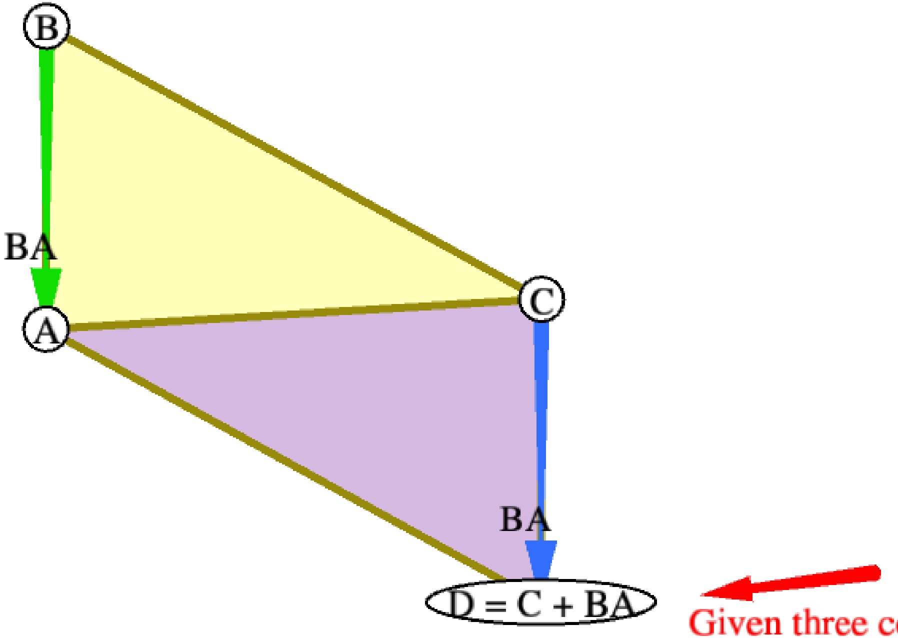
PLAYING WITH POINTS AND VECTORS



Given three corners of a parallelogram, calculate the fourth corner

CS3451 FALL 2020

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PHSE 1: Problem statement

Given three points (A, B, C), find a fourth point D so that the set of points {A,B,C,D} makes up the corner points of a parallelogram.

COMMENTS:

In order to accurately calculate the fourth point, we assume the following:

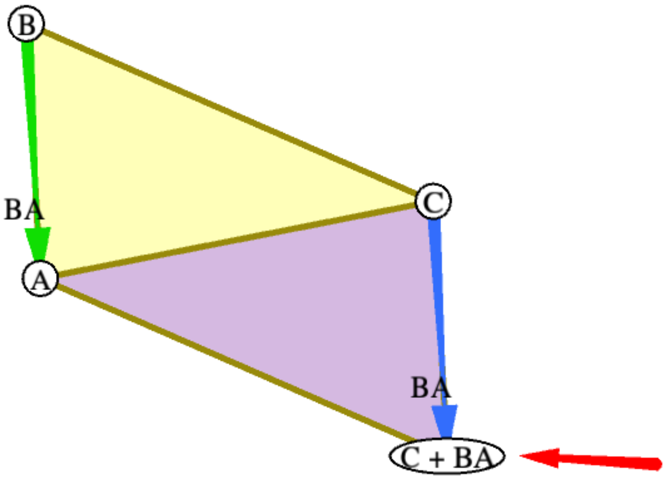
i. The initial points (A, B, C) are not colinear.

ii. The set of points {A, B, C, D} are ordered clockwise or anticlockwise around the parallelogram.

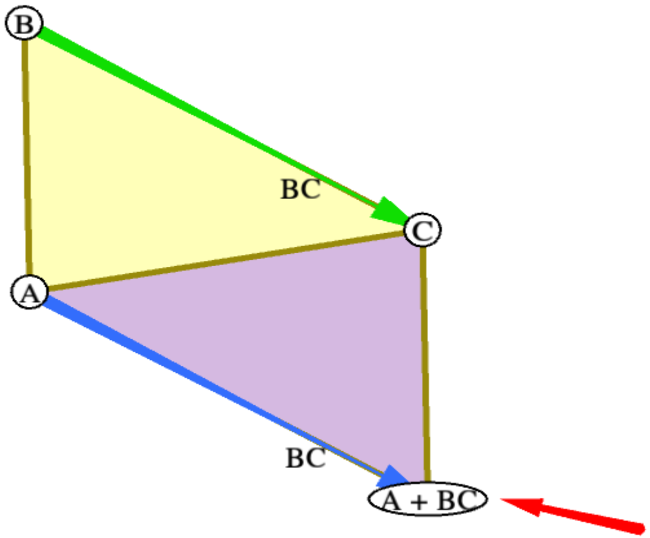
Here, there can be 3 possible parallelograms, but I have chosen to make the one with edge AC as common between the two halves. This is for simplicity.

