MA6201: Shimura varieties

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Abstract

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Contents

1 Modular curves

Let

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}.$$

There is a left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ given as follows: for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \quad \text{and} \quad z \in \mathbb{H},$$

define

$$g \cdot z := \frac{az+b}{cz+d}.$$

Note that this yields a surjection

$$\mathrm{SL}_2(\mathbb{R}) \twoheadrightarrow \mathrm{Aut}_{\mathrm{hol}}(\mathbb{H}),$$

where $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{H})$ is the holomorphic automorphism group of $\mathbb{H}$. We are interested in certain discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ whose actions produce interesting quotients of $\mathbb{H}$.

Definition 1.1. A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a *congruence subgroup* if it is a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma(N) \subset \Gamma$ with finite index for some $N \geq 1$, where

$$\Gamma(N) = \left\{ g \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

We will also usually assume that Γ is *small enough*, i.e., that $\Gamma \subset \Gamma(N)$ for some $N \geq 3$.

The group Γ (if small enough) acts freely and properly discontinuously on $\mathbb{H}$, and this implies that $\Gamma\backslash\mathbb{H}$ has a canonical complex manifold structure given by the complex structure on $\mathbb{H}$. Furthermore, the quotient map $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering.

Definition 1.2. The complex manifold $\Gamma\backslash\mathbb{H}$ is called a *modular curve*.

Proposition 1.3. A modular curve $\Gamma\backslash\mathbb{H}$ has the following properties:

(a) $\Gamma\backslash\mathbb{H}$ has the canonical structure of an algebraic variety over $\mathbb{C}$, which is compatible with the complex manifold structure.

(b) $\Gamma\backslash\mathbb{H}$ is the moduli space of elliptic curves over $\mathbb{C}$ with "Γ-level structure".

(c) The moduli interpretation in (b) also makes sense over some number field E (depending on Γ; e.g., $E = \mathbb{Q}(\zeta_N)$ if $\Gamma = \Gamma(N)$). This moduli problem over E is represented by a quasi-projective E-scheme whose base change to $\mathbb{C}$ recovers $\Gamma\backslash\mathbb{H}$ as a $\mathbb{C}$-scheme. We say that $\Gamma\backslash\mathbb{H}$ has a *model* over E.

(d) This moduli interpretation even extends integrally over $\mathbb{Z}[\zeta_N, 1/N]$, to produce a smooth scheme.

(e) The $\mathbb{C}$-scheme $\Gamma\backslash\mathbb{H}$ has a canonical compactification obtained by adding certain "special points" (cusps), giving a proper algebraic curve.

We will now explain how to see (a)-(d).

Remark 1.4. To show (a), even after giving a structure of a complex manifold to $\Gamma\backslash\mathbb{H}$ one cannot appeal to the usual GAGA equivalence between smooth projective curves over $\mathbb{C}$ and compact Riemann surfaces, because $\Gamma\backslash\mathbb{H}$ is not compact. However, there is a canonical compactification of $\Gamma\backslash\mathbb{H}$ which is a compact Riemann surface (the Baily–Borel compactification).

1.1 Elliptic curves over $\mathbb{C}$

Definition 1.5. An elliptic curve E over $\mathbb{C}$ is a smooth projective algebraic group of dimension 1.

Over $\mathbb{C}$, every elliptic curve arises as a quotient $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. More precisely, given an elliptic curve E, there is a holomorphic group homomorphism $\exp : \mathrm{Lie}E \to E$ coming from Lie group theory. Here $\mathrm{Lie}E$ is the tangent space at the identity O, and is a 1-dimensional complex vector space. (Note that exp is not algebraic.) Then $\ker(\exp)$ is a lattice in $\mathrm{Lie}E$ and $(\mathrm{Lie}E)/\ker(\exp) \xrightarrow{\sim} E$ is an isomorphism of elliptic curves. Notice also that (non-canonically) the left hand side is isomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$.

Remark 1.6. The map $\exp : \mathrm{Lie}E \to E$ is a universal covering. Hence we have the following canonical isomorphisms: $\ker(\exp) = \pi_1(E, O) = H^1(E, \mathbb{Z})$.

Suppose E and E' are elliptic curves. We have

$$\mathrm{Hom}(E, E') \cong \{f : \mathrm{Lie}E \to \mathrm{Lie}E' \mid f \text{ is } \mathbb{C}\text{-linear and } f\big(H^1(E,\mathbb{Z})\big) \subset H^1(E',\mathbb{Z})\}$$

where the assignment is given by $F \longmapsto dF\big|_{\mathrm{Lie}E}$. Combining the above facts, we have an equivalence of categories

$$\big((V,\Lambda),\ V \text{ a 1-dimensional } \mathbb{C}\text{-vector space and } \Lambda \subset V \text{ a } \mathbb{Z}\text{-lattice}\big) \xrightarrow{\sim} \big(\text{Elliptic curves}/\mathbb{C}\big)$$

given by

$$(V,\Lambda) \longmapsto V/\Lambda.$$

and the reverse map is given by

$$E \mapsto (\mathrm{Lie}E, H^1(E,\mathbb{Z})).$$

We first observe that two elliptic curves $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic if and only if there is a $\mathbb{C}$-linear holomorphic automorphism of $\mathbb{C}$ that takes Λ to Λ'. Every such holomorphic automorphism of $\mathbb{C}$ is given by multiplication by an element $\alpha \in \mathbb{C}^\times$. Indeed, any continuous group automorphism of a real vector space is necessarily an $\mathbb{R}$-linear map, and one checks that $\varphi(a+bi) = a\varphi(1) + b\varphi(i)$ is holomorphic if and only if $\varphi(i) = i\varphi(1)$. So $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ if and only if $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^\times$. This gives us:

$$\{\text{Lattices inside } \mathbb{C}\}/\text{homothety} \xrightarrow{\sim} \{\text{Elliptic curves over} \mathbb{C}\}/\text{isomorphism}.$$

Now we orient $\mathbb{C}$, as real vector space, so that $(1, i)$ is a positive orientation. We let

$$\mathcal{Z} = \{\text{pairs } (\omega, \omega') \text{ of positively oriented } \mathbb{R}\text{-bases of } \mathbb{C}\}.$$

The group $\mathrm{SL}(2, \mathbb{Z})$ acts on $\mathcal{Z}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\omega, \omega') = (a\omega + b\omega', c\omega + d\omega').$$

This action fixes the lattice $\mathbb{Z}\omega + \mathbb{Z}\omega' \subset \mathbb{C}$, and the quotient $\mathrm{SL}(2, \mathbb{Z})\backslash\mathcal{Z}$ is thus identified with the set of all lattices in $\mathbb{C}$. Thus we have a bijection

$$\mathrm{SL}(2, \mathbb{Z})\backslash\mathcal{Z}/\mathbb{C}^\times \xrightarrow{\sim} \{\text{Elliptic curves}\}/\text{isomorphism}.$$

where $\alpha \in \mathbb{C}^\times$ takes (ω, ω') to $(\alpha\omega, \alpha\omega')$, or equivalently takes the lattice Λ to $\alpha\Lambda$.

We further observe that

$$(\omega, \omega') \longmapsto \frac{\omega'}{\omega}$$

identifies $\mathcal{Z}/\mathbb{C}^\times$ with the upper half plane $\mathbb{H} \subset \mathbb{C}$:

$$\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}.$$

Thus, we have

$$\mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \{\text{Elliptic curves}\}/\text{isomorphism}.$$

Let $\mathbb{H}^\pm = \mathbb{C} \setminus \mathbb{R}$, the union of the upper and lower half planes. The group $\mathrm{GL}(2, \mathbb{Z})$ similarly acts by fractional linear transformations on $\mathbb{H}^\pm$ as above. Note that we further have an isomorphism

$$\mathrm{GL}(2, \mathbb{Z})\backslash\mathbb{H}^\pm \simeq \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \{\text{Elliptic curves}\}/\text{isomorphism}.$$

Let us now give these sets some complex structures. Recall that we have an $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic morphism $j : \mathbb{H}^+ \to \mathbb{C}$ inducing a bijection

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+ \xrightarrow{\sim} \mathbb{C}.$$

Here j corresponds to evaluating the classical j-invariant of an elliptic curve, which we recall is defined as follows:

Definition 1.7. Take an elliptic curve $E/\mathbb{C}$ and write it in Weierstrass form

$$y^2 = x^3 + ax + b.$$

The *j-invariant* is given by

$$j(E) = 1728\frac{4a^3}{4a^3 + 27b^2}.$$

We may use this map to identify the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+$ with $\mathbb{C}$ in order to give the former a complex manifold structure.

Note that $\mathbb{H}^+ \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+$ is a holomorphic map, but not a local isomorphism. In other words, this is not a covering map; there is ramification over the images of i and $e^{\frac{2\pi i}{3}}$ with branching of order 2 and 3 respectively. This is related to the fact that the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}^+$ is problematic in the following sense:

- $-I \in \mathrm{SL}_2(\mathbb{Z})$ acts trivially on $\mathbb{H}^+$. In particular, the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}^+$ is not free.
- The naive solution is to now consider the action of $\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ on $\mathbb{H}^+$. For this action, most points in $\mathbb{H}^+$ have trivial stabilizer, but points in the orbit of i and the orbit of $e^{\frac{2\pi i}{3}}$ have nontrivial stabilizers. So this is also not a solution.

This phenomenon exactly corresponds to the fact that for any elliptic curve E over $\mathbb{C}$ (or in fact any algebraically closed field of characteristic away from 2 or 3), the automorphism group of E is either:

1. $\mathbb{Z}/2\mathbb{Z}$, where the nontrivial automorphism is negation. This corresponds to the inclusion of $\{\pm I\}$ in all stabilizers.
2. $\mathbb{Z}/4\mathbb{Z}$. This automorphism group applies to a unique isomorphism class of elliptic curves.
3. $\mathbb{Z}/6\mathbb{Z}$. This automorphism group applies to a unique isomorphism class of elliptic curves.

Remark 1.8. The complex manifold structure we put on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+$ (using j) is the unique one such that the projection $\mathbb{H}^+ \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^+$ is holomorphic.

Remark 1.9. It is "more correct", in some sense, to define the orbifold (or Deligne-Mumford stack) quotient of $\mathbb{H}^+$ by $\mathrm{SL}_2(\mathbb{Z})$. This allows us to still form something like a fine moduli space of elliptic curves that remembers the automorphisms (including the generic $\mathbb{Z}/2\mathbb{Z}$-automorphisms). We will discuss this in more detail when we talk about moduli spaces.

1.2 Alternative point of view: Hodge structures

In the previous subsection, we classified elliptic curves $\mathbb{C}/\Lambda$ up to isomorphism by morally fixing the complex vector space $\mathbb{C}$ and varying Λ. We now introduce a different way to think about the upper half plane with its complex structure which is more amenable to generalization to higher dimensions. We may fix an abstract $\mathbb{Z}$-module Λ, finite free of rank 2, and ask how we could vary the $\mathbb{C}$-structure.

