MC

1. **C**

Solution: Since the sum of the interior angles of a triangle must equal 180°, the $x = 180 - 50 - 45 = 85$.

(Pascal 2014, #2)

2. **C**

Solution: $(1)^{10} + (-1)^{8} + (-1)^{7} + (1)^{5} = (1) + (1) + (-1) + (1) = 2$.

(Cayley 1997, #2)

3. **E**

Solution: Let the integer be n and let the result be r. Then $r = n * 2 * 5 = n * 10$. Thus n must end in a zero. 30 is the only option that satisfies this requirement.

(Cayley 1997, #3)

4. **D**

Solution: There are ten possible pairs of numbers that can be chosen: −3 and −1; −3 and 0; −3 and 2; −3 and 4; −1 and 0; −1 and 2; −1 and 4; 0 and 2; 0 and 4; 2 and 4. Each pair is equally likely to be chosen. Pairs that include 0 (4 pairs) have a product of 0; pairs that do not include 0 (6 of them) do not have a product of 0. Therefore, the probability that a randomly chosen pair has a product of 0 is 4/10 or 2/5.

(Cayley 2014, #19)

5. **E**

Solution: Suppose that Megan and Shana competed in exactly n races.
Since Shana won exactly 2 races, then Megan won exactly $n - 2$ races.
Since Shana won 2 races and lost $n - 2$ races, then she received $2x + (n - 2)y$ coins.
Thus, $2x + (n - 2)y = 35$.
Since Megan won $n - 2$ races and lost 2 races, then she received $(n - 2)x + 2y$ coins.
Thus, $(n - 2)x + 2y = 42$.
If we add these two equations, we obtain $(2x + (n - 2)y) + ((n - 2)x + 2y) = 35 + 42$ or
$nx + ny = 77$ or $n(x + y) = 77$.
Since n, x and y are positive integers, then n is a positive divisor of 77, so n = 1, 7, 11 or 77.
Subtracting $2x + (n - 2)y = 35$ from $(n - 2)x + 2y = 42$, we obtain
$((n - 2)x + 2y) - (2x + (n - 2)y) = 42 - 35$
or $(n - 4)x + (4 - n)y = 7$ or $(n - 4)(x - y) = 7$.
Since n, x and y are positive integers and $x > y$, then $n-4$ is a positive divisor of 7, so $n-4 = 1$ or $n - 4 = 7$, giving $n = 5$ or $n = 11$.
Comparing the two lists, we determine that n must be 11.
Thus, we have $11(x + y) = 77$ or $x + y = 7$.
Also, $7(x - y) = 7$ so $x - y = 1$.
Adding these last two equations, we obtain $(x + y) + (x - y) = 7 + 1$ or $2x = 8$, and so $x = 4$.

(Checking, if $x = 4$, then $y = 3$. Since $n = 11$, then Megan won 9 races and Shana won 2 races. Megan should receive $9(4) + 2(3) = 42$ coins and Shana should receive $2(4) + 9(3) = 35$ coins, which agrees with the given information.)

(Fermat 2013, #22)

Word Problems

1.

Solution: Since $a + b = 9 - c$ and $a + b = 5 + c$, then $9 - c = 5 + c$. Therefore, $9 - 5 = c + c$ or $2c = 4$ or $c = 2$.

(CIMC 2011, #A2)

2.

Solution: When the red die is rolled, there are 6 equally likely outcomes. Similarly, when the blue die is rolled, there are 6 equally likely outcomes.
Therefore, when the two dice are rolled, there are $6 \times 6 = 36$ equally likely outcomes for the combination of the numbers on the top face of each. (These outcomes are Red 1 and Blue 1, Red 1 and Blue 2, Red 1 and Blue 3, . . ., Red 6 and Blue 6.)
The chart below shows these possibilities along with the sum of the numbers in each case:

		Blue Die					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Red	3	4	5	6	7	8	9
Die	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Since the only perfect squares between 2 and 12 are 4 (which equals 2^2) and 9 (which equals 3^2), then 7 of the 36 possible outcomes are perfect squares. Since each entry in the table is equally likely, then the probability that the sum is a perfect square is $7/36$.

