Name: Siyuan Chai 3069923
Class: EECS 212
Instructor: Dr. Huck Bennett
Date: 2018-10-10

Homework 1

# Problem 1

**prove by induction:**

- base case

  when $n = 2$, it's obvious that we need at most one flip to make the stack in ascending order; thus, there's at most $2n - 3 = 1$ flip needed

- inductive case

  Assuming wee need at most $2k - 3$ flips when $k \geq 2$, we can move on to prove we need at most $2(k + 1) - 3$ flips to sort the stack when $n = k + 1$. Based on the Bring-to-Top algorithm, there are three cases to be discussed about extra operations:

  - the pancake with $k + 1$ value is initially at bottom
    No more operations needed to place the extra pancake.
  - the pancake with $k + 1$ value is initially at top
    Only one operation which flip the entire stack is needed.
  - the pancake with $k + 1$ value is in the mid of the stack
    We need to first flip from the position of the extra pancake and then flip the entire stack to place it at bottom.

  All in all, we need at most two more flips to place one pancake into its correct bottom position. Thus, we need at most $2k-3+2 = 2(k+1)-3$ flips to sort the entire stacks of $k + 1$ pancakes.

- conclusion

  Based on the cases above, the Bring-to-Top algorithm requires at most $2n - 3$ flips to sort stack of $n$ pancakes.

## Problem 2

**prove by induction:**

- base case

  When `size(A) == 0`, the sorting is trivial. When `size(A) == 1`, the return result contains just pivot `p`; thus, it's in ascending order itself. The base case is proved.

- strong inductive case

  Assume the algorithm works for all non-negative integers from 0 to $k$. That means `Quicksort(A)` returns a valid sorted array if `A.length` $\leq$ `k`. Based on the `Quicksort` algorithm, if the input is a array with length of $k+1$, $L_1$`.length` $\leq$ `k` and $L_2$`.length` $\leq$ `k` because there is one pivot $q$ extracted. According to our inductive assumption, $Quicksort(L_1)$ and $Quicksort(L_2)$ are sorted. Also, $\forall\, l_1 \in Quicksort(L_1)$ and $l_2 \in Quicksort(L_2)$, $l_1 < q < l_2$. While both $Quicksort(L_1)$ and $Quicksort(L_2)$ are sorted, the concat of $Quicksort(L_1)$, q and $Quicksort(L_2)$ are also sorted.

- conclusion

  Based on the base case and strong inductive case, the `Quicksort` algorithm works for arrays of given length.

## Problem 3

**Prove by induction**

- base case

  - $F_0 = 0$
  - $F_1 = 1$
  - $F_2 = 1$
  - $F_3 = 2$

  When $n = 2$, $F_1^2 + F_2^2 = 2 = F_2 F_3$. Thus, the theorem holds true in base case.

- strong inductive case

  Assume that it is true that

  $$F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1} \tag{1}$$

  where $k > 2$.
  We try to prove

  $$F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2 = F_{k+1} F_{k+2} \tag{2}$$

  We can add $F_{k+1}^2$ on both sides of equation (1), which turns out to be

  $$F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2 = F_k F_{k+1} + F_{k+1}^2 \tag{3}$$

  If we combine the terms on the right, it turns out be exactly equation (2):

  $$\begin{aligned} F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2 &= F_{k+1}(F_k + F_{k+1}) \\ F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2 &= F_{k+1} F_{k+2} \end{aligned}$$

  which finish the proof.

- conclusion

  Based the base case and inductive case, we prove that

  $$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

  where n is an integer larger than 2.

# Problem 4

### Prove by induction

- base case

We can try to find the value of sum and solve for $a\,b\,c\,d$.

$$\begin{aligned}
n = 0 \quad & \sum_{k=0}^{n}(3k^2 + 2k - 4) = & -4 \\
n = 1 \quad & \sum_{k=0}^{n}(3k^2 + 2k - 4) = & -3 \\
n = 2 \quad & \sum_{k=0}^{n}(3k^2 + 2k - 4) = & 9 \\
n = 3 \quad & \sum_{k=0}^{n}(3k^2 + 2k - 4) = & 38
\end{aligned}$$

Thus, an equation set can be established:

$$\begin{aligned}
d &= -4 \\
a + b + c + d &= -3 \\
8a + 4b + 2c + d &= 9 \\
27a + 9b + 3c + d &= 38
\end{aligned}$$

Solving the equation, we can get

$$\begin{aligned}
a &= 1 \\
b &= 5/2 \\
c &= -5/2 \\
d &= -4
\end{aligned}$$

These are the four numbers we find that works for the first four base cases in the scenario.

- inductive case

  Assume it is true that

$$\sum_{k=0}^{n}(3k^2 + 2k - 4) = n^3 + \frac{5}{2}n^2 - \frac{5}{2}n - 4 \tag{4}$$

  where n is an arbitrary natural number.
  We want to prove that

$$\sum_{k=0}^{n+1}(3k^2 + 2k - 4) = (n+1)^3 + \frac{5}{2}(n+1)^2 - \frac{5}{2}(n+1) - 4 \tag{5}$$

is also true.