As before, an elliptic curve is given by $E = (\mathrm{Lie}E)/H^1(E,\mathbb{Z})$. Also, we have a canonical isomorphism of 2-dimensional $\mathbb{R}$-vector spaces: $\mathrm{Lie}E \cong H^1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Notice of course that $\mathrm{Lie}E$ also has a complex structure. Thus in order to reconstruct E, we need the abstract $\mathbb{Z}$-module $H^1(E,\mathbb{Z})$ together with a complex structure on the $\mathbb{R}$-vector space $H^1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. In general, to define a complex structure on an $\mathbb{R}$-vector space V, it suffices to define multiplication by i such that $i^2 = -1$. In other words, a complex structure on V is exactly an element $J \in \mathrm{End}_{\mathbb{R}}(V)$ such that $J^2 = -1$, and this element corresponds to scalar multiplication by i. Then, we extend this to all $\mathbb{C}$ by setting:

$$(x + iy) \cdot v \;=\; x \cdot v \;+\; y \cdot J(v)$$

for all real numbers x, y. However, we will give a slightly different definition here:

Definition 1.10. A complex structure on $\mathbb{R}^2$ is a homomorphism

$$h : \mathbb{C}^\times \to \mathrm{GL}(2, \mathbb{R}) = \mathrm{Aut}(\mathbb{R}^2)$$

such that the eigenvalues of $h(z) \in \mathbb{C}^\times$ on $\mathbb{R}^2$ are z and $\bar{z}$.

The equivalence between the two definitions can be seen by taking $J = h(i)$, and observing that the condition on the eigenvalues forces the characteristic polynomial of J to be $X^2 + 1$.

Choosing the base point $e_0 = (1, 0) \in \mathbb{R}^2$, we see that any complex structure h defines an isomorphism $i_h : \mathbb{R}^2 \to \mathbb{C}$ of complex vector spaces, via $i_h^{-1}(z) = h(z) \cdot e_0$.

Now, denote $V = \mathbb{R}^2$, and let $h : \mathbb{C}^\times \to \mathrm{Aut}(\mathbb{R}^2)$ be a complex structure. Then for any $z \in \mathbb{C}$, $z \notin \mathbb{R}$, $h(z)$ is diagonalizable and by definition has two eigenvalues on $V \otimes \mathbb{C}$, namely z and $\bar{z}$. For some $z \in \mathbb{C}^\times$, let $V^{-1,0} = V_h^{-1,0}$, resp. $V^{0,-1} = V_h^{0,-1}$, denote the z-eigenspace, resp. the $\bar{z}$-eigenspace, for $h(z)$ on $V_{\mathbb{C}}$. Observe that since h is a homomorphism, the subspaces $V^{-1,0}, V^{0,-1}$ are independent of the choice of $z \in \mathbb{C}^\times \backslash \mathbb{R}$.

Example 1.11. We can define a complex structure by the homomorphism

$$h_0 : \mathbb{C}^\times \to \mathrm{GL}(2, \mathbb{R}) \quad \text{such that} \quad h_0(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix};$$

this obviously satisfies the hypothesis, and if we look at the eigenspaces for $z = i$, we get

$$V^{-1,0} = \mathbb{C} \cdot v_0, \qquad V^{0,-1} = \mathbb{C} \cdot v_0'$$

where

$$v_0 = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \qquad v_0' = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Proposition 1.12. Let $V^{-1,0}, V^{0,-1}$ as above. Then under the complex conjugation on $V \otimes \mathbb{R}$, we have

$$V^{-1,0} = \overline{V^{0,-1}}.$$

Proof. Let v, v' be the basis of $V \otimes \mathbb{C}$ such that $h(i)v = iv$, $h(i)v' = -iv'$. Thus $V^{-1,0} = \mathbb{C} \cdot v$, $V^{0,-1} = \mathbb{C} \cdot v'$. On the other hand, $h(i) \in \mathrm{Aut}(\mathbb{R}^2) = \mathrm{GL}(2, \mathbb{R})$ is a real matrix with eigenvalues i, $-i$, hence there is a real matrix γ such that

$$\gamma^{-1} h(i) \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = h_0(i).$$

Then we see that, with v_0 and v_0' as above, we have $\gamma\mathbb{C}\cdot v_0 = V_h^{-1,0}$, resp. $\gamma\mathbb{C}\cdot v_0' = V_h^{0,-1}$. Now, we see that the claim is true for the complex structure h_0, since $\bar{v_0} = v_0'$. Thus, since γ is real, the claim follows. $\square$

Now, we can relate the space of complex structures to $\mathbb{H}$ in the following way. Observe that $\mathrm{GL}(2,\mathbb{R})$ acts by fractional linear transformations on $\mathbb{H}^{\pm}$. The complex number $\tau_h = \gamma(i)$ then belongs to $\mathbb{H}^{\pm}$. Moreover we can define a map

$$\pi : \{\text{complex structures}\} \longrightarrow \mathbb{H}^{\pm}$$

by $\pi(h) = \tau_h$. This map may appear to depend on the choice of the matrix γ such that $h(i) = \gamma h_0(i)\gamma^{-1}$, but we have the following:

Proposition 1.13. The map π is well-defined.

Proof. We write $\tau_h(\gamma)$ to take provisional account of this dependence. Note first of all that h_0 and h both extend to algebra homomorphisms $\mathbb{C} \to M(2,\mathbb{R})$, and since i generates $\mathbb{C}$ as $\mathbb{R}$-algebra it follows that $\gamma h_0 \gamma^{-1} = h$. If γ' is another choice, then $k = \gamma'^{-1}\gamma$ belongs to the centralizer in $\mathrm{GL}(2,\mathbb{R})$ of h_0, i.e. to the centralizer of $h_0(\mathbb{C})$, which is just $h_0(\mathbb{C})$. Thus $k \in h_0(\mathbb{C})$, and if

$$k = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

we have $\tau_h(\gamma') = \gamma'(i) = \gamma(k(i))$. Since $k(i) = \dfrac{xi + y}{-yi + x} = i$, there is no dependence. $\square$

In other words, letting $K_\infty = h_0(\mathbb{C}^\times) \subset \mathrm{GL}(2,\mathbb{R})$, there is a sequence of identifications

$$\{\text{complex structures}\} \simeq \mathrm{GL}(2,\mathbb{R})/K_\infty \simeq \mathbb{H}^{\pm}.$$

The significance of this is that the final term has an obvious complex structure, hence so do the first two terms. Moreover, this complex structure is $\mathrm{GL}(2,\mathbb{R})$-invariant.

We can expland on this a little bit more, to define the Borel embedding. The function associating the normalized vector $v' = v_h' \in V_h^{0,-1}$ to h is compatible with the complex structure. Now $V_h^{0,-1} \subset V_{\mathbb{C}}$ is a variable line in $V_{\mathbb{C}}$, hence defines a variable point $p_h \in \mathbb{P}(V_{\mathbb{C}}) = \mathbb{P}^1(\mathbb{C})$. If $\binom{\alpha}{\beta}$ is the homogeneous coordinate of a point in $\mathbb{P}^1$, we use the standard inhomogeneous coordinate α/β. Then the inhomogeneous coordinate of $V_h^{0,-1}$ is just τ_h. We thus have a holomorphic embedding

$$\{\text{complex structures}\} \simeq \mathrm{GL}(2,\mathbb{R})/K_\infty \hookrightarrow \mathbb{P}(V_{\mathbb{C}})$$

obtained by associating the subspace $V_h^{0,-1}$ to h.

Now, we define a family of elliptic curves $\mathcal{E}$ over $\mathbb{H}$ which was given, for each complex structure h, some elliptic curve E_h given as $\mathbb{C}/i_h(\mathbb{Z}^2)$. Recall the formula for $i_h : \mathbb{R}^2 \simeq \mathbb{C}$:

$$i_h\big(h(z)e_0\big) = z \cdot i_h(e_0).$$

The map i_h extends by linearity to a surjective homomorphism

$$\mathbb{R}^2 \otimes \mathbb{C} = V_{\mathbb{C}} \longrightarrow \mathbb{C}.$$

The left hand side is $V^{-1,0} \oplus V^{0,-1}$, and since the formula shows that i_h commutes with the action of $\mathbb{C}^\times$ on both sides, it follows that the map $V_{\mathbb{C}} \to \mathbb{C}$ is the projection

$$V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}/V^{0,-1}.$$

In other words, the $\mathbb{C}$ in the numerator is identified with $V^{-1,0}$, and we have the formula

$$E_h = (V_{\mathbb{C}}/V_h^{0,-1})/i_h(\mathbb{Z}^2).$$

We can further check that $i_h(\mathbb{Z}^2)$ is given by $\mathbb{Z} \oplus \mathbb{Z} \cdot \tau_h$.

1.3 Moduli interpretation

We have constructed a family of elliptic curves $\mathcal{E}$ over $\mathbb{H}$, but as we saw above, elliptic curves over $\mathbb{C}$ are parametrized by $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. We want to say that the family $\mathcal{E}/\mathbb{H}$ descends to this quotient. This would imply the existence of a universal family over $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, and hence this would imply that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ a fine moduli space. (We will discuss this notion rigorously later.)

However, we see that $\mathcal{E}$ does not admit a quotient by $\mathrm{GL}(2,\mathbb{Z})$. More precisely, there is an action of $\mathrm{GL}(2,\mathbb{Z})$ on the family $\mathcal{E}$ preserving the subgroup $i_h(\mathbb{Z}^2)$; we simply let $g \in \mathrm{GL}(2,\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^2)$ act naturally on $i_h(\mathbb{Z}^2) \subset V_{\mathbb{C}}$. We run into the same issue as before: the element $-I_2 \in \mathrm{GL}(2,\mathbb{Z})$ acts as -1 on each $\mathcal{E}_h$ and the quotient is no longer a family of elliptic curves; and there are other elliptic fixed points in $\mathbb{H}$ (namely i and $e^{\frac{2\pi i}{3}}$) whose stabilizers define automorphisms of the corresponding elliptic curves.

Now, if we instead consider the group

$$\Gamma(N) = \{g \in \mathrm{GL}(2,\mathbb{Z}) \mid g \equiv I_2 \pmod N, \}$$

then there are no fixed points in $\mathbb{H}$ for any integer $N \geq 3$.

Proposition 1.14. For $N \geq 3$, $\Gamma(N)$ acts freely and properly discontinuously on $\mathbb{H}^+$.

Proof. We sketch the proof that the action is free. Suppose $\gamma \in \Gamma(N)$ has a fixed point in $\mathbb{H}^+$. Since the stabilizer of $i \in \mathbb{H}^+$ in $\mathrm{SL}_2(\mathbb{R})$ is $\mathrm{SO}_2(\mathbb{R})$ and since $\mathbb{H}^+$ is transitive under $\mathrm{SL}_2(\mathbb{R})$, we see that γ must lie in a $\mathrm{SL}_2(\mathbb{R})$-conjugate of $\mathrm{SO}_2(\mathbb{R})$. In particular γ must be semi-simple and its eigenvalues in $\mathbb{C}$ have absolute value 1. On the other hand, the characteristic polynomial of γ is monic with integer coefficients, so the eigenvalues of γ are algebraic integers. Combined with the previous fact, we see that the eigenvalues of γ must be roots of unity. In particular, we see that $\langle\gamma\rangle$ is a torsion subgroup of $\Gamma(N)$, but we can check that $\Gamma(N)$ is torsion free.

We omit the proof that $\Gamma(N)$ acts properly discontinuously. See [DS06, §2.1]. $\square$

In particular, this implies that $\Gamma(N)\backslash\mathbb{H}^+$ has the natural structure of a Riemann surface and $\mathbb{H}^+ \to \Gamma(N)\backslash\mathbb{H}^+$ is a covering. Further, this is obviously the universal covering, since $\mathbb{H}^+$ is simply connected.