PHSE 1: Solution outline

We can compute the required point D using a point plus vector approach. We need to pick point A or C and add the vector which is between the other two points. This vector will be equal to the required vector to finish the parallelogram.



Here, we get the fourth point
by adding BA to point C



Here, we get the fourth point
by adding BC to point A

PHSE 1: Solution math

D = A + BC = C + BA

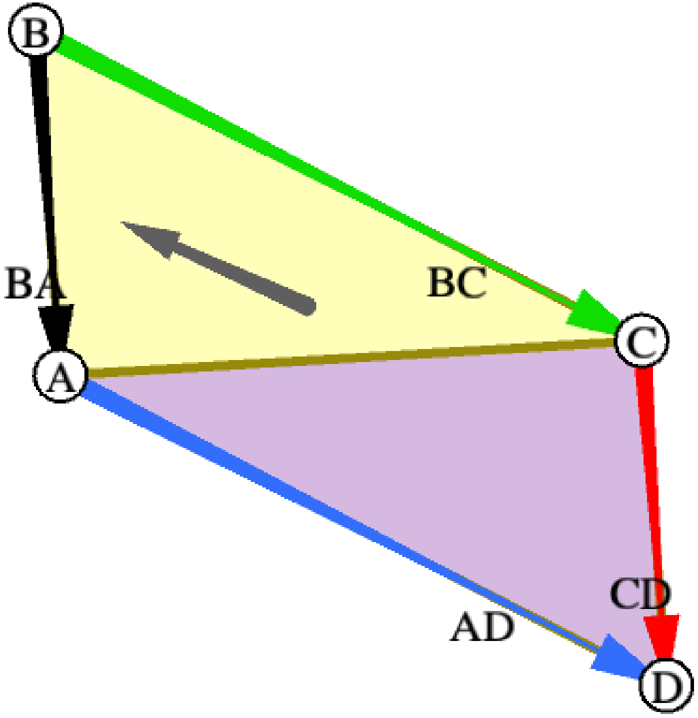
JUSTIFICATION:

We know the following:

- In a parallelogram with points A, B, C, D, we have vector equality AD = BC
- In a parallelogram with points A, B, C, D, we have vector equality BA = CD.

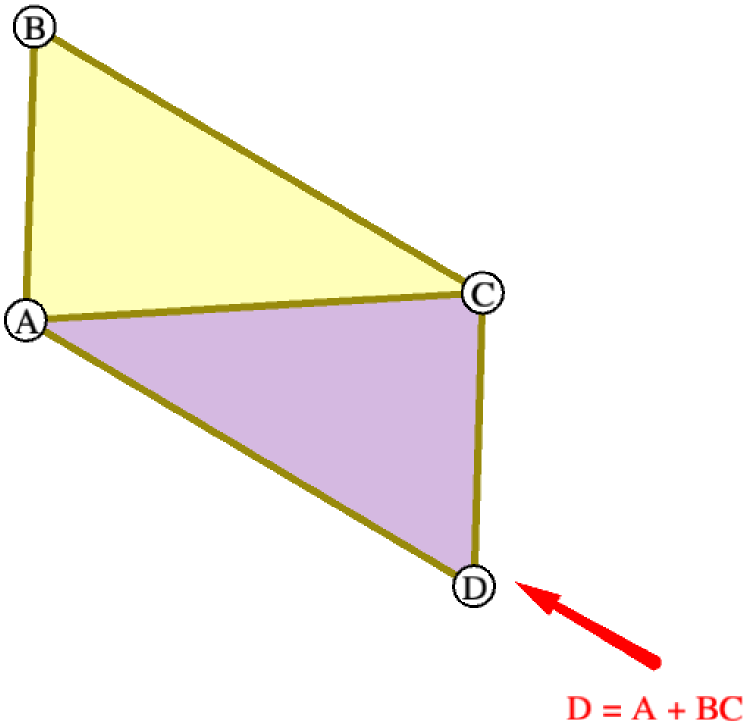
Hence, we get:

