the
little
attitude
guide

Definitions, formulas and derivations for attitude representation.

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Contents

Chapter 1

Introduction

This guide presents the main mathematical concepts used to represent the attitude of a rigid body in three-dimensional space. It presents a unified guide to attitude in a condensed form for easy reference. All the assumptions and standards used are explicitly stated for reference.

1.1 Nomenclature

The notion used throughout this reference is as follows:

- $\mathbf{A}$ represents a general $(m \times n)$ matrix.

- $\mathbf{x}$ represents a $(n \times 1)$ column vector matrix or $(1 \times n)$ row vector matrix.

- $\mathbf{\Phi}$ represents the attitude expressed in any representation of a frame relative to a reference frame. The convention used in this reference is that this attitude maps from the reference frame to the rotated frame of reference such that $\mathbf{\Phi} : \mathcal{F}^w \leftrightarrow \mathcal{F}^b$.

- $\mathcal{F}^a$ represents a frame of reference identified by the subscript a. $\mathcal{F}^w$ is commonly used to represent the world (fixed) reference frame and $\mathcal{F}^b$ is used to represent the body (rotated) reference frame.

- $\vec{x}$ represents a cartesian vector quantity.

- $\vec{x}^a$ represents the vector quantity $\vec{x}$ expressed in frame $\mathcal{F}^a$.

- $\vec{x}_a$ represents the vector quantity $\vec{x}$ describing a.

- $\mathbf{R}_a^b$ represents a rotation matrix that transforms a vector from $\mathcal{F}^a$ to $\mathcal{F}^b$. The convention used in this reference is that the rotation matrix is pre-multiplied by the column vector $\vec{x}^b = \mathbf{R}_a^b \vec{x}^a$.

Chapter 2

Linear Algebra

2.1 Matrices

A $(m \times n)$ matrix is a rectangular array of numbers or expressions arranged in m rows and n columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The elements in the matrix $\mathbf{A}$ can be denoted by their row i and column j index, such as:

$$\mathbf{A}_{ij} = a_{ij}$$

2.1.1 Matrix Addition and Subtraction

Matrices may be added or subtracted if they are compatible (i.e they have the same size). The result is simply the addition or subtraction of each element in the matrices individually.

$$\mathbf{C} = \mathbf{A} \pm \mathbf{B}$$

where the elements of $\mathbf{C}$ are:

$$c_{ij} = a_{ij} \pm b_{ij}$$

For example:

$$\mathbf{A}+\mathbf{B} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] + \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] = \left[\begin{array}{cc} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{array} \right]$$

$$\mathbf{A}-\mathbf{B} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] - \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] = \left[\begin{array}{cc} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{array} \right]$$

The matrices follow the typical commutative rules for addition and subtraction:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

2.1.2 Matrix Scalar Multiplication and Division

To multiply a matrix by a scalar or divide a matrix by a scalar, the operation is applied to each element of the matrix separately:

$$\mathbf{B} = k\mathbf{A}$$

where the elements of $\mathbf{B}$ are:

$$b_{ij} = ka_{ij}$$

The scalar multiplication and division operations are also commutative:

$$k\mathbf{A} = \mathbf{A}k$$

2.1.3 Matrix Transpose

The transpose of an $(m \times n)$ matrix $\mathbf{A}$ is the $(n \times m)$ matrix $\mathbf{A}^T$ given by:

$$\left(\mathbf{A}^T\right)_{ij} = \mathbf{A}_{ji}$$

For example:

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

The matrix transpose rules along with the matrix addition, matrix subtraction and scalar multiplication are commutative:

$$(k\mathbf{A})^T = k\left(\mathbf{A}^T\right)$$

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$\left(\mathbf{A}^T\right)^T = \mathbf{A}$$

2.1.4 Matrix Multiplication

Two matrices $\mathbf{A}$ and $\mathbf{B}$ of size $(m \times t)$ and $(t \times n)$ respectively can be multiplied together if they are conformable (such that the number of columns in the first matrix are equal to the number of rows in the second matrix), to give a resulting $(m \times n)$ matrix.

$$\mathbf{C} = \mathbf{AB}$$

where the elements of $\mathbf{C}$ are:

$$c_{ij} = \sum_{k=1}^{t} a_{ik} b_{kj}$$

for $j = 1, \ldots, t$ and $i = 1, \ldots, m$.
Matrix multiplication is **not** commutative in general:

$$\mathbf{AB} \neq \mathbf{BA}$$

It does however follow associativity and left and right distributivity relationships:

$$(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$$

$$(\mathbf{A} + \mathbf{B})\,\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

$$\mathbf{C}\,(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$$

2.1.5 Identity Matrix

An Identity matrix is a square matrix $\mathbf{I}$ of size $(m \times m)$, with the main diagonal components all equal to 1 and all other entries equal to 0.

$$\mathbf{I}_{ik} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

For example:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any matrix that is multiplied with a conformable Identity matrix will just be the original matrix:

$$\mathbf{AI} = \mathbf{A}$$

$$\mathbf{IB} = \mathbf{B}$$

2.1.6 Matrix Inverse

The inverse of a square $(n \times n)$ matrix $\mathbf{A}$, is the matrix $\mathbf{A}^{-1}$ that satisfies the condition:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Not all square matrices are invertible, a matrix is not-invertible if it is singular, a matrix is singular if and only if the $\det(\mathbf{A}) = 0$.

The matrix inverse of a (2×2) matrix A is given by:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2.1.7 Matrix Determinant

The determinant of a square matrix, written as $\det(\mathbf{A})$ or $|\mathbf{A}|$, is given by:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

if the matrix is a (2×2) matrix or given by:

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

if the matrix is a (3×3) matrix.

The main properties of Determinants are:

$$\det(\mathbf{A}) \neq 0 \text{ if and only if } \mathbf{A} \text{ is invertible}$$

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det()\mathbf{B})$$

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

2.1.8 Matrix Trace

The trace of a square $(n \times n)$ matrix, written as $\mathrm{tr}(\mathbf{A})$, is defined as:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

It has the following properties:

$$\mathrm{tr}(k\mathbf{A}) = k\ \mathrm{tr}(\mathbf{A})$$

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$

$$\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$$

$$\mathrm{tr}(\mathbf{A}^T) = \mathrm{tr}(\mathbf{A})$$

2.1.9 Eigenvalues and Eigenvectors

The **eigenvector** of a square $(n \times n)$ matrix $\mathbf{A}$ is a non-zero $(n \times 1)$ column vector matrix $\mathbf{x}$ such that:

$$\mathbf{Ax} = \lambda\mathbf{x}$$

where λ is a **eigenvalue** of $\mathbf{A}$. The **eigenvalues** are the roots of characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

which may be real or complex roots or repeated roots. The matrix will have at most n different roots. For each **eigenvalue** λ, the **eigenvector** is calculated from solving the equation:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

The set of all the solutions is called the λ-**eigenspace** of $\mathbf{A}$, which also includes the zero **eigenvector** solution $\mathbf{x} = \mathbf{0}$.

2.1.10 Orthogonal Matrix

An orthogonal matrix is a special form of a square $(n \times n)$ matrix that satisfies the condition:

$$\mathbf{A}^T = \mathbf{A}^{-1}$$

such that:

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$$

2.1.11 Symmetric and Skew-Symmetric Matrices

An symmetric matrix is a special form of a square $(n \times n)$ matrix that satisfies the condition:

$$\mathbf{A} = \mathbf{A}^T$$

while the skew-symmetric matrix instead satisfies the condition:

$$\mathbf{A} = -\mathbf{A}^T$$

2.2 Vectors

A vector $\vec{v}$ is a multi-dimensional quantity that has both a **magnitude** and **direction**. In 2 or 3 dimensions, it may be represented as the cartesian form $\vec{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors in the direction of the positive x, y, z axes respectively, and the coefficients x, y, z are the cartesian coordinate components of $\vec{v}$.

In general, a n-dimensional vector $\vec{v}$ may be written as a $(n \times 1)$ column vector matrix or a $(1 \times n)$ row vector matrix, such that:

$$\vec{v} = \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\vec{v} = \mathbf{v} = [x_1, x_2, \ldots, x_n]$$

Since a vector can be expressed in the column or row vector matrix form, it can be treated like a matrix in general and follows all the same algebraic rules as a normal matrix of the equivalent size.

