SEMISIMPLICITY OF ÉTALE COHOMOLOGY OF CERTAIN SHIMURA VARIETIES

SI YING LEE

ABSTRACT. Building on work of Fayad and Nekovář, we show that a certain part of the etale cohomology of some abelian-type Shimura varieties is semisimple, assuming the associated automorphic Galois representations exists, and satisfies some good properties. The proof combines an abstract semisimplicity criterion of Fayad-Nekovář with generalized Eichler-Shimura relations for abelian type Shimura varieties and partial Frobenii. We apply this to some symplectic and orthogonal Shimura varieties.

1. Introduction

Let G be a reductive group over $\mathbb{Q}$, and X be a conjugacy class of homomorphisms

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$$

such that (G, X) is a Shimura datum. Given a compact open subgroup $K \subset G(\mathbb{A}_f)$, we can form the associated Shimura variety

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

which is a complex manifold for K small enough, and has a canonical model over E, for some number field E.

Consider a representation valued in complex vector spaces

$$\xi : G_{\mathbb{C}} \to \mathrm{GL}(V_\xi)$$

such that $\xi(Z(\mathbb{Q}) \cap K) = 1$, where Z is the center of G. This gives rise to a locally constant sheaf of complex vector spaces

$$\mathcal{L}_\xi = G(\mathbb{Q})\backslash V_\xi \times X \times G(\mathbb{A}_f)/K.$$

Suppose now that G^{der} is anisotropic, so that the Shimura variety $\mathrm{Sh}_K(G, X)$ is compact. This assumption is not necessary for the statements of the theorems, as long as we replace the cohomology groups with the intersection cohomology; we assume this here for simplicity. We now consider the complex analytic cohomology of the tower of the Shimura variety, i.e.

$$H^i(\mathrm{Sh}(G, X)^{\mathrm{an}}, \mathcal{L}_\xi) := \varinjlim_{K} H^i(\mathrm{Sh}_K(G, X)^{\mathrm{an}}, \mathcal{L}_\xi).$$

Let A_G be the largest $\mathbb{Q}$-split subtorus of Z_G. Choose $h \in X$, and let K_∞ be the stabilizer of h in $G(\mathbb{R})$. By Matsushima's formula for L^2-cohomology established by Franke [Fra98], and (the proof of) Zucker's conjecture, we have a decomposition of $H^i(\mathrm{Sh}(G, X)^{\mathrm{an}}, \mathcal{L}_\xi)$ in terms of Lie algebra cohomology

$$H^i(\mathrm{Sh}(G, X)^{an}, \mathcal{L}_\xi) = \bigoplus_{\pi=\pi_\infty\otimes\pi^\infty} m(\pi) H^i(\mathfrak{a}_G\backslash\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi) \otimes (\pi^\infty),$$

where $\mathfrak{a}_G = \mathrm{Lie}(A_G)$, and π runs through unitary automorphic representations of $G(\mathbb{A})$, and $m(\pi)$ is the multiplicity of π appearing in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$, the space of measurable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which are square-integrable modulo center and with infinitesimal character ω.

Fix a prime ℓ, and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. The representation ξ gives rise to an ℓ-adic automorphic sheaf, in the following way. ξ gives rise to a representation valued in $\overline{\mathbb{Q}}_\ell$-vector spaces

$$\xi_\ell : G_{\overline{\mathbb{Q}}_\ell} \to \mathrm{GL}(V_{\xi,\ell}),$$

and similarly gives rise to a sheaf

$$\mathcal{L}_{\xi,\ell} = G(\mathbb{Q})\backslash V_{\xi,\ell} \times X \times G(\mathbb{A}_f)/K.$$

Since $\mathrm{Sh}_K(G,X)$ is defined over E, we can consider the étale cohomology of the tower

$$H^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}) := \varinjlim_{K} H^i_{\acute{e}t}(\mathrm{Sh}_K(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}),$$

then we have an isomorphism

$$H^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}) \simeq H^i(\mathrm{Sh}(G,X)^{\mathrm{an}}, \mathcal{L}_\xi) \otimes_\iota \overline{\mathbb{Q}}_\ell,$$

which is $G(\mathbb{A}_f)$-equivariant, and thus we have a decomposition

$$H^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,l}) = \bigoplus_{\pi^\infty} V^i(\pi^\infty) \otimes (\pi^\infty),$$

where $V^i(\pi^\infty) = \mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, H^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}))$. If $\mathrm{Sh}_K(G,X)$ is not compact, we consider instead

$$V^i(\pi^\infty) = \mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, IH^i_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}))$$

We now let σ_{π^∞} be the associated Galois representation

$$\sigma_{\pi^\infty} : \mathrm{Gal}(\bar{E}/E) \to V^i(\pi^\infty).$$

On the other hand, the global Langlands correspondence conjectures that to the automorphic representation π we have an associated semisimple Galois representation $\phi_\pi : \mathrm{Gal}(\bar{E}/E) \to {}^LG$, and we denote the composition

$$\rho_{\pi,\mu} : \mathrm{Gal}(\bar{E}/E) \xrightarrow{\phi_\pi} {}^LG \xrightarrow{r_{-\mu}} \mathrm{GL}(V_{-\mu}),$$

where μ is the minuscule Hodge cocharacter associated to the Shimura datum, and $V_{-\mu}$ is the vector space of the associated highest weight representation $r_{-\mu}$. Here, $r_{-\mu}$ is the highest weight representation associated with the dominant Weyl conjugate of μ. We remark here that this $-\mu$ is such that the action of the geometric Frobenius at the unramified places corresponds to the *left* Hecke action of $\mu^{-1}(p)$ under Artin reciprocity.

When the group G is of the form $\mathrm{Res}_{F/\mathbb{Q}}H$ for some connected reductive group H, and F is a totally real field of degree d, the global Langlands correspondence then further conjectures that we have a decomposition (perhaps after passing to a finite extension E' of E) of $\tilde{\rho}$ as

$$\rho_{\pi,\mu} = \bigotimes_{v|\infty} \rho_{\pi,\mu,v},$$

where the product runs over infinite places of F. Moreover, $\rho_{\pi,\mu,v}$ should have the following form. Observe that over $\mathbb{C}$, we have a decomposition

$$\mathrm{Res}_{F/\mathbb{Q}}H \simeq \prod_{v|\infty} H,$$

and the cocharacter μ also decomposes as $\prod_v \mu_v$. Then we should obtain $\rho_{\pi,\mu,v}$ as the composition

$$\rho_{\pi,\mu,v} = \rho_{\pi,\mu_v} : \mathrm{Gal}(\bar{E}'/E') \xrightarrow{\phi_\pi} {}^L H \xrightarrow{r_{-\mu_v}} \mathrm{GL}(V_{-\mu_v}),$$

where here we view π as an automorphic representation of H over F.

We expect a close relationship between σ_{π^∞} and $\rho_{\pi,\mu}$. In the case where π does not have endoscopy (for instance, if it is a twist of Steinberg at some finite place), then we expect to have

$$\sigma_{\pi^\infty} \simeq \rho_{\pi,\mu}^{\oplus m}$$

for some integer m, and thus if $\rho_{\pi,\mu}$ is irreducible, σ_{π^∞} semisimple. Such a result may be shown via the Langlands-Kottwitz method, in combination with the results of this paper.

The main theorem of this paper is the following:

Theorem 1.0.1. *Let (G, X) be a Shimura datum of abelian type such that $G = \mathrm{Res}_{F/\mathbb{Q}}H$ for some connected reductive group H, and totally real number field F. Let π be an automorphic representation of $G(\mathbb{A}_{\mathbb{Q},f}) = H(\mathbb{A}_{F,f})$. For all v, suppose that the ${}^L H$-valued Galois representation associated to π exists, and we consider for all v the composition with the highest weight representation $\rho_{\pi,\mu_v} : \mathrm{Gal}(\bar{F}/F) \to {}^L H \to \mathrm{GL}(V_{-\mu_v})$. Suppose that moreover we also know that*

(1) ρ_{π,μ_v} is strongly irreducible
(2) For all primes $\mathfrak{p}$ of E such that $\mathfrak{p}|l$, the Hodge-Tate weights of $\rho_{\pi,\mu_v}|_{D_\mathfrak{p}}$ are distinct.

Then σ_{π^∞} is a semisimple representation.

This result was previously known in the cases where the abelian type Shimura variety had a cover by an associated PEL type A Shimura variety, as shown in [FN19]. Note that if G is adjoint, then the group G is always a product of groups of the form $\mathrm{Res}_{F/\mathbb{Q}}H$ for some absolutely simple adjoint group H.

Remark 1.0.2. The statement of Theorem 1.0.1 is most interesting when $F \neq \mathbb{Q}$, and H is absolutely simple. Indeed, while the statement is still valid even when $F = \mathbb{Q}$, in this situation it is often the case that the Galois representations ϕ_π is constructed in the cohomology of $\mathrm{Sh}_K(G, X)$, and thus the statement is tautological. Moreover, the condition on ρ_{π,μ_v} to be strongly irreducible is rarely satisfied if H is not absolutely simple.

We can apply the above theorem to situations beyond the cases of unitary groups considered in [FN19]. In particular, we can consider now the cases where H is an inner form of the groups GSp_{2g} or GSO_{2n}, where the automorphic Galois representations have been constructed by Kret and Shin in [KS20b] and [KS20a]. In the following theorem, we will let G^* be one of the following groups:

symplectic: $G^* = \mathrm{GSp}_{2g}$
orthogonal, n **even:** $G^* = \mathrm{GSO}_{2n}$
orthogonal, n **odd:** G^* is a non-split quasi-split form of GSO_{2n} relative to E/F, a CM extension

We now assume that H is an inner form of G^*. Let d be the dimension of $\mathrm{Sh}(G,X)$.

