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3.2. Exercises #46

If $h(2) = 4$ and $h'(2) = -3$, find

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2} = \frac{h'(x)\cdot x - h(x)}{x^2}\bigg|_{x=2} = \frac{-3\cdot 2 - 4}{4} = -\frac{5}{2}$$

3.3. Exercises #22

Find an equation of the tangent line to the curve at the given point.

$y = e^x \cos x, \quad (0, 1)$

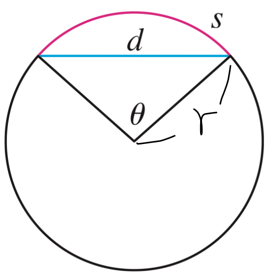
$y' = e^x \cdot \cos x - e^x \cdot \sin x \big|_{x=0} = 1$

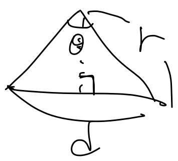
~~tangent line~~ : $y = 1\cdot(x-0)+1 = x+1$

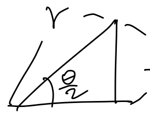
3.3 Exercises #57

The figure shows a circular arc of length s and a chord of length d, both subtended by a central angle θ. Find

$$\lim_{\theta \to 0^+} \frac{s}{d}$$







$\frac{d}{2} = r\sin\frac{\theta}{2}$

$$\begin{cases} s = r\theta \\ d = 2r\sin\frac{\theta}{2} \end{cases}$$

$$\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{\cancel{r}\theta}{2\cancel{r}\sin\frac{\theta}{2}} = \lim_{\theta\to 0^+}\frac{\frac{\theta}{2}}{\sin\frac{\theta}{2}} = 1$$

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3.4. Exercises #78

Find the 1000th derivative of $f(x) = xe^{-x}$.

$(x-0)e^{-x}$

$f^{(1)}(x) = e^{-x} + x(-1)(e^{-x}) = (1-x)e^{-x}$

$f^{(2)}(x) = (-1)e^{-x} + (1-x)(-1)e^{-x} = (x-2)e^{-x}$

$f^{(3)}(x) = e^{-x} + (x-2)(-1)e^{-x} = (3-x)e^{-x}$

<u>lemma</u> $f^{(n)}(x) = (-1)^n \cdot (x-n) \cdot e^{-x} \quad \forall n \in \mathbb{N} \cup \{0\}$

pf) We proove this inductively.

(i) (base case) $n=0$;

$f^{(0)}(x) = (-1)^0 \cdot (x-0) \cdot e^{-x} = x \cdot e^{-x}$ (the lemma holds)

(ii) (inductive step)

Suppose that the lemma holds for $n=k$.

Then, $f^{(k+1)}(x) = \dfrac{d f^{(k)}(x)}{dx} = \dfrac{d(-1)^k(x-k)e^{-x}}{dx}$

$= (-1)^k\left(e^{-x} + (x-k)(-1)e^{-x}\right) = (-1)^k(-x+k+1)e^{-x}$

$= (-1)^{k+1}(x-(k+1))e^{-x}$ (the lemma holds)

By (i) and (ii), We conclude that the lemma holds for $n \in \mathbb{N} \cup \{0\}$ ∎

$\therefore f^{(1000)}(x) = (-1)^{1000}(x-1000) \cdot e^{-x} = (x-1000)e^{-x}$ by the lemma.

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3.5. Exercises #39

If $xy + e^y = e$, find the value of y'' at the point where $x = 0$. $\to f(0)=1$

Let $F(x,y) = xy + e^y - e = 0$

Since $\frac{dF(x,y)}{dy} = x + e^y \Big|_{x=0} = e^{y} \neq 0$, ($e^y > 0$)

y is a function of x in a neighborhood of $x=0$.

Let $y=f(x)$ be an implicit function induced by the given equation.

