QUANTUM MECHANICS

Homepage: https://sj6219.github.io/GameProgrammingDocument/

Email: sj6219@hotmail.com

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양자역학에 관한 내용 정리

목차

VECTOR

양자역학에서 입자의 역학적 상태를 $\varphi: \mathbb{R} \to \mathbb{C}$ 함수로 표현한다.

φ는 Fourier transformable 해야 한다.

이때 무한 차원의 column vector 로 표현하는데, 이를 **ket vector** 라고 부른다. ket vector 의 집합을 $\mathbb{C}^{\mathbb{R}\times 1}$로 표기하겠다.

$$|\varphi\rangle = |\varphi(x)\rangle = \int \varphi(\alpha)\, |\delta(x-\alpha)\rangle\, d\alpha$$

이 vector 의 complex conjugate 를 **bra vector** 로 부르고, 아래와 같이 표기한다. bra vector 의 집합은 $\mathbb{C}^{1\times\mathbb{R}}$로 표기한다.

$$\langle\varphi| = |\varphi\rangle^* = \int \varphi^*(\alpha)\langle\delta(x-\alpha)|\ d\alpha$$

inner product 를 아래와 같이 정의한다.

$$\langle\phi|\varphi\rangle = \int \phi^*(\alpha)\,\varphi(\alpha)\,d\alpha$$

OPERATOR

ket vector 를 입력과 출력으로 하는 변환 $\Omega: \mathbb{C}^{\mathbb{R}\times 1} \to \mathbb{C}^{\mathbb{R}\times 1}$을 **operator** 라고 부른다. operator 의 집합을 $\mathbb{C}^{\mathbb{R}\times\mathbb{R}}$로 표기한다.

예를 들어, 위치를 나타내는 operator $x \in \mathbb{C}^{\mathbb{R}\times\mathbb{R}}$는 다음과 같이 표현할 수 있다.

$$x = \int |\delta(x-\alpha)\rangle\alpha\langle\delta(x-\alpha)|\,d\alpha$$

어떤 함수 $\varphi: \mathbb{R} \to \mathbb{C}$에 대해, $\varphi: \mathbb{C}^{\mathbb{R}\times\mathbb{R}} \to \mathbb{C}^{\mathbb{R}\times\mathbb{R}}$는 다음을 의미한다.

$$\varphi(x) = \int |\delta(x-\alpha)\rangle\,\varphi(\alpha)\,\langle\delta(x-\alpha)|\,d\alpha$$

ORTHONORMAL

operator Ω의 eigenvalue 들이 서로 다른 값을 가진다고 가정하자. eigenvalue ω' 에 대한 eigenvector 를 $|\Omega = \omega'\rangle$ 라고 표기할 때,

$$\langle\Omega = \omega'|\Omega = \omega''\rangle = \delta(\omega' - \omega'')$$

을 만족하면, 이런 eigenvector 들의 집합을 **orthonormal** 하다고 정의한다.

eigenvector $|\Omega = \omega'\rangle$ 의 집합이 orthonormal 할 때, 어떤 함수 $\varphi: \mathbb{R} \to \mathbb{C}$에 대해 $\varphi(\Omega)$, $|\varphi(\Omega)\rangle$ 를 다음과 같이 정의한다.

$$\varphi(\Omega) = \int |\Omega = \omega'\rangle\,\varphi(\omega')\,|\Omega = \omega'\rangle\,d\omega'$$

$$|\varphi(\Omega)\rangle = \int \varphi(\omega')\,|\Omega = \omega'\rangle\,d\omega'$$

다음 성질을 만족한다.

$$\varphi(\omega') = \langle \Omega = \omega' | \varphi(\Omega) \rangle$$

$$|\Omega = \omega'\rangle = \int \delta(\omega'' - \omega')\, |\Omega = \omega''\rangle\, d\omega'' = |\delta(\Omega - \omega')\rangle$$

COMPLETE SET

모든 ket vector 가 어떤 ket vector 집합에 대해 linear dependent 하면, 그 ket vector 집합을 (좁은 의미의) **complete set** 이라고 정의한다.
ket vector 집합의 모든 원소에 대해 어떤 operator 연산의 결과가 그 ket vector 집합에 linear dependent 하면, 그 연산에 대해 닫혀있다고 말한다. 어떤 ket vector 집합이 모든 연산에 닫혀있으면 그 ket vector 집합을 (넓은 의미의) complete set 이라고 정의한다.
예를 들어, $\{|\delta(x-\alpha)\rangle : \alpha \in \mathbb{R}\}$와 $\left\{\left|\frac{1}{\sqrt{2\pi}} e^{i\alpha x}\right\rangle : \alpha \in \mathbb{R}\right\}$는 좁은 의미의 complete set 이고, $\{|\delta(x-\alpha)\rangle : 0 < \alpha < 1\}$는 넓은 의미의 complete set 이다. 그리고, $\left\{\left|\frac{1}{\sqrt{2\pi}} e^{ix}\right\rangle\right\}$는 complete set 이 아니다.

IDENTITY OPERATOR

identity operator 1 은 임의의 $|\varphi(x)\rangle$ 에 대해 다음을 만족한다.

$$1\, |\varphi(x)\rangle = |\varphi(x)\rangle$$

$\int |\delta(x-\alpha)\rangle \langle \delta(x-\alpha)|\, d\alpha$은 identity operator 이다.

$$\int |\delta(x-\alpha)\rangle \langle \delta(x-\alpha)|\, d\alpha\ |\varphi(x)\rangle$$

$$= \int |\delta(x-\alpha)\rangle \langle \delta(x-\alpha) | \varphi(x)\rangle\, d\alpha$$

$$= \int |\delta(x-\alpha)\rangle\, \varphi(\alpha)\, d\alpha = |\varphi(x)\rangle$$

일반적으로, operator $\mathcal{E}$ 의 eigenvector 집합이 좁은 의미의 complete set 일 때,

$$\int |\delta(\mathcal{E} - \mathcal{E}')\rangle \langle \delta(\mathcal{E} - \mathcal{E}')|\, d\mathcal{E}' = 1$$

이다.

FOURIER TRANSFORM

$$\mathcal{F} = \int \left|\frac{1}{\sqrt{2\pi}} e^{i\alpha x}\right\rangle \langle \delta(x-\alpha)|\, d\alpha$$

$$\mathcal{F}^*\mathcal{F} = \iint |\delta(x-\alpha)\rangle \left\langle \frac{1}{\sqrt{2\pi}} e^{i\alpha x} \middle| \middle| \frac{1}{\sqrt{2\pi}} e^{i\beta x} \right\rangle \langle \delta(x-\beta)|\ d\alpha\, d\beta = 1$$

그러므로, $\mathcal{F}$는 unitary matrix 이다.

여기서

$$\int_{-\infty}^{\infty} e^{i\alpha x}\, dx = 2\pi\delta(\alpha)$$

을 증명해 보자.

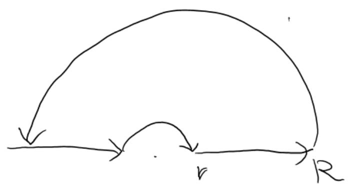
이것을 증명하려면, 임의의 함수 $f\colon \mathbb{R} \to \mathbb{R}$, 모든 양수 ε에 대해서,

$$\lim_{g\to\infty} \int_{-\varepsilon}^{\varepsilon} f(\alpha) \left(\int_{-g}^{g} e^{i\alpha x}\, dx \right) d\alpha = 2\pi f(0)$$

임을 증명하면 된다.

$$\lim_{g\to\infty} \int_{-\varepsilon}^{\varepsilon} f(\alpha) \left(\int_{-g}^{g} e^{i\alpha x}\, dx \right) d\alpha = \lim_{g\to\infty} \left(\int_{-\varepsilon g}^{\varepsilon g} f\left(\frac{z}{g}\right) \frac{e^{iz}}{iz} dz - \int_{-\varepsilon g}^{\varepsilon g} f\left(\frac{z}{g}\right) \frac{e^{-iz}}{iz} dz \right)$$