Theorem 1.0.3. *Let π be a cuspidal L-algebraic automorphic representation of $H(\mathbb{A}_F)$, satisfying*

(1) *There is a finite F-place v_{St} such that $H_{F_{v_{St}}}$ and $G^*_{F_{v_{St}}}$ are isomorphic, and under this isomorphism $\pi_{v_{St}}$ is the Steinberg representation of $G^*(F_{v_{St}})$ twisted by a character.*

(2) *$\pi_\infty|\mathrm{sim}|^{-n(n-1)/4}$ is ξ-cohomological for an irreducible algebraic representation $\xi = \otimes_{y:F\to\mathbb{C}}\xi_y$ of the group $(\mathrm{Res}_{F/\mathbb{Q}}G^*)_{\mathbb{C}}$, where sim is the similitude factor map $\mathrm{sim} : G^* \to \mathbb{G}_m$.*

(3) *The representation π_v is regular after composing with the representation $\mathrm{GSpin}_{2g+1} \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^g}$ if symplectic (resp. $\mathrm{GSpin}_{2n} \xrightarrow{\mathrm{std}} \mathrm{GL}_{2n}$ if orthogonal) at every infinite place v of F.*

If moreover the ℓ-adic Galois representation $\phi_\pi : \mathrm{Gal}(\bar{F}/F) \to \widehat{H}$ satisfies

(4) *The image of ϕ_π is Zariski dense in $\widehat{H}$,*

Then the Galois module

$$\mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, IH^d_{\acute{e}t}(\mathrm{Sh}(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}))$$

is semisimple.

To show this result, we first define a 'partial Frobenius isogeny' at primes p which are split in F, and then show the Eichler-Shimura congruence relations for these partial Frobenius for split groups, using results from [Lee20]. More precisely, we show the following:

Theorem 1.0.4. *Let (G,X) be a Shimura datum of abelian type, such that $G = \mathrm{Res}_{F/\mathbb{Q}}H$ for some connected reductive group G, and totally real number field F of degree d. Let p be a prime which is split in F and for which the group $G_{\mathbb{Q}_p}$ is split. Then for all $i = 1,\ldots,d$ we have a partial Frobenius correspondence $\mathrm{Frob}_{\mathfrak{p}_i}$ such that*

$$\mathrm{Frob} = \prod_i \mathrm{Frob}_{\mathfrak{p}_i}$$

where $\mathfrak{p}_i$ is a prime of F dividing p, and in the ring of algebraic correspondences of the mod p reduction of $\mathrm{Sh}_K(G,X)$,

$$H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0,$$

where H_i is the Hecke polynomial at $\mathfrak{p}_i$, which is a renormalized characteristic polynomial of the irreducible representation of $\widehat{H}$ with highest weight $\hat{\mu}_i$.

This is a refinement of the Eichler-Shimura relation considered for Hodge type Shimura varieties in [Lee20]. The Hecke polynomial $H_{G,X}$ defined there (which is valid for all Shimura varieties) will be a tensor product of the polynomials H_i, and in particular, under the same assumptions of splitting of the group G at the prime p, we get the Eichler-Shimura relations for the abelian type Shimura varieties considered in the theorem.

We also note that the construction of this partial Frobenius isogeny is some shadow of the plectic conjecture of Nekovář-Scholl, and while in this paper we construct it only with the additional assumption that $G_{\mathbb{Q}_p}$ is split, in fact the construction does not need this assumption.

Once we have the generalized Eichler-Shimura relations, we can combine the above theorem with a semisimplicity criterion for Lie algebras shown in [FN19], which allows us to deduce the main result. Roughly speaking, the generalized Eichler-Shimura relations, along with the condition of distinct Hodge-Tate weights, shows that we have many Frobenius elements, in particular a positive density of them, which are semisimple. This, together with a strong condition on irreducibility of the Galois representation coming from the Langlands correspondence, allows us to show that the image of Galois is a reductive group, and hence the representation was semisimple.

We emphasize that the main theorem here *cannot* be used to prove semisimplicity of the Galois representations constructed in, for instance, [BLGGT14], [KS20a] or [KS20b]. Instead, the main novelty of this result is when one knows, for instance from [KSZ21], the expected shape of the semisimplification of the Galois representation appearing in the cohomology of the Shimura variety, via the study of zeta functions of Shimura varieties and the Langlands-Kottwitz method. In this case, we can conclude that the equality holds even without taking semisimplification.

Acknowledgements. Many thanks to my advisor, Mark Kisin, for introducing this problem to me, and for many helpful conversations. Thanks also to Sug Woo Shin, for suggestions about applications in Section 4. I would also like to thank the referee for detailed feedback, and all the very helpful suggestions and comments.

2. Eichler-Shimura Relations

In this section, we will prove the key technical result Theorem 1.0.4, and the corresponding result on (intersection) cohomology.

For this entire section, we will fix (G, X) a Shimura datum of abelian type such that $G = \mathrm{Res}_{F/\mathbb{Q}} H$ with F totally real of degree d. Choose a prime p that is split in F such that $G_{\mathbb{Q}_p}$ is split. Let $p = \mathfrak{p}_1 \dots \mathfrak{p}_d$, for $\mathfrak{p}_i$ primes of F above p. In this case, we have

$$G_{\mathbb{Q}_p} = \prod_{\mathfrak{p}_i | p} H_i.$$

Fix a choice of Borel B and torus T of $G_{\mathbb{Q}_p}$. We thus get that the conjugacy class of cocharacters $\{\mu\}$ obtained from X is defined over $\mathbb{Q}_p$, and thus we have $E_v = \mathbb{Q}_p$, and moreover if we let μ be a dominant representative we can write $\mu = \prod_i \mu_i$.

Fix a prime $\ell \neq p$. For a choice of hyperspecial $K_p = \prod_i K_{p,i}$ we get a Hecke polynomial $H_{\mu_i} \in \mathcal{H}_i := \mathcal{H}(H_i(\mathbb{Q}_p)//K_{i,p}, \mathbb{Q})$ for each i.

Let $\mathbb{T}_f$ denote the adelic Hecke algebra. The main result of this section is the following:

Theorem 2.0.1. *There are operators* $\mathrm{Frob}_{\mathfrak{p}_i}$ *for* $i = 1, \dots, d$ *acting on* $IH^i(\mathrm{Sh}_K(G, X)_{\bar{E}}, \mathbb{Q}_\ell)$ *such that*

(1) *The action of any pair* $\mathrm{Frob}_{\mathfrak{p}_i}$, $\mathrm{Frob}_{\mathfrak{p}_j}$ *commute, as does the action of any* $\mathrm{Frob}_{\mathfrak{p}_i}$ *with the* $\mathbb{T}_f$*-action and the* $\mathcal{H}_i$*-action*
(2) $\mathrm{Frob}_{\mathfrak{p}_1} \circ \dots \mathrm{Frob}_{\mathfrak{p}_d} = \mathrm{Frob}_v$*;*
(3) $H_{\mu_i}(\mathrm{Frob}_{\mathfrak{p}_i})$ *acts as 0 on* $IH^i(\mathrm{Sh}_K(G, X)_{\bar{E}}, \mathbb{Q}_\ell)$.

2.1. **Integral model.** We first recall the construction in [Kis10, §3], of the integral model $\mathscr{S}_K(G, X)$ of $S_K(G, X)$ over O_{E_v}. Let (G_1, X_1) denote a Hodge type Shimura datum such that there exists a central isogeny

$$f : G_1^{\mathrm{der}} \to G^{\mathrm{der}}$$

which induces an isomorphism $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \simeq (G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

Fix a connected component $X^+ \subset X$. Let $\mathrm{Sh}_K(G,X)^+$ denote the connected component of $\mathrm{Sh}_K(G,X)$ containing $\{1\} \times X^+$. Consider now the connected component

$$\mathrm{Sh}_{K_{1p}}(G_1,X_1)^+ = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1p}K_1^p}(G_1,X_1)^+$$

of $\mathrm{Sh}_{K_{1p}}(G_1,X_1) = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1p}K_1^p}(G_1,X_1)$, where $K_{1p} = G_1(\mathbb{Z}_p)$. Let $\mathscr{S}_{K_{1p}}(G_1,X_1)^+$ be the Zariski closure of $\mathrm{Sh}_{K_{1p}}(G_1,X_1)^+$ in $\mathscr{S}_{K_{1p}}(G_1,X_1)$. Write $Z = Z_G$, and define

$$\mathscr{S}_{K_p}(G,X) = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1p}}(G_1,X_1)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ,$$

where

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}}) = G(\mathbb{A}_f^p)/Z(\mathbb{Z}_{(p)})^- *_{G(\mathbb{Z}_{(p)})^+/Z(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+,$$

and

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ = G(\mathbb{Z}_{(p)})_+^-/Z(\mathbb{Z}_{(p)})^- *_{G(\mathbb{Z}_{(p)})^+/Z(\mathbb{Z}_{(p)})} G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+.$$

For any abelian type Shimura datum unramified at p with level K which is hyperspecial at p, we denote by

$$\bar{\mathcal{S}}_K(G,X) := \mathscr{S}_K(G,X)_{k_v},$$

where k_v is the residue field of O_{E_v}.

We also recall the result of Lan-Stroh [LS18, Theorem 6.1]:

Theorem 2.1.1. *There is an isomorphism*

$$IH^i(\mathrm{Sh}_K(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell}) \simeq IH^i(\bar{\mathcal{S}}_K(G,X)_{\bar{k}_v}, \mathcal{L}_{\xi,\ell}),$$

where on the right-hand side we abuse notation and let $\mathcal{L}_{\xi,\ell}$ denotes the mod p fiber of the extension of the ℓ-adic local system $\mathcal{L}_{\xi,\ell}$ from the generic fiber. This isomorphism is equivariant for the action of the adelic Hecke algebra $\mathbb{T}_f$, and also the action of the Frobenius at p.

This theorem will allow us to deduce results about $IH^i(\mathrm{Sh}_K(G,X)_{\bar{E}}, \mathcal{L}_{\xi,\ell})$ from results on the special fiber.