Definition 1.15. The *modular curve* $Y(N)$ is the complex manifold

$$Y(N) := \bigsqcup_{j\in(\mathbb{Z}/N\mathbb{Z})^*} \Gamma(N)\backslash\mathbb{H}.$$

It also follows that the quotient $\Gamma(N)\backslash\mathcal{E}$ is a family of elliptic curves over $\Gamma(N)\backslash\mathbb{H}$.

We can ask what this space classifies. Observe that $\Gamma(N)$ fixes the group $N^{-1}\mathbb{Z}^2/\mathbb{Z}^2$, the basis of points of order N in $\mathcal{E}_h$ as defined by the generators

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

modulo N is fixed for all $h \in \mathbb{H}$. Thus, we see that points of $\Gamma(N)\backslash\mathbb{H}$ carry more than the data of the elliptic curve E_h: we also have an isomorphism

$$\alpha_N : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N],$$

where

$$E[N] := \{z \in E \mid z + \cdots + z \text{ (N times) } = 0\}.$$

Recall that $E[N]$ is non-canonically isomorphic to $(\mathbb{Z}/\mathbb{Z})^2 = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$ as $\mathbb{Z}/N\mathbb{Z}$-modules.

Definition 1.16. A choice of an isomorphism $\gamma : E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$ is called a *level-N structure* on E. Equivalently, this is a choice of an ordered basis (P, Q) of $E[N]$ as a free $\mathbb{Z}/N\mathbb{Z}$-module.

Now, we consider the following: For each $j \in (\mathbb{Z}/N\mathbb{Z})^\times$, fix once and for all an element $g_j \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ such that $\det(g_j) = j$. For instance, we may take $g_j = \begin{pmatrix} j & 0 \\ 0 & 1\end{pmatrix}$.

For each $j \in (\mathbb{Z}/N\mathbb{Z})^\times$ and each $h \in \mathbb{H}^+$, we can define an elliptic curve together with a level-N structure: $(E_h, \gamma_h = g_j \circ \alpha_N : E_h[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2)$. Moreover, we say that two tuples (E, γ) and (E', γ') are isomorphic if we have an isomorphism $f : E \mapsto E'$ such that we have a commutative square

$$\begin{tikzcd} E[N] \arrow[r, "f"] \arrow[d, "\gamma"] & E'[N] \arrow[d, "\gamma'"] \\ (\mathbb{Z}/N\mathbb{Z})^2 \arrow[r, equal] & (\mathbb{Z}/N\mathbb{Z})^2. \end{tikzcd}$$

Thus, we have a map

$$Y(N) \to \{\text{elliptic curves with level } N\text{-structure}\}/\text{isomorphism} \tag{1.3.1}$$

Proposition 1.17. The map (1.3.1) is a bijection.

Proof. We sketch here surjectivity: Let (E, γ) be an elliptic curve with level N structure. As before, we can identify E with a complex structure on given by some $h \in \mathbb{H}$ on $\Lambda = \mathbb{Z}^2$. Fix an isomorphism

$$u : \Lambda/N\Lambda \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2.$$

Observe that a level N structure on E induces an isomorphism (also denoted γ)

$$\gamma : \Lambda/N\Lambda \simeq (\mathbb{Z}/N\mathbb{Z})^2,$$

we compose u with some element of $\mathrm{GL}_2(\mathbb{Z})$ of determinant -1, and as a result we can always assume that $h \in \mathbb{H}^+$. Now let γ' be the composition

$$(\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{u^{-1}} \Lambda/N\Lambda \xrightarrow{\gamma} (\mathbb{Z}/N\mathbb{Z})^2.$$

Then $\gamma' \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. We note the following fact:

Fact. (Strong approximation for SL_2.) The natural map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.

For a proof, see [DS06, Exercise 1.2.2]. Note that the statement is not true if we replace SL_2 by GL_2, since elements of $\mathrm{GL}_2(\mathbb{Z})$ all have determinants ± 1.

Let $j = \det(\gamma')$, so $\gamma' g_j^{-1} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. By the above fact, we can compose u with a suitable element of $\mathrm{SL}_2(\mathbb{Z})$ to arrange that $\gamma' = g_j$. When we do this the element h we found in the above will be moved by the element of $\mathrm{SL}_2(\mathbb{Z})$. The image of such h under the above map will hence be the isomorphism (E, γ). □

Remark 1.18. Note that to make a distinguished choice of one connected component $\Gamma(N)\backslash\mathbb{H}$ in $Y_(N)$ amounts to choosing a primitive N-th root of unity. In particular, we see that $Y(N)$ is a more natural space to consider, since we are not required to make this choice. This difference will become important later when we want to define the canonical model of the modular curve.

It is natural to simply 'extend the moduli problem' to classify elliptic curves over arbitrary bases (at least away from primes dividing N), where we have $E[N]$ is (at least étale locally) isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$. We will make this precise later.

2 Shimura data

For the modular curve the complex upper half plane $\mathbb{H}$ played a significant role; our first step is to understand its generalization.

2.1 Hodge structures

Definition 2.1. Let V be a finite-dimensional $\mathbb{R}$-vector space. A (real) *Hodge structure* on V is a $\mathbb{C}$-linear direct sum decomposition $V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ such that the natural conjugate-linear action $c \otimes v \mapsto \overline{c} \otimes v$ of complex conjugation on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ swaps $V^{p,q}$ and $V^{q,p}$. The decreasing chain of subspaces

$$F^p = \bigoplus_{p' \geq p} V^{p',q'}$$

is called the *Hodge filtration* of $V_{\mathbb{C}}$, and if the (p,q) for which $V^{p,q} \neq 0$ all satisfy $p+q=n$ then we say that the Hodge structure is *pure of weight* n.

Note that in general $F^p \cap \overline{F}^q = \bigoplus_{p' \geq p, q' \geq q} V^{p',q'}$, so if the Hodge structure is pure of weight n then $F^p \cap \overline{F}^q = V^{p,q}$ whenever $p+q=n$ (so in the pure case the Hodge structure can be recovered from the Hodge filtration by means of the complex conjugation action on $V_{\mathbb{C}}$). Also, it is generally harmless to restrict attention to pure Hodge structures, since if V is equipped with a Hodge structure then for any $n \in \mathbb{Z}$ the $\mathbb{C}$-subspaces $\bigoplus_{p+q=n} V^{p,q}$ in $V_{\mathbb{C}}$ are stable under the complex conjugation action and hence descend to $\mathbb{R}$-subspaces $V_n \subset V$ such that $\bigoplus V_n \simeq V$ (as scalar extension to $\mathbb{C}$ recovers the Hodge decomposition of $V_{\mathbb{C}}$, with the $V^{p,q}$ collected according to the value of $p+q$). Each V_n is equipped with a pure Hodge structure of weight n, so we can study these separately.

Remark 2.2. The terminology (and motivation) for a Hodge structure that is pure of weight n occurs for $V_n = H^n(Y(\mathbb{C}), \mathbb{R})$ with a smooth proper variety Y over $\mathbb{C}$. Indeed, we have that

$$H^n(Y(\mathbb{C}), \mathbb{R}) \otimes \mathbb{C} \simeq H^n(Y(\mathbb{C}), \mathbb{C}),$$

and by the Poincaré Lemma, we can use the complex-analytic (Kahler manifold) structure on $Y(\mathbb{C})$ to define a decreasing *Hodge filtration* F^p (consisting of de Rham cohomology classes represented by closed $\mathbb{C}$-valued C^∞ differential forms whose local expression in terms of dz_j's and $d\overline{z}_k$'s only involves wedge products with at least p of the dz_j's; e.g., $F^0 = (V_n)_{\mathbb{C}}, F^n$ contains the space $H^0(Y(\mathbb{C}), \Omega^n_Y)$ of global holomorphic n-forms, and $F^{n+1} = 0$). It is a deep theorem in Hodge theory that for V_n as above, $V^{p,q} := F^p \cap \overline{F}^q$ defines a Hodge structure on V_n that is pure of weight n (e.g., $V^{p,q} = \emptyset$ whenever $p+q \neq n$). Traditionally this is proved by exhibiting another construction of $V^{p,q}$ in terms of "harmonic forms" (relative to a choice of Kähler metric, say when Y is projective), but the final output of the construction turns out to be metric-independent. More explanations can be found in [Voi02]. There also exists a purely algebraic proof by Deligne–Illusie.

The cohomology algebra $\bigoplus_{n \geq 0} V_n = H^*(Y(\mathbb{C}), \mathbb{R})$ naturally inherits a Hodge structure for which V_n is the associated subspace that is pure of weight n.

For instance, if $Y = E$ is an elliptic curve over $\mathbb{C}$ and we take $n = 1$ then $F^1 = H^0(E, \Omega^1_E)$ is the space of global 1-forms on E and it is the kernel of the natural map $F^0 = (V_n)_{\mathbb{C}} = H^1(E(\mathbb{C}), \mathbb{C}) \to H^1(E(\mathbb{C}), \mathcal{O}_E(\mathbb{C})) = H^1(E, \mathcal{O}_E)$ which turns out to be surjective.

Proposition 2.3. There is an equivalence of categories between pairs $(V, h : \mathbb{C}^\times \to \mathrm{Aut}(V_{\mathbb{R}}))$, where V is a rational vector space and h is a homomorphism of real algebraic groups, and Hodge structures on V.

Proof. Let V be a rational vector space with a Hodge structure. Define $h : \mathbb{C}^\times \to \mathrm{Aut}(V_{\mathbb{R}})$ by letting $h(z)$ act as $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$ (this is the convention). One then has to check that h actually gives us a homomorphism of real algebraic groups. □

Remark 2.4. Another way to say this is to let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Then we want an algebraic homomorphism of $\mathbb{R}$-groups $h : \mathbb{S}_{\mathbb{R}} \to \mathrm{Aut}(V_{\mathbb{R}})$.

Definition 2.5. A polarization for a Hodge structure $(V, V^{p,q})$ of weight n is a nondegenerate bilinear form

$$\Psi : V \times V \to \mathbb{R}$$

which extends to $V_{\mathbb{C}}$ by linearity and is symmetric if n is even and alternating if n is odd. We also require that Ψ be subject to the following relations:

$$\Psi\left(V^{p,q}, V^{p',q'}\right) = 0 \quad \text{if } (p', q') \neq (q, p)$$

$$i^{p-q}\,\Psi(x, \overline{x}) > 0 \quad \text{for nonzero } x \in V^{p,q}.$$

These conditions are called the Hodge-Riemann bilinear relations.

Remark 2.6. Another equivalent definition is as follows. We denote by $\mathbb{R}(-n)$ the Hodge structure defined by $z \mapsto |z|^{-n}$. Then a polarization is a morphism of Hodge structures

$$\Psi : V \times V \to \mathbb{R}(-n)$$

such that $\Psi(v, h(i)w)$ is symmetric and positive definite.

Example 2.7. Fix a compact Kähler manifold X with Kähler form ω. Fix an integer $k \geq 0$. Let $(V, V^{p,q})$ be the pure Hodge structure obtained from $H^k(X(\mathbb{C}), \mathbb{R})$. We use ω to define a nondegenerate bilinear form $Q : V \times V \to \mathbb{R}$ by

$$Q(\alpha, \beta) := (-1)^{k(k-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{\dim(X)-k}.$$

This form Q is a polarization for $(V, V^{p,q})$, so we get the polarized Hodge structure $(V, V^{p,q}, Q)$.

