SEMISIMPLICITY OF ÉTALE COHOMOLOGY OF CERTAIN SHIMURA VARIETIES

SI YING LEE

ABSTRACT. Building on work of Fayad and Nekovář, we show that a certain part of the etale cohomology of some abelian-type Shimura varieties is semisimple, assuming the associated automorphic Galois representations exists, and satisfies some good properties. The proof combines an abstract semisimplicity criterion of Fayad-Nekovář with the Eichler-Shimura relations.

1. INTRODUCTION

Let G be a reductive group over $\mathbb{Q}$, and X be a conjugacy class of homomorphisms

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$$

such that (G, X) is a Shimura datumn. Given a compact open subgroup $K \subset G(\mathbb{A}_f)$, we can form the associated Shimura variety

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K,$$

which is a complex manifold for K small enough, and has a canonical model over E, for some number field E.

Consider a representation valued in complex vector spaces

$$\xi : G_{\mathbb{C}} \to GL(V_\xi)$$

such that $\xi(Z(\mathbb{Q}) \cap K) = 1$, where Z is the center of G. This gives rise to a locally constant sheaf of complex vector spaces

$$\mathcal{L}_\xi = G(\mathbb{Q}) \backslash V_\xi \times X \times G(\mathbb{A}_f)/K.$$

We now assume that G^{der} is anisotropic, so that the Shimura variety $\mathrm{Sh}_K(G, X)$ is compact. Similar results should hold with étale cohomology replaced with intersection cohomology of the Baily-Borel compactification $\mathrm{Sh}_K^{BB}(G, X)$ of the Shimura variety.

We now consider the complex analytic cohomology of the tower of the Shimura variety, i.e.

$$H^i(Sh(G, X)^{an}, \mathcal{L}_\xi) := \varinjlim_{K} H^i(Sh_K(G, X)^{an}, \mathcal{L}_\xi).$$

Choose $h \in X$, and let K_∞ be the stabilizer of h in $G(\mathbb{R})$. By Matsushima's formula, we have a decomposition of $H^i(Sh(G, X)^{an}, \mathcal{L}_\xi)$ in terms of Lie algebra cohomology

$$H^i(Sh(G, X)^{an}, \mathcal{L}_\xi) = \bigoplus_{\pi = \pi_\infty \otimes \pi^\infty} m(\pi) H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi) \otimes (\pi^\infty),$$

where π runs through unitary automorphic representations of $G(\mathbb{A})$, and $m(\pi)$ is the multiplicity of π appearing in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)$, the space of measurable functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ which are square-integrable modulo center, where ω is the central character of π. Recall that π_∞ is

cohomological in degree i for ξ (i.e. $H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi) \neq 0$) if and only if the central character ω_ξ of ξ satisfies $\omega_\xi|_{Z(\mathbb{R})} = \omega_{\pi_\infty}^{-1}$.

Fix a prime l, and an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. The representation ξ gives rise to an l-adic automorphic sheaf, in the following way. ξ gives rise to a representation valued in $\overline{\mathbb{Q}}_l$-vector spaces

$$\xi_l : G_{\overline{\mathbb{Q}}_l} \to GL(V_{\xi,l}),$$

and similarly gives rise to a sheaf

$$\mathcal{L}_{\xi,l} = G(\mathbb{Q}) \backslash V_{\xi,l} \times X \times G(\mathbb{A}_f)/K.$$

If we consider the étale cohomology of the tower

$$H^i_{\text{ét}}(Sh(G,X)_{\overline{\mathbb{Q}}}, \mathcal{L}_{\xi,l}) := \varinjlim_{K} H^i_{\text{ét}}(Sh_K(G,X)_{\overline{\mathbb{Q}}}, \mathcal{L}_{\xi,l}),$$

then we have an isomorphism

$$H^i_{\text{ét}}(Sh(G,X)_{\overline{\mathbb{Q}}}, \mathcal{L}_{\xi,l}) \simeq H^i(Sh(G,X)^{an}, \mathcal{L}_\xi) \otimes_\iota \overline{\mathbb{Q}}_l,$$

which is $G(\mathbb{A}_f)$-equivariant, and thus we have a decomposition

$$H^i_{\text{ét}}(Sh(G,X)_{\overline{\mathbb{Q}}}, \mathcal{L}_{\xi,l}) = \bigoplus_{\pi^\infty} V^i(\pi^\infty) \otimes (\pi^\infty),$$

where $V^i(\pi^\infty) = \text{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, H^i_{\text{ét}}(Sh(G,X)_{\overline{\mathbb{Q}}}, \mathcal{L}_{\xi,l}))$. We define the Galois representation

$$\rho : \text{Gal}(\bar{E}/E) \to V^i(\pi^\infty).$$

The global Langlands correspondence conjectures that to the automorphic representation π we have an associated semisimple Galois representation $\text{Gal}(\bar{E}/E) \to {}^L G$, and we denote the composition

$$\tilde{\rho} : \text{Gal}(\bar{E}/E) \to {}^L G \xrightarrow{r_{-\mu}} GL(V_\mu),$$

where μ is the minuscule cocharacter associated to the Shimura datumn.

When the group G is of the form $\text{Res}_{F/\mathbb{Q}} G'$ for some connected reductive group G', and F is a totally real field of degree d, one a decomposition (perhaps after passing to a finite extension E' of E) of $\tilde{\rho}$ as

$$\tilde{\rho} = \bigotimes_{v|\infty} \tilde{\rho}_v.$$

Moreover, $\tilde{\rho}_v$ should have the following form. Observe that over $\mathbb{C}$, we have a decomposition

$$\text{Res}_{F/\mathbb{Q}} G' \simeq \prod_{v|\infty} G',$$

and the cocharacter μ also decomposes as $\prod_v \mu_v$. Then we should have

$$\tilde{\rho}_v : \text{Gal}(\bar{E}'/E') \xrightarrow{\phi_\pi} {}^L G' \xrightarrow{r_{-\mu_v}} GL(V_{\mu_v}),$$

where here we view π as an automorphic representation of G' over F.

The main theorem of this paper is the following:

Theorem 1.0.1. *Let (G,X) be a Shimura datumn of abelian type such that $G = \text{Res}_{F/\mathbb{Q}} G'$ for some connected reductive group G, and totally real number field F. Let π be an automorphic*

representation of $G(\mathbb{A}_{\mathbb{Q},f}) = G'(\mathbb{A}_{F,f})$. For all v, suppose that the Galois representation associated to π exists, and is given by $\tilde{\rho}_v : \operatorname{Gal}(\bar{E}/E) \to {}^L G' \to GL(V_{\mu_v})$. Suppose that moreover we also know that

- *(1) $\tilde{\rho}_v$ is strongly irreducible*
- *(2) For all primes v' of E such that $v'|l$, the Hodge-Tate weights of $\tilde{\rho}$, viewed as a $\operatorname{Gal}(\bar{E}_{v'}/E_{v'})$-representation, are distinct.*

Then ρ is a semisimple representation.