(CIMC 2011, #A4)

3.

Solution:

a) Expanding and simplifying, we obtain $(a - 1)(6a^2 - a - 1) = 6a^3 - a^2 - a - 6a^2 + a + 1 = 6a^3 - 7a^2 + 1$.
b) To solve the equation $6 \cos^3 \theta - 7 \cos^2 \theta + 1 = 0$, we first make the substitution $a = \cos \theta$. The equation becomes $6a^3 - 7a^2 + 1 = 0$.
   From (a), factoring the left side gives the equation $(a - 1)(6a^2 - a - 1) = 0$.
   We further factor $6a^2 - a - 1$ as $(3a + 1)(2a - 1)$.
   This gives $(a - 1)(3a + 1)(2a - 1) = 0$.
   Thus, $a = 1$ or $a = -1/3$ or $a = 1/2$.

This tells us that the solutions to the original equation are the values of θ in the range $-180^\circ < \theta < 180^\circ$ with $\cos\theta = 1$ or $\cos\theta = -1/3$ or $\cos\theta = 1/2$.
If $-180^\circ < \theta < 180^\circ$ and $\cos\theta = 1$, then $\theta = 0^\circ$.
If $-180^\circ < \theta < 180^\circ$ and $\cos\theta = -1/3$, then $\theta \approx 109.5^\circ$ or $\theta \approx -109.5^\circ$. (The positive value for θ can be obtained using a calculator. The negative value can be obtained by thinking about cos θ as an even function of θ or by picturing either the graph of $y = \cos\theta$ or the unit circle.)
If $-180^\circ < \theta < 180^\circ$ and $\cos\theta = 1/2$, then $\theta = 60^\circ$ or $\theta = -60^\circ$.
Therefore, the solutions to the equation $6\cos^3\theta - 7\cos^2\theta + 1 = 0$, rounded to one decimal place as appropriate, are 0°, 60°, -60°, 109.5°, -109.5°.

c) To solve the inequality $6\cos^3\theta - 7\cos^2\theta + 1 < 0$, we factor the left side to obtain $(\cos\theta - 1)(3\cos\theta + 1)(2\cos\theta - 1) < 0$
From b), we know the values of θ at which the left side equals 0, so we examine the intervals between these values and look at the sign (positive or negative) of each of the factors in these intervals. We complete the final column of the table below by noting that the product of three positive numbers is positive, the product of two positives with one negative is negative, the product of one positive and two negatives is positive, and the product of three negatives is negative.

Range of θ	Range of $\cos\theta$	$\cos\theta - 1$	$3\cos\theta + 1$	$2\cos\theta - 1$	Product
$-180^\circ < \theta < -109.5^\circ$	$-1 < \cos\theta < -\frac{1}{3}$	$-$	$-$	$-$	$-$
$-109.5^\circ < \theta < -60^\circ$	$-\frac{1}{3} < \cos\theta < \frac{1}{2}$	$-$	$+$	$-$	$+$
$-60^\circ < \theta < 0^\circ$	$\frac{1}{2} < \cos\theta < 1$	$-$	$+$	$+$	$-$
$0^\circ < \theta < 60^\circ$	$\frac{1}{2} < \cos\theta < 1$	$-$	$+$	$+$	$-$
$60^\circ < \theta < 109.5^\circ$	$-\frac{1}{3} < \cos\theta < \frac{1}{2}$	$-$	$+$	$-$	$+$
$109.5^\circ < \theta < 180^\circ$	$-1 < \cos\theta < -\frac{1}{3}$	$-$	$-$	$-$	$-$

From this analysis, the values of θ for which $6\cos^3\theta - 7\cos^2\theta + 1 < 0$ are $-180^\circ < \theta < -109.5^\circ$ and $-60^\circ < \theta < 0^\circ$ and $0^\circ < \theta < 60^\circ$ and $109.5^\circ < \theta < 180^\circ$. We could also have determined these intervals by looking only at positive values for θ and then using the fact that cos θ is an even function to determine the negative values.

(CSMC 2013, #2)