$\frac{d}{dx}$: $\frac{d(xy+e^y)}{dx} = \frac{de}{dx} \Rightarrow y + x\cdot y' + y'\cdot e^y = 0 \Rightarrow y' = -\frac{y}{x+e^y}$

$y' + y' + xy'' + y''\cdot e^y + y'\cdot y' e^y = 0 \Rightarrow (x+e^y)\, y'' = -(y')^2 e^y - 2y' \Rightarrow y'' = \frac{-(y')^2 e^y - 2y'}{x+e^y}$

$\therefore y''\big|_{x=0} = \frac{-\frac{1}{e^2}\cdot \cancel{e^1} - 2\left(-\frac{1}{e}\right)}{0+e} = \frac{\frac{1}{e}}{e} = \frac{1}{e^2}$

$f(0) = 1$, $f'(0) = -\frac{1}{0+e^1} = -\frac{1}{e}$

3.6. Exercises #44

Use logarithmic differentiation to find the derivative of the function.

$y = x^{\cos x}$

$\frac{d}{dx}$: $\log y = \log x^{\cos x} = \cos(x)\cdot \log x$

$\frac{y'}{y} = -\sin(x)\cdot \log x + \frac{\cos(x)}{x}$

$\Rightarrow y' = x^{\cos x}\left(-\sin(x)\cdot \log x + \cos(x)\cdot x^{-1}\right)$

3.11. Exercises #55

(a) Show that any function of the form

$$y = A \sinh mx + B \cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

$y' = mA\cosh(mx) + mB\sinh(mx) = m\left(A\cosh(mx) + B\sinh(mx)\right)$

$y'' = m\left(mA\sinh(mx) + mB\cosh(mx)\right) = m^2 y$

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(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

Let $y = A\sinh(mx) + B\cosh(mx)$.

$$\begin{cases} y(0) = A\cdot\sinh(0) + B\cdot\cosh(0) = B = -4 \\ y'(0) = m(A\cdot\cosh(0) + B\cdot\sinh(0)) = m\cdot A = 6 \\ y'' \overset{\text{by (a)}}{=} m^2 y = 9y \end{cases}$$

$\Rightarrow A = \pm 2,\ B = -4,\ m = \pm 3$ (복부호동순)

$\therefore y = 2\cdot\sinh(3x) - 4\cdot\cosh(3x)$ or $y = -2\cdot\sinh(-3x) - 4\cdot\cosh(-3x)$

4.1. Exercises #57

Find the absolute maximum and absolute minimum values of f on the given interval.

$f(t) = 2\cos t + \sin 2t$, $[0, \pi/2]$

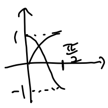
$\sin 2t \to 2\cdot\sin(t)\cdot\cos(t)$

Using Closed Interval Method.

(i) interior points of $[0, \pi/2]$

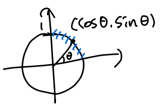
$f'(t) = 2(-\sin(t)) + 2\cos(2t) = 0$

※ $\begin{cases} \sin(A \pm B) = \sin A\cdot\cos B \pm \cos A\cdot\sin B \\ \cos(A \pm B) = \cos A\cdot\cos B \mp \sin A\cdot\sin B \end{cases}$

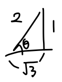


$\Rightarrow -\sin(t) + \cos(2t) = -\sin(t) + \cos^2(t) - \sin^2(t)$

$= -2\sin^2(t) - \sin(t) + 1 = 0$



$\Rightarrow \sin(t) = \frac{1}{2}$ or $\cancel{-1}$ $\because [0, \pi/2]$



$\therefore t = \frac{\pi}{6}$ is the only critical point

and critical value is $f(\frac{\pi}{6}) = 2\cos(\frac{\pi}{6}) + \sin(\frac{\pi}{3}) = 3\cdot\frac{\sqrt{3}}{2}$

$2 < \sqrt{\frac{27}{4}}$

(ii) boundary points of $[0, \pi/2]$

$f(0) = 2\cos(0) + \sin(0) = 2$

$f(\pi/2) = 2\cos(\pi/2) + \sin(\pi) = 0$

The largest number $\frac{3\sqrt{3}}{2}$ from (1) and (2) is the absolute maximum value

The smallest number 0 from (1) and (2) is the absolute minimum value

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4.2. Exercises #31

Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

(i) For $a=b$ where $a,b \in \mathbb{R}$

$|\sin a - \sin b| = 0$, $|a-b| = 0$

and so the given equation holds.

(ii) Take any $a<b$ where $a,b \in \mathbb{R}$.

Note that $\sin x \in C([a,b]) \cap D((a,b))$.

$\exists c \in (a,b)$ s.t. $\dfrac{\sin b - \sin a}{b-a} = \dfrac{d \sin x}{dx}\Big|_{x=c} = \cos c$ by MVT.