To show this result, we first define partial Frobenius isogeny at primes p which satisfy condition A, and then show the Eichler-Shimura congruence relations for partial Frobenius for split groups, using results from [Lee20]. More precisely, we show the following:

Theorem 1.0.2. *Let (G, X) be a Shimura datumn of abelian type, such that $G = \operatorname{Res}_{F/\mathbb{Q}} G'$ for some connected reductive group G, and totally real number field F of degree d. Let p be a prime satisfying the conditions in Proposition 2.9.4. Then for all $i = 1, \ldots, d$ we have a partial Frobenius correspondence $\operatorname{Frob}_{\mathfrak{p}_i}$ such that*

$$\operatorname{Frob} = \prod_i \operatorname{Frob}_{\mathfrak{p}_i}$$

and

$$H_i(\operatorname{Frob}_{\mathfrak{p}_i}) = 0,$$

where H_i is the renormalized characteristic polynomial of the irreducible representation of $\hat{G}'_i$ with highest weight $\hat{\mu}_i$.

This, combined with a semisimplicity criterion for Lie algebras shown in [FN19], allows us to deduce the main result. In the final section, we apply this result to some abelian-type Shimura varieties attached to similitude groups, following the construction of the automorphic Galois representations in [KS20].

Acknowledgements. Many thanks to my advisor, Mark Kisin, for introducing this problem to me, and for various helpful conversations. Thanks also to Sug Woo Shin, for suggestions about applications in Section 4.

2. EICHLER-SHIMURA RELATIONS

2.1. **p-divisible groups.** For the entirety of this subsection, we fix a prime $p > 2$, and let G be a connected reductive group over $\mathbb{Q}_p$. Let k be a perfect field of characteristic p. We denote by $L = W(\bar{\mathbb{F}}_p)[1/p]$ the maximal unramified extension of $\mathbb{Q}_p$.

Definition 2.1.1. A p-divisible group with G structure over k consists of a p-divisible group $\mathscr{G}/k$ and a collection of φ-invariant tensors $(s_{\alpha,0})$ which define a reductive subgroup of $GL(\mathbb{D}(\mathscr{G}))$ such that there exists a finite free $\mathbb{Z}_p$-module U and an isomorphism

$$(2.1.2) \qquad\qquad U \otimes_{\mathbb{Z}_p} W(k) \xrightarrow{\sim} \mathbb{D}(\mathscr{G})(W(k))$$

such that under this isomorphism $(s_{\alpha,0})$ correspond to tensors $(s_\alpha) \subset U^\otimes$. Moreover, these s_α define the reductive subgroup $G_{\mathbb{Z}_p} \subset GL(U)$.

Associated to any p-divisible group $\mathscr{G}$ with G-structure over $\bar{\mathbb{F}}_p$, we have a $G(W(\bar{\mathbb{F}}_p))$-σ-conjugacy class of elements $b \in G(L)$, such that under the isomorphism (2.1.2) the Frobenius on $\mathbb{D}(\mathscr{G})(W(\bar{\mathbb{F}}_p))$ is given by $b\sigma$.

Let μ be the minuscule cocharacter of G such that b lies in $G(W(\bar{\mathbb{F}}_p))p^\mu G(W(\bar{\mathbb{F}}_p))$ under the Cartan decomposition.

We now define the Rapoport-Zink space for the triple (G, b, μ) following [Kim18, Def. 4.6].

Definition 2.1.3. Let $\mathrm{RZ}(G, b, \mu)$ be the functor which assigns to any p-locally nilpotent smooth W-algebra R the set of isomorphism classes $((X, \rho, t_\alpha)$ such that

(1) (X, t_α) is a p-divisible group over R with tensors $t_\alpha \subset \mathbb{D}(X)^\otimes$, where (t_α) consists of morphisms of crystals $t_\alpha : 1 \to \mathbb{D}(X)^\otimes$ over $\mathrm{Spec}(R)$ such that

$$t_\alpha : 1[1/p] \to \mathbb{D}(X)^\otimes[1/p]$$

is Frobenius equivariant;

(2) $\rho : \mathbb{X}_{R/p} \to X_{R/p}$ is a quasi isogeny;

(3) For some nilpotent ideal $J \subset R$ containing (p), the pull-back of t_α over $\mathrm{Spec}(R/J)$ is identified with s_α under the isomorphism of isocrystals induced by ρ:

$$\mathbb{D}(X_{R/J})[1/p] \xrightarrow{\sim} \mathbb{D}(_{0,R/J})[1/p].$$

(4) For some (any) formally smooth p-adic W-lift $\tilde{R}$ of R, endowed with the standard PD-structure on $\ker(\tilde{R} \to R) = p^m \tilde{R}$ for some m, let $(t_\alpha(\tilde{R}))$ denote the $\tilde{R}$-section of (t_α). Then the $\tilde{R}$-scheme

$$P(\tilde{R}) := \mathrm{Isom}_{\tilde{R}}([\mathbb{D}(X)_{\tilde{R}}), (t_\alpha(\tilde{R}))], [\tilde{R} \otimes_{\mathbb{Z}_p} \Lambda^*, (1 \otimes s_\alpha)]),$$

classifying isomorphisms matching $(t_\alpha(\tilde{R}))$ and $(1 \otimes s_\alpha)$, is a G_W-torsor.

(5) The Hodge filtration $\mathrm{Fil}^1(X) \subset \mathbb{D}(X)(R)$ is a $\{\mu\}$-filtration with respect to $(t_\alpha(R)) \subset \mathbb{D}(X)(R)^\otimes$, where $\{\mu\}$ is the unique $G(W)$-conjugacy class of cocharacters such that $b \in G(W)p^v G(W)$.

If the group G admits a decomposition over $\mathbb{Q}_p$ as $G = G_1 \times G_2$, then the associated Rapoport-Zink spaces also decompose, as the following proposition [Kim18, Thm 4.9.1] shows:

Proposition 2.1.4. *Let $b = (b_1, b_2) \in G_1(L) \times G_2(L)$, and $\mu = \mu_1 \times \mu_2$. Then we have an isomorphism*

$$\mathrm{RZ}(G_1, b_1, \mu_1) \times_{\mathrm{Spf}W} \mathrm{RZ}(G_2, b_2, \mu_2) \xrightarrow{\sim} \mathrm{RZ}(G, b, \mu)$$

induced by taking product of p-divisible groups, and isogenies.

2.2. **Partial Frobenius for Hodge type.** Consider the situation where G' is a connected reductive group over $\mathbb{Q}$, and $G = \mathrm{Res}_{F/\mathbb{Q}} G'$, where F is a totally real field of degree r over $\mathbb{Q}$. Suppose that p is a prime which satisfies the following criterion:

(1) p splits in F

(2) The group G has good reduction at p

(3) The group $G_{\mathbb{Q}_p}$ is split.

Observe that since G is a reductive group, there is some finite extension K of $\mathbb{Q}$ over which the group G splits. For any prime p which splits in K, the group $G_{\mathbb{Q}_p}$ is split. Thus we see that there is a positive density of primes which satisfies the above three criterion.

We can define the partial Frobenius as follows. Observe that since p splits in F, we see that

$$G_{\mathbb{Q}_p} \simeq \prod G'_{i,\mathbb{Q}_p},$$

and since the group $G_{\mathbb{Q}_p}$ is split, so too are each of the factors G'_i.

