SEMISIMPLICITY OF ÉTALE COHOMOLOGY OF CERTAIN SHIMURA VARIETIES

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ABSTRACT. Building on work of Fayad and Nekovář, we show that a certain part of the etale cohomology of some abelian-type Shimura varieties is semisimple, assuming the associated automorphic Galois representations exists, and satisfies some good properties. The proof combines an abstract semisimplicity criterion of Fayad-Nekovář with generalized Eichler-Shimura relations for abelian type Shimura varieties and partial Frobenii. We apply this to some symplectic and orthogonal Shimura varieties.

1. INTRODUCTION

Let G be a reductive group over $\mathbb{Q}$, and X be a conjugacy class of homomorphisms

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$$

such that (G, X) is a Shimura datum. Given a compact open subgroup $K \subset G(\mathbb{A}_f)$, we can form the associated Shimura variety

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

which is a complex manifold for K small enough, and has a canonical model over E, for some number field E.

Consider a representation valued in complex vector spaces

$$\xi : G_{\mathbb{C}} \to \mathrm{GL}(V_\xi)$$

such that $\xi(Z(\mathbb{Q}) \cap K) = 1$, where Z is the center of G. This gives rise to a locally constant sheaf of complex vector spaces

$$\mathcal{L}_\xi = G(\mathbb{Q})\backslash V_\xi \times X \times G(\mathbb{A}_f)/K.$$

Suppose now that G^{der} is anisotropic, so that the Shimura variety $\mathrm{Sh}_K(G, X)$ is compact. This assumption is not necessary for the statements of the theorems, as long as we replace the cohomology groups with the intersection cohomology; we assume this here for simplicity. We now consider the complex analytic cohomology of the tower of the Shimura variety, i.e.

$$H^i(\mathrm{Sh}(G, X)^{\mathrm{an}}, \mathcal{L}_\xi) := \varinjlim_K H^i(\mathrm{Sh}_K(G, X)^{\mathrm{an}}, \mathcal{L}_\xi).$$

Let A_G be the largest $\mathbb{Q}$-split subtorus of Z_G. Choose $h \in X$, and let K_∞ be the stabilizer of h in $G(\mathbb{R})$. By Matsushima's formula for L^2-cohomology established by Franke [Fra98], and (the proof of) Zucker's conjecture, we have a decomposition of $H^i(\mathrm{Sh}(G, X)^{\mathrm{an}}, \mathcal{L}_\xi)$ in terms of Lie algebra cohomology

$$H^i(\mathrm{Sh}(G, X)^{an}, \mathcal{L}_\xi) = \bigoplus_{\pi = \pi_\infty \otimes \pi^\infty} m(\pi) H^i(\mathfrak{a}_G\backslash\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi) \otimes (\pi^\infty),$$

where $\mathfrak{a}_G = \mathrm{Lie}(A_G)$, and π runs through unitary automorphic representations of $G(\mathbb{A})$, and $m(\pi)$ is the multiplicity of π appearing in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$, the space of measurable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which are square-integrable modulo center and with infinitesimal character ω.

Fix a prime ℓ, and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. The representation ξ gives rise to an ℓ-adic automorphic sheaf, in the following way. ξ gives rise to a representation valued in $\overline{\mathbb{Q}}_\ell$-vector spaces

$$\xi_\ell : G_{\overline{\mathbb{Q}}_\ell} \to \mathrm{GL}(V_{\xi,\ell}),$$

and similarly gives rise to a sheaf

$$\mathcal{L}_{\xi,\ell} = G(\mathbb{Q})\backslash V_{\xi,\ell} \times X \times G(\mathbb{A}_f)/K.$$

Since $\mathrm{Sh}_K(G,X)$ is defined over E, we can consider the étale cohomology of the tower

$$H^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}) := \varinjlim_K H^i_{\acute{e}t}(\mathrm{Sh}_K(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}),$$

then we have an isomorphism

$$H^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}) \simeq H^i(\mathrm{Sh}(G,X)^{\mathrm{an}}, \mathcal{L}_\xi) \otimes_\iota \overline{\mathbb{Q}}_\ell,$$

which is $G(\mathbb{A}_f)$-equivariant, and thus we have a decomposition

$$H^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,l}) = \bigoplus_{\pi^\infty} V^i(\pi^\infty) \otimes (\pi^\infty),$$

where $V^i(\pi^\infty) = \mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, H^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}))$. If $\mathrm{Sh}_K(G,X)$ is not compact, we consider instead the intersection cohomology of the Bailey-Borel compactification $\mathrm{Sh}(G,X)^{\mathrm{min}}$,

$$V^i(\pi^\infty) = \mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, IH^i_{\acute{e}t}(\mathrm{Sh}(G,X)^{\mathrm{min}}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi,\ell}))).$$

We now let σ_{π^∞} be the associated Galois representation

$$\sigma_{\pi^\infty} : \mathrm{Gal}(\bar{E}/E) \to V^i(\pi^\infty).$$

On the other hand, the global Langlands correspondence conjectures that to the automorphic representation π we have an associated semisimple Galois representation $\phi_\pi : \mathrm{Gal}(\bar{E}/E) \to {}^L G$, and we denote the composition

$$\rho_{\pi,\mu} : \mathrm{Gal}(\bar{E}/E) \xrightarrow{\phi_\pi} {}^L G \xrightarrow{r_{-\mu}} \mathrm{GL}(V_{-\mu}),$$

where μ is the minuscule Hodge cocharacter associated to the Shimura datum, and $V_{-\mu}$ is the vector space of the associated highest weight representation $r_{-\mu}$. Here, $r_{-\mu}$ is the highest weight representation associated with the dominant Weyl conjugate of μ. We remark here that this $-\mu$ is such that the action of the geometric Frobenius at the unramified places corresponds to the *left* Hecke action of $\mu^{-1}(p)$ under Artin reciprocity.

When the group G is of the form $\mathrm{Res}_{F/\mathbb{Q}}H$ for some connected reductive group H, and F is a totally real field of degree d, the global Langlands correspondence then further conjectures that we have a decomposition (perhaps after passing to a finite extension E' of E) of $\tilde{\rho}$ as

$$\rho_{\pi,\mu} = \bigotimes_{v|\infty} \rho_{\pi,\mu,v},$$

where the product runs over infinite places of F. Moreover, $\rho_{\pi,\mu,v}$ should have the following form. Observe that over $\mathbb{C}$, we have a decomposition

$$\mathrm{Res}_{F/\mathbb{Q}}H \simeq \prod_{v|\infty} H,$$

and the cocharacter μ also decomposes as $\prod_v \mu_v$. Then we should obtain $\rho_{\pi,\mu,v}$ as the composition

$$\rho_{\pi,\mu,v} = \rho_{\pi,\mu_v} : \mathrm{Gal}(\bar{E}'/E') \xrightarrow{\phi_\pi} {}^L H \xrightarrow{r_{-\mu_v}} \mathrm{GL}(V_{-\mu_v}),$$

where here we view π as an automorphic representation of H over F.

We expect a close relationship between σ_{π^∞} and $\rho_{\pi,\mu}$. In the case where π does not have endoscopy (for instance, if it is a twist of Steinberg at some finite place), then we expect to have

$$\sigma_{\pi^\infty} \simeq \rho_{\pi,\mu}^{\oplus m}$$

for some integer m, and thus if $\rho_{\pi,\mu}$ is irreducible, σ_{π^∞} semisimple. Such a result may be shown via the Langlands-Kottwitz method, in combination with the results of this paper.

The main theorem of this paper is the following:

Theorem 1.0.1. *Let (G, X) be a Shimura datum of abelian type such that $G = \mathrm{Res}_{F/\mathbb{Q}}H$ for some connected reductive group H, and totally real number field F. Let π be an automorphic representation of $G(\mathbb{A}_{\mathbb{Q},f}) = H(\mathbb{A}_{F,f})$. For all v, suppose that the ${}^L H$-valued Galois representation associated to π exists, and we consider for all v the composition with the highest weight representation $\rho_{\pi,\mu_v} : \mathrm{Gal}(\bar{F}/F) \to {}^L H \to \mathrm{GL}(V_{-\mu_v})$. Suppose that moreover we also know that*

(1) ρ_{π,μ_v} is strongly irreducible
(2) For all primes $\mathfrak{p}$ of E such that $\mathfrak{p}|l$, the Hodge-Tate weights of $\rho_{\pi,\mu_v}|_{D_\mathfrak{p}}$ are distinct.

Then σ_{π^∞} is a semisimple representation.

This result was previously known in the cases where the abelian type Shimura variety had a cover by an associated PEL type A Shimura variety, as shown in [FN19]. Note that if G is adjoint, then the group G is always a product of groups of the form $\mathrm{Res}_{F/\mathbb{Q}}H$ for some absolutely simple adjoint group H.

Remark 1.0.2. The statement of Theorem 1.0.1 is most interesting when $F \neq \mathbb{Q}$, and H is absolutely simple. Indeed, while the statement is still valid even when $F = \mathbb{Q}$, in this situation it is often the case that the Galois representations ϕ_π is constructed in the cohomology of $\mathrm{Sh}_K(G, X)$, and thus the statement is tautological. Moreover, the condition on ρ_{π,μ_v} to be strongly irreducible is rarely satisfied if H is not absolutely simple.

We can apply the above theorem to situations beyond the cases of unitary groups considered in [FN19]. In particular, we can consider now the cases where H is an inner form of the groups GSp_{2g} or GSO_{2n}, where the automorphic Galois representations have been constructed by Kret and Shin in [KS20b] and [KS20a]. In the following theorem, we will let G^* be one of the following groups:

symplectic: $G^* = \mathrm{GSp}_{2g}$
orthogonal, n even: $G^* = \mathrm{GSO}_{2n}$
orthogonal, n odd: G^* is a non-split quasi-split form of GSO_{2n} relative to E/F, a CM extension

We now assume that H is an inner form of G^*. Let d be the dimension of $\mathrm{Sh}(G, X)$.

Theorem 1.0.3. *Let π be a cuspidal L-algebraic automorphic representation of $H(\mathbb{A}_F)$, satisfying*

(1) *There is a finite F-place v_{St} such that $H_{F_{v_{St}}}$ and $G^*_{F_{v_{St}}}$ are isomorphic, and under this isomorphism $\pi_{v_{St}}$ is the Steinberg representation of $G^*(F_{v_{St}})$ twisted by a character.*

(2) *$\pi_\infty |\mathrm{sim}|^{-n(n-1)/4}$ is ξ-cohomological for an irreducible algebraic representation $\xi = \otimes_{y:F\to\mathbb{C}} \xi_y$ of the group $(\mathrm{Res}_{F/\mathbb{Q}} G^*)_\mathbb{C}$, where sim is the similitude factor map $\mathrm{sim} : G^* \to \mathbb{G}_m$.*

(3) *The representation π_v is regular after composing with the representation $\mathrm{GSpin}_{2g+1} \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^g}$ if symplectic (resp. $\mathrm{GSpin}_{2n} \xrightarrow{\mathrm{std}} \mathrm{GL}_{2n}$ if orthogonal) at every infinite place v of F.*

If moreover the ℓ-adic Galois representation $\phi_\pi : \mathrm{Gal}(\bar{F}/F) \to \widehat{H}$ satisfies

(4) *The image of ϕ_π is Zariski dense in $\widehat{H}$,*

Then the Galois module

$$\mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, IH^d_{\text{ét}}(\mathrm{Sh}(G,X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi,\ell})))$$

is semisimple.

To show this result, we first define a 'partial Frobenius isogeny' at primes p which are split in F, and then show the Eichler-Shimura congruence relations for these partial Frobenius for split groups, using results from [Lee20]. More precisely, we show the following:

Theorem 1.0.4. *Let (G, X) be a Shimura datum of abelian type, such that $G = \mathrm{Res}_{F/\mathbb{Q}} H$ for some connected reductive group G, and totally real number field F of degree d. Let p be a prime which is split in F and for which the group $G_{\mathbb{Q}_p}$ is split. Then for all $i = 1, \ldots, d$ we have a partial Frobenius correspondence $\mathrm{Frob}_{\mathfrak{p}_i}$ such that*

$$\mathrm{Frob} = \prod_i \mathrm{Frob}_{\mathfrak{p}_i}$$

where $\mathfrak{p}_i$ is a prime of F dividing p, and in the ring of algebraic correspondences of the mod p reduction of $\mathrm{Sh}_K(G, X)$,

$$H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0,$$

where H_i is the Hecke polynomial at $\mathfrak{p}_i$, which is a renormalized characteristic polynomial of the irreducible representation of $\widehat{H}$ with highest weight $\hat{\mu}_i$.

This is a refinement of the Eichler-Shimura relation considered for Hodge type Shimura varieties in [Lee20]. The Hecke polynomial $H_{G,X}$ defined there (which is valid for all Shimura varieties) will be a tensor product of the polynomials H_i, and in particular, under the same assumptions of splitting of the group G at the prime p, we get the Eichler-Shimura relations for the abelian type Shimura varieties considered in the theorem.

We also remark that the construction of this partial Frobenius isogeny should be viewed as a shadow of the plectic conjecture of Nekovář-Scholl, and while in this paper we construct it only with the additional assumption that $G_{\mathbb{Q}_p}$ is split, in fact the construction does not need this assumption.

Once we have the generalized Eichler-Shimura relations, we can combine the above theorem with a semisimplicity criterion for Lie algebras shown in [FN19], which allows us to deduce the

main result. Roughly speaking, the generalized Eichler-Shimura relations, along with the condition of distinct Hodge-Tate weights, shows that we have many Frobenius elements, in particular a positive density of them, which are semisimple. This, together with a strong condition on irreducibility of the Galois representation coming from the Langlands correspondence, allows us to show that the image of Galois is a reductive group, and hence the representation was semisimple.

We emphasize that the main theorem here *cannot* be used to prove semisimplicity of the Galois representations constructed in, for instance, [BLGGT14], [KS20a] or [KS20b]. Instead, the main novelty of this result is when one knows, for instance from [KSZ21], the expected shape of the semisimplification of the Galois representation appearing in the cohomology of the Shimura variety, via the study of zeta functions of Shimura varieties and the Langlands-Kottwitz method. In this case, we can conclude that the equality holds even without taking semisimplification.

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2. EICHLER-SHIMURA RELATIONS

In this section, we will prove the key technical result Theorem 1.0.4, and the corresponding result on (intersection) cohomology.

For this entire section, we will fix (G, X) a Shimura datum of abelian type such that $G = \operatorname{Res}_{F/\mathbb{Q}} H$ with F totally real of degree d. Choose a prime $p > 2$ that is split in F such that $G_{\mathbb{Q}_p}$ is split. Let $p = \mathfrak{p}_1 \ldots \mathfrak{p}_d$, for $\mathfrak{p}_i$ primes of F above p. In this case, we have

$$G_{\mathbb{Q}_p} = \prod_{\mathfrak{p}_i | p} H_i.$$

Fix a choice of Borel B and torus T of $G_{\mathbb{Q}_p}$.

For any $h \in X$, we can define

$$\mu_h : \mathbb{C}^\times \xrightarrow{x \mapsto (x, \bar{x})} \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S} \to G(\mathbb{C}).$$

The conjugacy class $\{\mu_h\}$ is defined over the reflex field E. Let $v | p$ be a prime of E. In the setup above where $G_{\mathbb{Q}_p}$ is split, we thus get that the conjugacy class of cocharacters $\{\mu\}$ obtained from X is defined over $\mathbb{Q}_p$, and thus in this case we have $E_v = \mathbb{Q}_p$. Let $\mu \in X_*(T)$ be a dominant representative of this conjugacy class, and the decomposition of $G_{\mathbb{Q}_p}$ means we can write $\mu = \prod_i \mu_i$.

