SET THEORY

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COROLLAY 5.18

Theorem 5.24. *For every $\kappa \geq \omega$, κ^+ is regular. That is, $\aleph_{\alpha+1}$ is regular for all α.*

Remark. To help provide some intuition for the relationship of this theorem to Lemma 5.20, we can show the following special case, namely, that $\omega^+ = \aleph_1$ is regular.

Suppose otherwise, namely, that $\mathrm{cf}(\aleph_1) = \omega$ (note, by Lemma 5.23, that this is the only choice for $\mathrm{cf}(\aleph_1)$ if $\aleph_1$ is not regular). That is, for some $f : \omega \to \aleph_1$, $\mathrm{rng}(f)$ is cofinal in $\aleph_1$, *i.e.*, $\bigcup \mathrm{rng}(f) = \aleph_1$. Now, we note the following facts:

- $|\mathrm{rng}(f)| = \omega$. This is clear since $\mathrm{dom}(f) = \omega$.
- For every $\alpha \in \mathrm{rng}(f)$, $|\alpha| \leq \omega$. This follows since $\alpha \in \aleph_1$, so the biggest its cardinality could possibly be is $\aleph_0 = \omega$.

Hence $\bigcup \mathrm{rng}(f)$ is a countable union of countable sets—but we know this is countable, so it cannot be equal to $\aleph_1$.

Proof. We now give a general proof of Theorem 5.24; it follows much the same shape as the preceding remark.

For purposes of contradiction, suppose that $\aleph_{\alpha+1}$ is not regular, that is, there is some cofinal map $f : \aleph_\beta \to \aleph_{\alpha+1}$ where $\beta \leq \alpha$. Then $\bigcup \mathrm{rng}(f) = \aleph_{\alpha+1}$. If $\gamma \in \mathrm{rng}(f)$, then $|\gamma| < \aleph_{\alpha+1}$. Therefore, $|\mathrm{rng}(f)| = \aleph_\beta$ and $\sup(\mathrm{rng}(f)) = \aleph_\alpha$, so by Lemma 5.20, the cardinality of $\bigcup \mathrm{rng}(f)$ is $\alpha \times \beta = \max(\alpha, \beta) < \alpha + 1$, contradicting the cofinality of f. ☒

Lemma 4.5. *Let κ be a singular cardinal and suppose the continuum function below κ is eventually constant, that is, there is a $\gamma < \kappa$ such that $2^{<\kappa} = 2^\gamma$. Then $2^\kappa = 2^\gamma$.*

Proof. An argument similar to lemma 4.4 shows that for any κ we have $2^\kappa \leq (2^{<\kappa})^{\mathrm{cof}(\kappa)}$, and hence easily $2^\kappa = (2^{<\kappa})^{\mathrm{cof}(\kappa)}$. Thus, $2^\kappa = (2^{<\kappa})^{\mathrm{cof}(\kappa)} = (2^\gamma)^{\mathrm{cof}(\kappa)} = (2^\delta)^{\mathrm{cof}(\kappa)} = 2^\delta = 2^\gamma$, where $\gamma < \delta < \kappa$ and $\delta \geq \mathrm{cof}(\kappa)$. □

Theorem 4.6. *For any cardinal κ:*

1. *If κ is a successor cardinal, then $2^\kappa = \gimel(\kappa)$.*
2. *If κ is a limit cardinal and the continuum below κ is eventually constant, then $2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa)$.*
3. *If κ is a limit cardinal and the continuum below κ is not eventually constant, then $2^\kappa = \gimel(2^{<\kappa})$.*

