Effect of Network Topology in Opinion Formation Models

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Abstract. Simulations of consensus formation in networks of interacting agents have recently demonstrated that convergence to a small number of opinion clusters is more likely when the network is adaptive rather than static. In order to better model realistic social networks, we have extended an existing model of such a process, by the introduction of a parameter representing each agent's level of 'authority,' based on their opinion relative to the overall opinion distribution. Here we aim to determine the importance of initial network topology for opinion formation in this model, using two distinct initial network topologies: an Erdős-Rényi random network, and the Watts-Strogatz small-world network. It is shown that marked differences exist in statistics of the model after opinion convergence. These include the number of interactions between agents needed to reach consensus, as well as a clear influence of opinion tolerance on the network's clustering coefficient, mean shortest path, and degree distribution after convergence. This latter effect suggests some interesting possibilities regarding the topology of 'converged' networks.

Keywords: Opinion Formation, Consensus Formation, Adaptive Network, Social Network Analysis, Agent-based Model.

1 Introduction

Recent sociophysics simulations, aimed at investigating opinion formation and consensus in terms of the emergence of global phenomena based solely on local interactions, have illustrated the importance of adaptability in the connections between agents in social networks [1,2]. In this work a random graph network [3] is used to simulate interactions between individuals, to map the difference between social groups that are able to dynamically reform, and those which are not. As described in Section 2, nodes in the network represent agents and edges in the network represent a relationship between pairs of agents who are neighbours that may potentially communicate. Each agent has a continuously valued opinion that can be altered after an interaction between pairs of neighbours, provided the two neighbours' opinions are within some tolerance. Investigations within this model included varying population sizes, as well as levels of adaptability, i.e. the likelihood of an agent cutting a link to a neighbour and creating a

C. Guttmann, F. Dignum, and M. Georgeff (Eds.): CARE 2009/2010, LNAI 6066, pp. 114–124, 2011.

new one, as well as the number of interactions required to achieve the network's final state [1]. Simulation results indicated that global consensus can only be formed in a static model when agents have relatively high tolerance for opinion difference, whereas when the ability for nodes to break old relationships and form new ones is introduced, consensus with relatively small tolerances becomes likely.

Consideration of this model from a sociological view raised two interesting questions. Firstly, the affect on consensus of a more realistic 'variable' opinion-convergence factor, and secondly that of external determinants such as *perceived authority disparity* between nodes. In considering the first point, there is general acceptance [4,5] that when a consensus is reached between individuals (or indeed groups), as often as not that consensus is not the result of an absolute meeting of the minds. It is rather a working agreement, which originates from, as well as facilitating, a more subtle convergence of actual opinion. These adaptations were shown to provide notable, and arguably more realistic outcomes for opinion convergence modeling [6].

A further factor examined in this paper is the affect of network topology on the process of convergence, as well as any effect that process has on topology. Precisely how to model local and universal social networks has been a topic of research for some time. From Milgram, in the 60s [7], through Strogatz and Watts' seminal work [8] and to this day, it has generally been agreed that social networks are not entirely unstructured by nature. *Exactly* what form such networks take, and precisely how the formation paradigm can be defined, is as yet not clear.

The original topology in [1] (described in detail in the next section) was an Erdős-Rényi network model, which produces an artificial random network, in the sense that all nodes are likely to have the same number of neighbours, and there is no local clustering. In this paper we compare and contrast consensus formation in both the original Erdős-Rényi and the Watts-Strogatz [8] network models, using their network creation methods. We note differences in simulation data for each, both during the process of opinion convergence, and in the final state of the network after convergence.

Additionally we consider the effect of initial opinion distribution. In the previous model opinion was uniformly distributed. Here we compare this with a more polarized opinion distribution, whereby opinions are initially distributed with a preference for either extreme. It is assumed that this model will better represent more divisive issues faced by the population.

With the inclusion of these factors, the model considered has potentially greater scope for usefulness in a broad range of applications. This might include task allocation problems, where it has been shown that employment of an always optimistic method of estimating team efficiency will still result in an optimal solution for task assignment [9]. Another example is semi-autonomous sensor arrays with adaptable positioning and team membership capability, where tight cordon array patterns could enhance target acquisition and lock.