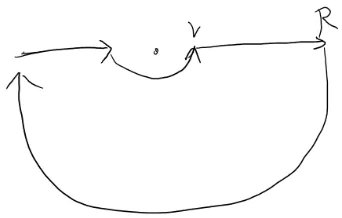
앞부분 적분은 아래그림처럼 complex integration 한다.



$$\lim_{R\to\infty} \int_0^{\pi} f\left(\varepsilon e^{i\theta}\right) e^{iR\cos\theta - R\sin\theta} d\theta \quad = 0$$

$$\lim_{r\to 0} \int_{\pi}^{0} f\left(\frac{re^{i\theta}}{g}\right) e^{ir\cos\theta - r\sin\theta} d\theta \quad = -\pi f(0)$$

뒷부분은 다음 그림처럼 적분한다.



$$\lim_{R\to\infty}\int_{2\pi}^{\pi} -f\left(\varepsilon e^{i\theta}\right)e^{-iR\cos\theta+R\sin\theta}\,d\theta \;\; = 0$$

$$\lim_{r\to 0}\int_{\pi}^{2\pi} -f\left(\frac{re^{i\theta}}{g}\right)e^{-ir\cos\theta+r\sin\theta}\,d\theta \;\; = -\pi f(0)$$

또한, $\lim\limits_{z\to 0}\frac{e^{iz}-e^{-iz}}{iz} = \lim\limits_{z\to 0}\frac{2\sin(z)}{z} \; = 2$ 이므로 $\lim\limits_{r\to 0}\int_{-r}^{r} f\left(\frac{z}{g}\right)\,\frac{e^{iz}-e^{-iz}}{iz}dz = 0$ 임을 알 수 있다.

$$\therefore \lim_{g\to\infty}\int_{-\infty}^{\infty} f(\alpha)\,d\alpha\int_{-g}^{g} e^{i\alpha x}\,dx = 2\pi f(0)$$

DIFFERENTIATION

$$\frac{d}{dx}|\varphi(x)\rangle$$

$$= \frac{d}{dx}(\mathcal{F}\mathcal{F}^*|\varphi(x)\rangle)$$

$$= \frac{d}{dx}\iint \left|\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}\right\rangle \langle\delta(x-\alpha)| \;\; |\delta(x-\beta)\rangle \left\langle\tfrac{1}{\sqrt{2\pi}}e^{i\beta x}\right| |\varphi(x)\rangle\,d\alpha\,d\beta$$

$$= \frac{d}{dx}\int \left|\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}\right\rangle \;\; \langle\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}|\varphi(x)\rangle\,d\alpha$$

$$= \int \frac{d}{dx}\left(\left|\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}\right\rangle\right)\langle\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}|\varphi(x)\rangle\,d\alpha$$

$$= \int \left|\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}\right\rangle \; i\alpha\,\langle\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}|\varphi(x)\rangle\,d\alpha$$

$$\therefore \frac{d}{dx} = \int \left|\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}\right\rangle i\alpha \left\langle\tfrac{1}{\sqrt{2\pi}}e^{i\alpha x}\right| \;\; d\alpha$$

POSITION

$$x = \int |\delta(x-\alpha)\rangle \alpha \langle \delta(x-\alpha)| d\alpha$$

상태 벡터가 $|\varphi(x)\rangle$인 입자의 위치가 x'와 $x' + \Delta x'$사이에 있을 확률은 다음과 같이 구한다.

$$\langle \varphi(x)|\chi_{(x',x'+\Delta x')}(x)|\varphi(x)\rangle = \int_{x'}^{x'+\Delta x'} \phi^*(x)\varphi(x)dx = \phi^*(x')\varphi(x')\Delta x' + o(\Delta x')$$

$\chi_{(x',x'+\Delta x')}$는 위키피디어에서 **Indicator Function** 을 참조하라.

MOMENTUM

operator $-i\hbar\frac{d}{dx}$ 를 momentum operator 라고 하고 p 로 표기한다.

$$p = -i\hbar\frac{d}{dx} = \int \left|\frac{1}{\sqrt{2\pi}}e^{i\alpha x}\right\rangle \hbar\alpha \left\langle \frac{1}{\sqrt{2\pi}}e^{i\alpha x}\right| d\alpha = \int \left|\frac{1}{\sqrt{2\pi\hbar}}e^{i\alpha x/\hbar}\right\rangle \alpha \left\langle \frac{1}{\sqrt{2\pi\hbar}}e^{i\alpha x/\hbar}\right| d\alpha$$

상태 벡터가 $|\varphi(x)\rangle$인 입자의 운동량이 p'와 $p' + \Delta p'$사이에 있을 확률은 다음과 같이 구한다.

$$\langle \varphi(x)|\chi_{(p',p'+\Delta p')}\,(p)\,|\varphi(x)\rangle$$

$$= \int \,\langle \varphi(x)| \left|\frac{1}{\sqrt{2\pi}}e^{i\alpha x}\right\rangle \chi_{(p',p'+\Delta p')}\,(\hbar\alpha)\, \left\langle \frac{1}{\sqrt{2\pi}}e^{i\alpha x}\right|\, |\varphi(x)\rangle\; d\alpha$$

$$= \int_{\frac{p'}{\hbar}}^{\frac{p'+\Delta p'}{\hbar}} \left|\int \frac{1}{\sqrt{2\pi}}e^{-i\alpha x}\varphi(\alpha)\,d\alpha\right|^2 dx$$

PRINCIPLE OF UNCERTAINTY

상태 벡터가 $|\frac{1}{\sqrt{\Delta q}}\,\chi_{(q-\frac{1}{2}\Delta q,q+\frac{1}{2}\Delta q)}(x)\,e^{ip'x}\rangle$인 입자의 운동량을 알아보자.

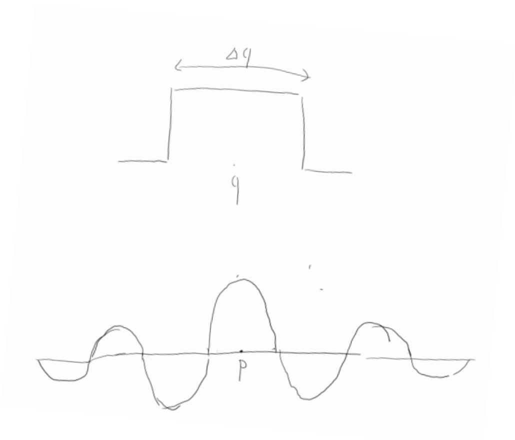
이 입자의 운동량이 x'와 $x' + \Delta x'$사이에 있을 확률은

$$\left\langle \frac{1}{\sqrt{\Delta q}}\,\chi_{(q-\frac{1}{2}\Delta q,q+\frac{1}{2}\Delta q)}(x)\,e^{ip'x}\right|\chi_{(x',x'+\Delta x')}\,(p)\,\left|\frac{1}{\sqrt{\Delta q}}\,\chi_{(q-\frac{1}{2}\Delta q,q+\frac{1}{2}\Delta q)}(x)\,e^{ip'x}\right\rangle$$

$$= \int_{\frac{x'}{\hbar}}^{\frac{x'+\Delta x'}{\hbar}} \left| \int \frac{1}{\sqrt{2\pi}} e^{-i\alpha x} \frac{1}{\sqrt{\Delta q}} \chi_{\left(q-\frac{1}{2}\Delta q, q+\frac{1}{2}\Delta q\right)}(\alpha)\, e^{ip'\alpha}\, d\alpha \right|^2 dx$$

Fourier transform 을 하면

$$\int_{q-\frac{1}{2}\Delta q}^{q+\frac{1}{2}\Delta q} \frac{1}{\sqrt{2\pi}} e^{-i\alpha x} \frac{1}{\sqrt{\Delta q}} e^{ip'\alpha} d\alpha = \sqrt{\frac{\Delta q}{2\pi}}\, e^{-iq(x-p')} \frac{\sin\left(\frac{\Delta q}{2}(x-p')\right)}{\frac{\Delta q}{2}(x-p')}$$



Δq가 작아질 수록 운동량은 p' 주위에 더 분산되어 분포된다.

