10.9

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Last time, $\varphi: \overset{\text{connected}}{G} \to H$ homo. with discrete kernel. Prove the kernel lies in the center of G. That implies the fundamental group of Lie group is abelian.

Pf. $\sigma \in \ker \varphi$. $\tau \in G$. WTS: $\tau^{-1}\sigma\tau = \sigma$.

$\exists$ path $\gamma: e \to \tau$.

$\gamma(t)\sigma\gamma^{-1}(t) \in \ker\varphi$.

discrete $\Rightarrow$ $\gamma(0)\sigma\gamma^{-1}(0) = \gamma(1)\sigma\gamma^{-1}(1)$.

$\tilde{G} \to G$ universal covering.

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{f} & \tilde{G} \\ \pi\downarrow & \circlearrowleft & \swarrow \pi \\ G & & \end{array}$$

deck transformation. $\cong \pi_1(G)$.

$\ker\pi$ discrete $\Rightarrow$ $\ker\pi$ in center of $\tilde{G}$.

Hence abelian. $\square$

Lecture in 10.9.

① Left-invariant vector field.

Def: $L_g: G \to G$, $h \mapsto gh$. A vector field is called left-invariant if $X_{gh} = dL_g(X_h)$ $\forall h \in G$.

Prop: 1. Left-invariant field are invariant under $[\cdot,\cdot]$

2. $\mathcal{I}$'s dim. $= \dim G$.

3. $\mathcal{I}$ is smooth

Remark: The left invariant vector field is 1-1 corresponding to $\mathrm{Lie}(G) \triangleq \{M \in gl_n(\mathbb{R}) \mid \exp(tM) \in G.\ \forall t \in \mathbb{R}\}$

② Differential form.

Def: germ. Co-tangent space T_p^*M $\{dx_i\}_{i=1}^n$

$V_{n,m} = \underbrace{V\otimes\cdots\otimes V}_{n} \otimes \underbrace{V^*\otimes\cdots\otimes V^*}_{m}$

$\Lambda_k(V) = V_{k,0}/\mathcal{I}_k$, where $\mathcal{I}_k = \mathrm{span}\{v\otimes v\} \cap V_{k,0}$.

$= \{\text{anti-symmetric } k\text{-tensor on } V\}^*$ ← dual

$\Lambda_k(T_pM) =: \Omega_p^k(M)$. $\Omega^k(M) = \coprod_{p\in M} \Omega_p^k(M)$. vector bundle.

Section $E^k(M) = \Gamma(\Omega^k(M))$. $dx^1\wedge\cdots\wedge dx^k$

Prop: $\exists!$ anti-derivation $d: E^k \to E^{k+1}$ s.t. $d^2=0$ and df is the usual derivation.

$d(v\wedge w) = dv\wedge w + (-1)^p v\wedge dw$. p is order of v.

Def: Left-invariant differential form $\delta L_g w = w$

↑ pull back

$E^*_{l\,inv}(G) = \sum_p \Sigma^p_{l\,inv}(G)$

③. Compute some Lie algebra of Lie group

$O(n) \longrightarrow o(n) = \{\text{anti-sym matrix}\}$