2.2.1 Vector Norm

The norm (or length) of the vector $\vec{v}$ where $\vec{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, written as $|\vec{v}|$, is given by:

$$|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$$

2.2.2 Unit Vector

The vector that has $|\vec{v}| = 1$ is called a **unit vector** and can be expressed as $\hat{\mathbf{v}}$. The unit vector in the direction of $\vec{v}$ is given by:

$$\hat{\mathbf{v}} = \frac{\vec{v}}{|\vec{v}|}$$

2.2.3 Vector Dot Product (Scalar Product)

Let $\vec{u} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$ and $\vec{v} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$, then the dot product of vector $\vec{u}$ with $\vec{v}$, expressed as $\vec{u} \cdot \vec{v}$, is given by:

$$\vec{u} \cdot \vec{v} = x_1x_2 + y_1y_2 + z_1z_2$$

The dot product of two vectors is also the angle $\cos(\theta)$ between the vectors when placed tail to tail, such that:

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}|\cos(\theta)$$

If the vectors are perpendicular then the dot product is equal to 0. The squared norm can also be written as the dot product of the vector with itself:

$$|\vec{u}|^2 = \vec{u} \cdot \vec{u}$$

2.2.4 Vector Cross Product (Vector Product)

Let $\vec{u} = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}$ and $\vec{v} = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}$, then the vector or cross product of vector $\vec{u}$ with $\vec{v}$, expressed as $\vec{u} \times \vec{v}$, is given by:

$$\vec{u} \times \vec{v} = \det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}\right)$$

$$\vec{u} \times \vec{v} = (y_1 z_2 - y_2 z_1)\hat{\mathbf{i}} - (x_1 z_2 - x_2 z_1)\hat{\mathbf{j}} + (x_1 y_2 - x_2 y_1)\hat{\mathbf{k}}$$

The cross product $\vec{u} \times \vec{v}$ has the property that the resulting vector is perpendicular to both $\vec{u}$ and $\vec{v}$. It also has the following relationships:

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin(\theta)$$

The cross product $\vec{u} \times \vec{v}$ may also be expressed as a matrix-vector multiplication such that:

$$\vec{u} \times \vec{v} = [\mathbf{u}]_\times \mathbf{v} = [\mathbf{v}]_\times^T \mathbf{u}$$

where $[\cdot]_\times$ is the skew-symmetric matrix given by:

$$[\vec{\mathbf{v}}]_\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Chapter 3

Attitude Representations

An attitude representation allows the allows the attitude (or orientation) of an object to be mathematically described relative to a reference frame. This mathematical description which links the orientation between two frames of reference together, allowing quantities in one frame to be express in the other frame.

3.1 Reference Frames and Coordinate Systems

A **coordinate system** is used to uniquely specify a point or quantity using a sequence of numbers (i.e. coordinates). An typical example of this is the 3D Cartesian Coordinate systems (X,Y,Z). Each axis is orthogonal to the other two (i.e a change in one will not affect the other). When a coordinate system is fixed to an origin or object, then it becomes a reference frame.

There are two important reference frames needed when describing attitude:

- World/Fixed (Reference) Coordinate System $\mathcal{F}^w$

  Fixed in space (stays at same location and attitude)

- Body (Rotated) Coordinate System $\mathcal{F}^b$

  Rigidly attached to the object whose attitude we would like to describe relative to the world frame

Let $\mathbf{\Phi}$ represent the attitude of an object or frame such that it maps quantities in one frame to the other frame:

$$\mathbf{\Phi} : \mathcal{F}^w \leftrightarrow \mathcal{F}^b$$

3.2 Attitude Representation Convention

There are two different conventions that could be taken:

1. The attitude $\mathbf{\Phi}$ expresses the rotation of the *rotated frame* $\mathcal{F}^b$ relative to the *reference frame* $\mathcal{F}^w$, such that:

$$\mathbf{\Phi}_1 : \mathcal{F}^w \rightarrow \mathcal{F}^b$$

2. The attitude $\mathbf{\Phi}$ expresses the rotation of the *reference frame* $\mathcal{F}^w$ relative to the *rotated frame* $\mathcal{F}^b$, such that:

$$\mathbf{\Phi}_2 : \mathcal{F}^b \rightarrow \mathcal{F}^w$$

In either convention, that the attitude can be inverted to find the opposite frame mapping or inverse rotation:

$$\mathbf{\Phi}_1 : \mathcal{F}^w \rightarrow \mathcal{F}^b$$

$$\mathbf{\Phi}_1^{-1} : \mathcal{F}^w \leftarrow \mathcal{F}^b$$

For the rest of the document it is assumed that the first convention $\mathbf{\Phi} : \mathcal{F}^w \rightarrow \mathcal{F}^b$ is used.

If the attitude is expressed in the second convention $\mathbf{\Phi}_2$ then it can be converted into the first convention by using the inverse attitude, for example:

$$\mathbf{\Phi}_1 = \mathbf{\Phi}_2^{-1}$$

and conversely:

$$\mathbf{\Phi}_2 = \mathbf{\Phi}_1^{-1}$$

3.3 Attitude Representations

There are different mathematical ways/techniques to represent the attitude $\mathbf{\Phi}$ such as using:

- Linear Transforms (Direct Cosine Matrix)
- Euler Angles
- Quaternions
- Rotation Vectors

Each representation method has advantages and disadvantages:

- Ease of User Interpretation and Interaction (can you mentally picture it and describe it)
- Storage (size in memory)
- Numerical Issues (stability, computation performance, uniqueness)
- Integration and Kinematics (how to describe a rotating object or changing attitude)
- Interpolation (computer graphics and animation, smoothly changing between two orientations)

3.4 Attitude Kinematics

Attitude kinematics allows the attitude to be expressed as a function of time $\mathbf{\Phi}(t)$. This is done by integrating the rate of change of attitude $\dot{\mathbf{\Phi}}(t)$ such that:

$$\mathbf{\Phi}(t) = \int \dot{\mathbf{\Phi}}(t)dt$$

The attitude rate $\dot{\mathbf{\Phi}}$ is a function of the current attitude $\mathbf{\Phi}$ and the instantaneous angular rotation rates $\vec{\omega}$ such that:

$$\dot{\mathbf{\Phi}}(t) = f\left(\mathbf{\Phi}, \vec{\omega}\right)$$

The instantaneous angular rotation rates $\vec{\omega}$ express the rotation rate around the (x, y, z)-axes at this instance in time. The angular rate express how quickly body frame is rotating with respect to the world frame $\mathcal{F}^w \leftrightarrow \mathcal{F}^b$ with respect to time.

This angular rate does not equal the attitude rate $\dot{\mathbf{\Phi}} \neq \omega$ as the attitude is represented by a different method, so a different mapping is required for each representation.

If the angular rate of $\mathcal{F}^b$ with respect to $\mathcal{F}^a$ is $\vec{\omega}_{b/a}$ then:

$$\vec{\omega}_{a/b} = -\vec{\omega}_{b/a}$$

The rotation of one frame with respect to another has the same effect as if it was the other frame that was rotating but in the opposite direction.

The angular rate $\vec{\omega}_{b/a}$ of $\mathcal{F}^b$ with respect to $\mathcal{F}^a$ may be expressed in any frame, such that $\vec{\omega}^b_{b/a}$ is the angular rate of $\mathcal{F}^b$ with respect to $\mathcal{F}^a$ expressed in $\mathcal{F}^b$.

Chapter 4

Rotation Matrices

4.1 Definition

The rotation transformation matrix can be described as an operation that projects the reference frame axes onto a rotated frame axes. We define a rotation transformation matrix $\mathbf{R}$ that when multiplied by a vector expressed in one coordinate frame transforms the same vector to a second coordinate frame.

For this rotation matrix $\mathbf{R}$ to be a valid transformation then it must be orthogonal such that:

$$\det(\mathbf{R}) = 1$$

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

This means that the transformed vector stays the same size and that the inverse mapping is simply the opposite operation so that the inverse matrix is just the transpose of the matrix.

This allows a rotation transformation matrix to be defined that encodes the attitude $\mathbf{\Phi}$ information so following the attitude convention, it gives:

$$\mathbf{R}_w^b = f(\mathbf{\Phi})$$

which $\mathbf{R}_w^b$ transforms a vector in the reference or world coordinate frame $\mathcal{F}^w$ to the body or rotated coordinate frame $\mathcal{F}^b$.

4.2 Pre-Multiplication vs Post-Multiplication

There can be two different standards when applying rotation matrix transformations to vectors. The rotation matrix can pre-multiply column vectors or post-multiply row vectors.

Let $\mathbf{R}_a^b$ be a rotation matrix that maps a vector from $\mathcal{F}^a$ to $\mathcal{F}^b$. When the vectors $\vec{v} = [x, y, z]^T$ are column vectors then pre-multiplication is used, such that:

$$\vec{v}^b = \mathbf{R}_a^b \vec{v}^a$$

$$\begin{bmatrix} x^b \\ y^b \\ z^b \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x^a \\ y^a \\ z^a \end{bmatrix}$$

$$\begin{aligned} x^b &= r_{11}x^a + r_{12}y^a + r_{13}z^a \\ y^b &= r_{21}x^a + r_{22}y^a + r_{23}z^a \\ z^b &= r_{31}x^a + r_{32}y^a + r_{33}z^a \end{aligned}$$

If the vectors $\vec{w} = [x, y, z]$ are row vectors then post-multiplication is used, such that:

$$\vec{w}^b = \vec{w}^a \widehat{\mathbf{R}}_a^b$$

$$\begin{bmatrix} x^b & y^b & z^b \end{bmatrix} = \begin{bmatrix} x^a & y^a & z^a \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{aligned} x^b &= r_{11}x^a + r_{21}y^a + r_{31}z^a \\ y^b &= r_{12}x^a + r_{22}y^a + r_{32}z^a \\ z^b &= r_{13}x^a + r_{23}y^a + r_{33}z^a \end{aligned}$$

It is important to note that the rotation transformation results are **different**. So that for a same rotation, the rotation matrices must be different $\mathbf{R}_a^b \neq \widehat{\mathbf{R}}_a^b$ depending on if the transformation is pre or post-multiplied.

Post-multiplication with row vectors $\vec{w} = \vec{v}^T$ is the *same* as pre-multiplication using column vectors $\vec{v}$ with the *inverse* or transposed rotation:

$$\begin{aligned} \mathbf{R}\vec{v} &= \left(\vec{w}\widehat{\mathbf{R}}\right)^T \\ \mathbf{R}\vec{v} &= \widehat{\mathbf{R}}^T\vec{w}^T \\ \mathbf{R}\vec{v} &= \widehat{\mathbf{R}}^T\vec{v} \\ \therefore \mathbf{R} &= \widehat{\mathbf{R}}^T \end{aligned}$$

The convention used in this reference is that all the transformations are pre-multiplied, such that $\vec{v}^b = \mathbf{R}_a^b\vec{v}^a$.

