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# Chapter 3

# Modern definitions in reliability engineering

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Abstracts: To better understand the parametric ALT that is a core of this book, this chapter will briefly review the basic definitions of reliability engineering that can be used widely for reliability testing. It consists of bathtub, reliability index, fundamentals in statistics and probability theory, statistical distributions like Weibull, and experimental design. When product is designed, engineer knows if final design has problems and is satisfied with the reliability target. Engineer should fully recognize the basic concepts that reliability testing is required. From customer's standpoint, reliability depends on the product design and can be explained as two separate concepts – product of lifetime and failure rate. Engineer for reliability theory may feel complex because it requires an extensive concepts of probability and statistics. To conduct the reliability testing of mechanical product and obtain the reasonable test data, mechanical engineer should understand the modern definitions that can be used. When product is subjected to repetitive stresses (sole factor) and there is design faulty in it, product will fail. However, there is no current methodology because product failures rarely happen in its lifetime. As an alternative, we might suggest parametric ALT in Chapter 8.

Keywords: Reliability concepts, Reliability testing, Probability theory, Statistics, Reliability testing, Robust design.

## 3.1 Introduction

The ability of an item is to perform a required function under given environmental and operational conditions for a stated period of time (ISO8402). Reliability theory developed apart from the mainstream of probability and statistics, which was used primarily as a tool to help nineteenth century maritime and life insurance companies compute profitable rates to charge their customers. The modern concepts in reliability engineering started from bathtub, which was found in the reliability study of vacuum tube – problematic parts in the WW2. Invented in 1904 by John Ambrose Fleming, vacuum tubes were a basic component for electronics throughout the first half of the twentieth century.

The reliability concepts except the quality control in product manufacture can focus on the study of quality itself in design. When mechanical system is subjected to random stress (or loads), mechanical structures are designed to withstand the loads with proper stiffness and strength. Requirements on stiffness, being the resistance against reversible deformation, may depend on their applications. Strength, the resistance against irreversible deformation, is always required to be high, because this deformation may lead to loss of functionality and even failure. Modern definitions in reliability engineering will be required to develop the methodology of reliability testing – parametric Accelerated Life Testing in Chapter 8 and 9. It will uncover the faulty designs of product and modify them. Finally, it confirms whether the reliability of final designs is achieved.

## 3.2 Reliability and Bathtub Curve

### 3.2.1 Reliability function and failure rate

Reliability is the probability that a mechanical product will properly operate for a design life under the environmental or operating conditions. If $T$ is a random variable denoting the time to failure, the reliability function at time $t$ can be expressed as

$$R(t) = P(T > t) \tag{3-1}$$

The cumulative distribution function (CDF) as the complement of R(t) can also be expressed as:

$$F(t) = 1 - R(t) \tag{3-2}$$

If the time to failure, *T*, has a probability density function *f(t)*, equations (3-1) and (3-2) can be rewritten as

$$R(t) = 1 - F(t) = 1 - \int_0^t f(\xi)d\xi \tag{3-3}$$

The failure rate in a time interval [$t_1$, $t_2$] can be defined as the probability that a failure rate unit time occurs in the interval given that no failure has occurred prior to $t_1$, the beginning of the interval. Thus, the failure rate is expressed as:

$$\frac{R(t_1) - R(t_2)}{(t_2 - t_1)R(t_1)} \tag{3-4}$$

If we replace $t_1$ by t and $t_2$ by t+Δt, we rewrite Equation (3-4) as

$$\frac{R(t) - R(t + \Delta t)}{\Delta t R(t)} \tag{3-5}$$

As $\Delta t$ approaches zero, the instantaneous failure rate can be redefined as:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t + \Delta t)}{\Delta t R(t)} = \frac{1}{R(t)}\left[-\frac{d}{dt}R(t)\right] = \frac{f(t)}{R(t)} \tag{3-6}$$

### 3.2.2 Bathtub curve

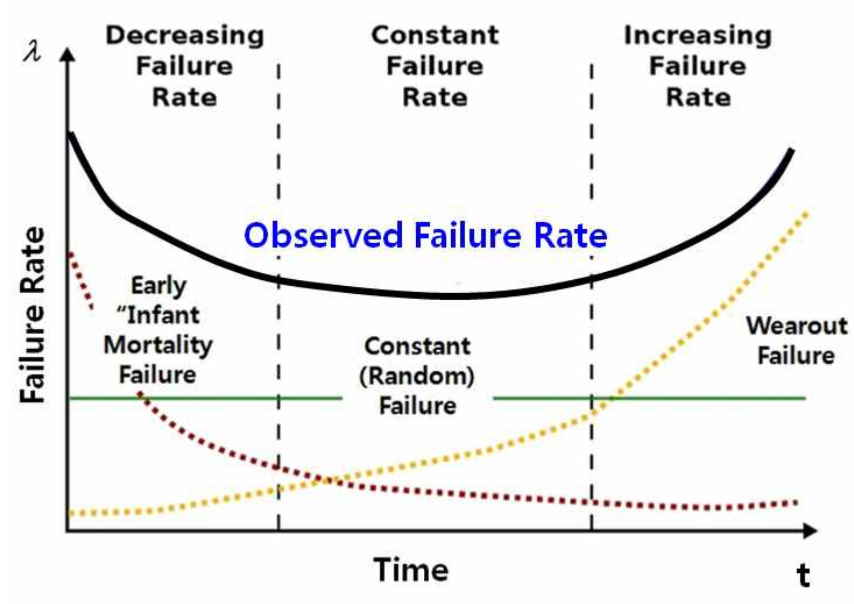



Fig. 3.1 Bathtub curve

The bathtub curve for the faulty product such as vacuum tube was created by mapping the rate of early "infant mortality" failures, the rate of random failures with constant failure rate during its "useful life", and finally the rate of "wear out" failures. As defective products are removed, the failure rate in the early life of a product is high but rapidly decreasing. Early sources of potential failure such as storage, handling and installation error are dominated and aging test can remove them immediately. In the mid-life of a product the failure rate is constant. In this period product may experience the random failure like the usage of end-users or overstress. However, if design problems in product exist, the failure of product will increase catastrophically. Engineer should be removed by proper testing methods such as parametric ALT. For bathtub there are three types of reliability testing in accordance with the failure rate (Figure 3.2).

- Early failures: Because it requires the short test time, it easily improve by aging test prior to shipment
- Random failures: Specific environmental tests related to shipping/usage/disposal

  - Shipping testing: storage, transportation by vehicle/ship/rail

  - Usage Testing: High voltage, lightning, shock, temperature & humidity, EMC (EMI);

  - Disposal (Low frequency, Pass/Fail testing)
- Catastrophic disasters: It often happens in 1-2 years after production, which comes from the design failure. If it is reproduced and corrected by reliability testing like parametric Accelerated Life Testing described in Chapter 8 and 9, it can be eliminated.

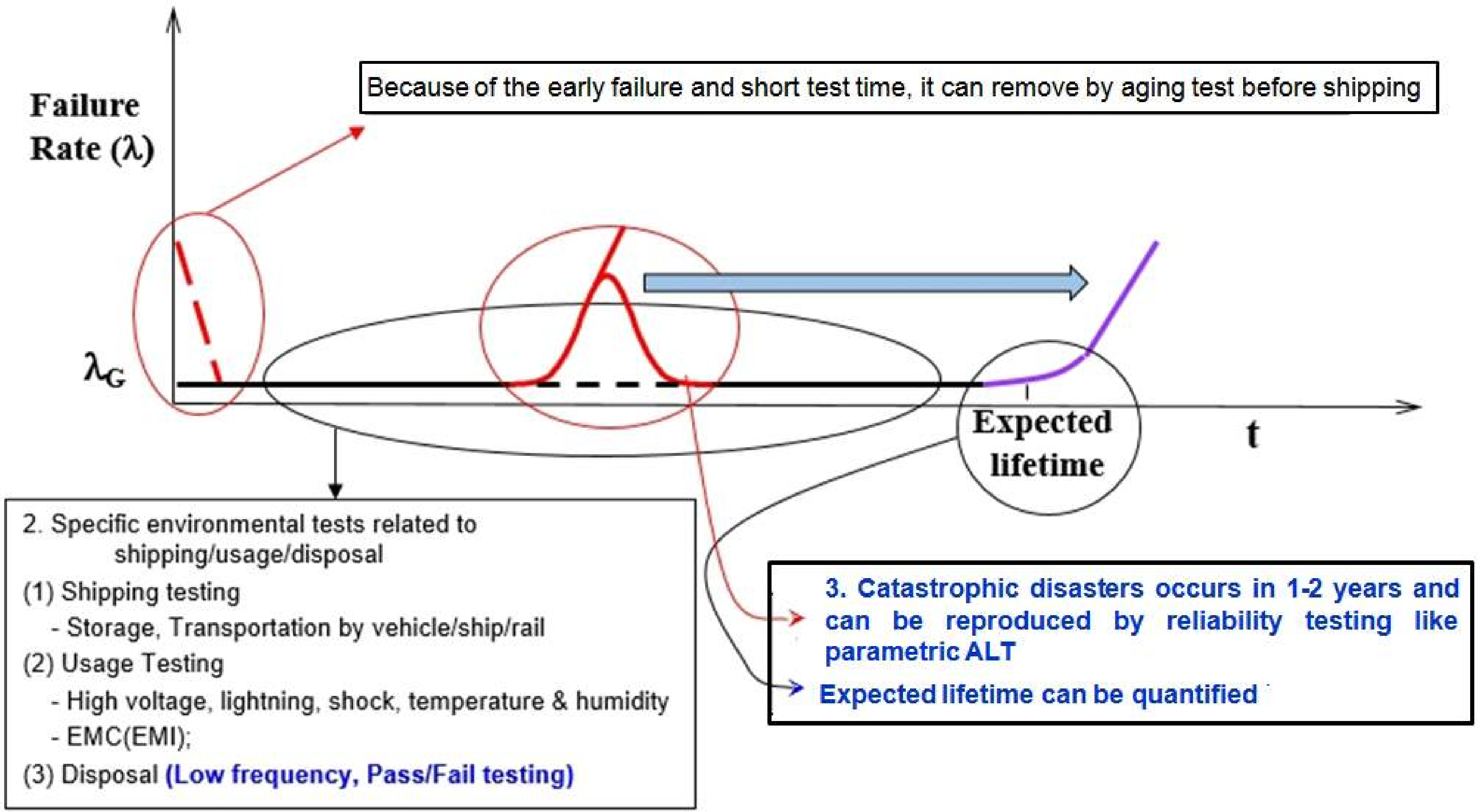



Fig. 3.2 Three types of reliability tests in accordance with the failure rate

## 3.3 Reliability lifetime metrics

An important goal for reliability designers is to assess lifetime from product failures or testing data. Reliability lifetime metrics are used to quantify a failure rate and the resulting time of expected performance. MTTF, MTBF, MTTR, FIT and BX% life are reliability lifetime metrics as follows:

- MTTF (Mean Time To Failure)
- MTBF (Mean Time Between Failure),
- MTTR (Mean Time To Repair),
- BX% life

### 3.3.1 Mean Time To Failure (MTTF)

A non-repairable system is one for which individual items that fail are removed permanently from the population. MTTF is a basic lifetime metric of reliability to specify the lifetime of non-repairable systems–"one-shot" devices like light bulbs. It is the mean time until a piece of equipment fails at first statistically. MTTF is the mean over a long period of time with a large unit (Figure 3.3).

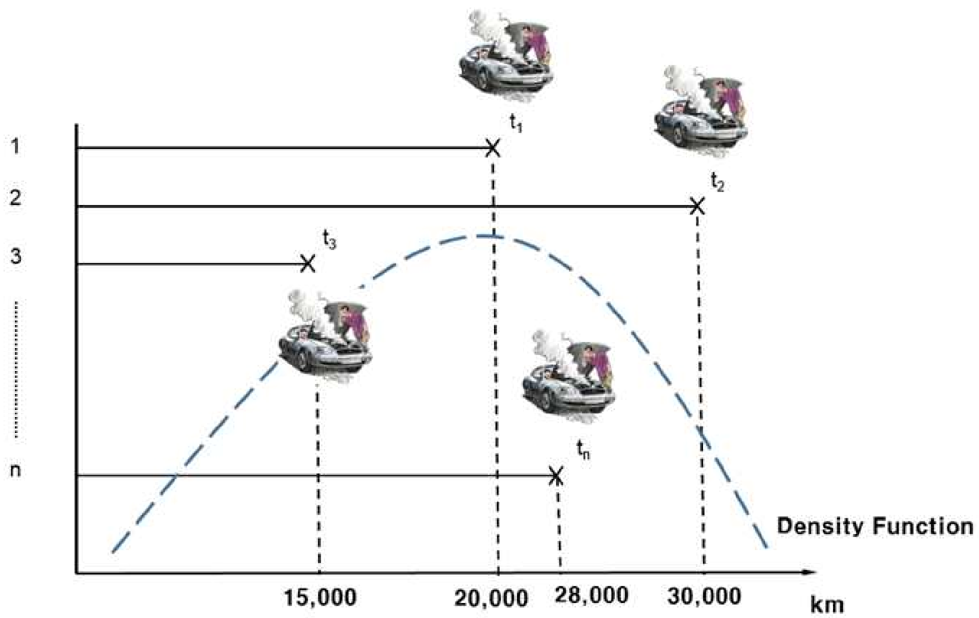



Fig. 3.3 Concept of Mean Time To Failure

$$MTTF = \frac{t_1 + t_2 + ...... + t_n}{n} \tag{3-7}$$

Because we know that MTTF in Figure 3.3 is 23,000 *km*, the MTTF can be described with other mathematical terms:

$$MTTF = E(T) = \int_0^\infty t \cdot f(t)dt = -\int_0^\infty t \frac{dR(t)}{dt}dt = \int_0^\infty R(t)dt \tag{3-8}$$

where $f(t) = \frac{d}{dt}F(t) = -\frac{d}{dt}R(t)$

**Example 3.1** Consider a system with reliability function

$$R(t) = \frac{1}{(0.2t+1)^2}, \quad \text{for } t > 0$$

Find the probability density function, failure rate, and MTTF

Probability density $f(t) = -\frac{d}{dt}R(t) = \frac{0.4}{(0.2t+1)^3}$

Failure rate $\lambda(t) = \frac{f(t)}{R(t)} = \frac{0.4}{(0.2t+1)}$

Mean time to failure $MTTF = \int_0^\infty R(t)dt = 5\ months$

### 3.3.2 Mean Time Between Failure (MTBF)

A repairable system is one which can be restored to satisfactory operation by any action, including parts replacements or changes to adjustable settings. Mean Time Between Failure (MTBF) is a reliability metric used to describe the mean lifetime of repairable components – automobiles, refrigerator, and airplanes. MTBF remains a basic measure of a systems' reliability for most products, though it still is debated and changed. MTBF still is more important for industries and integrators than for consumers (Figure 3.4).

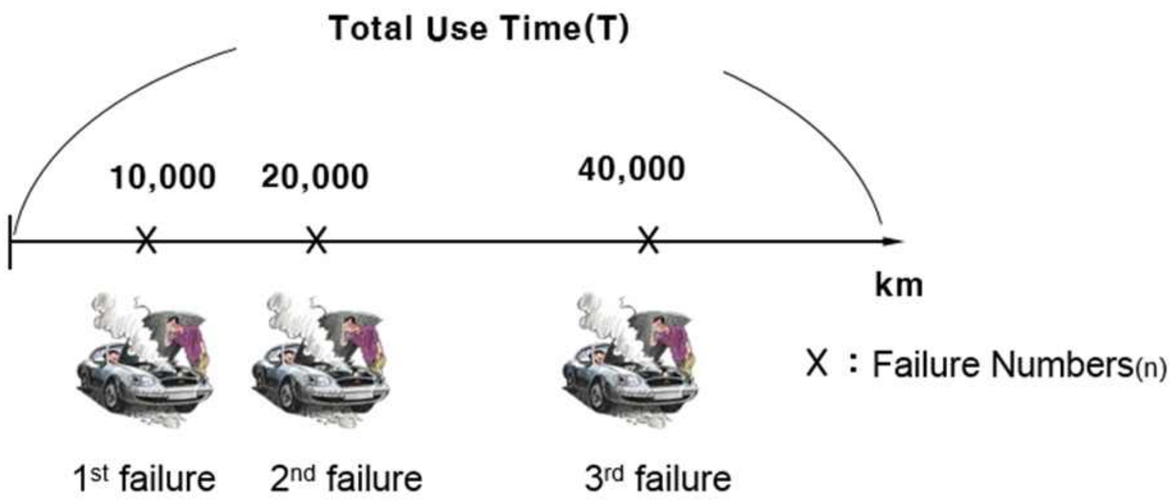



Fig. 3.4 Concept of Mean Time Between Failure

$$MTBF = \frac{T}{n} \tag{3-9}$$

MTBF value is equivalent to the expected number of operating hours (service life) before a product fails. There are several variables that can impact failures. Aside from component failures, customer use/installation can also result in failure. MTBF is often calculated based on an algorithm that factors in all of a product's components to reach the sum life cycle in hours. MTBF is considered a system failure. It is still regarded as a useful tool when considering the purchase and installation of a product.