Fix a prime $\ell \neq p$. Let $K \subset \mathbb{A}_f$ be an open compact subgroup which we fix for this entire paper, and for which K_p is hyperspecial. Let $\mathbb{T}_f^p$ denote the prime-to-p-adelic Hecke algebra $\mathcal{H}(G(\mathbb{A}_f^p) // K^p, \mathbb{Q})$ of K^p-bi-invariant compactly supported functions on $G(\mathbb{A}_f^p)$.

Let $\xi : G_{\bar{\mathbb{Q}}_\ell} \to \operatorname{GL}(V_\xi)$ be a representation such that $\xi(Z(\mathbb{Q}) \cap K) = 1$. As explained in the introduction, this gives rise to an ℓ-adic local system $\mathcal{L}_{\xi, \ell}$ on $\operatorname{Sh}_K(G, X)$.

For a choice of hyperspecial K_p which decomposes as $K_p = \prod_i K_{p,i}$ we get for each i Hecke polynomials $H_{\mu_i} \in \mathcal{H}_i := \mathcal{H}(H_i(\mathbb{Q}_p) // K_{i,p}, \mathbb{Q})$ the Hecke algebra of K_{ip} bi-invariant functions on $H_i(\mathbb{Q}_p)$. We define H_{μ_i} in 2.11. We also define the Hecke algebra at p, $\mathcal{H}_p := \mathcal{H}(G(\mathbb{Q}_p) // K_p, \mathbb{Q})$.

The main result of this section is the following:

Theorem 2.0.1. *There are operators* $\mathrm{Frob}_{\mathfrak{p}_i}$ *for* $i = 1, \ldots, d$ *acting on* $IH^j(\mathrm{Sh}_K(G, X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi, \ell}))$ *such that*

(1) *The action of any pair* $\mathrm{Frob}_{\mathfrak{p}_i}$, $\mathrm{Frob}_{\mathfrak{p}_j}$ *commute, as does the action of any* $\mathrm{Frob}_{\mathfrak{p}_i}$ *with the* $\mathbb{T}^p_f$*-action and the* $\mathcal{H}_p$*-action;*

(2) $\mathrm{Frob}_{\mathfrak{p}_1} \circ \cdots \circ \mathrm{Frob}_{\mathfrak{p}_d} = \mathrm{Frob}_v$, *where* Frob_v *is the action of the geometric Frobenius at* v;

(3) *For each* i, $H_{\mu_i}(\mathrm{Frob}_{\mathfrak{p}_i})$ *acts as 0 on* $IH^j(\mathrm{Sh}_K(G, X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi, \ell}))$.

In the theorem, $IH^j(\mathrm{Sh}_K(G, X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi, \ell}))$ denotes the (étale) intersection cohomology of the Bailey-Borel compactification $\mathrm{Sh}_K(G, X)^{\min}$ of $\mathrm{Sh}_K(G, X)$, with coefficients in the perverse intermediate extension of $\mathcal{L}_{\xi, \ell}$ to $\mathrm{Sh}_K(G, X)^{\min}$. More precisely, if we let $j : \mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_K(G, X)^{\min}$ be the open embedding, then intermediate extension gives us a perverse ℓ-adic sheaf $\mathrm{IC}(\mathcal{L}_{\xi, \ell}) := j_{!*}\mathcal{L}_{\xi, \ell}$, and $IH^j(\mathrm{Sh}_K(G, X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi, \ell}))$ is the j-th cohomology group of $\mathrm{Sh}_K(G, X)^{\min}$ with coefficients in $\mathrm{IC}(\mathcal{L}_{\xi, \ell})$.

2.1. **Integral model.** We first recall the construction in [Kis10, §3], of the integral model $\mathscr{S}_K(G, X)$ of $\mathrm{Sh}_K(G, X)$ over O_{E_v}. Let (G_1, X_1) denote a Hodge type Shimura datum which covers (G, X), i.e. such that there exists a central isogeny

$$f : G_1^{\mathrm{der}} \to G^{\mathrm{der}}$$

which induces an isomorphism on adjoint Shimura data $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \simeq (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. In fact, we may choose (G_1, X_1) as in the following lemma:

Lemma 2.1.1. *There exists a covering Hodge type datum* (G_1, X_1) *such that the local reflex field* $E_{1,v'}$ *of* (G_1, X_1) *for some prime* $v'|p$ *is* $\mathbb{Q}_p$.

Proof. Since $G_{\mathbb{Q}_p}$ is split this implies that G^{ad} is also split over $\mathbb{Q}_p$. We can now apply [KP18, Lemma 4.6.22]. More precisely, in the proof of loc. cit. it is shown that one may choose a covering Shimura datumn (G_1, X_1) of Hodge type such that any prime $v|p$ of the reflex field of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, denoted $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, splits in $E(G_1, X_1)$. In particular, since $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_v = \mathbb{Q}_p$, we may in fact choose any prime v' over v and we will have $E(G_1, X_1)_{v'} = \mathbb{Q}_p$. $\square$

Fix a connected component $X^+ \subset X$. Let $\mathrm{Sh}_K(G, X)^+$ denote the geometric connected component of $\mathrm{Sh}_K(G, X)$ containing $\{1\} \times X^+$. Consider now the geometric connected component

$$\mathrm{Sh}_{K_{1p}}(G_1, X_1)^+ = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1p}K_1^p}(G_1, X_1)^+$$

of $\mathrm{Sh}_{K_{1p}}(G_1, X_1) = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1p}K_1^p}(G_1, X_1)$, where $K_{1p} = G_1(\mathbb{Z}_p)$. Let $\mathscr{S}_{K_{1p}}(G_1, X_1)$ be the integral model of the Hodge type Shimura variety constructed in [Kis10, §2], and let $\mathscr{S}_{K_{1p}}(G_1, X_1)^+$ be the Zariski closure of $\mathrm{Sh}_{K_{1p}}(G_1, X_1)^+$ in $\mathscr{S}_{K_{1p}}(G_1, X_1)$. Write $Z = Z_G$, and define

$$\mathscr{S}_{K_p}(G, X) = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1p}}(G_1, X_1)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{\circ},$$

where

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}}) = G(\mathbb{A}_f^p)/Z(\mathbb{Z}_{(p)})^- *_{G(\mathbb{Z}_{(p)})^+/Z(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+,$$

and

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})^{\circ} = G(\mathbb{Z}_{(p)})^-_+/Z(\mathbb{Z}_{(p)})^- *_{G(\mathbb{Z}_{(p)})^+/Z(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+.$$

It is shown in [Kis10, Theorem 3.4.10] that $\mathscr{S}_{K_p}(G, X)$ as defined above is then the canonical integral model of $\mathrm{Sh}_{K_p}(G, X)$.

For any abelian type Shimura datum unramified at p with level K which is hyperspecial at p, we denote by

$$\bar{\mathscr{S}}_K(G, X) := \mathscr{S}_K(G, X)_{k_v},$$

where k_v is the residue field of O_{E_v}.

We also recall the result of Lan-Stroh [LS18, Corollary 4.10], which will allow us to deduce results about cohomology from results on the mod p fiber:

Theorem 2.1.2. *There is an isomorphism*

$$IH^i(\mathrm{Sh}_K(G, X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi, \ell})) \simeq IH^i(\bar{\mathscr{S}}_K(G, X)^{\min}_{\bar{k}_v}, \mathrm{IC}(\mathcal{L}_{\xi, \ell})),$$

where on the right-hand side we abuse notation and let $\mathcal{L}_{\xi, \ell}$ also denote the mod p fiber of the canonical extension of the ℓ-adic local system $\mathcal{L}_{\xi, \ell}$ from the generic fiber. This isomorphism is equivariant for the action of the prime-to-p Hecke algebra $\mathbb{T}^p_f$, and also the action of the Frobenius at p.

2.2. **Newton stratification.** We now recall the construction of Newton strata for Shimura varieties of abelian type, as constructed in [SZ17, §2]. Let $L = W(\bar{\mathbb{F}}_p)[1/p]$, with Frobenius given by σ, and $v = \sigma(\mu^{-1}) = \mu^{-1}$. Recall that $B(G)$ denotes the set of σ-conjugacy classes of $G(L)$. The Newton strata of the Shimura variety are indexed by elements in $B(G, v)$.

The following result of Kottwitz is a key input in the construction.

Lemma 2.2.1 ([Kot97, Section 6.5]). *Let G be a connected reductive group over $\mathbb{Q}_p$, and let $\{v\}$ be a conjugacy class of minuscule cocharacters. Then, projecting to the adjoint group G^{ad} induces a map $B(G) \to B(G^{\mathrm{ad}})$, and restricting this map to $B(G, \{v\})$ induces a bijection between $B(G, \{v\})$ and $B(G^{\mathrm{ad}}, \{v^{\mathrm{ad}}\})$.*

In [SZ17, §2.3], the Newton strata are first constructed for the adjoint Shimura variety. More precisely, let K_p^{ad} be the hyperspecial subgroup of $G^{\mathrm{ad}}(\mathbb{Q}_p)$ determined by K_p. (Equivalently, from the arguments in [Kis10, Corollary 3.4.14], this is also the hyperspecial subgroup determined by K_{1p}.) In [SZ17, 2.3.3, 2.3.4], it is shown that if we restrict the Newton stratification to the connected component $\bar{\mathscr{S}}_{K_{1p}}(G_1, X_1)^+ \hookrightarrow \bar{\mathscr{S}}_{K_{1p}}(G_1, X_1)$, and we denote by $\bar{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{[b_1], +}$ the strata corresponding to $[b_1] \in B(G_1, \{v_1\})$, then taking

$$\bar{\mathscr{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{[b^{\mathrm{ad}}]} = [\mathscr{A}(G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}) \times \bar{\mathscr{S}}_{K_{1,p}}(G_1, X_1)^{[b_1]+}] / \mathscr{A}(G_{1, \mathbb{Z}_{(p)}})^\circ$$

gives a well-defined stratification on $\bar{\mathscr{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Here, we let $[b^{\mathrm{ad}}]$ be the image of $[b_1]$ under the bijection of Lemma 2.2.1.

We have a natural map of integral models $\mathscr{S}_{K_p}(G, X) \to \mathscr{S}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ induced by the map of Shimura data $(G, X) \to (G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and the extension property, which may moreover be seen explicitly as follows.

$$\mathscr{S}_{K_p}(G, X) \simeq [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_p}(G, X)^+] / \mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \to [\mathscr{A}(G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_p}(G, X)^+] / \mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ,$$

where the last term is isomorphic to $\mathscr{S}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$. In particular, if we let

$$\Delta = \ker(\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}));$$

observe that this is a finite group, and we have that $\mathscr{S}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^+ \simeq \mathscr{S}_{K_p}(G, X)^+/\Delta$ is a finite étale quotient.

The Newton strata for $\bar{\mathscr{S}}_{K_p}(G, X)$ is then defined in [SZ17, 2.3.5] to be the pullback of the Newton strata for $\bar{\mathscr{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ via the natural map

$$\bar{\mathscr{S}}_{K_p}(G, X) \to \bar{\mathscr{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}).$$

On geometric connected components, the construction implies that we have the following compatibility of Newton strata. In particular, we see that the Newton strata are exactly pulled back along the maps

$$\bar{\mathscr{S}}_{K_{1p}}(G_1, X_1)^+ \to \bar{\mathscr{S}}_{K_p}(G, X)^+ \to \bar{\mathscr{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^+.$$

Moreover, as stated in [SZ17, 2.3.4] we can also realize the Newton strata for (G, X) from the one for (G_1, X_1) in terms of the action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$. In fact we have

$$\bar{\mathscr{S}}_{K_p}(G, X)^{[b]} = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \bar{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{[b],+}]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{\circ}.$$

We can take quotients by prime-to-p level K^p to obtain the Newton strata at finite level K.

Finally, observe that by construction we have μ_1 is defined over $\mathbb{Q}_p$, and thus the ordinary locus of $\bar{\mathscr{S}}_{K_{1p}}(G_1, X_1)$ is non-empty, and thus we will also refer to the maximal Newton strata for the Shimura varieties $\bar{\mathscr{S}}_{K_p}(G, X)$ and $\bar{\mathscr{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ as the ordinary locus, even though these are not parametrizing abelian varieties.

2.3. Rapoport-Zink spaces for abelian type.

For this entire section, we let $(G, [b], \{v\})$ be an unramified local Shimura datum, i.e. G is a reductive group over $\mathbb{Q}_p$ which has a reductive model over $\mathbb{Z}_p$, which we still denote by G, $[b] \in B(G, \{v\})$, and $\{v\}$ is a conjugacy class of minuscule cocharacters. If μ is the dominant Hodge cocharacter of Shimura varieties, then in the local Shimura data we should take $v = \sigma(\mu^{-1})$, as explained in [Kis17, 1.4.1].

2.3.1. We first recall the group-theoretic construction of affine Deligne-Lusztig varieties. Let LG and L^+G be the loop group (resp. positive loop group) defined in [XZ17, §3.1], and we consider the affine Grassmannian Gr_G defined as the étale quotient LG/L^+G. Fix a representative $b \in [b]$. The affine Deligne Lusztig variety $X_v^G(b)$ is the perfect closed subvariety inside Gr_G whose $\bar{\mathbb{F}}_p$-points are given by

$$X_v^G(b)(\bar{\mathbb{F}}_p) = \{g \in G(L)/G(W) \mid g^{-1}b\sigma(g) \in G(W)v(p)G(W)\}.$$

where $W = W(\bar{\mathbb{F}}_p)$.

2.3.2. We now want to recall results about Rapoport-Zink spaces, following Shen [She20].

Theorem 2.3.3 (Theorem 4.6,[She20])**.** *Let $(G, [b], \{v\})$ be an unramified local Shimura datum of abelian type. Fix a representative $b \in G(L)$ of $[b] \in B(G, \{v\})$. Then there exists a formal scheme $\mathrm{RZ}(G, b, \{v\})$, which is formally smooth, formally locally of finite type over W, such that moreover we have that the perfection of the reduced special fiber is isomorphic to the affine Deligne-Lusztig variety $X_v^G(b)$.*

In particular, given an element $g \in X_v^G(b)$, we will in the following also say that $g \in \mathrm{RZ}(G, b, \{v\})$.

2.3.4. Note that from the construction of $\mathrm{RZ}(G, b, \{v\})$ that if $M \subset G$ is a Levi subgroup which contains T, and we chose a representative b such that $b \in M(L)$, then we have a natural inclusion
$$\mathrm{RZ}(M, b, \{v\}) \hookrightarrow \mathrm{RZ}(G, b, \{v\})$$
as a closed formal subscheme.

Moreover, there is also a uniformization map on isogeny classes, from [She20, Theorem 6.7].

Theorem 2.3.5. *We have an isomorphism of formal schemes*
$$\Theta : \bigsqcup_{\phi, \phi^{\mathrm{ad}} = \phi_0} I_\phi(\mathbb{Q}) \backslash \mathrm{RZ}(G, b, \{v\}) \times G(\mathbb{A}_f^p) \xrightarrow{\sim} \widehat{S_{K_p / Z_{\phi_0}}}.$$

The group I_ϕ is defined in loc. cit. Here, ϕ_0 (resp. ϕ) is an admissible morphism of Galois gerbs for G^{ad} (resp. G), and $\widehat{S_{K_p / Z_{\phi_0}}}$ is the completion of $\mathscr{S}_{K_p}(G, X)$ along Z_{ϕ_0} a locally closed subspace of $\bar{\mathscr{S}}_{K_p}(G, X)$ defined in loc. cit. which is the union of mod p isogeny classes corresponding to ϕ_0. Since the exact definition is not needed for the rest of this paper, we do not recall the definition of an admissible morphism of Galois gerbs here, but it is defined in [Kis17, 3.3.6]. In this paper it suffices to understand this as an indexing set for isogeny classes.