4.3 Direct Cosine Matrix

The rotation transformation matrix can be described as an operation that projects the reference frame axes onto the rotated frame axes, so that it forms a mapping of how much each axis is transferred to all the other axes. This is called a 'Direct Cosine Matrix' (DCM) because each element of the matrix is the cosine of the unsigned angle between the frame axes.

$$\mathbf{R}_a^b = \begin{bmatrix} \cos(\theta_{x^a,x^b}) & \cos(\theta_{y^a,x^b}) & \cos(\theta_{z^a,x^b}) \\ \cos(\theta_{x^a,y^b}) & \cos(\theta_{y^a,y^b}) & \cos(\theta_{z^a,y^b}) \\ \cos(\theta_{x^a,z^b}) & \cos(\theta_{y^a,z^b}) & \cos(\theta_{z^a,z^b}) \end{bmatrix}$$

where θ_{x^a,x^b} is the unsigned angle between the x^a axis and the x^b axis. This process effectively rotates the reference frame via the projection, it does *not* rotate the vector, rather it views the same vector from a difference frame.

For a pure frame rotation around the z-axis, where the rotated frame $\mathcal{F}^b$ rotates by angle θ around the z-axis from the reference frame $\mathcal{F}^a$, the angles between the axes are:

$$\begin{aligned} \theta_{x^a,x^b} &= \theta_{y^a,y^b} = \theta \\ \theta_{y^a,x^b} &= \frac{\pi}{2} - \theta \\ \theta_{z^a,x^b} &= \theta_{z^a,y^b} = \theta_{x^a,z^b} = \theta_{y^a,z^b} = \frac{\pi}{2} \\ \theta_{x^a,y^b} &= \frac{\pi}{2} + \theta \end{aligned}$$

$$\mathbf{R}_a^b = \begin{bmatrix} \cos(\theta) & \cos(\frac{\pi}{2} - \theta) & \cos(\frac{\pi}{2}) \\ \cos(\frac{\pi}{2} + \theta) & \cos(\theta) & \cos(\frac{\pi}{2}) \\ \cos(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & \cos(0) \end{bmatrix}$$

$$\mathbf{R}_a^b = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.4 Single Axis Transformations

The single axis frame rotation transformations are given by:

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It should be noted again that these transformations rotate the reference frame, they do not rotate the vector itself.

4.5 Active or Passive Rotations

A rotation can be considered active or passive based on if it is the frame rotates (passive) or if the vector itself rotates (active).

When the frame rotates, it is considered a passive rotation since the vector does not change, rather it is the frame of reference that rotates. This is the standard used in this reference, as it is most closely aligned with engineering concents.

For example if a car is travelling directly north but you are facing the car from the west, you see the car travelling to the left, the actual velocity vector of the car has not changed, only the frame of reference you are viewing from.

Rotations have no absolute quantities, they are always relative to a reference. Therefore an active rotation can be just considered as a passive rotation, but in the opposite direction. So an passive rotation by θ is the same as an active rotation by $-\theta$.

Let $\mathbf{R}(\theta)$ be a passive rotation and let $\mathbf{R}'(\theta)$ be an active rotation. The same rotation transformation can be express using the relationship:

$$\mathbf{R}(\theta) = \mathbf{R}'(-\theta)$$

Applying this to the single axis *frame* rotations, gives the single axis *vector* rotations:

$$\mathbf{R}'_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}'_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}'_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These equations are very similar as the frame rotation matrices, care should be taken when referring other resources as they may use this as the convention.

4.6 Chaining Multiple Rotations

Multiple rotations can be chained together by multiplying the rotation matrices in order (right to left). Consider the rotation from $\mathcal{F}^a$ to $\mathcal{F}^c$ via the intermediate frame $\mathcal{F}^b$:

$$\begin{aligned}
\vec{x}^c &= \mathbf{R}_b^c \vec{x}^b \\
\vec{x}^b &= \mathbf{R}_a^b \vec{x}^a \\
\vec{x}^c &= \mathbf{R}_b^c \left(\mathbf{R}_a^b \vec{x}^a \right) \\
\vec{x}^c &= \mathbf{R}_a^c \vec{x}^a
\end{aligned}$$

So the composite rotation matrix $\mathbf{R}_a^c$ is formed by chaining the rotations together $\mathbf{R}_a^c = \mathbf{R}_b^c \mathbf{R}_a^b$ by multiplying the subsequent rotations on the left.

If a different standard is used, such as if the rotation matrix operates as post-multiplication on row vectors then the rotation order might need to be reconsidered.

4.7 Intrinsic vs Extrinsic Rotations

When multiple rotations are chained together by sequential operations, there are two different ways of viewing the operation, there are intrinsic or extrinsic rotation sequences.

In the **intrinsic** definition, each sequential operation is applied to the **intermediate** frame from the pervious rotation. Consider the example:

$$(x, y, z) \to \mathbf{R}_z \to \left(x', y', z\right) \to \mathbf{R}_{y'} \to \left(x'', y', z''\right)$$

The first rotation $\mathbf{R}_z$ rotates the (x, y, z)-axes around the z-axis, so that the z-axis stays the same and the (x,y)-axes shift. The next rotation $\mathbf{R}_{y'}$ is about the **new** y'-axis, so the (x',z)-axes now shift. This is the convention used so far in this reference.

In the **extrinsic** definition, each sequential operation is applied to the **original** un-rotated frame. Consider the example:

$$(x, y, z) \to \mathbf{R}_z \to \left(x', y', z\right) \to \mathbf{R}_y \to \left(x'', y'', z''\right)$$

The first rotation $\mathbf{R}_z$ rotates the (x, y, z)-axes around the z-axis, while the next rotation $\mathbf{R}_y$ is about the **original** y-axis, causing all the (x, y, z)-axes to change.

To convert the chained **intrinsic** rotation sequence $\mathbf{R} = \mathbf{R}_i\mathbf{R}_j\mathbf{R}_k$ into an **extrinsic** rotation sequence, the rotation sequence can be reversed $\mathbf{R}' = \mathbf{R}_k\mathbf{R}_j\mathbf{R}_i$. The opposite is also true, to convert an **extrinsic** rotation sequence into an **intrinsic** rotation sequence, reverse the sequence order of the rotations.

4.8 DCM Attitude Kinematics

The rate of change of the rotation matrix $\mathbf{R}$, expressed as $\dot{\mathbf{R}}$, is a function of the angular rates $\vec{\omega}$ and the current rotation:

$$\dot{\mathbf{R}} = f\left(\vec{\omega}, \mathbf{R}\right)$$

The rate of change of the rotation matrix $\mathbf{R}_a^b$ of $\mathcal{F}^a$ to $\mathcal{F}^b$ can be expressed in terms of the angular rates $\vec{\omega}_{b/a}^b$ of $\mathcal{F}^b$ relative to $\mathcal{F}^a$ expressed in frame $\mathcal{F}^b$ by the relationship:

$$\dot{\mathbf{R}}_a^b = -\mathbf{\Omega}_{b/a}^b\mathbf{R}_a^b$$

where $\mathbf{\Omega}$ is the skew-symmetric matrix defined by $\mathbf{\Omega} = [\vec{\omega}]_\times$.

This relationship can be derived from the transformation equation:

$$\vec{x}^b = \mathbf{R}_a^b \vec{x}^a$$

$$\mathbf{R}_b^a \vec{x}^b = \vec{x}^a$$

Taking the time derivative:

$$\dot{\mathbf{R}}_b^a \vec{x}^b + \mathbf{R}_b^a \frac{\partial \vec{x}^b}{\partial t} = 0$$

The vector transport theorem gives the known relationship for:

$$\frac{\partial \vec{x}^a}{\partial t} = \frac{\partial \vec{x}^b}{\partial t} + \vec{\omega}_{b/a}^b \times \vec{x}^b$$

$$0 = \frac{\partial \vec{x}^b}{\partial t} + \vec{\omega}_{b/a}^b \times \vec{x}^b$$

$$\frac{\partial \vec{x}^b}{\partial t} = -\vec{\omega}_{b/a}^b \times \vec{x}^b$$

Substituting this into the original equation gives:

$$\dot{\mathbf{R}}_b^a \vec{x}^b - \mathbf{R}_b^a \left(\vec{\omega}_{b/a}^b \times \vec{x}^b \right) = 0$$

Let $\mathbf{\Omega}^b = [\omega]_\times^b$ then:

$$\dot{\mathbf{R}}_b^a \vec{x}^b - \mathbf{R}_b^a \mathbf{\Omega}_{b/a}^b \vec{x}^b = 0$$

$$\left(\dot{\mathbf{R}}_b^a - \mathbf{R}_b^a \mathbf{\Omega}_{b/a}^b \right) \vec{x}^b = 0$$

Therefore:

$$\left(\dot{\mathbf{R}}_b^a - \mathbf{R}_b^a \mathbf{\Omega}_{b/a}^b \right) = 0$$

$$\dot{\mathbf{R}}_b^a = \mathbf{R}_b^a \mathbf{\Omega}_{b/a}^b$$

Taking the transpose of both sides gives:

$$\dot{\mathbf{R}}_a^b = -\mathbf{\Omega}_{b/a}^b \mathbf{R}_a^b$$

Chapter 5

Euler Angles

5.1 Definition

Euler angles describe any arbitrary rotation using a set of 3 angles, each angle representing a single axis rotation that is applied in an intrinsic sequence. Let $\mathbf{\Phi}$ be a set of Euler Angles consisting of the angles $\{\phi, \theta, \psi\}$. The rotation matrix is calculated with:

$$\mathbf{R}_{ijk}\left(\mathbf{\Phi}\right) = \mathbf{R}_i\left(\phi\right)\mathbf{R}_j\left(\theta\right)\mathbf{R}_k\left(\psi\right)$$

where $i, j, k = \{x, y, z\}$. There are 12 valid rotation sequences, that can be broken up into *Proper Euler Angles* which have a rotation axis repeated and *Tait-Bryan Angles* which have 3 distinct rotation axes.