SCHRÖDINGER EQUATION

시간이 t 일 때, 입자의 상태를 $|\varphi_t(x)\rangle$ 라고 하자. 양자역학에서는 $|\varphi_t(x)\rangle$가 $|\varphi_0(x)\rangle$에 대해 linear dependent 하다고 가정한다.

$$|\varphi_t(x)\rangle = T(x,p,t)\, |\varphi_0(x)\rangle$$

여기서 operator T는 φ_0 에 상관없이 일정하다.

그런데, 모든 t에 대하여 $\langle\varphi_t(x)|\varphi_t(x)\rangle = 1$이어야 하므로,

$$\langle\varphi_t(x)|\varphi_t(x)\rangle = \langle\varphi_0(x)|T^*(x,p,t)T(x,p,t)|\varphi_0(x)\rangle = 1$$

여기서, 모든 φ_0에 대해서 성립하기 위해서는 $T^*T = 1$이어야 하므로 T는 unitary matrix 이다.

또한 $\frac{\partial}{\partial t}(TT^*) = \frac{\partial}{\partial t}T\, T^* + T\frac{\partial}{\partial t}T^* = 0$ 이므로,

$$H = i\hbar\frac{\partial}{\partial t}T\,T^{*}$$

는 Hermitian matrix 이다.

$$i\hbar\frac{\partial}{\partial t}|\varphi_t(x)\rangle = i\hbar\frac{\partial}{\partial t}(T|\varphi_0(x)\rangle) = HT|\varphi_0(x)\rangle = H|\varphi_t(x)\rangle$$

인데, $i\hbar\frac{\partial}{\partial t}|\varphi_t(x)\rangle = H(x,p,t)\,|\varphi_t(x)\rangle$ 를 **Schrödinger equation** 이라고 한다.

예를 들어, $H(x,p) = \frac{p^2}{2m} + V(x)$ 라고 하면

$$\frac{d}{dt}\langle\varphi_t(x)|x|\varphi_t(x)\rangle$$

$$= \langle\varphi_t(x)|\frac{xH - Hx}{i\hbar}|\varphi_t(x)\rangle$$

$$= \langle\varphi_t(x)|\frac{[x,H]}{i\hbar}|\varphi_t(x)\rangle$$

$$= \langle\varphi_t(x)|\frac{p}{m}|\varphi_t(x)\rangle$$

$$\frac{d}{dt}\langle\varphi_t(x)|p|\varphi_t(x)\rangle$$

$$= \langle\varphi_t(x)|\frac{[p,H]}{i\hbar}|\varphi_t(x)\rangle$$

$$= \langle\varphi_t(x)| -\frac{d}{dx}V(x)|\varphi_t(x)\rangle$$

$$\therefore\ m\frac{d^2}{dt^2}\langle\varphi_t(x)|x|\varphi_t(x)\rangle = \langle\varphi_t(x)| -\frac{d}{dx}V(x)|\varphi_t(x)\rangle$$

[2] [31] 참조

$$|\varphi_t(x)\rangle = \left|A(x,t)e^{iS(x,t)/\hbar}\right\rangle$$

$$\left|i\hbar\frac{\partial A}{\partial t} - A\frac{\partial S}{\partial t}\right\rangle = \left(\frac{\left(p + \frac{\partial S}{\partial x}\right)^2}{2m} + V\right)|A(x,t)\rangle$$

$$\left|-A\frac{\partial S}{\partial t}\right\rangle = \left(\frac{(p)^2}{2m} + \frac{\left(\frac{\partial S}{\partial x}\right)^2}{2m} + V\right)|A(x,t)\rangle$$

$$-\frac{\partial S}{\partial t} = \left(\frac{\left(\frac{\partial S}{\partial x}\right)^2}{2m} + V\right) + \frac{-\hbar^2}{2mA}\frac{\partial^2 A}{\partial x^2}$$

$$\left| i\hbar \frac{\partial A}{\partial t} \right\rangle = \left(\frac{p \frac{\partial S}{\partial x} + \frac{\partial S}{\partial x} p}{2m} \right) |A(x,t)\rangle$$

$$\frac{\partial A}{\partial t} = \left(\frac{\frac{\partial^2 S}{\partial x^2} A + 2 \frac{\partial S}{\partial x} \frac{\partial A}{\partial x}}{2m} \right)$$

$$-\frac{\partial A^2}{\partial t} = \frac{\partial}{\partial x} \left(A^2 \frac{\frac{\partial S}{\partial x}}{m} \right)$$

DISPLACEMENT OPERATOR

$$\begin{aligned} D(\Delta x) &= \int |\delta(x-\alpha-\Delta x)\rangle\langle\delta(x-\alpha)| \; d\alpha \\ &= \int \left| \tfrac{1}{\sqrt{2\pi}} e^{i\alpha x} \right\rangle e^{-i\alpha\Delta x} \left\langle \tfrac{1}{\sqrt{2\pi}} e^{i\alpha x} \right| \; d\alpha \end{aligned}$$
$$d_x = \lim_{\Delta x \to 0} \frac{D(\Delta x) - 1}{\Delta x} = \int \left| \tfrac{1}{\sqrt{2\pi}} e^{i\alpha x} \right\rangle - i\alpha \left\langle \tfrac{1}{\sqrt{2\pi}} e^{i\alpha x} \right| \; d\alpha = -\frac{d}{dx}$$

$f(0) = 0$인 실수 함수 $f: \mathbb{R} \to \mathbb{R}$에 대해, 다음의 $\overline{D}(\Delta x)$ 역시 displacement operator 이다.

$$\overline{D}(\Delta x) = e^{i\, f(\Delta x)}\, D(\Delta x)$$
$$\overline{d_x} = \lim_{\Delta x \to 0} \frac{\overline{D}(\Delta x) - 1}{\Delta x} = d_x + i\, f'(0)$$

ANGULAR MOMENTUM

orbital angular momentum $\vec{L}$ 에 대해 다음 식이 성립한다.

$$\vec{L} = \vec{x} \times \vec{p}$$
$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} = \int_0^\infty d\alpha \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \left| \delta(r-\alpha) Y_{l'}^{m'}(\theta,\phi) \sqrt{r^2 \sin\theta} \right\rangle \hbar m' \left\langle \delta(r-\alpha) Y_{l'}^{m'}(\theta,\phi) \sqrt{r^2 \sin\theta} \right|$$
$$L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2$$
$$l = \sqrt{\frac{L^2}{\hbar^2} + \frac{1}{4}} - \frac{1}{2}$$

$$m_l = \frac{L_z}{\hbar}$$

$$L^2 = \sum_{l'=0}^{\infty} \sum_{m_{l'}=-l'}^{l'} \left|\delta_{l'}^{l} \delta_{m_{l'}}^{m_l}\right\rangle \hbar^2 l'(l'+1) \left\langle \delta_{l'}^{l} \delta_{m_{l'}}^{m_l}\right|$$

[1] [4.132],[4.32] 참조

spin angular momentum $\vec{S}$ 는 다음과 같다.

$$S_z = \sum_{s' \in \{0,0.5,1,1.5,\dots\}} \sum_{m_{s'} \in \{-s',-s'+1,-s'+2,\dots,s'\}} \left|\delta_{s'}^{s} \delta_{m_{s'}}^{m_s}\right\rangle \hbar m_s{}' \left\langle \delta_{s'}^{s} \delta_{m_{s'}}^{m_s}\right|$$

$$S^2 = \sum_{s' \in \{0,0.5,1,1.5,\dots\}} \sum_{m_{s'} \in \{-s',-s'+1,-s'+2,\dots,s'\}} \left|\delta_{s'}^{s} \delta_{m_{s'}}^{m_s}\right\rangle \hbar^2 s'(s'+1) \left\langle \delta_{s'}^{s} \delta_{m_{s'}}^{m_s}\right|$$

total angular momentum $\vec{J}$은 다음과 같다.

