) \supset \{\alpha \in \mathrm{End}(H_1) \mid \alpha \text{ commutes with the action of } T\}$, and so $\mathrm{End}^0(A) \otimes \mathbb{C} \supset \{\alpha \in \mathrm{End}(H_1) \mid \alpha \text{ commutes with the action of } T\} \otimes \mathbb{C} = \{\alpha \in \mathrm{End}(H_1 \otimes \mathbb{C}) \mid \alpha \text{ commutes with the action of } T_{\mathbb{C}}\}$. Because T is a torus, $H_1(A(\mathbb{C}), \mathbb{C}) = \bigoplus_{\chi \in X^*(T)} H_\chi$, and so $\mathrm{End}_T(H_1 \otimes \mathbb{C})$ contains an étale $\mathbb{C}$-algebra of degree $2 \dim A$. It follows that $\mathrm{End}^0(A)$ does also. One can check that $E := \mathrm{End}^0(A)$ is a CM-field. □

Corollary 5.28. If $(A, \lambda, \eta K)$ maps to $[x, a]_K$ under $\mathcal{M}_K \to \mathrm{Sh}_K$, then A is CM if and only if x is special.

Proof. Recall that if $(A, \lambda, \eta K) \mapsto [x, a]_K$, then there exists an isomorphism $H_1(A(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} V$ sending h_A to h_x. Thus, the statement follows from Proposition 5.27. □

Theorem 5.29. Suppose that $(G, X) = (\mathrm{GSp}_{2g}, \mathcal{H}^\pm)$, and $K = K(N) \subset G(\mathbb{A}_f)$ for $N \geq 3$. Let $M_K(G, X)$ be the model for $\mathrm{Sh}_K(G, X)$ over $\mathbb{Q}$ we constructed. Then $M_K(G, X)$ is canonical.

Proof. We first show that the reflex field for $(\mathrm{GSp}_{2g}, \mathcal{H}^\pm)$ is $\mathbb{Q}$. To see this, observe that since GSp_{2g} is split over $\mathbb{Q}$, so there is a representative μ_h which is defined over $\mathbb{Q}$.

Thus, it remains to check that an abelian variety which is CM satisfies the condition that

$$\mathrm{art}_{E(\mu_h)}(s) \cdot [h, a]_K \;=\; [h,\, r_h(s)\, a]_K.$$

For a special point $[x, a]_K$ corresponding to an abelian variety A with complex multiplication by a field E, this is an immediate consequence of the main theorem of complex multiplication, since $E^* = E(\mu_h)$. If A is a product of such abelian varieties, then this also follows. Otherwise, we replace A by an isogenous one which is a product, and observe that this isogeny can be represented on the rational adelic Tate modules by an element in $T(\mathbb{A}_f)$ for the torus T as in 5.27 (since the map respects the E-structure on both sides), which we note obviously commutes with the image of $r_h(s)$. □

Remark 5.30. What about the case beyond Hodge type Shimura varieties (including in particular those for the exceptional groups E_6, E_7?) This was settled by Milne in [Mil83]. We sketch the rough idea here. Firstly, from Theorem 5.6 and the proof, we see that we need to show:

(a) Sh_K is quasi-projective. This is known from the Baily-Borel compactification

(b) $f_\sigma : \sigma\mathrm{Sh}_K \to \mathrm{Sh}_K$ isomorphisms.

(c) A finite set of points, defined over $\mathbb{Q}$, such that the set of automorphisms of Sh_K which fixes these points is simply the identity, and such that for some finitely generated subfield K of $\mathbb{C}$, we have that $\mathrm{Aut}(\mathbb{C}/K)$ fix these points. As we have seen, these should be a finite set of special points.

The difficulty outside the case of Hodge types lies even in the construction of the isomorphism f_σ in these cases, as we do not have an explicit moduli problem to work with. Instead, Milne's approach to this problem was via a conjecture of Langlands on a construction of an isomorphism

$$g_\sigma : \sigma\mathrm{Sh}(G, X) \to \mathrm{Sh}(G^\sigma, X^\sigma),$$

where G^σ is an (explicitly constructed) connected reductive group over $\mathbb{Q}$ such that $G^\sigma(\mathbb{A}_f) \simeq G(\mathbb{A}_f)$. Assuming the existence of this isomorphism g_σ, we see that we can construct f_σ as the composition of the isomorphisms

$$\sigma\mathrm{Sh}(G, X) \xrightarrow{g_\sigma} \mathrm{Sh}(G^\sigma, X^\sigma) \simeq \mathrm{Sh}(G, X).$$

Part of Langlands' conjecture prescribes how special points behave under the isomorphism, so that this map does have the desired property that the model constructed will be canonical.