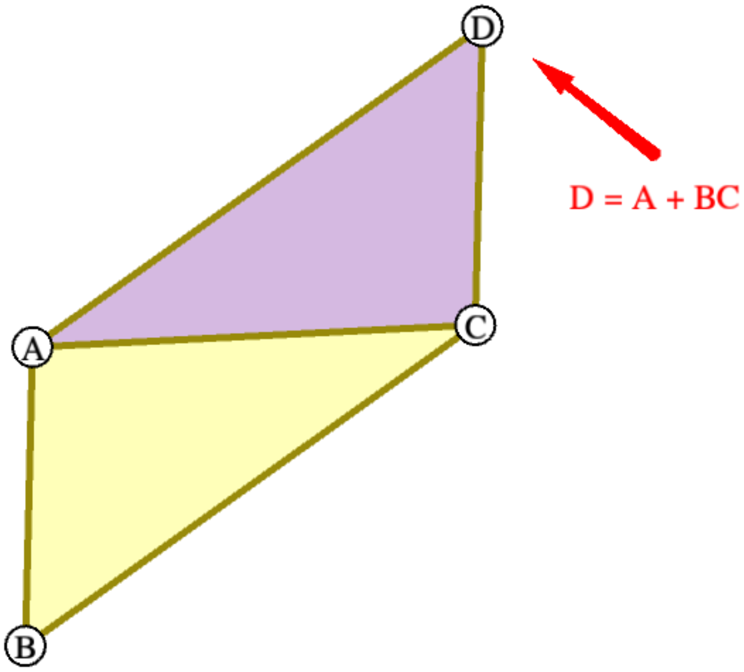
- D – A = BC => D = A + BC
- D – C = BA => D = C + BA



PHSE 1: Solution examples and limitations

My solutions works for clockwise and counterclockwise orientations of the initial points A, B, C





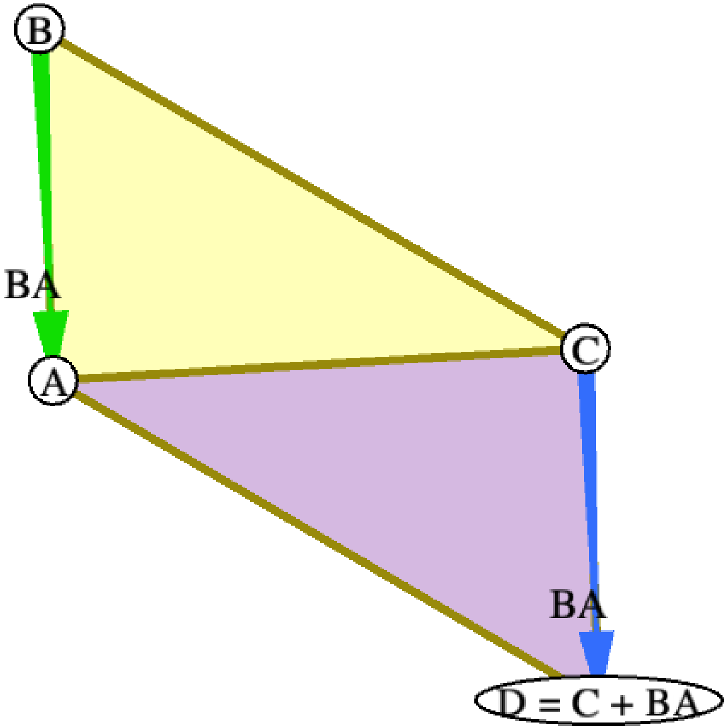
PHSE 1: Code

main GUI MyProject arrow circles ducks grid interpolation pts pv spiral triangles utilities

```

//============================================================================ PART 1
void showPart1(PNT A, PNT B, PNT C, PNT D) //
  {
  PartTitle[1] =   "Complete the parallelogram"; // https://en.wikipedia.org/wiki/Parallelogram
  // Making Vector BC and Point X (Point D of Parallelogram)
  VCT BA = V(B,A);
  PNT X = P(C, BA);

  // Making triange of initial points and coloring it
  cwfo(dgold,5,yellow,70);
  showLoop(A,B,C);
  // Making triange of initial points and new point X (Point D of Parallelogram)
  cwfo(dgold,5,dmagenta,70);
  showLoop(A,C,X);
  // Highlighting BA vector which is replicated to get point X
  show(B,BA,green,"BA");
  show(C,BA,blue,"BA");
  // Labeling our new point D
  X.circledLabel("D = C + BA");

  guide="MyProject keys: '0' through '9' to select project, 'a' to start/stop animation ";
  A.circledLabel("A"); B.circledLabel("B"); C.circledLabel("C"); // D.circledLabel("D");
  }

```

PHSE 1: Code

The code from the previous slide outputs the following:



PHSE 1: Sources

My solution is based on my own previous knowledge of parallelograms. In a parallelogram with points A, B, C, D, we have AD = BC and BA = CD.

This is derived from the following statements:

- Opposite sides are parallel.
- Opposite sides are equal in length.'

I found this information on Math is Fun at https://www.mathsisfun.com/geometry/parallelogram.html

PHSE 2: Problem statement

Given a triangle made of three points (A, B, C), find the point such that the total distance from the three vertices of the triangle to the point is the minimum possible. This will be the Fermat Point X.