Now, let us consider Hodge structures in families.

Definition 2.8. A variation of Hodge structures of weight n on a $\mathbb{C}$-scheme S is a $\mathbb{R}$-local system V on S, together with a decreasing filtration $F^pV_S \subset V_S := V \otimes \mathcal{O}_S$ by holomorphic vector bundles, such that:

1. The F^p define a Hodge structure on each fiber of V_S, that is, $V_S = F^p V_S \oplus \overline{F^{n-p+1} V_S}$.

2. (Griffiths Transversality) Let $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$. Then by the Riemann-Hilbert correspondence, $V_{\mathbb{C}}$ defines an integrable holomorphic connection

$$\nabla : V_S \to V_S \otimes \Omega^1_S.$$

We require that the F^p satisfy Griffiths transversality with respect to ∇, that is,

$$\nabla(F^p V_S) \subset F^{p-1} V_S \otimes \Omega^1_S.$$

Definition 2.9. A polarized variation of Hodge structure of weight n on S is a variation of Hodge structure $(V, F^p V_S)$ on S along with a bilinear form $Q : V \otimes V \to \mathbb{R}(-n)$, so that the restriction of $(V, F^p V_S, Q)$ to each fiber is a polarized Hodge structure.

Example 2.10. The key (and motivating) example of a variation of Hodge structures is the (relative) de Rham cohomology of the fibers of a morphism $f : X \to S$ that is the analytification of a smooth proper morphism between smooth complex algebraic varieties. More specifically, one defines"relative de Rham cohomology" sheaves $\mathcal{H}^n_{\mathrm{dR}}(X/S) = R^n f_*(f^{-1}\mathcal{O}_S) \simeq R^n f_*(\mathbb{C}) \otimes \mathcal{O}_S$, and homological methods provide a relative Hodge to de Rham spectral sequence that defines a natural decreasing filtration on $\mathcal{H}^n_{\mathrm{dR}}(X/S)$ by $\mathcal{O}_S$-submodules $\mathcal{F}$. One can show (eg. [Sta25, 0FK4], and Ehresmann's theorem) that these subsheaves are actually locally free with formation commuting with any base change, inducing on s-fibers exactly the traditional Hodge filtration on $H^n(X_s, \mathbb{C})$ determined by the complex structure on X_s. The connection ∇ here is the Gauss-Manin connection.

A variation of Hodge structures always admits a period morphism, which looks at how the stages of the Hodge filtration move within this fixed space. We saw this in the the case of the Weierstrass family of elliptic curves over $\mathbb{C} - \mathbb{R}$, for which the line $H^0(\mathcal{E}_\tau, \Omega^1) = \mathbb{C}dz$ moves within $H^1(\mathcal{E}_\tau, \mathbb{C}) = \mathbb{C} \otimes \Lambda = \mathbb{C}^2$ via the expression $dz = \tau e_1^* + e_2^*$ with varying τ, where we had the Borel embedding $\mathbb{H} \hookrightarrow \mathbb{P}^1(\mathbb{C})$. In general, given a variation of Hodge structures on S, for each p we can define a holomorphic map

$$S \to \mathrm{Gr}_d(n)$$

which is the Grassmannian of dimension d subspaces of a dimension n vector space, where d is the dimension of $\mathcal{F}^p$, and n is the dimension of V.

We now want to understand moduli spaces of polarized variation of Hodge structures, and relate them to Hermitian symmetric domains.

2.2 Hermitian symmetric domains

Let M be a complex manifold, and let $J_p : T_p M \to T_p M$ denote the action of i on the tangent space at a point p of M.

Definition 2.11. A *Hermitian metric* on M is a Riemannian metric g on the underlying smooth manifold of M such that J_p is an isometry for all p. Recall that a *Riemannian metric* on M is a positive-definite smooth contravariant 2-tensor field g on M. This means: g consists of positive-definite bilinear forms $g_p : T_p(M) \times T_p(M) \to \mathbb{R}$ for each point $p \in M$ such that for smooth vector fields X, Y on M, the function $g(X, Y) : p \mapsto g_p(X_p, Y_p)$ is smooth. In local coordinates $(x^i)_{1 \le i \le n}$ a Riemannian metric has the form

$$g_p = \sum_{i,j} g_{ij}(p) dx^i \otimes dx^j$$

for some symmetric positive-definite matrix $g_{ij}(p)$ that depends smoothly on p.

A *Hermitian manifold* is a complex manifold equipped with a hermitian metric g, and a *Hermitian symmetric space* is a connected homogeneous Hermitian manifold M that admits a symmetry at each point p, i.e., the group of holomorphic isometries acts transitively, and there exists an involution s_p having p as an isolated fixed point.

We denote by $\mathrm{Hol}(M)$ the group of holomorphic automorphisms of a Hermitian symmetric space M, and $\mathrm{Is}(M, g)$ the group of homomorphic isometries. These are real Lie groups and we denote by $\mathrm{Hol}(M)^+$, $\mathrm{Is}(M, g)^+$ the connected component of the identity.

Theorem 2.12. Every hermitian symmetric space M is a product of hermitian symmetric spaces of the following types:

- Noncompact type — the sectional curvature is negative
- Compact type — the sectional curvature is positive
- Euclidean type — the sectional curvature is zero.

In the first two cases, the space is simply connected. A hermitian symmetric space is *indecomposable* if it is not a product of two hermitian symmetric spaces of lower dimension.

A proof can be found in [Hel79, Chapter VIII].

Definition 2.13. A Hermitian symmetric space of non-compact type is called a Hermitian symmetric domain.

Example 2.14.
1. Let Λ be a discrete subgroup of $\mathbb{C}$. Then $\mathbb{C}/\Lambda$ is a Hermitian symmetric space. After identifying $\mathbb{C}$ with $\mathbb{R}^2$, the metric is given by the standard form $dx^2 + dy^2$. The translations act transitively on $\mathbb{C}/\Lambda$ and $x \mapsto -x$ is a symmetry at 0. Geodesics are images of straight lines in $\mathbb{C}$, and this has zero curvature.

2. The Riemann sphere $\mathbb{P}^1(\mathbb{C}) \cong S^2$ is a Hermitian symmetric space. The metric is induced from the standard metric $dx^2 + dy^2 + dz^2$ of ambient space $\mathbb{R}^3$ of S^2. Rotations act transitively on S^2 and symmetries are given by rotations by π around a diameter. Geodesics are great circles, and this has positive curvature.

3. The upper half-plane $\mathbb{H}_1$ is a Hermitian symmetric domain. After identifying $\mathbb{C}$ with $\mathbb{R}^2$, the metric is given by $\frac{dx^2+dy^2}{y^2}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}_1$ by Möbius transforms. This space has negative curvature.

We observe the following:

Proposition 2.15. Let M be a Hermitian symmetric space, $p \in M$, and denote by K_p the stabilizer subgroup of $\mathrm{Isom}(M, g)^+$. Then K_p is compact, and we have an isomorphism of smooth manifolds

$$\mathrm{Isom}(M, g)^+/K_p \simeq M.$$

We are interested in the situation where $\mathrm{Isom}(M, g)^+ \simeq G(\mathbb{R})^+$ for some connected reductive group G. One way to arrange for this is to mimic the GL_2 case and suppose that M is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$. (It will turn out that *every* Hermitian symmetric domain has such a form.)

2.3 Shimura data

We now suppose that G is a connected reductive group over $\mathbb{Q}$, and we let X be a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$.

We first explain how to get a variation of Hodge structures over X.

Firstly, from the previous section, we see that the points h serve as abstract encodings of G-valued Hodge structures, using a Tannakian perspective. More precisely, what we see is that given any algebraic representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{R}$, we may look at the compositions

$$\rho_{\mathbb{R}} \circ h : \mathbb{C}^\times \to \mathrm{Aut}(V)$$

(where $\rho_{\mathbb{R}}$ denotes ρ restricted to $\mathbb{R}$-points) and hence over X we get a "family" of Hodge structures on V. Now, there is a particularly special choice for (ρ, V), namely the adjoint representation

$$\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g}) \qquad \text{for } \mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}\big(G(\mathbb{R})\big).$$

One of the key observations we will see is that Hodge-theoretic conditions on the Hodge structures arising from every (ρ, V) should be expressed entirely in terms of the pair (G, X) and the Hodge structure on $\mathfrak{g}$ arising from $\rho = \mathrm{Ad}_G$.

Now, we fix a pair (ρ, V), and for each h we consider the $\mathbb{R}$-linear weight decomposition $\bigoplus_{n \in \mathbb{Z}} V_{n,h}$ on V arising from each $h \in X$ is (by definition) the decomposition arising from the $\mathbb{G}_m$-action $\rho_{\mathbb{R}} \circ h|_{\mathbb{R}^\times}$ on V over $\mathbb{R}$. Concretely, $V_{n,h}$ consists of those $v \in V$ such that $\rho(h(x))(v) = x^n v$ for all $x \in \mathbb{R}^\times$. We first wish to formulate a condition on (G, X) that is equivalent to the property that for any (ρ, V) the subspaces $V_{n,h} \subseteq V$ are the same for all $h \in X$.

Definition 2.16. Let $\mathbb{G}_m \to \mathbb{S}$ be the algebraic map which on $\mathbb{R}$-points sends $z \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$ to $z^{-1} \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. For any $h \in X$, we let $w_h : \mathbb{G}_m \to G$ be the composition

$$\mathbb{G}_m \to \mathbb{S} \xrightarrow{h} G,$$

and we call this the weight homomorphism.

Lemma 2.17. The subspaces $V_{n,h} \subseteq V$ are independent of $h \in X$ for every $\rho : G \to \mathrm{GL}(V)$ if and only if w_h has image in the center of G for some (equivalently, all) $h \in X$.

Proof. For any $g \in G(\mathbb{R})$, $\rho\big(w_{g,h}(x)\big) = \rho(g)\,\rho\big(w_h(x)\big)\,\rho(g)^{-1}$ for all $x \in \mathbb{R}^\times$. If we look at a faithful representation of G, then we see that the image of $\rho \circ w_h$ must lie in the center of $\rho(G)$, so the image of w_h is in Z_G. The reverse implication is clear. $\square$

Observe that we see that the composition $\mathrm{Ad}_G \circ h$ must be trivial when restricted to $\mathbb{R}^\times$, and thus it has weight 0. Now, we have not given any holomorphic structure on X, but it turns out we have the following:

Proposition 2.18. X has a unique complex structure for which each h defines a holomorphic family of Hodge structures.

