STS2005(09) 미적분학I (HW6) DueDate(2020/5/13)

학과 : 경제학과 학번 : 20160563 이름 : 송진아

11.1 Exercises #49

Determine whether the sequence converges or diverges.
If it converges, find the limit.

$$a_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\ln(2n^2+1) - \ln(n^2+1)\right) = \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right)$$

$$\underset{\ln \text{ is Continuous}}{=} \ln\left(\lim_{n\to\infty}\frac{2n^2+1}{n^2+1}\right) = \ln 2$$

$\therefore \{a_n\}$ converges to $\ln 2$

11.1 Exercises #81

Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

$3-\frac{2}{5}$ $\frac{13}{5}$

Observe that $a_1 = 1,\ a_2 = 2,\ a_3 = 2.5,\ a_4 = 2.6,\ \cdots$

Claim $0 < a_n < a_{n+1} < 3,\ \forall n \in \mathbb{N}$.

We prove this inductively.

(i) (base case) $n=1$;

$0 < a_1 = 1 < a_2 = 3 - \frac{1}{1} = 2 < 3$ (O.K.)

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(ii) (inductive step)

Suppose $0 < a_k < a_{k+1} < 3$. Then,

$$\begin{array}{rl} & a_{k+1} = 3 - \frac{1}{a_k} \\ - & a_{k+2} = 3 - \frac{1}{a_{k+1}} \\ \hline & a_{k+1} - a_{k+2} = \frac{1}{a_{k+1}} - \frac{1}{a_k} < 0 \end{array}$$

Hence, $a_{k+1} < a_{k+2}$.

By the hypothesis, we have $a_{k+1} > 0$ and so

$$a_{k+2} = 3 - \frac{1}{a_{k+1}} < 3.$$

Thus, $0 < a_{k+1} < a_{k+2} < 3$.

By the claim, $\{a_n\}$ is increasing and $a_n < 3 \ \forall n \in \mathbb{N}$.

MST gives that $\{a_n\}$ converges to $\sup\{a_n \mid n \in \mathbb{N}\}$.

Let $\alpha = \sup\{a_n \mid n \in \mathbb{N}\} = \lim_{n\to\infty} a_n \geq a_1 = 1 > 0$.

Taking limit to both sides of $a_{n+1} = 3 - \frac{1}{a_n}$,

we get $\alpha = 3 - \frac{1}{\alpha}$. $\alpha^2 - 3\alpha + 1 = 0$

$$\Rightarrow \alpha = \frac{3+\sqrt{5}}{2}, \cancel{\frac{3-\sqrt{5}}{2}} \quad (\because \alpha \geq 1)$$

Therefore, $\{a_n\}$ converges to $\frac{3+\sqrt{5}}{2}$ ▨

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11.2 Exercises #85

If $\Sigma\, a_n$ is convergent and $\Sigma\, b_n$ is divergent, show that the series $\Sigma\,(a_n + b_n)$ is divergent. [*Hint:* Argue by contradiction.]

Suppose to the contrary that $\Sigma(a_n+b_n)$ is convergent.

$\Sigma b_n = \underbrace{\Sigma(a_n+b_n)}_{\in\mathbb{R}} - \underbrace{\Sigma a_n}_{\in\mathbb{R}} \in\mathbb{R}$ ⊗

$\therefore\ \Sigma(a_n+b_n)$ is convergent.

11.3 Exercises #21

Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Let $f(x) = \dfrac{1}{x\cdot \ln x}$.

$u = \ln x$
$du = \frac{1}{x}dx$

Note that $f(x)$ is continuous, positive, decreasing function on $[2,\infty)$ and

$$\int_2^\infty f(x)\,dx = \lim_{t\to\infty}\int_2^t f(t)\,dt = \lim_{t\to\infty}\int_{\ln 2}^t \frac{1}{u}\,du \ (\text{where } u=\ln x)$$

$$= \lim_{t\to\infty}\left[\ln u\right]_{\ln 2}^t = \lim_{t\to\infty}(\ln t - \ln(\ln 2)) = \infty$$

$\therefore\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ is divergent since $\int_2^\infty f(x)\,dx$ is divergent. ($\because$ integral test)

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11.4 Exercises #5

Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$$

$$\frac{n+1}{n\sqrt{n}} > \frac{n}{n\sqrt{n}} > 0, \ \forall n \in \mathbb{N}.$$

$\sum_{n=1}^{\infty} \frac{n}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent by p-series test.

By comparison test, $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ is divergent.

11.4 Exercises #29

Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

$$\frac{1}{n!} = \frac{1}{\underbrace{1\cdot 2\cdot 3\cdots n}_{n}} \leq \frac{1}{\underbrace{1\ 2\cdot 2\cdots 2}_{n}} = \left(\frac{1}{2}\right)^{n-1}, \ \forall n \in \mathbb{N}$$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ converges to $\frac{1}{1-\frac{1}{2}} = 2$,

$\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by comparison test.

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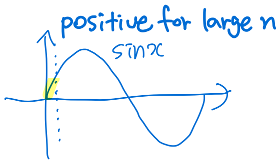
11.4 Exercises #45

If $\Sigma\, a_n$ is a convergent series with positive terms, is it true that $\Sigma \sin(a_n)$ is also convergent?

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1$$



By the limit comparison test,

$\Sigma \sin(a_n)$ is also convergent since $\Sigma\, a_n$ is convergent.

11.5 Exercises #11

Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$$

Note that $\displaystyle \frac{d}{dx} \frac{x^2}{x^3+4} = \frac{2x(x^3+4) - x^2(3x^2)}{(x^3+4)^2}$

$$= \frac{2x^4 + 8x - 3x^4}{(x^3+4)^2}$$

$$= \frac{x(2-x)(4+4x+x^2)}{(x^3+4)} \le 0 \text{ on } [2,\infty)$$

and so $\displaystyle \frac{x^2}{x^3+4}$ is monotone decreasing on $[2,\infty)$.

So that $0 \le b_{n+1} \le b_n$, $\forall n \in \mathbb{N} - \{1\}$, where $b_n = \dfrac{n^2}{n^3+4}$.

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Note also that $\lim_{n\to\infty} \frac{n^2}{n^3+4} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{4}{n^3}} = 0$.

By convergence test for alternating series,

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4} = \frac{1}{5} + \underbrace{\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}}_{\text{converges}} \text{ converges.}$$