COMMENTS:

In order to accurately calculate the Fermat point, we assume the following:

i. The initial points (A, B, C) form a triangle.

ii. The set of points {A, B, C} are ordered clockwise or anticlockwise around the triangle.

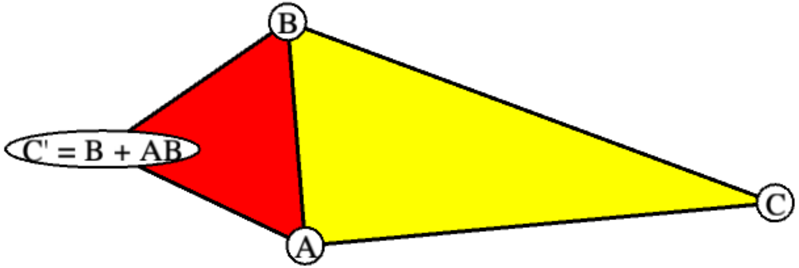
iii. No angle in triangle ABC is moe than 120 degrees ($2\pi/3$ radians)

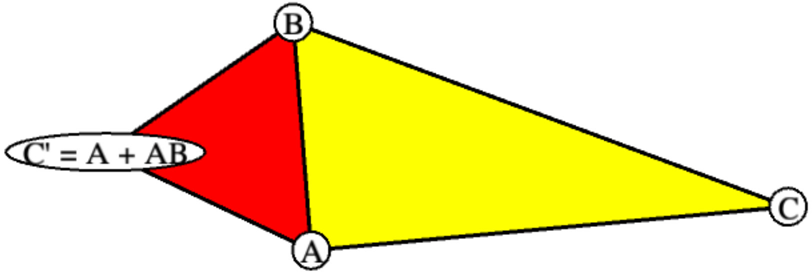
Here, there is no ambiguity.

PHSE 2: Solution outline

- To start calculating the Fermat point, we need to make equilateral triangles using each edge of the original triangle.

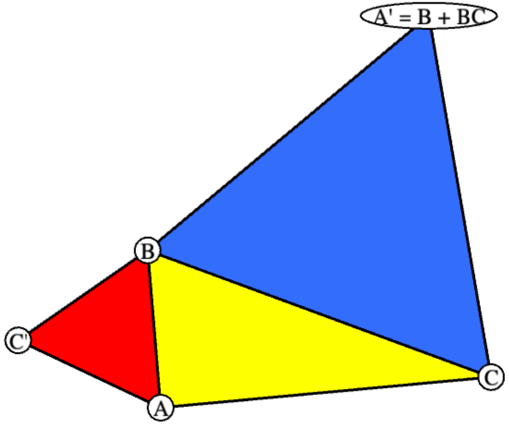
- Here, we make a point C' which forms an equilateral triangle ABC'. This is because the resultant vectors AB, BC' and C'A form a 60-degree (π/3 radians) angle with each other.