Proof. The first part is clear as successor cardinals are regular. Suppose that κ is a limit cardinal, and the continuum function is eventually constant below κ, say $2^{<\kappa} = 2^\gamma$. If κ is singular, then $2^\kappa = 2^\gamma$ by the previous lemma. If κ is regular, then $2^\kappa = \kappa^\kappa = \kappa^{\mathrm{cof}(\kappa)} = \gimel(\kappa)$. In the singular case, note that $2^\gamma = 2^{<\kappa} \geq \kappa$. Thus, $\gimel(\kappa) = \kappa^{\mathrm{cof}(\kappa)} \leq (2^\gamma)^{\mathrm{cof}(\kappa)} = (2^\delta)^{\mathrm{cof}(\kappa)} = 2^\delta = 2^\gamma$ (where $\mathrm{cof}(\kappa) < \delta < \kappa$). Thus, $2^{<\kappa} \cdot \gimel(\kappa) = 2^{<\kappa}$. If κ is regular, then $2^{<\kappa} \leq 2^\kappa = \kappa^\kappa = \gimel(\kappa)$. So, $2^{<\kappa} \cdot \gimel(\kappa) = \gimel(\kappa)$. In either case, $2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa)$. Finally, suppose κ is a limit cardinal and the continuum function below κ is not eventually constant. Then $\mathrm{cof}(2^{<\kappa}) = \mathrm{cof}(\kappa)$. Then $2^\kappa = (2^{<\kappa})^{\mathrm{cof}(\kappa)} = \gimel(2^{<\kappa})$. □

(5.22) HAUSDORFF FORMULA

Theorem 18 *Let κ and λ be infinite cardinals. Let $\tau = \sup_{\rho<\kappa} |\rho|^\lambda$. Then*

$$\kappa^\lambda = \begin{cases} 2^\lambda & \text{if } \kappa \leq 2^\lambda, \\ \kappa \cdot \tau & \text{if } \lambda < \mathrm{cf}(\kappa), \\ \tau & \begin{array}{l}\text{if } \mathrm{cf}(\kappa) \leq \lambda, 2^\lambda < \kappa, \text{ and} \\ \rho \mapsto |\rho|^\lambda \text{ is eventually constant below } \kappa,\end{array} \\ \kappa^{\mathrm{cf}(\kappa)} & \text{otherwise.} \end{cases}$$

PROOF: 1. If $\kappa \leq 2^\lambda$, then $2^\lambda \leq \kappa^\lambda \leq (2^\lambda)^\lambda = 2^\lambda$.

2. If $\lambda < \mathrm{cf}(\kappa)$ then any function $f : \lambda \to \kappa$ is bounded, so ${}^\lambda\kappa = \bigcup_{\alpha<\kappa} {}^\lambda\alpha$, and $\kappa^\lambda \leq \sum_\alpha |\alpha|^\lambda = \kappa \cdot \rho \leq \kappa^\lambda$.

3. Suppose that $\mathrm{cf}(\kappa) \leq \lambda$, $2^\lambda < \kappa$, and $\rho \mapsto |\rho|^\lambda$ is eventually constant (and equal to τ) as ρ approaches κ.

Since $\kappa > \mathrm{cf}(\kappa)$, κ is a singular cardinal and we can choose a strictly increasing sequence of cardinals $(\kappa_\alpha : \alpha < \mathrm{cf}(\kappa))$ cofinal in κ such that $\kappa_\alpha^\lambda = \tau$ for all $\alpha < \mathrm{cf}(\kappa)$. In particular, $\tau^\lambda = \tau$.

Note that for each $\alpha < \mathrm{cf}(\kappa)$, $\kappa_\alpha < \prod_{i\in\mathrm{cf}(\kappa)} \kappa_i$, so also $\kappa = \sup_\alpha \kappa_\alpha \leq \prod_\alpha \kappa_\alpha$. [Of course, it follows from Kõnig's lemma that in fact we have a strict inequality, but we only need this weaker estimate.]

We now have $\kappa^\lambda \leq (\prod_\alpha \kappa_\alpha)^\lambda = \prod_\alpha \kappa_\alpha^\lambda = \prod_\alpha \tau = \tau^{\mathrm{cf}(\kappa)} = \tau$. The other inequality is clear.

4. Finally, suppose that $\mathrm{cf}(\kappa) \leq \lambda$, $2^\lambda < \kappa$, and $\rho \mapsto |\rho|^\lambda$ is not eventually constant as ρ approaches κ.

Notice that if $\rho < \kappa$ then $|\rho|^\lambda < \kappa$. Otherwise, $\mu^\lambda = (\mu^\lambda)^\lambda \geq \kappa^\lambda \geq \mu^\lambda$ for any $\rho \leq \mu < \kappa$, and the map $\rho \mapsto |\rho|^\lambda$ would be eventually constant below κ after all. Hence, $\tau = \kappa$.