$$\vec{J} = \vec{L} + \vec{S}$$

$$J_z = \sum_{j' \in \{0,0.5,1,1.5,\dots\}} \sum_{m_{j'} \in \{-j',-j'+1,-j'+2,\dots,j'\}} \left|\delta_{j'}^{j} \delta_{m_{j'}}^{m_j}\right\rangle \hbar m_j{}' \left\langle \delta_{j'}^{j} \delta_{m_{j'}}^{m_j}\right|$$

$$= \sum_{l'=0}^{\infty} \sum_{m_l'=-l'}^{l'} \sum_{s' \in \{0,0.5,1,1.5,\dots\}} \sum_{m_s' \in \{-s',-s'+1,-s'+2,\dots,s'\}} \left|\delta_{l'}^{l} \delta_{m_{l'}}^{m_l} \delta_{s'}^{s} \delta_{m_{s'}}^{m_s}\right\rangle \hbar m_l{}' + \hbar m_s{}' \left\langle \delta_{l'}^{l} \delta_{m_{l'}}^{m_l} \delta_{s'}^{s} \delta_{m_{s'}}^{m_s}\right|$$

$$J_\pm = J_x \pm i J_y$$

$$J_\pm \left|\delta_{j'}^{j} \delta_{m_{j'}}^{m_j}\right\rangle = \hbar \sqrt{j'(j'+1) - m_j{}'(m_j{}' \pm 1)} \left|\delta_{j'}^{j} \delta_{m_{j'}\pm 1}^{m_j}\right\rangle$$

[1] [4.121] 참조

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[6] [5.1] 참조

$$\Psi(x',0) = (\pi\Delta^2)^{-\frac{1}{4}} e^{ipx'/\hbar} e^{-x'^2/2\Delta^2}$$
$$U(x,t,x',0) = \left(\frac{m}{2\pi\hbar i t}\right)^{\frac{1}{2}} e^{im(x-x')^2/2\hbar t}$$
$$\Psi(x,t) = \int U(x,t,x',0)\Psi(x',0)dx'$$

$$= \int m^{\frac{1}{2}}(2\pi\hbar it)^{-\frac{1}{2}}(\pi\Delta^2)^{-\frac{1}{4}}\, e^{\frac{imx^2}{2\hbar t}-\frac{imxx\prime}{\hbar t}+\frac{imx\prime^2}{2\hbar t}-\frac{x\prime^2}{2\Delta^2}+\frac{ipx\prime}{\hbar}} dx'$$

$$= \int m^{\frac{1}{2}}(2\pi\hbar it)^{-\frac{1}{2}}(\pi\Delta^2)^{-\frac{1}{4}}\, e^{\frac{im}{2\hbar t}(1+i\hbar t/m\Delta^2)\left(x'+\left(\frac{-imx}{\hbar t}+\frac{ip}{\hbar}\right)\frac{\hbar t}{im(1+i\hbar t/m\Delta^2)}\right)^2+\frac{imx^2}{2\hbar t}+\left(\frac{mx}{\hbar t}-\frac{p}{\hbar}\right)^2\frac{\hbar t}{2im(1+i\hbar t/m\Delta^2)}} dx'$$

$$= \pi^{\frac{1}{2}}\left(\frac{-im}{2\hbar t}(1+i\hbar t/m\Delta^2)\right)^{-\frac{1}{2}} m^{\frac{1}{2}}(2\pi\hbar it)^{-\frac{1}{2}}(\pi\Delta^2)^{-\frac{1}{4}}\, e^{\frac{imx^2}{2\hbar t}-\frac{imx^2}{2\hbar t(1+i\hbar t/m\Delta^2)}+\frac{ixp}{\hbar(1+i\hbar t/m\Delta^2)}-\frac{ip^2t}{2m\hbar(1+i\hbar t/m\Delta^2)}}$$

$$= \pi^{-\frac{1}{4}}\left((\Delta+i\hbar t/m\Delta)\right)^{-\frac{1}{2}}\, e^{\frac{-x^2}{2\Delta^2(1+i\hbar t/m\Delta^2)}+\frac{ixp}{\hbar(1+i\hbar t/m\Delta^2)}-\frac{ip^2t}{2m\hbar(1+i\hbar t/m\Delta^2)}}$$

$$= \pi^{-\frac{1}{4}}\left((\Delta+i\hbar t/m\Delta)\right)^{-\frac{1}{2}}\, e^{\frac{-(x-pt/m)^2}{2(\Delta^2+\hbar^2t^2/m^2\Delta^2)}(1-i\hbar t/m\Delta^2)+\frac{ip}{\hbar}(x-pt/2m)}$$

HYDROGEN WAVE FUNCTION

[1] [4.9] 참조

$$|\Psi_{nlm}(r,\theta,\phi,t)\rangle = \left|\psi_{nlm}(r,\theta,\phi)e^{-iE_nt/\hbar}\right\rangle$$

$$n = 1, 2, \ldots$$

$$l = 0, 1, \ldots, n-1$$

$$m = -l, -l+1, \ldots, +l$$

[1] [4.70] 참조

$$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2}$$

[1] [4.72] 참조

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

[1] [4.75] 참조

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_l^m(\theta,\phi)$$

[1] [4.89] 참조

$$R_{nl}(r) = \sqrt{\left(\frac{2}{na}\right)^3\frac{(n-l-1)!}{2n\left((n+l)!\right)^3}}\, e^{-r/na}\left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1}(2r/na)$$

[1] [4.32] 참조

$$Y_l^m(\theta,\phi) = \epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\, e^{im\phi}P_l^m(\cos\theta)$$

CLEBSCH-GORDAN COEFFICIENTS

[1] [4.185] 참조

$$|s\ m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m}\, |s_1\ m_1\rangle |s_2\ m_2\rangle$$

$$\left(\sum_{a=1}^{min(s_1,s_2+m)+min(s_1,s_2-m)+1} \sum_{b=1}^{min(s_1,s_2+m)+min(s_1,s_2-m)+1} C^{s_1 s_2 (s_1+s_2-b+1)}_{(min(s_1,s_2+m)-a+1)(m-(min(s_1,s_2+m)-a+1))m} \overleftarrow{e_a}\, \overleftarrow{e_b}^T \right) is\ unitary\ matrix$$

$$|s\ m-1\rangle = \frac{S_-|s\ m\rangle}{\hbar\sqrt{s(s+1)-m(m-1)}}$$

$$= \sum_{m_1+m_2=m} \frac{C^{s_1 s_2 s}_{m_1 m_2 m}\left(\hbar\sqrt{s_1(s_1+1)-m_1(m_1-1)}|s_1\ m_1-1\rangle|s_2\ m_2\rangle+\hbar\sqrt{s_2\ (s_2+1)-m_2(m_2-1)}|s_1\ m_1\rangle|s_2\ m_2-1\rangle\right)}{\hbar\sqrt{s(s+1)-m(m-1)}}$$

RELATIVITY

[1] [6.49] 참조

$$H = m\sqrt{1+\left(\frac{\boldsymbol{p}}{m}\right)^2}$$

ELECTRODYNAMICS

[1] [4.204] 참조

$$H = \frac{(\boldsymbol{p}-e\boldsymbol{A})^2}{2m} + eA^0$$

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[4] [7.141] 참조

$$L = -m\sqrt{1-\vec{v}\cdot\vec{v}} - eA^0(t,\vec{x}) + e\vec{A}(t,\vec{x})\cdot\vec{v} - V(\vec{x})$$
$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\vec{v}\cdot\vec{v}}} + e\vec{A}$$
$$\frac{d}{dt}\left(\frac{mv^i}{\sqrt{1-\vec{v}\cdot\vec{v}}}\right) + \left(eA^i{}_{,0} + eA^i{}_{,j}v^j\right) - \left(-eA^0{}_{,i} + eA^j{}_{,i}v_{\mathrm{j}} + V^{,i}\right) = 0$$
$$\frac{d}{dt}\left(\frac{mv^i}{\sqrt{1-\vec{v}\cdot\vec{v}}}\right) = e\left(-A^i{}_{,0} - A^0{}_{,i}\right) + e\left(-A^i{}_{,j} + A_j{}^{,i}\right)v^j + V^{,i}$$
$$= e\left(\vec{\boldsymbol{E}} + \vec{v}\times\vec{\boldsymbol{B}}\right) + V^{,i}$$