Proof. (Sketch) This is [Mil05, 2.14(a)], and the rough idea is that after composing with a faithful representation of G, we can use the Borel embedding to embed X into a flag variety by looking at the Hodge filtration F^p for each p. We then check that the tangent spaces at each h define a complex subspace of the tangent space of the corresponding point in the Grassmannian. $\square$

We want this family of Hodge structures to be a polarizable variation of Hodge structures. We first look at the condition to be a variation of Hodge structures:

Proposition 2.19. Fix $\rho : G \to \mathrm{GL}(V)$ be a faithful representation. Let V_n be the holomorphic bundles on X corresponding to the weight filtration on the Hodge structure. Denote by the Hodge filtration by $\{F^p_{n,h}\}$ on V_n as h varies in X, and $\mathcal{H}_n = (V_n)_\mathbb{C} \otimes \mathcal{O}_X$. Define the connection

$$\nabla = 1 \otimes d : \mathcal{H}_n \otimes \mathcal{O}_X \to \mathcal{H}_n \otimes \Omega^1_X$$

Then V_n defines a variation of Hodge structures, i.e.

$$\nabla_\rho\big(\mathcal{F}^p_n\big) \subseteq \mathcal{F}^{p-1}_n \otimes_{\mathcal{O}_X} \Omega^1_X \qquad \text{for all } p \in \mathbb{Z}$$

if and only if for every $h \in X$ the weight-0 Hodge structure $\{V_h^{p,-p}\}_{p\in\mathbf{Z}}$ on $\mathfrak{g}$ associated to $\mathrm{Ad}_G \circ h$ is of type

$$\{(-1,1),(0,0),(1,-1)\}.$$

Proof. (Sketch) We use the construction as in the previous proof of the Borel embedding, which we denote by ϕ. Note that on tangent spaces, we have a map

$$d\phi : \mathfrak{g}/\mathfrak{g}^{00} \to \mathrm{End}(V)/\mathrm{End}(V)^{00} \simeq \mathrm{End}(V_\mathbb{C})/F^0\mathrm{End}(V_\mathbb{C})$$

and Griffiths transversality corresponds to the condition that the image of $d\phi$ lies in the coset classes of those $T \in \mathrm{End}(V_\mathbb{C})$ such that $T(F^p) \subseteq F^{p-1}$ for all $p \in \mathbb{Z}$, which is exactly $F^{-1}(\mathrm{End}(V_\mathbb{C}))$. Now, since $d\phi$ is injective, this is true if and only if we have $\mathfrak{g}_\mathbf{C} = F^{-1}(\mathfrak{g}_\mathbf{C})$, which is to say that the Hodge structure on $\mathfrak{g}$ defined by h makes $\mathfrak{g}^{p,q}_\mathbb{C}$ vanish whenever $p < -1$. Since complex conjugation on $\mathfrak{g}_\mathbb{C}$ swaps $\mathfrak{g}^{q,p}_\mathbb{C}$ and $\mathfrak{g}^{p,q}_\mathbb{C}$, we conclude that the transversality condition on fibers at $h \in X$ is equivalent to the vanishing of $\mathfrak{g}^{p,-p}_\mathbb{C}$ whenever $p \neq 0, \pm 1$, which is to say that the Hodge structure on $\mathfrak{g}$ defined by h has type $\{(1,-1),(0,0),(-1,1)\}$. $\square$

We now turn our attention to getting a polarization. The following result is [Del79, 1.1.14(iii)].

Theorem 2.20. In order that the variation of Hodge structures above admits a polarization, it is necessary and sufficient that the following two conditions hold.

1. Let G_1 be the minimal connected closed $\mathbb{R}$-subgroup in G such that all maps $h : \mathbb{C}^\times \to G(\mathbb{R})$ factor through $G_1(\mathbb{R})$. Then G_1 is reductive.

2. Let G_1' be the connected semisimple derived group of G_1 (e.g., $G_1' = \mathrm{SL}_n$ when $G_1 = \mathrm{GL}_n$). Then the intersection of the fixed-point locus of ι_h in $G(\mathbb{R})$ with $G_1'(\mathbb{R})^0$ in a maximal compact subgroup for some (equivalently, any) $h \in X$.

Here, we note that since for any $h \in X$, the conjugation action by $h(i)$ on G is independent of the choice of $i = \sqrt{-1} \in \mathbb{C}$ and is an involution since $h(-1)$ is central in G (by the weight-0 hypothesis), we obtain an involution denoted by ι_h.

Note that if we assumed that G was reductive, then the first condition clearly holds. Moreover, for the second condition, we can equivalently look at the image in the adjoint group $G_{\mathbb{R}}^{\mathrm{ad}}$, which we require to be maximal compact.

Definition 2.21. A involution θ of G is called a Cartan involution if the real form G^θ is compact. Here G^θ is a real algebraic group whose $\mathbb{R}$-points are

$$G^\theta(\mathbb{R}) = \{\, g \in G(\mathbb{C}) \mid \theta(g) = \overline{g} \,\}\,.$$

Remark 2.22. Observe that condition (2) implies that $\theta = i_h$ is a Cartan involution of G, and thus G^θ is hence an inner form of G.

To summarize, to get a Hermitian symmetric domain, we require the pair (G, X) to satisfy:

SV1: For all $h \in X$, the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{Ad}_G \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\};$$

SV2: For all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G_{\mathbb{R}}^{\mathrm{ad}}$,

Definition 2.23. A *Shimura datum* is a pair (G, X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

SV1: For all $h \in X$, the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{Ad}_G \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\};$$

SV2: For all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G_{\mathbb{R}}^{\mathrm{ad}}$,

SV3: G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Remark 2.24. SV3 is a mild extra condition that ensures some 'minimality' of the group G. There are situations (eg. quaternionic Shimura sets) where we ignore this condition.

Remark 2.25. Observe that for $n \geq 3$, GL_n cannot admit a Shimura datum because the quotient $\mathrm{GL}_n(\mathbb{R})/K_\infty$ where $K_\infty = \mathbb{R}^\times K_\infty^\circ$, and K_∞° is a maximal compact in $\mathrm{GL}_n(\mathbb{R})$, is not a Hermitian symmetric domain. The easiest way to see this is that if this were the case $\mathrm{GL}_{n,\mathbb{R}}$ must admit an inner form which is compact, and $\mathrm{GL}_{n,\mathbb{R}}$ does not for $n \geq 3$

Remark 2.26. One can ask about quotients $G(\mathbb{R})/K_\infty$ which are not Hermitian symmetric domains, merely Riemannian symmetric domains. An example of this is the arithmetic hyperbolic 3-manifold, for the group $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, for F an imaginary quadratic field. It turns out that some of these locally symmetric spaces can be found in the boundary of the Borel-Serre compactification of some Shimura variety, and this observation has been used quite successfully to prove results in this case, c.f. [?Scholzetorsion].

2.4 List of Hermitian symmetric domains

We can give an exhaustive list of the possible Hermitian symmetric domains, using the above characterization. Indeed, from [Lan17, §3]we have the following list of possible Hermitian symmetric domain $\mathcal{D} = G(\mathbb{R})/K$:

1. $G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}), K \cong U_n, \mathcal{D} = \mathcal{H}_n$, where
$$\mathcal{H}_n := \{\, Z \in \mathrm{Sym}_n(\mathbb{C}) : \mathrm{Im}(Z) > 0 \,\}$$
known as the Siegel upper half space, and
$$K := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{R}) \right\} \simeq U_n := \{\, g = A + iB : {}^t\overline{g}\, g = 1_n \,\}.$$

2. For all $a, b \geq 1$,
$$G(\mathbb{R}) = U_{a,b} := \{\, g \in \mathrm{GL}_{a+b}(\mathbb{C}) : {}^t\overline{g}\, 1_{a,b}\, g = 1_{a,b} \,\},$$
where $1_{a,b} := \begin{pmatrix} 1_a & 0 \\ 0 & -1_b \end{pmatrix}$,
$$\mathcal{D}_{a,b} := \left\{ U \in M_{a,b}(\mathbb{C}) : {}^t\begin{pmatrix} \overline{U} \\ 1 \end{pmatrix} \begin{pmatrix} 1_a & 0 \\ 0 & -1_b \end{pmatrix} \begin{pmatrix} U \\ 1 \end{pmatrix} = {}^t\overline{U}\, U - 1_b < 0 \right\},$$
$$K := \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in U_{a,b} \right\} \simeq U_a \times U_b$$

3. $G(\mathbb{R}) = \mathrm{SO}^*_{2n}, K \cong U_n, \mathcal{D} = \mathcal{H}_{\mathrm{SO}^*_{2n}}$, with $n \geq 2$:

4. $G(\mathbb{R})^+ = \mathrm{SO}_{a,2}(\mathbb{R})^+, K \cong \mathrm{SO}_a(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R}), \mathcal{D}^+ = \mathcal{H}^+_{\mathrm{SO}_{a,2}}$, with $a \geq 1$ but $a \neq 2$:

5. Lie $G \cong \mathfrak{e}_{6(-14)}$, Lie $K \cong \mathfrak{so}_{10} \oplus \mathbb{R}, \mathcal{D} = \mathcal{H}_{E_6}$:

6. Lie $G \cong \mathfrak{e}_{7(-25)}$, Lie $K \cong \mathfrak{e}_6 \oplus \mathbb{R}, \mathcal{D} = \mathcal{H}_{E_7}$:

Example 2.27. Let B be a quaternion algebra over a totally real field F, and let G be the algebraic group over $\mathbb{Q}$ with $G(\mathbb{Q}) = B^\times$. Then $B \otimes_\mathbb{Q} F = \prod_\nu B \otimes_{F,\nu} \mathbb{R}$, where ν runs over the embeddings of F into $\mathbb{R}$. We have

$$B \otimes_\mathbb{Q} \mathbb{R} \approx \mathbb{H} \times \cdots \times \mathbb{H} \times M_2(\mathbb{R}) \times \cdots \times M_2(\mathbb{R})$$

$$G(\mathbb{R}) \approx \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times \times \mathrm{GL}_2(\mathbb{R}) \times \cdots \times \mathrm{GL}_2(\mathbb{R})$$

$$h(a+ib) = 1 \quad \cdots \quad 1 \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \cdots \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Let X denote the $G(\mathbb{R})$-conjugacy class of h. Then (G, X) satisfies SV1 and SV2, and so it is a Shimura datum provided B splits (i.e., becomes isomorphic to $M_2(\mathbb{R})$) at at least one real prime of F. It then satisfies SV3 because G^{ad} is simple (as an algebraic group over $\mathbb{Q}$). Let $I = \mathrm{Hom}(F, \mathbb{Q}^a) = \mathrm{Hom}(F, \mathbb{R})$, and let I_{nc} be the set of ν such that $B \otimes_{F,\nu} \mathbb{R}$ is split. If B is non-split at all real places, then this forms a quaternionic Shimura set.

Example 2.28. Let $G = \mathrm{GSp}_{2n}/\mathbb{Q}$, and $X = \mathcal{H}_n^\pm$. Then this is a Shimura datum, corresponding to the *Siegel modular variety.*

Finally, we can ask how to realize the original SL_2 formulation of the modular curve in this picture; these form connected Shimura varieties.

Definition 2.29. A *connected Shimura datum* is a pair (G, X^+) consisting of a semisimple algebraic group over $\mathbb{Q}$ and a $G^{ad}(\mathbb{R})^+$-conjugacy class of homomorphisms $h : \mathbb{S} \to G^{ad}_\mathbb{R}$ satisfying

1. For all $h \in X^+$, the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{Ad}_G \circ h$ is of type
$$\{(-1, 1), (0, 0), (1, -1)\};$$

2. For all $h \in X^+$, $\mathrm{ad}(h(i))$ is a Cartan involution on $G^{ad}_\mathbb{R}$.

3. G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

2.5 Adelic description

Finally, we want to explain how the adelic description of $\mathrm{Sh}_K(G, X)$ has the structure of a locally symmetric space. We define

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K,$$

for an open compact subgroup. We want to show that this is a locally symmetric space, that is, we can write it in the form $\Gamma \backslash X$ for a discrete subgroup Γ of the automorphisms of X. For simplicity, we will only cover the case of the modular curve, as the proof in general is quite similar.