2.3.6. Since we would like to understand how this map works with a fixed base-point, let us briefly recall the construction. For the Hodge type Shimura datum (G_1, X_1) and a point $x \in \mathscr{S}_{K_p}(G_1, X_1)^b(\bar{\mathbb{F}}_p)$ if we choose the representative $b_1 \in [b_1]$ associated to x as in [Kis17, 1.4.1], we have a uniformization map
$$\Theta_{b_1} : \mathrm{RZ}(G_1, b_1, \{v_1\}) \times G_1(\mathbb{A}_f^p) \to \widehat{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{[b_1]},$$
sending the identity element $1 \in \mathrm{RZ}(G_1, b_1, \{v_1\})$ to x. In fact, this induces an injective map
$$\Theta_{x, b_1} : I_{\phi_1}(\mathbb{Q}) \backslash \mathrm{RZ}(G_1, b_1, \{v_1\}) \times G_1(\mathbb{A}_f^p) \to \widehat{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{[b_1]},$$
where $\widehat{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{[b_1]}$ is the completion of $\bar{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{[b_1]}$ along $\mathscr{S}_{K_{1p}}(G_1, X_1)$.

Then, as described in [She20, §6.3], there is a relation between the isogeny classes of G_1 and G, which can be upgraded into a uniformization map for G. For ϕ an admissible morphism of Galois gerbs for G, we define the formal scheme
$$\widehat{S}(G, \phi) := I_\phi(\mathbb{Q}) \backslash \mathrm{RZ}(G, b, \{v\}) \times G(\mathbb{A}_f^p),$$
and similarly for G_1. Fix ϕ_0 an admissible morphism of Galois gerbs for G^{ad}. Let
$$\widehat{S}(G, \phi_0) = \bigsqcup_{[\phi], \phi^{\mathrm{ad}} = \phi_0} \widehat{S}(G, \phi).$$
Denote by $\widehat{S}(G_1, \phi_1)^+$ the fiber of Θ_{x, b_1} over the connected component $\widehat{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{[b_1]+}$ (note that this is just the open and closed formal subscheme whose mod p fiber lies in $\mathscr{S}_{K_{1p}}(G_1, X_1)^+$.), and similarly for $\widehat{S}(G_1, \phi_0)^+$. From [She20, Proposition 6.6] we know that
$$\widehat{S}(G, \phi_0) \simeq [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \widehat{S}(G_1, \phi_0)^+] / \mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$
Now, given a point $x \in \mathscr{S}_{K_p}(G, X)^{[b]}(\bar{\mathbb{F}}_p)$, from [Kis10, 2.2.5] we may act on it by some element $g^p \in G(\mathbb{A}_f^p)$ such that $g^p x$ lies in $\mathscr{S}_{K_p}(G, X)^{[b]+}(\bar{\mathbb{F}}_p)$. Thus, there exists some $x' \in$

$\bar{\mathcal{S}}_{K_1}(G_1, X_1)^{[b_1]+}(\bar{\mathbb{F}}_p)$ lifting $g^p x$. We thus have a map with base point x'

$$\Theta^+_{x', b_1} : \widehat{S}(G_1, \phi_1)^+ \hookrightarrow \widehat{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{[b_1]+},$$

which on taking product with $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ and quotient by $\mathscr{A}(G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}})^\circ$, thus gives us a map

$$\Theta_{g^p x, b} : \widehat{S}(G, \phi) \hookrightarrow \widehat{S}(G, \phi_0) \hookrightarrow \widehat{\mathscr{S}}_{K_p}(G, X)^{[b]},$$

for some ϕ such that $g^p x$ lies in Z_ϕ. Now, we can translate by $(g^p)^{-1}$ and quotient by K^p to get an injective map

$$\Theta_{x, b} : \widehat{S}(G, \phi) \to \widehat{\mathscr{S}}_K(G, X)^{[b]}.$$

2.3.7.   Note here that this map depends on some auxillary choices that we fix once and for all when we are defining the Rapoport-Zink space for $(G, b, \{v\})$ from $(G_1, b_1, \{v_1\})$, where b_1 is as defined in 2.3.6. More precisely, what we see is that from the proof of [She20, Theorem 4.6] if we let $\mathrm{RZ}(G^{\mathrm{ad}}_1, b^{\mathrm{ad}}_1, \{v^{\mathrm{ad}}_1\})^+$ denote the connected component of $\mathrm{RZ}(G^{\mathrm{ad}}_1, b^{\mathrm{ad}}_1, \{v^{\mathrm{ad}}_1\})$ containing 1, we have an isomorphism

$$\mathrm{RZ}(G^{\mathrm{ad}}_1, b^{\mathrm{ad}}_1, \{v^{\mathrm{ad}}_1\})^+ \simeq \mathrm{RZ}(G, b, \{v\})^+,$$

of connected components containing 1, and also an isomorphism

$$\mathrm{RZ}(G, b, \{v\}) \simeq (J_b(\mathbb{Q}_p) \times \mathrm{RZ}(G^{\mathrm{ad}}_1, b^{\mathrm{ad}}_1, \{v^{\mathrm{ad}}_1\})^+)/J_b(\mathbb{Q}_p)^+$$

where $J_b(\mathbb{Q}_p)^+ \subset J_b(\mathbb{Q}_p)$ is the stabilizer of the connected component containing 1 in $X^G_v(b)$.

Thus, if we let g be the image of $1 \in \mathrm{RZ}(G^{\mathrm{ad}}_1, b^{\mathrm{ad}}_1, \{v^{\mathrm{ad}}_1\})^+$ under these isomorphism, observe that the map $\Theta_{x, b}$ sends $(g, 1) \in \mathrm{RZ}(G, b, \{v\}) \times G(\mathbb{A})^p_f$ to x. Moreover, if we replace b by $b' = g'^{-1}b\sigma(g')$, then the map $\Theta_{x, b'}$ instead sends $(g(g')^{-1}, 1) \in \mathrm{RZ}(G, b', \{v\}) \times G(\mathbb{A}^p_f)$ to x.

Observe now that we can compare the $\Theta_{x, b}$ maps for different base-points. For the next proposition, and in the rest of the paper, we will use $\Theta_{x, b}(g)$ to mean $\Theta_{x, b}(g, 1)$, that is, we implicitly set the second element in $G(\mathbb{A}^p_f)$ to be 1.

Lemma 2.3.8. *Let $g' \in X^G_v(b)$, and denote by $y = \Theta_x(g')$. Then if we let $b' = (g')^{-1}b\sigma(g')$, we have for all $h \in X^G_v(b')$, we have*

$$\Theta_{y, b'}(h) = \Theta_{x, b}(hg').$$

2.3.9.   Consider now the adic generic fiber $\mathcal{M}$ of $\mathrm{RZ}(G, b, \{v\})$. Observe that the uniformization map in Theorem 2.3.5 implies that we have an isomorphism of adic spaces

(2.3.10) $$\bigsqcup_{\phi, \phi^{\mathrm{ad}} = \phi_0} I_\phi(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}^p_f) \xrightarrow{\sim} \mathcal{S}_{\phi_0}.$$

where $\mathcal{S}_{\phi_0}$ denotes the adic generic fiber of $\widehat{\mathscr{S}}_{K_p/Z_{\phi_0}}$. As constructed in [She20, Theorem 5.20], $\mathcal{M} = \mathcal{M}_{K_p}$ fits into a tower $(\mathcal{M}_{K'_p})$ for $K'_p \subset G(\mathbb{Q}_p)$ compact open, and there is an action of $G(\mathbb{Q}_p)$ on this tower.

Moreover, from [She20, Corollary 6.10] for any open compact subgroup $K'_p \subset G(\mathbb{Q}_p)$ we also have an isomorphism of adic spaces

$$\bigsqcup_{\phi, \phi^{\mathrm{ad}} = \phi_0} I_\phi(\mathbb{Q}) \backslash \mathcal{M}_{K'_p} \times G(\mathbb{A}^p_f) \xrightarrow{\sim} \mathcal{S}_{\phi_0, K'_p},$$

where $\mathcal{S}_{\phi_0, K'_p} \to (\mathrm{Sh}_{K'_p}(G, X))^{\mathrm{ad}}$ is the pullback of $\mathcal{S}_{\phi_0} \to (\mathrm{Sh}_{K_p}(G, X))^{\mathrm{ad}}$ under the projection $(\mathrm{Sh}_{K'_p}(G, X))^{\mathrm{ad}} \to (\mathrm{Sh}_{K_p}(G, X))^{\mathrm{ad}}$. In particular, this implies that on the generic fiber the

map Θ_x is equivariant for the Hecke action of $G(\mathbb{Q}_p)$ on $\mathcal{M}$ and the usual Hecke action on $(\mathrm{Sh}_{K_p}(G, X))^{\mathrm{ad}}$.

2.3.11.   Now, observe that $\mathrm{RZ}(G, b, \{v\})$ has by constrution an action of $J_b(\mathbb{Q}_p)$, which on the reduced special fiber agrees with the action of $J_b(\mathbb{Q}_p)$ on $X_v^G(b)$. In particular, since $Z_G(\mathbb{Q}_p) \subset J_b(\mathbb{Q}_p)$, we have an action of central elements. Moreover, we can see that action agrees with the Hecke action of $Z_G(\mathbb{Q}_p)$ on $\mathcal{M}$, since from [She20, Theorem 5.20] we can check this after passing to the associated diamond $\mathcal{M}^\diamond$, which is isomorphic to the moduli space of local shtukas $\mathrm{Sht}_{K_p}(G, b, \{v\})$.

Moreover, we see that $\mathscr{S}_K(G, X)$ also has an action by $Z_G(\mathbb{Q}_p)$, acting by Hecke correspondences, which also induces an action on $\widehat{\mathscr{S}_K}(G, X)$. We have the following:

Proposition 2.3.12. *The map $\Theta_{x,b}$ is equivariant for the action of $Z_G(\mathbb{Q}_p)$.*

Proof. Since on both sides $z \in Z_G(\mathbb{Q}_p)$ acts as isomorphisms, it suffices to check this equivariance on the generic fibers, where both are constructed via the Hecke action. $\square$

2.4. **Quasi-isogenies for abelian type.** We now review the meaning of a quasi-isogeny at p for points lying on a Shimura variety of abelian type. We will only define it on characteristic zero or $\bar{\mathbb{F}}_p$ valued points, using the Rapoport-Zink spaces defined above and we avoid in this paper the delicate question of what an isogeny means outside this setting.

In characteristic zero, for any $g \in G(\mathbb{Q}_p)$, we may consider the covering map from

$$\pi_{K_p} : S_{K^p}(G, X) \to S_K(G, X),$$

given by taking limit over all K_p (trivializing the structures at p), thus if y is the preimage of any $x \in S_K(G, X)$, we may look at $x' := \pi_{K_p}(gy)$. In such a case we say x, x' are isogenous with quasi-isogeny given by g. Note that there are only finitely many such x' isogenous to x via g.

In characteristic p, we use the construction of the uniformization map above. More precisely, for any $x \in \mathscr{S}_K(G, X)^{[b]}(\bar{\mathbb{F}}_p)$ we consider the uniformization map $\Theta_{x,b}$ considered above to construct the isogeny class. Suppose that $\Theta_{x,b}(g) = x$. Then, for any $g' \in X_v^G(b)(\bar{\mathbb{F}}_p)$, we look at the image $x' := \Theta_{x,b}(g')$, and we say x, x' are isogenous via $(g')^{-1}g$.

In general, we cannot say much about the relation between the quasi-isogeny in characteristic zero and the quasi-isogeny in characteristic p, but if $z \in Z_G(\mathbb{Q}_p)$ we can do so, from Proposition 2.3.12, which gives us the following Corollary:

Corollary 2.4.1. *Let $x \in \mathscr{S}_{K_p}(G, X)^{[b]}(\mathbb{F}_p)$, and x' a lift of x. Suppose that we have $\Theta_{x,b}(g, 1) = x$ for some $g \in X_v(b)$. Then, for any $z \in Z_G(\mathbb{Q}_p)$ the mod p reduction of zx' is $\Theta_{x,b}(zg, 1)$.*

2.5. **Models for Hecke correspondences.** For Shimura varieties of Hodge type, the main integral model we consider for Hecke correspondences is $p - \mathrm{Isog}$, a moduli space of p-power (quasi)-isogenies preserving extra structures, which was defined and studied in [Lee20, §6]. However, for general abelian type Shimura varieties, we do not have a moduli interpretation in terms of abelian varieties, and thus it is unclear how to define $p - \mathrm{Isog}$ in this setting. (This, however, will be settled in forthcoming work with Keerthi Madapusi.)

Thus, in this paper we work entirely with middle dimensional algebraic cycles in $\mathscr{S}_{K_p}(G, X) \times \mathscr{S}_{K_p}(G, X)$ or $\mathrm{Sh}_{K_p}(G, X) \times \mathrm{Sh}_{K_p}(G, X)$ and view Hecke correspondences as elements in this middle dimensional Chow group. We denote the space of such correspondences on the generic fiber, the integral model and the special fiber as $\mathrm{Cor}(\mathrm{Sh}_{K_p}(G, X), \mathrm{Sh}_{K_p}(G, X))$, $\mathrm{Cor}(\mathscr{S}_{K_p}(G, X), \mathscr{S}_{K_p}(G, X))$

and $\mathrm{Cor}(\bar{\mathscr{S}}_{K_p}(G,X), \bar{\mathscr{S}}_{K_p}(G,X))$ respectively. Note that all these groups have a multiplicative structure induced by intersection product, as detailed in [Bö02, Appendix A].

Remark 2.5.1. We use the convention here that composition is from right to left, for coherence with our convention that the Hecke action is on the left for the Shimura variety.

By construction, for any such algebraic correspondence C, we have proper maps $p_1, p_2 : C \to \mathrm{Sh}_{K_p}(G,X)$ given by projection onto the first (resp. second) factor, and similarly for the integral model/special fiber.

Let $1_{K_p g K_p}$ be a double coset in $\mathcal{H}_p$, and we let $C_g \subset \mathrm{Sh}_{K_p}(G,X) \times \mathrm{Sh}_{K_p}(G,X)$ be the associated Hecke correspondence, which is given as the image of the projection map

$$\mathrm{Sh}(G,X) \xrightarrow{(\pi \circ g, \pi)} \mathrm{Sh}_{K_p}(G,X) \times \mathrm{Sh}_{K_p}(G,X).$$

By construction, this consists of all pairs of (x', x) with a quasi-isogeny from x to x' given by g. Note that this cycle depends only on $K_p g K_p$ and not on the choice of g, but we use C_g for notational simplicity. Note again that we take $(\pi \circ g, \pi)$ rather than $(\pi, \pi \circ g)$ because all our actions are left actions.

The same argument as in [Bö02, §4.3] show that this association defines an algebra homomorphism

$$h : \mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p), \mathbb{Q}) \to \mathrm{Cor}(\mathrm{Sh}_{K_p}(G,X), \mathrm{Sh}_{K_p}(G,X)),$$

and by taking Zariski closures and restricting to the special fiber, we get an induced homomorphism, which we also denote by h,

(2.5.2) $\qquad h : \mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p), \mathbb{Q}) \to \mathrm{Cor}(\bar{\mathscr{S}}_{K_p}(G,X), \bar{\mathscr{S}}_{K_p}(G,X)).$

Denote by $\mathscr{C}_g$ the Zariski closure of C_g in $\mathscr{S}_K(G,X) \times \mathscr{S}_{K_p}(G,X)$. For any correspondence $\mathscr{C}_g$ over $\mathscr{S}_{K_p}(G,X)$, we will let $C_{g,0}$ denote the special fiber, which is an algebraic correspondence over $\bar{\mathscr{S}}_{K_p}(G,X)$. We have a similar construction for Hecke correspondences for the groups G_1, G^{ad}. Let g^{ad} denote the image of $g \in G(\mathbb{Q}_p)$ in G^{ad}. Let $C_{g^{\mathrm{ad}}}^{\mathrm{ad}} \subset \mathrm{Sh}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \times \mathrm{Sh}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ the associated Hecke correspondence given by $1_{K_p^{\mathrm{ad}} g^{\mathrm{ad}} K_p^{\mathrm{ad}}}$ on the generic fiber. We further denote by $\mathscr{C}_{\mathrm{ad}}^{\mathrm{ad}}$ be the closure and let $C_{g^{\mathrm{ad}},0}^{\mathrm{ad}}$ denote the special fiber.