2.2. **Newton stratification.** We now recall the construction of Newton strata for Shimura varieties of abelian type, as constructed in [SZ17, §2]. Observe that for any connected reductive group G, and conjugacy class of minuscule cocharacters $\{\mu\}$, we have an isomorphism $B(G,\{\mu\}) = B(G^{\mathrm{ad}},\{\mu^{\mathrm{ad}}\})$, from [Kot97, Section 6.5]. In [SZ17, §2], the Newton strata is first constructed for adjoint groups, and thus we have a stratification on the mod p fiber. More precisely, the condition that $G_{\mathbb{Q}_p}$ is split implies the same for $G^{\mathrm{ad}}_{\mathbb{Q}_p}$, hence the mod p fibers are all defined over $\mathbb{F}_p$.

The Newton strata for $\bar{\mathcal{S}}_{K_p}(G,X)$ is then defined to be the pullback of the Newton strata for $\bar{\mathcal{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$ via the natural map

$$\bar{\mathcal{S}}_{K_p}(G,X) \to \bar{\mathcal{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}}).$$

Moreover, we can also describe the Newton strata using connected components. Note that the splitting condition on $G_{\mathbb{Q}_p}$ also implies that $G_{1,\mathbb{Q}_p}$ is also split. For any $b \in B(G,\mu)$, let $\bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)^{b,+}$ be the pullback of $\bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)^b$ under the inclusion $\bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)^+ \hookrightarrow \bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)$. Observe that the Newton strata of $\bar{\mathcal{S}}_{K_p}(G,X)^+$ and $\bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)^+$ is exactly that pulled back along the maps

$$\bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)^+ \to \bar{\mathcal{S}}_{K_p}(G,X)^+ \to \bar{\mathcal{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}})^+.$$

Moreover, we can also realize the Newton strata in terms of the action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$. In fact we have

$$\bar{\mathcal{S}}_{K_p}(G,X)^b = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)^{b,+}]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^{\circ}.$$

We can take quotients by prime-to-p level K^p to obtain the Newton strata at finite level K.

Finally, observe that the splitting condition on p also implies that the ordinary locus of $\bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)$ is non-empty, and thus we will also refer to the maximal Newton strata for the Shimura varieties $\bar{\mathcal{S}}_{K_p}(G,X)$ and $\bar{\mathcal{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$ as the ordinary locus, even though these are not parametrizing abelian varieties.

2.3. **Rapoport-Zink uniformization for abelian type.** We now want to recall results about Rapoport-Zink spaces, following Shen [She20].

Theorem 2.3.1 (Theorem 4.6,[She20])**.** *Let $(G,[b],\{\mu\})$ be an unramified local Shimura datum of abelian type. Fix a representative $b \in G(L)$ of $[b] \in B(G,\{\mu\})$. Then there exists a formal scheme $\mathrm{RZ}(G,b,\{\mu\})$, which is formally smooth, formally locally of finite type over O_{E_v}, such that moreover we have that the perfection of the reduced special fiber is isomorphic to the affine Deligne-Lusztig variety $X_\mu^G(b)$.*

Moreover, there is also a uniformization map on isogeny classes, from [She20, Theorem 6.7].

Theorem 2.3.2. *We have an isomorphism of formal schemes*

$$\Theta : \bigsqcup_{\phi,\phi^{\mathrm{ad}}=\phi_0} I_\phi(\mathbb{Q})\backslash \mathrm{RZ}(G,b,\{\mu\} \times G(\mathbb{A}_f)/K^p \xrightarrow{\sim} \widehat{S_K}_{/Z_{\phi_0,K^p}}.$$

Here, ϕ_0 (resp. ϕ) is an admissible morphism of Galois gerbs for G^{ad} (resp. G), and for Z_{ϕ_0,K^p} a locally closed subspace of $\bar{\mathcal{S}}_K(G,X)$ defined in loc. cit. Since we would like to understand how this map works with a fixed base-point, let us briefly recall the construction.

For the Hodge type Shimura datum (G_1,X_1) and a point $x \in \bar{\mathcal{S}}_K(G_1,X_1)(\bar{\mathbb{F}}_p)$ we have a uniformization map

$$\Theta = \Theta_x : \mathrm{RZ}(G_1,b_1,\{\mu_1\}) \times G_1(\mathbb{A}_f^p) \to \widehat{\mathscr{S}_{K_{1p}}}(G_1,X_1)^{b_1},$$

which in fact induces an injective map

$$\Theta : \hat{S}(G_1,\phi_1) := I_{\phi_1}(\mathbb{Q})\backslash \mathrm{RZ}(G_1,b_1,\{\mu_1\}) \times G_1(\mathbb{A}_f^p) \to \widehat{\mathscr{S}_{K_{1p}}}(G_1,X_1)^{b_1}.$$

Then, as described in [She20, §6.3], there is a relation between the isogeny classes of G_1 and G, which can be upgraded into a uniformization map for G. Fix $\phi_0 : \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ad}}}$ an admissible morphism of Galois gerbs, and let

$$\widehat{S}(G,\phi_0) = \bigsqcup_{[\phi],\phi^{\mathrm{ad}}=\phi_0} \widehat{S}(G,\phi),$$

where

$$\widehat{S}(G,\phi) := I_\phi(\mathbb{Q})\backslash \mathrm{RZ}(G,b,\{\mu\}) \times G(\mathbb{A}_f^p).$$

Denote by $\widehat{S}(G_1,\phi_1)$ the fiber of Θ over the connected component $\widehat{\mathscr{S}_{K_{1p}}}(G_1,X_1)^{b_1+}$ (note that this is just the open and closed formal subscheme whose mod p fiber lies in $\bar{\mathcal{S}}_{K_{1p}}(G_1,X_1)^+$.) Then if we denote

$$\widehat{S}(G_1,\phi_0)^+ = \bigsqcup_{[\phi_1],\phi_1^{\mathrm{ad}}=\phi_0} \widehat{S}(G_1,\phi_1)^+,$$

we moreover have [She20, Proposition 6.6], that

$$\widehat{S}(G,\phi_0) \simeq [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \widehat{S}(G_1,\phi_0)^+]/\mathscr{A}(G_{1\mathbb{Z}_{(p)}})^\circ.$$

Now, given a point $x \in \bar{\mathcal{S}}_K(G,X)^b(\bar{\mathbb{F}}_p)$, from [Kis10, 2.2.5] we may act on it by some element $g^p \in G(\mathbb{A}_f^p)$ such that $g^p x$ lies in $\bar{\mathcal{S}}_K(G,X)^{b+}(\bar{\mathbb{F}}_p)$. Thus, there exists some $x' \in \bar{\mathcal{S}}_{K_1}(G_1,X_1)^{b_1+}(\bar{\mathbb{F}}_p)$ lifting $g^p x$. We thus have a map with base point x'

$$\Theta^+_{x'} : \widehat{S}(G_1,\phi_1)^+ \to \widehat{\mathscr{S}_{K_{1p}}}(G_1,X_1)^{b_1+},$$

which thus induces a map

$$\Theta_{g^p x} : \widehat{S}(G,\phi) \to \widehat{\mathscr{S}_{K_p}}(G,X),$$

for some ϕ, from which we can translate by $(g^p)^{-1}$ and quotient by K^p to get a map

$$\Theta_x : \widehat{S}(G,\phi) \to \widehat{\mathscr{S}_K}(G,X).$$

2.4. **Quasi-isogenies for abelian type.** We now review the meaning of a quasi-isogeny at p for points lying on a Shimura variety of abelian type. We will only define it on characteristic zero or $\bar{\mathbb{F}}_p$ valued points, using the Rapoport-Zink spaces defined above and we avoid in this paper the delicate question of what an isogeny means outside this setting.

In characteristic zero, for any $g \in G(\mathbb{Q}_p)$, we may consider the covering map from

$$\pi_{K_p} : S_{K^p}(G,X) \to S_K(G,X),$$

given by taking limit over all K_p (trivializing the structures at p), thus if y is the preimage of any $x \in S_K(G,X)$, we may look at $x' := \pi(gy)$. In such a case we say x, x' are isogenous with quasi-isogeny given by g. Note that there are only finitely many such x' isogenous to x.

In characteristic p, we use the construction of the uniformization map above. More precisely, consider any $x \in \bar{\mathcal{S}}_K(G_1,X_1)^b(\bar{\mathbb{F}}_p)$ we consider the uniformization map Θ_x considered above to construct the isogeny class. In particular, for any $g \in X_\mu^G(b)(\bar{\mathbb{F}}_p)$, we look at the image $x' := \Theta_x(g,1)$, and we say x, x' are isogenous with quasi-isogeny given by g.

2.5. **Models for Hecke correspondences.** For general abelian type Shimura varieties, we do not have a moduli interpretation in terms of abelian varieties, and thus it is unclear how to define $p-\mathrm{Isog}$ in this setting. (This, however, will be settled in forthcoming work with Keerthi Madapusi.) Thus, in this paper we work entirely with dimension d algebraic cycles in $\mathscr{S}_K(G,X)\times\mathscr{S}_K(G,X)$, where d is the dimension of $\mathrm{Sh}_K(G,X)$, and view Hecke correspondences as such algebraic cycles.

Consider the Hecke correspondence $C \subset \mathrm{Sh}_K(G,X) \times \mathrm{Sh}_K(G,X)$, which is given as the pushforward via the map $\mathrm{Sh}_{K^p}(G,X) \xrightarrow{(\pi,\pi\circ g)} \mathrm{Sh}_K(G,X)\times\mathrm{Sh}_K(G,X)$. This consists of all pairs of (x,x') which are isogenous with isogeny given by g, and we label the corresponding algebraic cycle by $1_{K_p g K_p}$, since this is the element in the spherical Hecke algebra which corresponds to this.

Denote by $\mathscr{C}$ the Zariski closure of C in $\mathscr{S}_K(G,X) \times \mathscr{S}_K(G,X)$. For any correspondence $\mathscr{C}$ over $\mathscr{S}_K(G,X)$, we will let C_0 denote the special fiber, which is an algebraic correspondence over $\bar{\mathcal{S}}_K(G,X)$. We have a similar construction for Hecke correspondences for the groups G_1, G^{ad}. Let g^{ad} denote the image of $g \in G(\mathbb{Q}_p)$ in G^{ad}. Let $C^{\mathrm{ad}} \subset \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}}) \times \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$ the associated Hecke correspondence given by $1_{K_p^{\mathrm{ad}} g^{\mathrm{ad}} K_p^{\mathrm{ad}}}$ on the generic fiber. We further denote by $\mathscr{C}^{\mathrm{ad}}$ be the closure and let C_0^{ad} denote the special fiber.