2 Model Formulation and Extensions

The consensus formation model of [1] is predicated on a simple and virtually axiomatic principle. If two agents (network nodes) each have opinions that differ only within a tolerance that would make communication productive, and they are offered the opportunity to communicate, the disparity between the opinions will be reduced.

Several variables are integral to the model of [1], which is initially an Erdős-Rényi random graph. We assume the network has size N, and initially has average degree $\bar{k} = 10$. In each interaction, a randomly chosen pair of neighbours is selected. After the t-th interaction of the entire network, agent i has opinion, $o(i,t)$ $(i \in 1,..,N)$, which is an initially uniformly random value between 0 and 1, representing each node's opinion. In this paper we consider only a single opinion. Tolerance, d, is an arbitrarily set value used to determine if opinions are close enough to make communication between neighbours productive, or far enough apart to require dynamically altering the network, by breaking a link between those two neighbours, and the formation of a new one. Communication factor w is an arbitrarily chosen constant between 0 and 1, such that with probability w there will be an opportunity for communication between nodes in any given interaction and with probability $1-w$ the connection will be broken and a new one created, if the opinion difference is larger than d.

In [1] the opinions of interacting nodes i and j update whenever $|o(i,t) - o(j,t)| \leq d$ according to

$$\begin{aligned} o(i,t+1) &= o(i,t) + \mu\left[o(j,t) - o(i,t)\right] \\ o(j,t+1) &= o(j,t) - \mu\left[o(j,t) - o(i,t)\right], \end{aligned} \tag{1}$$

where μ is an opinion-convergence factor. Interactions continue until the final state of the network is reached, which is defined as the point at which further interactions will not result in any change to the current status of the network.

Throughout [1], the convergence factor is set at $\mu = 0.5$, and therefore a complete local agreement is reached every time two interacting agents alter their opinions. When discussing this approach, [1] make it clear that use of this complete agreement model, rather than one which makes room for variable convergence, was chosen primarily to simplify the process. However as mentioned in the introduction, we aim to adapt the model to be more analogous to 'real life', by employing a randomly chosen value $\mu \in [0, 0.5]$.

While the adoption of a variable opinion-convergence factor provides a further measure of realism, it is argued that this does not go far enough, as it only provides a representation of internal sociological factors. As far back as Aristotle, it has been recognised that there are three modes of persuasion: pathos (appealing to the target's emotions), logos (applying a logical argument), and ethos (affect of the 'persuader's' perceived authority) [10]. Indeed [11] argues that all three are present in any effective form of persuasive discourse. The first two are catered for in the model of [1], as it is argued that they can be considered internal in nature, provided a random opinion-convergence factor is used, but not the effect of authority, which is a determinant derived, at least in part, from external factors. There is a considerable body of evidence to support the argument that the relative authority of the agents that are communicating would have an impact on the convergence of opinions.

In order to model this, we introduce a new variable to represent the 'authority level' of the i–th node, $a(i,t) \in [0,1]$—see [6] for more details on how $a(i,t)$ is calculated. Authority is determined by the relative popularity of a node's opinion within its peer group, and the size of that peer group in relation to the total population [6]. The magnitude of the difference in authority between two communicating nodes, $|A_{i,j}| := |a(i,t) - a(j,t)| \in [0,1]$, is used to skew the altered opinions toward the more authoritative node's opinion, via the following extension of Eqn. (1). When two neighbours interact, they alter their opinions only if $|o(i,t) - o(j,t)| \leq d + |A_{i,j}/3|$, in which case

$$\begin{aligned} o(i,t+1) &= o(i,t) + \mu(1 - A_{i,j})\,[o(j,t) - o(i,t)] \\ o(j,t+1) &= o(j,t) - \mu(1 + A_{i,j})\,[o(j,t) - o(i,t)]\,. \end{aligned} \tag{2}$$

In our definition of authority difference, if node i has more authority, then $A_{i,j}$ is positive, while if node j has more authority, $A_{i,j}$ is negative. Hence, in the extreme case where $A_{i,j} = 1$, node i's opinion is unchanged, while node j's is changed to $o(j,t+1) = o(j,t)(1-2\mu) + 2\mu o(i,t))$. Thus, when $\mu = 0.5$, node j takes on node i's opinion exactly. Otherwise, node j's opinion change is doubled relative to the model of [1] with the same μ. By the criteria for convergence, if both nodes have the same relative authority, then tolerance is simply d. Otherwise, note that authority disparity is also used to affect the level of tolerance, which is increased by one third of the authority difference.