For repairable complex systems, failures are considered to be those out of design conditions which place the system out of service and into a state for repair. Technically, MTBF is used only in reference to a repairable item, while MTTF is used for non-repairable items like electric components.

### 3.3.3 Mean Time To Repair (MTTR)

Mean Time To Repair (MTTR) is the average lifetime needed to fix a problem. In an operational system, repair generally means replacing a failed hardware part. Thus, hardware MTTR could be viewed as mean time to replace a failed hardware module. Taking too long to repair a product drives up the cost of the installation in the long run, due to down time until the new part arrives and the possible window of time required scheduling the installation. To avoid MTTR, many companies purchase spare products so that a replacement can be installed quickly. Generally, however, customers will inquire about the turn-around time of repairing a product, and indirectly, that can fall into the MTTR category. And relationship among MTTF, MTBF and MTTR can be described in Figure 3.5.

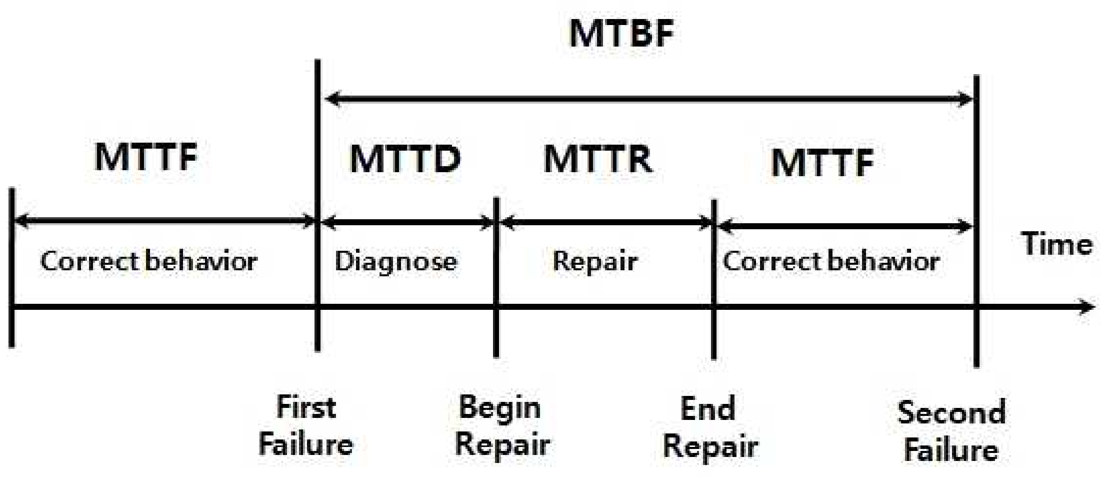



Fig. 3.5 A schematic diagram of MTTF, MTTR, and MTBF

### 3.3.4 BX% life

The BX life metric originated in the ball and roller bearing industry, but has become a product lifetime metric used across a variety of industries today. It’s particularly useful in establishing warranty periods for a product. The BX% life is the lifetime metric which takes to fail X% of the units in a population. For example, if an item has a B10 life of 1000km, then 10% of the population will have failed by 1000 km of operation.

Alternatively, the B10% life has the 90% reliability of a population at a specific point in product lifetime. The “BX” or “Bearing Life” nomenclature refers to the time at which X% of items in a population will fail. The B10 life metric became popular among product industries due to the industry’s strict requirement. Now B1, B10 and B50 lifetime values serve as a measurement for the reliability of a product (Figure 3.6).

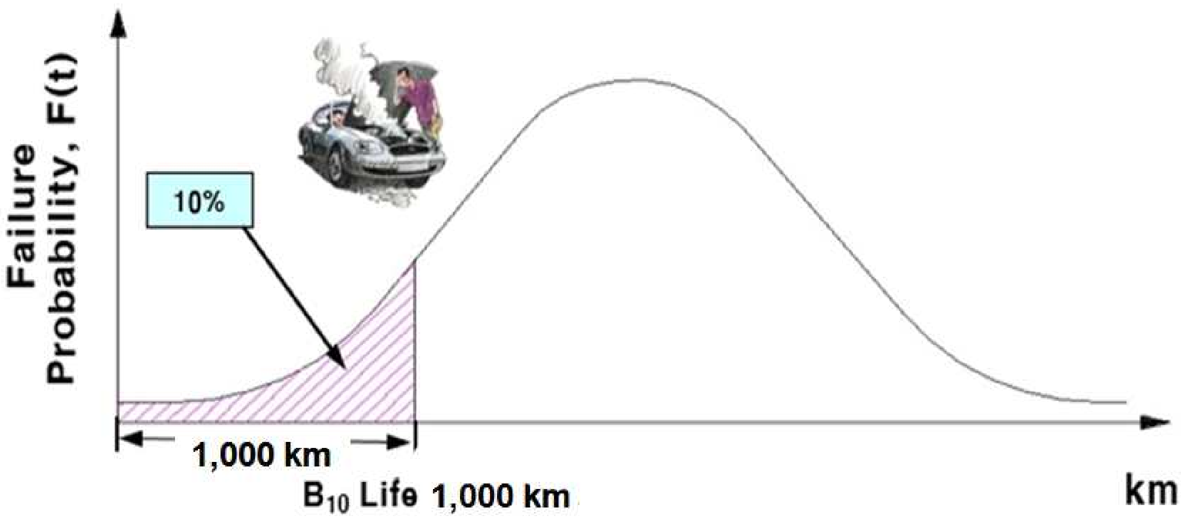



Fig. 3.6 Concept of $B_X$% life

### 3.3.5 The inadequacy of the MTTF (or MTBF) and the alternative metric BX life

Two representative metrics of reliability may describe product lifetime and the failure rate. The failure rate is adequate for understanding situations that include unit periods, such the annual failure rate. But the lifetime is frequently indexed using the mean time to failure.

The MTTF are misinterpreted. For instance, assume that the MTTF of a printed circuit assembly for television is 40,000 hours. Annual usage reaches 40,000 divided by 2,000 and become 20 years, which is regarded as the lifetime of the unit. The average lifetime of the television PCA is assumed to be 20 years. But because actual customer experience is that the lifetime of a television is a 10 years, this can lead to misjudgments or overdesign that wastes material.

MTTF is often assumed to be the same as lifetime because customers understand the MTTF as, literally, the average lifetime because customers understand the average lifetime of their appliances, so they suppose products will operate well with until they reach the MTTF. In reality, this does not happen. By definition, the MTTF is an arithmetic mean; specifically, it equals the period from the start of usage to the time that the 63rd item fails among 100 sets of one production lot when arranged in the sequence of failure times.

Under this definition, the number of failed televisions before the MTTF is reached would be so high that customers would never accept the MTTF as a lifetime index in the current competitive market. The products of first-class companies have fewer failures in a lifetime than would occur at the MTTF. In the case of home appliances, customers expect no failure for the 10 years. The failure of the TV is accepted from the customer’s perspective in the later time. Customers would expect the failure of all televisions once the expected use time is exceeded – 12 years in the case of a television set – but they will not accept major problems within the first 10 years,

The MTTF is inappropriate as a lifetime index of product design. Alternatively, it is reasonable to define the lifetime as the point in time when the accumulated failure rate has reached X %. This is called the BX life. The value X may vary from product to product, but for home appliance, the time to achieve a 10 to 30% cumulative failure rate failure rate, B20~30 life, exceeds 10 years. Thus, an average annual failure rate equals to 1 ~ 3%.

Now let’s calculate the B10 life from the MTTF of 40,000hours. Since the annual usage is 2,000 hours, the B10 life is 2 years, which means that the yearly failure rate would be 5%. The reliability level of this television, then, would not be acceptable in light of the current annual failure rate of 1~3%. The misinterpretation of reliability using an MTTF of 20 years would lead to higher service expenses if the product were released into the market without further improvement. The lifetime of a television is 12 to 14 years, not 20 years. Since random failure cannot account for the sharply increasing failure rate, the MTTF based on random failure or on an exponential distribution is obviously not the same as the design lifetime of product (Table 3.1).

Table 3.1 Results of 1987 Army SINCGARS Study*

| Vendor | *MIL - HDBK-217* MTBF(hour) | *Actual Test* MTBF(hour) |
|---|---|---|
| A | 2840 | 1160 |
| B | 1269 | 74 |
| C | 2000 | 624 |
| D | 1845 | 2174 |
| E | 2000 | 51 |
| F | 2304 | 6903 |
| G | 3080 | 3612 |
| H | 811 | 98 |
| I | 2450 | 472 |

* The transition from statistical field failure based models to physics-of-failure based models for reliability assessment of electronic packages, EEP-Vol. 10-2, 619-625, Advances in Electronic Packaging ASME 1995, - T.J. Stadterman et al

## 3.4 Fundamentals in Statistics and Probability Theory

When new product is launched, engineer obtains the failure data of products in marketplace. Company analyzes the failure behavior of product and its components through the statistics and probability theory. We can find the MTTF, MTBF, and BX life from these random events in field. For example, if thousand aircraft engine controllers are operating in service, collect all the times to failure data and analyze them. Finally, engineer estimates the lifetime of product from failure data.

### 3.4.1 Statistics

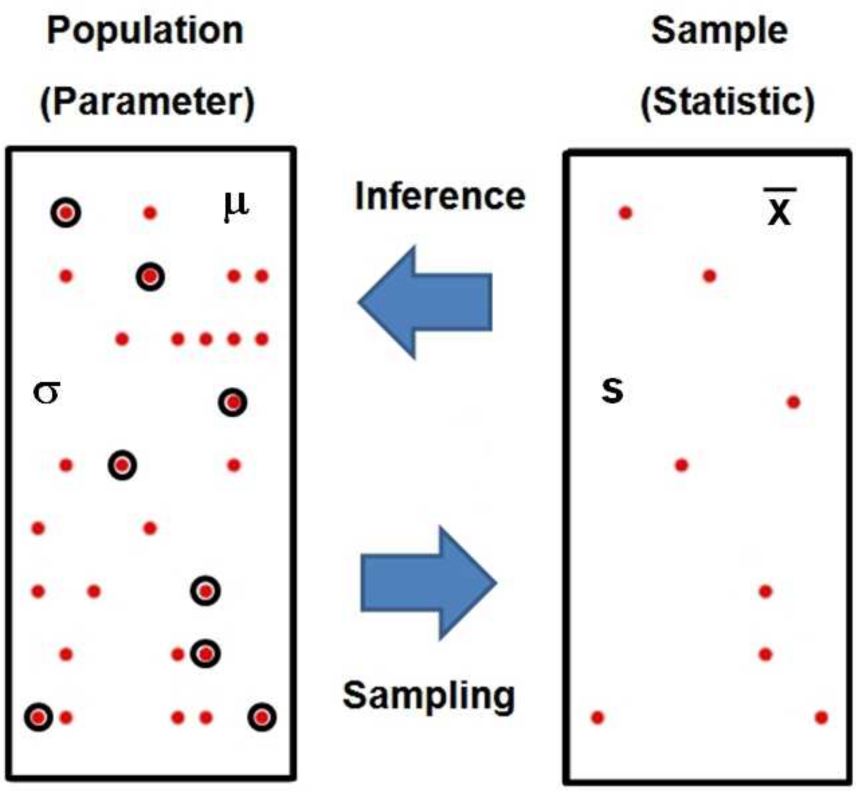



Fig. 3.7 Statistics concepts

As seen in Figure 3.7, statistics in engineering is a kind of the methodology for collecting, analyzing, interpreting and drawing conclusions from field information (or testing data). In other words, it is the methodology which engineering principles and statistical knowledge have been combined for interpreting and drawing conclusions from collected data. Everything that deals with the collection, processing, interpretation and presentation of field data belongs to the domain of statistics in designing new products. And it will be a basis of decision-making for new design.

There are descriptive statistics and inferential statistics in statistics. Descriptive statistics is a kind of methods for organizing and summarizing information. It includes the construction of graphs, charts, and tables, and the calculation of various descriptive measures such as averages, measures of variation, and percentiles. On the other hands, inferential statistics is concerned with using sample data to make an inference about a population of data. It includes methods like point estimation, interval estimation and hypothesis testing which are all based on probability theory.

To improve the product designs from test data, statistics in engineering is more than just the tabulation of numbers and its graphical presentation. Major factor in experimental design can find the fundamental principle of mechanical engineering – load analysis in Chapter 5. The statistical methods therefore are used to provide the solutions for 1) experimental design, 2) description: summarizing and exploring data, 3) inference: confirming whether product meets the reliability targets – 10 years of B1 life that will be the accumulated failure rate 1% (Chapter 8).

### 3.4.2 Probability

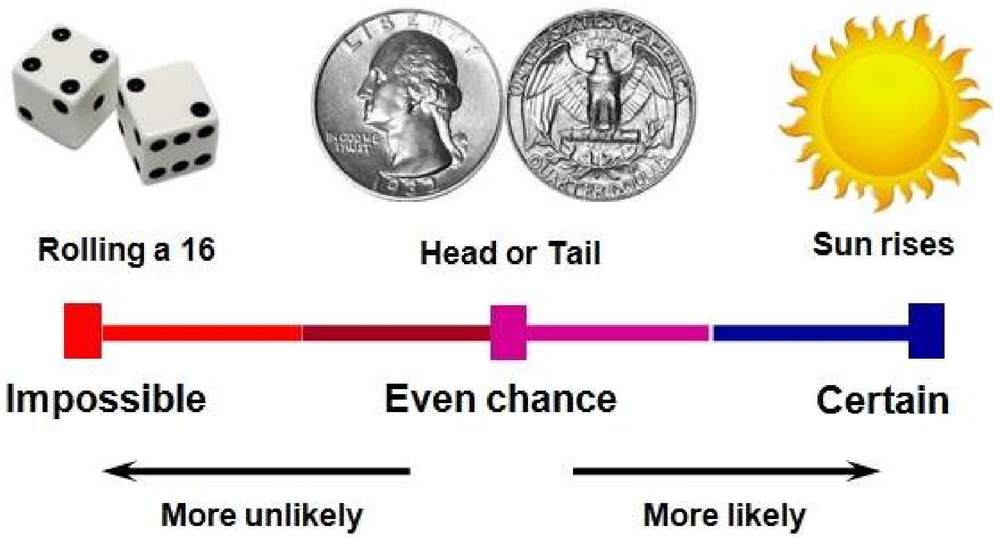



Fig. 3.8 Probability line

Lots of similar phenomena in our modern life happen frequently or repeatedly with chance, not certainly (Figure 3.8). The probability was originally established by gamblers who were interested in high stakes. To answer the basic question “how probable”, a game of gambling occurs. An early mathematician, Laplace and Pascal, invented the probability. That is, when *N* is the number of times that *X* occurs in the n repeated experiments, the probability of occurrence of event *X*, *P(X)*, can be defined as:

$$P(X) = \lim_{n\to\infty}\left(\frac{\text{number of cases favorable to X}}{\text{number of all possible cases}}\right) = \lim_{n\to\infty}(N/n) \qquad (3\text{-}10)$$

where *X* is a random variable

For example, if trials *n* approaches ∞, the probability of rolling a 1 with a die is:

$$P(X = 3) = \frac{1}{6} = 0.167 \qquad (3\text{-}11)$$

In modern theory, axiomatic probability has the assumptions:

- Each random variable *X* has $0 \leq P(X) \leq 1$.
- The area under the curve is equal to 1: $\int P(X)\,dX = 1$, where $0 \leq X \leq \infty$
- If $X_1, X_2, X_3, \ldots$ are random variables, then $P(X_1 \cup X_2 \cup \ldots) = P(X_1) + P(X_2) + \cdots$

A random variable, usually written *X*, is a variable whose possible values are numerical outcomes of a random phenomenon from statistical experiment. A random variable is a function from a sample space S into the real numbers (Figure 3.9).

