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1) Prove $f_1(n) = \log(n^3)$ $f_2(n) = (\log n)^3$ $f_1(n) = o(f_2(n))$

$$\lim_{n\to\infty} \frac{\log(n^3)}{(\log n)^3} = \lim_{n\to\infty} \frac{3\log n}{(\log n)^3} = \lim_{n\to\infty} \frac{3\ln n/\ln 2}{(\ln n)^3/(\ln 2)^3}$$

L'Hôpital

$$= \lim_{n\to\infty} \frac{3/n}{3(\ln n)^2/(\ln 2)^2 \cdot \frac{1}{n}}$$

$$= 0$$

$\therefore \log(n^3) = o((\log n))^3$

2) $f_2(n) = (\log n)^3$, $f_3(n) = 2^{\sqrt{\log n}}$

Prove $f_2(n) = o(f_3(n))$

let $x = \log n$

L'Hôpital

$$\lim_{n\to\infty} \frac{(\log n)^3}{2^{\sqrt{\log n}}} = \lim_{x\to\infty} \frac{x^3}{2^{\sqrt{x}}} = \lim_{x\to\infty} \frac{3x^2}{2^{\sqrt{x}} \cdot \ln 2 \cdot x^{-\frac{1}{2}} \cdot \frac{1}{2}}$$

$$= \lim_{x\to\infty} \frac{3 \cdot \frac{5}{2} x^{\frac{3}{2}}}{2^{\sqrt{x}} (\ln 2/2)^2 \cdot x^{-\frac{1}{2}}}$$

After 4 more times of apply L'Hôpital's rule,

$$= \lim_{x\to\infty} \frac{3 \cdot \frac{5}{2} \cdot 2 \cdot \frac{3}{2} \cdots \frac{1}{2}}{2^{\sqrt{x}} (\ln 2/2)^6} = 0$$

$\therefore (\log n)^3 = o(2^{\sqrt{\log n}})$

3) $f_3(n) = 2^{\sqrt{\log n}}$, $f_4(n) = 2^{\ln n}$

Prove $f_3(n) = o(f_4(n))$

$\log n = \ln n / \ln 2$

Let $x = \ln n$

$$\lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{2^{\ln n}} = \lim_{x\to\infty} \frac{2^{\sqrt{\ln 2}\cdot\sqrt{x}}}{2^{x}} = \lim_{x\to\infty} 2^{\sqrt{\ln 2 \cdot x} - x} = 0$$

~~Lim~~ Thus, $f_3(n) = o(f_4(n))$

4) $f_4(n) = 2^{\ln n}$ $f_5(n) = \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$

Prove $f_4(n) = o(f_5(n))$

$$2^{\ln n} = 2^{\log_2 n/\log_2 e} = n^{\frac{1}{\log_2 e}}$$

$\because \frac{1}{\log_2 e} < 1$ $\therefore \forall n > 1$, $f_4(n) < n \Rightarrow f_4(n) = o(n)$

$$\lim_{n\to\infty} \frac{n}{f_5(n)} = \lim_{n\to\infty} \frac{n}{\frac{n(n-1)(n-2)}{6}} \xrightarrow{\text{L'Hôpital}} \lim_{n\to\infty} \frac{1}{\frac{3n^2-6n-2}{6}} = 0.$$

$\therefore n = o(f_5(n))$

Thus, $f_4(n) = o(f_5(n))$

5). $f_5(n) = \binom{n}{3}$ $\quad f_6(n) = 5n^3$

Prove $f_5(n) = \Theta(f_6(n))$

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)/3}{5n^3}$$

Apply L'Hôpital's rule three times

$$= \lim_{n\to\infty} \frac{1}{5\cdot 3\cdot 2} = \frac{1}{30}$$

Thus, $f_5(n) = \Theta(f_6(n))$

6) $f_6(n) = 5n^3$ $\quad f_7(n) = (\log n)^n$

$\forall n > 3$, $\because \log n > \log 3 > 1$

$\therefore (\log n)^n > (\log 3)^n$

Thus $f_7(n) = \omega((\log 3)^n)$

$$\lim_{n\to\infty} \frac{(\log 3)^n}{5n^3} \xrightarrow{\text{L'Hôpital's Rule for three times}} \frac{(\log_3)^n \cdot [\ln(\log 3)]^3}{5\cdot 3\cdot 2} = 0$$

$\therefore 5n^3 = o((\log 3)^n)$ while $(\log 3)^n = o((\log n)^n)$

Using the transitivity, we can know

$f_6(n) = o(f_7(n))$

7) $f_7(n) = (\log n)^n \qquad f_8(n) = n!$

Prove $f_7(n) = o(f_8(n))$

$$n! = n(n-1)(n-2)\cdots(n/2)\cdots 1$$
$$\geq n(n-1)(n-2)\cdots(n/2) \geq (n/2)^{n/2} \quad \therefore (n/2)^{n/2} = O(n!)$$

$$\lim_{n\to\infty} \frac{(\log n)^n}{(n/2)^{n/2}} = \lim_{n\to\infty} \frac{[(\log n)^{2}]^{n/2}}{(n/2)^{n/2}} = \lim_{n\to\infty} \left(\frac{\log^2 n}{n/2}\right)^{n/2}$$