Proposition 2.30. We have a bijection:

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{A}_f)/K(N) \cong \Gamma\backslash\mathbb{H}.$$

where $K(N) = \ker(\mathrm{SL}_2(\hat{\mathbb{Z}}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$.

Proof. We first consider $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_f)/K(N)$. Observe that the natural inclusion

$$\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \subset \mathrm{SL}_2(\mathbb{A}_f)/K(N).$$

induces an inclusion

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \subset \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_f)/K(N),$$

since $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Q}) \cap \mathrm{SL}_2(\hat{\mathbb{Z}})$.

Now, observe that we have an equality $\mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(\mathbb{Q})\mathrm{SL}_2(\hat{\mathbb{Z}})$, so this inclusion is surjective hence a bijection. But by definition

$$\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \cong \mathrm{SL}_2(\mathbb{Z}/NZ),$$

so we conclude using strong approximation

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{Z}/NZ) \cong \{1\}.$$

It follows that

$$\mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(\mathbb{Q})K(N).$$

Consequently, we see that

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{A}_f)/K(N) \cong (\mathrm{SL}_2(\mathbb{Q}) \cap K(N))\backslash\mathbb{H} \cong \Gamma\backslash\mathbb{H},$$

where the first isomorphism follows because every representative (h, g) with $h \in \mathbb{H}$ and $g \in \mathrm{SL}_2(\mathbb{A}_f)$ is equivalent to a representative of the form (h', k), for $h' \in \mathbb{H}$ and $k \in K(N)$, and the first map is induced by $(h, g) \mapsto h'$. Moreover, $(h_1, k_1) \sim (h_2, k_2)$ if and only if there exists $\gamma \in \mathrm{SL}_2(\mathbb{Q}) \cap K(N)$ such that $\gamma h_1 = h_2$. □

As explained above, we would like to get the whole $Y(N)$, rather than just one connected component. To do this, it turns out that we have to work with the group $G = \mathrm{GL}_2$ instead of SL_2. Then denote by $\mathrm{GL}_2(\mathbb{R})_+ \subset \mathrm{GL}_2(\mathbb{R})$ the subset which stabilizes $\mathbb{H}^+ \subset \mathbb{H}^\pm$ (namely those with positive determinant).

Proposition 2.31.

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K'(N) \cong Y(N).$$

where $K'(N) = \ker(\mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$.

Proof. Let $\mathrm{GL}_2(\mathbb{Q})_+ = \mathrm{GL}_2(\mathbb{R})_+ \cap \mathrm{GL}_2(\mathbb{Q})$. Then we can argue as above to see that we have an isomorphism

$$\mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{A}_f)/K'(N) \cong \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

Moreover, we have an isomorphism

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathbb{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_f)/K'(N) \simeq \mathrm{GL}_2(\mathbb{Q})_+\backslash \mathbb{H} \times \mathrm{GL}_2(\mathbb{A}_f)/K'(N) = \bigsqcup_{(\mathbb{Z}/NZ)^\times} \Gamma\backslash\mathbb{H}.$$

The first isomorphism follows from the fact that since $\mathrm{GL}_2(\mathbb{Q})$ is dense in $\mathrm{GL}_2(\mathbb{R})$ (from Real approximation, see [Mil05, Appendix A] for a proof) and $\mathrm{GL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$, every $h \in \mathbb{H}^\pm$ is of the form qh^+ with $q \in \mathrm{GL}_2(\mathbb{Q})$ and $h^+ \in \mathbb{H}^+$. This shows that the map is surjective. We leave injectivity as an exercise. □

2.6 Hodge type Shimura varieties

Definition 2.32. We define a morphism of Shimura data $(G, X) \to (G', X')$ to be a algebraic group homomorphism $f : G \to G'$ which maps the conjugacy class X to X'.

Given any morphism of Shimura data, and a level $K \subset G(\mathbb{A}_f)$ whose image lies in some $K' \subset G'(\mathbb{A}_f)$, we get a map of Shimura varieties

$$\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G', X').$$

If the map f is a closed embedding, we can even have the following:

Proposition 2.33 ([Del71, Proposition 1.15])**.** Let $(G, X) \hookrightarrow (G', X')$. For any compact open $K \subset G(\mathbb{A}_f)$, there exists a compact open $K' \subset G'(\mathbb{A}_f)$ such that the map

$$\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G', X')$$

is a closed immersion of schemes.

Definition 2.34. We say a Shimura datum (G, X) is of Hodge type of there is a closed embedding $(G, X) \hookrightarrow (\mathrm{GSp}_{2\mathrm{n}}, \mathcal{H}_n^\pm)$ for some n.

We will focus on Hodge type Shimura varieties in this class, since we are able to deduce results about them from results about Siegel modular varieties.

For groups of type A,B and C, it is always possible, up to central isogeny of G, to assume that the Shimura datum is of Hodge type.

3 Abelian schemes

We would like to explain how to get a canonical model and integral model of Shimura varieties. In order to understand what structures we need to pay attention to, we first analyze what happens for generalizations of the modular curves to the higher-dimensional *Siegel modular varieties*. These are moduli spaces of polarized abelian schemes with full level N structure. Our next goal is to prove that such a moduli functor is indeed representable over a base like $\mathbb{Z}[1/N]$. Our main references here are [Mum08, MFK94]. Much of the exposition here is also from [Zhu22]

From now on, all schemes are assumed to be locally noetherian.

Definition 3.1. Let S be a scheme. An *abelian scheme* A over S is a smooth proper group scheme over S all of whose geometric fibers are connected.

In particular, for any geometric point s of S, A_s is an abelian variety over $k(s)$.

3.1 Abelian varieties over $\mathbb{C}$

We first want to understand abelian varieties over $\mathbb{C}$. We first see that by Serre's GAGA, the analytification is a complex torus:

Definition 3.2. A complex torus is a connected compact Lie group over $\mathbb{C}$.

Example 3.3. If $V \simeq \mathbb{C}^g$ and $\Lambda \subset V$ is a lattice (a discrete, co-compact subgroup) then V/Λ is a complex torus.

Theorem 3.4. Every complex torus A is commutative and the holomorphic exponential map $\exp\colon T_0(A) \to A$ is a surjective homomorphism with kernel $\Lambda \subset T_0(A)$ a lattice. Hence $A \simeq T_0(A)/\Lambda$.

Proof. See [Mum08, pg. 1-2]. To see commutativity, the key is to study the adjoint representation of A acting on $T_0(A)$. This map $a \mapsto dc_a(e)$ (with $c_a(x) = axa^{-1}$) is a holomorphic map $A \to \mathrm{GL}(T_0(A))$ from a connected compact complex manifold into an open submanifold of a Euclidean space, so it must be constant (by the maximum principle in several complex variables), and by setting $x = a$ we see that this must be $\mathrm{id}_{T_0(A)}$.

Now, the surjectivity of exp follows because the image is a subgroup of A which contains an open neighbourhood of 0, and by connectedness must necessarily be A. Finally, let Λ be the kernel of exp. It must be discrete because the exp is a local isomorphism at 0. Finally, we see that the induced homomorphism $T_0(A)/\Lambda \to A$ is holomorphic by the definition of the structure of complex manifold on $T_0(A)/\Lambda \to A$, and is thus an isomorphism. Since lattices are the only discrete subgroups of vector spaces with compact quotient, Λ must be a lattice. $\square$

Remark 3.5. Note that analytification does *not* induce an equivalence of categories between abelian varieties of dimension g over $\mathbb{C}$ and complex tori of dimension g, for $g \geq 2$.

Definition 3.6. Let $X = V/L$ be a complex torus of dimension g, and let E be a skew-symmetric form $E\colon L \times L \to \mathbb{Z}$. Since $L \otimes_{\mathbb{Z}} \mathbb{R} = V$, we can extend E to a skew-symmetric $\mathbb{R}$-bilinear form $E_{\mathbb{R}}\colon V \times V \to \mathbb{R}$. We call E a *Riemann form* if

(a) $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$;

(b) the associated Hermitian form $H(x, y) = E(ix, y) + iE(x, y)$ is positive definite.

Note that there is a natural bijection between Hermitian forms H on V and skew-symmetric $\mathbb{R}$ bilinear forms E on V satisfying $E(ix, iy) = E(x, y)$. The correspondence is given by

$$H \mapsto E = \operatorname{Im}(H) \quad \text{and} \quad E \mapsto H(x, y) = E(ix, y) + i\,E(x, y).$$

Observe that H is positive definite if and only if $E_{\mathbb{R}}(iv, v) > 0$ for all $v \neq 0$.

Remark 3.7. If X has dimension 1, then $\bigwedge^2 L \simeq \mathbb{Z}$, and so there is a skew-symmetric form $E\colon L \times L \to \mathbb{Z}$ such that every other such form is an integral multiple of it. The form E is uniquely determined up to sign, and exactly one of $\pm E$ is a Riemann form.

We say that X is *polarizable* if it admits a Riemann form.

Remark 3.8. Note that the complex analytic structure on a complex torus X induces a Hodge structure of weight $(-1, 0), (0, -1)$, since we have $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{C}^g$, exactly as in the case of elliptic curves. Note that E is a Riemann form exactly when $-E$ defines a polarization on this Hodge structure.

Theorem 3.9. A complex torus X is of the form $A(\mathbb{C})$ if and only if it is polarizable.

Proof. (Rough Sketch) The main idea here is that X is the analytification of an abelian variety if and only if it is algebraizable, and to show that it is algebraizable we need an embedding into projective space. This embedding is induced by sections of an ample line bundle.

More precisely, for a Hermitian form H, and $\alpha : \Lambda \to \{z \in \mathbb{C}^\times : |z| = 1\}$ be a map with

$$\alpha(\lambda_1 + \lambda_2) = e^{i\pi E}(\lambda_1, \lambda_2)\alpha(\lambda_1)\alpha(\lambda_2)$$

we let $\mathcal{L}(H, \alpha)$ be the line bundle given by the quotient of $\mathbb{C} \times V$ by the action of Λ defined by for all $\lambda \in \Lambda$,

$$\lambda \cdot (z, w) = \big(\alpha(\lambda)\, e^{\pi H(w,\lambda) + \frac{1}{2}\pi H(\lambda,\lambda)} \cdot z,\ w + \lambda\big).$$

Then we have the following theorem of Appell and Humbert that classifies all line bundles on X:

Theorem 3.10 (Appell-Humbert)**.** Let $\mathcal{L}$ be a line bundle on the complex torus X. Then $\mathcal{L}$ is isomorphic to a $\mathcal{L}(H, \alpha)$ for a uniquely determined (H, α) as defined above.