2.5.3.   In the Hodge type case for (G_1, X_1) we have an alternative definition of Hecke correspondences, introduced in [Lee20, §6] in terms of a moduli space of p-power quasi-isogenies, denoted by $p - \mathrm{Isog}$, or $p - \mathrm{Isog}_{G_1}$ when we want to emphasize the group. This is not a scheme, but rather an ind-scheme. Moreover, we also take limits over the prime-to-p-level. However, for the purposes of this paper we may take a very large power of p^n, and look at the space $p - \mathrm{Isog}_n$ of quasi-isogenies ρ such that $p^n \rho$ is an honest isogeny, and this will be a scheme, though not of finite type. This suffices for us to make sense of irreducible components. As explained in 6.1.6 of loc. cit, for every $1_{K_{1p} g_1 K_{1p}}$ we have a union $h(1_{K_{1p} g_1 K_{1p}})$ of top-dimensional irreducible components of $p - \mathrm{Isog}$. Moreover, we have proper maps $s, t : p - \mathrm{Isog} \to \mathscr{S}_{K_{1p}}(G_1, X_1)$, and thus we may also consider the image $\mathscr{C}'_{g_1}$ of $h(1_{K_{1p} g_1 K_{1p}})$ under the map

$$(s,t) : p - \mathrm{Isog} \to \mathscr{S}_{K_{1p}}(G_1, X_1) \times \mathscr{S}_{K_{1p}}(G_1, X_1).$$

Observe that these two possible Hecke correspondences $\mathscr{C}'_{g_1} = \mathscr{C}_{g_1}$ are the same, since they are both flat and agree on the generic fiber.

2.5.4.   Observe that the Hecke operators for G, G^{ad} are related as follows. For any $g \in G(\mathbb{Q}_p)$, we have that $\mathscr{C}_{g^{\mathrm{ad}}}^{\mathrm{ad}}$ is the image of $\mathscr{C}_g$ under the (finite étale) projection map

$$\mathscr{S}_{K_p}(G, X) \times \mathscr{S}_{K_p}(G, X) \to \mathscr{S}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \times \mathscr{S}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}).$$

Thus, we see that the map on special fibers $C_{g,0} \to C_{g^{\mathrm{ad}},0}^{\mathrm{ad}}$ is also a finite étale map.

Since we want to be able to consider Hecke correspondences which differ by an element of the center, observe that we have the following:

Proposition 2.5.5. *Let $z \in Z_G(\mathbb{Q}_p)$, and let $\mathscr{C}_z$ denote the integral model of the Hecke correspondence corresponding to 1_{zK_p}. Then the projection maps $p_1, p_2 : \mathscr{C}_z \to \mathscr{S}_{K_p}(G, X)$ are isomorphisms.*

Proof. We first observe that these projection maps p_1, p_2 are isomorphisms on the generic fiber. Observe that we can check this after base change to $\mathbb{C}$, and if we let $x = (g, h) \in \mathrm{Sh}_{K_p}(G, X)(\mathbb{C})$ for $g \in G(\mathbb{A}_f)$ and $h \in X$, then observe that if the pair $(x', x) \in C_z$, then we must have $x' = (h, zg) \in \mathrm{Sh}_K(G, X)(\mathbb{C})$. In particular, we see that 'multiplication-by-z' defines an isomorphism $f_z : \mathrm{Sh}_{K_p}(G, X) \to \mathrm{Sh}_{K_p}(G, X)$, and we can similarly define an isomorphism $f_{z^{-1}}$. By the extension property of canonical integral models, these extend to maps $f_z, f_{z^{-1}} : \mathscr{S}_{K_p}(G, X) \to \mathscr{S}_{K_p}(G, X)$ which are inverses to each other, hence we see that $\mathscr{C}_z$ is simply the graph of f_z, which we have shown to be an isomorphism. $\square$

2.6. **Mod p Correspondences for abelian type.** We will now analyze the mod p Hecke correspondences, using results from the correspondences for $p - \mathrm{Isog}_{G_1}$ for (G_1, X_1). We now let C_0 denote any algebraic correspondence on $\bar{\mathscr{S}}_{K_p}(G, X)$ which arises in the special fiber of some $1_{K_p g K_p}$.

Firstly, we can still define a Newton stratification on any such mod p correspondence C_0. For any $[b] \in B(G, \upsilon)$, it is the locally closed $C_0^{[b]}$ of C_0 defined by the preimage via p_1 of $\bar{\mathscr{S}}_{K_p}(G, X)^{[b]}$. We first make the following observation, which implies that we could also have taken the preimages via p_2:

Proposition 2.6.1. *The image of $C_0^{[b]}$ under p_2 also lies in $\bar{\mathscr{S}}_{K_p}(G, X)^{[b]}$.*

Proof. Suppose that we have a pair $(x, x') \in C_0^{[b]}(\bar{\mathbb{F}}_p)$. By construction, such a pair lifts to an O_K-point $(\tilde{x}, \tilde{x}')$, where K is some finite extension of L, where $\tilde{x}, \tilde{x}'$ are quasi-isogenous via $g \in G(\mathbb{Q}_p)$. We can view $\tilde{x}$ as a point on the adic generic fiber $\mathcal{S}_{\phi_0}$, as defined in 2.3.9, and we see that the compatibility with Hecke actions on the generic fiber implies that $\tilde{x}'$ also determines a point on $\mathcal{S}_{\phi_0}$, and hence its mod p reduction lies in $\bar{\mathscr{S}}_{K_p}(G, X)^{[b]}$. $\square$

Following the discussion of Newton strata under the map to adjoint Shimura varieties in §2.2 shows that $C_0^{\mathrm{ad},\mathrm{ord}}$ is the image of the projection to $\bar{\mathcal{S}}_K(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\mathrm{ord}} \times \bar{\mathcal{S}}_K(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\mathrm{ord}}$ of C_0^{ord}, and also $C_{1,0}^{\mathrm{ord}}$, assuming that g^{ad} lifts to an element $g_1 \in G_1(\mathbb{Q}_p)$.

2.6.2.   We now want to recall some facts about lifting cocharacters from G to G_1. Let $\tilde{G}$ be the simply connected cover of G_1^{der}. Let Z_G denote the center of the group G. Recall that we have a central isogeny

$$Z \times \tilde{G} \to G,$$

and thus, for a maximal torus T of G defined over $\mathbb{Q}$, we have a injective map with finite cokernel

$$X_*(Z_G) \oplus X_*(T^{\mathrm{der}}) \hookrightarrow X_*(T)$$

where T^{der} in $\tilde{G}$ is a maximal torus.

In particular, observe that for any cocharacter $\lambda \in X_*(T)$, there exists some positive integer m such that λ^m lifts (up to some cocharacter in $X_*(Z_G)$) to a cocharacter of $\tilde{G}$, and hence to G_1^{der}.

Proposition 2.6.3. *For any $g \in G(\mathbb{Q}_p)$, consider the correspondence C associated with $1_{K_p g K_p}$ as above. Then we have C_0 has a dense ordinary locus. Moreover, the projection map p_1, p_2 are finite.*

Proof. Note that if there exits some g_1 such that $g_1^{\mathrm{ad}} = g^{\mathrm{ad}}$, then $C_{0,1}$ has a dense ordinary locus by an equivalent result for $p - \mathrm{Isog}(G_1, X_1)$ Proposition 2.6.4 quoted below. In particular, since C_0^{ad} is the image of $C_{1,0}$ under a finite étale map, and from §2.2 we know that the Newton strata under this map is compatible, we know that the Hecke correspondence C_0^{ad} has a dense ordinary locus.

Thus, since the image of C_0 under a finite map to $\bar{\mathscr{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \times \bar{\mathscr{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has a dense ordinary locus, the original Hecke correspondence C_0 must have had a dense ordinary locus as well.

Otherwise, we use the Cartan decomposition to replace g with some $\lambda(p)$ for a dominant cocharacter λ, and consider some $g' = \lambda^n(p)\lambda'(p)$, such that λ' is central, and we know from above that this lifts to G_1. Observe that from Proposition 2.5.5 the correspondences $C_{g',0}$ and $C_{\lambda^n(p),0}$ are isomorphic. Thus, it now suffices to show that if $g = \lambda(p)$ does not have a dense ordinary locus, neither does the mod p fiber of the correspondence associated with the double coset $1_{K_p \lambda^n(p) K_p}$. To see this, we suppose that there is some irreducible component of $C_{g,0}$ which does not intersect the ordinary locus. Let $\bar{\eta}$ be the geometric generic point of $C_{g,0}$, and observe that we can lift $\bar{\eta}$ over some local ring R with residue field $\bar{\eta}$. Over $F = \mathrm{Frac}R$ we hence have a pair (x', x), or more precisely there is some lift of x to $y \in \mathrm{Sh}(G, X)$ such that $\pi_{K_p}(gy) = x'$. We can hence consider the pair $(x' = \pi_{K_p}(gy), \pi_{K_p}((g^{-1})^{n-1}y))$ which is a point on the generic fiber of the correspondence associated with the double coset $1_{K_p \lambda^n(p) K_p}$, and if we look at the mod p reduction of the Zariski closure of this in $\mathscr{S}_{K_p}(G, X) \times \mathscr{S}_{K_p}(G, X)$, we clearly have some irreducible component whose projection via p_1 lies outside the ordinary locus as well, and hence this component also does not have a dense ordinary locus.

Now, to show that p_1, p_2 are finite, first observe that if suffices to show that the map p_1 is finite, since otherwise we can look at the dual correspondence $C_0^\vee$, given as the space of pairs (y, x) for $(x, y) \in C_0$. Note that if C_0 appears as a component of $C_{g,0}$, then such a $C_0^\vee$ will appear as a component of $C_{g^{-1},0}$. If the map p_1 is not finite, then we see that this implies that $p_1(C_0)$ is a proper subset of $\bar{\mathscr{S}}_{K_p}(G, X)$ of strictly smaller dimension.

The same analysis as above allows us to see that if g^{ad} admits a lifting to g_1, then we will have some component $C_{1,0}$ which under the projection map p_1 has an image of strictly smaller dimension, and hence the same must be true of $p - \mathrm{Isog}_{G_1}$, which is not possible.

Otherwise, we can again replace g with $\lambda(p)$, and we see that we must have a component of $C_{\lambda^n(p),0}$ with this property that the image under p_1 is of strictly smaller dimension, which again is not possible, again from applying Proposition 2.6.4 below. $\square$

Proposition 2.6.4. *For any correspondence $C_{1,0}$ appearing in the mod p fiber of some Hecke correspondence of $\bar{\mathscr{S}}_{K_1}(G_1, X_1)$, the ordinary locus $C_{1,0}^{\mathrm{ord}}$ is dense in $C_{1,0}$. Moreover, we have that the projection maps $p_1, p_2 : C_{1,0}^{\mathrm{ord}} \to \bar{\mathscr{S}}_{K_{1p}}(G_1, X_1)^{\mathrm{ord}}$ are finite.*

Proof. From 2.5.3, we see that it suffices to consider the corresponding statement for $p-\mathrm{Isog}_{G_1} \otimes \kappa$, the mod p fiber of $p - \mathrm{Isog}$. This follows from [Lee20, Remark 6.1.22], or more precisely the observation there is that the ordinary density condition holds when the only unramified element in $B(G_1, \{v_1\})$ is ordinary. Moreover, for a split group G, only the unramified element in $B(G, \{v\})$ is ordinary. In paticular, since $[b]$ is unramified if and only if $[b^{\mathrm{ad}}]$ is unramified, and we have bijections $B(G, \{v\}) \simeq B(G^{\mathrm{ad}}, \{v^{\mathrm{ad}}\}) \simeq B(G_1, \{v_1\})$, we see that only the ordinary σ-conjugacy class $[b_1^{\mathrm{ord}}]$ in $B(G_1, \{v_1\})$ is unramified, and thus the density statement follows. It remains to see that the projection maps $s, t : p - \mathrm{Isog}_{G_1} \otimes \kappa \to \bar{\mathscr{S}}_K(G_1, X_1)$ are finite when restricted to the ordinary locus, but this follows from the fact that the fibers of s, t are a subvariety of $X_{v_1}^{G_1}(b_1^{\mathrm{ord}})$ for some choice of b_1^{ord} in the ordinary σ-conjugacy class, and $\dim X_{v_1}^{G_1}(b_1^{\mathrm{ord}}) = 0$ □

2.7. Action of correspondences on points.

We briefly recall what it means for a Hecke correspondence to act on a geometric point of the Shimura variety. Let $C \in \mathrm{Cor}(\mathrm{Sh}_{K_p}(G, X), \mathrm{Sh}_{K_p}(G, X))$, such that the projection maps are generically finite and let $x \in \mathrm{Sh}_{K_p}(G, X)(\mathbb{C})$. We can view $[x] \in \mathrm{Ch}_0(\mathrm{Sh}_{K_p}(G, X))$, the Chow group of zero-dimensional cycles on $\mathrm{Sh}_{K_p}(G, X)$, and we can let C act on $[x]$ as defined in [BÖ2, Definition A.4], which we slightly modify to have a left action instead. Let us briefly recall how this goes. We have a refined Gysin map

$$i^! : \mathrm{Ch}_d(C \times_{\mathbb{F}_p} \bar{\mathscr{S}}_{K_p}(G, X)) \to \mathrm{Ch}_0(C \times_{\bar{\mathscr{S}}_{K_p}(G,X)} \bar{\mathscr{S}}_{K_p}(G, X))$$

and a pushforward map

$$p_{1*} : \mathrm{Ch}_0(C) \to \mathrm{Ch}_0(\bar{\mathscr{S}}_{K_p}(G, X)),$$

and we define $C \cdot x = p_{1*}i^!([C \times \{x\}])$.

In particular, from the definition we see that if the correspondence C has projections p_1, p_2 finite, then since the intersection of $p_2^{-1}(x)$ and C is proper, the resultant cycle $i^!([C \times \{x\}])$ in $\mathrm{Ch}_0(C)$ is simply $p_2^{-1}(x)$, thus $C \cdot x$ is simply given by a finite sum

$$C \cdot x = \sum_i a_i [x_i],$$

where x_j's are elements of the finite set $p_1(p_2^{-1}(x))$, and a_i is the multiplicity of the fiber $p_1^{-1}(x_i)$.

Specializing this to the case of Hecke correspondences give us the following lemma:

Lemma 2.7.1. *If $C = C_g$ for a double coset $1_{K_p g K_p}$, and we can write $K_p g K_p$ in terms of left coset decomposition as a finite disjoint union*

$$K_p g K_p = \bigsqcup_j b_j g_j K_p,$$

where $b_j \in \mathbb{Z}$ and $g_j \in G(\mathbb{Q}_p)$, then for any $x \in \mathrm{Sh}_{K_p}(G, X)$ and any preimage $\tilde{x}$ of x under the projection map $\pi_{K_p} : \mathrm{Sh}(G, X) \to \mathrm{Sh}_{K_p}(G, X)$, we have

$$C \cdot x = \sum_j b_j [\pi_{K_p}(g_j \tilde{x})]$$

We also may define an action of a mod p Hecke correspondence, $C_0 \in \mathrm{Cor}(\bar{\mathscr{S}}_{K_p}(G, X), \bar{\mathscr{S}}_{K_p}(G, X))$, and let $x \in \bar{\mathscr{S}}_{K_p}(G, X)(\bar{\mathbb{F}}_p)$. We can again view $[x] \in \mathrm{Ch}_0(\bar{\mathscr{S}}_{K_p}(G, X))$ as a zero-dimensional cycle on $\bar{\mathscr{S}}_{K_p}(G, X)$, and the definition [BÖ2, Definition A.4] works as well in this situation to define an action of correspondences on points. Now, we may compare the actions on the generic fiber and special fiber:

Proposition 2.7.2. *Suppose that we have $C, D \in \mathrm{Cor}(\mathrm{Sh}_{K_p}(G, X), \mathrm{Sh}_{K_p}(G, X))$, and let C_0, D_0 be the mod p reduction. Let $x_0 \in \bar{\mathscr{S}}_{K_p}(G, X)^{\mathrm{ord}}(\bar{\mathbb{F}}_p)$ be an ordinary point, and suppose that $x \in \mathrm{Sh}_{K_p}(G, X)(\mathbb{C})$ is a lift of x_0. Then we have the equalities*

$$\overline{C \cdot D} = C_0 \cdot D_0$$

$$\overline{C \cdot x} = C_0 \cdot x_0,$$

where $\overline{(-)}$ means taking the mod p reduction.