Let $\mathrm{Cor}(\bar{\mathcal{S}}_{K_p}(G,X),\bar{\mathcal{S}}_{K_p}(G,X))$ denote the Chow group of d-dimensional algebraic cycles in $\bar{\mathcal{S}}_{K_p}(G,X),\bar{\mathcal{S}}_{K_p}(G,X)$, with composition given by taking intersection products, as defined in [B02, Appendix]. We refer to elements of $\mathrm{Cor}(\bar{\mathcal{S}}_{K_p}(G,X),\bar{\mathcal{S}}_{K_p}(G,X))$ as correspondences.

The map taking $1_{K_pgK_p}$ to C_0 defined above hence gives a ring homomorphism

$$h : \mathcal{H}(G(\mathbb{Q}_p)//K_p,\mathbb{Q}) \to \mathrm{Cor}(\bar{\mathcal{S}}_{K_p}(G,X),\bar{\mathcal{S}}_{K_p}(G,X)).$$

Observe that the Hecke operators for G, G^{ad} are related as follows. C^{ad} is the image of C under the (finite étale) projection map

$$\mathscr{S}_K(G,X)\times\mathscr{S}_K(G,X)\to\mathscr{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}})\times\mathscr{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}}).$$

Thus, we see that $C_0\to C_0^{\mathrm{ad}}$ is also a finite étale map.

2.6. **Mod p Correspondences for abelian type.** We will now analyze the mod p Hecke correspondences, using results from the correspondences for $p-\mathrm{Isog}$ for (G_1,X_1). We now let C_0 denote any dimension d algebraic cycle in $\bar{\mathcal{S}}_K(G,X)$.

Firstly, we can still define a Newton stratification on any mod p correspondence C_0 which arises in the special fiber of some $1_{K_pgK_p}$. Namely, it is the locally closed C_0^b of C_0 defined by the fiber of p_1 above $\bar{\mathcal{S}}_K(G,X)^b$, where p_1 is the projection to the first factor of inclusion $C_0\subset\bar{\mathcal{S}}_K(G,X)\times\bar{\mathcal{S}}_K(G,X)$. We first make the following observation:

Proposition 2.6.1. *The image of C_0^b under p_2 also lies in $\bar{\mathcal{S}}_K(G,X)^b$.*

Proof. Suppose on the contrary that there exists some $\bar{\mathbb{F}}_p$ point (x,x') on C_0^b such that $[b(x)]\neq[b(x')]$. By construction, there exists a lift of (x,x') to an O_K-point $(\tilde{x},\tilde{x}')$, where K is some finite extension of $\breve{\mathbb{Q}}_p$, such that over K we have that the two points are quasi-isogenous with quasi-isogeny given by g. We can now consider the associated $G^c(\mathbb{Q}_p)$-valued crystalline Galois representation over the generic fiber of $\tilde{x}$ and $\tilde{x}'$, and we see that these two $G^c(\mathbb{Q}_p)$-Galois representations are isomorphic since the integral $\mathbb{Z}_p$ lattices differ by an element of $G^c(\mathbb{Q}_p)$. Hence, if we look the image of this under the exact tensor functor D_{crys} from crystalline Galois representations to filtered F-crystals, then the associated F-isocrystals on x,x', considered via the crystalline canonical model constructed in [Lov17, §3.4], are the same. Finally, we see from [SZ17, 5.4.3] that the Newton stratification coming from the classification of G^c-F-isocrystals from [Lov17] is the same as defined in Section 2.2. □

Following the discussion of Newton strata under the map to adjoint Shimura varieties above shows that $C_0^{\mathrm{ad,ord}}$ is the image of the projection to $\bar{\mathcal{S}}_K(G^{\mathrm{ad}},X^{\mathrm{ad}})^{\mathrm{ord}}\times\bar{\mathcal{S}}_K(G^{\mathrm{ad}},X^{\mathrm{ad}})^{\mathrm{ord}}$ of C_0^{ord}, and also $C_{1,0}^{\mathrm{ord}}$, assuming that g^{ad} lifts to an element $g_1\in G_1(\mathbb{Q}_p)$.

2.6.2. We now want to recall some facts about lifting cocharacters from G to G_1. Let $\tilde{G}$ be the simply connected cover of G_1^{der}. Let Z_G denote the center of the group G. Recall that we have a central isogeny

$$Z\times\tilde{G}\to G,$$

and thus, for a maximal torus T of G defined over $\mathbb{Q}$, we have a injective map with finite cokernel

$$X_*(Z_G)\oplus X_*(T^{\mathrm{der}})\hookrightarrow X_*(T)$$

where T^{der} in $\tilde{G}$ is a maximal torus.

In particular, observe that for any cocharacter $\lambda \in X_*(T)$, there exists some positive integer m such that λ^m lifts (up to some cocharacter in $X_*(Z_G)$) to a cocharacter of $\tilde{G}$, and hence to G_1^{der}.

Proposition 2.6.3. *For any $g \in G(\mathbb{Q}_p)$, consider the correspondence C associated with $1_{K_pgK_p}$ as above. Then we have C_0 has a dense ordinary locus.*

Proof. Note that if there exits some g_1 such that $g_1^{\mathrm{ad}} = g^{\mathrm{ad}}$, then we are done since we have an isomorphism $B(G,\mu) \simeq B(G^{ad},\mu^{ad})$, the condition that G is split implies that $C_{0,1}$ has a dense ordinary locus by the result for $p-\mathrm{Isog}(G_1,X_1)$ Proposition 2.6.4 quoted below, since we have a natural map $p-\mathrm{Isog}(G_1,X_1)\otimes\kappa \to \bar{\mathcal{S}}_K(G_1,X_1)\times\bar{\mathcal{S}}_K(G_1,X_1)$, which respects the Newton stratifications and is finite when restricted to any connected component of $p-\mathrm{Isog}(G_1,X_1)\otimes\kappa$.

Thus, by the above arguments, we know that the Hecke correspondence C_0^{ad} has a dense ordinary locus, since it is the image under a finite étale map of the Hecke correspondence $C_{1,0}$ on $\bar{\mathcal{S}}_{K_1}(G_1,X_1)$. Moreover, we know that $C_{1,0}$ will have a dense ordinary locus since the image of $p-\mathrm{Isog}(G_1,X_1)\otimes\kappa$ has a dense μ-ordinary locus.

Thus, since the image of C_0 under a finite map to $\bar{\mathcal{S}}_K(G^{\mathrm{ad}},X^{\mathrm{ad}})\times\bar{\mathcal{S}}_K(G^{\mathrm{ad}},X^{\mathrm{ad}})$ has a dense ordinary locus, the original Hecke correspondence must C_0 must have had a dense ordinary locus as well.

Otherwise, we use the Cartan decomposition to replace g with some $\lambda(p)$ for a dominant cocharacter λ, and consider some $\lambda^n(p)\lambda'(p)$, such that λ' is central, and we know from above that this lifts to G_1. Observe that the correspondences C_0 we get do not change when we modify by an element of the center. Thus, we now want to say that if $g=\lambda(p)$ does not have a dense ordinary locus, neither does the mod p fiber of the correspondence associated with the double coset $1_{K_p\lambda^n(p)K_p}$. To see this, we suppose that there is some irreducible component of C_0 which does not intersect the ordinary locus. Let $\bar{\eta}$ be the geometric generic point of C_0, and observe that we can lift $\bar{\eta}$ over some local ring R with residue field $\bar{\eta}$. Over $F=\mathrm{Frac}R$ we hence have a pair (x,x'), or more precisely there is some lift of x to $y\in \mathrm{Sh}_{K^p}(G,X)$ such that $\pi_{K_p}(gy)=x'$. We can hence consider the pair $(\pi_{K_p}((g^{-1})^{n-1}y), x'=\pi_{K_p}(gy))$ which is a point on the generic fiber of the correspondence associated with the double coset $1_{K_p\lambda^n(p)K_p}$, and if we look at the mod p reduction of the Zariski closure of this in $\mathscr{S}_K(G,X)\times\mathscr{S}_K(G,X)$, we clearly have some irreducible component whose projection via π_2 lies outside the ordinary locus as well, and hence we do not have a dense ordinary locus for this. □

We recall the following result from [Lee20, Remark 6.1.22]:

Proposition 2.6.4. *When G is split over $\mathbb{Q}_p$, the ordinary locus is dense in $p-\mathrm{Isog}(G,X)\otimes\kappa$.*

Here, $p-\mathrm{Isog}(G,X)\otimes\kappa$ is the mod p fiber of the ind-scheme which is the moduli of p-power quasi-isogenies respecting G-structure.

2.7. **Canonical liftings of ordinary points.** Consider now any ordinary point $x\in\mathscr{S}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$. We suppose now that the point x lies in $S_{K_p}(G,X)^+$. Note that this is always possible up to the action of some $g^p\in G(\mathbb{A}_f^p)$, since by [Kis10, 2.2.5] $G(\mathbb{A}_f^p)$ acts transitively on $S_{K_p}(G,X)$.

We suppose that x is the image of some ordinary point $x_1\in\mathscr{S}_{K_{1,p}}(G_1,X_1)^+(\overline{\mathbb{F}}_p)$. We know from [SZ16, Theorem 3.5] that for every ordinary point in $\mathscr{S}_{K_{1,p}}(G_1,X_1)^+(\overline{\mathbb{F}}_p)$, there exists a special point lifting $\tilde{x}_1$, with associated Shimura datum (T_1,h_1) and associated cocharacter $\mu_{\tilde{x}_1}$ satisfying $\mu_{\tilde{x}_1,\mathbb{Q}_p}=\mu_1$.