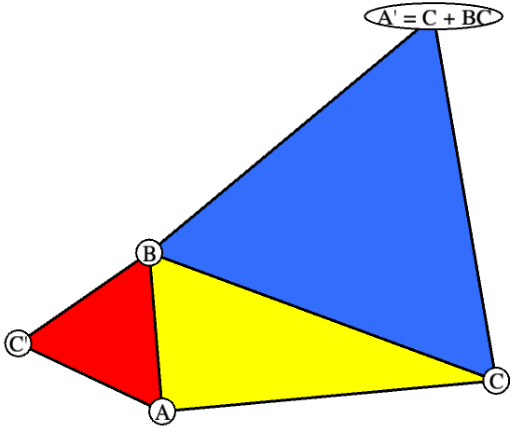




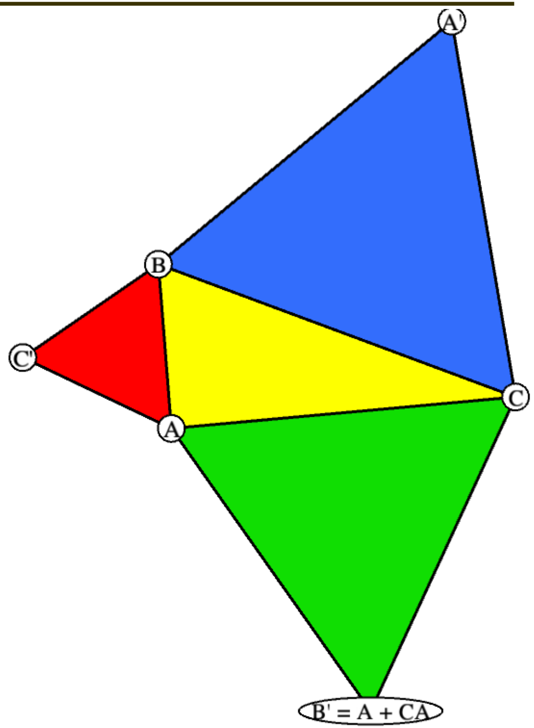
PHSE 2: Solution outline

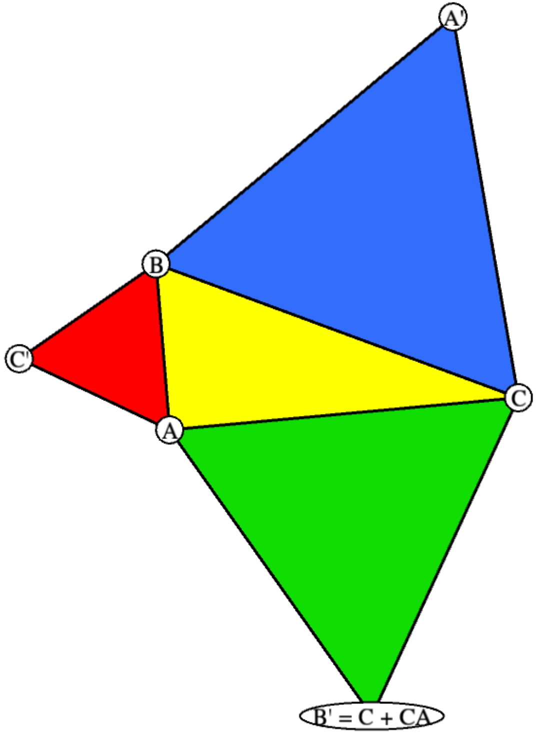
- Next, we make a point A’ which forms an equilateral triangle BCA’. This is because the resultant vectors BC, CA’ and A’B form a 60-degree ($\pi/3$ radians) angle with each other.





PHSE 2: Solution outline



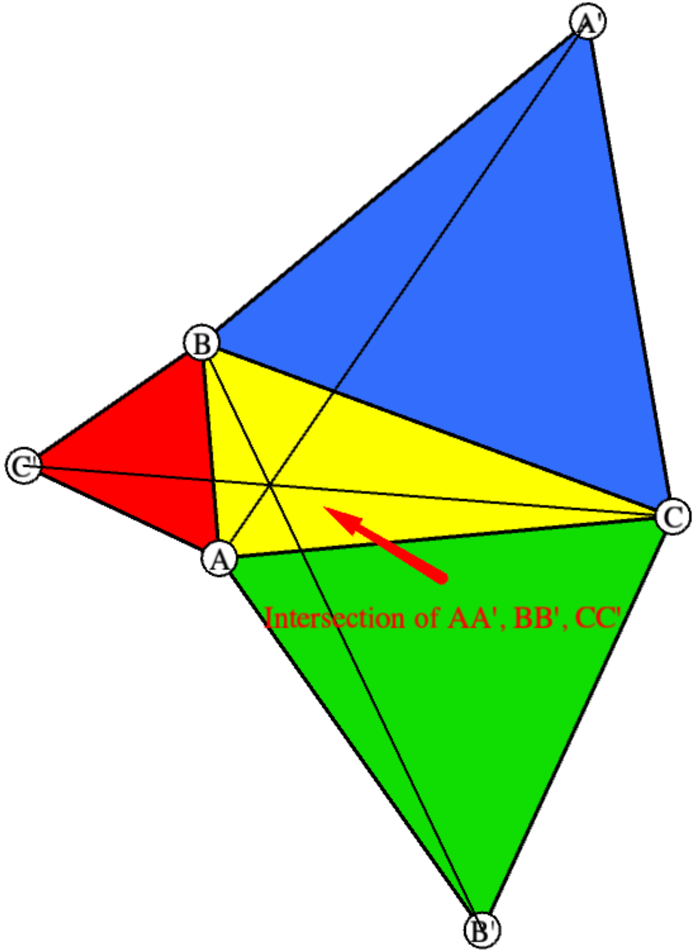


- Next, we make a point B’ which forms an equilateral triangle ACB’. This is because the resultant vectors AC, CB’ and B’A form a 60-degree (π/3 radians) angle with each other.