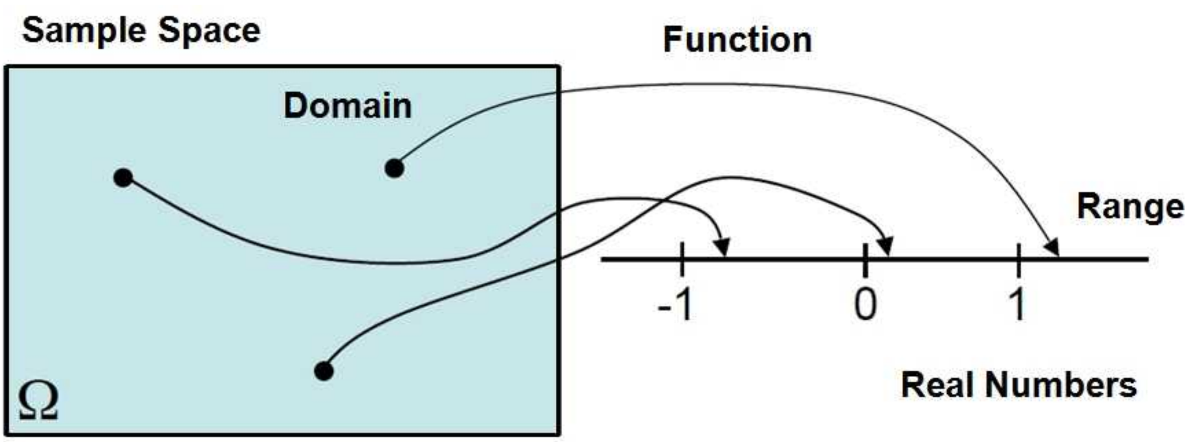



Fig. 3.9 Random variable

There are two types of random variables–discrete and continuous. A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4,   Examples of discrete random variables are wrong typing per page, defective product per lot in manufacturing, the number of children in a family, the number of defective light bulbs in a box of ten. On the other hands, a continuous random variable is one which takes an infinite number of possible values in real interval. Examples include height, weight, departing time of airplane, the time required to run a mile.

For a data set, mean refers to one measure of the central tendency either of a probability distribution or of the random variable characterized by that distribution. If we have a data set containing the failure times $t_1, t_2, \dots, t_n$, the mean is defined by the formula:

$$t_m = \frac{t_1 + t_2 + t_3 + \cdots + t_n}{n} = \frac{\sum_{i=1}^{n} t_i}{n} \tag{3-12}$$

The mean describes the parameter where the middle of the failure times approximately locates. The mathematical mean is affected to the lowest or highest failure times.

Median is the number separating the higher half of a data sample. In reliability testing, the median is the time in the middle of failure data. The median may be determined by the cumulative distribution function *F(t)*.

$$F(t_{median}) = 0.5 \tag{3-13}$$

The mathematical median is not affected to the lowest or highest failure times.

The mode is the value that appears most often in a set of data. In reliability testing, the mode is the most frequent failure time. The mode is the maximum value of the density function *f(t)*. It can be expressed as:

$$f'(t_{\mathrm{mod}\,e}) = 0 \tag{3-14}$$

The mathematical median is not affected to the lowest or highest failure times.

In statistics, the standard deviation (SD) is used to quantify the variation amount of a set of data values. In reliability testing, the standard deviation is the square root of the variance. This is expressed by

$$\sigma = \left[ \frac{\sum_{i=1}^{n} (t_i - t_m)^2}{n} \right]^{1/2} \tag{3-15}$$

The standard deviation has the same dimension as the failure times$t_i$

In probability theory, the expected value of a random variable is the average value of the experiment repetitively. For example, the expected value in rolling a six-sided die is 3.5. Let *X* be a continuous random variable with range [a, b] and probability density function *f(x)*. The expected value of *X* is defined as:

$$E(X) = \int_a^b x f(x) dx \tag{3-16}$$

As before, the expected value is also called the mean or average. In general, this is adequate for gambling. In technical reality, the failure probabilities happen to vary amounts in time. Because not all data is normally distributed, other distributions – Weibull is especially suited to analysis of product failures.

If select the failure data and draw histogram, we can find the skewed right (or left) histogram. We know that the probability concepts –mean, median and mode depend on the histogram of data (Figure 3.10).

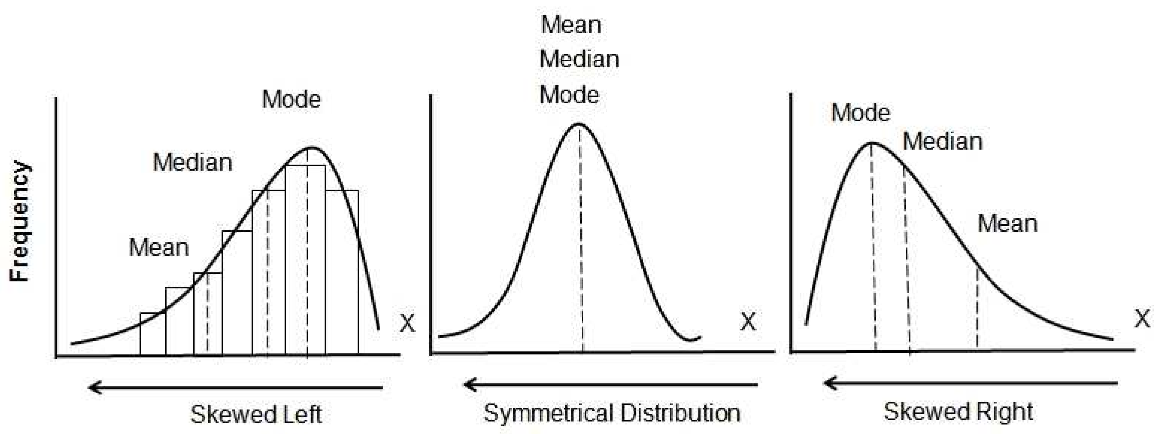



Fig. 3.10 Mean, median and mode for skewed left/right, and symmetric distribution

## 3.5 Probability Distributions

### 3.5.1 Introduction

A probability distribution is a mathematical function that provides the probabilities of occurrence of different possible outcomes in an experiment. Probability distributions are generally divided into two classes. A discrete probability distribution can be applicable to the scenarios where the set of possible outcomes is discrete. For example, there are binomial distribution and Poisson distribution. On the other hand, a continuous probability distribution can be applicable to the scenarios where the set of possible outcomes can take on values in a continuous range, such as the temperature on a given day. The normal distribution is a commonly encountered continuous probability distribution. And there are typical lifetime distribution models that can model failure times arising from a wide range of products, such as exponential distribution and Weibull distribution.

### 3.5.2 Binomial Distribution

Binomial distribution happens in everyday life. Typical occasions are in the following: 1) the number of heads/tails in a sequence of coin flips, 2) vote counts for two different candidates in an election, 3) the number of male/female employees in a company, 4) the number of successful sales calls, and 5) the number of defective products in manufacturing line.

There is several assumptions which accounts for a binomial distribution: 1) *n* fixed statistical experiments are conducted, 2) each trial is one of two outcomes – a success or a failure (Bernoulli trial), 3) the probability of "success" *p* is the same for each outcome, 4) the outcomes of different trials are independent, 5) we are interested in the total number of successes in these *n* trials.

Under the above assumptions, let random variable *X* be the total number of successes, the probability distribution of *X* is called the binomial distribution. Probability is expressed as:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, 2, ..., n. \tag{3-17}$$

And binomial Mean and Variance can be shown as:

$$\mu = E(X) = np, \quad Var(X) = np(1-p) \tag{3-18}$$

where the values of *n* and *p* are called the parameters of the binomial distribution.

When $p = 0.5$, the binomial distribution is symmetrical – the mean and median are equal. Even when

$p < 0.5$ (or $p > 0.5$), the shape of the distribution becomes more and more symmetrical the larger the value of $N$. Because the binomial distribution can quickly become unwieldy, there are approximations to the binomial that can be much easier to use when $N$ is large (Figure 3.11).

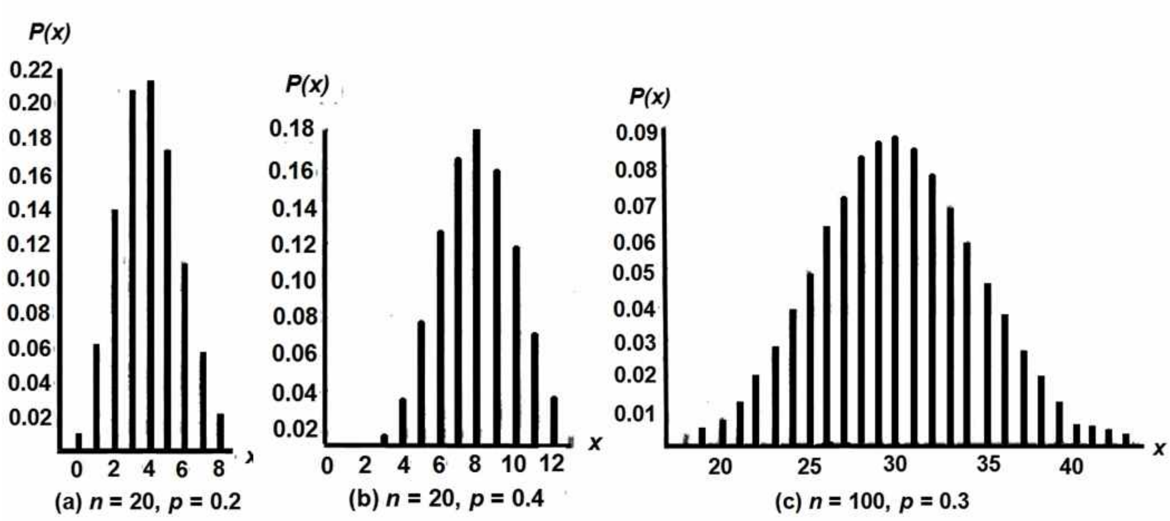



Fig. 3.11 Shape of the binomial distribution according to $n$ and $p$.

### 3.5.3 Poisson Distributions

The Poisson distribution is named after Simeon Poisson (1781-1840), a French mathematician, and used in situations where big declines in a time period occurs with a specific average rate, regardless of the time that has elapsed. More specifically, this distribution is used when the number of possible events is large, but the occurrence probability over a specified time period is small (Figure 3.12).

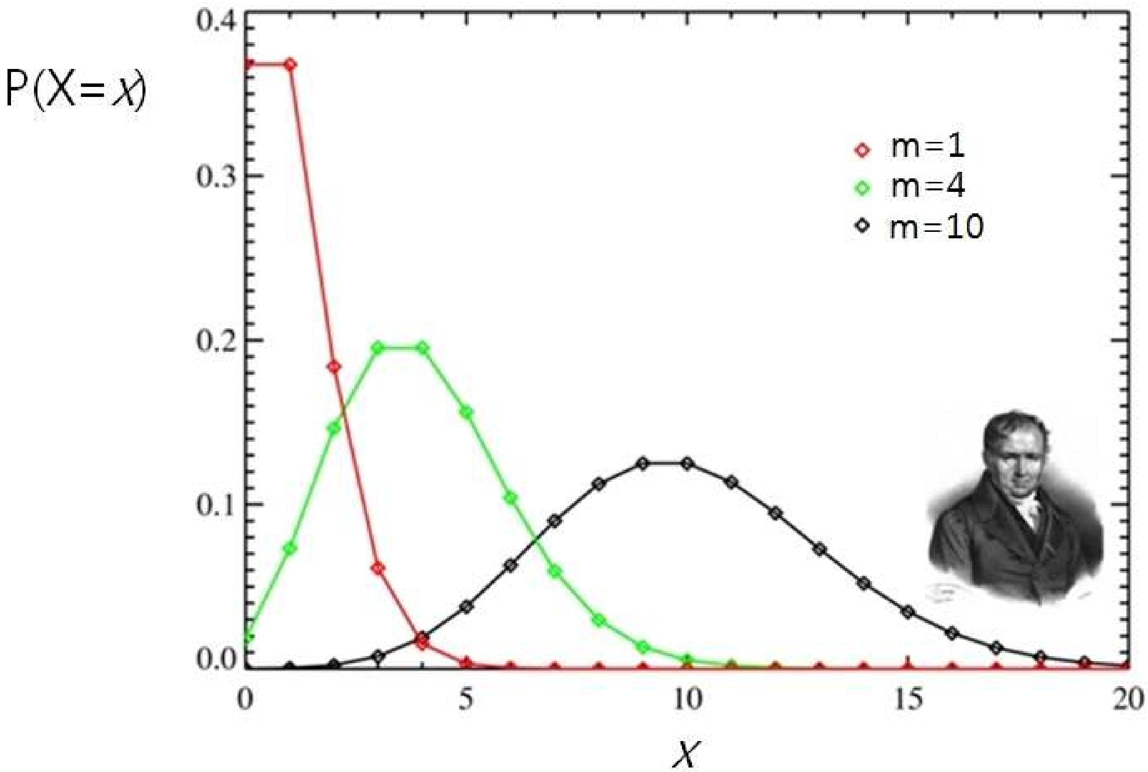



Fig. 3.12 Poisson Distributions

Two examples of such a situation are as follows:

- A store that rents books has an average rental of 200 books every Saturday night. Using this data, you can predict the probability that more books will sell (perhaps 300 or 400) on the following Saturday nights.
- Another example is the number of diners in a certain restaurant every day. If the average number of diners for seven days is 500, you can predict the probability of a certain day having more customers.

A Poisson distribution has the following properties: 1) The experiment results in outcomes that can be classified as successes or failures. 2) The average number of successes (μ) that occurs in a specified region is known. 3) The probability that a success will occur is proportional to the size of the region. 4) The probability that a success will occur in an extremely small region is virtually zero.

This distribution also has applications in many reliability areas when one is interested in the occurrence of a number of events that are of the same type. Each event's occurrence is denoted as a time scale and each event represents a failure.

If probability *p* is very small and trial is far enough, Poisson probability function from Equation (3-17) can be approximated. That is,

$$
\begin{aligned}
{}_nC_x p^x (1-p)^{n-x} &= \frac{n(n-1)\cdots(n-x+1)}{x!}\left(\frac{m}{n}\right)^x \left(1-\frac{m}{n}\right)^n \Big/ \left(1-\frac{m}{n}\right)^x \\
&= \frac{m^x}{x!}\underbrace{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}_{A}\underbrace{\left(1-\frac{m}{n}\right)^n}_{B} \Big/ \underbrace{\left(1-\frac{m}{n}\right)^x}_{C}
\end{aligned}
\tag{3-19}
$$

As *n* increases, *A*~*C* can be rearranged.

$$A = 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\xrightarrow[n\to\infty]{} 1 \tag{3-20a}$$

$$B = \left(1-\frac{m}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-m} \tag{3-20b}$$

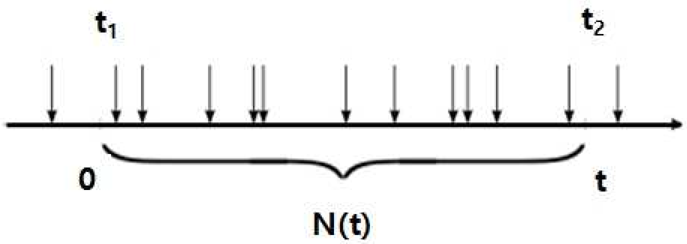
$$C = \left(1-\frac{\lambda}{n}\right)^x \xrightarrow[n\to\infty]{} 1 \tag{3-20c}$$

Therefore, we can summarize as following:

$$P(X = x) = \frac{(m)^x e^{-m}}{x!} \tag{3-21}$$

**Poisson process:** The Poisson process is one of the most important random processes in probability theory. It is widely used to model random “points” in time and space. Several important probability distributions arise naturally from the Poisson process such as the exponential distribution. That is, the constant failure rate in bathtub can be described by a Poisson process.

In the following it is instructive to think that the Poisson process we consider represents discrete failure time.




Mathematically the process is described by the so called counter process $N_t$ or *N(t)*. The counter tells the number of failure that have occurred in the interval (0, *t*) or, more generally, in the interval ($t_1$, $t_2$).

*N(t)* = number of failure in the interval (0,*t*) (the stochastic process)

$N(t_1, t_2)$ = number of arrival in the interval ($t_1$, $t_2$) (the increment process $N(t_2) - N(t_1)$

A counting process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda$ if all the following conditions hold:

(i) $N(0) = 0$,

(ii) *N(t)* has independent increments,

(iii) $N(t) - N(s) \sim$ Poisson $(\lambda(t - s))$ for $s < t$.

### 3.5.4 Normal Distribution

Normal distribution was first introduced by French mathematician Abraham de Moivre (1667 –1754).