$$H = m\sqrt{1+\left(\frac{\vec{p}-e\vec{A}}{m}\right)^2} + eA^0 + V$$

****

$$L = -m\sqrt{-g_{\alpha\beta}v^\alpha v^\beta} + eA_\lambda v^\lambda - V$$
$$p_i = \frac{mg_{i\alpha}v^\alpha}{\sqrt{-g_{\mu\nu}v^\mu v^\nu}} + eA_i$$
$$p_0 = \frac{mg_{0\alpha}v^\alpha}{\sqrt{-g_{\mu\nu}v^\mu v^\nu}} + eA_0$$
$$(p_0 - eA_0)^2 g^{00} + 2(p_0 - eA_0)(p_i - eA_i)g^{0i} + (p_i - eA_i)\left(p_j - eA_j\right)g^{ij} = -m^2$$
$$p_0 = \frac{-(p_i - eA_i)g^{0i} + \sqrt{\left((p_i - eA_i)g^{0i}\right)^2 - g^{00}\left((p_i - eA_i)\left(p_j - eA_j\right)g^{ij} + m^2\right)}}{g^{00}} + eA_0$$
$$H = \frac{-mg_{0\alpha}v^\alpha}{\sqrt{-g_{\alpha\beta}v^\alpha v^\beta}} - eA_0 + V = -p_0 + V$$

[7] [12.4.1],[12.1.6] 참조

$$\int L_M\left(x, x_n(x^0), x_{n,0}(x^0)\right)dx^0$$
$$L_M = \sum_n\left(-m_n\sqrt{-g_{\alpha\beta}(x_n)\,x_n{}^\alpha{}_{,0}\,x_n{}^\beta{}_{,0}} + e_nA_\lambda(x_n)\,x_n{}^\lambda{}_{,0}\right)$$
$$\delta L_M = \left(\frac{-1}{2}\frac{m_n g_{\alpha\beta,i}x_n{}^\alpha{}_{,0}x_n{}^\beta{}_{,0}}{\sqrt{-g_{\mu\nu}x_n{}^\mu{}_{,0}\,x_n{}^\nu{}_{,0}}} + e_nA_{\lambda,i}\,x_n{}^\lambda{}_{,0}\right)\delta x_n{}^i + \left(\frac{m_n g_{i\lambda}x_n{}^\lambda{}_{,0}}{\sqrt{-g_{\mu\nu}x_n{}^\mu{}_{,0}\,x_n{}^\nu{}_{,0}}} + e_n\,A_i\right)\delta x_n{}^i{}_{,0}$$
$$+\int\left(\sum_n\delta^3(x-x_n)\left(e_n\,x_n{}^\alpha{}_{,0}\right)\right)\delta A_\alpha\,d^3x$$

$$+\int\left(\sum_n \delta^3(x-x_n)\left(\frac{1}{2}\frac{m_n x_n{}^{\alpha}{}_{,0} x_n{}^{\beta}{}_{,0}}{\sqrt{-g_{\mu\nu}x_n{}^{\mu}{}_{,0}\, x_n{}^{\nu}{}_{,0}}}\right)\right)\delta g_{\alpha\beta}\, d^3x$$

$$=\left(\frac{-1}{2}m_n g_{\alpha\beta,i}\frac{dx_n{}^{\alpha}}{d\tau_n}x_n{}^{\beta}{}_{,0}+e_n A_{\lambda,i}\, x_n{}^{\lambda}{}_{,0}\right)\delta x_n{}^{i}+\left(m_n g_{i\lambda}\frac{dx_n{}^{\lambda}}{d\tau_n}+e_n\, A_i\right)\delta x_n{}^{i}{}_{,0}$$

$$+\int\left(\sqrt{g}J^{\alpha}\right)\delta A_{\alpha}\, d^3x$$

$$+\int\left(\frac{\sqrt{g}}{2}T_M{}^{\alpha\beta}\right)\delta g_{\alpha\beta}\, d^3x$$

$$\frac{d}{dx^0}\left(\frac{\partial L_M}{\partial x_n{}^{i}{}_{,0}}\right)=\left(m_n g_{i\mu}\left(\frac{dx_n{}^{\mu}}{d\tau_n}\right)_{,0}+m_n g_{i\mu,\lambda}x_n{}^{\lambda}{}_{,0}\frac{dx_n{}^{\mu}}{d\tau_n}+e_n\, A_{i,\lambda}x_n{}^{\lambda}{}_{,0}\right)$$

$$\frac{d}{dx^{\mu}}\left(\frac{\partial L_M}{\partial x_n{}^{i}{}_{,\mu}}\right)-\frac{\partial L_M}{\partial x_n{}^{i}}=m_n g_{i\alpha}\left(\left(\frac{dx_n{}^{\alpha}}{d\tau_n}\right)_{,0}+\Gamma^{\alpha}_{\beta\,\lambda}\,\frac{dx_n{}^{\beta}}{d\tau_n}x_n{}^{\lambda}{}_{,0}+e_n F^{\alpha}{}_{\lambda}x_n{}^{\lambda}{}_{,0}\right)$$

$$H_M=\sum_n\left(-m_n g_{0\mu}\frac{dx_n{}^{\mu}}{d\tau_n}-e_n\, A_0\right)=\int -\sqrt{g}g_{0\mu}T_M{}^{0\mu}-\sqrt{g}J^0 A_0\, d^3x$$

$$\int L_E\left(x, A_{\alpha}(x)\boldsymbol{e^{\alpha}}, A_{\alpha,\beta}(x)\boldsymbol{e^{\alpha}e^{\beta}}\right)d^4x$$

$$L_E=-\frac{\sqrt{g(x)}}{4}F_{\alpha\beta}F^{\alpha\beta}=\frac{\sqrt{g}}{2}(E_iE^i-B_iB^i)$$

$$F_{\alpha\beta}=A_{\beta,\alpha}-A_{\alpha,\beta}$$

$$\delta F^{\alpha\beta}=g^{\alpha\bar{\alpha}}g^{\beta\bar{\beta}}\delta F_{\bar{\alpha}\bar{\beta}}+\delta g^{\alpha\mu}g^{\beta\bar{\beta}}F_{\mu\bar{\beta}}+g^{\alpha\bar{\alpha}}\delta g^{\beta\mu}F_{\bar{\alpha}\mu}$$

$$=g^{\alpha\bar{\alpha}}g^{\beta\bar{\beta}}\delta F_{\bar{\alpha}\bar{\beta}}-g^{\alpha\bar{\alpha}}g^{\beta\bar{\beta}}g^{\mu\bar{\mu}}F_{\mu\bar{\beta}}\delta g_{\bar{\alpha}\bar{\mu}}-g^{\alpha\bar{\alpha}}g^{\beta\bar{\beta}}g^{\mu\bar{\mu}}F_{\bar{\alpha}\mu}\delta g_{\bar{\beta}\bar{\mu}}$$

$$=g^{\alpha\bar{\alpha}}g^{\beta\bar{\beta}}\delta F_{\bar{\alpha}\bar{\beta}}-g^{\alpha\bar{\alpha}}F^{\mu\beta}\delta g_{\bar{\alpha}\mu}-g^{\beta\bar{\beta}}F^{\alpha\mu}\delta g_{\bar{\beta}\mu}$$

$$\delta L_E=\left(\sqrt{g}F^{\alpha\beta}\right)\delta A_{\alpha,\beta}+\left(-\frac{\delta\sqrt{g}}{4}F_{\alpha\beta}F^{\alpha\beta}\right)+\left(-\frac{\sqrt{g}}{4}F_{\alpha\beta}\delta F^{\alpha\beta}\right)$$

$$=\left(\sqrt{g}F^{\alpha\beta}\right)\delta A_{\alpha,\beta}+\left(-\frac{\sqrt{g}}{8}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right)\delta g_{\alpha\beta}+\left(\frac{\sqrt{g}}{2}F_{\mu}{}^{\alpha}F^{\mu\beta}\right)\delta g_{\alpha\beta}$$

