SS2857 Probability and Statistics 1
Fall 2024

Lecture 7

3.1 Random Variables

Definition

For a given sample space $\mathcal{S}$ of some experiment, a random variable (abbreviated rv or RV) is a rule associates a number with each outcome in $\mathcal{S}$.

Mathematically, a random variable is a function that maps elements of $\mathcal{S}$ to $\mathbb{R}$.

Notation
We traditionally use upper case letters to denote the random variable (function) and lower case letters to denote the observed value (value of the function) associated with a specific outcome.

Purpose

Random variables simply provide a new way that we can conveniently define these events without listing out all the outcomes.

Consider an experiment in which we toss a coin 20 times and count the number of heads. We can describe events in two ways:

1) Naming every possible outcome:
   Let A_0 be the event that there are no heads, A_1 the event that there is one head, A_2 the event that there are two heads, etc.

2) Defining a random variable:
   Let X be the number of heads. We can consider events $X = 0$, $X = 1$, $X = 2$, etc.

Purpose

Random variables also make it easier to describe more complex events.

Using the previous notation, the event that the less than 10 heads are observed is

1) $A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_9$
2) $X < 9$

Example 7.1: Random Variables

Approximately 79% of world's population has brown eyes[1].

Suppose that we sample 5 people from the population at random with replacement and record their eye-colour as brown or not brown. Let X denote the number of people in the sample with brown eyes.

a) What are the possible outcomes in the sample space?

b) List the outcomes in the event that exactly 3 people have brown eyes. Write this event in terms of the random variable X.

c) List the outcomes in the event that no more than 3 people have brown eyes. Write this event in terms of the random variable X.

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[1] https://www.worldatlas.com/articles/
which-eye-color-is-the-most-common-in-the-world.html#targetText=Approximately%2079%
25%20of%20the%20world's,include%20gray%20and%20red%2Fviolet.

Bernoulli Random Variable

Any random variable whose possible values are 0 and 1 is called a Bernoulli random variable.

Any random variable, X, with only two outcomes, say x_1 and x_2, can be converted to a Bernoulli random variable by defining

$$Y = \begin{cases} 0 & X = x_1 \\ 1 & X = x_2 \end{cases}$$

Bernoulli Random Variable

Bernoulli random variables can also be used to model experiments with two non-numeric outcomes by mapping the outcomes to numbers.

1) True and False: False $= 0$, True $= 1$
2) Assigned sex: Male $= 0$, Female $= 1$
3) Eye colour: Brown $= 1$, Not brown $= 0$
4) Side of the force: Dark-side $=0$, Light-side $= 1$

Discrete Random Variables

A random variable is discrete if the set of possible values is countable.

A set is countable if it is finite or if the elements can be mapped to the natural numbers.

1) Then integers are countable:

$$0 \to 0, \quad 1 \to 1, \quad -1 \to 2, \quad 2 \to 3, \quad 3 \to 4, \cdots$$

2) The set of ordered pairs of natural numbers is countable:

$$(m, n) \to 2^m 3^n$$

3) The rational numbers are countable.
4) Any subset of a countable set is countable.
5) Any interval on the real line is not countable.

Continuous Random Variables

A random variable is continuous if

1. The set of possible values consists of a union of disjoint intervals with length greater than zero.
2. No single value has positive probability ($P(X = x) = 0$ for all $x \in \mathbb{R}$.

Note There are random variables that are neither discrete nor continuous. We will not discuss these.

Example 7.2: Continuous vs Discrete RVs

Which of the following random variables are discrete and which are continuous.

a) The year of birth of a randomly selected student.

b) The time it takes a randomly selected student to drive to school.

c) The number of blue candies in a box of Smarties.

d) The minimum of your shoe size and the distance you live from Western in kilometres.

Questions?

Exercise 7.1

Identify the following random variables as discrete or continuous. Which are Bernoulli random variables.

a) Whether or not it rains tomorrow with $1 =$ Rain and $0 =$ No rain.

b) The number of birds in a flock.

c) The wavelength of light measured from a distant star.

d) Whether or not you live past 80.

e) Height of a randomly selected building in metres.

f) Height of a randomly selected building in floors.