$\Rightarrow \sin b - \sin a = (b-a)\cos c$

$\Rightarrow |\sin b - \sin a| = |(b-a)\cos c|$

$= |b-a| \cdot |\cos c| \leq |b-a| \quad (\because 0 \leq |\cos c| \leq 1)$

$\therefore$ By (i) and (ii), we conclude that the given equation holds for all $a,b \in \mathbb{R}$

4.3. Exercises #17

(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

$$f(x) = x^2 - x - \ln x \quad (0,\infty)$$

(a) $f'(x) = 2x - 1 - \frac{1}{x} = 0 \Rightarrow \overset{2}{\overset{1}{2}}x^2 - \overset{1}{\overset{-1}{x}} - 1 = 0 \Rightarrow x = 1$ (critical point)

$f'(x) = \frac{1^{+}}{x}(2x^{+}+1)(x^{-}-1) < 0$ for $0<x<1$

$f'(x) = \frac{1^{+}}{x}(2x^{+}+1)(x^{+}-1) > 0$ for $x>1$

$\therefore f$ is increasing on $(1,\infty)$, decreasing on $(0,1)$

(b) $f(1) = 1-1-0 = 0$

$\therefore$ 0 is the local minimum value of f by (a) (there's no local maximum values)

(c) $f''(x) = 2 + x^{-2} > 0$ for $x>0$

$\therefore f$ is concave upwards without inflection points.

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4.3. Exercises #57

Suppose the derivative of a function f is $f'(x) = (x+1)^2(x-3)^5(x-6)^4$. On what interval is f increasing? $(-\infty, \infty)$

Critical points : $x=-1$, $x=3$, $x=6$

$f'(x) = (x\overset{+}{+}1)^2(x\overset{-}{-}3)^5(x\overset{+}{-}6)^4 < 0$ for $x<-1$

$f'(x) = (x\overset{+}{+}1)^2(x\overset{-}{-}3)^5(x\overset{+}{-}6)^4 < 0$ for $-1<x<3$

$f'(x) = (x\overset{+}{+}1)^2(x\overset{+}{-}3)^5(x\overset{+}{-}6)^4 > 0$ for $x>3$

$\therefore f$ is increasing on $(3,6)\cup(6,\infty)$

4.4. Exercises #27

Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

$$\lim_{x\to 0}\frac{(e^{x}{}^{\to 1} - 1 - x^{\to 0})^{\to 0}}{x^2{}_{\to 0}}$$

Let $f(x) = e^x - 1 - x$ and $g(x) = x^2$.

Note that (i) $f(x)$ and $g(x)$ are diff$\underline{\text{ble}}$ on $\mathbb{R}-\{0\}$

(ii) $g'(x) = 2x \neq 0 \quad \forall x \in \mathbb{R}-\{0\}$

(iii) $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$

(iv) $\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{e^x - 1^{\to 0}}{2x_{\to 0}}$

Similary, the assumption (i) to (iii) of l'Hospital's Rule holds for $f'(x)$ and $g'(x)$

and $\lim_{x\to 0}\frac{f''(x)}{g''(x)} = \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$

$\therefore \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)} = \frac{1}{2}$ by l'Hospital's Rule.

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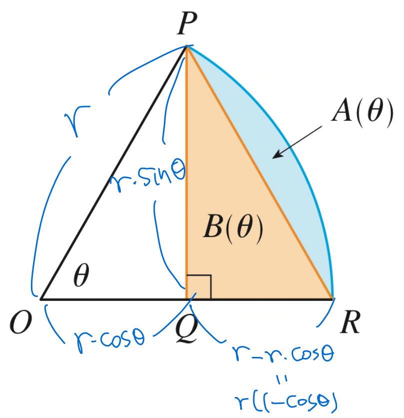
4.4. Exercises #84

The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta\to0^+} A(\theta)/B(\theta)$.