The Euler sequence **XYZ** is commonly called 'Cardan Angles' or 'Nautical Angles'. It is also the sequence that is typically used when people refer to 'Euler Angles'. The angles in this sequence (ϕ, θ, ψ) are commonly called Roll, Pitch and Yaw.

Proper Euler Angles	Tait-Bryan Angles
• ZXZ • XYX • YZY • ZYZ • XZX • YXY	• XYZ • YZX • ZXY • XZY • ZYX • YXZ

The 'Proper Euler Angles' sequences are the sequences that Leonhard Euler derived for gyroscope motion analysis and representation. In particular the sequence **ZXZ** referred to as 'X-convention', is commonly used for gyroscopic motion with the angles (ϕ, θ, ψ) are referred to as Precession, Nutation and Spin.

5.2 Angle Conventions and Ranges

Angles are defined according to the right hand rule. Positive values are clockwise when viewed along the axis direction.

The range for ϕ and ψ are defined by *modulo* 2π so that:

$$-\pi \leq \{\phi, \psi\} \leq \pi \quad \text{or} \quad 0 \leq \{\phi, \psi\} \leq 2\pi$$

The range for θ is defined between the ranges:

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \text{or} \quad 0 \leq \theta \leq \pi$$

5.3 Euler Angle and DCM Conversions

The following are the conversion functions between Euler Angles and Direct Cosine Matrix attitude representations. Let $c_{(\cdot)}$ and $s_{(\cdot)}$ be the $\sin(\cdot)$ and $\cos(\cdot)$ operations respectively.

5.3.1 Euler Angles (XYZ)

$$\mathbf{R}_{xyz} = \mathbf{R}_x(\phi)\,\mathbf{R}_y(\theta)\,\mathbf{R}_z(\psi)$$

$$\mathbf{R}_{xyz} = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\theta s_\phi \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\theta c_\phi \end{bmatrix}$$

$$\mathbf{\Phi}_{xyz} = \begin{bmatrix} \operatorname{arctan2}(r_{23}, r_{33}) \\ -\arcsin(r_{13}) \\ \operatorname{arctan2}(r_{12}, r_{11}) \end{bmatrix}$$

5.3.2 Euler Angles (ZXZ)

$$\mathbf{R}_{zxz} = \mathbf{R}_z(\phi)\,\mathbf{R}_x(\theta)\,\mathbf{R}_z(\psi)$$

$$\mathbf{R}_{zxz} = \begin{bmatrix} c_\phi c_\psi - s_\phi c_\theta s_\psi & c_\phi s_\psi + s_\phi c_\theta c_\psi & s_\phi s_\theta \\ -s_\phi c_\psi - c_\phi c_\theta s_\psi & -s_\phi s_\psi + c_\phi c_\theta c_\psi & c_\phi s_\theta \\ s_\theta s_\psi & -s_\theta c_\psi & c_\theta \end{bmatrix}$$

$$\mathbf{\Phi}_{zxz} = \begin{bmatrix} \operatorname{arctan2}(r_{13}, r_{23}) \\ \arccos(r_{33}) \\ \operatorname{arctan2}(r_{31}, -r_{32}) \end{bmatrix}$$

5.3.3 Euler Angles (XYX)

$$\mathbf{R}_{xyx} = \mathbf{R}_x(\phi)\,\mathbf{R}_y(\theta)\,\mathbf{R}_x(\psi)$$

$$\mathbf{R}_{xyx} = \begin{bmatrix} c_\theta & s_\theta s_\psi & -s_\theta c_\psi \\ s_\phi s_\theta & c_\phi c_\psi - s_\phi c_\theta s_\psi & c_\phi s_\psi + s_\phi c_\theta c_\psi \\ c_\phi s_\theta & -s_\phi c_\psi - c_\phi c_\theta s_\psi & -s_\phi s_\psi + c_\phi c_\theta c_\psi \end{bmatrix}$$

$$\mathbf{\Phi}_{xyx} = \begin{bmatrix} \operatorname{arctan2}(r_{21}, r_{31}) \\ \arccos(r_{11}) \\ \operatorname{arctan2}(r_{12}, -r_{13}) \end{bmatrix}$$

5.3.4 Euler Angles (YZX)

$$\mathbf{R}_{yzx} = \mathbf{R}_y(\phi)\,\mathbf{R}_z(\theta)\,\mathbf{R}_x(\psi)$$

$$\mathbf{R}_{yzx} = \begin{bmatrix} c_\phi c_\theta & c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & c_\theta c_\psi & c_\theta s_\psi \\ s_\phi c_\theta & s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi \end{bmatrix}$$

$$\mathbf{\Phi}_{yzx} = \begin{bmatrix} \operatorname{arctan2}(r_{31}, r_{11}) \\ -\arcsin(r_{21}) \\ \operatorname{arctan2}(r_{23}, r_{22}) \end{bmatrix}$$

5.3.5 Euler Angles (YZY)

$$\mathbf{R}_{yzy} = \mathbf{R}_y(\phi)\,\mathbf{R}_z(\theta)\,\mathbf{R}_y(\psi)$$

$$\mathbf{R} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & c_\theta s_\theta & -c_\phi c_\theta s_\psi - s_\phi c_\psi \\ -s_\theta c_\psi & c_\theta & s_\theta s_\psi \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & s_\phi s_\theta & -s_\phi c_\theta s_\psi + c_\phi c_\psi \end{bmatrix}$$

$$\mathbf{\Phi}_{yzy} = \begin{bmatrix} \operatorname{arctan2}(r_{32}, r_{12}) \\ \arccos(r_{22}) \\ \operatorname{arctan2}(r_{23}, r_{21}) \end{bmatrix}$$

5.3.6 Euler Angles (ZXY)

$$\mathbf{R}_{zxy} = \mathbf{R}_z\left(\phi\right)\mathbf{R}_x\left(\theta\right)\mathbf{R}_y\left(\psi\right)$$

$$\mathbf{R}_{zxy} = \begin{bmatrix} c_\phi c_\psi + s_\phi s_\theta s_\psi & s_\phi c_\theta & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\phi c_\psi + c_\phi s_\theta s_\psi & c_\theta c_\phi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ c_\theta s_\psi & -s_\theta & c_\theta c_\psi \end{bmatrix}$$

$$\mathbf{\Phi}_{zxy} = \begin{bmatrix} \arctan2(r_{12,r_{22}}) \\ -\arcsin(r_{32}) \\ \arctan2(r_{31}, r_{33}) \end{bmatrix}$$

5.3.7 Euler Angles (ZYZ)

$$\mathbf{R}_{zyz} = \mathbf{R}_z\left(\phi\right)\mathbf{R}_y\left(\theta\right)\mathbf{R}_z\left(\psi\right)$$

$$\mathbf{R}_{zyz} = \begin{bmatrix} -c_\phi c_\theta c_\psi - s_\phi s_\psi & c_\phi c_\theta s_\psi + s_\phi c_\psi & -c_\phi s_\theta \\ -s_\phi c_\theta c_\psi - c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

$$\mathbf{\Phi}_{zyz} = \begin{bmatrix} \arctan2(r_{23}, -r_{13}) \\ \arccos(r_{33}) \\ \arctan2(r_{32}, r_{31}) \end{bmatrix}$$

5.3.8 Euler Angles (XZY)

$$\mathbf{R}_{xzy} = \mathbf{R}_x\left(\phi\right)\mathbf{R}_z\left(\theta\right)\mathbf{R}_y\left(\psi\right)$$

$$\mathbf{R}_{xzy} = \begin{bmatrix} c_\theta c_\psi & s_\theta & -c_\theta s_\psi \\ -c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi c_\theta & c_\phi s_\theta s_\psi + s_\phi c_\psi \\ s_\phi s_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta & -s_\phi s_\theta s_\psi + c_\phi c_\psi \end{bmatrix}$$

$$\mathbf{\Phi}_{xzy} = \begin{bmatrix} \arctan2(-r_{32}, r_{22}) \\ \arcsin(r_{12}) \\ \arctan2(-r_{13}, r_{11}) \end{bmatrix}$$

5.3.9 Euler Angles (ZYX)

$$\mathbf{R}_{zyx} = \mathbf{R}_z(\phi)\,\mathbf{R}_y(\theta)\,\mathbf{R}_x(\psi)$$

$$\mathbf{R}_{zyx} = \begin{bmatrix} c_\phi c_\theta & s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi - c_\phi s_\theta c_\psi \\ -s_\phi c_\theta & c_\phi c_\psi - s_\phi s_\theta s_\psi & c_\phi s_\psi + s_\phi s_\theta c_\psi \\ s_\theta & -s_\psi c_\theta & c_\psi c_\theta \end{bmatrix}$$

$$\mathbf{\Phi}_{zyx} = \begin{bmatrix} \mathrm{arctan2}(-r_{21}, r_{11}) \\ \arcsin(r_{31}) \\ \mathrm{arctan2}(-r_{32}, r_{33}) \end{bmatrix}$$