$$=\left(\sqrt{g}F^{\alpha\beta}\right)\delta A_{\alpha,\beta}+\left(\frac{\sqrt{g}}{2}T_E{}^{\alpha\beta}\right)\delta g_{\alpha\beta}$$

$$\frac{\partial L_E}{\partial A_{\alpha,\beta}}=\sqrt{g}F^{\alpha\beta}$$

$$\frac{d}{dx^{\beta}}\left(\frac{\partial L_E}{\partial A_{\alpha,\beta}}\right)=\left(-\sqrt{g}F^{\beta\alpha}\right)_{,\beta}$$

$$\frac{d}{dx^{\beta}}\left(\frac{\partial L_{M+E}}{\partial A_{\alpha,\beta}}\right)-\frac{\partial L_{M+E}}{\partial A_{\alpha}}=\sqrt{g}\left(-F^{\beta\alpha}{}_{;\beta}-J^{\alpha}\right)$$

$$(H_E)^{\alpha}_{\beta}=\frac{\partial L_E}{\partial A_{\mu,\alpha}}A_{\mu,\beta}-\delta^{\alpha}_{\beta}L_E=\sqrt{g}F^{\mu\alpha}A_{\mu,\beta}+\frac{\sqrt{g}}{4}\delta^{\alpha}_{\beta}F_{\mu\nu}F^{\mu\nu}$$

$$=-\sqrt{g}g_{\beta\bar{\beta}}\left(F^{\mu\alpha}F_{\mu}^{\bar{\beta}}-\frac{g^{\alpha\bar{\beta}}}{4}F_{\mu\nu}F^{\mu\nu}\right)-\frac{d}{dx^{\mu}}\left(\sqrt{g}F^{\mu\alpha}A_{\beta}\right)+\sqrt{g}J^{\alpha}A_{\beta}$$

$$\frac{d}{dx^{\mu}}(H_E)^{\mu}_{\beta}=\frac{\partial L_M}{\partial A_{\mu}}A_{\mu,\beta}=\sqrt{g}J^{\mu}A_{\mu,\beta}$$

$$H_{M+E}=H_M+\int (H_E)^0_0 d^3x=\int -\sqrt{g}g_{0\mu}T_{M+E}{}^{0\mu}\, d^3x$$

$$\int L_G\left(g_{\alpha\beta}(x)\boldsymbol{e^{\alpha}e^{\beta}}, g_{\alpha\beta,\mu}(x)\boldsymbol{e^{\alpha}e^{\beta}e^{\mu}}, g_{\alpha\beta,\mu\nu}(x)\boldsymbol{e^{\alpha}e^{\beta}e^{\mu}e^{\nu}}\right)d^4x$$

$$L_G = \frac{-1}{16\pi G}\sqrt{g}R$$
$$g = -\det\left(g_{\alpha\beta}\boldsymbol{e}^{\alpha}\boldsymbol{e}^{\beta}\right)$$
$$R = g^{\alpha\bar{\alpha}}R_{\alpha\bar{\alpha}}$$
$$R_{\alpha\beta} = R^{\lambda}{}_{\alpha\lambda\beta}$$
$$R_{\alpha\beta\mu\nu} = \Gamma_{\alpha\beta\mu,\nu} - \Gamma_{\alpha\beta\nu,\mu} + \Gamma^{\lambda}_{\beta\mu}\Gamma_{\alpha\nu\lambda} - \Gamma^{\lambda}_{\beta\nu}\Gamma_{\alpha\mu\lambda}$$
$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}\mu\nu}$$
$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}\left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\right)$$

$$\delta g = g g^{\alpha\bar{\alpha}}\delta g_{\alpha\bar{\alpha}}$$
$$\delta g^{\mu\nu} = -g^{\mu\bar{\mu}}g^{\nu\bar{\nu}}\delta g_{\bar{\mu}\bar{\nu}}$$
$$\delta\Gamma^{\lambda}_{\mu\nu} = \delta g^{\lambda\kappa}\Gamma_{\kappa\mu\nu} + g^{\lambda\bar{\lambda}}\delta\Gamma_{\bar{\lambda}\mu\nu}$$
$$= \frac{1}{2}g^{\lambda\bar{\lambda}}\left(\delta g_{\bar{\lambda}\mu,\nu} - \Gamma^{\kappa}_{\bar{\lambda}\nu}\delta g_{\kappa\mu} - \Gamma^{\kappa}_{\mu\nu}\delta g_{\bar{\lambda}\kappa} + \delta g_{\bar{\lambda}\nu,\mu} - \Gamma^{\kappa}_{\bar{\lambda}\mu}\delta g_{\kappa\nu} - \Gamma^{\kappa}_{\mu\nu}\delta g_{\bar{\lambda}\kappa} - \delta g_{\mu\nu,\bar{\lambda}} + \Gamma^{\kappa}_{\mu\bar{\lambda}}\delta g_{\kappa\nu} + \Gamma^{\kappa}_{\nu\bar{\lambda}}\delta g_{\kappa\mu}\right)$$
$$= \frac{1}{2}g^{\lambda\bar{\lambda}}\left(\delta g_{\bar{\lambda}\mu;\nu} + \delta g_{\bar{\lambda}\nu;\mu} - \delta g_{\mu\nu;\bar{\lambda}}\right)$$
$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\lambda;\nu} - \delta\Gamma^{\lambda}_{\mu\nu;\lambda}$$
$$\delta\left(\sqrt{g}R\right) = \delta\left(\sqrt{g}R\right) + \sqrt{g}R_{\alpha\beta}\delta g^{\alpha\beta} + \sqrt{g}g^{\alpha\bar{\alpha}}\delta R_{\alpha\bar{\alpha}}$$
$$\sqrt{g}g^{\alpha\bar{\alpha}}\delta R_{\alpha\bar{\alpha}} = g^{\alpha\bar{\alpha}}\sqrt{g}\left(\delta\Gamma^{\lambda}_{\alpha\lambda;\bar{\alpha}} - \delta\Gamma^{\lambda}_{\alpha\bar{\alpha};\lambda}\right)$$
$$= \left(\sqrt{g}g^{\alpha\bar{\alpha}}\delta\Gamma^{\lambda}_{\bar{\alpha}\lambda}\right)_{;\alpha} - \left(\sqrt{g}g^{\alpha\bar{\alpha}}\delta\Gamma^{\lambda}_{\alpha\bar{\alpha}}\right)_{;\lambda} = \left(\sqrt{g}g^{\alpha\bar{\alpha}}\delta\Gamma^{\lambda}_{\bar{\alpha}\lambda}\right)_{,\alpha} - \left(\sqrt{g}g^{\alpha\bar{\alpha}}\delta\Gamma^{\lambda}_{\alpha\bar{\alpha}}\right)_{,\lambda} = 0$$
$$\delta\left(\sqrt{g}R\right) = \left(\frac{1}{2}\sqrt{g}g^{\alpha\bar{\alpha}}\delta g_{\alpha\bar{\alpha}}\right)R + \sqrt{g}R_{\alpha\beta}\left(-g^{\alpha\bar{\alpha}}g^{\beta\bar{\beta}}\delta g_{\bar{\alpha}\bar{\beta}}\right)$$
$$= -\sqrt{g}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\delta g_{\alpha\beta}$$
$$\delta L_G = \frac{1}{16\pi G}\sqrt{g}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\delta g_{\alpha\beta}$$
$$\frac{\partial L_{M+E+G}}{\partial g_{\alpha\beta}} - \frac{d}{dx^{\mu}}\frac{\partial L_{M+E+G}}{\partial g_{\alpha\beta,\mu}} + \frac{d}{dx^{\mu}}\frac{d}{dx^{\nu}}\frac{\partial L_{M+E+G}}{\partial g_{\alpha\beta,\mu\nu}} = \frac{1}{16\pi G}\sqrt{g}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R + 8\pi G T_{M+E}{}^{\alpha\beta}\right)$$