Proof. This follows from [BÖ2, Lemma B.1]. More precisely, if we consider the open subscheme $\mathscr{S}' = \mathscr{S}_K(G, X) \backslash \bar{\mathscr{S}}_{K_p}(G, X)^{\mathrm{non-ord}}$, given by removing the non-ordinary part of the special fiber, the restriction of the Zariski closure $\mathscr{C} \in \mathrm{Cor}(\mathscr{S}_{K_p}(G, X), \mathscr{S}_{K_p}(G, X)$ of C (similarly $\mathscr{D}$ of D) to this open satisfies the condition that p_1, p_2 is finite using Proposition 2.6.3. We can then take the Zariski closure of $C \cdot x$ in $\mathscr{S}'$, or $C \cdot D \in \mathscr{S}' \times \mathscr{S}'$ and we are in a situation to apply [BÖ2, Lemma B.1]. $\qquad\square$

Finally, note that the same argument as in [BÖ2, Lemma A.3] implies the following associativity result:

Lemma 2.7.3. *Let C, D be correspondences in $\mathrm{Cor}(\mathrm{Sh}_{K_p}(G, X), \mathrm{Sh}_{K_p}(G, X))$ such that the projection maps p_1, p_2 are generically finite when restricted to C, D. Then*

$$C \cdot (D \cdot x) = (C \cdot D) \cdot x$$

2.8. **Canonical liftings of ordinary points.** In the following, we will fix the element $b = v(p)$ as a representative of the ordinary σ-conjugacy class, and so the uniformization maps $\Theta_{x,b}$ will be for this choice of b, hence we will suppress the b.

Proposition 2.8.1. *Let x be an ordinary point in $\bar{\mathscr{S}}_{K_p}(G, X)^{\mathrm{ord}}(\bar{\mathbb{F}}_p)$. Then, there exists some special point $\tilde{x}$ given by $(\mathbf{T}, h)$ which satisfies $\mathbf{T}_{\mathbb{Q}_p} \simeq T$, and under this isomorphism we have $\mu_{h, \mathbb{Q}_p} = \mu$.*

Proof. Note that there exists some g^p such that the point x lies in $\bar{\mathscr{S}}_{K_p}(G, X)^+$, since by [Kis10, 2.2.5] $G(\mathbb{A}_f^p)$ acts transitively on connected components of $\mathscr{S}_{K_p}(G, X)$.

From the discussion in §2.2 we can find an ordinary point $x_1 \in \mathscr{S}_{K_{1,p}}(G_1, X_1)^+(\bar{\mathbb{F}}_p)$ which maps to x. We know from [SZ16, Theorem 3.5] that for every ordinary point in $\mathscr{S}_{K_{1,p}}(G_1, X_1)^+(\bar{\mathbb{F}}_p)$, there exists a special point lifting $\tilde{x}_1$, with associated Shimura datum $(\mathbf{T}_1, h_1)$ such that $\mathbf{T}_{1, \mathbb{Q}_p}$ is conjugate to T_1 and under this association we have $\mu_{h_1, \mathbb{Q}_p} = \mu_1$

Consider now the image $\tilde{x}$ of $\tilde{x}_1$ under the map $\mathscr{S}_{K_{1p}}(G_1, X_1)^+ \to \mathscr{S}_{K_p}(G, X)^+$, then $\tilde{x}$ is a special point. Denote the special point datum associated with $\tilde{x}$ by $(\mathbf{T}, h)$. Observe that we have $\mathbf{T}_{\mathbb{Q}_p}^{\mathrm{ad}}$ is conjugate to T^{ad}, so $\mathbf{T}_{\mathbb{Q}_p}$ is quasi-split and hence up to conjugating $\mathbf{T}$ we may assume $\mathbf{T}_{\mathbb{Q}_p} = T$. Now, we want to show that the cocharacter μ_h satisfies $\mu_{h, \mathbb{Q}_p} = \mu$. We first observe that from 2.6.2 that any cocharacter λ, is determined by the map to the adjoint group λ^{ad}, since for any cocharacter $\lambda : \mathbb{G}_m \to G$, it is determined by the induced maps to G/G^{der} and G^{ad}. Observe that since G/G^{der} is commutative, the quotient map $\mu_h : \mathbb{G}_m \to G \to G/G^{\mathrm{der}}$ is constant for all elements $h \in X^+$. Thus, we see that the associated cocharacter μ_h satisfies $\mu_{h, \mathbb{Q}_p} = \mu$, since $\mu_h^{\mathrm{ad}} = \mu_{h_1}^{\mathrm{ad}} = \mu^{\mathrm{ad}}$. $\qquad\square$

2.8.2. Let $M := M_\mu$ be the Levi centralizer of μ. In fact, this special point lift $\tilde{x}$ is an M-adapted lift, a notion we now define. As in the proof of Proposition 2.6.1 any O_K lift $\tilde{x}$ of x,

where K is some finite extension of L, defines a point on the adic space $\mathcal{S}_{\phi_0}$, and from 2.3.10 we have an isomorphism

$$\mathcal{S}_{\phi_0} \simeq \bigsqcup_{\phi, \phi^{\mathrm{ad}} = \phi_0} I_\phi(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}_f^p).$$

Let $\mathcal{N} \subset \mathcal{M}$ be the closed adic space corresponding to the closed formal subscheme

$$\mathrm{RZ}(M, b, \{v\}) \subset \mathrm{RZ}(G, b, \{v\}),$$

where recall we require that the representative b lies in $M(\mathbb{Q}_p)$, which holds since we took $b = v(p)$. We say that $\tilde{x}$ is M-adapted if under the isomorphism above, we have

$$\tilde{x} \in \bigsqcup_{\phi, \phi^{\mathrm{ad}} = \phi_0} I_\phi(\mathbb{Q}) \backslash \mathcal{N} \times G(\mathbb{A}_f^p).$$

Remark 2.8.3. Another possible equivalent way to define this would be to look at the complete local ring at x, and define a closed formal subscheme corrsponding to lifts with M-structure, by identifying the complete local rings for $\mathscr{S}_K(G, X)$ and $\mathscr{S}_{K_1}(G, X)$, and applying the definition for M_{μ_1}-adapted liftings for p-divisible groups as was done in [Lee20, §3.1].

Proposition 2.8.4. *The lift $\tilde{x}$ constructed is Proposition 2.8.1 is M-adapted.*

Proof. Firstly, note that the special point lifting $\tilde{x}_1$ constructed in [SZ16, Theorem 3.5] is M_{μ_1}-adapted. Now, it remains to check that if $\tilde{x}_1 \in I_{\phi_1} \backslash (\mathrm{RZ}(M_1, b_1, \{v_1\})^{+\mathrm{ad}} \times G(\mathbb{A}_f^p))$, where $\mathrm{RZ}(M_1, b_1, \{v_1\})^{+,\mathrm{ad}}$ denotes the adic generic fiber of the connected component of $\mathrm{RZ}(M_1, b_1, \{v_1\})^+$ containing 1, then the image $\tilde{x}$ must lie in $\bigsqcup_{\phi, \phi^{\mathrm{ad}} = \phi_0} I_\phi(\mathbb{Q}) \backslash \mathcal{N} \times G(\mathbb{A}_f^p)$. But this follows from the fact that under the isomorphism

$$\mathcal{M} \simeq (J_b(\mathbb{Q}_p) \times \mathrm{RZ}(G_1^{\mathrm{ad}}, b_1^{\mathrm{ad}}, \{v_1^{\mathrm{ad}}\})^{+,\mathrm{ad}}) / J_b(\mathbb{Q}_p)^+,$$

we have $\mathcal{N} \simeq (J_b(\mathbb{Q}_p) \times \mathrm{RZ}(M_1^{\mathrm{ad}}, b_1^{\mathrm{ad}}, \{v_1^{\mathrm{ad}}\})^{+,\mathrm{ad}}) / J_b(\mathbb{Q}_p)^+$. $\qquad\square$

2.8.5. We now consider $\tilde{x}$ as in the previous proposition. Let $g \in G(\mathbb{A}_f)$ be such that $\tilde{x}$ lies in the image of $g \cdot \mathrm{Sh}(\mathbf{T}, h)$. Then, we may choose a preimage $\tilde{x}'$ of $\tilde{x}$ under the projection map

$$\pi_{K_p} : \mathrm{Sh}(G, X) \to \mathrm{Sh}_{K_p}(G, X)$$

which also lies in the image of $g \cdot \mathrm{Sh}(\mathbf{T}, h)$. We want to relate the Frobenius mod p with the action of $j = \mu_h^{-1}(p)$.

Proposition 2.8.6. *Let x be an ordinary point in $\mathscr{S}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$, and $\tilde{x}'$ the special point lift constructed in 2.8.5. Let $j = \mu_h^{-1}(p)$. Then, the mod p reduction of $\pi_{K_p}(j\tilde{x}) \in \mathrm{Sh}_{K_p}(G, X)$ is exactly the image of the relative Frobenius of $\mathscr{S}_{K_p}(G, X)$ acting on x.*

Proof. This follows from the theory of special points. More precisely, observe that the condition that $\mu_{h, \mathbb{Q}_p} = \mu$ implies that $\tilde{x}$ is defined over $\mathbb{Q}_p$. By construction, we have that the action of the geometric Frobenius $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}_p / \mathbb{Q}_p)$ on $\tilde{x}$ acts by left multiplication by $\mu_h^{-1}(p)$. Thus, we see that the mod p reduction of $\pi_{K_p}(j\tilde{x})$ is the image of x under the relative Frobenius. $\qquad\square$

We now want to show the following proposition, which is a generalization of [BÖ2, Lemma 4.5], which determines the action mod p of $u \in U(\mathbb{Q}_p)$ on such canonical liftings.

Proposition 2.8.7. *Let x be an ordinary point in $\mathscr{S}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$, and let $\tilde{x}' \in \mathrm{Sh}(G, X)$ the special point lift constructed in 2.8.5. Let U denote the unipotent radical of the parabolic*

subgroup P of G associated to μ. Then for any $u \in U(\mathbb{Q}_p)$, we have that the mod p reduction of $\pi_{K_p}(u\tilde{x}') \in \mathscr{S}_{K_p}(G, X)$ is just x.

Proof. We first show this for a similarly defined special-point lift $\tilde{x}'_1 \in \mathrm{Sh}(G_1, X_1)$ for x_1 an ordinary point on the Hodge type cover. To see this, note that from [Kis17, 1.2.18] we have a map $G_1(\mathbb{Q}_p)/G_1(\mathbb{Z}_p) \to G_1(L)/G_1(O_L)$ that sends u to the image of the reduction mod p of the isogeny given by u on $\tilde{x}_1$. Let $g_0 \in X_{v_1}^{G_1}(b_1^{\mathrm{ord}})$ be the image, where here we choose $b_1^{\mathrm{ord}} = v_1(p)$, since by [Wor13, §7] we can always choose a trivialization of the p-divisible group $\mathscr{G}_{x_1}$ over x_1 such that the Frobenius on $\mathbb{D}(\mathscr{G}_{x_1})$ is $b_1^{\mathrm{ord}}\sigma$. We want to say that this element g_0 is simply 1, and to see this we can apply the result of [Lee20, Prop 2.5.4], to see that since $\tilde{x}_1$ is M_1-adapted, where $M_1 = M_{1\mu_1}$, we must have $g_0 \in U(L)$.

Moreover, observe that with this choice of representative $b_1^{\mathrm{ord}} = v(p)$, we can identify $X_{v_1}^{G_1}(b_1^{\mathrm{ord}})$ with the discrete set $M_1(\mathbb{Q}_p)/M_1(\mathbb{Z}_p)$, from [CKV15, Proposition 2.5.4]. More precisely, we can see this by using the Iwasawa decomposition to write any element $g_0 \in X_{v_1}^{G_1}(b_1^{\mathrm{ord}})$ as $g \cdot G_1(O_L)$ for $g \in P(L)$, and sending $g = um$ for $u \in U(L), m \in M(L)$ to $mM_1(O_L)$. Then [CKV15, Proposition 2.5.4] shows that this induces a bijection from $X_{v_1}^{G_1}(v(p))$ to $X_{v_1}^{M_1}(v(p))$, and since v is central in M_1 we have $X_{v_1}^{M_1}(v(p)) \simeq M_1(\mathbb{Q}_p)/M_1(\mathbb{Z}_p)$. In particular, the condition that $g_0 \in U(L)$ implies that g_0 is 1 under all these identifications.

Up to the action of some $g^p \in G(\mathbb{A}_f^p)$, we may assume that $x \in \mathscr{S}_{K_p}(G, X)^+(\overline{\mathbb{F}}_p)$. Observe that we have an isomorphism of root systems $\Phi(G, T) = \Phi(G^{\mathrm{ad}}, T^{\mathrm{ad}}) = \Phi(G_1, T_1)$. Hence if we consider U_1 the unipotent radical of the standard parabolic subgroup of G_1 corresponding to μ_1, then we can identify U_1 with U. In particular, since this result is true for the lift $\tilde{x}_1$, for any $u \in U_1(\mathbb{Q}_p)$, the same is true for the action of $u \in U(\mathbb{Q}_p)$ on $\tilde{x}$. Finally, we see that the action of $G(\mathbb{A}_f^p)$ commutes with the action of $G(\mathbb{Q}_p)$, and with mod p reduction, hence the result we want is also true for any x. $\qquad\square$

We can also precisely determine the mod p reduction of $\pi_{K_p}(\mu_i^{-1}(p)\tilde{x}')$. Recall that we have a map $\Theta_x := \Theta_{x,b=v(p)}$ constructed in the previous section. Note that by [Wor13, §7] we may also arrange the isomorphisms of connected components such that $1 \in X_{v_1}^{G_1}(v_1(p))$ goes to $1 \in X_v^G(b)$, and hence we assume that $\Theta_{x,b}(1) = x$. Moreover, note that with this choice of b, given an element $z \in Z_M(\mathbb{Q}_p)$ where Z_M is the center of M, we have $z \in X_v^G(b)$, and hence from the remark following Theorem 2.3.3 we view $z \in \mathrm{RZ}(G, b, \{v\})$.