Note that if we consider the image $\tilde{x}$ of $\tilde{x}_1$ in $\mathscr{S}_{K_p}(G,X)^+$, then $\tilde{x}$ is a special point since it is the image of (T_1,h_1), whose reduction mod p is the point x. Moreover, note that the cocharacter $\mu_{\tilde{x}}$ is determined by the map to the adjoint group $\mu_{\tilde{x}}^{\mathrm{ad}}$, since for any cocharacter $\mathbb{G}_m \to G$, it is determined by the induced maps to G/G^{der} and G^{ad}, and since G/G^{der} is commutative, the map $\mathbb{G}_m \to G/G^{\mathrm{der}}$ is constant for all elements $h \in X^+$. Thus, we see that the associated cocharacter $\mu_{\tilde{x}}$ satisfies $\mu_{\tilde{x},\mathbb{Q}_p} = \mu$, since $\mu^{\mathrm{ad}} = \mu_1^{\mathrm{ad}}$. We let the Shimura datum for this canonical lift be (T,h).

Define the point $j\tilde{x}$ to be $\pi_{K_p}(j\tilde{x}')$ for any lift $\tilde{x}'$ of $\tilde{x}$ to a point of $\mathrm{Sh}(G,X)$ which is moreover in the image of $\mathrm{Sh}(T,h)$, as the image $\pi_{K_p}(j\tilde{x}')$ does not depend on $\tilde{x}'$. Thus, we have the following proposition:

Proposition 2.7.1. *Let x be an ordinary point in $\mathscr{S}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$. Then x admits a lifting to a special point $\tilde{x}$ over $W(\bar{\mathbb{F}}_p)[\frac{1}{p}]$. Moreover, if we look at the reduction mod p of $j\tilde{x}$, where*

$$j = \mu^{-1}(p) \in T(\mathbb{Q}_p),$$

then this is the image of the geometric Frobenius acting on x.

Proof. The only thing we need to show is the statement about the Frobenius. Note that by functoriality of Shimura varieties and integral models, we only need to understand this image in $\mathrm{Sh}_{K_p \cap T(\mathbb{A}_f)}(T,h)$, which is of dimension 0. To see this, observe that over x we have an F-crystal with $T_{\mathbb{Q}_p}$-structure, and by construction we know the Frobenius on this is given by $\mu(p)$ since $\mu_{\tilde{x},\mathbb{Q}_p} = \mu$. □

We now want to show the following proposition, which is a generalization of a result of Bültel [BÖ2, Lemma 4.5], which determines the action mod p of $u \in U(\mathbb{Q}_p)$ on such canonical liftings.

Proposition 2.7.2. *Let x be an ordinary point in $\mathscr{S}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$, and let $\tilde{x}$ be the canonical lifting constructed in Proposition 2.7.1. We let $\tilde{x}' \in \mathrm{Sh}(T,h)$ be a preimage of $\tilde{x}$ under π. Let U denote the unipotent radical of the parabolic subgroup of G associated to μ. Then for any $u \in U(\mathbb{Q}_p)$, we have that the mod p reduction of $\pi_{K_p}(u\tilde{x}') \in \mathscr{S}_{K_p}(G,X)$ is just x.*

Proof. We first show this for the lift $\tilde{x}_1 \in \mathrm{Sh}_{K_p}(G_1,X_1)$ for the Hodge type cover. To see this, suppose that the image of the reduction mod p of the isogeny given by u on $\tilde{x}_1$, is given by an element $g_0 \in X_\mu^G(b)$. We want to say that this element is simply 1, and to see this we can apply the result of [Lee20, Prop 2.5.4], to see that $g_0 \in U(L)$. Moreover, we have that under the identification of $X_\mu^G(b) \subset \mathrm{Gr}_G$ with the discrete set $M_{1\mu_1}(\mathbb{Q}_p)/M_{1\mu_1}(\mathbb{Z}_p)$, we have that the image of any element in $U(L)/G(\mathbb{Z}_p)$ goes to 1, as desired.

Up to the action of some $g^p \in G(\mathbb{A}_f^p)$, we may assume that $x \in \mathscr{S}_{K_p}(G,X)^+(\overline{\mathbb{F}}_p)$. Observe that we have an isomorphism of root systems $\Phi(G,T) = \Phi(G^{\mathrm{ad}},T^{\mathrm{ad}}) = \Phi(G_1,T_1)$. Hence if we consider U_1 the unipotent radical of the standard parabolic subgroup of G_1 corresponding to μ_1, then we can identify U_1 with U. In particular, since this result is true for the lift $\tilde{x}_1$, for any $u \in U_1(\mathbb{Q}_p)$, the same is true for the action of $u \in U(\mathbb{Q}_p)$ on $\tilde{x}$. Finally, we see that the action of $G(\mathbb{A}_f^p)$ commutes with the action of $G(\mathbb{Q}_p)$, and with mod p reduction, hence the result we want is also true for any x. □

2.8. **Partial Frobenius for abelian type.** We would now like to define the partial Frobenius to be the p-power quasi-isogeny represented by $\mu_i^{-1}(p)$. Since we are in the abelian type case, we cannot work directly with p-divisible groups. Instead, here we will define the partial Frobenius correspondence, at least over an open subscheme of the ordinary locus, and take the Zariski closure.

2.8.1. We first recall the notion of a central leaf for a Shimura variety of Hodge type. Consider a geometric point $x \in \bar{\mathcal{S}}_{K_{1,p}}(G_1, X_1)(\bar{\mathbb{F}}_p)$, and let $\mathbb{X}$ denote the associated p-divisible group with $G_{1,\mathbb{Z}_p}$-structure.

We consider the following subset

$$\mathscr{C}_{\mathbb{X}} := \{x \in \bar{\mathcal{S}}_{K_{1p}}(G_1, X_1) : \exists \text{ isomorphism } \rho : \mathcal{A}_x[p^\infty] \times k(\bar{x}) \simeq \mathbb{X} \times k(\bar{x}) \text{ preserving } G_{1,\mathbb{Z}_p}\text{-structure}\}.$$

It is shown in [Man05, Proposition 1] that this is a locally closed subset of $\bar{\mathcal{S}}_{K_{1p}}(G_1, X_1)$, and that we can give this subset the induced reduced scheme structure which makes the associated scheme $\mathscr{C}_{\mathbb{X}}$ smooth.

Observe that the Zariski closure of the image of the $G_1(\mathbb{A}_f^p)$-orbit of x on $\bar{\mathcal{S}}_{K_{1p}}(G_1, X_1)$, is the union of some components of $\mathscr{C}_{\mathbb{X}}$ (conjecturally all). Moreover, we know using the main result of [Wor13] that all ordinary points have isomorphic p-divisible groups, hence when x is ordinary $\bar{\mathcal{S}}_{K_{1p}}(G_1, X_1)^{\mathrm{ord}} = \mathscr{C}_{\mathbb{X}}$.

Thus, we define Frob_i as the Zariski closure of all the pairs $(x, \Theta_x(\mu_i))$ for all $x \in \bar{\mathcal{S}}_{K_p}(G, X)^{\mathrm{ord}}$. Observe that since $\Theta_{g^p x}(\mu_i) = g^p \Theta_x(\mu_i)$, this Zariski closure is the union of some closed irreducible subschemes of $\bar{\mathcal{S}}_{K_p}(G, X) \times \bar{\mathcal{S}}_{K_p}(G, X)$ of dimension d, where $d = \dim \bar{\mathcal{S}}_{K_p}(G, X)$. Note that by construction, the restriction of Frob_i to $\bar{\mathcal{S}}_{K_p}(G, X)^{\mathrm{ord}} \times \bar{\mathcal{S}}_{K_p}(G, X)^{\mathrm{ord}}$ satisfies that the projection maps are proper, and on $\bar{\mathbb{F}}_p$-points consists exactly of the points $(x, \Theta_x(\mu_i))$. More precisely, we can look at the perfections $\bar{\mathcal{S}}_{K_p}(G, X)^{\mathrm{ord,perf}} \subset \bar{\mathcal{S}}_{K_p}(G, X)^{\mathrm{perf}}$, and we see that under the projection maps $\mathrm{Frob}_i^{\mathrm{ord,perf}} \xrightarrow{p_1^{\mathrm{perf}}} \bar{\mathcal{S}}_{K_p}(G, X)^{\mathrm{ord,perf}}$ is an isomorphism.

2.9. **Composition of algebraic correspondence.** Recall that we let $\mathrm{Cor}(\bar{\mathcal{S}}_{K_p}(G, X), \bar{\mathcal{S}}_{K_p}(G, X))$ denote the Chow group of d-dimensional algebraic cycles in $\bar{\mathcal{S}}_{K_p}(G, X), \bar{\mathcal{S}}_{K_p}(G, X)$, with composition given by taking intersection products.

We first recall the following result to show vanishing of algebraic cycles in the situation where we know the projections are generically finite.

Proposition 2.9.1 ([Bö2, Lemma A.6])**.** *Let X and Y be smooth d-dimensional varieties over an algebraically closed field k. Let C be a generically finite correspondence from X to Y. Assume that there are given open dense subvarieties $X^\circ \subset X$ and $Y^\circ \subset Y$, such that the restriction C° of C to $X^\circ \times Y^\circ$ has projection maps p_1, p_2 finite. Assume that $x \cdot C$ vanishes for all closed points $x \in X^\circ$. Then C vanishes.*

We can apply the above to $\bar{\mathcal{S}}_{K_p}(G, X)^{\mathrm{ord}}$, and hence to show equality of correspondences we only need to show equality for the action on ordinary points. Moreover, it is easy to that the action of Frob_i commutes with the action of $G(\mathbb{A}_f^p)$, since this holds over the ordinary locus by construction. Thus, we have the following proposition, which establishes the first two parts of Theorem 2.0.1 in conjunction with Theorem 2.1.1.