5.3.10 Euler Angles (XZX)

$$\mathbf{R}_{xzx} = \mathbf{R}_x(\phi)\,\mathbf{R}_z(\theta)\,\mathbf{R}_x(\psi)$$

$$\mathbf{R}_{xzx} = \begin{bmatrix} c_\theta & c_\psi s_\theta & s_\psi s_\theta \\ -c_\phi s_\theta & c_\phi c_\theta c_\psi - s_\phi s_\psi & c_\phi c_\theta s_\phi + s_\phi c_\psi \\ s_\phi s_\theta & -s_\phi c_\theta c_\psi - c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi \end{bmatrix}$$

$$\mathbf{\Phi}_{xzx} = \begin{bmatrix} \mathrm{arctan2}(r_{31}, -r_{21}) \\ \arccos(r_{11}) \\ \mathrm{arctan2}(r_{13}, r_{12}) \end{bmatrix}$$

5.3.11 Euler Angles (YXY)

$$\mathbf{R}_{yxy} = \mathbf{R}_y(\phi)\,\mathbf{R}_x(\theta)\,\mathbf{R}_y(\psi)$$

$$\mathbf{R}_{yxy} = \begin{bmatrix} c_\phi c_\psi - s_\phi c_\theta s_\psi & s_\theta s_\phi & -c_\phi s_\psi - s_\phi c_\theta c_\psi \\ s_\theta s_\psi & c_\theta & s_\theta c_\psi \\ s_\phi c_\psi + c_\phi c_\theta s_\psi & -s_\theta c_\phi & -s_\phi s_\psi + c_\phi c_\theta c_\psi \end{bmatrix}$$

$$\mathbf{\Phi}_{yxy} = \begin{bmatrix} \mathrm{arctan2}(r_{12}, -r_{32}) \\ \arccos(r_{22}) \\ \mathrm{arctan2}(r_{21}, r_{23}) \end{bmatrix}$$

5.3.12 Euler Angles (YXZ)

$$\mathbf{R}_{yxz} = \mathbf{R}_y\left(\phi\right)\mathbf{R}_x\left(\theta\right)\mathbf{R}_z\left(\psi\right)$$

$$\mathbf{R}_{yxz} = \begin{bmatrix} c_\phi c_\psi - s_\phi s_\theta s_\psi & c_\phi s_\psi + s_\phi s_\theta c_\psi & -c_\theta s_\phi \\ -c_\theta s_\psi & c_\theta c_\psi & s_\theta \\ s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi - c_\phi s_\theta c_\psi & c_\phi c_\theta \end{bmatrix}$$

$$\mathbf{\Phi}_{yxz} = \begin{bmatrix} \text{arctan2}(-r_{13}, r_{33}) \\ \arcsin(r_{23}) \\ \text{arctan2}(-r_{21}, r_{22}) \end{bmatrix}$$

5.4 Euler Angle Kinematics

The rate of change of the attitude when represented with Euler Angles can be calculated from the relationship:

$$\dot{\mathbf{\Phi}} = E_{ijk}\left(\mathbf{\Phi}\right)\vec{\omega}^b_{b/w}$$

where $\dot{\mathbf{\Phi}} = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$ is a vector of the rate of change of the Euler Angles, $\vec{\omega}$ is the angular rate and $E_{ijk}(\mathbf{\Phi})$ is a specific matrix for the Euler Angle sequence. For the Euler Angle Sequence **XYZ**, this relationship is:

$$E_{xyz}\left(\mathbf{\Phi}\right) = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{bmatrix}$$

This relationship can be derived from:

$$\vec{\omega}^b_{b/w} = \vec{\omega}_{\dot{\phi}} + \vec{\omega}_{\dot{\theta}} + \vec{\omega}_{\dot{\psi}}$$

Inspecting the Euler Angle sequence:

$$\mathbf{R}_{xyz} = \mathbf{R}_x(\phi)\mathbf{R}_y(\theta)\mathbf{R}_z(\psi)$$

The angular rate can be described as transformations of unit vectors though the rotation transform:

$$\vec{\omega}_{b/w}^{b} = \dot{\phi}\hat{\mathbf{i}} + \mathbf{R}_x(\phi)\dot{\theta}\hat{\mathbf{j}} + \mathbf{R}_x(\phi)\mathbf{R}_y(\theta)\dot{\psi}\hat{\mathbf{k}}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_x(\phi)\left(\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{R}_y(\theta)\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}\right)$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \cos(\theta)\sin(\phi) \\ 0 & -\sin(\phi) & \cos(\theta)\cos(\phi) \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Taking the inverse relationship:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

which can be written as:

$$\dot{\mathbf{\Phi}} = E_{xyz}\left(\mathbf{\Phi}\right)\vec{\omega}_{b/w}^{b}$$

5.4.1 Euler Angles Kinematics (XYZ)

$$E_{xyz}\left(\mathbf{\Phi}\right) = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{bmatrix}$$

5.4.2 Euler Angles Kinematics (ZXZ)

$$E_{zxz}\left(\mathbf{\Phi}\right) = \begin{bmatrix} -\sin(\phi)\cot(\theta) & -\cos(\phi)\cot(\theta) & 1 \\ \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi)\csc(\theta) & \cos(\phi)\csc(\theta) & 0 \end{bmatrix}$$

5.4.3 Euler Angles Kinematics (XYX)

$$E_{xyx}\left(\mathbf{\Phi}\right) = \frac{1}{s_\theta}\begin{bmatrix} s_\theta & -s_\phi c_\theta & c_\phi c_\theta \\ 0 & c_\phi s_\theta & -s_\phi s_\theta \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

5.4.4 Euler Angles Kinematics (YZX)

$$E_{yzx}\left(\mathbf{\Phi}\right) = \frac{1}{c_\theta}\begin{bmatrix} c_\phi s_\theta & c_\theta & s_\phi s_\theta \\ -s_\phi c_\theta & 0 & c_\phi c_\theta \\ c_\phi & 0 & s_\phi \end{bmatrix}$$

5.4.5 Euler Angles Kinematics (YZY)

$$E_{yzy}\left(\mathbf{\Phi}\right) = \frac{1}{s_\theta}\begin{bmatrix} -c_\phi c_\theta & s_\theta & -s_\phi c_\theta \\ -s_\phi s_\theta & 0 & c_\phi s_\theta \\ c_\phi & 0 & s_\phi \end{bmatrix}$$

5.4.6 Euler Angles Kinematics (ZXY)

$$E_{zxy}\left(\mathbf{\Phi}\right) = \frac{1}{c_\theta}\begin{bmatrix} s_\phi s_\theta & c_\phi s_\theta & c_\theta \\ c_\theta c_\phi & -s_\phi c_\theta & 0 \\ s_\phi & c_\phi & 0 \end{bmatrix}$$

5.4.7 Euler Angles Kinematics (ZYZ)

$$E_{zyz}\left(\mathbf{\Phi}\right) = \frac{1}{s_\theta}\begin{bmatrix} c_\phi c_\theta & -s_\phi c_\theta & s_\theta \\ s_\phi s_\theta & c_\phi s_\theta & 0 \\ -c_\phi & s_\phi & 0 \end{bmatrix}$$

5.4.8 Euler Angles Kinematics (XZY)

$$E_{xzy}(\mathbf{\Phi}) = \frac{1}{c_\theta}\begin{bmatrix} c_\theta & -c_\phi s_\theta & s_\phi s_\theta \\ 0 & s_\phi c_\theta & c_\phi c_\theta \\ 0 & c_\phi & -s_\phi \end{bmatrix}$$

5.4.9 Euler Angles Kinematics (XZX)

$$E_{xzx}(\mathbf{\Phi}) = \frac{1}{s_\theta}\begin{bmatrix} s_\theta & c_\phi c_\theta & -s_\phi c_\theta \\ 0 & s_\phi s_\theta & c_\phi s_\theta \\ 0 & -c_\phi & s_\phi \end{bmatrix}$$

5.4.10 Euler Angles Kinematics (YXY)

$$E_{yxy}(\mathbf{\Phi}) = \frac{1}{s_\theta}\begin{bmatrix} -s_\phi c_\theta & s_\theta & c_\phi c_\theta \\ s_\theta c_\phi & 0 & s_\theta s_\phi \\ s_\phi & 0 & -c_\theta \end{bmatrix}$$

5.4.11 Euler Angles Kinematics (YXZ)

$$E_{yxz}(\mathbf{\Phi}) = \frac{1}{c_\theta}\begin{bmatrix} s_\phi s_\theta & c_\theta & -c_\phi s_\theta \\ c_\phi c_\theta & 0 & s_\phi c_\theta \\ -s_\phi & 0 & c_\phi \end{bmatrix}$$

Chapter 6

Quaternions

A quaternion $\mathbf{q} \in \mathbb{H}$ is an extended number system for complex numbers. It may be represented as a vector:

$$\mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T = \begin{bmatrix} q_0 \\ \mathbf{q}_{1:3} \end{bmatrix}$$

Quaternions are generally represented in the vector form:

$$\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k$$

where q_0 is the real scalar part and q_1, q_2 and q_3 are the real components of the quaternion vector unit components i, j and k.