After that, German mathematician and physicist Johann Carl Friedrich Gauss (1777 – 1855) made significant contributions to many fields – physics and astronomy. The normal distribution function or Gaussian distribution function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \quad \text{for } -\infty < x < \infty \tag{3-22}$$

Thus, the curve has two parameters – mean $\mu$ and standard deviation $\sigma$, which has bell shaped and is symmetric around the mean $\mu$. If a random variable $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$, the notation will be represented by $X \sim N(\mu, \sigma^2)$. The characteristics of such normal distribution can be summarized as:

- Bell shaped and continuous for all values of $X$ between $-\infty$ and $\infty$.
- Symmetric around the mean $\mu$. Probability for the left and right of mean is 0.5 separately.
- Depends on parameters – $\mu$ and $\sigma$, there are infinite normal distributions.
- Probability for interval $[\mu - \sigma \leq X \leq \mu + \sigma]$ is 0.6826. Probability for interval $[\mu - 2\sigma \leq X \leq \mu + 2\sigma]$ is 0. 9544. And probability for interval $[\mu - 3\sigma \leq X \leq \mu + 3\sigma]$ is 0. 997. That is, most of data in normal distribution locate around mean and there is very little data at more than three times of standard deviation.

The normal distribution is the most important distribution in statistics, since it arises naturally in numerous applications. The key reason is that large sums of (small) random variables often turn out to be normally distributed. When random variable X follows N(μ, σ2), probability for interval [a, b] will be the area of f(x) that is enclosed by a and b on x axis. The mathematical area is given by:

$$P(a \le X \le b) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} dx \tag{3-23}$$

However, this integral is very difficult. Fortunately, in case of a normal random variable *X* with arbitrary parameters *μ* and *σ*, we can transform it into a standardized normal random variable *Z* with parameters 0 and 1 (Figure 3.13).

$$Z = \frac{X-\mu}{\sigma} \tag{3-24}$$

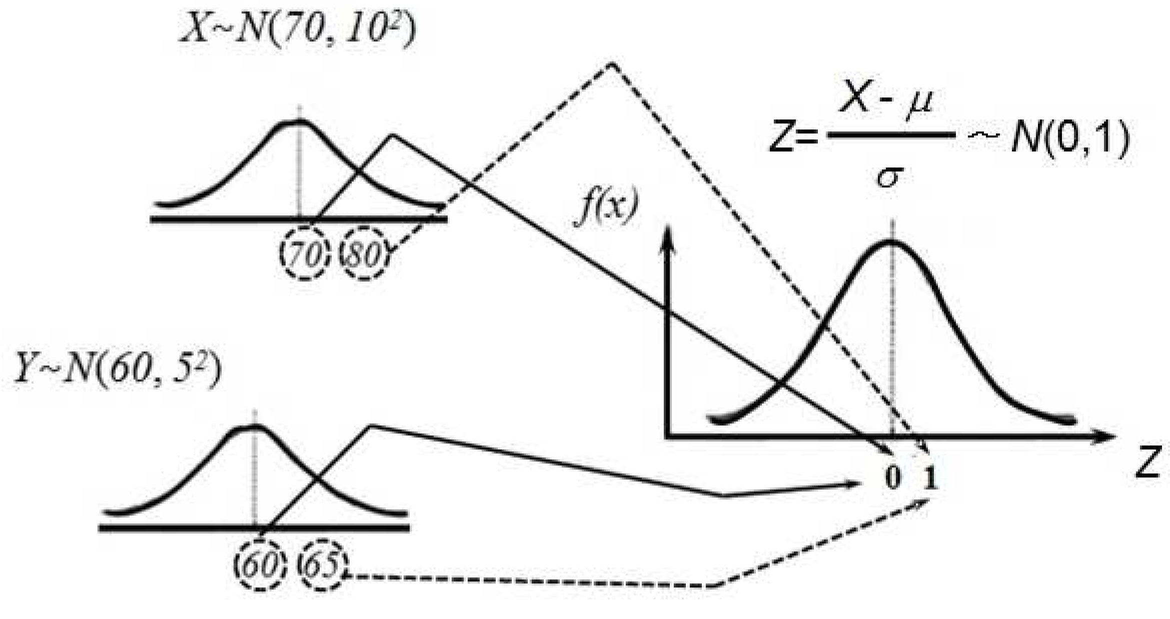



Fig. 3.13 Standardization normal distribution

### 3.5.5 Exponential Distributions

The exponential distribution, with only one unknown parameter, is the simplest of all life distribution models in reliability engineering. Many engineering modules exhibit constant failure rate during the product lifetime if they follows the exponential distribution. Also, it is relatively easy to handle in performing reliability analysis. The key equations for the exponential are shown below:

From Poisson distribution Equation (3-21), let $N(t)$ be a Poisson process with rate $\lambda$.

Let $X_1$ be the time of the first failure.

$$R(t) = P(X_1 > t) = P(\text{no failure} \quad \text{in } (0,t]\,) = \frac{(m)^0 e^{-m}}{0!} = e^{-m} = e^{-\lambda t} \tag{3-25}$$

So the cumulative distribution function also is obtained as:

$$F(t) = 1 - e^{-\lambda t} \tag{3-26}$$

If the cumulative distribution function is differentiated, the probability density function is obtained as:

$$f(t) = \lambda e^{-\lambda t} \qquad t \geq 0,\ \lambda > 0 \tag{3-27}$$

Failure rate $\lambda(t)$ is defined by:

$$\lambda(t) = f(t)/R(t) = \lambda e^{-\lambda t}/e^{-\lambda t} = \lambda \tag{3-28}$$

Note that the failure rate reduces to the constant $\lambda$ for any time. The exponential distribution is the only distribution to have a constant failure rate. Also, another name for the exponential mean is the

Mean Time To Fail or MTTF and we have MTTF = $1/\lambda$. In general, if product follows the exponential distribution, mean time to failure (MTTF) is 0.63 at $1/\lambda$. Exponential distribution model is useful as following:

1. Because of its constant failure rate property, the exponential distribution is an excellent model for the long flat "useful life" portion of the Bathtub Curve. Since most mechanical systems spend most of their lifetimes in this portion of the Bathtub Curve, this justifies frequent use of the exponential distribution.
2. Just as it is often useful to approximate a curve by piecewise straight line segments, we can approximate any failure rate curve by week-by-week or month-by-month constant rates that are the average of the actual changing rate during the respective time durations.
3. Some natural phenomena have a constant failure rate (or occurrence rate) property. The exponential model works well for inter arrival times while the Poisson distribution describes the total number of events in a given period (Figure 3.14).

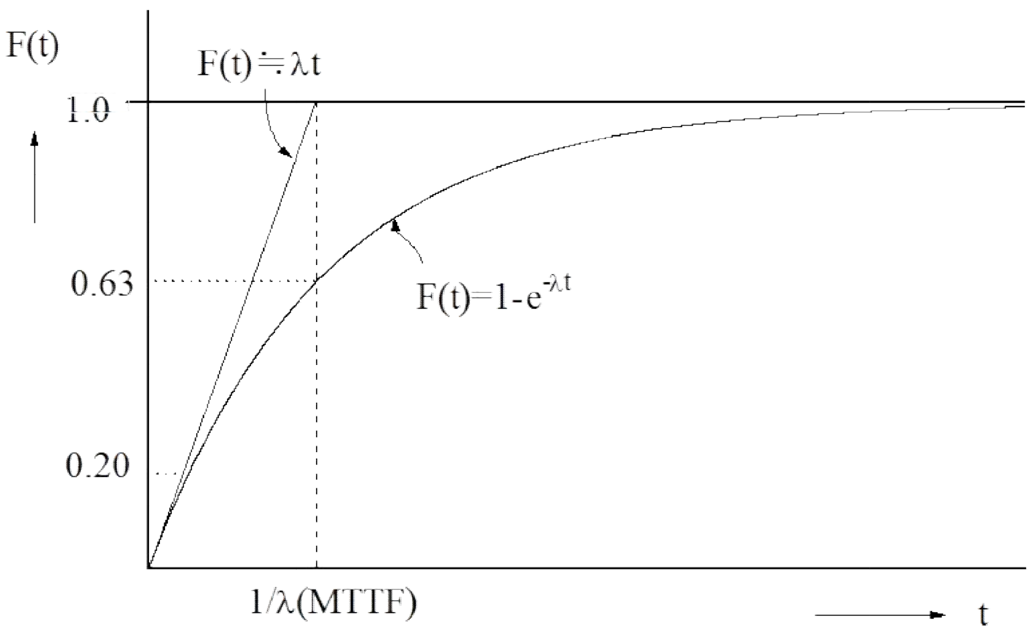



Fig.3.14 Cumulative distribution function *F(t)* of exponential distribution

**Example 3.2** In a certain region, the number of traffic accidents averages one per two days happens. Find the probability that $x$ = 0, 1, 2 accidents will occur in a given day.

So the number of traffic accidents averages one per two days, $m = \lambda t = 0.5$,

X=0, $f(0) = \dfrac{(0.5)^0 e^{-0.5}}{0!} = 0.606$,

Accident days = 365 day $\times$ 0.606 = 221 day

X=1, $f(1) = \dfrac{(0.5)^1 e^{-0.5}}{1!} = 0.303$,

Accident days = 365 day $\times$ 0.303 = 110 day

X=2, $f(2) = \dfrac{(0.5)^2 e^{-0.5}}{2!} = 0.076$,

Accident days = 365 day $\times$ 0.076 = 27 day

**Example 3.3** TV is selling in a certain area and average failure rate is 1%/2000hr. If 100 TV units are sampling and testing for 2,000 hours, find the probability that no accidents, $x$ = 0, will occur.

$$m = n \cdot \lambda \cdot t = 100 \times 0.01/2000 \times 2000 = 1$$

Because no accident, the probability is

$$X = 0.\ f(0) = \frac{(1)^0 e^{-1}}{0!} = 0.36$$

We can estimate the confidence level is 63% for 100 TV units. If no accidents, x = 0, the confidence level would like to increase to 90%, how many TV units will it requires?

$$X = 0,\ f(0) = \frac{(m)^0 e^{-m}}{0!} = 0.1$$

So if *m*=2.3, the required sample size *n* = 230 will be obtained as

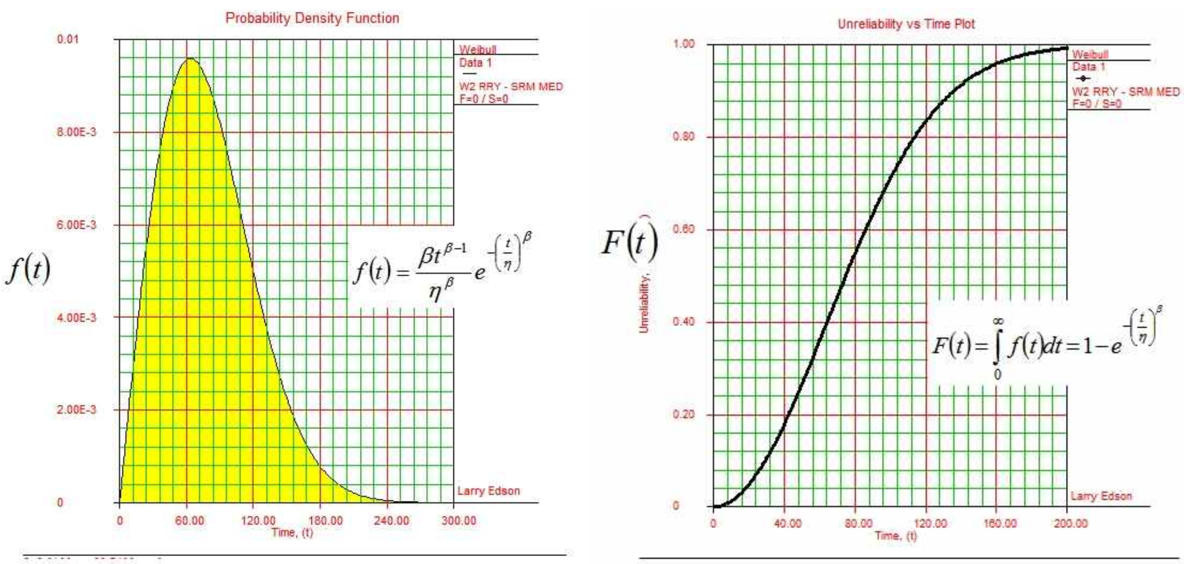
$$m = n \cdot \lambda \cdot t = n \times 0.01/2000 \times 2000 = 2.3$$

## 3.6 Weibull distributions and its applications

### 3.6.1 Weibull Parameter Estimation

The main challenge of fitting distributions to reliability data is to find the type of distribution and the values of the parameters that give the highest probability of producing the observed data. One of the most common probability density functions used in industry is the Weibull Distribution. It was invented by W. Weibull in 1937, who found it to be so flexible that it effectively worked on a very wide range of problems. Many other extensions of the Weibull distribution have been proposed to enhance its capability to fit diverse lifetime data since 1970s.

This is mainly due to its weakest link properties, but other reasons have its increasing failure rate with component age and the variety of distribution shapes. The increasing failure rate accounts to some extent for fatigue failures. The density function depend upon the shape parameter $\beta$. For low $\beta$ values ($\beta$ < 1), the failure behavior can be similar to the exponential distribution. For $\beta$ >1, the density function always begins at *f(t)* = 0, reaches a maximum with increasing lifetime and decreasing slowly again. Two-parameter fit is more common in reliability testing and more efficient with the same sample size. If the shape parameter is assumed to be known, it can reduce fit to one parameter (Figure 3.15).




(a) Probability density function (b) Cumulative distribution function

Fig. 3.15 Probability density and cumulative distribution function on the Weibull distributions

Because the Weibull is a very flexible life distribution model with two parameters, the probability density function is defined as:

$$f(t) = \frac{\beta t^{\beta-1}}{\eta^{\beta}} e^{-\left(\frac{t}{\eta}\right)^{\beta}}, \ t \geq 0,\ \eta > 0,\ \beta > 0 \tag{3-29}$$

where $\eta$ and $\beta$ are characteristic life and shape parameters, respectively.

When Equation (3.29) is integrated, the cumulative distribution function is obtained as:

$$F(t) = \int_{0}^{\infty} f(t)dt = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}, \quad t > 0 \tag{3-30}$$

Reliability function *R(t)* is defined as:

$$R(t) = 1 - F(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}, \quad t > 0 \tag{3-31}$$

Hazard (or failure) rate function $\lambda(t)$ can be described by:

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta}, \quad t > 0 \tag{3-32}$$

For $\beta$ =1 and 2, the exponential and Rayleigh distributions are especially called in Weibull distribution, respectively.

Abernathy (1996) recommended Weibayes as best practice for all small samples, 20 failures or less, if a reasonable point estimate of $\beta$ is available. Using Weibayes method, $\hat{\beta}$ can be assumed from some historical data or prior knowledge, leaving $\hat{\eta}$ as the single parameter, which can be estimated using maximum likelihood as:

$$\hat{\eta} = \left[ \sum_{i=1}^{N} \left( \frac{t_i^{\beta}}{r} \right)^{\frac{1}{\beta}} \right] \tag{3-33}$$

where $t$ is testing time, r is the number of failed sample, $N$ is the total number of failures

The various failure rates of the Weibull distribution specified in bathtub curve can be divided into three regions.

- β < 1.0: Failure rates decrease with increasing lifetime (early failure)
- β = 1.0: Failure rates are constant.
- β > 1.0: Failure rates increase with increasing lifetime

The characteristic lifetime $\eta$ is assigned to the cumulative distribution function *F(t)* = 63.2% for exponential distribution.

A shape parameter estimated from the data affects the shape of a Weibull distribution, but does not affect the location or scale of its distribution. The spread of the shape parameters represents the confidence intervals and a dependency of the stress level. A summary of the determined shape parameters is approximately described as:

- High temperature, high pressure, high stress: 2.5 < $\beta$ <10
  - Low cycle fatigue : depend on cycle times
  ex) disk, shaft, turbine
- Low temperature, low pressure, low stress: 0.7 < $\beta$ < 2
  - Degradation : depend on use time
  ex) electrical appliance, pump, fuel control value

Shape parameter $\beta$ of a certain component would be invariable, but its characteristic life $\eta$ varies according to use condition and material status. Thus, shape parameter ($\beta$) would be estimable and then will be confirmed after test. The density function and hazard rate function for the Weibull distribution range from shape parameters $\beta \approx 1.0 \sim 5.0$.

### 3.6.2 Weibull Parameter Estimation

There are two widely used general methods that can estimate life distribution parameters from a particular data set: 1) Graphical estimation in Weibull plotting, 2) Maximum Likelihood Estimation (MLE) and median rank regression (MRR). Weibull plotting is a graphical method for informally checking on the assumption of Weibull distribution model and also for estimating the two Weibull parameters– shape parameter and characteristic life.