Our refined model of opinion convergence is applied in two distinct initial network topologies. We then analyze changes in network topology caused by opinion convergence, by considering *clustering coefficient*, as defined by Watts and Strogatz [8], *mean shortest path*, and *degree distribution.*

We consider undirected graphs. The first topology we consider is created by the Erdős-Rényi method where n nodes are randomly connected with a probability k. The resultant average degree of the network (the average number of connections from a node to any other node) is denoted as $\bar{k}$. The second topology is the Watts-Strogatz 'small world' (SW) model [8]. Creation of this topology begins with a lattice network of n nodes, where each node i is connected to k nodes $i_1, i_2 ... \, i_{\frac{k}{2}}$ and $i_{-1}, i_{-2} ... \, i_{-\frac{k}{2}}$. Random re-wirings are then applied to the network with a constant probability of b. Higher b produces a final network topology that is less clustered and more like the Erdős-Rényi model. This process creates a so-called 'small world' network, because it results in a network with a higher clustering coefficient than an Erdős-Rényi random network model, and an average shortest path only slightly longer than the Erdős-Rényi random network model.

Our initial results indicated a profound effect on topology by the consensus formation process, and thus a further extension to the model was deemed necessary. The ability to repeat the process of consensus formation—without

Table 1. Quick reference guide for variables

Variables/Acronyms	**Definitions**
Opinion, $o(i,t)$	Continuous variable between 0 and 1, representing the opinion of node i after t interactions, on one specific issue. The initial opinions, $o(i,0)$ may be uniformly or Beta distributed.
Tolerance, d	The maximum difference between node opinions before communication becomes disallowed.
Convergence factor, μ	A uniform continuous variable between 0 and 0.5. Determines the degree of convergence prior to adjustment for authority (that is, how each node's opinion changes), when communication has occurred.
Average degree, k	Average number of connections from a node to other nodes in the network.
Communication factor, w	Determines the probability with which a link between nodes is broken, and a new link formed to another randomly chosen node when two nodes' opinions are not within tolerance.
Authority, $a(i,t)$	Node i's authority in relation to all other nodes, after t interactions. Derived from the popularity of a node's opinion within its peer group, and the size of that peer group in relation to the sample population.
MSP	Mean shortest path: the average of all shortest paths between all node pairs in a network .
CC	Clustering coefficient: the average of all local clustering coefficients within a network as defined by [8].
DD	Degree distribution: the frequency of node degree occurrences within a network, where the degree for a single node is the total number of connections from that node to other nodes.
Time slice, t	Term introduced by [1] to refer to a single iteration of the potential communication process.
Topology	Refers to either the Erdős-Rényi random network model [1] (ER), or the Watts-Strogatz small-world model (SW) [8].

resetting to the original topologies—was added. This involves the replacement of 'converged' opinions with a new set of randomly chosen opinions. This models the effect of introducing a new topic for consideration, while maintaining the topology changes which occurred in the original process of opinion convergence.

A further factor is introduced to offer a comparison with more polarized opinion distributions. We achieve this by use of the transformation $o = 0.5 + 0.5\sin((r-0.5)\pi)$, where $r \in [0,1]$ is a uniformly distributed variable. This results in opinions that have a Beta distribution, $\beta(0.5, 0.5)$.

Table 1 provides a quick reference guide for terms and variables used in this model.

3 Results

This section contains simulation results that illustrate the effect of altering the various parameters described in Section 2. All results are for a network with $N = 1000$ agents, $w = 0.25$ and $\bar{k} = 10$, and ensemble-averaged over 50 different runs.