Proposition 2.8.8. *Let x be an ordinary point in $\mathscr{S}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$, and let $\tilde{x}' \in \mathrm{Sh}(G, X)$ the special point lift constructed in 2.8.5. Then, for any $\lambda \in X_*(T)$ such that $\lambda(p)$ lies in $Z_M(\mathbb{Q}_p)$ where Z_M denotes the center of M, we have the mod p reduction of $\pi_{K_p}(\lambda(p)\tilde{x}')$ is $\Theta_x(\lambda(p))$.*

Proof. We can see this by analyzing the Hecke action on the tower of adic spaces $(\mathcal{N}_K)_K$ for $K \subset M(\mathbb{Q}_p)$ a compact open subgroup. More precisely, we see that the Hecke actions of $M(\mathbb{Q}_p)$ on $(\mathcal{N}_K)_K$ is simply the restriction of the $G(\mathbb{Q}_p)$ action on the tower $(\mathcal{M})_{K_p'}$, and since the lift $\tilde{x}$ is M-adapted, and $\lambda(p) \in T(\mathbb{Q}_p) \subset M(\mathbb{Q}_p)$, it suffices to study the Hecke action on $(\mathcal{N}_K)_K$. Now, since $\lambda(p)$ is central in M, we have that this Hecke action agrees with the usual action on the formal scheme $\mathrm{RZ}(M, b, \{v\})$ where we view $Z_M(\mathbb{Q}_p)$ as a subset of $J_b(\mathbb{Q}_p)$, as explained in 2.3.11. In particular, the mod p reduction of this action just sends, when looking at the affine Deligne-Lusztig variety, an element $g \in X_v^G(b)$ to $\lambda(p)g$. Thus, we see that for any $\tilde{x}$ which is M-adapted, the mod p reduction of $\pi_{K_p}(\lambda(p) \cdot \tilde{x}')$ is just $\Theta_x(\lambda(p))$, as desired. $\qquad\square$

2.9. **Partial Frobenius for abelian type.** We would now like to define the partial Frobenius to be the p-power quasi-isogeny represented by $\mu_i^{-1}(p)$. Since we are in the abelian type case, we cannot work directly with p-divisible groups. Instead, here we will define the partial Frobenius correspondence, at least over a dense set of points of the ordinary locus, and take the Zariski closure. Again, we continue to fix the element $b = v(p)$ as a representative of the ordinary σ-conjugacy class, and thus we see that $\mu_i^{-1}(p) \in X_v^G(b)$.

Define Frob_i as the Zariski closure in $\bar{\mathscr{S}}_{K_p}(G, X) \times \bar{\mathscr{S}}_{K_p}(G, X)$ of all the pairs $(\Theta_x(\mu_i^{-1}(p)), x)$ for all $x \in \bar{\mathscr{S}}_{K_p}(G, X)^{\mathrm{ord}}$. Since $\Theta_{g^p x}(\mu_i^{-1}(p)) = g^p \Theta_x(\mu_i^{-1}(p))$ for all $g \in G(\mathbb{A}_f^p)$, we see that this Zariski closure is the union of some closed irreducible subschemes of $\bar{\mathscr{S}}_{K_p}(G, X) \times \bar{\mathscr{S}}_{K_p}(G, X)$ of dimension d, where $d = \dim \bar{\mathscr{S}}_{K_p}(G, X)$. We have the following.

Proposition 2.9.1. *The closed points of the restriction $\mathrm{Frob}_i^{\mathrm{ord}}$ of Frob_i to the ordinary locus consists exactly of pairs $(\Theta_x(\mu_i^{-1}(p)), x)$.*

Proof. $\mathrm{Frob}_i^{\mathrm{ord}}$ clearly contains all such points, and we want to show that we have no other closed points. We first observe that by generic flatness, there is some open dense $U \subset \bar{\mathscr{S}}_{K_p}(G, X)$ such that the map $p_2^{-1}(U) \to U$ is flat, and the fibers over $x \in U$ (ignoring multiplicity) are all a single point. Let $x \in \bar{\mathscr{S}}_K(G, X)^{\mathrm{ord}}$. We want to show that translating by $g \in G(\mathbb{A}_f^p)$ will allow us to translate any small enough open neighborhood of x into U. Observe that this would follow if we know that the Zariski closure of the $G(\mathbb{A}_f^p)$-orbit of x is dense in $\bar{\mathscr{S}}_K(G, X)^{\mathrm{ord}}$. We may see this by first noting that up to translating by $g \in G(\mathbb{A}_f^p)$ we may assume that $x \in \bar{\mathscr{S}}_K(G, X)^{\mathrm{ord},+}$, and let x_1 be a lift to $\bar{\mathscr{S}}_{K_1}(G_1, X_1)^{\mathrm{ord},+}$. Then, we see that from the ordinary Hecke orbit conjecture for Hodge type as proved by van Hoften in [vH24], the intersection of the Zariski closure of the $G_1(\mathbb{A}_f^p)$-orbit of x_1 and $\bar{\mathscr{S}}_{K_1}(G_1, X_1)^{\mathrm{ord},+}$ is the whole $\bar{\mathscr{S}}_{K_1}(G_1, X_1)^{\mathrm{ord},+}$. This implies that the intersection of the Zariski closure of the $G^{\mathrm{ad}}(\mathbb{A}_f^p)$-orbit of $x^{\mathrm{ad}} \in \bar{\mathscr{S}}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\mathrm{ord},+}$ and $\bar{\mathscr{S}}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\mathrm{ord},+}$ is the whole $\bar{\mathscr{S}}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\mathrm{ord},+}$ as well, since the projection map is finite. Thus, we also see that the Zariski closure of the $G(\mathbb{A}_f^p)$-orbit of $x \in \bar{\mathscr{S}}_K(G, X)^{\mathrm{ord},+}$ and $\bar{\mathscr{S}}_K(G, X)^{\mathrm{ord},+}$ is the whole $\bar{\mathscr{S}}_K(G, X)^{\mathrm{ord},+}$, so we are done.

In particular, the relation $\Theta_{g^p x}(\mu_i^{-1}(p)) = g^p \Theta_x(\mu_i^{-1}(p))$ for all $g \in G(\mathbb{A}_f^p)$ means that after translating into this open U the fibers of $p_2^{-1}(g^p x)$ just contain a single point, so $p_2^{-1}(x)$ also contains a single point (ignoring multiplicity). In fact, the argument shows that if we look at the perfections $\bar{\mathscr{S}}_{K_p}(G, X)^{\mathrm{ord},\mathrm{perf}} \subset \bar{\mathscr{S}}_{K_p}(G, X)^{\mathrm{perf}}$, and we see that the projection maps $\mathrm{Frob}_i^{\mathrm{ord},\mathrm{perf}} \xrightarrow{p_2^{\mathrm{perf}}} \bar{\mathscr{S}}_{K_p}(G, X)^{\mathrm{ord},\mathrm{perf}}$ is an isomorphism, since it is a universal homeomorphism. A similar statement is also true for p_1. $\qquad\square$

2.10. **Composition of algebraic correspondence.** We first recall the following result we will continually use to show vanishing of algebraic cycles in the situation where we know the projections are generically finite.

Proposition 2.10.1 ([Bö02, Lemma A.6]). *Let X and Y be smooth d-dimensional varieties over an algebraically closed field k. Let C be a generically finite correspondence from X to Y. Assume that there are given open dense subvarieties $X^\circ \subset X$ and $Y^\circ \subset Y$, such that the restriction C° of C to $X^\circ \times Y^\circ$ has projection maps p_1, p_2 finite. Assume that $C \cdot x$ vanishes for all closed points $x \in X^\circ$. Then C vanishes.*

We can apply the above to $X = Y = \bar{\mathscr{S}}_K(G, X)$ and $X^\circ = Y^\circ = \bar{\mathscr{S}}_K(G, X)^{\mathrm{ord}}$, and hence to show equality of correspondences we only need to show equality for the action on ordinary points. Moreover, it is easy to see that the action of Frob_i commutes with the action of $G(\mathbb{A}_f^p)$, since this holds over the ordinary locus by construction. We first show the following:

Lemma 2.10.2. *For each i, the projection map p_2 is étale, and $p_1 : \mathrm{Frob}_i^\circ \to \mathscr{S}_{K_p}(G, X)$ is purely inseparable of degree $p^{\langle 2\rho, \mu_i \rangle}$.*

Proof. Observe that as shown in the proof of Propositon 2.9.1 the projection maps are bijections on geometric points, so the inseparable degree is the same as the degree of the respective maps. We first project Frob_i to a correspondence $\mathrm{Frob}_i^{\mathrm{ad}}$ on $\bar{\mathcal{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, and observe that since the map $\mathrm{Frob}_i \to \mathrm{Frob}_i^{\mathrm{ad}}$ is finite étale, we only need to check the inseparable degrees of the projection maps of $\mathrm{Frob}_i^{\mathrm{ad}}$. More precisely, we have a commutative square

$$
\begin{array}{ccc}
\mathrm{Frob}_i & \longrightarrow & \mathrm{Frob}_i^{\mathrm{ad}} \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1^{\mathrm{ad}}} \\
\bar{\mathscr{S}}_K(G, X) & \longrightarrow & \bar{\mathscr{S}}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}),
\end{array}
$$

where the horizontal maps are all finite étale, thus we see that the inseparable degree of p_1 is the same as p_1^{ad}, and similarly for p_2.

Now, observe that from 2.6.2 there exists some positive integer m such that $(\mu_i^{\mathrm{ad}})^{-m}$ can be lifted to a cocharacter λ_i in $X_*(T_1)$, up to a central cocharacter. Note that this central cocharacter does not change the inseparable degrees, from Proposition 2.5.5.

The same argument as above shows that we only need to determine the inseparable degrees of an algebraic correspondence D_{λ_i} on $\bar{\mathscr{S}}_{K_1}(G_1, X_1)$ which we may define similarly to the definition of Frob_i, that is, we look at the Zariski closure of all pairs $(\Theta_x(\lambda_i), x)$ for $x \in \bar{\mathscr{S}}_{K_1}(G_1, X_1)^{\mathrm{ord}}$. In particular, we want to show that for $D_{\lambda_i(p)}$, we have p_2 is étale, while p_1 is inseparable of degree $p^{\langle 2\rho, -\lambda_i \rangle}$. Indeed, if we let p^a be the inseparable degree of p_2 for Frob_i, and p^b be the inseparable degree of p_1 for Frob_i, then we see that the inseparable degree of p_2 for $D_{\lambda_i(p)}$ is p^{ma} (resp. for p_1 is p^{mb}), and so we deduce $a = 0$, $b = \langle 2\rho, \mu_i \rangle$.

Using 2.5.3, we may first compare it with the following algebraic cycle in $p - \mathrm{Isog}_{G_1} \otimes \mathbb{F}_p$. Following [Lee20, 6.1.14], there is an algebra homomorphism

$$
\bar{h} : \mathcal{H}(M_1(\mathbb{Q}_p)//M_1(\mathbb{Z}_p), \mathbb{Q}) \to \mathbb{Q}[p - \mathrm{Isog}_{G_1} \otimes \mathbb{F}_p],
$$

where $\mathbb{Q}[p - \mathrm{Isog}_{G_1}^{\mathrm{ord}} \otimes \mathbb{F}_p]$ is the Chow group of top-dimensional cycles in $p - \mathrm{Isog}_{G_1} \otimes \mathbb{F}_p$. In particular, we can consider the cycle $\bar{h}(1_{\lambda_i(p)M_1(\mathbb{Z}_p)})$. By definition, points on $\bar{h}(1_{\lambda_i(p)M_1(\mathbb{Z}_p)})$ correspond to quasi-isogenies of points on $\bar{\mathscr{S}}_{K_1}(G_1, X_1)^{\mathrm{ord}}$ which are given, on the canonical lift, by $\lambda_i(p)$.

Now, we argue similarly to [Wed00, 5.7] to determine the degrees. Consider a $\mathbb{Z}_p$-lattice Λ with $M_{1,\mathbb{Z}_p}$-structure: more precisely we can decompose $\Lambda = \Lambda_0 \oplus \Lambda_1$, where Λ_0 is the eigenspace where $\mu_1(z)$ acts trivially, and Λ_1 is the eigenspace where $\mu_1(z)$ acts as z^{-1}. Now, we can use Serre-Tate coordinates on the complete local ring at an ordinary point to determine the degrees of the maps. This complete local ring is a formal torus with character group N which is a sublattice of $\mathrm{Hom}(\Lambda_0, \Lambda_1)$. Write $\lambda_i(p) = (m_0, m_1)$, where m_0 stabilizes Λ_0, and m_1 stabilizes Λ_1. Note that the complete local ring of D_{λ_i} over any ordinary point will correspond to the subspace N' of N^2 consisting of maps (f_1, f_2) such that $f_1 \circ m_0 = m_1 \circ f_2$. Moreover, we see

that the inseparable degree of s, t is the size of the preimage under the projection from N' to the first (resp. second) copy of N.

We hence see that since $\lambda_i(p)$ is, up to some central element, a power of $\mu_i^{-1}(p)$, we see that m_0 acts trivially on Λ_0, and so the inseparable degree of t is 1. For s, observe that since N is free of rank $\langle 2\rho, \mu \rangle$, and post-composing with m_1 acts as p on $\langle 2\rho, \mu_i \rangle$ of the basis elements of N, we have this inseparable degree is $p^{\langle 2\rho, \mu_i \rangle}$.

Now, observe that under the map

$$(s, t) : p - \mathrm{Isog}^{\mathrm{ord}} \otimes \kappa \to \bar{\mathscr{S}}_{K_{1p}}(G_1, X_1) \times \bar{\mathscr{S}}_{K_{1p}}(G_1, X_1),$$

the image of $\bar{h}(1_{\lambda_i(p)M_1(\mathbb{Z}_p)})$ is D_{λ_i}, since the corresponding image of this isogeny is between $(\Theta_x(\lambda_i), x)$. Moreover, we note that by Grothendieck-Messing theory this map is locally an isomorphism, since deforming an isogeny is just a compatible deformation of the source and the target, hence the inseparable degree of s is the same as p_1, and similarly the inseparable degree of t is the same as p_2, which gives us the desired result. $\qquad\square$

Thus, we have the following proposition, which establishes the first two parts of Theorem 2.0.1 on the level of algebraic correspondences.

Proposition 2.10.3. *The partial Frobenius correspondence* $\mathrm{Frob}_{\mathfrak{p}_i} := \mathrm{Frob}_i$ *commute for each* $i \neq j$, *and satisfy that*

(2.10.4)
$$\mathrm{Frob}_{\mathfrak{p}} = \prod_i \mathrm{Frob}_{\mathfrak{p}_i},$$

where $\mathrm{Frob}_{\mathfrak{p}}$ *is the dual graph of the relative Frobenius at* $\mathfrak{p}$ *on* $\bar{\mathcal{S}}_{K_p}(G, X)$.

Proof. Let $x \in \bar{\mathscr{S}}_{K_p}(G, X)^{\mathrm{ord}}(\bar{\mathbb{F}}_p)$, and to apply Proposition 2.10.1, we first observe that by construction of Frob_i and the previous proposition

$$\mathrm{Frob}_i \cdot x = p^{\langle 2\rho, \mu_i \rangle}[\Theta_x(\mu_i^{-1}(p))],$$

since p_2 for Frob_i is étale is totally inseparable of degree $p^{\langle 2\rho, \mu_i \rangle}$. Now, observe that if we let $x_i = \Theta_x(\mu_i^{-1}(p))$, $x_j = \Theta_x(\mu_j^{-1}(p))$, then we have

$$\Theta_{x_i}(\mu_j^{-1}(p)) = \Theta_x(\mu_j^{-1}(p) \cdot \mu_i^{-1}(p)) = \Theta_{x_j}(\mu_i^{-1}(p)),$$

from the construction of Θ_x, the fact that μ_i, μ_j are central in M_μ, and the comparison between different base-points 2.3.8. In particular, this implies that

$$\mathrm{Frob}_i \cdot \mathrm{Frob}_j \cdot x = p^{\langle 2\rho, \mu_i + \mu_j \rangle}[\Theta_x(\mu_j(p)\mu_i(p))] = \mathrm{Frob}_i \cdot \mathrm{Frob}_j \cdot x,$$

and thus we may conclude that $\mathrm{Frob}_j \cdot \mathrm{Frob}_i - \mathrm{Frob}_i \cdot \mathrm{Frob}_j = 0$.