For a Weibull probability plot draw a horizontal line from the y-axis to the fitted line at the 62.3 percentile point. That estimation line intersects the line through the points at a time that is the estimate of the characteristic life parameter $\eta$. In order to estimate the slope of the fitted line (or the shape parameter $\beta$), choose any two points on the fitted line and divide the change in the y variable by the change in x variable.

There are several different methods of estimating Weibull parameters such as Maximum Likelihood and median rank regression (MRR). Olteanu and Freeman [1] have investigated the performance of MLE and MRR methods and concluded that the median rank regression method is the best combination of accuracy and ease of interpretation when the sample size and number of suspensions are small. This method is popular in industry because fitting can be easily visualized.

The Median Ranks method is used to obtain an estimate of the unreliability for each failure.

First, we will examine the fitting of two-parameter Weibull using median rank regression method. Median rank regression determines the best-fit straight line by least squares regression curve fitting. This method proceeds as follows:

1) Obtain failure data

2) The cumulative distribution function *F(t)* has an *S*-like shaped curve (Figure 3.16(a)). With a Weibull Probability Paper, If plotted the function *F(t)* in Weibull Probability Paper, it is useful to evaluate the lifetime of mechanical system in reliability testing.

After taking inverse number and logarithmic transformation from reliability equation (3-31), it can be expressed as:

$$\ln(1 - F(t))^{-1} = \left(\frac{t}{\eta}\right)^{\beta} \tag{3-34}$$

After taking logarithmic transformation one more time, it can be expressed as:

$$\ln\left(\ln\frac{1}{1 - F(t)}\right) = \beta \cdot \ln t - \beta \ln \eta \tag{3-35}$$

If *F* is sufficiently small, then equation (3-35) can be modified as:

$$\ln\left(\ln\frac{1}{1 - F(t)}\right) \cong \ln F(t) = \beta \cdot \ln t - \beta \ln \eta \tag{3-36}$$

Equation (3-36) corresponds to a linear equation in the form. That is,

$$y = ax + b \tag{3-37}$$

with the variables

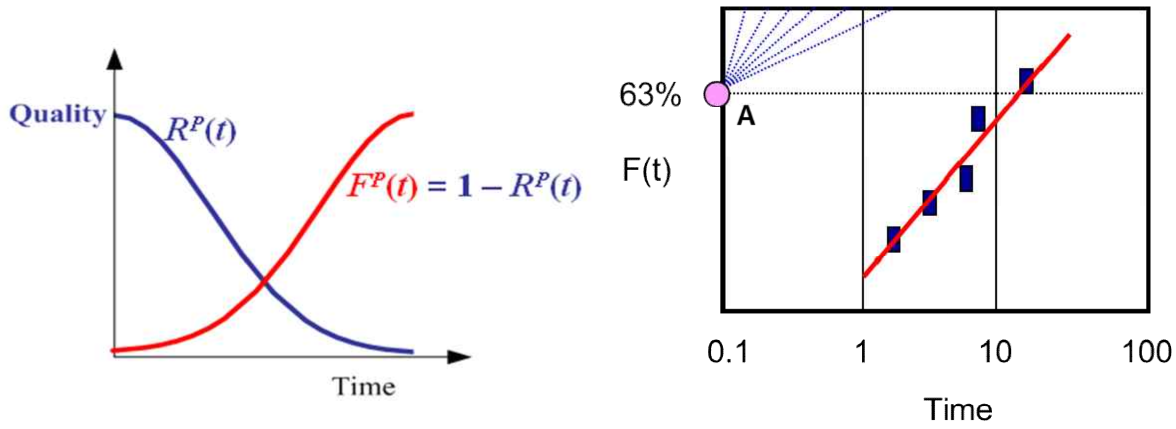
$a = \beta$ (slope) (3-38a)

$b = -\beta \ln \eta$ (axis intersection) (3-38b)

$x = \ln t$ (abscissa scaling) (3-38c)

$y = \ln(-\ln(1 - F(t)))$ (ordinate scaling) (3-38d)

That is, two parametric Weibull distribution can be expressed as a straight line on the Weibull Probability Paper. The slope of its straight line becomes the shape parameter $\beta$ (See Figure 3.16 (b)).




(a) $R(t) = 1 - F(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}$ (b) $\ln\left(\ln \frac{1}{1 - F(t)}\right) \cong \ln F(t) = \beta \cdot \ln t - \beta \ln \eta$

Fig. 3.16 A plotting of Weibull probability paper

3) Calculate median ranks: Rank failure times in ascending order. Mean ranks are less accurate for the skewed Weibull distribution, therefore median ranks are preferable. Median ranks can be calculated as follows:

$$\sum_{k=i}^{N} \binom{N}{k} (MR)^k (1-MR)^{N-k} = 0.5 = 50\% \tag{3-39}$$

Bernard used an approximation of it as follows:

$$F(t_i) \approx \frac{i-0.3}{n+0.4} \times 100 \tag{3-40}$$

where *i* is failure order number, *N* is total sample size

Step 1: Rank the times-to-failure in according to ascending order $t_1 < t_2 \ldots. < t_n$

| i | 1 | 2 | 3 | ……… | r-1 | r |
|---|---|---|---|---|---|---|
| $t_i$ | $t_1$ | $t_2$ | $t_3$ | | $t_{r-1}$ | $t_r$ |

By ordering the failure times, an overview is won over the timely progression of the failure times. In addition, the ordered failure times are required in the next analysis step and are referred to as order statistics. Their index corresponds to their rank.

Step 2: Determine the failure probability $F(t_i)$ of the individual order statistics from equation (3-51)

Step 3: Enter the coordinate ($t_i$, $F(t_i)$) in the Weibull probability paper.

Step 4: Approximate sketch the best fit straight line through the entered points and determine the

Weibull parameters $\hat{\beta}$. At the F(t) = 63.2% ordinate point, draw a straight horizontal line until this line intersects the fitted straight line. Draw a vertical line through this intersection until it crosses the abscissa. The value at the intersection of the abscissa is the estimate of $\hat{\eta}$ (Figure 3.17)

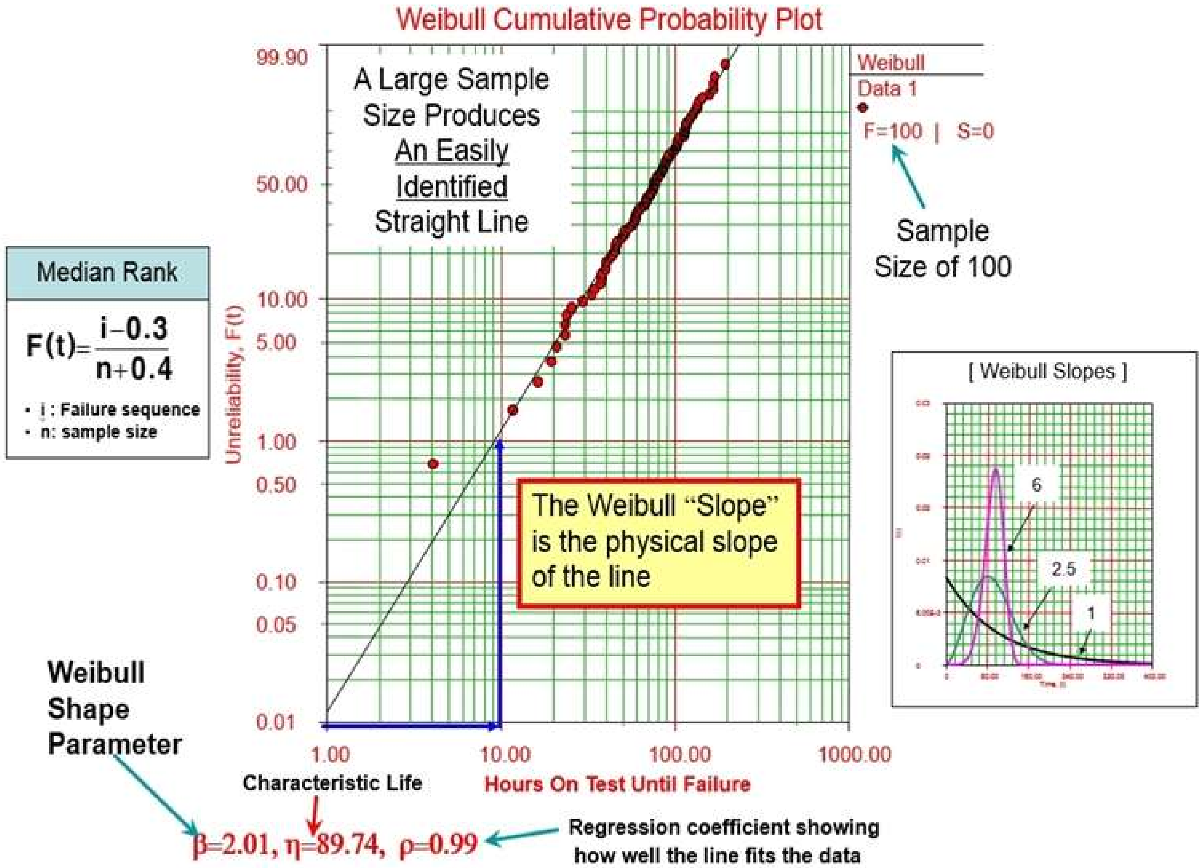



Fig.3.17 A typical characteristics of Weibull plot with a large sample size

The best fit line in least squares is defined as the one that minimizes the sum of squared differences between the true and estimated values. Suppose we have a set of points ($x_1$, $y_1$), ($x_2$, $y_2$),…,($x_n$, $y_n$), obtained by linearization of life data. We can use the ordinary least squares to estimate the slope $\hat{b}$ and the intercept $\hat{a}$ of the straight line defined by the equation $\bar{y} = \hat{a} + \hat{b}\bar{x}$ as follows:

$$\hat{b} = \frac{\sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{N}}{\sum_{i=1}^{n} x_i^2{}_i - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{N}} \tag{3-41}$$

$$\hat{a} = \frac{\sum_{i=1}^{n} y_i}{N} - \hat{b}\frac{\sum_{i=1}^{n} x_i}{N} = \bar{y} - \hat{b}\bar{x} \tag{3-42}$$

**Example 3.4** Assume that six automobile units are tested. All of these units fail during the test after operating the following number of hours $t_i$: 93, 34, 16, 120, 53 and 75. Estimate the values of the parameters for a two-parameter Weibull distribution and determine the reliability of the units at a time of 15 hours.

First, rank the times-to-failure in ascending order as shown next.

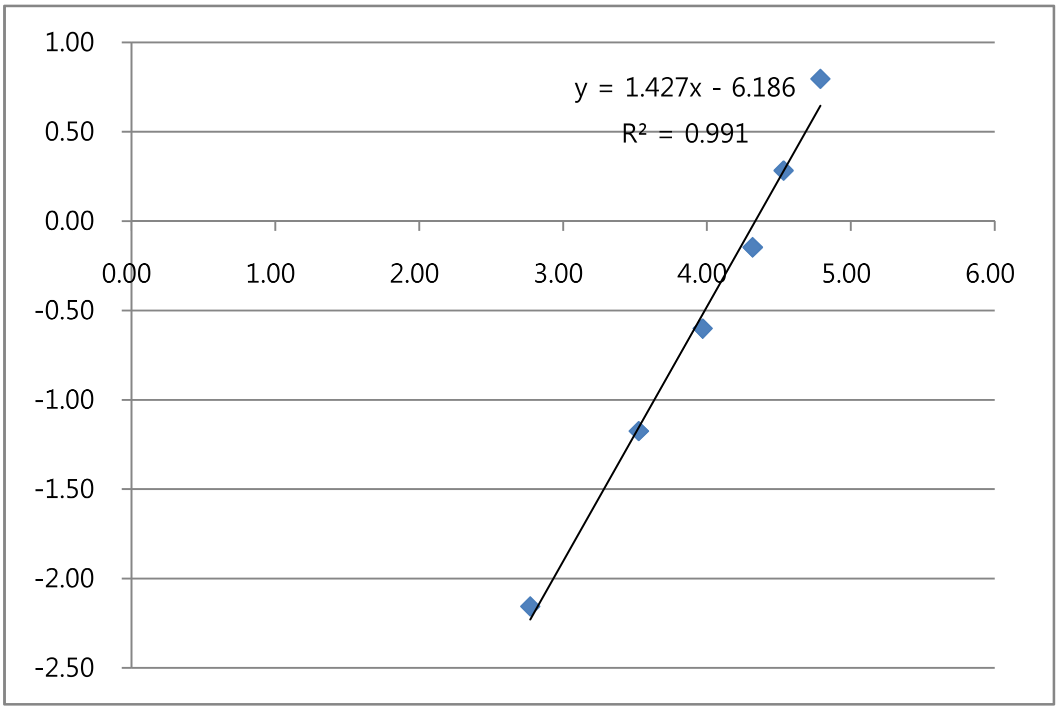
| Failure order ( i) | Time-to-failure, hours ( $t_i$ ) | F($t_i$), % |
|---|---|---|
| 1 | 16 | 10.94 |
| 2 | 34 | 26.56 |
| 3 | 53 | 42.19 |
| 4 | 75 | 57.81 |
| 5 | 93 | 73.44 |
| 6 | 120 | 89.06 |

Second, by using Excel, approximate sketch the best fit straight line through the entered points (ln($t_i$), ln[-ln(1-F(t))]).

| Failure order ( i) | $\ln(t_i)$ | $\ln[-\ln(1-F(t))]$ |
|---|---|---|
| 1 | 2.77 | -2.16 |
| 2 | 3.53 | -1.18 |
| 3 | 3.97 | -0.60 |
| 4 | 4.32 | -0.15 |
| 5 | 4.53 | 0.28 |
| 6 | 4.79 | 0.79 |

We can obtain the estimated shape parameter $\hat{\beta}$ = slope = 1.427, estimated characteristic life $\hat{\eta}$ (Q(t) is 63.2% ordinate point) = $e^{\frac{6.187}{1.427}} = 76.3226$ hours

where ln[-ln(1-0.63)] = -0.00576, -0.00576 = 1.427x - 6.186




A Weibull distribution with the shape parameter $\beta$ = 1.427 and $\eta$ = 76.32 hour is drawn on the Weibull Probability Paper. The cumulative distribution function is described as

$$F(t) = 1 - e^{-\left(\frac{t}{76.32}\right)^{1.43}}$$

In result a straight line is sketched with slope $\beta$ =1.4 on the Weibull Probability Paper. The characteristic lifetime is 76.0 hour when the cumulative distribution function, i *F(t)* is 63% (Figure 3.18).

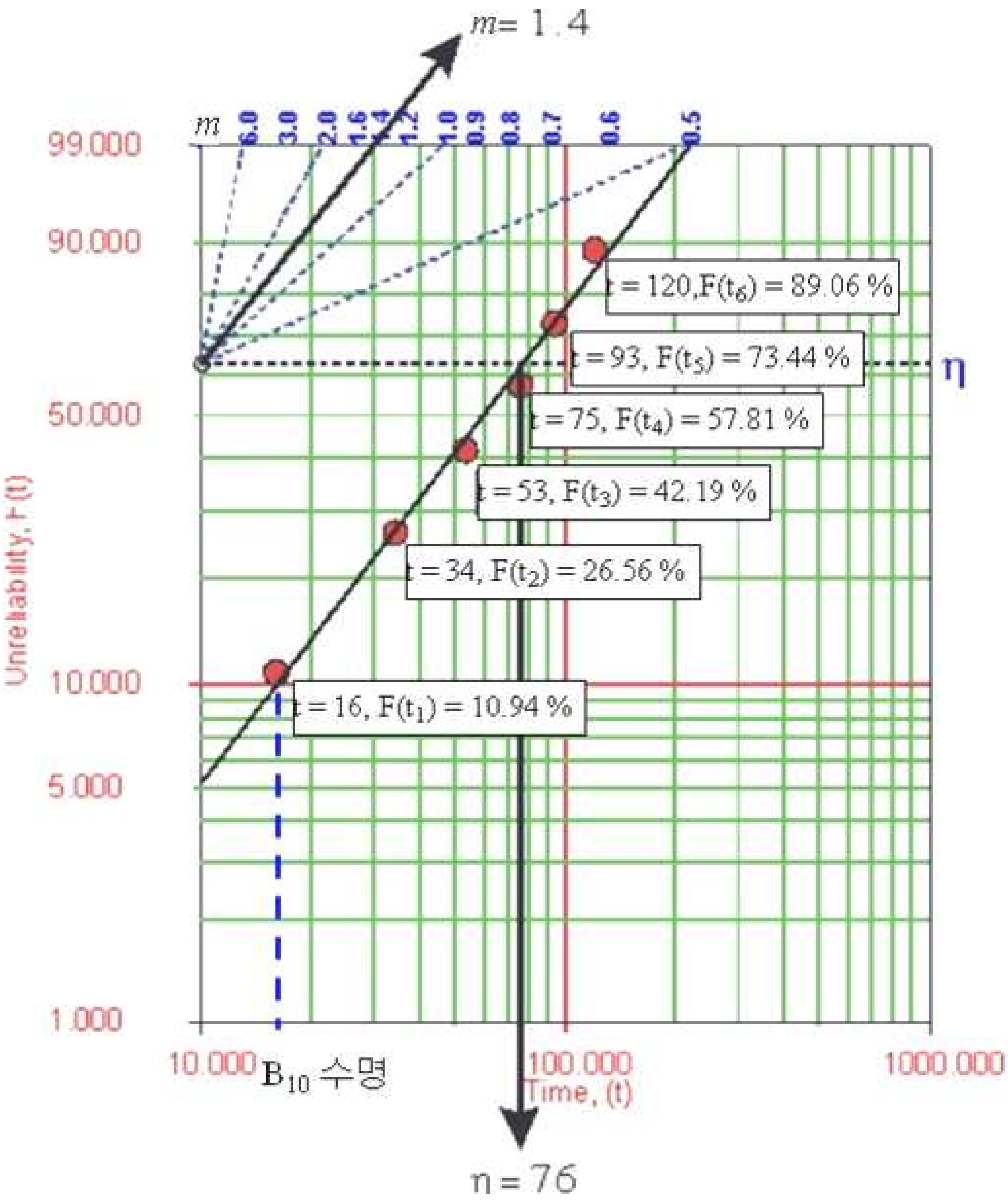



Fig. 3.18 How to use Weibull CDF

### 3.6.3 Confidence Interval

In statistics the purpose of taking a random sample from population is to approximate the mean of the population. Because life data analysis results are estimates based on the observed lifetimes of a sampling of units, there is uncertainty in the results due to the limited sample sizes. How well the sample statistic estimates the underlying population value is always an issue. A confidence interval addresses this issue because it provides a range of values which is likely to contain the population parameter of interest (Figure 3.19).