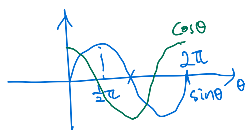
$r^2\cancel{\pi}\cdot\frac{\theta}{2\cancel{\pi}}$

$$B(\theta)=\frac{r^2\sin\theta(1-\cos\theta)}{2}$$

$$A(\theta)=\frac{r^2\theta}{2}-\frac{r^2\sin\theta}{2}$$

$$\lim_{\theta\to0^+}\frac{A(\theta)}{B(\theta)}=\frac{\overset{\to0}{\theta}-\overset{\to0}{\sin\theta}}{\underset{\to0}{\sin\theta}\,\underbrace{(1-\cos\theta)}_{\to0}}$$

Let $f(\theta)=\theta-\sin\theta$ and $g(\theta)=\sin\theta(1-\cos\theta)$



Note that (i) $f(\theta)$ and $g(\theta)$ are diff$^{\underline{ble}}$ on $\overset{\theta\text{ near }0}{(0,\frac{1}{2}\pi)}\subset\overset{\text{domain}}{(0,2\pi)}$

(ii) $g'(\theta)=\overset{0<}{\cos\theta}\underbrace{(\overset{<1}{1}-\overset{0<}{\cos\theta}^{<1})}_{0<\ <1}+\overset{0<\ <1}{\sin^2\theta}\neq0$ on $(0,\frac{1}{2}\pi)$

(iii) $\lim_{\theta\to0^+}f(\theta)=0=\lim_{\theta\to0^+}g(\theta)$

(iv) $\lim_{\theta\to0^+}\frac{f'(\theta)}{g'(\theta)}=\lim_{\theta\to0^+}\frac{\cancel{1-\cos\theta}}{\cos\theta\cancel{(1-\cos\theta)}+\underset{1+\cos\theta}{\cancel{(1-\cos^2\theta)}}}=\lim_{\theta\to0^+}\frac{1}{1+2\cdot\cos\theta}=\frac{1}{3}$

$\therefore\lim_{\theta\to0^+}\frac{f(x)}{g(x)}=\lim_{\theta\to0^+}\frac{f'(x)}{g'(x)}=\frac{1}{3}$ by l'Hospital's Rule

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4.5. Exercises #46

Use the guidelines of this section to sketch the curve.

$y = x - \ln x$

A. Domain : $(0, \infty)$

B. Intercepts : None

C. Symmetry : None

D. Asymptotes : $\lim_{x \to 0+} (\overset{\nearrow 0}{x} - \overset{\nearrow -\infty}{\ln x}) = \infty \Rightarrow x = 0$ is a vertical asymptote.

E. Intervals of Increase or Decrease

: $y' = 1 - \frac{1}{x} = 0 \Rightarrow$ Critical point is $x = 1$

$\left.\begin{array}{l} y' = 1 - \frac{1}{x} < 0 \text{ for } x \in (0,1) \\ y' = 1 - \frac{1}{x} > 0 \text{ for } x \in (1, \infty) \end{array}\right\} \Rightarrow$ y is increasing on $(1, \infty)$ and decreasing on $(0,1)$

F. Local Maximum or Minimum Values

: $y|_{x=1} = 1$

$\Rightarrow y = 1$ is the only minimum value and there's no maximum values.

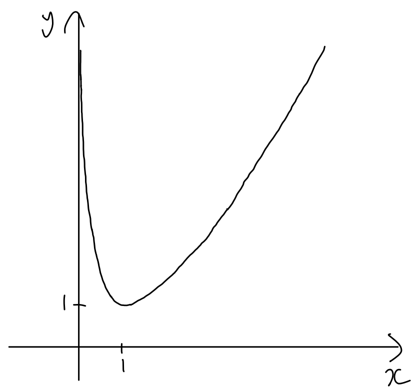
G. Concavity and Points of Inflection

: $y'' = \frac{1}{x^2} \neq 0$

$y'' = \frac{1}{x^2} > 0$ for $x \in (0, \infty)$

$\Rightarrow$ y is concave upward without inflection point

H.



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4.9. Exercises #41

Find f.

$$f''(\theta) = \sin\theta + \cos\theta, \quad f(0) = 3, \quad f'(0) = 4$$

$$\left.\begin{aligned} f'(\theta) &= -\cos\theta + \sin\theta + C_1 \\ f'(0) &= -1 + 0 + C = 4 \end{aligned}\right\} C_1 = 5$$

$$\left.\begin{aligned} f(\theta) &= -\sin\theta - \cos\theta + 5\theta + C_2 \\ f(0) &= 0 - 1 + 0 + C_2 = 3 \end{aligned}\right\} C_2 = 4$$

$$\therefore f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$$