6.1 Quaternion Algebra

6.1.1 Arithmetic

Quaternion addition and subtraction is applied element-wise:

$$\mathbf{q} + \mathbf{p} = \begin{bmatrix} q_0 + p_0 \\ \mathbf{q}_{1:3} + \mathbf{p}_{1:3} \end{bmatrix}$$

$$\mathbf{q} - \mathbf{p} = \begin{bmatrix} q_0 - p_0 \\ \mathbf{q}_{1:3} - \mathbf{p}_{1:3} \end{bmatrix}$$

6.1.2 Multiplication

Quaternion multiplication is applied by multiplication of the complete form:

$$\mathbf{q} \cdot \mathbf{p} = (q_0 + q_1 i + q_2 j + q_3 k)\,(p_0 + p_1 i + p_2 j + p_3 k)$$

with the following imaginary rules applied:

$$\begin{aligned} i^2 = j^2 = k^2 = ijk &= -1 \\ ij = -ji &= k \\ jk = -kj &= i \\ ki = -ik &= j \end{aligned}$$

The expanded multiplication operation can be written as:

$$\mathbf{q} \cdot \mathbf{p} = \begin{bmatrix} q_0 p_0 - \mathbf{q}_{1:3}^T \mathbf{p}_{1:3} \\ q_0 \mathbf{p}_{1:3} + p_0 \mathbf{q}_{1:3} + \mathbf{q}_{1:3} \times \mathbf{p}_{1:3} \end{bmatrix}$$

$$\mathbf{q} \cdot \mathbf{p} = \begin{bmatrix} (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) \\ (p_0 q_1 + p_1 q_0 - p_2 q_3 + p_3 q_2) \\ (p_0 q_2 + p_1 q_3 + p_2 q_0 - p_3 q_1) \\ (p_0 q_3 - p_1 q_2 + p_2 q_1 + p_3 q_0) \end{bmatrix}$$

In general quaternion multiplication is **not** commutative:

$$\mathbf{q} \cdot \mathbf{p} \neq \mathbf{p} \cdot \mathbf{q}$$

It is only commutative in certain subsets of pure real multiplication or pure real and pure imaginary multiplication.

Quaternion multiplication can also be written more compactly as a matrix/vector multiplication, that is:

$$\mathbf{q} \cdot \mathbf{p} = Q(\mathbf{q})\mathbf{p} = \bar{Q}(\mathbf{p})\mathbf{q}$$

where the quaternion matrix $Q(\cdot)$ is defined by:

$$Q(\mathbf{q}) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

$$\bar{Q}(\mathbf{q}) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}$$

which also yields the relationships:

$$Q(\bar{\mathbf{q}}) = Q(\mathbf{q})^T$$
$$\bar{Q}(\bar{\mathbf{q}}) = \bar{Q}(\mathbf{q})^T$$

6.1.3 Conjugate

The conjugate of a quaternion is the negative of the imaginary term (i.e the vector component of the quaternion).

$$\bar{\mathbf{q}} = \begin{bmatrix} q_0 \\ -\mathbf{q}_{1:3} \end{bmatrix}$$

The following conjugate properties hold:

$$\overline{\mathbf{p} \cdot \mathbf{q}} = \bar{\mathbf{q}} \cdot \bar{\mathbf{p}}$$

6.1.4 Norm

The norm of a quaternion is the same as the Euclidean norm of the vector:

$$\|\mathbf{q}\| = \sqrt{\mathbf{q} \cdot \bar{\mathbf{q}}} = \sqrt{\bar{\mathbf{q}} \cdot \mathbf{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

6.1.5 Inverse

The inverse of the quaternion or division of a quaternion is calculated as:

$$\mathbf{q}^{-1} = \frac{\bar{\mathbf{q}}}{\|\mathbf{q}\|}$$

which exists for every non-zero quaternion.

6.1.6 Unit Quaternion

A unit quaternion has a unity norm $\|\mathbf{q}\| = 1$, so that the follow properties are valid:

$$\mathbf{q} \cdot \bar{\mathbf{q}} = 1$$

$$\mathbf{q}^{-1} = \bar{\mathbf{q}}$$

6.2 Unit Quaternion and Attitude Representation

A unit quaternion can be used to represent the attitude of a rigid body or a rotation transformation. The attitude can be represented as an rotation axis (the vector part of the quaternion) while the scalar part of the quaternion represents the rotation angle around the rotation axis.

The quaternion can be formed by the relationship:

$$\mathbf{q} = \begin{bmatrix} \cos\left(\frac{1}{2}D\right) \\ \cos(A)\sin\left(\frac{1}{2}D\right) \\ \cos(B)\sin\left(\frac{1}{2}D\right) \\ \cos(C)\sin\left(\frac{1}{2}D\right) \end{bmatrix}$$

where A, B and C are the angles between the rotation axis vector and the x, y and z axes respectively, and D is the rotation angle around the rotation axis vector.

With rotations, reversing the axis of rotation or reversing the angle of rotation, reverses the transformation. Reversing both at the same time leaves the total transformation unchanged.

$$\begin{aligned} \mathbf{q} &: \mathcal{F}^a \to \mathcal{F}^b \\ -\mathbf{q} &: \mathcal{F}^a \to \mathcal{F}^b \\ \bar{\mathbf{q}} &: \mathcal{F}^a \leftarrow \mathcal{F}^b \\ -\bar{\mathbf{q}} &: \mathcal{F}^a \leftarrow \mathcal{F}^b \end{aligned}$$

The rotation transformation can be applied using the following relationship. Consider a vector $\vec{x} \in \mathbb{R}^3$, with $\vec{x}^a$ being the vector in frame $\mathcal{F}^a$ and $\vec{x}^b$ being the same vector represented in frame $\mathcal{F}^b$, then the following relationship holds:

$$\begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} = \mathbf{q} \cdot \begin{bmatrix} 0 \\ \vec{x}^a \end{bmatrix} \cdot \mathbf{q}^{-1}$$

where $\mathbf{q}$ is the unit quaternion that represents the rotation between frame $\mathcal{F}^a$ and frame $\mathcal{F}^b$.

6.3 Unit Quaternion and DCM Conversions

The quaternion rotation transformation relationship can be expanded to find the equivalent DCM matrix in terms of the quaternion components:

$$\begin{aligned}
\begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} &= \mathbf{q} \cdot \begin{bmatrix} 0 \\ \vec{x}^a \end{bmatrix} \cdot \mathbf{q}^{-1} \\
&= \mathbf{q} \cdot \begin{bmatrix} 0 \\ \vec{x}^a \end{bmatrix} \cdot \bar{\mathbf{q}} \\
&= \bar{Q}(\mathbf{q})^T Q(\mathbf{q}) \begin{bmatrix} 0 \\ \vec{x}^a \end{bmatrix} \\
&= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} 0 \\ \vec{x}^a \end{bmatrix}
\end{aligned}$$

where the DCM rotation matrix $\mathbf{R}(\mathbf{q})$ is:

$$\mathbf{R}(\mathbf{q}) =$$
$$\begin{bmatrix} \left(q_0^2 + q_1^2 - q_2^2 - q_3^2\right) & 2q_1q_2 + 2q_0q_3 & 2q_1q_3 - 2q_0q_2 \\ 2q_1q_2 - 2q_0q_3 & \left(q_0^2 - q_1^2 + q_2^2 - q_3^2\right) & 2q_2q_3 + 2q_0q_1 \\ 2q_1q_3 + 2q_0q_2 & 2q_2q_3 - 2q_0q_1 & \left(q_0^2 - q_1^2 - q_2^2 + q_3^2\right) \end{bmatrix}$$

The reverse transformation mapping is slightly more complicated. Inspection of the DCM matrix yields the following inverse mappings for the quaternion components:

$$\mathbf{q}_0(\mathbf{R}) = \frac{1}{2}\begin{bmatrix} \sqrt{(1+r_{11}+r_{22}+r_{33})} \\ (r_{23}-r_{32})\,/\sqrt{(1+r_{11}+r_{22}+r_{33})} \\ (r_{31}-r_{13})\,/\sqrt{(1+r_{11}+r_{22}+r_{33})} \\ (r_{12}-r_{21})\,/\sqrt{(1+r_{11}+r_{22}+r_{33})} \end{bmatrix}$$

$$\mathbf{q}_1(\mathbf{R}) = \frac{1}{2}\begin{bmatrix} (r_{23}-r_{32})\,/\sqrt{(1+r_{11}-r_{22}-r_{33})} \\ \sqrt{(1+r_{11}-r_{22}-r_{33})} \\ (r_{12}+r_{21})\,/\sqrt{(1+r_{11}-r_{22}-r_{33})} \\ (r_{31}+r_{13})\,/\sqrt{(1+r_{11}-r_{22}-r_{33})} \end{bmatrix}$$

$$\mathbf{q}_2(\mathbf{R}) = \frac{1}{2}\begin{bmatrix} (r_{31}-r_{13})\,/\sqrt{(1-r_{11}+r_{22}-r_{33})} \\ (r_{12}+r_{21})\,/\sqrt{(1-r_{11}+r_{22}-r_{33})} \\ \sqrt{(1-r_{11}+r_{22}-r_{33})} \\ (r_{23}+r_{32})\,/\sqrt{(1-r_{11}+r_{22}-r_{33})} \end{bmatrix}$$

$$\mathbf{q}_3(\mathbf{R}) = \frac{1}{2}\begin{bmatrix} (r_{12}-r_{21})\,/\sqrt{(1-r_{11}-r_{22}+r_{33})} \\ (r_{31}+r_{13})\,/\sqrt{(1-r_{11}-r_{22}+r_{33})} \\ (r_{23}+r_{32})\,/\sqrt{(1-r_{11}-r_{22}+r_{33})} \\ \sqrt{(1-r_{11}-r_{22}+r_{33})} \end{bmatrix}$$