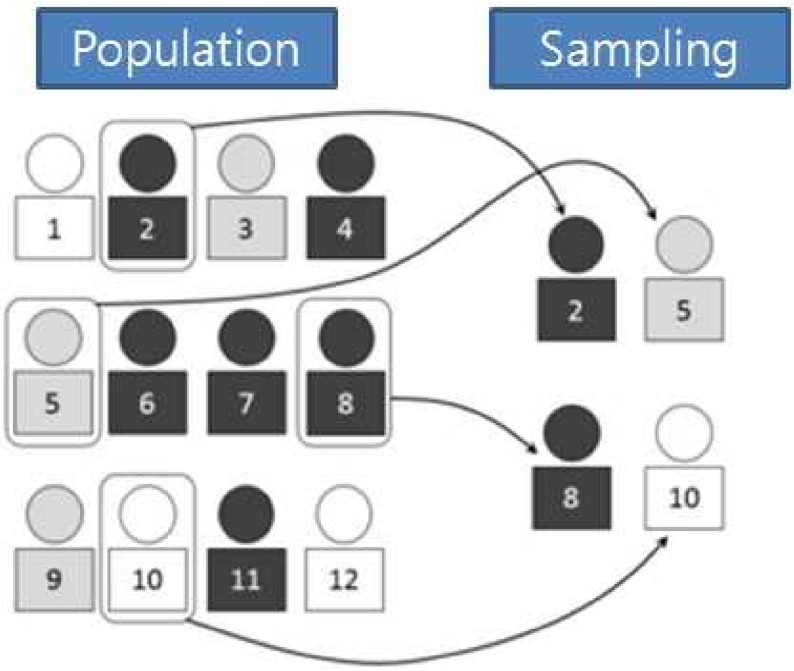



Fig.3.19 Concept of confidence interval

A confidence interval (CI) is characterized as the probability that a random value lies within a certain range. CI is represented by a percentage. For example, a 90% confidence interval implies that in 90 out of 100 cases, the observed value falls within this certain interval. After any particular sample is taken, the population parameter is either in the interval realized or not. The desired level of confidence is set by the researcher. A 90% confidence interval reflects a significance level of $\alpha$ = 0.1. The confidence level also depends on the product field.

The average of failure times can often deviate within a certain range. The Weibull line may describe experimental results. If the median is used to determine $F(t_i)$, 50% of the experimental results lie below the Weibull line. To know the truth of the Weibull line, it is necessary to determine its confidence interval.

Over an observation of several test samples, the Weibull line drawn in Figure 3.20 is the most probable in the middle – median values and its confidence intervals. The line in the middle represents the population mean – observed over several test specimens – thus 50% of the cases lie above and 50% lie below this line.

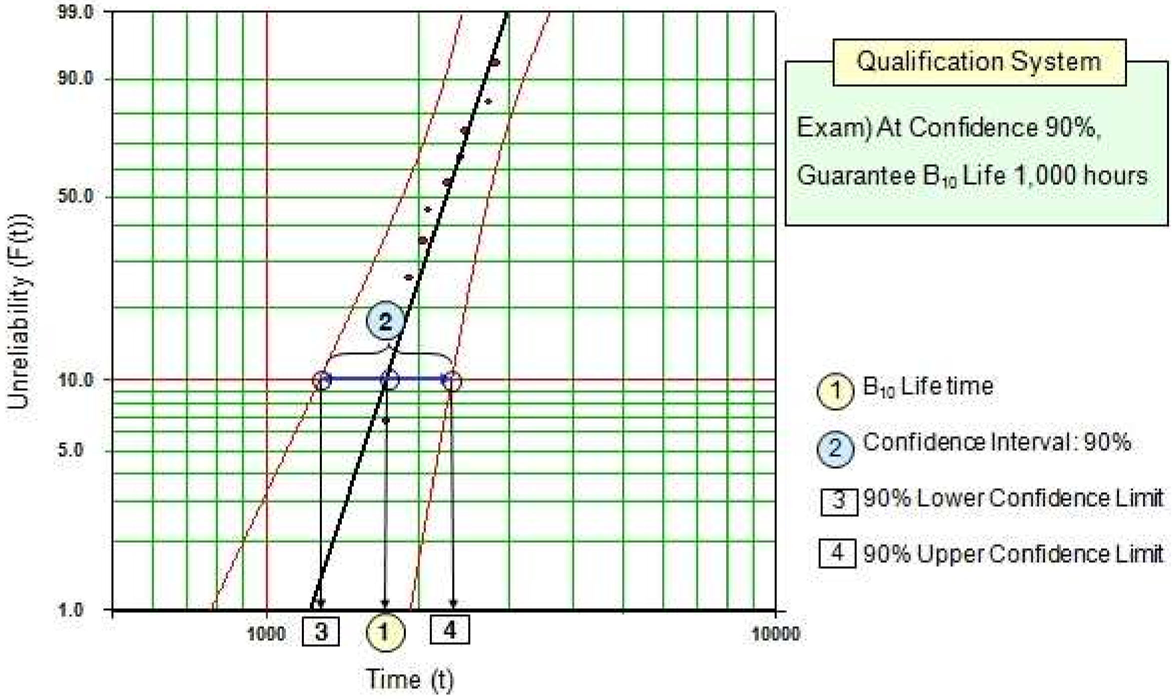



Fig.3.20 Weibull plot of five failures with 90% confidence interval

## 3.7 Sample Distributions

### 3.7.1 Introduction

Until now, we have studied to analyze the sample on the premise the characteristics of population are known. However, in real life or academic study, we have frequently confronted the situations we have to search out its characteristics in case populations are unknown.

If we want to know mean capacity of battery produced in a company, we have to investigate its whole products. To complete this case, it is impossible because a lot of time and cost are required. After we choose a proper sample, we will compute its statistics. Based on that, we will figure out parameters of populations – mean and standard deviations. In the same manner, selecting sample from population and leading the estimation or conclusion on population, we call it statistical inference.

These statistics vary for each different random sample we select. That is, they are random variables. If the sampling is done randomly, the value of a statistic will be random. Since statistics are random variables, they have the sampling distribution. It provides the following information: 1) What values of the statistic can occur, 2) What is the probability of each value to occur (Figure 3.21).

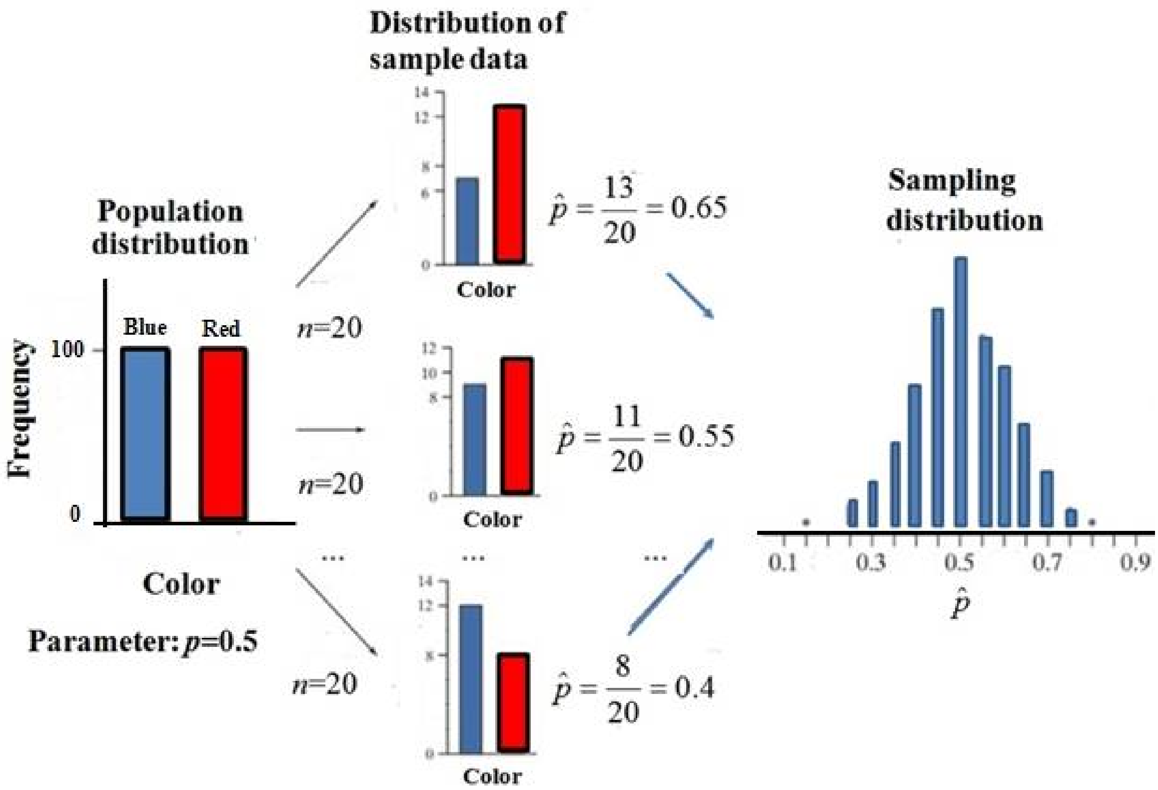



Fig. 3.21 Sample distributions in case of the known population

### 3.7.2 The distribution of sample mean

Sample mean $\bar{x}$ that is computed from a large sample tends to be closer to $\mu$ than does $\bar{x}$ based on a small *n*. Suppose $X_1$,….., $X_n$ are random variables with the same distribution with mean $\mu$ and population standard deviation $\sigma$. Now look at the random variable $\bar{X}$.

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) \tag{3-43}$$

$$E(\bar{X}) = \frac{1}{n}(E(X_1) + E(X_2) + \cdots E(X_n)) = \frac{1}{n} \cdot n\mu = \mu \tag{3-44}$$

$$V(\bar{X}) = \frac{1}{n^2}(V(X_1) + V(X_2) + \cdots + V(X_n)) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n} \tag{3-45}$$

The expectation value is $E(\bar{X}) = \mu$ and variation is $V(\bar{X}) = \sigma^2/n$. That is,

1. The population mean of $\bar{X}$, denoted $\mu_{\bar{X}}$, is equal to $\mu$.

2. The population standard deviation of $\bar{X}$, denoted $\sigma_{\bar{X}}$, is equal to

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \tag{3-46}$$

This means that the sampling distribution of $\bar{x}$ is always centered at $\mu$ and the second statement gives the rate the spread of the sampling distribution (sampling variability) decreases as *n* increases. The standard deviation of a statistic is called the standard error of the statistic. The standard error gives the precision of statistic for estimating a population parameter. The smaller the standard error, the higher the precision.

The standard error of the mean $\bar{X}$ is

$$SE(\overline{X}) = \sigma/\sqrt{n} \tag{3-47}$$

Now that we learned about the mean and the standard deviation of the sampling distribution of the sample mean, we might ask, if there is anything we can tell about the shape of the density curve of this distribution.

If population is infinite and sample size *n* is large enough, we know the distribution of sample mean is approximately normal distributed regardless the population characteristics. Namely, the Central Limit Theorem states that under rather general conditions, means of random samples drawn from one population tend to have an approximately normal distribution. We find that it does not matter which kind of distribution we find in the population. It even can be discrete or extremely skewed (Figure 3.22).

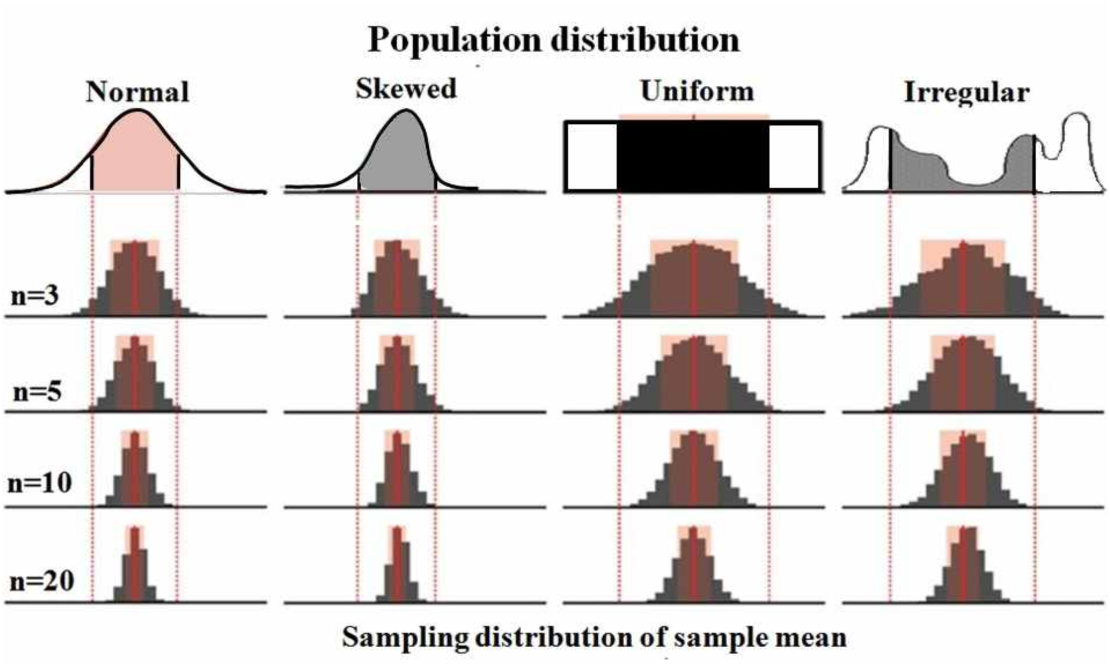



Fig. 3.22 Central Limit Theorem

**Central Limit Theorem:** From any population with finite mean $\mu$ and standard deviation $\sigma$, when *n* is large, if random samples of *n* observations are chosen, the sampling distribution of the mean $\bar{X}$ is approximately normal distributed, with mean $\mu$ and standard deviation $\sigma/\sqrt{n}$. That is,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \tag{3-48}$$

The Central Limit Theorem becomes very substantial one in modern statistics. Now look at a binomial distributed random variable *X*. With probability *p* (Bernoulli trial), we can carry out the experiment of *n* trials. Using Central Limit Theorem, as *n* increase infinitely, the random variable *X* will be distributed normally.