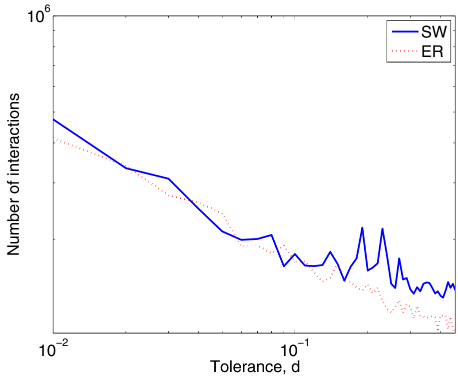


Fig. 1. Comparison of total number of required interactions for both topologies as a function of tolerance, d

Our first finding, as illustrated in Fig. 1, is that more time slices (node interactions) are required for the SW network to reach convergence than the ER network, for d larger than about $d = 0.12$. For d smaller than this, there is negligible difference.

We now compare the clustering coefficient, mean shortest path, and degree distribution before and after the opinion convergence process, for the two different initial network topologies and initial opinion distributions.

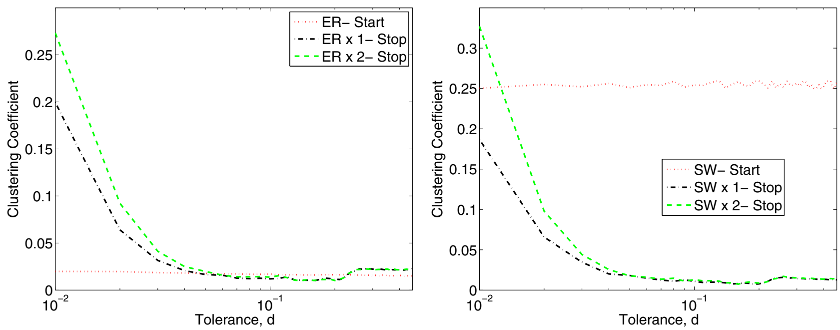
3.1 Topology Before and After the Opinion Formation Process

Figs. 2, 3 and 4 show that the clustering coefficient (CC), degree distribution (DD) and mean shortest path (MSP) all are markedly changed by the opinion convergence process, with final values highly dependent on the tolerance, d, for both initial topologies.

As d increases, at $d \simeq 0.035$ the final clustering coefficient, as shown in Fig. 2—see also Fig. 5(a)—for both topologies drops below the Erdős-Rényi starting point. This reduction continues with increasing d, until $d \simeq 0.12$ when the random network maintains a rough parity with its initial level, with the SW slightly higher.

When the opinion convergence process is run twice, the final clustering coefficient for both topologies is higher than when the process is run once. However, this difference decreases as d increases, and at $d \simeq 0.03$ the difference all but disappears.

Fig. 3 shows the degree distribution for the network before and after opinion formation, for each topology, for several values of d. As is expected, the degree distribution in the initial state is spread wider for the ER topology (most node degrees between 0 and 20), than it is for a SW network (between 5 and 15). We find that the final degree distribution (after opinion formation) for *both* initial topologies has an inverse relationship to tolerance (in the sense that the

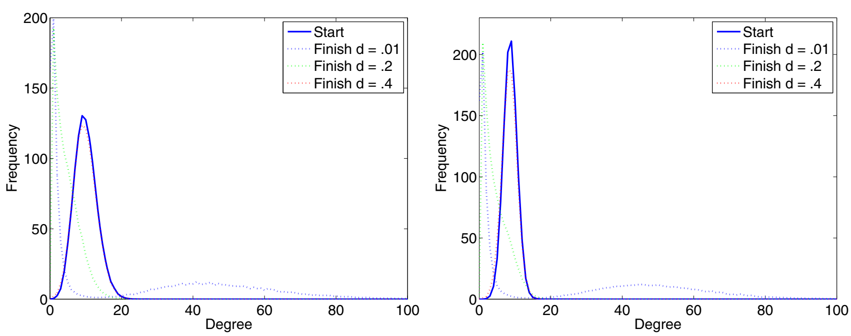


(a) ER network (b) SW network

Fig. 2. Initial and final clustering coefficients for single and double consensus process runs, as a function of tolerance, d