PHSE 2: Solution outline

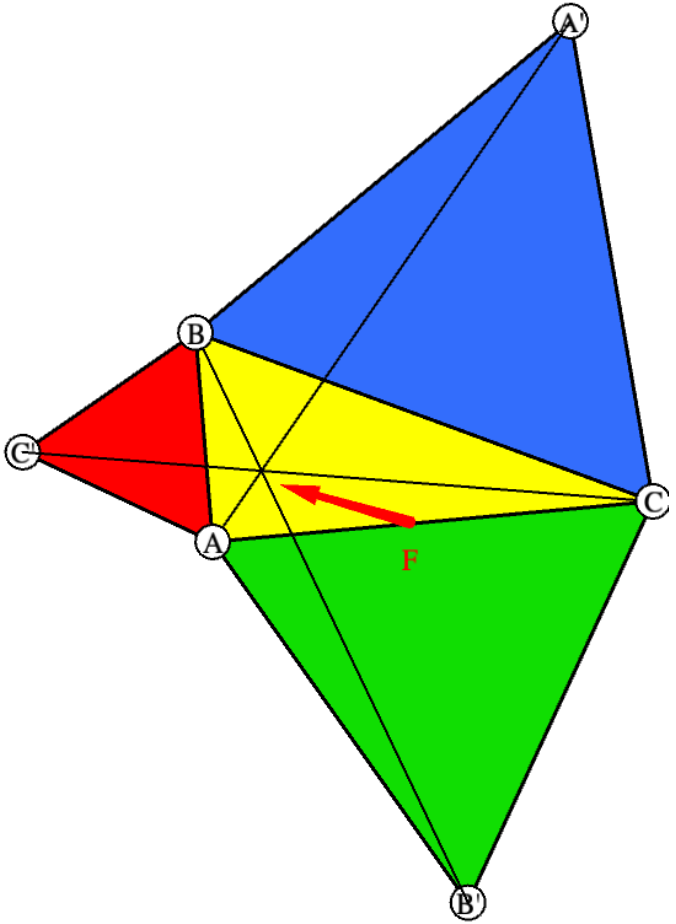
- Finally, we draw lines from each new vertex to the opposite vertex of the original triangle. So we get lines AA’, BB’ and CC’.

- The point of intersection that is formed here is the required Fermat point.



PHSE 2: Solution math

The point F formed by the intersection of the apexes of the created equilateral triangles and their opposite vertices is the Fermat point of the triangle ABC.



JUSTIFICATION:

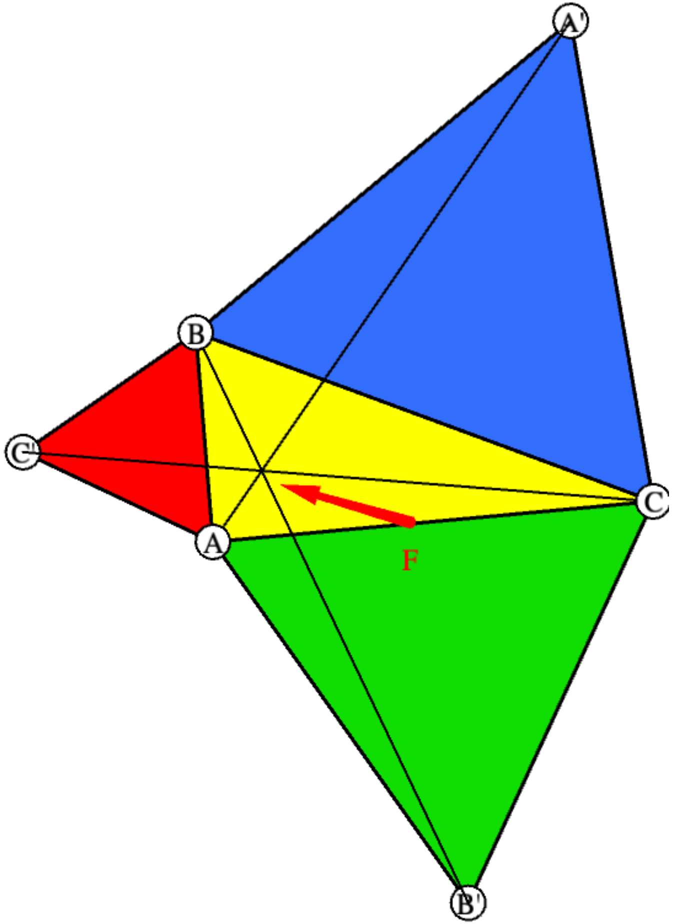
We know that for F to be the Fermat point, the following must be true:

- AF + BF + CF = Minimum possible distance

PHSE 2: Solution math

- Here, let our origin be the point F. Since our three constructed triangles are equilateral, we have ∠BFA = ∠AFC = ∠CFB = 120 degrees ($2\pi/3$ radians).