The decomposition of the group G induces a decomposition of the Rapoport-Zink space. We thus have an isomorphism of Rapoport-Zink spaces

$$\mathrm{RZ}(G, b) = \prod_i \mathrm{RZ}(G'_i, b_i),$$

and given an isogeny f over a characteristic p ring R, we can decompose the isogeny $f = \prod f_i$ for some $f_i \in \mathrm{RZ}(G_i, b_i)(R)$.

In particular, we can apply the above construction to the Frobenius isogeny, to get quasi-isogenies $\mathrm{Frob}_{\mathfrak{p}_i}$ for $i = 1, \ldots, r$ and by construction we have the following relationship between the partial Frobenius and the actual Frobenius

$$\mathrm{Frob}_p = \prod_i \mathrm{Frob}_{\mathfrak{p}_i}$$

Remark 2.2.1. The construction of the partial Frobenius works just assuming conditions (1) and (2), but this is not necessary for the rest of the paper.

2.3. **The moduli space** $p - \mathrm{Isog}$. We now suppose that the Shimura variety is of Hodge type, and recall the constructions in [Lee20] of the associated moduli space $p - \mathrm{Isog}$.

Let T be a scheme over $O_{E,(v)}$, and consider any two points x, y lying in $\mathscr{S}_K(G, S)(T)$. For any geometric point t of T, let x_t, y_t be the pullback of x, y to t. From the main construction in [Kis10], we have l-adic étale and de Rham tensors $(s_{x_t,\alpha,l})$ for $l \neq p$, and $(s_{x_t,\alpha,dR})$ (respectively $(s_{y_t,\alpha,l}), (s_{y_t,\alpha,dR})$ for y_t). Observe that $k(t)$, the residue field at t, could be of either characteristic 0 or characteristic p. Suppose $k(t)$ is a field of characteristic 0, i.e. it is an extension of E. Then, we also have p-adic étale tensors $(s_{x_t,\alpha,p}), (s_{y_t,\alpha,p})$. Otherwise, if $k(t)$ is of characteristic p, it is an extension of κ. Similarly, we have crystalline tensors $(s_{x_t,\alpha,0}), (s_{y_t,\alpha,0})$.

We define a quasi-isogeny between x, y to be a quasi-isogeny $f : \mathcal{A}_x \to \mathcal{A}_y$ of abelian schemes over T, such that for any geometric point t, the induced quasi-isogeny $f_t : \mathcal{A}_{x_t} \to A_{y_t}$ of abelian varieties over $k(t)$ preserves all the tensors described above.

We define a p-quasi-isogeny between x, y to be a quasi-isogeny as defined above, such that the isomorphism on the rational prime-to-p Tate modules $f : \hat{V}^p(\mathcal{A}_x)_{\mathbb{Q}} \xrightarrow{\sim} \hat{V}^p(\mathcal{A}_y)_{\mathbb{Q}}$, induced by the quasi-isogeny $\mathcal{A}_x \to \mathcal{A}_y$, respects the prime to p-level structures $\varepsilon_x^p, \varepsilon_y^p$, i.e. ε_y^p is given by the composition

$$V_{\mathbb{A}_p^f} \xrightarrow[\varepsilon_x^p]{\sim} \hat{V}^p(\mathcal{A}_x)_{\mathbb{Q}} \xrightarrow[f]{\sim} \hat{V}^p(\mathcal{A}_y)_{\mathbb{Q}}.$$

In particular, we see that the weak polarizations on $\mathcal{A}_x, \mathcal{A}_y$ differ by some power of p.

Let $p - \mathrm{Isog}$ be the *fppf*-sheaf of groupoids of p-quasi-isogenies between points on $\mathscr{S}_{K_p}(G, X)$. Concretely, for any $O_{E,(v)}$-scheme T, points of $p - \mathrm{Isog}$ are pairs (x, f), where $x \in \mathscr{S}_{K_p}(G, X)(T)$, and f is a p-quasi-isogeny $f : \mathcal{A}_x \to \mathcal{A}_y$, where $y \in \mathscr{S}_{K_p}(G, X)(T)$. For $K^p \subseteq G(\mathbb{A}_p^f)$, we can define $p - \mathrm{Isog}_{K^p}$ in a similar way, by setting $K = K^p K_p$ and considering p-quasi-isogenies between points on $\mathscr{S}_K(G, X)$ instead. For small enough K^p such that $\mathscr{S}_K(G, X)$ is a scheme, $p - \mathrm{Isog}_{K^p}$ is in fact also a scheme over $O_{E,(v)}$. In the following, we always assume sufficient level structure K^p such that $p - \mathrm{Isog}_{K^p}$ is a scheme, and for notational simplicity we will simply denote this by $p - \mathrm{Isog}$.

We have can define projection maps back to $\mathscr{S}_{K_p}(G, X)$ sending a p-quasi isogeny (x, f) to x (respectively y)

$$s : p - \text{Isog} \to \mathscr{S}_{K_p}(G, X) \qquad t : p - \text{Isog} \to \mathscr{S}_{K_p}(G, X).$$

These maps s, t are proper, and surjective.

Consider the closure $\mathscr{J}$ of the generic fiber $p - \text{Isog} \otimes E$ in $p - \text{Isog}$. We abuse notation and still denote the special fiber of $\mathscr{J}$ by $p - \text{Isog} \otimes \kappa$, and the $\mathbb{Q}$-vector space of irreducible components by $\mathbb{Q}[p - \text{Isog} \otimes \kappa]$. Since $\mathscr{J}$ is flat over $O_{E,(v)}$, Irreducible components of $p - \text{Isog} \otimes \kappa$ are hence of dimension $2\langle \rho, \mu \rangle$.

We now consider the μ-ordinary locus $p - \text{Isog}^{ord} \otimes \kappa$. This is the subspace of $p - \text{Isog} \otimes \kappa$ which maps to the μ-ordinary locus under the map s (equivalently, t). The following argument can be extracted from [Lee20]:

Proposition 2.3.1. *When G is split over $\mathbb{Q}_p$, the μ-ordinary locus is dense in $p - \text{Isog} \otimes \kappa$*

Proof. The discussion in [Lee20] shows that for any irreducible component in $p - \text{Isog} \otimes \kappa$, it has a dense open subset which corresponds to the Newton strata for some unramified $[b] \in B(G, v)$. If G is split over $\mathbb{Q}_p$, then the only unramified element in $B(G, v)$ is the μ-ordinary σ-conjugacy class, since we have an isomorphism $B(G, v) \simeq B(G^{ad}, v^{ad})$, which maps unramified elements to each other. From [XZ17, 4.2.11], we see that the identity element represents the basic element in $B(M_{ad}, \mu_{ad})$ if and only if $[1] \in B(M_{ad}, \mu_{ad})$, i.e. $\mu_{ad} = 0$ in $\pi_1(M_{ad})_\Gamma$. If M_{ad} is split, then $\mu_{ad} = 0$ in $\pi_1(M_{ad})_\Gamma$ implies that μ_{ad} is the sum of coroots of M_{ad}, and hence μ_{ad} is either the identity or it cannot be minuscule. $\square$