The selection of which $\mathbf{q}_i(\mathbf{R})$ to use depends on the values in $\mathbf{R}$ as some combinations will give complex results. The correct $\mathbf{q}_i(\mathbf{R})$ mapping to use is decided using the following function:

$$\mathbf{q}(\mathbf{R}) = \begin{cases} \mathbf{q}_0(\mathbf{R}) & \text{if } r_{22} > -r_{33},\ r_{11} > -r_{22},\ r_{11} > -r_{33} \\ \mathbf{q}_1(\mathbf{R}) & \text{if } r_{22} < -r_{33},\ r_{11} > r_{22},\ r_{11} > r_{33} \\ \mathbf{q}_2(\mathbf{R}) & \text{if } r_{22} > r_{33},\ r_{11} < r_{22},\ r_{11} < -r_{33} \\ \mathbf{q}_3(\mathbf{R}) & \text{if } r_{22} < r_{33},\ r_{11} < -r_{22},\ r_{11} < r_{33} \end{cases}$$

6.4 Unit Quaternion and Euler Angle Conversions

The conversion to Euler Angles from a unit quaternion can be found by substituting the unit quaternion to DCM conversion into the appropriate DCM to Euler Angle conversion with the desired Euler Angle sequence such that:

$$\mathbf{\Phi}_{ijk} = \mathbf{\Phi}_{ijk}\left(\mathbf{R}\left(\mathbf{q}\right)\right)$$

The conversion of a unit Quaternion from Euler Angles can be found in a similar way as generating a DCM matrix from Euler Angles by chaining successive single axis rotations:

$$\mathbf{q}\left(\mathbf{\Phi}_{ijk}\right) = \mathbf{q}_i\left(\phi\right) \cdot \mathbf{q}_j\left(\theta\right) \cdot \mathbf{q}_k\left(\psi\right)$$

where the single axis rotations are:

$$\mathbf{q}_x\left(\theta\right) = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{q}_y\left(\theta\right) = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ 0 \\ \sin(\frac{\theta}{2}) \\ 0 \end{bmatrix}$$

$$\mathbf{q}_z\left(\theta\right) = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ 0 \\ 0 \\ \sin(\frac{\theta}{2}) \end{bmatrix}$$

6.4.1 Unit Quaternions and Euler Angles (XYZ) Conversions

The conversion to Euler Angles (XYZ) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{xyz}(\mathbf{q}) = \begin{bmatrix} \operatorname{arctan2}\left(2q_2q_3 + 2q_0q_1,\ q_0^2 - q_1^2 - q_2^2 + q_3^2\right) \\ -\arcsin\left(2q_1q_3 - 2q_0q_2\right) \\ \operatorname{arctan2}\left(2q_1q_2 + 2q_0q_3,\ q_0^2 + q_1^2 - q_2^2 - q_3^2\right) \end{bmatrix}$$

The conversion of a Euler Angles (XYZ) sequence to a unit quaternion is:

$$\mathbf{q}(\mathbf{\Phi}_{xyz}) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.2 Unit Quaternions and Euler Angles (ZXZ) Conversions

The conversion to Euler Angles (ZXZ) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{zxz}(\mathbf{q}) = \begin{bmatrix} \operatorname{arctan2}(2q_1q_3 - 2q_0q_2,\ 2q_2q_3 + 2q_0q_1) \\ \arccos\left(q_0^2 - q_1^2 - q_2^2 + q_3^2\right) \\ \operatorname{arctan2}(2q_1q_3 + 2q_0q_2,\ -2q_2q_3 + 2q_0q_1) \end{bmatrix}$$

The conversion of a Euler Angles (ZXZ) sequence to a unit quaternion is:

$$\mathbf{q}\left(\mathbf{\Phi}_{zxz}\right) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.3 Unit Quaternions and Euler Angles (XYX) Conversions

The conversion to Euler Angles (XYX) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{xyx}\left(\mathbf{q}\right) = \begin{bmatrix} \operatorname{arctan2}\left(2q_1q_2 - 2q_0q_3,\ 2q_1q_3 + 2q_0q_2\right) \\ \arccos\left(q_0^2 + q_1^2 - q_2^2 - q_3^2\right) \\ \operatorname{arctan2}\left(2q_1q_2 + 2q_0q_3,\ -2q_1q_3 + 2q_0q_2\right) \end{bmatrix}$$

The conversion of a Euler Angles (XYX) sequence to a unit quaternion is:

$$\mathbf{q}\left(\mathbf{\Phi}_{xyx}\right) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.4 Unit Quaternions and Euler Angles (YZX) Conversions

The conversion to Euler Angles (YZX) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{yzx}(\mathbf{q}) = \begin{bmatrix} \operatorname{arctan2}\left(2q_1q_3 + 2q_0q_2,\ q_0^2 + q_1^2 - q_2^2 - q_3^2\right) \\ -\arcsin\left(2q_1q_2 - 2q_0q_3\right) \\ \operatorname{arctan2}\left(2q_2q_3 + 2q_0q_1,\ q_0^2 - q_1^2 + q_2^2 - q_3^2\right) \end{bmatrix}$$

The conversion of a Euler Angles (YZX) sequence to a unit quaternion is:

$$\mathbf{q}(\mathbf{\Phi}_{yzx}) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.5 Unit Quaternions and Euler Angles (YZY) Conversions

The conversion to Euler Angles (YZY) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{yzy}(\mathbf{q}) = \begin{bmatrix} \operatorname{arctan2}\left(2q_2q_3 - 2q_0q_1,\ 2q_1q_2 + 2q_0q_3\right) \\ \arccos\left(q_0^2 - q_1^2 + q_2^2 - q_3^2\right) \\ \operatorname{arctan2}\left(2q_2q_3 + 2q_0q_1,\ -2q_1q_2 + 2q_0q_3\right) \end{bmatrix}$$

The conversion of a Euler Angles (YZY) sequence to a unit quaternion is:

$$\mathbf{q}\left(\mathbf{\Phi}_{yzy}\right) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.6 Unit Quaternions and Euler Angles (ZXY) Conversions

The conversion to Euler Angles (ZXY) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{zxy}\left(\mathbf{q}\right) = \begin{bmatrix} \operatorname{arctan2}\left(2q_1q_2 + 2q_0q_3,\ q_0^2 - q_1^2 + q_2^2 - q_3^2\right) \\ -\arcsin\left(2q_2q_3 - 2q_0q_1\right) \\ \operatorname{arctan2}\left(2q_1q_3 + 2q_0q_2,\ q_0^2 - q_1^2 - q_2^2 + q_3^2\right) \end{bmatrix}$$

The conversion of a Euler Angles (ZXY) sequence to a unit quaternion is:

$$\mathbf{q}\left(\mathbf{\Phi}_{zxy}\right) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.7 Unit Quaternions and Euler Angles (ZYZ) Conversions

The conversion to Euler Angles (ZYZ) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{zyz}(\mathbf{q}) = \begin{bmatrix} \operatorname{arctan2}(2q_2q_3 + 2q_0q_1,\ -2q_1q_3 + 2q_0q_2) \\ \arccos\left(q_0^2 - q_1^2 - q_2^2 + q_3^2\right) \\ \operatorname{arctan2}(2q_2q_3 - 2q_0q_1,\ 2q_1q_3 + 2q_0q_2) \end{bmatrix}$$

The conversion of a Euler Angles (ZYZ) sequence to a unit quaternion is:

$$\mathbf{q}(\mathbf{\Phi}_{zyz}) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.8 Unit Quaternions and Euler Angles (XZY) Conversions

The conversion to Euler Angles (XZY) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{xzy}(\mathbf{q}) = \begin{bmatrix} \operatorname{arctan2}\left(-2q_2q_3 + 2q_0q_1,\ q_0^2 - q_1^2 + q_2^2 - q_3^2\right) \\ \arcsin(2q_1q_2 + 2q_0q_2) \\ \operatorname{arctan2}\left(-2q_1q_3 + 2q_0q_2,\ q_0^2 + q_1^2 - q_2^2 - q_3^2\right) \end{bmatrix}$$

The conversion of a Euler Angles (XZY) sequence to a unit quaternion is:

$$\mathbf{q}\left(\mathbf{\Phi}_{xzy}\right) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.9 Unit Quaternions and Euler Angles (ZYX) Conversions

The conversion to Euler Angles (ZYX) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{zyx}\left(\mathbf{q}\right) = \begin{bmatrix} \text{arctan2}\left(-2q_1q_2 + 2q_0q_3,\ q_0^2 + q_1^2 - q_2^2 - q_3^2\right) \\ \arcsin\left(2_q1_q3 + 2q_0q_2\right) \\ \text{arctan2}\left(-2q_2q_3 + 2q_0q_1,\ q_0^2 - q_1^2 - q_2^2 + q_3^2\right) \end{bmatrix}$$

The conversion of a Euler Angles (ZYX) sequence to a unit quaternion is:

$$\mathbf{q}\left(\mathbf{\Phi}_{zyx}\right) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.10 Unit Quaternions and Euler Angles (XZX) Conversions