- Now, consider a point in the plane X. Let *a* = VCT A, *b* = VCT B, *c* = VCT C and *x* = VCT X and *i, j, k* be the unit vectors along *a, b* and *c.*



PHSE 2: Solution math

- Due to this, we get:

$|a| = a * i = (a - x) * i + x * \mathrm{i} \leq |\mathrm{a} - x| + x * i$

$|b| = b * j = (b - x) * j + x * j \leq |\mathrm{b} - x| + x * j$

$|c| = a * k = (c - x) * k + x * k \leq |\mathrm{c} - x| + x * k$

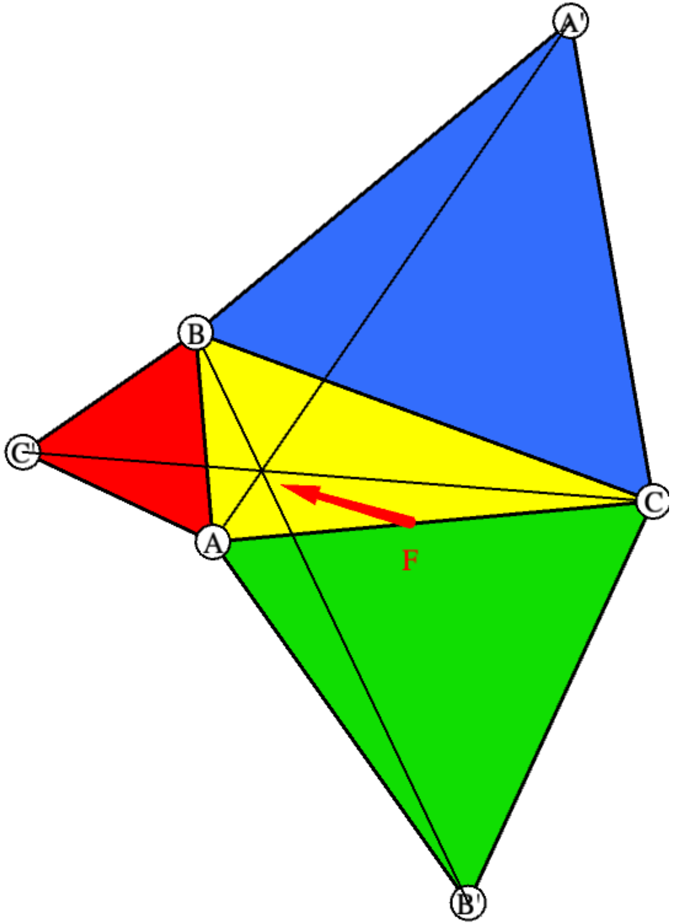
- Noting that $i + j + k = 0$ and adding, we see that:

$|a| + |b| + |c| \leq |a - x| + |b - x| + |c - x|,$

OR

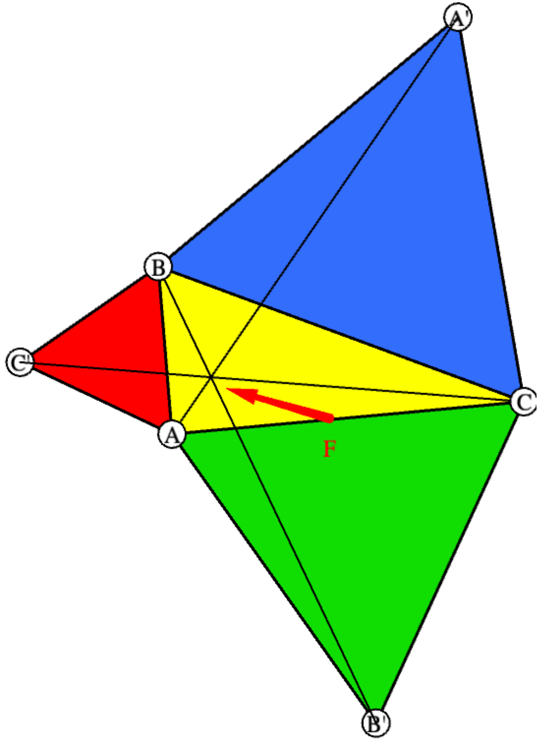
AF + BF + CF ≤ AX + BX + CX

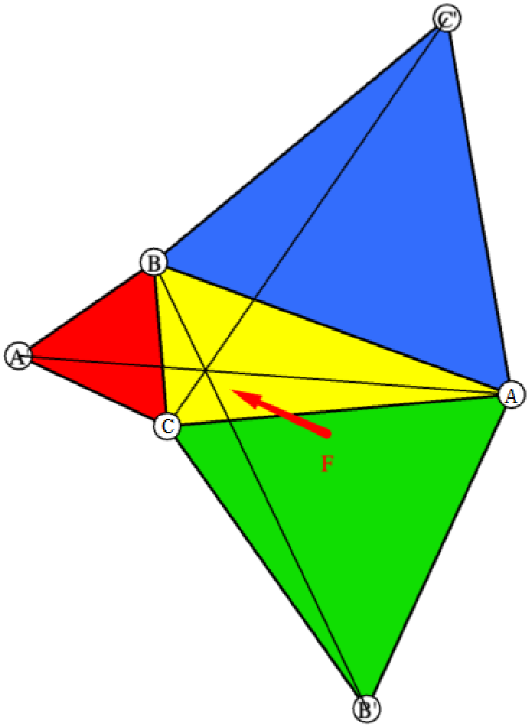
Thus, the origin or point F is the desired point.