We can start playing with the left hand side of equation(5):

$$\begin{aligned}
&\sum_{k=0}^{n+1}(3k^2+2k-4)\\
&=\sum_{k=0}^{n}(3k^2+2k-4)+3(n+1)^2+2(n+1)-4
\end{aligned}$$

Applying equation (4) here and do bit more math:

$$\begin{aligned}
&=n^3+\frac{5}{2}n^2-\frac{5}{2}n-4+3(n+1)^2+2(n+1)-4\\
&=n^3+\frac{5}{2}n^2-\frac{5}{2}n-4+3(n+1)^2-\frac{5}{2}(n+1)-4+\frac{9}{2}(n+1)\\
&=n^3+\frac{5}{2}n^2+2n+\frac{1}{2}+3(n+1)^2-\frac{5}{2}(n+1)-4\\
&=n^3+\frac{5}{2}n^2+2n+\frac{1}{2}+\frac{5}{2}(n+1)^2-\frac{5}{2}(n+1)-4+\frac{1}{2}(n+1)^2\\
&=n^3+3n^2+3n+1+\frac{5}{2}(n+1)^2-\frac{5}{2}(n+1)-4\\
&=(n+1)^3+\frac{5}{2}(n+1)^2-\frac{5}{2}(n+1)-4
\end{aligned}$$

Calculation turns out to be exactly the right hand side of equation (5)

- conclusion

  Based on both the base and inductive cases, it could be concluded that

$$\exists a,b,c,d\in\mathbb{R}\quad st.\sum_{k=0}^{n}(3k^2+2k-4)=an^3+bn^2+cn+d$$

  where $n\in\mathbb{N}$ and

$$\begin{aligned}
a&=1\\
b&=5/2\\
c&=-5/2\\
d&=-4
\end{aligned}$$

# Problem 5

### Prove by induction

- base case

  Base case is obvious when $n = 1$,

  $$(1 + 2x) > 2x$$

  regardless of the value of $x$.

- inductive case

  Assume it is true that

  $$(1 + 2x)^n > 2nx \tag{6}$$

  where $n \in \mathbb{Z}^+$ and $x \geq -\frac{1}{2}$.
  We wish to prove that

  $$(1 + 2x)^{n+1} > 2(n + 1)x \tag{7}$$

  where $x \geq -\frac{1}{2}$.
  We can start playing with the left hand side of equantion (7):

  $$\begin{aligned}(1 + 2x)^{n+1} =& (1 + 2x)^n(1 + 2x)\\ =& (1 + 2x)^n + 2x(1 + 2x)^n\\ (1 + 2x)^{n+1} - 2(n + 1)x =& ((1 + 2x)^n - 2nx) \quad \texttt{(term 1)}\\ &+ (2x(1 + 2x)^n - 2x) \quad \texttt{(term 2)}\end{aligned}$$

  Applying equation (6), we know `(term 1)` $> 0$.
  For `(term 2)`

  - when $x > 0$

    $$\begin{aligned}(1 + 2x)^n &> 1\\ (1 + 2x)^n - 1 &> 0\\ 2x((1 + 2x)^n - 1) &> 0\\ (2x(1 + 2x)^n - 2x) &> 0\end{aligned}$$

    where $n \in \mathbb{Z}^+$

  - when $-\frac{1}{2} \leq x \leq 0$,

    $$\begin{aligned}(1 + 2x)^n - 1 &\leq 0\\ 2x((1 + 2x)^n - 1) &\geq 0 \quad (x \leq 0)\\ (2x(1 + 2x)^n - 2x) &\geq 0\end{aligned}$$

    where $n \in \mathbb{Z}^+$

$$\begin{aligned} \text{because} \quad & \texttt{(term 1)} > 0 \quad \text{and} \quad \texttt{(term 2)} \geq 0 \\ \text{therefore} \quad & (1+2x)^{n+1} - 2(n+1)x > \texttt{(term 1)} + \texttt{(term 2)} > 0 \\ & (1+2x)^{n+1} > 2(n+1)x \end{aligned}$$

The last relationship we derived is exactly what in equation (7).

- conclusion

  Based on the base case and conclusion case above, it's sufficient to conclude that
  $$(1+2x)^n > 2nx$$
  where $x \geq -\frac{1}{2}$ and $x \in \mathbb{Z}^+$.

# Problem 6

**Prove by induction** We try to prove that it is not possible to get into the situation in which only one girl sitting on the left and another boy sitting on right. We can prove by showing the correctness of a stronger predicate that no odd number of boys are on the right of the rightmost girl, regardless of how many operations we go.

Define the predicate $P(k)$ : after k rounds of arriving and leaving, the number of boys on the right of the rightmost girl on bench is not odd.

- base case

  When $k = 0$, there is zero boys on the right of the rightmost girl;thus, $P(0)$ is obviously true here.

- inductive case

  Assume after k rounds of arriving and leaving, $P(k)$ is still true. We can move on to prove the correctness of $P(k+1)$.
  We know that there are even numbers of boys on the right of the rightmost girl after $k$ rounds. For the $k+1$ round, if a pair of boys sitting next to each on the right edge leave, there are still even numbers of boys on the right of the rightmost girl. If some boys attempt to join the bench, because they must sit next to each other, the number of boys on the right edge can only add 2, which is still even.
  All in all, as long as $P(k)$ is true, there will still be even number of boys on the right of the rightmost girl after $k+1$ rounds.

- conclusion

  Based on the base and inductive cases above, it's invariant that even numbers of boys will appear on the right of the rightmost girls. The correctness of the predicate eliminates the possibility in which only one girl on the left and one boy on the right.