2.4. **Abstract Eichler-Shimura Relations.** We have isomorphisms of Hecke algebras

$$\mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \simeq \bigotimes_i \mathcal{H}(G_i(\mathbb{Q}_p)//K_p, \mathbb{Q}).$$

Similarly, since $G_{\mathbb{Q}_p}$ admits a decomposition, we can write

$$(2.4.1) \qquad\qquad\qquad\qquad \mu = \prod_i \mu_i$$

where μ_i is a minuscule cocharacter of G_i. If we let M be the centralizer of μ in G, then similarly we also have

$$M_{\mathbb{Q}_p} = \prod_i M_i$$

where M_i is the centralizer of μ_i in M. Thus, we also have an isomorphism of Hecke algebras

$$(2.4.2) \qquad\qquad \mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q}) \simeq \bigotimes_i \mathcal{H}(M_i(\mathbb{Q}_p)//M_p, \mathbb{Q}).$$

For a quasi-split reductive group G with standard parabolic subgroup P and Levi subgroup M, we can define following algebra homomorphism, known as the twisted Satake homomorphism

$$\dot{\mathcal{S}}_M^G : \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \to \mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q}),$$

defined as follows. Write $P = NM$, for N the unipotent radical of P, and given a function $f \in \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q})$, we have

$$\dot{\mathcal{S}}_M^G(f)(m) = \int_{n \in N} f(nm) dn.$$

The twisted Satake isomorphism also factors: we have an isomorphism

$$\dot{\mathcal{S}}_M^G = \bigotimes_i \dot{\mathcal{S}}_{M_i}^{G_i}.$$

Consider now the representation $\rho_i : \hat{G} \to GL(V_{\mu_i})$ of $\hat{G}$ with highest weight cocharacter $(1, \ldots, \mu_i, \ldots, 1)$, where μ_i is in the i-th position. Observe that $\rho_\mu = \otimes_i \rho_{\mu_i}$. Define the polynomial

$$H_i(x) \in \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q})(x)$$

as the polynomial given by

(2.4.3) $$H_i(x) = \det(x - p^{nr}\rho_{\mu_i}(\sigma \ltimes \hat{g})).$$

Note that since μ is central in M, μ_i is also central in M, hence we can consider the element $1_{\mu_i(p)M_c} \in \mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q})$.

Proposition 2.4.4. *We have the following equality in $\mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q})$: for all i,*

$$H_i(1_{\mu_i(p)M_c}) = 0.$$

Proof. This follows from the same proof as [BÖ2, Prop 3.4], and the observation that under the decomposition (2.4.2), we see that $H_i(1_{\mu_i(p)M_c})$ corresponds to the polynomial with coefficients in $\mathcal{H}(M_i(\mathbb{Q}_p)//M_{i,c}, \mathbb{Q})$ defined similarly as in (2.4.3), where we instead take the determinant of highest weight representation of $\hat{G}_i$ corresponding to μ_i. $\square$

2.5. **Newton stratification.** From now on, we will drop the assumption that the Shimura datumn is of Hodge type, and consider general abelian type Shimura datum. For any abelian type Shimura datumn, we will let (G_1, X_1) denote the Hodge type Shimura datumn such that there exists a central isogeny

$$f : G_1^{der} \to G^{der}$$

which induces an isomorphism $(G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})$.

We now recall the construction of Newton strata for Shimura varieties of abelian type, as constructed in [SZ17]. Observe that for any connected reductive group G, and minuscule cocharacter v, we have an isomorphism $B(G, v) = B(G^{ad}, v^{ad})$. In [SZ17], the Newton strata is first constructed for adjoint groups, and thus we have a stratification on $\mathscr{S}_{K_p^{ad}}(G^{ad}, X^{ad})$. The Newton strata for $\mathscr{S}_{K_p}(G, X)$ is then defined to be the pullback of the Newton strata for $\mathscr{S}_{K_p^{ad}}(G^{ad}, X^{ad})$ via the natural map

$$\mathscr{S}_{K_p}(G, X) \to \mathscr{S}_{K_p^{ad}}(G^{ad}, X^{ad}).$$

Fix a connected component X^+ of X, and a connected component X_1^+ of X_1, such that their images in X^{ad} are equal to some connected component $X^{ad,+}$. Let $\mathrm{Sh}_{K_p}(G, X)^+$ denote the connected component of $\mathrm{Sh}_{K_p}(G, X)$ containing $\{1\} \times X^+$, and similarly let $\mathrm{Sh}_{K_{1,p}}(G_1, X_1)^+$ denote the connected component of $\mathrm{Sh}_{K_{1,p}}(G_1, X_1)$ containing $\{1\} \times X_1^+$.

We observe that the Newton strata of $S_{K_p}(G, X)^+$ and $S_{K_{1,p}}(G_1, X_1)^+$ is exactly that pulled back along the maps

$$S_{K_{1,p}}(G_1, X_1)^+ \to S_{K_p}(G, X)^+ \to S_{K_p^{ad}}(G^{ad}, X^{ad})^+$$

in particular, we see that the μ_1-ordinary locus of $S_{K_{1,p}}(G_1, X_1)^+$ is exactly the preimage of the μ-ordinary locus of $S_{K_p}(G, X)^+$.

2.6. **Model for the Hecke correspondences.** For general abelian type Shimura varieties, we do not have a moduli interpretation in terms of abelian varieties, and thus we do not have the general formalism of $p-\mathrm{Isog}$. However, we can still define models for the Hecke correspondences, defined as follows.

Consider the Hecke correspondence $C \subset \mathrm{Sh}_K(G, X) \times \mathrm{Sh}_K(G, X)$ given by $1_{K_p g K_p}$ on the generic fiber, and let $\mathscr{C}$ be the closure of C in $\mathscr{S}_K(G, X) \times \mathscr{S}_K(G, X)$. For any correspondence $\mathscr{C}$ over $\mathscr{S}_K(G, X)$, we will let C_0 denote the special fiber, which is a correspondence over $\mathscr{S}_K(G, X)_\kappa$. We have a similar construction for Hecke correspondences for the groups G_1, G^{ad}. Observe that the Hecke operators for G, G^{ad} are related as follows. Let g^{ad} denote the image of $g \in G(\mathbb{Q}_p)$ in G^{ad}. $C^{ad} \subset \mathrm{Sh}_{K^{ad}}(G^{ad}, X^{ad}) \times \mathrm{Sh}_{K^{ad}}(G^{ad}, X^{ad})$ given by $1_{K_p^{ad} g^{ad} K_p^{ad}}$ on the generic fiber. Let $\mathscr{C}$ be the closure of C^{ad} in $\mathscr{S}_{K^{ad}}(G^{ad}, X^{ad}) \times \mathscr{S}_{K^{ad}}(G^{ad}, X^{ad})$, and let C_0^{ad} denote the special fiber.

Our key observation is the following: C^{ad} is the image of C under the (finite) projection maps

$$\mathrm{Sh}_K(G, X) \times \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K^{ad}}(G^{ad}, X^{ad}) \times \mathrm{Sh}_{K^{ad}}(G^{ad}, X^{ad}).$$

Thus, we see that C_0^{ad} is the image of the projection of C_0 to a correspondence on $\mathscr{S}_{K^{ad}}(G^{ad}, X^{ad})_\kappa$.