INVARIANT PROBLEM

[3] (6.2.23) (6.2.26) 참조

$$P_j(t,\vec{x},\vec{v}) = \frac{\partial L}{\partial \dot{x}^j}(t,\vec{x},\vec{v})$$
$$V^j(t,\vec{x},\vec{p}) = \frac{\partial H}{\partial p_j}(t,\vec{x},\vec{p})$$
$$V^j\left(t,\vec{x},\vec{P}(t,\vec{x},\vec{v})\right) = v^j$$
$$P_j\left(t,\vec{x},\vec{V}(t,\vec{x},\vec{p})\right) = p_j$$

[3] (6.2.25) 참조

$$H(t,\vec{x},\vec{p}) = -L\left(t,\vec{x},\vec{V}(t,\vec{x},\vec{p})\right) + p_j V^j(t,\vec{x},\vec{p})$$
$$L(t,\vec{x},\vec{v}) = -H\left(t,\vec{x},\vec{P}(t,\vec{x},\vec{v})\right) + v^j P_j(t,\vec{x},\vec{v})$$

[3] (6.2.27) (6.2.28) 참조

$$\frac{\partial H}{\partial x^j}(t,\vec{x},\vec{p}) = -\frac{\partial L}{\partial x^j}\left(t,\vec{x},\vec{V}(t,\vec{x},\vec{p})\right)$$

$$\frac{\partial H}{\partial t}(t,\vec{x},\vec{p}) = -\frac{\partial L}{\partial t}\left(t,\vec{x},\vec{V}(t,\vec{x},\vec{p})\right)$$

[3] (6.2.41) 참조

example

$$S\left(t,\Gamma^j(t)\right) = \int_{t_{min}}^{t} L\left(t,\vec{\Gamma}(t),\frac{d\vec{\Gamma}}{dt}(t)\right)dt$$
$$\frac{\partial S}{\partial t}(t,\vec{x}) = -H\left(t,\vec{x},\boldsymbol{\frac{\partial S}{\partial x^j}(t,\vec{x})\overleftarrow{e}^j}\right)$$

[3] (6.2.43) 참조

$$\frac{d}{dt}P_j\left(t,\vec{\Gamma}(t),\frac{d\vec{\Gamma}}{dt}(t)\right) = -\frac{\partial H}{\partial x^j}\left(t,\vec{\Gamma}(t),\vec{P}\left(t,\vec{\Gamma}(t),\frac{d\vec{\Gamma}}{dt}(t)\right)\right) = \frac{\partial L}{\partial x^j}\left(t,\vec{\Gamma}(t),\frac{d\vec{\Gamma}}{dt}(t)\right)$$
$$\frac{d}{dt}\Gamma^j(t) = \frac{\partial H}{\partial p_j}\left(t,\vec{\Gamma}(t),\vec{P}\left(t,\vec{\Gamma}(t),\frac{d\vec{\Gamma}}{dt}(t)\right)\right)$$

[3] (6.4.5) 참조

$$\begin{aligned}\omega(t,\vec{u}) &= p_j(t,\vec{u})dx^j - H\big(t,\vec{x}(t,\vec{u}),\vec{p}(t,\vec{u})\big)dt \\ &= p_j(t,\vec{u})\frac{\partial x^j}{\partial u^h}(t,\vec{u})du^h + L\left(t,\vec{x}(t,\vec{u}),\frac{\partial \vec{x}}{\partial t}(t,\vec{u})\right)dt\end{aligned}$$

[3] (6.4.21) 참조

$$\int_{\partial g}\omega = \int_{t_{min}}^{t_{max}} p_j\big(t,\vec{u}(t)\big)\frac{\partial x^j}{\partial u^h}\big(t,\vec{u}(t)\big)\frac{du^h}{dt}(t)dt = \int_{\vec{u}(t_{min})}^{\vec{u}(t_{max})} p_j(t_{min},\vec{u})\frac{\partial x^j}{\partial u^h}(t_{min},\vec{u})du^h$$

[3] (6.5.32) 참조

$$G_\Sigma = \left\{\overleftarrow{S}\left(\overleftarrow{t},\vec{x}(\overleftarrow{t})\right) : \overleftarrow{t} \in G\right\}$$
$$\int_G \Delta\left(\overleftarrow{t},\vec{x}(\overleftarrow{t}),\frac{dx^m}{dt^\beta}(\overleftarrow{t})\boldsymbol{\vec{e}_m\overleftarrow{e}^\beta}\right)d\overleftarrow{t} = \int_{G_\Sigma} d\overleftarrow{\Sigma}$$

[3] (6.5.36) 참조

$$\min_{v_\beta^m\vec{e}_m\overleftarrow{e}^\beta}\left(L\big(\overleftarrow{t},\vec{x},v_\beta^m\boldsymbol{\vec{e}_m\overleftarrow{e}^\beta}\big) - \Delta\big(\overleftarrow{t},\vec{x},v_\beta^m\boldsymbol{\vec{e}_m\overleftarrow{e}^\beta}\big)\right) = L\big(\overleftarrow{t},\vec{x},\psi_\beta^m(\overleftarrow{t},\vec{x})\boldsymbol{\vec{e}_m\overleftarrow{e}^\beta}\big) - \Delta\big(\overleftarrow{t},\vec{x},\psi_\beta^m(\overleftarrow{t},\vec{x})\boldsymbol{\vec{e}_m\overleftarrow{e}^\beta}\big) = 0$$

[3] (6.5.43) 참조

$$H_\beta^\alpha\big(\overleftarrow{t},\vec{x},v_r^m\boldsymbol{\vec{e}_m\overleftarrow{e}^\gamma}\big) = -L\big(\overleftarrow{t},\vec{x},v_r^m\boldsymbol{\vec{e}_m\overleftarrow{e}^\gamma}\big)\delta_\beta^\alpha + \frac{\partial L}{\partial \dot{x}_\alpha^j}\big(\overleftarrow{t},\vec{x},v_r^m\boldsymbol{\vec{e}_m\overleftarrow{e}^\gamma}\big)v_\beta^j$$
$$H_\beta^\alpha\big(\overleftarrow{t},\vec{x},\psi_\beta^m(\overleftarrow{t},\vec{x})\boldsymbol{\vec{e}_m\overleftarrow{e}^\beta}\big) = -C_\varepsilon^\alpha\big(\overleftarrow{t},\vec{x},\psi_\beta^m(\overleftarrow{t},\vec{x})\boldsymbol{\vec{e}_m\overleftarrow{e}^\beta}\big)\frac{\partial S^\varepsilon}{\partial t^\beta}(\overleftarrow{t},\vec{x})$$

[3] (6.5.40) 참조

example

$$S^1\left(\overleftarrow{t},\vec{\Gamma}(\overleftarrow{t})\right)=t^1$$
$$S^2\left(\overleftarrow{t},\vec{\Gamma}(\overleftarrow{t})\right)=\int_{t_{min}^2}^{t^2} L\left(t^1\vec{\boldsymbol{e}}_1+\tilde{t}\vec{\boldsymbol{e}}_2,\vec{\Gamma}(t^1\vec{\boldsymbol{e}}_1+\tilde{t}\vec{\boldsymbol{e}}_2),\frac{d\Gamma^m}{dt^\beta}(t^1\vec{\boldsymbol{e}}_1+\tilde{t}\vec{\boldsymbol{e}}_2)\vec{\boldsymbol{e}}_m\overleftarrow{\boldsymbol{e}}^\beta\right)d\tilde{t}$$
$$H_\beta^\alpha\left(\overleftarrow{t},\vec{x},\boldsymbol{\psi}_{\boldsymbol{\beta}}^{\boldsymbol{m}}(\overleftarrow{\boldsymbol{t}},\vec{\boldsymbol{x}})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\beta}}\right)=-C_\varepsilon^\alpha\left(\overleftarrow{t},\vec{x},\boldsymbol{\psi}_{\boldsymbol{\beta}}^{\boldsymbol{m}}(\overleftarrow{\boldsymbol{t}},\vec{\boldsymbol{x}})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\beta}}\right)\frac{\partial S^\varepsilon}{\partial t^\beta}(\overleftarrow{t},\vec{x})$$
$$\frac{\partial L}{\partial\dot{x}_\alpha^j}\left(\overleftarrow{t},\vec{x},\psi_\beta^m(\overleftarrow{t},\vec{x})\vec{e}_m\overleftarrow{e}^\beta\right)=C_\varepsilon^\alpha\left(\overleftarrow{t},\vec{x},\psi_\beta^m(\overleftarrow{t},\vec{x})\vec{e}_m\overleftarrow{e}^\beta\right)\frac{\boldsymbol{\partial S^\varepsilon}}{\boldsymbol{\partial x^j}}(\overleftarrow{\boldsymbol{t}},\vec{\boldsymbol{x}})$$