Choose an increasing sequence of cardinals $(\kappa_\alpha : \alpha < \mathrm{cf}(\kappa))$ cofinal in κ. Then

$$\kappa^\lambda \leq (\prod_\alpha \kappa_\alpha)^\lambda = \prod_\alpha \kappa_\alpha^\lambda \leq \prod_\alpha \kappa = \kappa^{\mathrm{cf}(\kappa)}.$$

The other inequality is clear. □

Corollary 19 (Hausdorff) $(\kappa^+)^\lambda = \kappa^+ \cdot \kappa^\lambda$.

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PROOF: The result is immediate from Theorem 18 if $\lambda < \kappa^+ = \mathrm{cf}(\kappa^+)$. If, on the other hand, $\lambda \geq \kappa^+$ then certainly $\kappa^+ \leq 2^\lambda$, and it is easy to see that both sides of the equation we want to prove equal 2^λ. $\square$

THEOREM 5.20

(i) $\kappa \leq 2^\lambda \to \kappa^\lambda = 2^\lambda$

(ii) $\mu < \kappa \wedge \mu^\lambda \geq \kappa \to \kappa^\lambda = \mu^\lambda$

(iii) $\forall\mu\,(\mu < \kappa \to \mu^\lambda < \kappa) \to \lambda < \kappa$

(iii) (a) $\lambda < cf\ \kappa \wedge \forall\mu\,(\mu < \kappa \to \mu^\lambda < \kappa) \to \kappa^\lambda = \kappa$

(iii) (b) $cf\ \kappa \leq \lambda \wedge \forall\mu\,(\mu < \kappa \to \mu^\lambda < \kappa) \to \kappa^\lambda = \kappa^{cf\ \kappa}$

(5.23) SINGULAR STRONG LIMIT CARDINAL

Let λ be a singular strong limit cardinal.

Prove that $2^\lambda = \lambda^{\mathrm{cf}\lambda}$.

It has been a while since I had to prove anything relating to cardinals, and I am not sure where to start. A little push in the right direction would be very appreciated.

(Not homework)

(set-theory) (cardinals)

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Let $\langle\lambda_\xi\rangle_{\xi<\mathrm{cf}\,\lambda}$ be a cofinal sequence of λ. Since $\sum_{\xi<\mathrm{cf}\,\lambda}\lambda_\xi = \lambda$, we get

$$2^\lambda = 2^{\sum\lambda_\xi} = \prod_{\xi<\mathrm{cf}\,\lambda} 2^{\lambda_\xi} \le \prod_{\xi<\mathrm{cf}\,\lambda}\lambda = \lambda^{\mathrm{cf}\,\lambda}$$

since $2^{\lambda_\xi} < \lambda$ for all $\xi < \mathrm{cf}\,\lambda$. (Check that λ is a strong limit.) In other direction, by König's lemm we get

$$\lambda^{\mathrm{cf}\,\lambda} = \left(\sum_{\xi<\mathrm{cf}\,\lambda}\lambda_\xi\right)^{\mathrm{cf}\,\lambda} \le \left(\prod_{\xi<\mathrm{cf}\,\lambda} 2^\lambda\right)^{\mathrm{cf}\,\lambda} = 2^{\lambda\cdot\mathrm{cf}\,\lambda}.$$

It is easy to check that $\lambda_\xi < 2^\lambda$ for all $\xi < \mathrm{cf}\,\lambda$ and $2^{\lambda\cdot\mathrm{cf}\,\lambda} = 2^\lambda$.

THEOREM 5.22 (II)

κ 가 limit cardinal 일 때, $2^\lambda < \kappa \wedge \nu < \kappa \to \nu^\lambda < \kappa$ 를 증명한다.

$\nu \le 2^\lambda$ 인 경우는 간단하다. $\nu^\lambda \le (2^\lambda)^\lambda = 2^\lambda < \kappa$

$2^\lambda < \nu$의 경우, 재귀적 귀납법에 의해 $\nu^\lambda \leq \max(\nu, \nu^+) = \nu^+ < \kappa$