We can also define the μ-ordinary locus C_0^{ord} of C_0 to be the subspace of C_0 which maps to the μ-ordinary locus under the natural projection maps to $\mathscr{S}_K(G, X)$. The discussion of Newton strata above shows that $C_0^{ad,ord}$ is the image of the projection to $\mathscr{S}_{K^{ad}}(G^{ad}, X^{ad})_\kappa^{ord} \times \mathscr{S}_{K^{ad}}(G^{ad}, X^{ad})_\kappa^{ord}$ of C_0^{ord}, and also $C_{1,0}^{ord}$.

Proposition 2.6.1. *Let G be split over $\mathbb{Q}_p$, and suppose $g \in G(\mathbb{Q}_p)$ is such that there exists $g_1 \in G_1(\mathbb{Q}_p)$ such that $g^{ad} = g_1^{ad} \in G^{ad}(\mathbb{Q}_p)$. Consider the correspondence C associated with $1_{K_p g K_p}$ as above. Then we have C_0 has a dense μ-ordinary locus.*

Proof. Note that since we have an isomorphism $B(G, \mu) \simeq B(G^{ad}, \mu^{ad})$, the condition that G is split implies that we also have exactly one unramified σ-conjugacy class in $B(G_1, \mu_1)$, and thus $p - \mathrm{Isog}(G_1, X_1) \otimes \kappa$ has a dense μ-ordinary locus, by Proposition 2.3.1.

By the above arguments, we know that the Hecke correspondence C_0^{ad} has a dense μ-ordinary locus, since it is the image under a finite map of the Hecke correspondence $C_{1,0}$ on $\mathrm{Sh}_{K_1}(G_1, X_1)$, which will have a dense μ-ordinary locus since $p - \mathrm{Isog}(G_1, X_1) \otimes \kappa$ has a dense μ-ordinary locus.

Thus, since the image of C_0 under a finite map to $\mathscr{S}_{K^{ad}}(G^{ad}, X^{ad})_\kappa \times \mathscr{S}_{K^{ad}}(G^{ad}, X^{ad})_\kappa$ has a dense μ-ordinary locus, the original Hecke correspondence must C_0 must have a dense μ-ordinary locus as well. $\qquad\square$

Observe that from the definition of the $H_{G,X}(t)$ that the Hecke correspondences which appear as coefficients in $H_{G,X}(t)$ are closed subschemes of Hecke correspondences lying in the subring R of $\mathcal{H}(G(\mathbb{Q}_p)//K_p)$ generated by $1_{K_p \mu(p) K_p}$. Let μ_1 be the cocharacter of $G_{1,\mathbb{Q}_p}$ associated to X_1. Observe that we have $\mu^{ad} = \mu_1^{ad}$ when after projecting to cocharacters of G^{ad}. Thus, if the group G is split, then the proposition holds for the coefficients of the Hecke polynomial.

2.7. **Canonical liftings of μ-ordinary points.** Consider now any μ-ordinary point $x \in \mathscr{S}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$. We suppose now that the point x lies in $S_{K_p}(G, X)^+$. Note that this is always possible up to the action of some $g^p \in G(\mathbb{A}_f^p)$, since by [Kis10, 2.2.5] $G(\mathbb{A}_f^p)$ acts transitively on $S_{K_p}(G, X)$.

We suppose that x is the image of some μ-ordinary point $x_1 \in \mathscr{S}_{K_{1,p}}(G_1, X_1)^+(\overline{\mathbb{F}}_p)$. We know from [SZ16] that for every ordinary point in $\mathscr{S}_{K_{1,p}}(G_1, X_1)^+(\overline{\mathbb{F}}_p)$, there exists a special point lifting $\tilde{x}_1$, with associated cocharacter $\mu_{\tilde{x}_1}$ satisfying $\mu_{\tilde{x}_1, \mathbb{Q}_p} = \mu_1$.

Note that since the map $X \to X^{ad}$ is injective, and takes a special point $x \in X$ to a special point in X^{ad}, if we consider the image $\tilde{x}$ of $\tilde{x}_1$ in $\mathscr{S}_{K_p}(G, X)^+$, then $\tilde{x}$ is a special point whose reduction mod p is the point x. Moreover, note that the cocharacter $\mu_{\tilde{x}}$ is determined by the map to the adjoint group $\mu_{\tilde{x}}^{ad}$, since for any map $\mathbb{G}_m \to G$, it is determined by the induced maps to G/G^{der} and G^{ad}, and since G/G^{der} is commutative, the map $\mathbb{G}_m \to G/G^{der}$ is constant for all elements $x \in X$. Thus, we see that the associated cocharacter $\mu_{\tilde{x}}$ satisfies $\mu_{\tilde{x}, \mathbb{Q}_p} = \mu$, since $\mu^{ad} = \mu_1^{ad}$. Thus, we have the following corollary:

Corollary 2.7.1. *Let x be a μ-ordinary point in $\mathscr{S}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$. Then x admits a lifting to a special point $\tilde{x}$, and the p^n-Frobenius map on $\tilde{x}$ is given by*

$$j = \mu(p)\sigma(\mu(p)) \dots \sigma^{n-1}(\mu(p)) \in G(\mathbb{Q}_p).$$

We now want to show the following proposition, which is a generalization of [BÖ2, Lemma 4.5].

Proposition 2.7.2. *Let x be a μ-ordinary point in $\mathscr{S}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$, and let $\tilde{x} = [h \times gK_p]$ be the lifting constructed in Corollary 2.7.1. Let U denote the unipotent radical of the parabolic subgroup of G associated to μ. Then for any $u \in U(\mathbb{Q}_p)$, we have that the mod p reductions of the points*

$$\overline{[h \times gK_p]} = \overline{[h \times guK_p]}$$

are equal.

Proof. Up to the action of some $g^p \in G(\mathbb{A}_f^p)$, we may assume that $x \in \mathscr{S}_{K_p}(G, X)^+(\overline{\mathbb{F}}_p)$. Observe that we have an isomorphism of root systems $\Phi(G, T) = \Phi(G^{ad}, T^{ad}) = \Phi(G_1, T_1)$. Hence if we consider U_1 the unipotent radical of the standard parabolic subgroup of G_1 corresponding to μ_1, then we can identify U_1 with U. In particular, since this result is true for the lift $\tilde{x}_1$, for any $u \in U_1(\mathbb{Q}_p)$, the same is true for the action of $n \in U(\mathbb{Q}_p)$ on $\tilde{x}$. $\square$

2.8. We now let $\tilde{G}$ be the simply connected cover of G_1^{der}. Let Z_G denote the center of the group G. Recall that we have a central isogeny

$$Z \times \tilde{G} \to G,$$

and thus, for a maximal torus T of G defined over $\mathbb{Q}$, we have a injective map with finite cokernel

$$X_*(Z_G) \oplus X_*(T^{der}) \hookrightarrow X_*(T)$$

where $\tilde{T}$ in $\tilde{G}$ is a maximal torus of G^{der}.