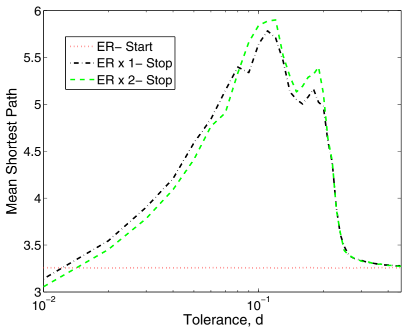
distribution is skewed towards lower degrees for smaller d), resulting in a more 'power-law-like' distribution with very small d. With $d = 0.01$, less than one node (on average) has a degree higher than 100, and most nodes have a degree within the range 0–20. Both topologies also display a rise and fall in the number of nodes with a degree approximately in the range 38 – 82, peaking at approximately 20 nodes with degree of of 60. For a larger tolerance ($d = 0.2$), a similar skewing of the degree distribution can be seen. However for larger tolerance $d = 0.4$, there is almost no change from the initial network, which is to be expected, since for large tolerance there are very few chances for nodes to alter their neighbours.

Fig. 4 shows that the effect of tolerance is more dramatic for MSP than it is for CC and DD. For a SW topology MSP is decreased by the opinion convergence process at low tolerance, as it is to a lesser extent for an ER topology. However, there is a sharp rise in MSP in comparison with the initial MSP for an ER

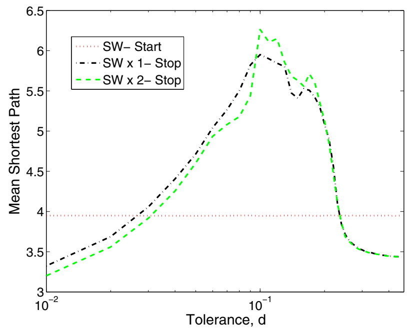


(a) ER network (b) SW network

Fig. 3. Comparison of final degree distributions, as a function of tolerance, d



(a) ER network



(b) SW network

Fig. 4. Initial and final mean shortest paths for single and double consensus process runs, as a function of tolerance, d

network, and a fall in MSP for a SW network, as d increases from its smallest value. Then for larger d, the MSP becomes larger than the initial value for both topologies, before dropping off sharply at the point of phase transition near $d = 0.3$. This results in similar MSPs after convergence for all d for both topologies. The value $d \simeq 0.3$ corresponds to a point of phase transition, above which the final network usually reaches a global consensus, as noted in [6].

It is also notable that the final MSP for the both topologies are both slightly lower when the process is run twice, compared to when it is run once.

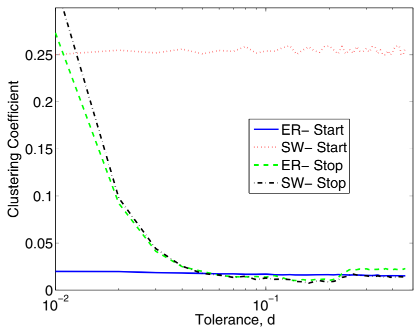
3.2 Erdős-Rényi (ER) Network versus Small World (SW) Network

Fig. 5 shows the final CC and MSP after the opinion formation process, with both initial topologies on the same axes. It is clear that the final topology is almost the same for each initial topology, although for small d the final MSP appears to be slightly smaller for the initially ER network than it is for the initially SW network, and a similar result is apparent for CC.

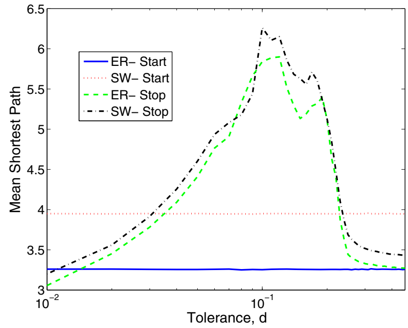
These results indicate that the final network topology due to opinion formation may be nearly independent of the initial network topology.