The random variable *X* is binomial distributed *B*(*n*, *p*) with mean *pn* and standard deviation equals $\sqrt{p(1-p)}$. Since $\hat{p}$ is simply the value of *X* expressed as an proportion, the sampling distribution of $\hat{p}$ is identical to the probability distribution of *X*. Then is $\hat{p} = \sum X_i/n = \bar{X}$. Using the Central Limit Theorem, as *n* increase infinitely, it will be normally distributed *N*(*np*, *np*(1-*p*)), which mean is *np* and variation is *np*(1-*p*). That is, as *n* increases infinitely, the random variable *X* is normally distributed:

$$\frac{X - np}{\sqrt{np(1-p)}} \sim N(0,1) \tag{3-49}$$

### 3.7.3 The distribution of sample variation

If relationship between population variation and sample variation is known, it will be helpful to estimate the unknown population variation. We would figure out the distribution of sample variance with Example 3.4.

**Example 3.4** There are five salesmen in a company. The working periods are 6, 2, 4, 8, and 10.

1) Select a random sample of size 2 with replacement and observe the working period. And find each

sample variation and compare the mean and variation of total sample variation with population variation.

2) Write down the frequency distribution table and draw the bar graph.

Solution) 1) The mean of this population is $\mu$ = 6 and its variation is $\sigma^2$=8. The variation of all possible samples with replacement can be summarized in Table 3.8. We know that some match that of population variation and the other have gaps like 0 or 32. The mean of all sample variation can be found as:

$$E\left(s^2\right) = \frac{0\times5+2\times8+8\times6+18\times4+32\times2}{25} = 8,$$

That is, we can see that the mean of all possible sample variation is equal to variation of population. We can call the sample variation the *unbiased estimator* of population variation.

Table 3.2 The variation of all possible samples with n=2 that can be chosen from the population N=5

| Sample | $s^2$ | Sample | $s^2$ | Sample | $s^2$ | Sample | $s^2$ | Sample | $s^2$ |
|---|---|---|---|---|---|---|---|---|---|
| 2, 2 | 0 | 4, 2 | 2 | 6, 2 | 8 | 8, 2 | 18 | 10, 2 | 32 |
| 2, 4 | 2 | 4, 4 | 0 | 6, 4 | 2 | 8, 4 | 8 | 10, 4 | 18 |
| 2, 6 | 8 | 4, 6 | 2 | 6, 6 | 0 | 8, 6 | 2 | 10, 6 | 8 |
| 2, 8 | 18 | 4, 8 | 8 | 6, 8 | 2 | 8, 8 | 0 | 10, 8 | 2 |
| 2, 10 | 32 | 4, 10 | 10 | 6, 10 | 8 | 8, 10 | 2 | 10, 10 | 0 |

2) Figure 3.23 is its bar graph. We call them the distribution of sample variation. What we see from the above figure and table is that the small sample variation is large and the large sample variation is small. That is, it is distributed asymmetrically. The mean of all possible sample variation ($E(s^2)$) is equal to variation of population ($\sigma^2$). The sample variation is the *unbiased estimator* of population

variation.

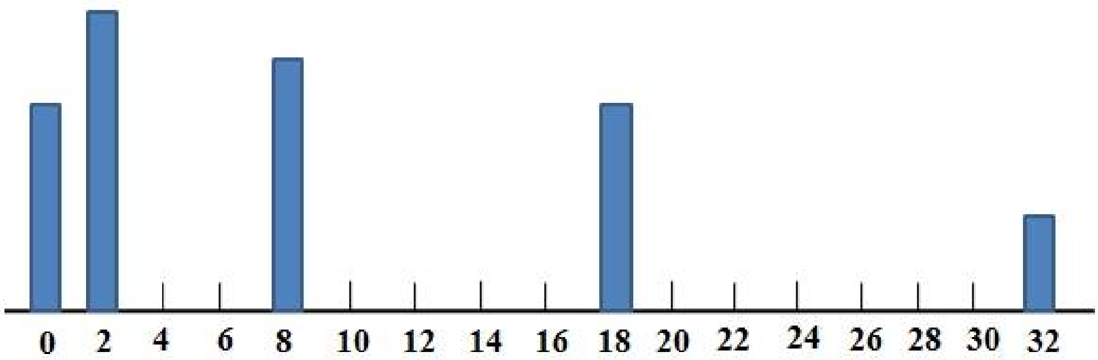



Fig. 3.23 Bar graph of the distribution of sample variation

As seen in Example 3.4, the sample variation is asymmetrically distributed. That is, the frequency at small sample variation is large. On the other hands, the frequency at large sample variation is small. Supposed that population is normally distributed and population variation is $\sigma^2$, sample variation follows chi-squared distribution with *k* degrees of freedom. Depending on integer *k* (or degrees of freedom), Chi-squared distribution is a kind of family, written as $\chi^2(1)$ with one degree of freedom, $\chi^2(2)$ with two degree of freedom, ····, and $\chi^2(27)$ with twenty seven degree of freedom. As seen in Figure 3.24, according to *k* degrees of freedom, Chi-squared distribution is asymmetrically distributed.

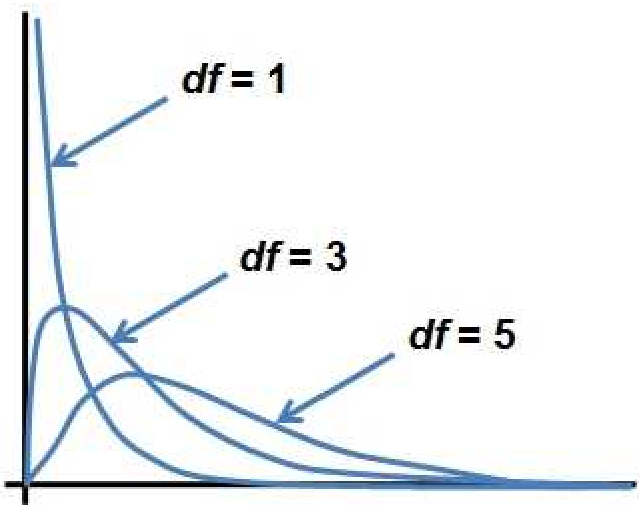



Fig. 3.24 Chi-squared distribution according to k degrees of freedom

**Distribution of sample variation:** Supposed that population with variation $\sigma^2$ is normally distributed, if $n$ sample is chosen randomly, $(n-1)S^2/\sigma^2$ follows Chi-squared distribution with ($n$ - 1) degrees of freedom. That is,

$$(n-1)S^2/\sigma^2 \sim \chi^2(n-1) \tag{3-50}$$

$$\text{where} \quad S^2 = \frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n-1} \tag{3-51}$$

## 3.8 Relationship between reliability and cumulative distribution function

To define the reliability function from the standpoint of load-strength interference, the strength of a product is modeled as a random variable *S.* The product is exposed to a load *L* that is also modeled as a random variable. The distributions of the strength and the load at a specific time *t* are illustrated in Figure 3.25. A failure will occur as soon as the load is higher than the strength. The reliability *R* of the item is defined as the probability that the strength is greater than the load,

$$R(t) = P(S > L) \tag{3-52}$$

where Pr(A) denotes the probability of event A.

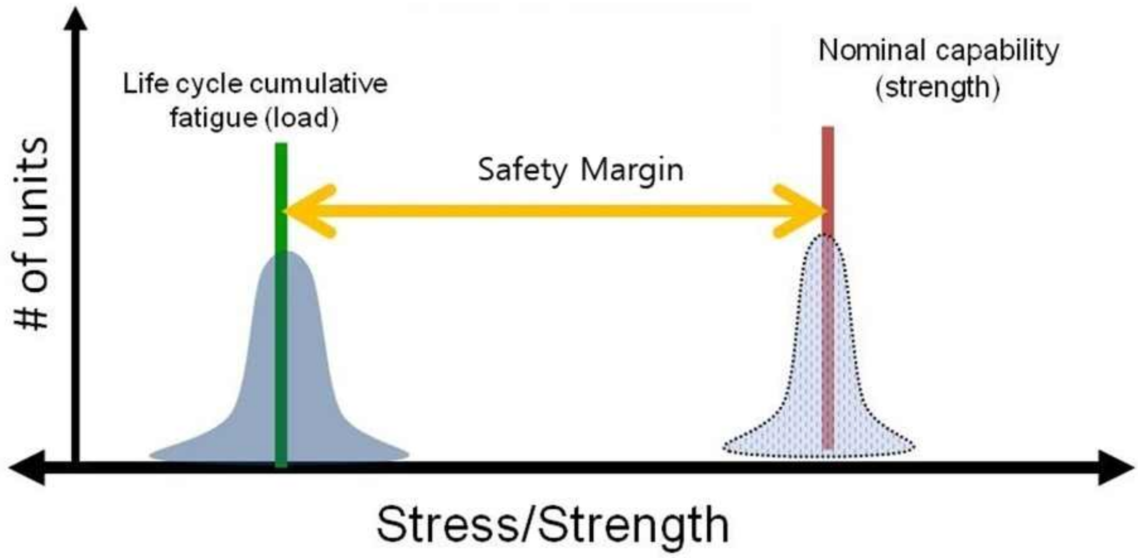



Fig. 3.25 Load and the strength distributions

The load will usually vary with time and may be modeled as a time-dependent variable *L(t)*. The product will deteriorate with time, due to failure mechanisms like fatigue, fracture, and corrosion. The strength of the product will therefore also be a function of time, *S(t)*. The time to failure T of the product is the (shortest) time until *S(t)* < *L(t)*,

$$T = \min\{t : S(t) \prec L(t)\} \quad (3\text{-}53)$$

and the reliability *R(t)* of the item may be expressed as:

$$R(t) = P(T > t) = 1 - F(t) = \int_t^{\infty} f(x)dx \quad (3\text{-}54)$$

*R(t)* is the probability that the item will not fail in the interval (0, t]. *R(t)* is the probability that it will survive at least until time *t* – it is sometimes called the survival function (Figure 3.26).

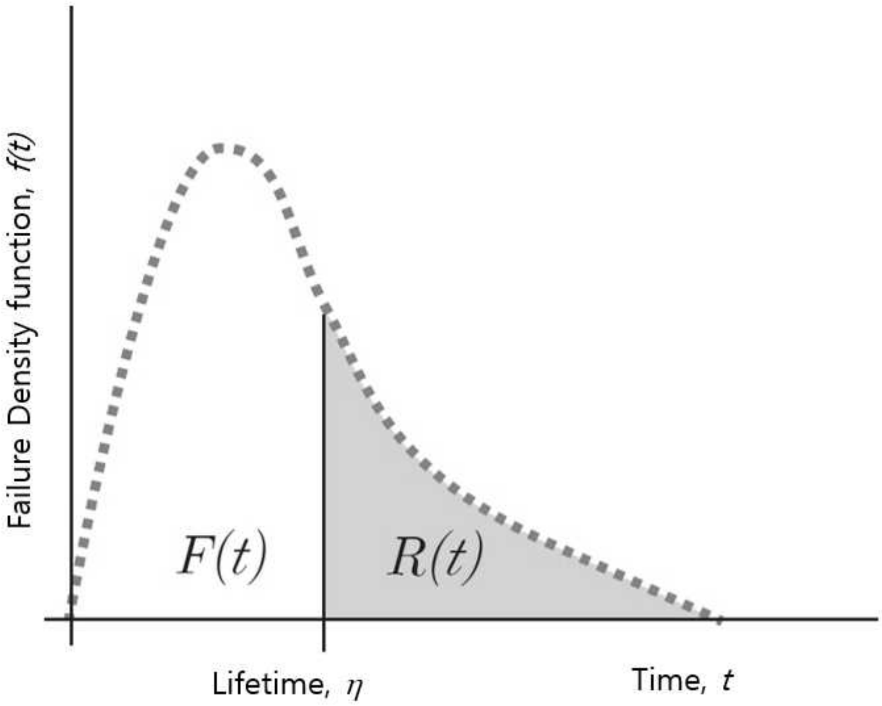



Fig. 3.26 Cumulative distribution function *F(t)* and Reliability function *R(t)*

The cumulative distribution function (CDF) is the probability that the variable *t* takes a value less than or equal to *T*. CDF associated with the time to failure *T* is expressed as:

$$F(t) = P(T \leq t) \tag{3-55}$$

which is the probability that the system fails within the time interval (0; t]. If T is a continuous random variable, the probability function is related to its probability density function *f (t)* by

$$F(t) = \int_0^t f(x)dx \tag{3-56}$$

In probability theory, a probability density function (PDF) is a function that describes the relative likelihood for this random variable to take on a given value. In reliability testing, density function *f(t)* is defined by:

$$\frac{dF(t)}{dt} = \frac{d\int_0^t f(x)dx}{dt} = f(t) \tag{3-57}$$

Failure rate (or Hazard rate function) is the frequency with which an engineered system or component fails. Consider the conditional probability:

$$P(t < T < t + \Delta t | T > t) = \frac{P(t < T \leq t + \Delta t)}{R(t)} = \frac{F(t + \Delta t) - F(t)}{R(t)} \tag{3-58}$$

In reliability engineering, failure rate (or hazard rate function) $\lambda(t)$ is defined by:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(t < T < t + \Delta t | T > t)}{\Delta t} = \frac{f(t)}{R(t)} \tag{3-59}$$

$\lambda(t)\ dt$ is the probability that the system will fail during the period(*t; t* + *dt*], given that it has survived until time *t*.

A survival and hazard function is to analyze the expected duration of time until one or more events happen, such as failure in mechanical systems. Cumulative hazard rate function $\Lambda(t)$ is defined by:

$$\Lambda(t) = \int_0^t \lambda(x)dx \tag{3-60}$$

Suppose the failure rate $\lambda(t)$ is known. Then it is possible to obtain *f (t)*, *F(t)*, and *R(t)*.

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} \Rightarrow \lambda(t) = -\frac{dR/dt}{R} \tag{3-61}$$

If Equation (3-61) is integrated, then reliability function becomes

$$R(t) = \exp\left[-\int_0^t \lambda(\tau)d\tau\right] \tag{3-62}$$

So the density function and cumulative distribution function are defined as:

$$f(t) = \lambda(t)\exp\left[-\int_0^t \lambda(\tau)d\tau\right] \tag{3-63}$$

$$F(t) = 1 - \exp\left[-\int_0^t \lambda(\tau)d\tau\right] \tag{3-64}$$

Relationship between reliability function *R(t)* and cumulative distribution function *F(t)* can be summarized in Figure 3.27.

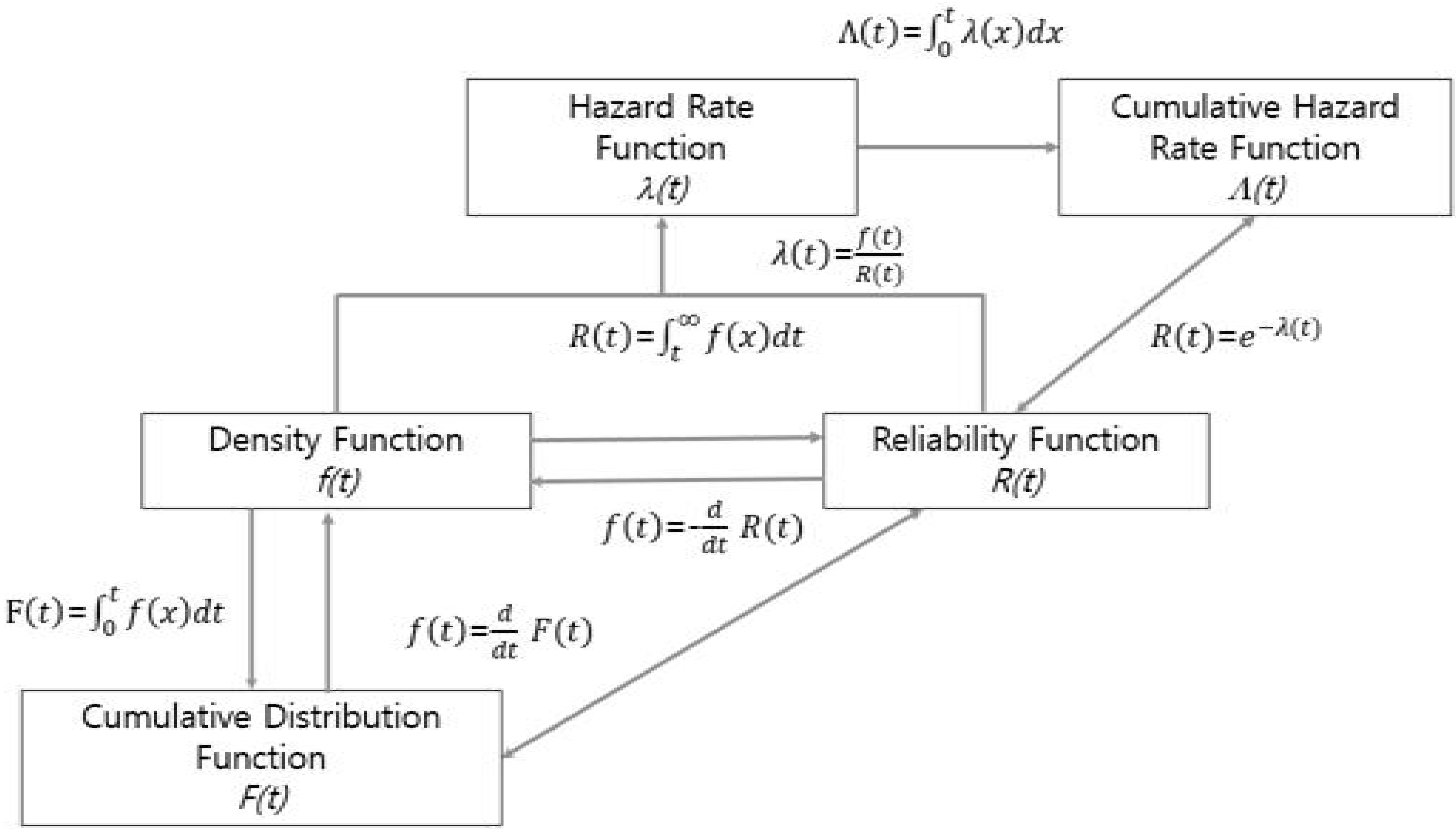



Fig. 3.27 Relationship between Reliability function *R(t)* and Cumulative distribution function *F(t)*

## 3.9 Design of experiment (DOE)

Engineering discovers often come from performing experiments. To verify them, we use the statistical methodology. Engineering learning is an iterative process, as represented in Figure 3.28.