Roughly speaking the existence of f_σ comes from a *rigidity property* of Shimura varieties (actually really of Hermitian symmetric domains), but we also need a deep result from arithmetic dynamics, namely Margulis superrigidity, which allows us to see that $\sigma\mathrm{Sh}(G, X)$ also has to be a Shimura variety.

Finally, checking the property for special points can reduced to looking at Shimura subvarieties given by embeddings of the group $\mathrm{SL}_2 \hookrightarrow G$.

6 p-divisible groups

We now turn our attention to understanding deformations of abelian schemes, as this will be necessary to understand integral models. We fix a base scheme $S = \operatorname{Spec} R$.

Definition 6.1. A finite flat group scheme G over R is an affine group scheme, represented by a finite flat R-algebra A. The order of G is the locally constant function $\#G$: that is, at a point $\mathfrak{p}$ by the rank of the free $R_{\mathfrak{p}}$-module $A_{\mathfrak{p}}$.

Example 6.2. Here are some examples.

1. The group μ_n is represented by $R[X]/(X^n - 1)$ with comultiplication, unit and inverse given by the maps
$$\Delta(X) = X \otimes X, \qquad \varepsilon(X) = 1, \qquad S(X) = X^{-1}.$$
Thus μ_n is finite flat of order n. In fact, the representing algebra is even a free R-module of rank n.

2. If G is a finite group, the constant group scheme $\underline{G}$ over R is represented by $\prod_{g \in G} R$ and hence finite flat of order the order of G.

3. In characteristic $p > 0$ the affine group scheme α_p is represented by $R[X]/(X^p = 0)$ with
$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \varepsilon(X) = 0, \qquad S(X) = -X.$$
Hence α_p is finite flat of order p.

4. An isogeny is a group homomorphism $\varphi\colon G \to H$ which corresponds to a finite flat map $\varphi^*\colon A \to B$ of the corresponding representing R-algebras. The kernel of an isogeny is a finite flat group scheme, as $B \otimes_A R$ is finite and flat over R by the preservation of the properties finite and flat under base change.

Definition 6.3. A p-divisible group of height h over R is an inductive system $G = \big(\{G_v\}, G_v \xrightarrow{i_v} G_{v+1}\big)_{v \geq 1}$ of group schemes over R satisfying the following:

(a) G_v is a finite flat group schemes over S of order p^{hv}.

(b) The sequence $0 \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p]} G_v \longrightarrow 0$ is exact (i.e., $i_v(G_v) = \ker([p]\colon G_{v+1} \to G_v)$ and $[p]$ is surjective).

A homomorphism of p-divisible groups $f\colon G = (G_v, i_v) \longrightarrow H = (H_v, i'_v)$ is a system of homomorphisms of group schemes $f_v\colon G_v \to H_v$ which are compatible with the structure maps: $i'_v \circ f_v = f_{v+1} \circ i_v$.

Let $i_{v,m}\colon G_v \to G_{v+m}$ denote the closed immersion $i_{v,m} = i_{v+m-1} \circ i_{v+m-2} \circ \cdots \circ i_{v+1} \circ i_v$. One can check that the sequence $0 \longrightarrow G_m \xrightarrow{i_{v,m}} G_{v+m} \xrightarrow{j_{m,v}} G_v \longrightarrow 0$ is exact (i.e. the closed immersion $\operatorname{coker}(i_{v,m}) \to G_v$ is an isomorphism).

Example 6.4. (a) The simplest p-divisible group over S of height h is the constant group:

$$(\mathbb{Q}_p/\mathbb{Z}_p)^h \;=\; \big((\mathbb{Z}/p^v\mathbb{Z})^h \xrightarrow{i_v} (\mathbb{Z}/p^{v+1}\mathbb{Z})^h\big)_{v\geq 1}.$$

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