The conversion to Euler Angles (XZX) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{xzx}(\mathbf{q}) = \begin{bmatrix} \operatorname{arctan2}(2q_1q_3 + 2q_0q_2,\ -2q_1q_2 + 2q_0q_3) \\ \arccos\left(q_0^2 + q_1^2 - q_2^2 - q_3^2\right) \\ \operatorname{arctan2}(2q_1q_3 - 2q_0q_2,\ -2q_1q_2 + 2q_0q_3) \end{bmatrix}$$

The conversion of a Euler Angles (XZX) sequence to a unit quaternion is:

$$\mathbf{q}(\mathbf{\Phi}_{xzx}) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.11 Unit Quaternions and Euler Angles (YXY) Conversions

The conversion to Euler Angles (YXY) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{yxy}(\mathbf{q}) = \begin{bmatrix} \operatorname{arctan2}(2q_1q_2 + 2q_0q_3,\ -2q_2q_3 + 2q_0q_1) \\ \arccos\left(q_0^2 - q_1^2 + q_2^2 - q_3^2\right) \\ \operatorname{arctan2}(2q_1q_2 - 2q_0q_3,\ 2q_2q_3 + 2q_0q_1) \end{bmatrix}$$

The conversion of a Euler Angles (YXY) sequence to a unit quaternion is:

$$\mathbf{q}\left(\mathbf{\Phi}_{yxy}\right) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.4.12 Unit Quaternions and Euler Angles (YXZ) Conversions

The conversion to Euler Angles (YXZ) sequence from a unit quaternion is:

$$\mathbf{\Phi}_{yxz}\left(\mathbf{q}\right) = \begin{bmatrix} \operatorname{arctan2}\left(-2q_1q_3 + 2q_0q_2,\ q_0^2 - q_1^2 - q_2^2 + q_3^2\right) \\ \arcsin\left(2q_2q_3 + 2q_0q_1\right) \\ \operatorname{arctan2}\left(-2q_1q_2 + 2q_0q_3,\ q_0^2 - q_1^2 + q_2^2 - q_3^2\right) \end{bmatrix}$$

The conversion of a Euler Angles (YXZ) sequence to a unit quaternion is:

$$\mathbf{q}\left(\mathbf{\Phi}_{yxz}\right) = \begin{bmatrix} c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) - s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \\ c\left(\frac{\phi}{2}\right)c\left(\frac{\theta}{2}\right)s\left(\frac{\psi}{2}\right) + s\left(\frac{\phi}{2}\right)s\left(\frac{\theta}{2}\right)c\left(\frac{\psi}{2}\right) \end{bmatrix}$$

6.5 Quaternion Attitude Kinematics

The rate of change of the attitude when represented with Unit Quaternions can be calculated from the relationship:

$$\dot{\mathbf{q}}_{b/a} = \frac{1}{2} W(\mathbf{q}_{b/a}) \vec{\omega}^{b}_{b/a}$$

where $W(\mathbf{q})$ is the reduced $Q(\mathbf{q})$ matrix:

$$W(\mathbf{q}) = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & q_3 & -q_2 \\ -q_3 & q_0 & q_1 \\ q_2 & -q_1 & q_0 \end{bmatrix}$$

The relationship between the frame angular rates and the quaternion derivative can be formulated from the vector transformation relationship:

$$\begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} = \mathbf{q} \cdot \begin{bmatrix} 0 \\ \vec{x}^a \end{bmatrix} \cdot \mathbf{q}^{-1}$$

Taking the time derivative:

$$\begin{bmatrix} 0 \\ \dot{\vec{x}}^b \end{bmatrix} = \dot{\mathbf{q}} \cdot \begin{bmatrix} 0 \\ \vec{x}^a \end{bmatrix} \cdot \mathbf{q}^{-1} + \mathbf{q} \cdot \begin{bmatrix} 0 \\ \vec{x}^a \end{bmatrix} \cdot \dot{\mathbf{q}}^{-1}$$

Expressing $\vec{x}^a$ in terms of $\vec{x}^b$ gives:

$$\begin{bmatrix} 0 \\ \dot{\vec{x}}^b \end{bmatrix} = \dot{\mathbf{q}} \cdot \mathbf{q}^{-1} \cdot \begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} \cdot \mathbf{q} \cdot \mathbf{q}^{-1} + \mathbf{q} \cdot \mathbf{q}^{-1} \cdot \begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} \cdot \mathbf{q} \cdot \dot{\mathbf{q}}^{-1}$$

Using the identity $\mathbf{q} \cdot \mathbf{q}^{-1} = 1$ gives:

$$\begin{bmatrix} 0 \\ \dot{\vec{x}}^b \end{bmatrix} = \dot{\mathbf{q}} \cdot \mathbf{q}^{-1} \cdot \begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} \cdot \mathbf{q} \cdot \dot{\mathbf{q}}^{-1}$$

Using the relationship that the scalar part of $\dot{\mathbf{q}}\cdot\mathbf{q}^{-1}$ must equal 0 since the norm is unity:

$$\begin{bmatrix} 0 \\ \dot{\vec{x}}^b \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{\mu} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{x}^b \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -\vec{\mu} \end{bmatrix}$$

where $\vec{\mu}$ is the required vector component. Inspecting the vector components gives:

$$\begin{aligned} \dot{\vec{x}}^b &= \vec{\mu} \times \vec{x}^b + \vec{x}^b \times -\vec{\mu} \\ &= \vec{\mu} \times \vec{x}^b - (-\vec{\mu} \times \vec{x}^b) \\ &= 2\vec{\mu} \times \vec{x}^b \end{aligned}$$

Using the known relationship $\dot{\vec{x}}^b = \vec{\omega}^a \times \vec{x}^b$, then $\vec{\mu} = \frac{1}{2}\vec{\omega}^a$ such that:

$$\dot{\mathbf{q}} \cdot \mathbf{q}^{-1} = \frac{1}{2} \begin{bmatrix} 0 \\ \vec{\omega}^a \end{bmatrix}$$

So that the quaternion derivative $\dot{\mathbf{q}}$ must be:

$$\dot{\mathbf{q}} = \frac{1}{2} \begin{bmatrix} 0 \\ \vec{\omega}^a \end{bmatrix} \cdot \mathbf{q}$$

Expressing the angular rates $\vec{\omega}^a$ in the frame $\vec{\omega}^b$ gives:

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{1}{2}\mathbf{q} \cdot \begin{bmatrix} 0 \\ \vec{\omega}^b \end{bmatrix} \cdot \mathbf{q}^{-1} \cdot \mathbf{q} \\ &= \frac{1}{2}\mathbf{q} \cdot \begin{bmatrix} 0 \\ \vec{\omega}^b \end{bmatrix} \end{aligned}$$

So the final kinematics relationship can be expressed in full as:

$$\dot{\mathbf{q}}_{b/a} = \frac{1}{2}\mathbf{q}_{b/a} \cdot \begin{bmatrix} 0 \\ \vec{\omega}^b_{b/a} \end{bmatrix}$$

$$\dot{\mathbf{q}}_{b/a} = \frac{1}{2} Q(\mathbf{q}_{b/a}) \begin{bmatrix} 0 \\ \vec{\omega}^b_{b/a} \end{bmatrix}$$

$$\dot{\mathbf{q}}_{b/a} = \frac{1}{2} W(\mathbf{q}_{b/a}) \vec{\omega}^b_{b/a}$$

where $W(\mathbf{q})$ is the reduced $Q(\mathbf{q})$ matrix:

$$W(\mathbf{q}) = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & q_3 & -q_2 \\ -q_3 & q_0 & q_1 \\ q_2 & -q_1 & q_0 \end{bmatrix}$$

6.6 Quaternion Attitude Interpolation

Spherical Linear Interpolation (SLERP) interpolates the quaternion along the surface of a unit sphere (rather than a straight line in a plane like linear interpolation). This means that the interpolation takes the shortest rotation or path between the two rotations, and that it rotates with a constant angular rate. The SLERP between two quaternions can be calculated from the equation:

$$\mathbf{q}(t) = \frac{\sin((1-t)\theta)}{\sin(\theta)} \mathbf{q}_0 + \frac{\sin(t\theta)}{\sin(\theta)} \mathbf{q}_1$$

where $\mathbf{q}_0$ and $\mathbf{q}_1$ are the two quaternions to interpolate between for $t = 0$ and $t = 1$ respectively. The angle θ is the angle between the two quaternions which can be calculated from the dot product of the unit quaternions treated like vectors:

$$\theta = \arccos\left(\vec{\mathbf{q}}_0 \cdot \vec{\mathbf{q}}_1\right)$$

Chapter 7

Other Useful Information

7.1 Greek Alphabet

Symbol	Name	Symbol	Name
α	alpha	ξ	xi
β	beta	Ξ	Capital xi
γ	gamma	o	o
Γ	Capital gamma	π	pi
δ	delta	ρ	rho
Δ	Capital delta	σ	sigma
ϵ	epsilon	Σ	Capital sigma
ζ	zeta	τ	tau
η	eta	υ	upsilon
θ	theta	Υ	Capital upsilon
Θ	Capital theta	ϕ	phi
ι	iota	Φ	Capital phi
κ	kappa	χ	chi
λ	lambda	ψ	psi
Λ	Capital lambda	Ψ	Capital psi
μ	mu	ω	omega
ν	Capital nu	Ω	Capital omega