[3] (6.5.52) 참조

$$\frac{d}{dt^\alpha}H_\beta^\alpha\left(\overleftarrow{t},\vec{x}(\overleftarrow{t}),\frac{dx^m}{dt^\gamma}(\overleftarrow{t})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^\gamma\right)$$
$$=-\frac{\partial L}{\partial t^\beta}\left(\overleftarrow{t},\vec{x}(\overleftarrow{t}),\frac{dx^m}{dt^\gamma}(\overleftarrow{t})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^\gamma\right)+\left(E_j(L)\right)\left(\overleftarrow{t},\vec{x}(\overleftarrow{t}),\frac{dx^m}{dt^\gamma}(\overleftarrow{t})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^\gamma,\frac{d^2x^m}{dt^\gamma dt^\varepsilon}(\overleftarrow{t})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^\gamma\overleftarrow{\boldsymbol{e}}^\varepsilon\right)\frac{dx^j}{dt^\beta}(\overleftarrow{t})$$

[3] (6.5.56) 참조

$$\Delta\left(\overleftarrow{t},\vec{x},v_r^m\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^\gamma\right)$$
$$=L^{1-m}\left(\overleftarrow{t},\vec{x},\psi_r^m(\overleftarrow{t},\vec{x})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^\gamma\right)$$
$$det\left[\left(L\left(\overleftarrow{t},\vec{x},\psi_r^m(\overleftarrow{t},\vec{x})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^\gamma\right)\delta_\beta^\alpha+\frac{\partial L}{\partial\dot{x}_\alpha^j}\left(\overleftarrow{t},\vec{x},\psi_r^m(\overleftarrow{t},\vec{x})\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^\gamma\right)\left(v_\beta^j-\psi_\beta^j(\overleftarrow{t},\vec{x})\right)\right)\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\beta}}\,\vec{\boldsymbol{e}}_{\boldsymbol{\alpha}}{}^{\boldsymbol{T}}\right]$$

[3] (6.7.13) 참조

$$H_\beta^\alpha\left(\overleftarrow{t},\vec{x},\dot{x}_\eta{}^m\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}},\ddot{x}_{\eta\,\theta}^{\;m}\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\theta}},\dddot{x}_{\eta\,\theta\,\lambda}^{\;m}\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\theta}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\lambda}}\right)$$
$$=-L\left(\overleftarrow{t},\vec{x},\dot{x}_\eta{}^m\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}},\ddot{x}_{\eta\,\theta}^{\;m}\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\theta}}\right)\delta_\beta^\alpha+\frac{\partial L}{\partial\dot{x}_\alpha^j}\left(\overleftarrow{t},\vec{x},\dot{x}_\eta{}^m\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}},\ddot{x}_{\eta\,\theta}^{\;m}\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\theta}}\right)\dot{x}_\beta^j$$
$$-\frac{d}{dt^\gamma}\left(\frac{\partial L}{\partial\ddot{x}_{\alpha\,\gamma}^{\;j}}\left(\overleftarrow{t},\vec{x},\dot{x}_\eta{}^m\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}},\ddot{x}_{\eta\,\theta}^{\;m}\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\theta}}\right)\right)\dot{x}_\beta^j+\frac{\partial L}{\partial\ddot{x}_{\alpha\,\gamma}^{\;j}}\left(\overleftarrow{t},\vec{x},\dot{x}_\eta{}^m\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}},\ddot{x}_{\eta\,\theta}^{\;m}\vec{\boldsymbol{e}}_{\boldsymbol{m}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\eta}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\theta}}\right)\ddot{x}_{\beta\,\gamma}^{\;j}$$

Wiki Euler-Lagrange Equation 참조

$$L\left(\overleftarrow{t},\vec{x},x_{\alpha_1}^j\vec{\boldsymbol{e}}_{\boldsymbol{j}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\alpha_1}},\dots,x_{\alpha_1\dots\alpha_n}^j\vec{\boldsymbol{e}}_{\boldsymbol{j}}\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\alpha_1}}\dots\overleftarrow{\boldsymbol{e}}^{\boldsymbol{\alpha_n}}\right)$$
$$x_{\alpha_1\dots\alpha_n}^j(\overleftarrow{t})=\frac{d^n}{dt^{\alpha_1}\dots dt^{\alpha_n}}\left(x^j(\overleftarrow{t})\right)$$
$$\left(E_j(L)\right)=-\frac{\partial L}{\partial x^j}+\frac{d}{dt^{\alpha_1}}\left(\frac{\partial L}{\partial x_{\alpha_1}^j}\right)\dots-(-1)^n\frac{d^n}{dt^{\alpha_1}\dots dt^{\alpha_n}}\left(\frac{\partial L}{\partial x_{\alpha_1\dots\alpha_n}^j}\right)$$

CANONICAL TRANSFORMATION

[5] (5.82) 참조

어떤 변환 $\vec{Q}(\vec{q},\vec{p},t),\vec{P}(\vec{q},\vec{p},t)$에서 다음 식을 만족하는 $K(\vec{Q},\vec{P},t)$가 존재한다면, 그 변환을 Canonical Transformation 이라고 한다.

$$\dot{Q}^j = \frac{\partial K}{\partial P_j}(\vec{Q}, \vec{P}, t)$$
$$\dot{P}_j = \frac{\partial K}{\partial Q^j}(\vec{Q}, \vec{P}, t)$$

[5] (5.88) 참조

$(\vec{q}, \vec{p})$ 좌표계에서 Poison bracket 과 $(\vec{Q}, \vec{P})$ 좌표계에서 Poison bracket 이 다음 조건을 만족한다면, 변환 $\vec{Q}(\vec{q}, \vec{p}, t), \vec{P}(\vec{q}, \vec{p}, t)$ 는 Canonical transformation이다.

$$\forall f, g \in \mathcal{F}(T^*\mathbb{Q}): [f, g]_{(\vec{Q}, \vec{P})} = a[f, g]_{(\vec{q}, \vec{p})}$$

QUANTUM FIELD THEORY

Fourier decomposition of the Dirac(electron positron) field

$$\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=1}^{2} \left(f_s(p) u_s(p) e^{-ip\cdot x} + \hat{f}_s^\dagger(p) v_s(p) e^{ip\cdot x} \right)$$

$$E_p = \sqrt{\mathbf{p}^2 + m^2}$$

[8] (4.96) 참조

$$u_r^\dagger(p) u_s(p) = v_r^\dagger(p) v_s(p) = 2E_p \delta_{rs}$$
$$u_r^\dagger(p) u_s(-p) = v_r^\dagger(p) v_s(-p) = 0$$

[8] (4.49) 참조

$$\sum_{s=1}^{2} \left(u_s(p) \overline{u}_s(p) \right) = \not{p} + m$$
$$\sum_{s=1}^{2} \left(v_s(p) \overline{v}_s(p) \right) = \not{p} - m$$

[8] (4.53) 참조

Fourier decomposition of the electromagnetic field

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} \left(\epsilon_r^\mu(k) a_r(k) e^{-ik\cdot x} + \epsilon^{*\mu}_{\ r}(k) a_r^\dagger(k) e^{ik\cdot x} \right)$$

$$\omega_k = \mathbf{k}$$

[8] (8.45) 참조

$$\sum_{r=0}^{3} \left(-g_{rr} \epsilon_r^\mu(k) \epsilon^{*\nu}_{\ r}(k) \right) = -g^{\mu\nu}$$

[8] (8.49) 참조

$$\left[a_r(k), a_s^\dagger(k')\right]_- = -g_{rs}\delta^3(k-k')$$

[8] (8.56) 참조

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