In particular, observe that for any cocharacter $\lambda \in X_*(T)$, there exists some positive integer m such that λ^m lifts (up to some cocharacter in $X_*(Z_G)$) to a cocharacter of $\tilde{G}$.

By [MS82, 3.4], there exists a Shimura variety $\mathrm{Sh}(G', X')$ and a map with central kernel $G' \to G_1$ such that $G'^{der} = \tilde{G}$ and via the composition map

$$\tilde{G} \to G_1^{der} \to G^{der}$$

there is an isomorphism of Shimura data $(G'^{ad}, X'^{ad}) \simeq (G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})$.

Now, we consider the Shimura variety $\mathrm{Sh}(\tilde{G}, \tilde{X})$. Since $\tilde{G}$ is simply connected, observe that the action of any $g' \in \tilde{G}(\mathbb{Q}_p)$ preserves connected components. We now choose a connected

component X'^+ of X' which maps to $X^{ad,+}$, and let $\mathrm{Sh}(G, X)^+$ be the connected component which contains $X'^+ \times \{1\}$. In particular, we see that g' maps $\mathrm{Sh}(G', X')^+$ back to itself.

Let $g \in G^{der}(\mathbb{Q}_p)$ be the image of g' under the central isogeny $\tilde{G} \to G^{der}$. Moreover, note that on geometrically connected components we have

$$\mathrm{Sh}_K(G, X)^+ = \mathrm{Sh}_{K'}(G', X')^+ / \Delta$$

where $\Delta = \ker(\mathscr{A}(G')^\circ \to \mathscr{A}(G))$, where the groups $\mathscr{A}(G')^\circ$ and $\mathscr{A}(G)$ are as defined in [Kis10, §3.3]. Thus, we see that since the action of g' preserves $\mathrm{Sh}(G', X')^+$, so too does the action of g preserve connected components of $\mathrm{Sh}_K(G, X)$, and moreover the action of g on $\mathrm{Sh}_K(G, X)^+$ is exactly the quotient by Δ of the action of g' on $\mathrm{Sh}(G', X')^+$.

If we let $g_1 \in G_1^{der}(\mathbb{Q}_p)$ be the image of g' under the central isogeny $\tilde{G} \to G_1^{der}$, a similar result holds for the action of g_1.

Moreover, we see that from [Kis17, 3.7.10] we have a surjective map

$$Z_G(\mathbb{Q}) \twoheadrightarrow Z_G(\mathbb{Q}_p)/Z_G(\mathbb{Z}_p).$$

2.9. **Partial Frobenius for abelian type.** Similar to the situation for the partial Frobenius for Shimura varieties of Hodge type, we would like to define the partial Frobenius to be the reduction mod p of the 'quasi-isogeny' given by $\mu_i(p)$. We will do this in this section.

Suppose that p is a prime which satisfies the following criterion:

(1) p splits in F
(2) The group G has good reduction at p
(3) The group $G_{\mathbb{Q}_p}$ is split.

Note that this implies the adjoint group G^{ad} is also split.

2.9.1. Firstly we will assume the group G is adjoint. By [Kis17, 4.6.6], there is a Hodge type Shimura datum (G_1, X_1) such that

(1) $(G_1^{ad}, X_1^{ad}) \xrightarrow{\sim} (G, X)$ and Z_{G_1} is a torus;
(2) if (G, X) has good reduction at p, then (G_1, X_1) in (1) can be chosen to have good reduction at p, and such that $E(G, X)_p = E(G_1, X_1)_p$.

Since Z_{G_1} is an unramified torus, by [Ama69, Corollary 2], we know that $H^1(\mathbb{Q}_p, Z_{G_1})$ is trivial, and thus we have a surjective map

$$G_1(\mathbb{Q}_p) \twoheadrightarrow G(\mathbb{Q}_p).$$

More precisely, it tells us that the element $\mu_i(p) \in G(\mathbb{Q}_p)$ lifts to an element $\tilde{\mu}_i(p) \in G_1(\mathbb{Q}_p)$ for some cocharacter $\tilde{\mu}_i$ of G_1. Now, observe that $\tilde{m}u_i(p)$ lies in the center of M_1, the centralizer of μ_1. Thus, we may consider the section of $p-\mathrm{Isog}_{G_1}$ given by the image of $1_{\tilde{\mu}_i(p)M_1(\mathbb{Z}_p)}$. This gives me a correspondence on $\mathscr{S}(G_1, X_1)_\kappa$, which we project to get a correspondence on $\mathscr{S}(G, X)_\kappa$. This is the partial Frobenius correspondence $\mathrm{Frob}_{\mathfrak{p}_i}$. Observe that by construction we have

$$\mathrm{Frob} = \prod_i \mathrm{Frob}_{\mathfrak{p}_i},$$

since the product of the images of $1_{\tilde{\mu}_i(p)M_1(\mathbb{Z}_p)}$ is the image of $1_{\tilde{\mu}(p)M_1(\mathbb{Z}_p)}$, where $\tilde{\mu}$ is a cocharacter whose image in $G(\mathbb{Q}_p)$ is $\mu(p)$, which corresponds to the Frobenius over the ordinary locus.

2.9.2. More generally, if G is not adjoint, then we will consider the Hecke correspondence on $\mathscr{S}(G, X)_\kappa$ given by $h(1_{K_p\mu_i(p)K_p})$, and its image in $\mathscr{S}(G^{ad}, X^{ad})_\kappa$. The image is given by

the Hecke correspondence $h(1_{K_p^{ad}\mu_i^{ad}(p)K_p^{ad}})$ in $\mathscr{S}(G^{ad}, X^{ad})_\kappa$, which in particular contains the partial Frobenius correspondence $\mathrm{Frob}_{\mathfrak{p}_i}^{ad}$ as an closed subscheme. Then, we define $\mathrm{Frob}_{\mathfrak{p}_i}$ to be the preimage of $\mathrm{Frob}_{\mathfrak{p}_i}^{ad}$ under the projection map

$$h(1_{K_p\mu_i(p)K_p}) \twoheadrightarrow h(1_{K_p^{ad}\mu_i^{ad}(p)K_p^{ad}}).$$

It remains to check that

(2.9.3) $$\mathrm{Frob} = \prod_i \mathrm{Frob}_{\mathfrak{p}_i},$$

which we can check over the ordinary locus. This follows from the corresponding observation for $\mathrm{Frob}^{\circ,ad}$, and noting that over the ordinary locus closed points in $\mathrm{Frob}_{\mathfrak{p}_i}$ consist of the reduction mod p of pairs $(\tilde{x}, \mu_i(p) \cdot \tilde{x})$, where $\tilde{x}$ is the special point lift of x constructed previously. Thus, we have the following proposition:

Proposition 2.9.4. *Let (G, X) be a Shimura datumn of abelian type, and let (G_1, X_1) be a Hodge type Shimura datumn constructed in (2.9.1) for (G^{ad}, X^{ad}). Let p be a prime such that*

(1) p splits in F
(2) The group G has good reduction at p
(3) The group $G_{\mathbb{Q}_p}$ is split

Then we have a partial Frobenius correspondence $\mathrm{Frob}_{\mathfrak{p}_i}$ such that

(2.9.5) $$\mathrm{Frob} = \prod_i \mathrm{Frob}_{\mathfrak{p}_i}.$$

Remark 2.9.6. Note that there is a positive density of primes p satisfying all 4 conditions. Indeed, if we take the compositum of the splitting fields of G, G_1 and F, then any prime which splits in the galois closure satisfies these conditions (1),(3),(4).