We now proceed to show

(2.10.5)
$$\mathrm{Frob}_{\mathfrak{p}} \cdot x = \left(\prod_i \mathrm{Frob}_{\mathfrak{p}_i} \right) \cdot x,$$

where x is some ordinary point. Note that by Proposition 2.8.6, we see that over the ordinary locus closed points in $\mathrm{Frob}_{\mathfrak{p}}^\circ$ consist exactly of pairs $(\Theta_x(\mu^{-1}(p)), x)$, while we see that

$$\left(\prod_i \mathrm{Frob}_i \right) \cdot x = \prod_i p^{\langle 2\rho, \mu_i \rangle} \Theta_x(\mu_1^{-1} \cdots \mu_d^{-1}(p)) = p^{\langle 2\rho, \mu \rangle} \Theta_x(\mu(p)) = \mathrm{Frob} \cdot x.$$

where we note that the relative Frobenius has degree $\dim \bar{\mathscr{S}}_K(G, X) = p^{\langle 2\rho, \mu \rangle}$, and so we are done. $\square$

Remark 2.10.6. Since we do not have a moduli interpretation for general abelian type Shimura varieties, outside of the μ-ordinary locus, it is not clear what $\mathrm{Frob}_{\mathfrak{p}_i}$ has to do with partial Frobenii. However, the above result at least suggests that the Frobenius isogeny, viewed as an algebraic correspondence, admits a decomposition into factors for each prime $\mathfrak{p}$ above p.

2.11. Abstract Eichler-Shimura Relations. We have isomorphisms of Hecke algebras

$$\mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \simeq \bigotimes_i \mathcal{H}(G_i(\mathbb{Q}_p)//K_{i,p}, \mathbb{Q}).$$

Similarly, since $G_{\mathbb{Q}_p}$ admits a decomposition, we can write

$$(2.11.1) \qquad\qquad\qquad\qquad \mu = \prod_i \mu_i$$

where μ_i is a minuscule cocharacter of G_i. If we let M be the centralizer of μ in G, then similarly we also have

$$M_{\mathbb{Q}_p} = \prod_i M_i$$

where M_i is the centralizer of μ_i in M. Thus, we also have an isomorphism of Hecke algebras

$$(2.11.2) \qquad\qquad \mathcal{H}(M(\mathbb{Q}_p)//M(\mathbb{Z}_p), \mathbb{Q}) \simeq \bigotimes_i \mathcal{H}(M_i(\mathbb{Q}_p)//M_i(\mathbb{Z}_p), \mathbb{Q}).$$

For a quasi-split reductive group G with standard parabolic subgroup P and Levi subgroup M, we can define following algebra homomorphism, known as the twisted Satake homomorphism

$$\dot{\mathcal{S}}_M^G : \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \to \mathcal{H}(M(\mathbb{Q}_p)//M_(\mathbb{Z}_p), \mathbb{Q}),$$

defined as follows. Write $P = NM$, for N the unipotent radical of P, and given a function $f \in \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q})$, we have

$$\dot{\mathcal{S}}_M^G(f)(m) = \int_{n \in N} f(nm) dn.$$

The twisted Satake homomorphism also factors: we have an isomorphism

$$\dot{\mathcal{S}}_M^G = \bigotimes_i \dot{\mathcal{S}}_{M_i}^{H_i}.$$

Consider now the representation $\rho_{\mu_i} : \hat{G} \to \mathrm{GL}(V_{\mu_i})$ of $\hat{G}$ with highest weight cocharacter the dominant Weyl conjugate of $(1, \ldots, \mu_i^{-1}, \ldots, 1)$, where μ_i^{-1} is in the i-th position. Define the polynomial

$$H_i(x) \in \mathcal{H}(H_i(\mathbb{Q}_p)//H_i(\mathbb{Z}_p), \mathbb{Q})[x]$$

as the polynomial given by

$$(2.11.3) \qquad\qquad\qquad H_i(x) = \det(x - p^{n_i} \rho_{\mu_i}(\sigma \ltimes \hat{g})),$$

where $n_i = \langle \rho_i, \mu_i \rangle$, where ρ_i is the half sum of positive roots of G_i. Note that since μ is central in M, μ_i^{-1} is also central in M_i, hence we can consider the element $1_{\mu_i^{-1}(p)M_i(\mathbb{Z}_p)} \in \mathcal{H}(M_i(\mathbb{Q}_p)//M_i(\mathbb{Z}_p), \mathbb{Q})$.

Proposition 2.11.4. *We have the following equality in* $\mathcal{H}(M_i(\mathbb{Q}_p)//M_i(\mathbb{Z}_p), \mathbb{Q})$*: for all* i*,*

$$H_i(p^{\langle 2\rho, \mu_i \rangle} 1_{\mu_i^{-1}(p) M_i(\mathbb{Z}_p)}) = 0.$$

Proof. This follows from the same proof as [BÖ2, Prop 3.4], and the observation that under the decomposition (2.11.2), we see that $H_i(x)$ corresponds to the polynomial with coefficients in $\mathcal{H}(M_i(\mathbb{Q}_p)//M_i(\mathbb{Z}_p), \mathbb{Q})$ defined similarly as in (2.11.3), where we instead take the determinant of highest weight representation of $\hat{G}_i$ corresponding to the dominant Weyl conjugate of μ_i^{-1}. The $p^{\langle 2\rho, \mu_i \rangle}$ factor is because of the dot Weyl action defined in [Wed00, 1.8] to pass from the dominant Weyl conjugate of μ_i^{-1} to μ_i^{-1}. $\square$

2.12. Proof of Eichler-Shimura relations.

We will now show the last part of Theorem 2.0.1 on the level of algebraic correspondences.

Proposition 2.12.1. *Let* (G, X) *be a Shimura datum of abelian type, such that* $G = \mathrm{Res}_{F/\mathbb{Q}} H$*, and* p *a prime satisfying the conditions in Proposition 2.10.3. Let* $H_i(t)$ *be the polynomial defined in (2.11.3), viewed as a polynomial with coefficients in* $\mathrm{Cor}(\bar{\mathcal{S}}_{K_p}(G, X), \bar{\mathcal{S}}_{K_p}(G, X))$ *via* h*. Then we have the equality*

$$H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0.$$

in the ring $\mathrm{Cor}(\bar{\mathcal{S}}_{K_p}(G, X), \bar{\mathcal{S}}_{K_p}(G, X))$*.*

Proof. Firstly, observe that from Proposition 2.6.3, all the terms appearing in $H_i(\mathrm{Frob}_{\mathfrak{p}_i})$ have a dense ordinary locus, hence it suffices to apply Proposition 2.10.1 to show that $H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0$, if we can show that

$$H_i(\mathrm{Frob}_{\mathfrak{p}_i}) \cdot x = 0$$

for all $x \in \mathscr{S}_{K_p}(G, X)(\bar{\mathbb{F}}_p)^{\mathrm{ord}}$.

Let the Hecke polynomial be

$$H_i(t) = \sum_j A_j t^j,$$

for elements $A_j \in \mathcal{H}(G_i(\mathbb{Q}_p)//G_i(\mathbb{Z}_p))$. Using the map h defined in 2.5.2, we can let $h(A_j)$ denote the mod p algebraic cycle in $\mathrm{Cor}(\bar{\mathscr{S}}_{K_p}(G, X), \bar{\mathscr{S}}_{K_p}(G, X))$ corresponding to A_j. Thus, we want to show that

$$(2.12.2) \qquad \left(\sum_j h(A_j) \cdot \mathrm{Frob}_{\mathfrak{p}_i}^j \right) \cdot x = 0.$$

The proof then follows as in [BÖ2, Thm 4.7]. We write the coefficients of the Hecke polynomial A_j in terms of left $G_i(\mathbb{Z}_p)$-cosets

$$A_j = \sum_k n_k^{(j)} m_k^{(j)} G_i(\mathbb{Z}_p),$$

where we further apply the Iwasawa decomposition so that $n^{(j)}$ lies in $U_i(\mathbb{Q}_p)$, and $m^{(j)}$ lies in $M_i(\mathbb{Q}_p)$. Here U_i is the unipotent radical of the standard parabolic subgroup P_i corresponding to μ_i.

If we let $\tilde{x}'$ be the special point lift of x as constructed in 2.8.5, then we may apply Proposition 2.7.2 relating the actions on the generic fiber and the special fiber to rewrite the LHS of (2.12.2) as follows. Firstly, Proposition 2.9.1 allows us to replace $\mathrm{Frob}_i^j \cdot x$ with $p^{\langle 2\rho, \mu_i \rangle} [\Theta_x(\mu_i^{-1}(p^j))]$, while Proposition 2.8.8 shows us that the mod p reduction of $\mu_i^{-1}(p^j) \tilde{x}'$ is the same as $\Theta_x(\mu_i^{-1}(p^j))$.

Moreover, we can apply Lemma 2.7.1 to write the action of the correspondences $h(A_j)$ in characteristic zero in terms of cosets. In particular, we see that the LHS of (2.12.2) is the same as the mod p reduction of

$$(2.12.3) \qquad \sum_{j,k} p^{j\langle 2\rho, \mu_i \rangle} \pi_{K_p}(n_k^{(j)} m_k^{(j)} \mu_i^{-1}(p^j)\tilde{x}').$$

Moreover, we see from Proposition 2.8.7 that we have an equality of mod p reductions

$$\sum_{j,k} p^{j\langle 2\rho, \mu_i \rangle} \left[\overline{\pi_{K_p}(n_k^{(j)} m_k^{(j)} \mu_i^{-1}(p^j)\tilde{x})} \right] = \sum_{j,k} p^{j\langle 2\rho, \mu_i \rangle} \left[\overline{\pi_{K_p}(m_k^{(j)} \mu_i^{-1}(p^j)\tilde{x})} \right],$$

since the $n_k^{(j)} \in U_i(\mathbb{Q}_p)$ do not change the mod p reduction. Now, the abstract Eichler-Shimura relation in Proposition 2.11.4 implies since $\dot{\mathcal{S}}_{M_i}^{H_i}(A_j) = \sum_k m_k^{(j)} M_i(\mathbb{Z}_p)$, we may write the polynomial $H_i(p^{\langle 2\rho, \mu_i \rangle} 1_{\mu_i^{-1}(p)M_c})$ as a sum of left $M_i(\mathbb{Z}_p)$-cosets,

$$\sum_{j,k} p^{j\langle 2\rho, \mu_i \rangle} m_k^{(j)} \mu_i^{-1}(p^j) M_i(\mathbb{Z}_p),$$

and since this sum is zero, together with the fact that $M_i(\mathbb{Z}_p)$ acts trivially on $\tilde{x}'$, we can conclude.

$\square$

2.13. Passage to intersection cohomology.

We now explain here how to get Theorem 2.0.1 on the action on intersection cohomology, from the previous result on algebraic correspondences. This is well known, but we include it here for the reader's convenience. For this, we are using the formalism of cohomological correspondence, explained for instance in [XZ17, A.2].

2.13.1.   In the following, all functors will be derived functors for notational simplicity. We first note that as explained in [Far04, §7.1], for each Hecke correspondence C_g on the generic fiber, we have an associated isomorphism of complexes $u_g : p_2^* \mathcal{L}_{\xi,\ell} \xrightarrow{\sim} p_1^* \mathcal{L}_{\xi,\ell} \simeq p_1^! \mathcal{L}_{\xi,\ell}$ supported on C_g. (Note that the correspondence acts from right to left).

Taking nearby cycles functor Ψ, as constructed in [Far04, 6.3.2] since p_1, p_2 are proper we get a map

$$\Psi(u_g) : p_2^* \Psi \mathcal{L}_{\xi,\ell} \to \Psi p_2^* \mathcal{L}_{\xi,\ell} \xrightarrow{\Psi(u_g)} \Psi p_1^! \mathcal{L}_{\xi,\ell} \to p_1^! \Psi \mathcal{L}_{\xi,\ell}$$

of complexes of sheaves on $C_{g,0}$.

In the proof of 2.1.2, Lan-Stroh show that since $\bar{\mathscr{S}}_K(G, X)$ is smooth, $\Psi \mathcal{L}_{\xi,\ell}$ is isomorphic to $\mathcal{L}_{\xi,\ell}$, where here we abuse notation to denote by $\mathcal{L}_{\xi,\ell}$ the special fiber of canonical extension of $\mathcal{L}_{\xi,\ell}$ to $\mathscr{S}_K(G, X)$. Thus, this gives us an associated map of complexes $\Psi(u_g) : p_2^* \mathcal{L}_{\xi,\ell} \to p_1^! \mathcal{L}_{\xi,\ell}$.

Let $C_{g,0}^{\min}$ denote the Zariski closure of $C_{g,0}$ in $\bar{\mathscr{S}}_K(G, X)^{\min} \times \bar{\mathscr{S}}_K(G, X)^{\min}$, and we let $j_g : C_{g,0} \hookrightarrow C_{g,0}^{\min}$, $j : \bar{\mathscr{S}}_K(G, X) \hookrightarrow \bar{\mathscr{S}}_K(G, X)^{\min}$ the respective open embeddings. Note that, as explained in [Wu22, §3], using Morel's work on t-structures [Mor06], since $\mathcal{L}_{\xi,\ell}$ is pure of some weight m_ξ, we can extend $\Psi(u_g)$ to a map

$$j_{g!*}\Psi(u_g) : p_2^* \mathrm{IC}(\mathcal{L}_{\xi,\ell}) \to p_1^! \mathrm{IC}(\mathcal{L}_{\xi,\ell})$$

which from [Wu22, Lemma 3.4] is moreover compatible with composition of cohomological correspondences.

We now define the cohomological correspondence on Frob_i. As explained in [LS18, §3], the ℓ-adic local system $\mathcal{L}_{\xi,\ell}$ is induced by the $G(\mathbb{Q}_\ell)$-action on the tower of Shimura varieties. More precisely, if we let V_ξ be the $\bar{\mathbb{Q}}_\ell$-vector space of the representation ξ, we can look at a $\bar{\mathbb{Z}}_\ell$-lattice Λ of V_ξ, and for each m we can look at $\Lambda_{0,m} = \Lambda_0/\ell^m\Lambda_0$. Let $U_m = \ker(G(\mathbb{Z}_\ell) \to G(\mathbb{Z}_\ell/\ell^m\mathbb{Z}_\ell))$ and define the torsion étale sheaf V_{ℓ^m} of sections of the contracted product

$$\mathscr{S}_K(G,X) \times^{G(\mathbb{Q}_\ell)/U_m} \underline{\Lambda_{0,m}},$$

where $\underline{\Lambda_{0,m}}$ is the associated constant group scheme over O_{E_v}. we then have

$$\mathcal{L}_{\xi,\ell} \simeq \lim_m V_{\ell^m}.$$

In particular, since $\ell \neq p$, we see that over Frob_i° we have an isomorphism $p_2^*\mathcal{L}_{\xi,\ell} \xrightarrow{\sim} p_1^*\mathcal{L}_{\xi,\ell}$. It remains to observe that the cycle class of $[\text{Frob}_i]$ induces, via the identification with Borel-Moore homology explained in [XZ17, A.2.2], a map $\bar{\mathbb{Q}}_\ell \to p_1^!\bar{\mathbb{Q}}_\ell$, which after tensoring with the isomorphism above gives us a cohomological correspondence

$$u_{\text{Frob}_i} : p_2^*\mathcal{L}_{\xi,\ell} \to p_1^!\mathcal{L}_{\xi,\ell},$$

supported on Frob_i° which we can also take intermediate extension to $\text{Frob}_i^{\min}$, the Zariski closure of Frob_i in $\mathscr{S}_K(G,X)^{\min} \times \mathscr{S}_K(G,X)^{\min}$ to get a cohomological correspondence

$$j_{\text{Frob}!*}u_{\text{Frob}_i} : p_2^*\text{IC}(\mathcal{L}_{\xi,\ell}) \to p_1^!\text{IC}(\mathcal{L}_{\xi,\ell}).$$

Note here that taking intermediate extensions from $\bar{\mathscr{S}}_K(G,X)^{\text{ord}}$ to $\bar{\mathscr{S}}_K(G,X)^{\min}$ is the same as taking intermediate extensions from $\bar{\mathscr{S}}_K(G,X)$ to $\bar{\mathscr{S}}_K(G,X)^{\min}$, since both $\bar{\mathscr{S}}_K(G,X)^{\text{ord}}$ and $\bar{\mathscr{S}}_K(G,X))$ are smooth.

