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11.6 Exercises #13

Use the Ratio Test to determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$$

Using the Ratio Test with $a_n = \dfrac{10^n}{(n+1)4^{2n+1}}$, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{10^{\cancel{n+1}}}{(n+2)4^{\cancel{2n+3}\,2}}}{\dfrac{\cancel{10^n}}{(n+1)\cancel{4^{2n+1}}}}\right| = \lim_{n\to\infty}\frac{10(n+1)}{16(n+2)} = \frac{10}{16} < 1$$

Thus $\displaystyle\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent.

11.6 Exercises #45

(a) Show that $\sum_{n=0}^{\infty} x^n/n!$ converges for all x.

Using the Ratio Test with $a_n = \dfrac{x^n}{n!}$, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{x^{\cancel{n+1}}}{(n+1)\cancel{!}}}{\dfrac{\cancel{x^n}}{\cancel{n!}}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1 \quad \forall x \in \mathbb{R}.$$

Thus $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is absolutely convergent for all x.

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(b) Deduce that $\lim_{n\to\infty} x^n/n! = 0$ for all x.

Since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges by (a), $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

11.8 Exercises #3

Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty} (-1)^n n x^n$$

Let $a_n = (-1)^n n x^n$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{\cancel{n}+1}(n+1)x^{\cancel{n}+1}}{\cancel{(-1)^n} n \cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x}{n} \right| = |x|$$

By the Ratio Test, the series converges if $|x|<1$ and diverges if $|x|>1$.

For $|x|=1$, consider the following cases :

(i) $x=1$;

$$\sum_{n=1}^{\infty}(-1)^n n(1)^n = \sum_{n=1}^{\infty}(-1)^n n \text{ diverges } (\because \lim_{n\to\infty}(-1)^n n \neq 0)$$

(ii) $x=-1$;

$$\sum_{n=1}^{\infty}(-1)^n n(-1)^n = \sum_{n=1}^{\infty}(-1)^{2n} n = \sum_{n=1}^{\infty} n \text{ diverges } (\because \lim_{n\to\infty} n \neq 0)$$

Therefore, the radius of convergence is 1 and the interval of convergence is $(-1, 1)$.

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11.8 Exercises #15

Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$$

Let $a_n = \frac{(x-2)^n}{n^2+1}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)^2+1}}{\frac{\cancel{(x-2)^n}}{n^2+1}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)(n^2+1)}{(n+1)^2+1}\right| = |x-2|$$

By the Ratio Test, the series converges if $|x-2|<1$ and diverges if $|x-2|>1$.

For $|x-2|=1$, consider the following cases

(i) $x=3$; $\sum_{n=0}^{\infty}\frac{1^n}{n^2+1} = \sum_{n=0}^{\infty}\frac{1}{n^2+1}$

Note that $0<\frac{1}{n^2+1}\le\frac{1}{n^2}$ $\forall n\in\mathbb{N}\cup\{0\}$.

Since $\sum_{n=0}^{\infty}\frac{1}{n^2}$ is p-series with $p=2$, $\sum_{n=0}^{\infty}\frac{1}{n^2}$ converges.

Then $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ converges by the Comparison Test.

(ii) $x=1$; $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$

Note that $0\le\frac{1}{(n+1)^2+1}\le\frac{1}{n^2+1}$ $\forall n\in\mathbb{N}\cup\{0\}$ and $\lim_{n\to\infty}\frac{1}{n^2+1}=0$.

By the Alternating Series Test, $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$ converges.

Therefore, the radius of convergence is 1 and the interval of convergence is $[1,3]$.

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11.9 Exercises #5

Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{2}{3-x}$$

$$= \frac{\frac{2}{3}}{1-\frac{x}{3}}$$ (Sum of Geometric Series with $a=\frac{2}{3}$, $r=\frac{x}{3}$ when $\left|\frac{x}{3}\right|<1$)

$$= \sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n$$

It converges when $\left|\frac{x}{3}\right|<1$

so the interval of convergence is $(-3,3)$.

11.9 Exercises #15

Find a power series representation for the function and determine the radius of convergence.

$$f(x) = \ln(5-x)$$

$$= \int \frac{1}{5-x}\,dx$$ ($\frac{1}{5-x}$ is the sum of Geometric Series when $|x|<1$)

$$= \int \sum_{n=0}^{\infty} -\frac{1}{5}\left(\frac{x}{5}\right)^n dx$$

interval of convergence of $f'(x)$

$$= \sum_{n=0}^{\infty} \int -\frac{x^n}{5^{n+1}}\,dx$$ ($\because$ Term-by-Term Integration Theorem)

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} + C$$

By putting $x=0$ both sides of the equation, we get $C=\ln 5$.

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Let $a_n = \frac{x^{n+1}}{5^n(n+1)}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+2}}{5^{n+1}(n+2)}}{\frac{x^{n+1}}{5^n(n+1)}}\right| = \left|\frac{x(n+1)}{5(n+2)}\right| \to \frac{x}{5} \text{ as } n\to\infty$$

By the Ratio Test, the series converges if $\left|\frac{x}{5}\right| < 1$, diverges if $\left|\frac{x}{5}\right| > 1$.

Therefore, $f(x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} + \ln 5$ and the radius of convergence is 5.

11.10 Exercises #18

Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \longrightarrow 0$.] Also find the associated radius of convergence.

$f(x) = \cosh x$

Since $f^{(2n+1)}(x) = \sinh x$, $f^{(2n)}(x) = \cosh x$ where $n \in \mathbb{N} \cup \{0\}$, we have

$$f^{(n)}(x) = \frac{e^x + (-1)^n e^{x}}{2} \quad \text{and so} \quad f^{(n)}(0) = \frac{1+(-1)^n}{2}.$$

Thus the Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2\cdot n!} x^n$$

Let $a_n = \frac{1+(-1)^n}{2\cdot n!} x^n$. Then

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$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{1+(-1)^{\cancel{n+1}}x^{\cancel{n+1}}}{\cancel{2}(n+1)\cancel{!}}}{\dfrac{1+\cancel{(-1)^n}\cancel{x^n}}{\cancel{2}\,\cancel{n!}}}\right| = \left|\frac{1-x}{2(n+1)}\right| \to 0 \text{ as } n\to\infty$$

By the Ratio Test, the series converges $\forall x \in \mathbb{R}$ and the radius of convergence is ∞.

11.10 Exercises #61

Use series to evaluate the limit.

$$\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}$$

Note that the Maclaurin Series for $\ln(1+x)$ is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$

$$\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2} = \lim_{x\to 0}\frac{x-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}}{x^2} = \lim_{x\to 0}\left(\frac{1}{x}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-2}}{n}\right)$$

$$= \lim_{x\to 0}\left(\cancel{\frac{1}{x}}-\left(\cancel{\frac{1}{x}}-\frac{1}{2}+\underbrace{\frac{x}{3}-\frac{x^2}{4}+\cdots}_{0}\right)\right)$$

$$= \frac{1}{2}$$