For each such $\mathcal{L}$, we can look at sections of $\mathcal{L}$, which are certain θ-functions on $\mathbb{C}^g$: ie. they satisfy

$$\theta(z + \lambda) = \alpha(\lambda)e^{\pi H(z,\lambda) + \frac{1}{2}\pi H(\lambda,\lambda)}\theta(z).$$

Theorem 3.11 (Lefschetz)**.** Let X be a complex torus V/Λ with $\mathcal{L} = \mathcal{L}(H, \alpha)$ the associated line bundle as above. The Riemann form H is positive-definite if and only if $\mathcal{L}$ is ample, and in this case the space of holomorphic sections of $\mathcal{L}^{\otimes n}$ gives an embedding of X as a closed complex submanifold in a projective space for each $n \geq 3$; i.e., a holomorphic map $\Theta : X \to \mathbb{CP}^d$ that is injective and induces an injective map on tangent spaces.

Assuming the two results above, we see that if X is the analytification of an abelian variety over $\mathbb{C}$, then by properness we get an embedding $A \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$. Therefore analytification gives us $\iota : X \hookrightarrow \mathbb{CP}^n$. There is an ample line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$, which pulls back via ι to an ample line bundle $\iota^*\mathcal{O}(1)$ on X. Applying Appell–Humbert Theorem then provides the positive-definite Riemann form H that we seek.

Conversely, suppose we are given a positive-definite Riemann form H on X. We thus have an ample line bundle on X. Thus, we have an embedding $X \hookrightarrow \mathbb{CP}^n$. Chow's Theorem gives that $X = A^{\mathrm{an}}$ for a smooth A such that $A \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ a closed embedding. GAGA gives that the group laws on X are algebraizable and come from the group laws on A. Therefore, we obtain that A is an abelian variety. ☐

In the case of abelian schemes over a general base, we first will need a rigidity result.

Theorem 3.12 (Rigidity Lemma)**.** Let S be a scheme, and G be a group scheme over S and separated over S. Let $f : X \to S$ be a scheme morphism such that

1. f is flat.
2. either f is proper, or f is closed and admits a section.
3. For each $s \in S$, the $k(s)$-vector space $H^0(X_s, \mathcal{O}_{X_s})$ is 1-dimensional.

Then for any two S-morphisms $\phi, \phi' : X \to G$, if ϕ and ϕ' agree on one geometric fiber (or equivalently, on one fiber) for each connected component of S, then ϕ and ϕ' differ by multiplication by a section in $G(S)$.

Proof. This is [MFK94, Proposition 6.1]. ☐

Corollary 3.13. Let X and G over S be as in Theorem 3.12. Assume either that $X \to S$ is proper, or that it is universally closed and admits a section. Let Y be a connected scheme over S and assume that $Y \to S$ admits a section ϵ. Then for any S-scheme morphism $\varphi : X \times_S Y \to G$, there are S-scheme morphisms $g : X \to G$ and $h : Y \to G$ such that φ is given by $(x, y) \mapsto g(x) \cdot h(y)$.

Proof. Let $f : X \to S$ be the structure map. We consider the following commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[rr, shift left, "\Phi"] \arrow[rr, shift right, "\Phi'"'] \arrow[dr] & & G \times_S Y \arrow[dl] \\ & Y & \end{tikzcd}$$

where we define

$$\Phi(x,y) := \big(\varphi(x,y),\, y\big), \qquad \Phi'(x,y) := \big(\varphi\big(x, \epsilon(f(x))\big),\, y\big).$$

The Y-scheme $X \times_S Y$ and the Y-group scheme $G \times_S Y$ satisfy the hypotheses of Theorem 3.12. Now Φ, Φ' are Y-morphisms and for any $y_0 \in \mathrm{im}(\epsilon)$, the morphisms Φ and Φ' agree on the fiber of $X \times_S Y$ over y_0. Hence by Theorem 3.12 we know that Φ and Φ' differ by multiplication by a section of $G \times_S Y \to Y$, which is of the form $y \mapsto (h(y), y)$ for some S-map $h : Y \to G$. Then we have $\varphi(x,y) \;=\; \varphi\big(x, \epsilon(f(x))\big) \cdot h(y)$. Setting $g(x) := \varphi\big(x, \epsilon(f(x))\big)$ we can conclude the proof. $\square$

Corollary 3.14. Suppose X/S is an abelian scheme and G/S is a separated group scheme. Then any S-map $\varphi : X \to G$ preserving the identity section is a group homomorphism. In particular, the group structure on X is determined by the identity section.

Remark 3.15. The corollary shows that the group structure on an abelian variety is uniquely determined by the choice of a zero element (as in the case of an elliptic curve).

Proof. We may assume that S is connected. Then X is connected, since $X \to S$ is closed, surjective, and has connected fibers. Consider the composition $\Phi : X \times_S X \xrightarrow{\mu} X \xrightarrow{\varphi} G$ where μ is the multiplication map, i.e., $\Phi(x,y) = \varphi(x \cdot y)$. Then by Corollary 3.13, we have $\varphi(x \cdot y) \;=\; g(x) \cdot h(y)$ for some $g : X \to G$ and $h : X \to G$. Now observe that $e \;=\; \varphi(e \cdot e) \;=\; g(e)\, h(e)$. This implies that $h(e) = g(e)^{-1}$. Then we have

$$\varphi(x) \;=\; \varphi(x \cdot e) \;=\; g(x)\, h(e) \;=\; g(x)\, g(e)^{-1}.$$

Also,

$$\varphi(x) \;=\; \varphi(e \cdot x) \;=\; g(e)\, h(x).$$

So we have $g(x) \;=\; \varphi(x)\, g(e), \qquad h(x) \;=\; g(e)^{-1}\, \varphi(x)$. Hence we have

$$\varphi(x \cdot y) \;=\; g(x)\, h(y) \;=\; \varphi(x)\, g(e)\, g(e)^{-1}\, \varphi(y) \;=\; \varphi(x)\, \varphi(y).$$

$\square$

Corollary 3.16. Suppose X/S is an abelian scheme. Then the group structure is commutative.

Proof. Apply the previous result to the inversion $X \to X$, $x \mapsto x^{-1}$. $\square$

Corollary 3.17. Every map $A \to B$ of abelian varieties is the composite of a homomorphism with a translation.

Proof. The map will send the k-rational point 0 of A to a k-rational point b of B. After composing with translation by $-b$, we get a map which preserves the identity. Applying the corollary above, we can conclude that this map is a homomorphism. $\square$

3.2 Polarization

3.2.1 Picard schemes

We first recall the following from algebraic geometry. Given a scheme X, we can consider the contravariant functor

$$h_X : (\mathrm{Sch})^{\mathrm{op}} \longrightarrow (\mathrm{Set}),$$
$$Y \mapsto \mathrm{Mor}(Y, X).$$

A morphism $Y \to Z$ gives a map of sets $\mathrm{Mor}(Z, X) \to \mathrm{Mor}(Y, X)$, and hence h_X is really a functor. This is called the functor of points.

Lemma 3.18 (Yoneda's Lemma)**.** Let X, X' be two schemes, and h_X and $h_{X'}$ the associated functors. If h_X and $h_{X'}$ are naturally isomorphic, then X and X' are canonically isomorphic.

Definition 3.19. A functor h is representable (in the category of schemes) if there is a scheme X and a natural isomorphism of functors $h \to h_X$.

For a scheme X, we define $\mathrm{Pic}(X)$ to be the abelian group of isomorphism classes of invertible $\mathcal{O}_X$-modules. A morphism $f : X \to Y$ of schemes gives a group homomorphism

$$f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X), \qquad \mathcal{L} \longmapsto f^*\mathcal{L}.$$

Thus, we may consider the *absolute Picard functor*

$$\mathrm{Pic} : (\mathrm{Sch})^{\mathrm{op}} \longrightarrow (\mathrm{Ab}), \qquad X \mapsto \mathrm{Pic}(X).$$

Suppose $f : X \to S$ is a scheme morphism. It will be easier to work with the *relative Picard functor*, defined as

$$\mathrm{Pic}_{X/S} : (S\text{-schemes})^{\mathrm{op}} \longrightarrow (\mathrm{Ab}), \qquad T \longmapsto \mathrm{Pic}(X_T)/f_T^*\mathrm{Pic}(T),$$

where $X_T := X \times_S T$ and $f_T : X_T \to T$ is the base change of f.

Theorem 3.20 (Grothendieck)**.** Suppose $X \to S$ is a flat projective morphism with all geometric fibers integral (irreducible and reduced). Also assume that X/S has a section. Then $\mathrm{Pic}_{X/S}$ is representable by a commutative group scheme over S which is locally of finite type and separated over S.

Remark 3.21. If $e \in X(S)$ is a section, then $\mathrm{Pic}_{X/S} \cong \mathrm{Pic}_{X/S,e}$, where $\mathrm{Pic}_{X/S,e}$ is the *rigidified Picard functor* sending each S-scheme T to the group of isomorphism classes of pairs $(\mathcal{L}, \rho)$, where $\mathcal{L}$ is a line bundle on X_T and ρ is an isomorphism $e_T^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_T$ (called a *rigidification* of $\mathcal{L}$ along e_T). Here, e_T denotes the section of $X_T \to T$ induced by the section e of $X \to S$. More explicitly, we have inverse bijections $\mathrm{Pic}(X_T)/f_T^*\,\mathrm{Pic}(T) \longleftrightarrow \mathrm{Pic}_{X/S,e}(T)$

$$\mathcal{L} \longleftarrow (\mathcal{L}, \rho)$$
$$\mathcal{L} \longmapsto \big(\mathcal{L} \otimes f_T^*e_T^*\mathcal{L}^{-1},\ \text{canonical } \rho\big).$$

Here the canonical ρ is defined by noting that

$$e_T^*\big(\mathcal{L} \otimes f_T^*e_T^*\mathcal{L}^{-1}\big) \cong e_T^*\mathcal{L} \otimes e_T^*\mathcal{L}^{-1} \cong \mathcal{O}_T.$$

Definition 3.22. Recall that a morphism $f : X \to S$ is called *projective*, if the S-scheme X is isomorphic to a closed subscheme of the projective bundle $\mathbb{P}(\mathcal{E})$ over S attached to some coherent $\mathcal{O}_S$-module $\mathcal{E}$ on S.

Remark 3.23. Note that in general this is not the same as requiring that X is isomorphic to a closed subscheme of $\mathbb{P}^n_S$ for some n. However, when S admits an ample invertible sheaf (e.g. when S is affine), the two definitions are the same; see for instance [Sta25, Tag 0B45].

If $f : X \to S$ is projective, then it is proper, and there is an open covering (U_i) of S such that $X|_{U_i}$ is U_i-isomorphic to a closed subscheme of $\mathbb{P}^{n_i}_{U_i}$ for each i (see [Sta25, Tag 01WB]). The converse is not true. Thus a locally projective morphism (i.e., one that becomes projective after passing to an open covering of the target) need not be projective.

Theorem 3.24 (Grothendieck)**.** Let $f : X \to S$ be a flat projective morphism whose geometric fibers are integral. Assume that f admits a section, and that f is smooth. Assume that S is noetherian. Then there is a closed and open subgroup scheme $\mathrm{Pic}^\tau_{X/S}$ of $\mathrm{Pic}_{X/S}$ (over S), called the *torsion component*, satisfying the following conditions:

1. For each $s \in S$, the fiber of $\mathrm{Pic}^\tau_{X/S}$ over s consists of the torsion connected components of $(\mathrm{Pic}_{X/S})_s$. Here we say that a connected component is torsion if its image under the multiplication-by-n map $[n] : (\mathrm{Pic}_{X/S}) \to (\mathrm{Pic}_{X/S})$ lies in the identity connected component for some $n \geq 1$.