3.3 Effect of Initial Opinion Distribution

We found that changing the distribution of initial opinions resulted in a significant change in the value of tolerance above which the network always reaches global convergence. This is illustrated in Fig. 6, which shows (following [1]) the fraction of the nodes in the network contained in the two largest opinion clusters in the final state of the network, for an ER network. An almost identical result (not shown) was obtained for an initially SW network.



(a) Clustering coefficient



(b) Mean shortest path

Fig. 5. Comparison of the effect of initial topology on mean shortest path and clustering coefficient, with double opinion convergence, as a function of tolerance, d

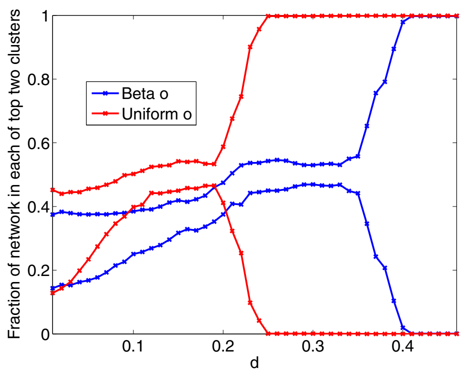


Fig. 6. The effect of initial opinion distribution on the top two opinion cluster sizes, as a function of d, for an ER network

It is clear that introducing a more polarized initial opinion distribution results in a markedly larger tolerance level being required before complete convergence of opinion is reached for *both* initial topologies. Complete opinion convergence is achieved for $d \simeq 0.24$ for uniformly distributed initial opinions, and at $d \simeq 0.36$ using a beta distribution.

4 Discussion

It is clear from our results that initial network topology may have only a limited role to play in the consensus formation process. It is equally clear however that the process itself affects the final topology. We begin by considering the impact of tolerance level. In our original paper [6], and in [1], the primary aspect of study was the formation of consensus. It was shown that a larger tolerance was able to compensate for the restrictions of a static network, and consensus required less iterations in an adaptive network.

In this paper we are concerned with topology, and consider results across a range of metrics, *i.e.* clustering coefficient (Fig. 2), degree distribution (Fig 3) and mean shortest path (Fig. 4). These have shown that the lower the tolerance level the more impact there is both by and on topology. To this end it is considered advantageous to maintain a low tolerance when deliberately trying to manipulate network topology. This makes sense from a sociological perspective; if tolerance is generally low within a given population, it follows that they tend to form relatively tight cliques within the broader population, maintaining rather narrow opinion divergency.

Another outcome of note is that for a low tolerance, the ER topology requires only one consensus process to reach a 'real world' like outcome. This may be defined loosely as a network with a small world topology (high clustering coefficient compared to the initial topology) and a 'power-law-like' degree distribution. In contrast, a small world topology requires two iterations of the process to get this effect at small tolerance—see Fig. 2(b). The first opinion formation process results in a shorter MSP, but it is only the second process that results in a higher clustering coefficient than the initial SW network. This suggests the possibility that the process of convergence, if run enough times and with an appropriately low tolerance, results in a specific topology type, which is robust in terms of initial topology type.

Much research has been carried out into methods for 'growing' network topologies which have a power law degree distribution. Obviously there are benefits in such networks, particularly in terms of path length. Also such networks are statistically unlikely to be overly affected by removal of a randomly chosen node. The one major drawback is that in the extreme case of loss of 'super-nodes' (nodes with very high degree), the results can be catastrophic. If a network was able to form or re-form such a topology, without adding or removing nodes—an outcome the process of convergence seems to provide—networks required to display the properties inherent with this type of topology would be considerably more robust.

The effect of initial opinion distribution is not surprising from a sociological viewpoint. What the results show is that if a network has two polarized opinion groups, then more interactions and a higher tolerance will be required to reach a complete consensus. However, it is an interesting phenomena from a topology manipulation perspective, suggesting that manipulation of specific variables can be used to change or maintain the network architecture.

While more extensions to our work are required, our results suggest that precise topologies may be created by introduction of specific types of manipulation algorithms to existing node variables in adaptive networks.

Acknowledgment

Mark D. McDonnell is funded by an Australian Research Council Post Doctoral Fellowship, DP0770747.

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