Proposition 2.9.2. *The partial Frobenius correspondence* $\mathrm{Frob}_{\mathfrak{p}_i} := \mathrm{Frob}_i$ *commute for each $i \neq j$, and satisfy that*

$$\mathrm{Frob}_{\mathfrak{p}} = \prod_i \mathrm{Frob}_{\mathfrak{p}_i}, \tag{2.9.3}$$

where $\mathrm{Frob}_{\mathfrak{p}}$ *is the algebraic correspondence induced by the relative Frobenius at $\mathfrak{p}$ on $\bar{\mathcal{S}}_{K_p}(G, X)$.*

Proof. It remains to check that

$$\mathrm{Frob}^{\circ}_{\mathfrak{p}} = \prod_i \mathrm{Frob}^{\circ}_{\mathfrak{p}_i}, \tag{2.9.4}$$

where $\circ$ denotes the restriction to the ordinary locus. Note that by Proposition 2.7.1 shows that over the ordinary locus closed points in $\mathrm{Frob}_{\mathfrak{p}}$ consist of pairs $(x, \mu^{-1}(p)x)$. Thus, the main issue with showing the equality is to check the equality of inseparable degrees of the projection map $p_{2,i}$ for each Frob_i, since the underlying algebraic cycle of the RHS is just the one which consists of pairs $(x, \prod_i \mu_i^{-1}(p)x)$, and we have by construction $\mu^{-1}(p) = \prod_i \mu_i^{-1}(p)$. To check inseparable degrees for μ_i, we first project Frob_i to a correspondence $\mathrm{Frob}_i^{\mathrm{ad}}$ on $\bar{\mathcal{S}}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, and observe that since this map is finite étale, we only need to check the inseparable degrees of the projection maps of $\mathrm{Frob}_i^{\mathrm{ad}}$. Now, observe that some $m_i\mu_i$ can be lifted to a cocharacter in $X_*(T_1)$, up to a central cocharacter. In particular, if we let $m = lcm(m_i)$, then let μ_{i1} be the lift of μ_i^m. Consider the following algebraic correspondence on $\bar{\mathcal{S}}_{K_1p}(G_1, X_1)$. We first consider the Zariski closure of in $p-\mathrm{Isog}_{G_1} \otimes \mathbb{F}_p$ given as the image of $1_{\mu_1^{-m}(p)M_1(\mathbb{Z}_p)}$ via the map $\bar{h}$ as constructed in [Lee20, 6.1.14], and then look at the pushforward of this algebraic cycle via the proper map $p-\mathrm{Isog}_{G_1} \otimes \mathbb{F}_p \to \bar{\mathcal{S}}_{K_1p}(G_1, X_1) \times \bar{\mathcal{S}}_{K_1p}(G_1, X_1)$, which we denote by $\bar{h}(1_{\mu_1^{-m}(p)M_1(\mathbb{Z}_p)})$. We can do this similarly for each $1_{\mu_{1i}(p)M_1(\mathbb{Z}_p)}$.

In particular, from [Lee20, Proposition 6.1.13], commutativity of the diagram in loc. cit. , together with the observation that as elements in the Hecke algebra $\mathcal{H}(M_1(\mathbb{Q}_p))//M_1(\mathbb{Z}_p), \mathbb{Q})$ we have

$$1_{\mu_1^{-m}(p)M_1(\mathbb{Z}_p)} = \prod 1_{\mu_{1i}(p)M_1(\mathbb{Z}_p)}$$

that (where on the right hand side we are looking at composition of algebraic cycles)

$$\bar{h}(1_{\mu_1^{-m}(p)M_1(\mathbb{Z}_p)}) = \prod_i \bar{h}(1_{\mu_{1i}(p)M_1(\mathbb{Z}_p)}), \tag{2.9.5}$$

since each μ_{1i} is central in M_1, and central cocharacters of G_1 simply give the diagonal Δ in $\bar{\mathcal{S}}_{K_1p}(G_1, X_1) \times \bar{\mathcal{S}}_{K_1p}(G_1, X_1)$. Now, observe that we can project all these algebraic cycles via a finite étale map to $\bar{\mathcal{S}}_{K^{\mathrm{ad}}p}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \times \bar{\mathcal{S}}_{K_1p}(G_1, X_1)$, and this does not change the inseparable degrees of the projection maps p_2. The image of the projection of the LHS of 2.9.5 is $(\mathrm{Frob}^{\mathrm{ad}})^m$, and the image of the RHS of is the m-fold product of $(\mathrm{Frob}_i^{\mathrm{ad}})$ for all i, which implies that m times the inseparable degree of $\mathrm{Frob}^{\mathrm{ad}}$ is the same as m times the sum of inseparable degrees of $\mathrm{Frob}_i^{\mathrm{ad}}$, as desired. □

Remark 2.9.6. Since we do not have a moduli interpretation for general abelian type Shimura varieties, outside of the μ-ordinary locus, it is not clear what $\mathrm{Frob}_{\mathfrak{p}_i}$ has to do with partial Frobenii. However, the above result at least suggests that the Frobenius isogeny, viewed as an algebraic correspondence, admits a decomposition into factors for each prime $\mathfrak{p}$ above p.

2.10. **Abstract Eichler-Shimura Relations.** We have isomorphisms of Hecke algebras

$$\mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \simeq \bigotimes_i \mathcal{H}(G_i(\mathbb{Q}_p)//K_p, \mathbb{Q}).$$

Similarly, since $G_{\mathbb{Q}_p}$ admits a decomposition, we can write

$$\mu = \prod_i \mu_i \tag{2.10.1}$$

where μ_i is a minuscule cocharacter of G_i. If we let M be the centralizer of μ in G, then similarly we also have

$$M_{\mathbb{Q}_p} = \prod_i M_i$$

where M_i is the centralizer of μ_i in M. Thus, we also have an isomorphism of Hecke algebras

$$\mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q}) \simeq \bigotimes_i \mathcal{H}(M_i(\mathbb{Q}_p)//M_p, \mathbb{Q}). \tag{2.10.2}$$

For a quasi-split reductive group G with standard parabolic subgroup P and Levi subgroup M, we can define following algebra homomorphism, known as the twisted Satake homomorphism

$$\dot{\mathcal{S}}_M^G : \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q}) \to \mathcal{H}(M(\mathbb{Q}_p)//M_p, \mathbb{Q}),$$

defined as follows. Write $P = NM$, for N the unipotent radical of P, and given a function $f \in \mathcal{H}(G(\mathbb{Q}_p)//K_p, \mathbb{Q})$, we have

$$\dot{\mathcal{S}}_M^G(f)(m) = \int_{n \in N} f(nm)dn.$$

The twisted Satake isomorphism also factors: we have an isomorphism

$$\dot{\mathcal{S}}_M^G = \bigotimes_i \dot{\mathcal{S}}_{M_i}^{G_i}.$$

Consider now the representation $\rho_i : \hat{G} \to GL(V_{\mu_i})$ of $\hat{G}$ with highest weight cocharacter the dominant Weyl conjugate of $(1, \dots, \mu_i^{-1}, \dots, 1)$, where μ_i^{-1} is in the i-th position. Observe that $\rho_\mu = \otimes_i \rho_{\mu_i}$. Define the polynomial

$$H_i(x) \in \mathcal{H}(G_i(\mathbb{Q}_p)//G_i(\mathbb{Z}_p), \mathbb{Q})(x)$$

as the polynomial given by

$$H_i(x) = \det(x - p^{n_i}\rho_{\mu_i}(\sigma \ltimes \hat{g})), \tag{2.10.3}$$

where $n_i = \langle \rho_i, \mu_i \rangle$, where ρ_i is the half sum of positive roots of G_i. Note that since μ is central in M, μ_i^{-1} is also central in M_i, hence we can consider the element $1_{\mu_i^{-1}(p)M_i(\mathbb{Z}_p)} \in \mathcal{H}(M_i(\mathbb{Q}_p)//M_i(\mathbb{Z}_p), \mathbb{Q})$.

Proposition 2.10.4. *We have the following equality in $\mathcal{H}(M_i(\mathbb{Q}_p)//M_i(\mathbb{Z}_p), \mathbb{Q})$: for all i,*

$$H_i(1_{\mu_i^{-1}(p)M_i(\mathbb{Z}_p)}) = 0.$$

Proof. This follows from the same proof as [B02̈, Prop 3.4], and the observation that under the decomposition (2.10.2), we see that $H_i(1_{\mu_i^{-1}(p)M_i(\mathbb{Z}_p)})$ corresponds to the polynomial with coefficients in $\mathcal{H}(M_i(\mathbb{Q}_p)//M_i(\mathbb{Z}_p), \mathbb{Q})$ defined similarly as in (2.10.3), where we instead take the determinant of highest weight representation of $\hat{G}_i$ corresponding to the dominant Weyl conjugate of $-\mu_i$. □

2.11. **Proof of Eichler-Shimura relations.**

We will now show the last part of Theorem 2.0.1.

Proposition 2.11.1. *Let (G, X) be a Shimura datumn of abelian type, such that $G = \mathrm{Res}_{F/\mathbb{Q}} G'$, and p a prime satisfying the conditions in Proposition 2.9.2. Let $H_i(t)$ be the polynomial defined in* (2.10.3)*, viewed as a polynomial with coefficients in* $\mathrm{Cor}(\bar{\mathcal{S}}_{K_p}(G, X), \bar{\mathcal{S}}_{K_p}(G, X))$ *via h. Then we have the equality*

$$H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0.$$

in the ring $\mathrm{Cor}(\bar{\mathcal{S}}_{K_p}(G,X), \bar{\mathcal{S}}_{K_p}(G,X))$.

Proof. Firstly, observe that from Proposition 2.6.3, all the terms appearing in $H_i(\mathrm{Frob}_{\mathfrak{p}_i})$ have a dense μ-ordinary locus, hence it suffices to show the result where we restrict all the terms to the μ-ordinary locus. Applying [BÖ2, A.6], to show that $H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0$, it suffices to show that

$$x \cdot H_i(\mathrm{Frob}_{\mathfrak{p}_i}) = 0$$

for all $x \in \mathscr{S}_{K_p}(G,X)(\overline{\mathbb{F}}_p)^{\mathrm{ord}}$.

Let the Hecke polynomial be

$$H_i(t) = \sum_j A_j t^j,$$

for elements $A_j \in \mathcal{H}(G_i(\mathbb{Q}_p)//G_i(\mathbb{Z}_p))$. Let $h(A_j)$ denote the mod p algebraic cycle in $\mathrm{Corr}(\mathscr{S}_\kappa, \mathscr{S}_\kappa)$ corresponding to A_j. Thus, we want to show that

$$x \cdot \left(\sum_j \mathrm{Frob}_{\mathfrak{p}_i}^j \cdot h(A_j) \right) = 0 \tag{2.11.2}$$

where $\mathrm{Frob}_{\mathfrak{p}_i}$ is the correspondence defined above.