2. $\mathrm{Pic}^\tau_{X/S}$ is projective over S.

3.2.2 Dual abelian schemes

Now if X/S is an abelian scheme, then all the assumptions in Theorem 3.20 are satisfied. If we assume that X/S is projective and that S is noetherian, then the assumptions in Theorem 3.24 are satisfied as well. Furthermore we have the following result:

Theorem 3.25. Let X/S be a projective abelian scheme, and assume that S is noetherian. Then $\mathrm{Pic}^\tau_{X/S}$ is smooth and has connected geometric fibers. Hence in view of Theorem 3.24 we know that $\mathrm{Pic}^\tau_{X/S}$ is a projective abelian scheme.

Remark 3.26. To show for the above theorem that $\mathrm{Pic}^\tau_{X/S}$ has connected geometric fibers, we reduce to the case where S is the spectrum of an algebraically closed field (since the formation of $\mathrm{Pic}^\tau_{X/S}$ commutes with base change). Then this is a fundamental result in the theory of abelian varieties over a field; see [Mum08, §13]. The proof that $\mathrm{Pic}^\tau_{X/S}$ is smooth is found in [MFK94, Prop. 6.7].

Definition 3.27. In the setting of Theorem 3.25, we call $\mathrm{Pic}^\tau_{X/S}$ the *dual abelian scheme* of X, and denote it by $X^\vee$.

Definition 3.28. Let A, B be two abelian schemes over an arbitrary (locally noetherian) S. By an *isogeny*, we mean an S-group scheme homomorphism $A \to B$ that is surjective and quasi-finite.

Lemma 3.29. Any isogeny $\phi : A \to B$ is finite and flat.

Proof. Since both A and B are proper over S, we know that ϕ is proper. But a proper and quasi-finite map is finite ([Sta25, Tag 02LS]), so ϕ is finite.

Since both A and B are flat and finite-type over S, we can use the fiberwise criterion for flatness to reduce to the case where S is the spectrum of a field k. We may also assume that k is algebraically closed since flatness satisfies fpqc descent. Now ϕ is a surjective map between two finite-type schemes over a field, and the target is integral. Hence we can make use of *generic flatness* : There exists a non-empty open subscheme $U \subset B$ over which ϕ is flat.

Since ϕ is a group homomorphism, we can use the group structure on B to translate U, in order to obtain an open covering of B such that ϕ is flat over each member of the covering. (For this step we need to use that k is algebraic closed.) It follows that ϕ is flat, as desired. $\square$

3.2.3 The Mumford Λ-construction

Let S be a noetherian scheme, and $f : A \to S$ a projective abelian scheme over S. For any line bundle L on A, we define the Mumford line bundle $\mathcal{M}(L)$ on $A \times_S A$ by $\mathcal{M}(L) := \mu^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$, where p_1, p_2 are the two projections $A \times_S A \to A$, and μ is the group law $A \times_S A \to A$.

Recall that for any S-scheme T to give an S-map $T \to \mathrm{Pic}_{A/S}$ is the same as to specify an element of $\mathrm{Pic}(A_T)/f_T^*\mathrm{Pic}(T)$, where $A_T = A \times_S T$. Thus for $T = A$, the Mumford line bundle $\mathcal{M}(L)$ on $A \times_S A = A_A$ gives rise to an S-map $\Lambda(L) : A \longrightarrow \mathrm{Pic}_{A/S}$.

Lemma 3.30. The S-map $\Lambda(L)$ takes the identity section of A to the identity section of $\mathrm{Pic}_{A/S}$.

Proof. Let $e \in A(S)$ be the identity section. To compute $\Lambda(L) \circ e$, we need to compute the pullback of $\mathcal{M}(L)$ under

$$A = A \times_S S \xrightarrow{(\mathrm{id},e)} A \times_S A, \qquad x \longmapsto (x, e(f(x))).$$

Note that the compositions of the above map followed by $\mu, p_1, p_2 : A \times_S A \to A$ respectively are $\mathrm{id}, \mathrm{id}, e \circ f$. Hence the pullback of $\mathcal{M}(L)$ under the above map is isomorphic to $f^*e^*L^{-1}$. This line bundle on A represents the zero element of $\mathrm{Pic}_{A/S}(S) = \mathrm{Pic}(A)/f^*\mathrm{Pic}(S)$. Hence $\Lambda(L) \circ e$ is the identity section of $\mathrm{Pic}_{A/S}$. $\square$

As a consequence, we know that $\Lambda(L) : A \to \mathrm{Pic}_{A/S}$ is a group homomorphism by the Rigidity Lemma (see Corollary 3.14). Moreover, by the fiberwise connectedness of A, we know that the image of $\Lambda(L)$ lies in $A^\vee$, so we get a homomorphism $A \to A^\vee$. Observe also that for two line bundles L, M on A we have

$$\Lambda(L \otimes M^{\pm 1}) = \Lambda(L) \pm \Lambda(M).$$

We now assume that A is an abelian variety over an algebraically closed field k. In this case A is automatically projective. Note that $\mathrm{Pic}_{A/k}(k) = \mathrm{Pic}(A)$, since $\mathrm{Pic}(k)$ is trivial.

Let L be a line bundle on A. Similar to the proof above, we see that at the level of k-points the map $\Lambda(L)$ is given by

$$A(k) \longrightarrow A^\vee \subset \mathrm{Pic}_{A/k}(k) = \mathrm{Pic}(A), \qquad x \longmapsto t_x^* L \otimes L^{-1},$$

where $t_x : A \to A$ is translation by x. The fact that $\Lambda(L)$ is a group homomorphism thus implies the following:

Theorem 3.31 (Theorem of the Square)**.** For any line bundle L on A and any $x, y \in A(k)$, we have an isomorphism of line bundles

$$t_{x+y}^* L \otimes L \;\cong\; t_x^* L \otimes t_y^* L.$$

We make two further observations.

Lemma 3.32. Let L be a line bundle on A. Then $\Lambda(L) = 0$ if and only if the Mumford line bundle $\mathcal{M}(L)$ is trivial.

Proof. The "if" direction is clear from definition. Suppose now that $\Lambda(L) = 0$. Then we see from definition of $\Lambda(L)$ that since the base change of f along $A \to \operatorname{Spec} k$ is p_2, this implies $\mathcal{M}(L)$ lies in $p_2^*\mathrm{Pic}(A)$. Thus suppose M is a line bundle on $A \times_k T$ which isomorphic to $p_2^*(N)$ for some line bundle N on T. Write e_T for the map $T \to A \times_k T, t \mapsto (e, t)$. Then $p_2^* e_T^* M \cong p_2^* e_T^* p_2^* N \cong p_2^* N \cong M$, where the second isomorphism is because $p_2 \circ e_T = \mathrm{id}_T$. In particular, $M \otimes p_2^* e_T^* M^{-1} \cong \mathcal{O}_{A \times T}$. Applying this to $M = \mathcal{M}(L)$, we know that $\mathcal{M}(L) \otimes p_2^*(e, \mathrm{id})^* \mathcal{M}(L)^{-1} \cong \mathcal{O}_{A \times A}$. One can check that the left hand side is isomorphic to $\mathcal{M}(L) \otimes e_2^* L$, where e_2 is the map $A \times_k A \to A, (x, y) \mapsto e$. But $e_2^* L$ is trivial since e_2 factors through $\operatorname{Spec} k$ and $\mathrm{Pic}(k) = 0$. Hence $\mathcal{M}(L)$ is trivial. $\square$

Since A/k is projective, it has ample line bundles.

Lemma 3.33. Let L be an ample line bundle on A. Then $\ker(\Lambda(L))$ is a finite subgroup scheme of A.

Proof. Suppose not. Then one can find a positive-dimensional abelian subvariety $B \subset A$ contained in $\ker(\Lambda(L))$. Note that $\Lambda(L)|_B = \Lambda(L|_B)$, and $L|_B$ is ample on B. Thus by considering $B = A$, we are in the case where $\Lambda(L) = 0$. By the previous lemma, $\mathcal{M}(L)$ is trivial. The pullback of $\mathcal{M}(L)^{-1}$ along the "anti-diagonal"

$$(\mathrm{id}, [-1]) : A \longrightarrow A \times_k A, \qquad x \longmapsto (x, -x)$$

is $L \otimes [-1]^* L$, and it must be trivial on A. Since L is ample and $[-1]$ is an automorphism of A, $L \otimes [-1]^* L$ is ample. Thus the trivial line bundle on A is ample, a contradiction with the fact that A is projective and positive-dimensional. $\square$

The following is the "main theorem" for line bundles on an abelian variety.

Theorem 3.34. Fix an ample line bundle L on A. For any line bundle M on A, we have $\Lambda(M) = 0$ if and only if $M \cong \Lambda(L)(x) = t_x^* L \otimes L^{-1}$ for some $x \in A(k)$.

Definition 3.35. Let M, M' be two line bundles on A. We say that M is algebraically equivalent to 0, if there exists a connected k-scheme T and a line bundle on $A \times_k T$ specializing to M and $\mathcal{O}_A$ at two k-points of T. We way M, M' are algebraically equivalent if $M^{-1} \otimes M'$ is algebraically equivalent to 0.

Lemma 3.36. Two line bundles M and M' are algebraically equivalent if and only if the isomorphism class of $M \otimes M'^{-1}$ lies in $A^\vee(k) \subset \mathrm{Pic}_{A/k}(k) = \mathrm{Pic}(A)$.

Corollary 3.37. Let M be a line bundle on A. Then M is algebraically equivalent to zero if and only if $\Lambda(M) = 0$.

Proof. Suppose M is algebraically equivalent to zero. Then there is a connected k-scheme T and a line bundle $\widetilde{M}$ on $A \times_k T$ which specializes to M and to $\mathcal{O}_A$ at two points $t_1, t_2 \in T(k)$. Consider the T-group scheme homomorphism $\Lambda(\widetilde{M}) : A_T = A \times_k T \longrightarrow \mathrm{Pic}_{A_T/T}$. On the fiber of A_T over t_2, the map induced by $\Lambda(\widetilde{M})$ is $\Lambda(\mathcal{O}_A) = 0$. Thus $\Lambda(\widetilde{M})$ and the zero homomorphism agree on one fiber over T. Since they both preserve the identity section, they must be equal by the Rigidity Lemma . Thus $\Lambda(\widetilde{M}) = 0$. But on the fiber over t_1, the map induced by $\Lambda(\widetilde{M})$ is $\Lambda(M)$. Hence $\Lambda(M) = 0$.

Conversely, suppose that $\Lambda(M) = 0$. Then by Theorem 3.34, there exists an (ample) line bundle L on A and a point $x \in A(k)$ such that $M = \Lambda(L)(x)$. But $\Lambda(L)(A(k)) \subset A^\vee(k)$, so $M \in A^\vee(k)$. Thus M is algebraically equivalent to zero by Lemma 3.36. □

Corollary 3.38. Let L be an ample line bundle on A. Then $\Lambda(L) : A \to A^\vee$ is an isogeny.

Proof. By Lemma 3.33, $\Lambda(L)$ is quasi-finite. To see that it is surjective, let $M \in A^\vee(k)$. Then $\Lambda(M) = 0$ by Corollary 3.37. Hence $M \in \mathrm{im}(\Lambda(L))$ by Theorem 3.34. □

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