We can find $\lim_{n\to\infty} \frac{\log^2 n}{n/2} = \lim_{n\to\infty} \frac{2\log n}{n/2} = \lim_{n\to\infty} \frac{2}{n/2} = 0$

Thus, $\lim_{n\to\infty} \left(\frac{\log^2 n}{n/2}\right)^{n/2} = 0$.

$$\therefore (\log n)^n = o\left((n/2)^{n/2}\right) \qquad (n/2)^{n/2} = O(n!)$$

$$\therefore (\log n)^n = o(n!)$$

In Conclusion:

$f_1(n) = \log(n^3)$

$f_2(n) = (\log n)^3$

$f_3(n) = 2^{\sqrt{\log n}}$

$f_4(n) = 2^{\ln n}$

$f_5(n) = \binom{n}{3}$

$f_6(n) = 5n^3$

$f_7(n) = (\log n)^n$

$f_8(n) = n!$

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Problem 2. $T(n) = T(n/2) + T(n/4) + cn$

Part 1 show $T(n) = \Omega(n)$

Base case: ~~Assume~~ $T(n) \geq \frac{1}{2}n$

when $n=1$ $T(1) = 1 \geq \frac{1}{2}$

when $n=2$ $T(2) = 2 > 1$

when $n=4$ $T(4) = T(2) + T(1) + c = 3 + c > 2$

Inductive case: Assume $\forall k \in [1, n]$ k is power of 2 $T(k) \geq \frac{1}{2}k$

$T(2n) = T(n) + T(n/2) + 2cn$

$\geq \frac{n}{2} + \frac{n}{4} + 2cn \geq n$ $(c \geq 1)$

Thus, $\exists c_0 = \frac{1}{2}$, st. $\forall n > 0$, n a power of 2, $T(n) \geq c_0 n \Rightarrow T(n) = \Omega(k)$

Part 2 show $T(n) = O(n)$

~~Assume~~ Show $T(k) \leq 10ck$ where k is power of 2. where c is the given constant $c \geq 1$

Base Case. when $n=1$ $T(1) = 1 \leq 10c$

$n=2$ $T(2) = 2 \leq 20c$

$n=4$ $T(4) = T(2) + T(1) + Cn = 3 + C \leq 40C$

Inductive case: Assume $\forall k \in [1, n]$ $T(k) \leq 10Ck$ where $k \in [1, n]$ power of 2.

$T(2n) = T(n) + T(n/2) + 2cn$

$= 10cn + 5cn + 2cn \leq 20cn$

Thus. $\exists C_0 = 8c$ st. $\forall n > 0$, n a power of 2, $T(n) \leq C_0 n \Rightarrow$ $T(n) = O(n)$

Based on Part 1 and Part 2, $T(n) = \Theta(n)$.

Problem 3

a. $4^n - 3^n$

If we can choose four letters per spot, there are 4^n choices. If there's no 'C' as option, ∴ there are only three options per spot, ∴ 3^n.

b. $3^{n-30} \cdot \binom{n}{30}$

Imagine there's n spots, we have $\binom{n}{30}$ ways to select 30 spots. For the remaining n-30 spots, each have 3 choices; Thus, $3^{n-30} \cdot \binom{n}{30}$ ways in total.

c) $H(1) = 4$, $H(2) = 16$.

$H(n) = 3H(n-1) + 3H(n-2)$

Consider a sequence of n legth.

If the last term is not C, there are $T(n-1)$ options for the first n-1 terms, and 3 options for the last one.

If the last term is C, the second last term can not be C, there are three options. For the first n-2 terms, $T(n-2)$ options are there.

Two cases are mutually exclusive, we can just add the product of them altogether.

Problem 4

a) $\frac{(2n)!}{2n} = (2n-1)!$

If it's to arrange people on bench, there are $(2n)!$ ways to arrange. Yet, there's actually 2n times fewer the arranging it on bench because the table can be rotated.

b) $2^n \frac{n!}{n} = 2^n (n-1)!$

If we n couples as n units, following the discussion of a), there are $\frac{n!}{n} = (n-1)!$ ways to do that. Yet, for each couple, they can switch with themselves which give rise to 2^n more times arrangement. Thus, in total $2^n(n-1)!$

Problem 5. Consider n side polygon, there will be a intersection, for each distinct group of 4 points. Because we don't consider the case in which more than 2 chords intersect at one point, there are $\binom{n}{4}$ ways to choose groups of 4 points, thus $\binom{n}{4}$ intersections.

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