PHSE 2: Solution examples and limitations

My solutions works for clockwise and counterclockwise orientations of the initial points A, B, C. The Fermat point calculated is the same regardless.





PHSE 2: Code

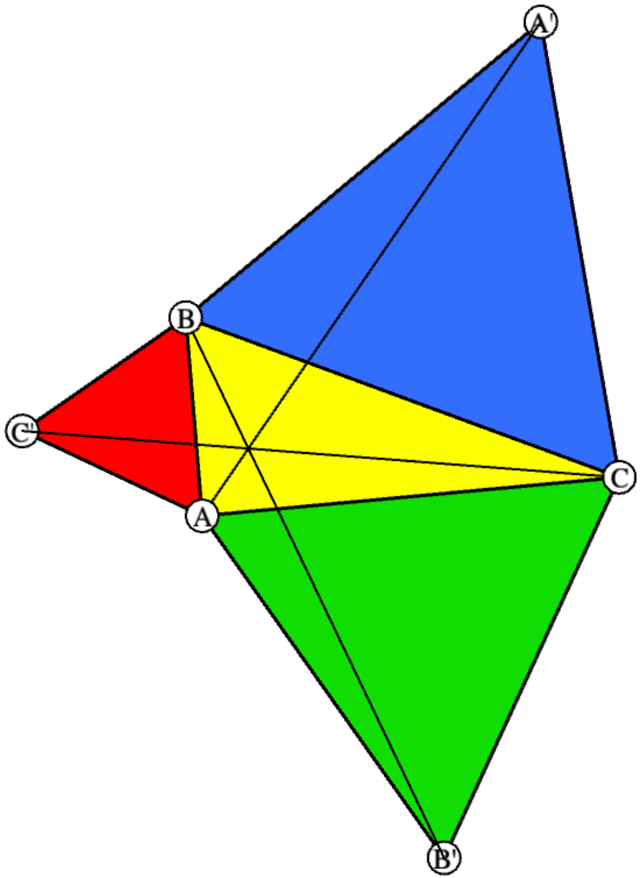
```
//================================================================ PART 2
void showPart2(PNT A, PNT B, PNT C) //
  {
  PartTitle[2] = "Fermat Point"; // https://en.wikipedia.org/wiki/Napoleon%27s_theorem
  cwf(black,3,yellow);
  showLoop(A,B,C);

  // Show the first equilateral triangle ABC'
  VCT BA = V(B, A);
  // Turn BA by PI/6 to get BC' (BX)
  VCT BX = R(BA, 2 * PI/6);
  PNT X = P(B, BX);
  cwf(black,3,red);
  showLoop(A, B, X);
  X.circledLabel("C'");

  // Show the second equilateral triangle BCA'
  VCT CB = V (C, B);
  // Turn CB by PI/6 to get CA' (CY)
  VCT CY = R(CB, 2 * PI/6);
  PNT Y = P(C, CY);
  cwf(black, 3, blue);
  showLoop(B, C, Y);
  Y.circledLabel("A'");
```

PHSE 2: Code (Continued)

```
// Show the third equilateral triangle CAB'
VCT AC = V (A, C);
// Turn AC by PI/6 to get AB' (AZ)
VCT AZ = R(AC, 2 * PI/6);
PNT Z = P(A, AZ);
cwf(black, 3, green);
showLoop(C, A, Z);
Z.circledLabel("B'");

// Show the lines from created vertices to opposite vertex of pre-existing triangle. Intersection point is F
show(A,Y);
show(B,Z);
show(C,X);

guide="MyProject keys: '0' through '9' to select project, 'a' to start/stop animation ";
A.circledLabel("A"); B.circledLabel("B"); C.circledLabel("C"); // D.circledLabel("D");
}
```

PHSE 2: Code

The code from the previous slide outputs the following:



PHSE 2: Sources

My solution is based on my own previous knowledge of equilateral triangles.

I researched on Fermat points from https://en.wikipedia.org/wiki/Fermat_point

From here, I derived:

- Fermat Point is a point such that the total distance from the three vertices of the triangle to the point is the minimum possible.
- The Fermat point of a triangle with largest angle at most 120° is simply its first isogonic center or X(13), which is constructed as follows:
  1. Construct an equilateral triangle on each of two arbitrarily chosen sides of the given triangle.
  2. Draw a line from each new vertex to the opposite vertex of the original triangle.
  3. The two lines intersect at the Fermat point.
- I also used http://jwilson.coe.uga.edu/EMAT6680Fa07/Shih/AS06/WU06.htm for reference.