The proof then follows as in [BÖ2, Thm 4.7]. As constructed above, we let $\tilde{x}$ be the special point lift of x. We write the coefficients of the Hecke polynomial A_j in terms of left $G_i(\mathbb{Z}_p)$-cosets

$$A_j = \sum_k n_k^{(j)} m_k^{(j)} G_i(\mathbb{Z}_p),$$

where we further apply the Iwasawa decomposition so that $n^{(j)}$ lies in $U(\mathbb{Q}_p)$, and $m^{(j)}$ lies in $M_i(\mathbb{Q}_p)$.

It remains for us to show that the mod p reduction of

$$\sum_{j,k} n_k^{(j)} g_k^{(j)} \mu_i^{-1}(p^{-k})\tilde{x}, \tag{2.11.3}$$

vanishes, since the mod p reduction is just the LHS of (2.11.2), and we observe that a lift of the partial Frobenius acting on $\tilde{x}$ is by construction given by $\mu_i^{-1}(p)$. Moreover, we see from Proposition 2.7.2 that we have an equality of mod p reductions

$$\sum_{j,k} \overline{n_k^{(j)} g_k^{(j)} \mu_i^{-1}(p^{-k})\tilde{x}} = \sum_{j,k} \overline{g_k^{(j)} \mu_i^{-1}(p^{-k})\tilde{x}},$$

(here the overline indicates the mod p reduction) and the RHS vanishes from the abstract Eichler-Shimura relation in Proposition 2.10.4, since in $\mathcal{H}(M_i(\mathbb{Q}_p)//M_i(\mathbb{Z}_p), \mathbb{Q})$ the polynomial $H_i(1_{\mu_i^{-1}(p)M_c})$ is, written as left $M_i(\mathbb{Z}_p)$-cosets,

$$\sum_{j,k} g_k^{(j)} \mu_i^{-1}(p^{-k}) M_i(\mathbb{Z}_p)$$

and hence is just 0. $\square$

3. Semisimplicity Criterion

3.1. **An abstract semisimplicity criterion.** We first recall the following theorem of Fayad and Nekovář [FN19, Theorem 1.7]. Let $\Gamma_E = \mathrm{Gal}_E$.

Definition 3.1.1. A continuous representation $\rho : \Gamma_E \to \mathrm{GL}(V)$ is said to be *strongly irreducible* if for any open finite index subgroup $U \subset \Gamma_E$, $\rho|_U$ is irreducible.

Theorem 3.1.2. *Let Γ be a profinite group, $V, W_1, ..., W_r$ non-zero vector spaces of finite dimension over $\bar{\mathbb{Q}}_\ell$. Let $\rho : \Gamma \to \mathrm{GL}(V)$ and $\rho_i : \Gamma \to \mathrm{GL}(W)$ be continuous representations of Γ with Lie algebras*

$$\mathfrak{g}_i = \mathrm{Lie}(\rho_i(\Gamma)), \qquad \mathfrak{g} = \mathrm{Lie}(\rho(\Gamma)).$$

We denote $\bar{\mathfrak{g}}_i = \mathfrak{g}_i \otimes \bar{\mathbb{Q}}_\ell$, $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \bar{\mathbb{Q}}_\ell$. If the following three conditions hold, then the representation $\rho = \rho^{ss}$ is semisimple.

1. *Each ρ_i is strongly irreducible (which implies that each $\mathfrak{g}_i$ is a reductive $\bar{\mathbb{Q}}_\ell$-Lie algebra and each element of its centre acts on W_i by a scalar).*
2. *For each $i = 1, \ldots, r$, every (equivalently, some) Cartan subalgebra $\mathfrak{h}_i$ of $\mathfrak{g}_i$ acts on W_i without multiplicities (i.e., all weight spaces of $\mathfrak{h}_i$ on W_i are one-dimensional).*
3. *There exists an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$ there exists a finite dimensional vector space over $\bar{\mathbb{Q}}_\ell$ (depending on g) $V(g) \supset V$ and elements $u_1, \ldots, u_r \in \mathrm{GL}(V(g))$ such that $u_i u_j = u_j u_i$, $P_{\rho_i(g)}(u_i) = 0$ for all i, $j = 1, ..., r$, and V is stable under $u_1 \ldots u_r$ and $u_1 \ldots u_r|_V = \rho(g)$*

Here, $P_{\rho_i(g)}$ is the characteristic polynomial of $\rho_i(g)$.

We state here a theorem of Sen [Sen73, Theorem 1] which we will use to find representations which satisfy condition (2) of the theorem above, given a Galois representation ρ.

Proposition 3.1.3. *Let $\mathfrak{g} = \bar{\mathbb{Q}}_l \cdot \mathrm{Lie}(\rho(\Gamma_E)) \subset \mathfrak{gl}(n, \bar{\mathbb{Q}}_l)$ be the $\bar{\mathbb{Q}}_l$-Lie algebra generated by the image of ρ. If $\rho|_{G_{E_\tau}}$ is Hodge-Tate for any $\tau : E \hookrightarrow \bar{\mathbb{Q}}_l$, then any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts on $\bar{\mathbb{Q}}_l^n$ by the n Hodge-Tate weights of $\rho|_{G_{E_\tau}}$.*

3.2. **Proof of Main Theorem.** We prove in this subsection Theorem 1.0.1. Let us recall the setup: we have a representation $\sigma_{\pi^\infty} : \Gamma_E \to V^i(\pi^\infty)$ appearing in the intersection cohomology of some Shimura variety (G, X), where $G = \mathrm{Res}_{F/\mathbb{Q}} H$, associated with some automorphic representation π of $H(\mathbb{A}_F, f)$. We also assume that we have attached via the Langlands correspondence $\rho_{\pi,\mu_i} : \Gamma_E \to {}^L G \xrightarrow{r_{-\mu_i}} \mathrm{GL}(V_{-\mu_i})$. We restate Theorem 1.0.1 here:

Theorem 3.2.1. *Let (G, X) be a Shimura datum of abelian type such that $G = \mathrm{Res}_{F/\mathbb{Q}} H$ for some connected reductive group H, and totally real number field F. Let π be an automorphic representation of $G(\mathbb{A}_{\mathbb{Q},f}) = H(\mathbb{A}_{F,f})$. For all v, suppose that the ${}^L H$-valued Galois representation associated to π exists, and we consider for all $v|\infty$ the composition with the highest weight representation $\rho_{\pi,\mu_v} : \mathrm{Gal}(\bar{F}/F) \to {}^L H \to \mathrm{GL}(V_{-\mu_v})$. Suppose that moreover we also know that*

1. *ρ_{π,μ_v} is strongly irreducible*
2. *For all primes $\mathfrak{p}$ of E such that $\mathfrak{p}|l$, the Hodge-Tate weights of $\rho_{\pi,\mu_v}|_{D_\mathfrak{p}}$ are distinct.*

Then σ_{π^∞} is a semisimple representation.

Proof. In the notation of Theorem 3.1.2, we have $\Gamma = \Gamma_E$, $\rho = \sigma_{\pi^\infty}$, $V = V^i(\pi^\infty)$, and each ρ_i is ρ_{π,μ_i}, $W_i = V_{-\mu_i}$. In order to apply this criterion, we want the elements g to be the Frobenius elements at the primes Frob_p which are split, and the u_i will be the partial Frobenius $\mathrm{Frob}_{\mathfrak{p}_i}$. Moreover, to show that the representation ρ is semisimple, it suffices to show that $\rho|_U$ is semisimple for any open finite index subgroup of $\mathrm{Gal}(\bar{E}/E)$, and thus, the fact that such primes p with this splitting condition have a positive density is sufficient.

From Theorem 2.0.1, observe that

$$H_{\mu_i}(\mathrm{Frob}_{\mathfrak{p}_i}|V^i(\pi^\infty)) = 0.$$

Note that each summand $V^i(\pi^\infty) \otimes (\pi^\infty)^K$ in $IH^i(\mathrm{Sh}_K(G,X), \mathcal{L}_{\xi,\ell})$ is stable for the action of $\mathrm{Frob}_{\mathfrak{p}_i}$, since the action of $G(\mathbb{A}_f^p)$ and $\mathcal{H}_i$-actions commute with the $\mathrm{Frob}_{\mathfrak{p}_i}$-action.

Replacing each element of the Hecke algebra $\mathcal{H}_i$ with its eigenvalue on ${\pi_p}^{K_p}$, we obtain

$$H_{\mu_i}|_{{\pi_p}^{K_p}}(\mathrm{Frob}_{\mathfrak{p}_i}|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K}) = 0.$$

Observe that the polynomial on the right hand side is, by the definition of the Hecke polynomial H_{μ_i}, the characteristic polynomial $\det(t - \rho_i(\mathrm{Frob}_p)) =: P_{\rho_i(g)}$.

In order to show semisimplicity of the endomorphism given by Frob_p, it suffices to show that the P_{ρ_i} has distinct roots. This is an open condition on $\mathrm{Gal}(\bar{E}/E)$, hence it suffices to exhibit an element $u \in \mathrm{Gal}(\bar{E}/E)$ such that the characteristic polynomial of $\rho_i(u)$ has distinct roots. To see this, observe that Proposition 3.1.3 applies, hence the Lie algebra

$$\bar{\mathfrak{g}} = \bar{\mathbb{Q}}_l \cdot \mathrm{Lie}(\rho_i(\Gamma_E))$$

contains a semisimple element whose eigenvalues on $V_i(\pi^\infty)$ act by the Hodge-Tate weights of $\rho_i|_{D_v}$, where v is a prime dividing ℓ. By assumption (2), the Hodge-Tate weights of $\rho_i|_{D_v}$ are all distinct. This implies that there is an open subset of $\mathrm{Gal}(\bar{E}/E)$ where the characteristic polynomial of $\rho_i(u)$ has distinct roots.