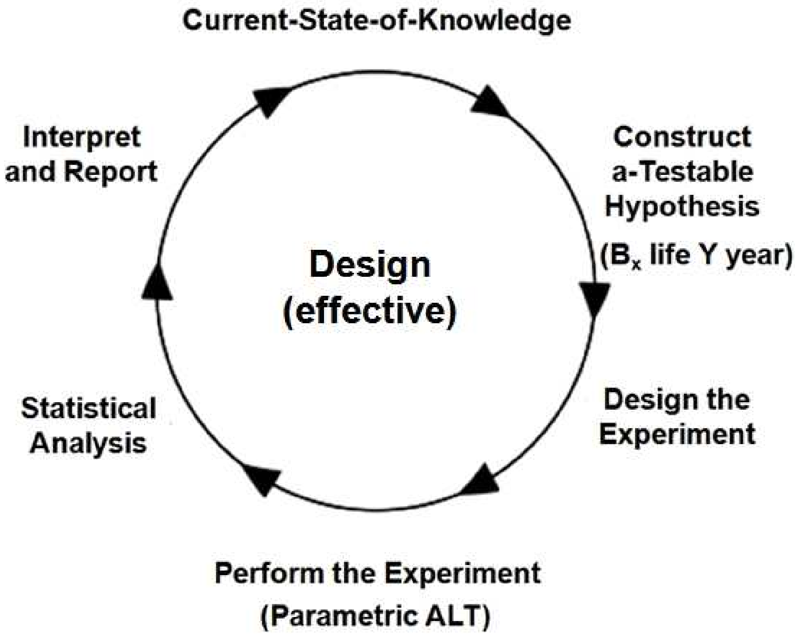



Fig.3.28 Engineering learning is an iterative process

If we start at current knowledge, the next step is choosing a new knowledge and constructing a testable hypothesis. Statistical theory focuses on “null hypothesis” which often represents the exact opposite of what an experimenter expects. On the other hands, ″alternative hypothesis” that is contrary to the null hypothesis is suggested. The next step in the cycle is to “Design an Experiment” and “Perform the Experiment.” Finally, we perform statistical analyses and interpret it, which leads to possible modification of the “Current State of Knowledge.”

Design of experiment (DOE) is a systematic method to determine the relationship between factors affecting a process and its output. The designing of experiment and the analysis like ANOVA from obtained data are inseparable. If the experiment is properly designed, the data can carry out the statistical inferences. It is important to understand first the basic terminologies used in the experimental design. For mechanical system the factor that product lifetime influences is stress due to the repetitive loads. If product is to estimate its lifetime, we have to carry out parametric ALT. Accelerated loads reveals the faulty designs at each ALT stage. If the mission cycle is achieved, we can evaluate the product lifetime.

### 3.9.1 Terminologies

**Experimental unit:** For conducting an experiment, the experimental material is divided into smaller parts and each part is referred to as experimental unit. The experimental unit is randomly assigned to a treatment is the experimental unit. The phrase “randomly assigned” is very important in this definition.

**Experiment:** A way of getting an answer to a question which the experimenter wants to know.

**Treatment:** Different objects or procedures compared in an experiment are called treatments.

**Factor:** A factor is a variable defining a categorization. A factor can be fixed or random in nature. A factor is termed as fixed factor if all the levels of interest are included in the experiment. A factor is termed as random factor if all the levels of interest are not included in the experiment.

**Replication:** It is the repetition of the experimental situation by replicating the experimental unit.

**Experimental error:** The unexplained random part of variation in any experiment is termed as experimental error. An estimate of experimental error can be obtained by replication.

**Treatment design:** A treatment design is the manner in which the levels of treatments are arranged in an experiment.

**Design of experiment:** One of the main objectives of designing an experiment is how to verify the hypothesis in an efficient and economical way. In the contest of the null hypothesis of equality of several means of normal populations having same variances, the analysis of variance technique can be used. Note that such techniques are based on certain statistical assumptions. If these assumptions are violated, the outcome of the test of hypothesis then may also be faulty and the analysis of data may be meaningless. So the main question is how to obtain the data such that the assumptions are met and the data is readily available for the application of tools like analysis of variance. The designing of such mechanism to obtain such data is achieved by the design of experiment. After obtaining the sufficient experimental unit, the treatments are allocated to the experimental units in a random fashion. Design of experiment provides a method by which the treatments are placed at random on the experimental units in such a way that the responses are estimated with the utmost precision possible.

**Principles of experimental design:** There are three basic principles of design – 1) Randomization, 2) Replication, and 3) Local control (error control).

**Completely randomized design (CRD):** For completely randomized designs, the levels of the primary factor are randomly assigned to the experimental units. By randomization, the run sequence of the experimental units is determined randomly. Following steps are needed to design a CRD:

- Divide the entire experimental material or area into a number of experimental units, say *n*.
- Fix the number of replications for different treatments in advance (for given total number of available experimental units).
- No local control measure is provided as such except that the error variance can be reduced by choosing a homogeneous set of experimental units.

### 3.9.2 Analysis

There is only one factor which is affecting the outcome – treatment effect. So the setup of one way analysis of variance (ANOVA) is to be used.

$y_{ij}$: Individual measurement of $j^{th}$ experimental units for $i^{th}$ treatment $i = 1,2,...,v$ , $j = 1,2,..., n_i$

Independently distributed following with $N(\mu+\alpha, \sigma^2)$ with $\sum_{i=1}^{v} n_i \alpha_i = 0$.

$\mu$: overall mean

$\alpha_i$: $i^{th}$ treatment effect

$H_0$: $\alpha_1=\alpha_2=....=\alpha_v=0$

$H_1$: All $\alpha'_i$ s are not equal.

The data set is arranged as follows:

**Table 3.3** Data set with one factor which is affecting the outcome – treatment effect

| Treatments | | | |
|---|---|---|---|
| 1 | 2 | …….. | ν |
| $y_{11}$ | $y_{21}$ | | $y_{\nu 1}$ |
| $y_{12}$ | $y_{22}$ | | $y_{\nu 2}$ |
| …. | …. | | …. |
| $y_{1n}$ | $y_{2n}$ | | $y_{\nu n}$ |
| $T_1$ | $T_2$ | | $T_\nu$ |

where $T_i = \sum_{i=1}^{n} y_j$ is the treatment total due to $i^{th}$ effect, $G = \sum_{i=1}^{\nu} T_i = \sum_{i=1}^{\nu}\sum_{j=1}^{n} y_j$ is the grand total of all the observations.

In order to derive the test for $H_0$, we can use either the likelihood ratio test or the principle of least squares. Since the likelihood ratio test has already been derived earlier, so we choose to demonstrate the use of least squares principle.

The linear model under consideration is

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad i = 1, 2, ....., n \tag{3-65}$$

where $\varepsilon_{ij}$'s are identically and independently distributed random errors with mean 0 and variance $\sigma^2$.

The normality assumption of ε's is not needed for the estimation of parameters but will be needed for deriving the distribution of various involved statistics and in deriving the test statistics. ε's is not needed for the estimation of parameters but will be needed for deriving the distribution of various involved statistics and in deriving the test statistics.

Let $$S = \sum_{i=1}^{v}\sum_{j=1}^{n} \varepsilon_{ij}^2 = \sum_{i=1}^{v}\sum_{j=1}^{n} \left(y_{ij} - \mu - \alpha_i\right)^2 \tag{3-66}$$

Minimizing *S* with respect to $\mu$ and $\alpha_i$, the normal equations are obtained as

$$\frac{\partial S}{\partial \mu} = 0 \Rightarrow n\mu + \sum_{i=1}^{v} n_i \alpha_i = 0 \tag{3-67a}$$

$$\frac{\partial S}{\partial \alpha_i} = 0 \Rightarrow n_i\mu + n_i\alpha_i = \sum_{j=1}^{n} y_{ij}, \quad i = 1,2,\cdots,v. \tag{3-67b}$$

Solving them using $\sum_{i=1}^{v} n_i \alpha_i = 0$, we get

$$\hat{\mu} = \bar{y}_{oo} \tag{3-67c}$$

$$\hat{\alpha}_i = \bar{y}_{io} - \bar{y}_{oo} \tag{3-67d}$$

where $\bar{y}_{io} = \frac{1}{n_i}\sum_{j=1}^{n} \bar{y}_{ij}$ is the mean of observation receiving the i<sup>th</sup> treatment and $\bar{y}_{oo} = \frac{1}{n}\sum_{i=1}^{v}\sum_{j=1}^{n} \bar{y}_{ij}$ is the mean of all the observations.

The fitted model is obtained after substituting the estimate $\hat{\mu}$ and $\hat{\alpha}_1$ in the linear model, we get

$$y_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\varepsilon}_{ij} \tag{3-68}$$

Squaring both sides and summing over all the observation, we have

$$\sum_{i=1}^{v}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{oo}\right)^2 = \sum_{i=1}^{v} n_i\left(\bar{y}_{io} - \bar{y}_{oo}\right)^2 + \sum_{i=1}^{v}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{oo}\right)^2 \tag{3-69}$$

Total sum of squares = Sum of squares due to treatment effects + Sum of squares due to error

Since $\sum_{i=1}^{v}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{oo}\right) = 0$, so *TSS* is based on the sum of (*n*-1) squared quantities. The *TSS* carries only (n−1) degrees of freedom.

Since $\sum_{i=1}^{v} n_i\left(y_{io} - \bar{y}_{oo}\right) = 0$, so *SST* is based on the sum of (ν-1) squared quantities. The *TSS* carries only (ν−1) degrees of freedom. Since $\sum_{i=1}^{n} n_i\left(\bar{y}_{ij} - \bar{y}_{io}\right) = 0$ for all *i* = 1, 2,..., *v*, so *SSE* is based on the sum of squaring n quantities like $\left(\bar{y}_{ij} - \bar{y}_{io}\right)$ with *v* constraints $\sum_{j=1}^{n}\left(\bar{y}_{ij} - \bar{y}_{io}\right) = 0$.

So SSE carries (*n* – *v*) degrees of freedom. *TSS* = *SSTr* + *SSE* with degrees of freedom partitioned as (n – 1) = (*v* - 1) + (n – *v*). Moreover, the equality in *TSS* = *SSTr* + *SSE* has to hold exactly. In order to ensure that the equality holds exactly, we find one of the sum of squares through subtraction. Generally, it is recommended to find SSE by subtraction as:

$$TSS = \sum_{i=1}^{v}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{io}\right)^2 = \sum_{i=1}^{v}\sum_{j=1}^{n} y_{ij}^2 - \frac{G^2}{n} \tag{3-70}$$

where $G = \sum_{i=1}^{v}\sum_{j=1}^{n} y_{ij}$

$$SSTr = \sum_{i=1}^{n} n_i \left(\bar{y}_{io} - \bar{y}_{oo}\right)^2 = \sum_{i=1}^{v} \left(\frac{T_i^2}{n}\right) - \frac{G^2}{n}$$ (3-71)

where $T_i = \sum_{j=1}^{n} y_{ij}$ , $\frac{G^2}{n}$: correction factor

Under $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_v = 0$, the model become

$$Y_{ij} = \mu + \varepsilon_{ij}$$ (3-71)

and minimizing $S = \sum_{i=1}^{v} \sum_{j=1}^{n} \varepsilon_{ij}^2$ with respect to *μ* gives

$$\frac{\partial S}{\partial \mu} = 0 \Rightarrow \hat{\mu} = \frac{G}{n} = \bar{y}_{oo}$$ (3-72)

The *SSE* under $H_0$ becomes

$$SSE = \sum_{i=1}^{v} \sum_{j=1}^{n} \left(y_{ij} - \bar{y}_{oo}\right)^2$$ (3-73)

and thus *TSS* = *SSE*

This *TSS* under $H_0$ contains the variation only due to the random error whereas the earlier *TSS* = *SSTr* + *SSE* contains the variation due to treatments and errors both. The difference between the two will provides the effect of treatments in terms of sum of squares as

$$SSTr = \sum_{i=1}^{v} n\left(\bar{y}_i - \bar{y}_{oo}\right)^2$$ (3-74)

The expectations are given as:

$$E(SSE)=\sum_{i=1}^{v}\sum_{j=1}^{n}E\left(y_{ij}-y_{io}\right)^{2}$$
$$=\sum_{i=1}^{v}\sum_{j=1}^{n}\left(\varepsilon_{ij}-\bar{\varepsilon}_{io}\right)^{2}=\sum_{i=1}^{v}\sum_{j=1}^{n}E\left(\varepsilon_{ij}^{2}\right)-\sum_{i=1}^{v}n_{i}E\left(\bar{\varepsilon}_{io}^{2}\right)=n\sigma^{2}-\sum_{i=1}^{v}n_{i}\frac{\sigma^{2}}{n_{i}}=(n-v)\sigma^{2} \tag{3-75}$$

$$E(MSE)=E\left(\frac{SSE}{n-v}\right)=\sigma^{2} \tag{3-76}$$

$$E(SSTr)=\sum_{i=1}^{v}nE\left(\bar{y}_{io}-\bar{y}_{oo}\right)^{2}=\sum_{i=1}^{v}nE\left(\alpha_{i}+\bar{\varepsilon}_{io}-\bar{\varepsilon}_{oo}\right)^{2}$$
$$=\sum_{i=1}^{v}n_{i}\alpha_{i}^{2}+\left[\sum_{i=1}^{v}n_{i}\frac{\sigma^{2}}{n_{i}}-n\frac{\sigma^{2}}{n}\right]=\sum_{i=1}^{v}n_{i}\alpha_{i}^{2}+(v-1)\sigma^{2} \tag{3-77}$$

$$E(MSTr)=E\left(\frac{SStr}{v-1}\right)=\frac{1}{v-1}\sum_{i=1}^{v}n\alpha_{i}^{2}+\sigma^{2} \tag{3-78}$$

In general $E(MSTr) \neq \sigma^2$ but under $H_0$, all $\alpha_i = 0$ and so

$$E(MSTr)=\sigma^{2} \tag{3-79}$$

Using the normal distribution property of $\varepsilon_{ij}$‘s we find that $y_{ij}$*’s* are also normal as they are the linear combination of $\varepsilon_{ij}$*‘s*

$$-\frac{SSTr}{\sigma^{2}}\sim\chi^{2}(v-1) \text{ under } H_o \tag{3-80}$$

$$-\frac{SSE}{\sigma^{2}}\sim\chi^{2}(n-v) \text{ under } H_o \tag{3-81}$$

*SSTr* and *SSE* are independently distributed

$$-\frac{MStr}{MSE} \sim F(v-1, n-v) \text{ under } H_o \qquad (3\text{-}82)$$

Reject $H_0$ at $\alpha^*$ level of significance if $F > F_{\alpha;v-1,n-v}$

The analysis of variance table is as follows

**Table 3.4** Analysis of variance (ANOVA) table

| Source of variation | Degrees of freedom | Sum of squares | Mean sum of squares | $F$ |
|---|---|---|---|---|
| Between treatments | $v$-1 | $SSTr$ | $MSTr$ | $\frac{MSTr}{MSE}$ |
| Errors | $n$-$v$ | $SSE$ | $MSE$ | |
| Total | n-1 | TSS | | |

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