Remark 2.9.7. Since we do not have a moduli interpretation for general abelian type Shimura varieties, outside of the μ-ordinary locus, it is not clear what $\mathrm{Frob}_{\mathfrak{p}_i}$ has to do with partial Frobenii. However, from the discussion in Section 2.2, if the Shimura variety is of Hodge type, then we have a well-defined notion of a partial Frobenius isogeny (at least at closed points), and indeed the correspondence $\mathrm{Frob}_{\mathfrak{p}_i}$ is given as the section defined by the partial Frobenius at i for all closed points.

2.10. **Proof of Eichler-Shimura relations.**

Proposition 2.10.1. *Let (G, X) be a Shimura datumn of abelian type, such that $G = \mathrm{Res}_{F/\mathbb{Q}}G'$, and p a prime satisfying the conditions in Proposition 2.9.4. Let $H_i(t)$ be the polynomial defined in (2.4.3), viewed as a polynomial with coefficients in $\mathrm{Corr}(\mathscr{S}_\kappa, \mathscr{S}_\kappa)$ via the map h. Then we have the equality*

$$H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0.$$

in the ring $\mathrm{Corr}(\mathscr{S}_\kappa, \mathscr{S}_\kappa)$.

Proof. Firstly, observe that from Proposition 2.6.1, all the terms appearing in $H_i(\mathrm{Frob}_{\mathfrak{p}_i})$ have a dense μ-ordinary locus, hence it suffices to show the result where we restrict all the terms to the μ-ordinary locus. Applying [BÖ2, A.6], to show that $H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0$, it suffices to show that

$$x \cdot H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0$$

for all $x \in \mathscr{S}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$.

Let the Hecke polynomial be

$$H_i(t) = \sum_j A_j t^j,$$

for elements $A_j \in \mathcal{H}(G(\mathbb{Q}_p)//K_p)$. Let $h(A_j)$ denote the mod p algebraic cycle in $\mathrm{Corr}(\mathscr{S}_\kappa, \mathscr{S}_\kappa)$ corresponding to A_j. Thus, we want to show that

$$x \cdot \left(\sum_j \mathrm{Frob}_{\mathfrak{p}_i}^j \cdot h(A_j) \right) = 0$$

where $\mathrm{Frob}_{\mathfrak{p}_i}$ is the correspondence defined above.

Let x' be a lift of x, and $\tilde{x}'$ be the canonical special point lift of x'. Then its image under the projection map is also a special point $\tilde{x}$, and the partial Frobenius acting on $\tilde{x}$ is given by $\mu_i(p)$. The proof then follows as in [Bö2, Thm 4.7]. $\qquad\square$

3. Semisimplicity Criterion

3.1. An abstract semisimplicity criterion.

Definition 3.1.1. A representation $\rho : \Gamma_E \to GL(V)$ is *strongly irreducible* if the restriction to any open finite index subgroup U ρ_U is still irreducible.

Theorem 3.1.2. *Let Γ be a profinite group, $V, W_1, ..., W_r$ non-zero vector spaces of finite dimension over $\mathbb{Q}$. Let $\rho : \Gamma \to Aut_{\mathbb{Q}}(V)$ and $\rho_i : \Gamma \to Aut_{\mathbb{Q}}(W)$ be representations of Γ with Lie algebras*

$$\mathfrak{g}_i = Lie(\rho_i(\Gamma)), \qquad \mathfrak{g} = Lie(\rho(\Gamma)).$$

We denote $\bar{\mathfrak{g}}_i = \mathfrak{g}_i \otimes \bar{\mathbb{Q}}, \bar{\mathfrak{g}} = \mathfrak{g} \otimes \bar{\mathbb{Q}}$. If the following three conditions hold, then the representation $\rho = \rho^{ss}$ is semisimple.

(1) *Each ρ_i is strongly irreducible (which implies that each $\mathfrak{g}_i$ is a reductive $\bar{\mathbb{Q}}$-Lie algebra and each element of its centre acts on W_i by a scalar).*

(2) *For each $i = 1, \ldots, r$, every (equivalently, some) Cartan subalgebra $\mathfrak{h}_i$ of $\mathfrak{g}_i$ acts on W_i without multiplicities (i.e., all weight spaces of $\mathfrak{h}_i$ onW_i are one-dimensional).*

(3) *There exists an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$ there exists a finite dimensional vector space over $\bar{\mathbb{Q}}$ (depending on g) $V(g) \supset V$ and elements $u_1, \ldots, u_r \in \mathrm{Aut}\mathbb{Q}(V(g))$ such that $u_i u_j = u_j u_i$, $P_{\rho_i(g)}(u_i) = 0$ for all i, $j = 1, ..., r$, and V is stable under $u_1 \ldots u_r$ and $u_1 \ldots u_r|_V = \rho(g)$*

The following is [Sen73, Theorem 1].

Proposition 3.1.3. *Let $\mathfrak{g} = \bar{\mathbb{Q}}_l \cdot Lie(\tilde{\rho}(Gal_E)) \subset \mathfrak{gl}(n, \bar{\mathbb{Q}}_l)$ be the $\bar{\mathbb{Q}}_l$-Lie algebra generated by the image of $\tilde{\rho}$. Then any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts on $\bar{\mathbb{Q}}_l^n$ by the n Hodge-Tate weights of $\tilde{\rho}|_{G_{F_\tau}}$, for any $\tau : F \hookrightarrow \bar{\mathbb{Q}}_l$.*

3.2. Proof of Main Theorem.

Proof. Observe that to show that the representation ρ is semisimple, it suffices to show that $\rho|_U$ is semisimple for any open finite index subgroup of $\mathrm{Gal}(\bar{E}/E)$. Thus, it suffices to find a set of primes p of positive density such that $\rho(\mathrm{Frob}_p)$ is a semisimple endomorphism of $V^i(\pi^\infty)$.

From the Eichler-Shimura relation for split primes, observe that if we let H be the Hecke polynomial H, then we have

$$H(\mathrm{Frob}_p | H^i(\mathrm{Sh}_K)) = 0$$

as endomorphisms of $H^i(\mathrm{Sh}_K)$. Applying the decomposition of $H^i(\mathrm{Sh}_K)$, observe that since each summand $V^i(\pi^\infty) \otimes (\pi^\infty)^K$ is Frobenius stable, we can consider Frob_p as an endomorphism of $V^i(\pi^\infty)$.

Replacing each element of the Hecke algebra with its eigenvalue on $\pi_p^{K_p}$, we obtain

$$H|_{\pi_p^{K_p}}(\mathrm{Frob}_p|_{V^i(\pi^\infty) \otimes (\pi^\infty)^K}) = 0.$$

Observe that the polynomial on the right hand side is, by the definition of the Hecke polynomial, $\det(t - \tilde{\rho}'(\mathrm{Frob}_p)) =: P_{\tilde{\rho}}$.