Finally, we conclude using the criterion in Theorem 3.1.2, since we have constructed a Zariski dense set of elements which are semisimple, that ρ is semisimple. $\square$

4. Applications to some Shimura varieties

We discuss here some examples of Shimura varieties where Theorem 1.0.1 applies. Let F be a totally real field and consider the Shimura variety associated to the group $G = \mathrm{Res}_{F/\mathbb{Q}} H$, where H is some inner form of G^*, defined as one of the following groups:

symplectic: $G^* = \mathrm{GSp}_{2g}$
orthogonal, n even: $G^* = \mathrm{GSO}_{2n}$
orthogonal, n odd: G^* is a non-split quasi-split form of GSO_{2n} relative to E/F, a CM extension

Let π be a cuspidal L-algebraic automorphic representation of $G(\mathbb{A}_F)$, and we assume that π satisfies

(1) There is a finite F-place v_{St} such that $H_{F_{v_{St}}}$ and $G^*_{F_{v_{St}}}$ are isomorphic, and under this isomorphism $\pi_{v_{St}}$ is the Steinberg representation of $G^*(F_{v_{St}})$ twisted by a character.
(2) $\pi_\infty|\mathrm{sim}|^{-n(n-1)/4}$ is ξ-cohomological for an irreducible algebraic representation $\xi = \otimes_{y:F\to\mathbb{C}}\xi_y$ of the group $(\mathrm{Res}_{F/\mathbb{Q}}G^*)_{\mathbb{C}}$, where sim is the similitude factor map $\mathrm{sim}: G^* \to \mathbb{G}_m$

Under these conditions on π, Kret and Shin [KS20b, Theorem A] and [KS20a, Theorem A] construct the Galois representation

symplectic: $\rho_\pi : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GSpin}_{2n+1}(\mathbb{Q}_\ell)$
orthogonal: $\rho_\pi : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GSpin}_{2n}(\mathbb{Q}_\ell)$

associated to π. In fact, from their arguments, we may also extract the following proposition about concentration in middle degree:

Proposition 4.0.1. *Suppose that π is a cuspidal L-algebraic automorphic representation of $G(\mathbb{A}_f)$ which satisfies that there is a finite F-place v_{St} such that $\pi_{v_{\mathrm{St}}}$ is a twist of Steinberg. Then we have $IH^i_{\acute{e}t}(\mathrm{Sh}(G,X),\mathcal{L}_{\xi,\ell})[\pi]=0$ for all $i\neq d:=\dim\mathrm{Sh}(G,X)$.*

Proof. For the case of symplectic groups, this is [KS20b, Lemma 2.7 and Proposition 8.2], while for orthogonal groups, this is [KS20a, Proposition 9.6 and Lemma 10.1]. □

Remark 4.0.2. This result relies on Arthur's results in [Art13], and is conditional on the twisted weighted fundamental lemma in general.

4.1. **Proof of Theorem 1.0.3.** Now, we explain how to use Theorem 1.0.1 to deduce Theorem 1.0.3. As explained, above, the first two conditions, which are (1) and (2) as state above, are necessary for the construction of the corresponding Galois representation. Now, we see that if we have condition

(4) The image of ϕ_π is Zariski dense in $\widehat{H}$,

then at all places $v|\infty$ where the group is not compact modulo center, the associated representation ρ_{π,μ_v} will be strongly irreducible. This is because ρ_{π,μ_v} is irreducible, since it does not factor through any Levi subgroup, and if we look at the Zariski closure of the image in ϕ_π of any finite index open subgroup, it must also be $\widehat{H}$, since $\widehat{H}$ is connected, and thus also irreducible.

We remark here that such a condition should hold for 'most' representations, for example, this is the generalization of the non-CM condition of [Nek18].

Finally, we see that the condition

(3) The representation π_v is regular after composing with the representation $\mathrm{GSpin}_{2g+1}\xrightarrow{\mathrm{spin}}\mathrm{GL}_{2^g}$ if symplectic (resp. $\mathrm{GSpin}_{2n}\xrightarrow{\mathrm{std}}\mathrm{GL}_{2n}$ if orthogonal) at every infinite place v of F.

implies that the Hodge-Tate weights of $\tilde{\rho}_{\pi,v}$ are distinct. To see this, we first recall the definition of regularity: this means that if we let $\varphi: W_{\mathbb{R}}\to\mathrm{GSpin}_m(\mathbb{C})$ be the Langlands parameter associated with π_v, then the composition

$$\mathbb{C}^\times\subset W_{\mathbb{R}}\to\mathrm{GSpin}_{2n+1}(\bar{\mathbb{Q}}_\ell)\xrightarrow{\mathrm{spin}}\mathrm{GL}_{2^n}(\bar{\mathbb{Q}}_\ell)$$

(symplectic case) or

$$\mathbb{C}^\times\subset W_{\mathbb{R}}\to\mathrm{GSpin}_{2n}(\bar{\mathbb{Q}}_\ell)\xrightarrow{\mathrm{std}}\mathrm{GL}_{2n}(\bar{\mathbb{Q}}_\ell)$$

(orthogonal case) is conjugate to the cocharacter $z\mapsto\mu_1(z)\mu_2(\bar{z})$ given by some $\mu_1,\mu_2\in X_*(\hat{T})\otimes_{\mathbb{Z}}\mathbb{C}=X^*(T)\otimes_{\mathbb{Z}}\mathbb{C}$ such that $\mu_1-\mu_2\in X^*(T)$. Then φ is regular if μ_1 is regular as a character of GL_m, that is, $\langle\alpha^\vee,\mu_1\rangle\neq 0$ for all coroots $\alpha^\vee$. In particular, we see that the Hodge-Tate cocharacter (as defined in [BG14, §2.4]) of the composition is μ_1, which is regular, and hence the Hodge-Tate weights are distinct.

Remark 4.1.1. We may also impose a condition in terms of ξ, the representation of $G_{\bar{\mathbb{Q}}_\ell}$ such that π appears in the cohomology of $\mathcal{L}_{\xi,\ell}$. The Hodge cocharacter μ_{Hodge} in this case will be $\lambda_\xi+\rho$, where λ_ξ is the dominant highest weight character for the representation of ξ, seen as a cocharacter for the dual group. Under the recipe for the Hodge-Tate cocharacter μ_{HT} from Theorem A in both [KS20a],[KS20b], we want the composition of μ_{HT} with spin or std respectively to be regular.

We thus satisfy all the conditions to apply Theorem 1.0.1, and we can deduce that the Galois module

$$\mathrm{Hom}_{G(\mathbb{A}_f)}(\pi^\infty, IH^i_{et}(\mathrm{Sh}(G,X), \mathcal{L}_{\xi,\ell}))$$

(which is finite-dimensional over $\bar{\mathbb{Q}}_\ell$) is semisimple for any i.

We may apply Proposition 4.0.1 to restrict just to middle degree where $i = d$.

References

[Art13] James Arthur. *The endoscopic classification of representations*, volume 61 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[BÖ2] Oliver Bültel. The congruence relation in the non-PEL case. *J. Reine Angew. Math.*, 544:133–159, 2002.

[BG14] Kevin Buzzard and Toby Gee. The conjectural connections between automorphic representations and Galois representations. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 135–187. Cambridge Univ. Press, Cambridge, 2014.

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor. Potential automorphy and change of weight. *Ann. of Math. (2)*, 179(2):501–609, 2014.

[FN19] Karam Fayad and Jan Nekovář. Semisimplicity of certain Galois representations occurring in étale cohomology of unitary Shimura varieties. *American Journal of Mathematics*, 141:503 – 530, 2019.

[Fra98] Jens Franke. Harmonic analysis in weighted L_2-spaces. *Ann. Sci. École Norm. Sup. (4)*, 31(2):181–279, 1998.

[Kis10] Mark Kisin. Integral models for Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 23(4):967–1012, 2010.

[Kot97] Robert E. Kottwitz. Isocrystals with additional structure. II. *Compositio Math.*, 109(3):255–339, 1997.

[KS20a] Arno Kret and Sug Woo Shin. Galois representations for even general special orthogonal groups. `https://arxiv.org/abs/2010.08408`, 2020.

[KS20b] Arno Kret and Sug Woo Shin. Galois representations for general symplectic groups. To appear in JEMS., 2020.

[KSZ21] Mark Kisin, Sug Woo Shin, and Yihang Zhu. The stable trace formula for Shimura varieties of abelian type. `https://arxiv.org/abs/2110.05381`, 2021.

[Lee20] Si Ying Lee. Eichler-Shimura relations for Shimura varieties of Hodge type. `arxiv.org:2006.11745`, 2020.

[Lov17] Tom Lovering. Filtered F-crystals on Shimura varieties of abelian type. `arXiv:1702.06611`, 2017.

[LS18] Kai-Wen Lan and Benoît Stroh. Nearby cycles of automorphic étale sheaves. *Compos. Math.*, 154(1):80–119, 2018.

[Man05] Elena Mantovan. On the cohomology of certain PEL-type Shimura varieties. *Duke Math. J.*, 129(3):573–610, 2005.

[Nek18] Jan Nekovar. Eichler-Shimura relations and semisimplicity of étale cohomology of quaternionic shimura varieties. *Ann. Sci. École Norm. Sup. (4)*, 51(5):1179–1252, 2018.

[Sen73] Shankar Sen. Lie algebras of Galois groups arising from Hodge-Tate modules. *Annals of Mathematics*, 97:160, 1973.

[She20] Xu Shen. On some generalized Rapoport–Zink spaces. *Canadian Journal of Mathematics*, 72(5):1111–1187, 2020.

[SZ16] Ananth N. Shankar and Rong Zhou. Serre-Tate theory for Shimura varieties of Hodge type. `arXiv:1612.06456`, 2016.

[SZ17] Xu Shen and C. Zhang. Stratifications in good reductions of Shimura varieties of abelian type. `arxiv.org:1707.00439`, 2017.

[Wor13] Daniel Wortmann. The μ-ordinary locus for Shimura varieties of Hodge type. `arXiv:1310.6444`, 2013.