Finally, to conclude cohomological vanishing, let $j_{!*}u_{H_i(\text{Frob}_i)}$ denote the cohomological correspondence given by taking sums, compositions of $j_{\text{Frob}!*}u_{\text{Frob}_i}$ and $j_{g!*}\Psi(u_g)$ as in the expression for $H_i(\text{Frob}_i)$. Observe that since as algebraic cycles on $\bar{\mathscr{S}}_K(G,X)^{\min} \times \bar{\mathscr{S}}_K(G,X)^{\min}$ we have $H_i(\text{Frob}_i) = 0$, the underlying support of the cohomological correspondence vanishes, and thus $j_{!*}u_{H_i(\text{Frob}_i)}$ must also vanish.

3. Semisimplicity Criterion

3.1. An abstract semisimplicity criterion.
We first recall the following theorem of Fayad and Nekovář [FN19, Theorem 1.7]. Let $\Gamma_E = \text{Gal}_E$.

Definition 3.1.1. A continuous representation $\rho : \Gamma_E \to \text{GL}(V)$ is said to be *strongly irreducible* if for any open finite index subgroup $U \subset \Gamma_E$, $\rho|_U$ is irreducible.

Theorem 3.1.2. *Let Γ be a profinite group, $V, W_1, ..., W_r$ non-zero vector spaces of finite dimension over $\bar{\mathbb{Q}}_\ell$. Let $\rho : \Gamma \to \text{GL}(V)$ and $\rho_i : \Gamma \to \text{GL}(W)$ be continuous representations of Γ with Lie algebras*

$$\mathfrak{g}_i = \text{Lie}(\rho_i(\Gamma)), \qquad \mathfrak{g} = \text{Lie}(\rho(\Gamma)).$$

We denote $\bar{\mathfrak{g}}_i = \mathfrak{g}_i \otimes \bar{\mathbb{Q}}_\ell, \bar{\mathfrak{g}} = \mathfrak{g} \otimes \bar{\mathbb{Q}}_\ell$. If the following three conditions hold, then the representation $\rho = \rho^{ss}$ is semisimple.

- *(1) Each ρ_i is strongly irreducible (which implies that each $\mathfrak{g}_i$ is a reductive $\bar{\mathbb{Q}}_\ell$-Lie algebra and each element of its centre acts on W_i by a scalar).*
- *(2) For each $i = 1, \ldots, r$, every (equivalently, some) Cartan subalgebra $\mathfrak{h}_i$ of $\mathfrak{g}_i$ acts on W_i without multiplicities (i.e., all weight spaces of $\mathfrak{h}_i$ on W_i are one-dimensional).*

(3) *There exists an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$ there exists a finite dimensional vector space over $\bar{\mathbb{Q}}_\ell$ (depending on g) $V(g) \supset V$ and elements $u_1, \ldots, u_r \in \mathrm{GL}(V(g))$ such that $u_i u_j = u_j u_i$, $P_{\rho_i(g)}(u_i) = 0$ for all i, $j = 1, \ldots, r$, and V is stable under $u_1 \ldots u_r$ and $u_1 \ldots u_r|_V = \rho(g)$*

Here, $P_{\rho_i(g)}$ is the characteristic polynomial of $\rho_i(g)$.

We state here a theorem of Sen [Sen73, Theorem 1] which we will use to find representations which satisfy condition (2) of the theorem above, given a Galois representation ρ.

Proposition 3.1.3. *Let $\mathfrak{g} = \bar{\mathbb{Q}}_l \cdot \mathrm{Lie}(\rho(\Gamma_E)) \subset \mathfrak{gl}(n, \bar{\mathbb{Q}}_\ell)$ be the $\bar{\mathbb{Q}}_\ell$-Lie algebra generated by the image of ρ. If $\rho|_{G_{E_\tau}}$ is Hodge-Tate for any $\tau : E \hookrightarrow \bar{\mathbb{Q}}_\ell$, then any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts on $\bar{\mathbb{Q}}_\ell^n$ by the n Hodge-Tate weights of $\rho|_{G_{E_\tau}}$.*

3.2. **Proof of Main Theorem.** We prove in this subsection Theorem 1.0.1. Let us recall the setup: we have a representation $\sigma_{\pi^\infty} : \Gamma_E \to V^i(\pi^\infty)$ appearing in the intersection cohomology of some Shimura variety (G, X), where $G = \mathrm{Res}_{F/\mathbb{Q}} H$, associated with some automorphic representation π of $H(\mathbb{A}_F, f)$. We also assume that we have attached via the Langlands correspondence $\rho_{\pi,\mu_i} : \Gamma_E \to {}^L G \xrightarrow{r_{-\mu_i}} \mathrm{GL}(V_{-\mu_i})$. We restate Theorem 1.0.1 here:

Theorem 3.2.1. *Let (G, X) be a Shimura datum of abelian type such that $G = \mathrm{Res}_{F/\mathbb{Q}} H$ for some connected reductive group H, and totally real number field F. Let π be an automorphic representation of $G(\mathbb{A}_{\mathbb{Q},f}) = H(\mathbb{A}_{F,f})$. For all v, suppose that the ${}^L H$-valued Galois representation associated to π exists, and we consider for all $v|\infty$ the composition with the highest weight representation $\rho_{\pi,\mu_v} : \mathrm{Gal}(\bar{F}/F) \to {}^L H \to \mathrm{GL}(V_{-\mu_v})$. Suppose that moreover we also know that*

(1) *ρ_{π,μ_v} is strongly irreducible*
(2) *For all primes $\mathfrak{p}$ of E such that $\mathfrak{p}|l$, the Hodge-Tate weights of $\rho_{\pi,\mu_v}|_{D_\mathfrak{p}}$ are distinct.*

Then σ_{π^∞} is a semisimple representation.

Proof. In the notation of Theorem 3.1.2, we have $\Gamma = \Gamma_E = \mathrm{Gal}(\bar{E}/E)$, $\rho = \sigma_{\pi^\infty}$, $V = V^i(\pi^\infty)$, and each ρ_i is ρ_{π,μ_i}, $W_i = V_{-\mu_i}$. We will apply this criterion where the elements g are the Frobenius elements at the primes Frob_p which are split, and the u_i will be the partial Frobenius $\mathrm{Frob}_{\mathfrak{p}_i}$. Moreover, to show that the representation ρ is semisimple, it suffices to show that $\rho|_U$ is semisimple for any open finite index subgroup of $\mathrm{Gal}(\bar{E}/E)$, and thus, we may restrict to primes p where $G_{\mathbb{Q}_p}$ is split, since such primes have a positive density.

From Theorem 2.0.1, observe that

$$H_{\mu_i}(\mathrm{Frob}_{\mathfrak{p}_i}|V^i(\pi^\infty)) = 0.$$

Note that each summand $V^i(\pi^\infty) \otimes (\pi^\infty)^K$ in $IH^i_{\text{ét}}(\mathrm{Sh}_K(G, X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi,\ell}))$ is stable for the action of $\mathrm{Frob}_{\mathfrak{p}_i}$, since the action of $G(\mathbb{A}_f^p)$ and $\mathcal{H}_i$-actions commute with the $\mathrm{Frob}_{\mathfrak{p}_i}$-action.

Replacing each element of the Hecke algebra $\mathcal{H}_i$ with its eigenvalue on $\pi_p^{K_p}$, we obtain

$$H_{\mu_i}\big|_{\pi_p^{K_p}}(\mathrm{Frob}_{\mathfrak{p}_i}|V^i(\pi^\infty)\otimes(\pi^\infty)^K) = 0.$$

Observe that the polynomial on the right hand side is, by the definition of the Hecke polynomial H_{μ_i}, the characteristic polynomial $\det(t - \rho_i(\mathrm{Frob}_p)) =: P_{\rho_i(g)}$.

In order to show semisimplicity of the endomorphism given by Frob_p, it suffices to show that the P_{ρ_i} has distinct roots. This is an open condition on $\mathrm{Gal}(\bar{E}/E)$, hence it suffices to exhibit

an element $u \in \mathrm{Gal}(\bar{E}/E)$ such that the characteristic polynomial of $\rho_i(u)$ has distinct roots. To see this, observe that Proposition 3.1.3 applies, hence the Lie algebra

$$\bar{\mathfrak{g}} = \bar{\mathbb{Q}}_l \cdot \mathrm{Lie}(\rho_i(\Gamma_E))$$

contains a semisimple element whose eigenvalues on $V_i(\pi^\infty)$ act by the Hodge-Tate weights of $\rho_i|_{D_v}$, where v is a prime dividing ℓ. By assumption (2), the Hodge-Tate weights of $\rho_i|_{D_v}$ are all distinct. This implies that there is an open subset of $\mathrm{Gal}(\bar{E}/E)$ where the characteristic polynomial of $\rho_i(u)$ has distinct roots.

Finally, we conclude using the criterion in Theorem 3.1.2, since we have constructed a Zariski dense set of elements which are semisimple, that ρ is semisimple. □

4. Applications to some Shimura varieties

We discuss here some examples of Shimura varieties where Theorem 1.0.1 applies.

4.1. **Proof of Theorem 1.0.3.** Let F be a totally real field and consider the Shimura variety associated to the group $G = \mathrm{Res}_{F/\mathbb{Q}}H$, where H is some inner form of G^*, defined as one of the following groups:

symplectic: $G^* = \mathrm{GSp}_{2g}$
orthogonal, n even: $G^* = \mathrm{GSO}_{2n}$
orthogonal, n odd: G^* is a non-split quasi-split form of GSO_{2n} relative to E/F, a CM extension

We recall the statement of Theorem 1.0.3 for the reader's convenience:

Theorem 4.1.1. *Let π be a cuspidal L-algebraic automorphic representation of $H(\mathbb{A}_F)$, satisfying*

(1) *There is a finite F-place v_{St} such that $H_{F_{v_{St}}}$ and $G^*_{F_{v_{St}}}$ are isomorphic, and under this isomorphism $\pi_{v_{St}}$ is the Steinberg representation of $G^*(F_{v_{St}})$ twisted by a character.*

(2) *$\pi_\infty |\mathrm{sim}|^{-n(n-1)/4}$ is ξ-cohomological for an irreducible algebraic representation $\xi = \otimes_{y:F\to\mathbb{C}}\xi_y$ of the group $(\mathrm{Res}_{F/\mathbb{Q}}G^*)_\mathbb{C}$, where sim is the similitude factor map $\mathrm{sim}: G^* \to \mathbb{G}_m$.*

(3) *The representation π_v is regular after composing with the representation $\mathrm{GSpin}_{2g+1} \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^g}$ if symplectic (resp. $\mathrm{GSpin}_{2n} \xrightarrow{\mathrm{std}} \mathrm{GL}_{2n}$ if orthogonal) at every infinite place v of F.*

If moreover the ℓ-adic Galois representation $\phi_\pi: \mathrm{Gal}(\bar{F}/F) \to \widehat{H}$ satisfies

(4) *The image of ϕ_π is Zariski dense in $\widehat{H}$,*

Then the Galois module

$$\mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, IH^d_{\text{ét}}(\mathrm{Sh}(G,X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi,\ell})))$$

is semisimple.

Firstly, note that under condition (1) and (2) on π, Kret and Shin [KS20b, Theorem A] and [KS20a, Theorem A] construct the Galois representation

symplectic: $\rho_\pi: \mathrm{Gal}(\bar{F}/F) \to \mathrm{GSpin}_{2n+1}(\mathbb{Q}_\ell)$
orthogonal: $\rho_\pi: \mathrm{Gal}(\bar{F}/F) \to \mathrm{GSpin}_{2n}(\mathbb{Q}_\ell)$

associated to π. In fact, from their arguments, we may also extract the following proposition about concentration in middle degree:

Proposition 4.1.2. *Suppose that π is a cuspidal L-algebraic automorphic representation of $G(\mathbb{A}_f)$ which satisfies that there is a finite F-place v_{St} such that $\pi_{v_{\mathrm{St}}}$ is a twist of Steinberg. Then we have $IH^i_{\acute{e}t}(\mathrm{Sh}(G,X)^{\min}_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi,\ell}))[\pi] = 0$ for all $i \neq d := \dim \mathrm{Sh}(G,X)$.*

Proof. For the case of symplectic groups, this is explained in the last paragraph of [KS20b, Proposition 8.2], while for orthogonal groups, this is the last two parargraphs of [KS20a, Proposition 9.6]. □

Remark 4.1.3. This result relies on Arthur's results in [Art13], and is conditional on the twisted weighted fundamental lemma in general.

Now, we explain how to use Theorem 1.0.1 to deduce Theorem 1.0.3. As explained above, the first two conditions are necessary for the construction of the corresponding Galois representation. Now, we see that if we have condition (4), then at all places $v|\infty$ where the group is not compact modulo center, the associated representation ρ_{π,μ_v} will be strongly irreducible. This is because ρ_{π,μ_v} is irreducible, since it does not factor through any Levi subgroup, and if we look at the Zariski closure of the image in ϕ_π of any finite index open subgroup, it must also be $\widehat{H}$, since $\widehat{H}$ is connected, and thus also irreducible.

We remark here that such a condition should hold for 'most' representations, for example, this is the generalization of the non-CM condition of [Nek18].

Finally, we see that condition (3) implies that the Hodge-Tate weights of ρ_{π,μ_v} are distinct. To see this, we first recall the definition of regularity. Let $\varphi : W_{\mathbb{R}} \to \mathrm{GSpin}_m(\mathbb{C})$ be the Langlands parameter associated with π_v, then the composition

$$\mathbb{C}^\times \subset W_{\mathbb{R}} \to \mathrm{GSpin}_{2n+1}(\bar{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^n}(\bar{\mathbb{Q}}_\ell)$$

(symplectic case) or

$$\mathbb{C}^\times \subset W_{\mathbb{R}} \to \mathrm{GSpin}_{2n}(\bar{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{std}} \mathrm{GL}_{2n}(\bar{\mathbb{Q}}_\ell)$$

(orthogonal case) is conjugate to the cocharacter $z \mapsto \mu_1(z)\mu_2(\bar{z})$ given by some $\mu_1, \mu_2 \in X_*(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{C} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ such that $\mu_1 - \mu_2 \in X^*(T)$. Then we say φ is regular if μ_1 is regular as a character of GL_m, that is, $\langle \alpha^\vee, \mu_1 \rangle \neq 0$ for all coroots $\alpha^\vee$. In particular, we see that the Hodge-Tate cocharacter (as defined in [BG14, §2.4]) of the composition is μ_1, which is regular, and hence the Hodge-Tate weights are distinct.

Remark 4.1.4. We may also impose a condition in terms of ξ, the representation of $G_{\bar{\mathbb{Q}}_\ell}$ such that π appears in the cohomology of $\mathcal{L}_{\xi,\ell}$. The Hodge cocharacter μ_{Hodge} in this case will be $\lambda_\xi + \rho$, where λ_ξ is the dominant highest weight character for the representation of ξ, seen as a cocharacter for the dual group. Under the recipe for the Hodge-Tate cocharacter μ_{HT} from Theorem A in both [KS20a],[KS20b], we want the composition of μ_{HT} with spin or std respectively to be regular.

We thus satisfy all the conditions to apply Theorem 1.0.1, and we can deduce that the Galois module

$$\mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, IH^i_{et}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathrm{IC}(\mathcal{L}_{\xi,\ell})))$$

(which is finite-dimensional over $\bar{\mathbb{Q}}_\ell$) is semisimple for any i.

We may apply Proposition 4.1.2 to restrict just to middle degree where $i = d$.

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