In order to show semisimplicity of the endomorphism given by Frob_p, it suffices to show that the $P_{\tilde{\rho}}$ has distinct roots. This is an open condition on $\mathrm{Gal}(\bar{E}/E)$, hence it suffices to exhibit an element $u \in \mathrm{Gal}(\bar{E}/E)$ such that the characteristic polynomial of $\rho(u)$ has distinct roots. To see this, observe that Proposition 3.1.3 applies, hence the Lie algebra

$$\bar{\mathfrak{g}} = \bar{\mathbb{Q}}_l \cdot \mathrm{Lie}(\tilde{\rho}(\mathrm{Gal}(\bar{E}/E)))$$

contains a semisimple element whose eigenvalues on $V_i(\pi^\infty)$ act by the Hodge-Tate weights of ρ_l. By assumption (2), the Hodge-Tate weights of ρ_l are all dinstinct. This implies that there is an open subset of $\mathrm{Gal}(\bar{E}/E)$ where the characteristic polynomial of $\rho(u)$ has distinct roots.

Finally, we conclude using the criterion in Theorem 3.1.2, since we have constructed a Zariski dense set of elements which are semisimple, that ρ is semisimple. □

4. Applications to some Shimura varieties

We discuss here some examples of Shimura varieties where the above theorem applies. Consider the Shimura variety associated to the group $\mathrm{Res}_{F/\mathbb{Q}}G$, where G is some inner form of $GSp_{2n,F}$ which is compact modulo center for at least one place $v|\infty$ of F. Let π be a cuspidal L-algebraic automorphic representation of $G(\mathbb{A}_F)$, and we assume that π satisfies

  (1) There is a finite F-place v_{St} such that $\pi_{v_{St}}$ is the Steinberg representation of $GSp_{2n}(F_{v_{St}})$ twisted by a character.
  (2) $\pi^\infty |\mathrm{sim}|^{n(n+1)/4}$ is ξ-cohomological for an irreducible algebraic representation $\xi = \otimes_{y:F\to\mathbb{C}}\xi_y$ of the group $(\mathrm{Res}_{F/\mathbb{Q}}GSp_{2n})_\mathbb{C}$, where sim is the similitude factor map $\mathrm{sim} : GSp_{2n} \to \mathbb{G}_m$.

Under these conditions on π, Kret and Shin [KS20, Theorem A] construct the Galois representation

$$\rho_\pi : \mathrm{Gal}(\bar{F}/F) \to GSpin_{2n+1}(\mathbb{Q}_l)$$

associated to π. If we moreover assume that the Zariski closure of the image of ρ_π maps onto SO_{2n+1} (which should hold generically), then at all places $v|\infty$ where the group is not compact modulo center, the associated representation

$$\tilde{\rho}_{\pi,v} : \mathrm{Gal}(\bar{F}/F) \to GSpin_{2n+1}(\mathbb{Q}_l) \xrightarrow{\mathrm{spin}} GL(V),$$

$\tilde{\rho}_{\pi,v}$ will be strongly irreducible, since it is irreducible and has connected image, hence the Zariski closure of the image of any finite index open subgroup is also SO_{2n+1}, and hence irreducible.

If we moreover assume that

  (1) The representation π_v is spin-regular at every infinite place v of F,

then [KS20, Theorem C] implies π is potentially automorphic. We will further assume that π is automorphic. Moreover, spin-regularity implies that the Hodge-Tate weights of $\tilde{\rho}_{\pi,v}$ are distinct. Under all these conditions, we have may apply the Main Theorem, and we can deduce the following:

Theorem 4.0.1. *Let π be a cuspidal L-algebraic automorphic representation of $G(\mathbb{A}_F)$, satisfying*

(1) *There is a finite F-place v_{St} such that $\pi_{v_{St}}$ is the Steinberg representation of $GSp_{2n}(F_{v_{St}})$ twisted by a character.*

(2) *$\pi^\infty |\mathrm{sim}|^{n(n+1)/4}$ is ξ-cohomological for an irreducible algebraic representation $\xi = \otimes_{y:F\to\mathbb{C}}\xi_y$ of the group $(\mathrm{Res}_{F/\mathbb{Q}}GSp_{2n})_{\mathbb{C}}$, where sim is the similitude factor map $\mathrm{sim} : GSp_{2n} \to \mathbb{G}_m$.*

(3) *The representation π_v is spin-regular at every infinite place v of F*

. *If moreover the l-adic Galois representation $\rho_\pi : \mathrm{Gal}(\bar{E}/E) \to GSpin_{2n+1}$ satisfies*

(1) $\mathrm{spin} \circ \rho_\pi$ *is automorphic*

(2) *The Zariski closure of the image of ρ_π maps onto SO_{2n+1},*

Then the Galois module

$$\mathrm{Hom}_{\mathbb{G}(\mathbb{A}_f)}(\pi^\infty, H^*_{et}(\mathrm{Sh}(G,X), \mathbb{Q}_l))$$

(which is finite-dimensional over $\mathbb{Q}_l$ is semisimple.

References

[Ama69]   Kazuo Amano. A Note on Galois Cohomology Groups of Algebraic Tori. *Nagoya Mathematical Journal*, 34:121–127, 1969.

[Bö02]   Oliver Bültel. The congruence relation in the non-PEL case. *J. Reine Angew. Math.*, 544:133–159, 2002.

[FN19]   Karam Fayad and Jan Nekovář. Semisimplicity of certain galois representations occurring in étale cohomology of unitary shimura varieties. *American Journal of Mathematics*, 141:503 – 530, 2019.

[Kim18]   Wansu Kim. Rapoport-Zink Spaces of Hodge Type. *Forum of Mathematics, Sigma*, 6, 2018.

[Kis10]   Mark Kisin. Integral models for Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 23(4):967–1012, 2010.

[Kis17]   Mark Kisin. Mod p points on Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 30(3):819–914, 2017.

[KS20]   Arno Kret and Sug Woo Shin. Galois representations for general symplectic groups. To appear in JEMS., 2020.

[Lee20]   Si Ying Lee. Eichler-shimura relations for shimura varieties of hodge type. `arxiv.org:2006.11745`, 2020.

[MS82]   James S. Milne and Kuang-yen Shih. Conjugates of shimura varieties. In *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*, pages 280–356. Springer-Verlag, Berlin-New York, 1982.

[Sen73]   Shankar Sen. Lie algebras of galois groups arising from hodge-tate modules. *Annals of Mathematics*, 97:160, 1973.

[SZ16]   Ananth N. Shankar and Rong Zhou. Serre-Tate theory for Shimura varieties of Hodge type. `arXiv: 1612.06456`, 2016.

[SZ17]   Xu Shen and C. Zhang. Stratifications in good reductions of shimura varieties of abelian type. `arxiv.org: 1707.00439`, 2017.

[XZ17]   Liang Xiao and Xinwen Zhu. Cycles on Shimura varieties via geometric Satake. `arXiv:1707.05700`, 2017.

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

Email address: `sylee@math.harvard.edu`