STAT 2857A – Lecture 16 Examples and Exercises

Solutions – Revised 7/11/24

Review

a) FALSE

b) FALSE

c) TRUE

d) This is a trick question. All of these are equal.

Example 16.1

a)

$$f(x) = \frac{1}{3\sqrt{2\pi}} \exp\left(-\frac{(x-64)^2}{18}\right)$$

for all $x \in \mathfrak{R}$.

b) Note that 5~ft is equivalent to 60 inches and 6~ft to 72 inches. Plugging these values into the calculator on stattrek we find

$$P(X < 60) = .09121$$

and

$$P(X > 72) = .00383.$$

Finally

$$P(60 < X < 72) = .99612 - .09121 = .9049.$$

c) Let η_p denote the $(100p) - th$ percentile of the distribution, as we did before. Then

$$P(X \le \eta_p) = p$$

and

$$P(X \le \eta_{.95+p}) = .95 + p$$

for any $0 < p < .05$. Hence

$$P(\eta_p < X < \eta_{.95+p}) = (.95 + p) - p = .95.$$

The shortest interval will occur when the interval is symmetric about the mean since it will then contain the points with highest density. This occurs when $p = .025$. In that cas

$$l = \eta_{.025} = 58.120 \text{ and } u = \eta_{.0975} = 69.880.$$

d)

$$\begin{aligned}
P(X \le 60) &= P\left(\frac{X-64}{3} \le \frac{60-64}{3}\right) \\
&= P\left(Z \le \frac{60-64}{3}\right) \\
&= P(Z \le -4/3) \\
&= .09121.
\end{aligned}$$

$$\begin{aligned}
&P(X \ge 72) \\
&\qquad = 1 - P(X \le 72) \\
&\qquad = 1 - P(\left(\frac{X-64}{3} \le \frac{72-64}{3}\right) \\
&\qquad = 1 - P\left(Z \le \frac{72-64}{3}\right) \\
&\qquad = 1 - P(Z \le 8/3) \\
&\qquad = 1 - .99617 \\
&\qquad = .00383
\end{aligned}$$

e) Let $0 < \alpha < .05$. Then

$$P(Z > z_\alpha) = \alpha$$

and

$$P(Z > z_{.95+\alpha}) = P(Z > z_{.95+\alpha}) = .95 + \alpha.$$

Then

$$P(z_{.95+\alpha} < Z < z_\alpha) = (.95 + \alpha) - \alpha = .95.$$

Reversing the standardization, this means that

$$P(\mu + (z_{.95+\alpha})\sigma < X < \mu + z_\alpha \sigma) = .95.$$

Once again, the shortest interval will occur when the points are symmetric about the mean which happens when $\alpha = .025$. Applying the empirical rule[1],

$$z_{.975} \approx -2 \text{ and } z_{.025} \approx 2.$$

Then

$$l \approx 64 + 3(-2) = 58 \text{ and } u \approx 64 + 3(2) = 70.$$

Example 16.2

a) We know from the previous lecture that

$$X \sim \text{Binomial}(200, .486).$$

Then

$$\begin{aligned} P(95 \leq X \leq 105) &= \sum_{x=95}^{105} \binom{200}{y} .486^y .514^{200-y} \\ &= 0.053515 + 0.0554514 + \cdots + 0.0359802 + 0.0311648 \\ &= 0.53066 \end{aligned}$$

b) Alternatively, if we let Z denote a standard normal random variable then

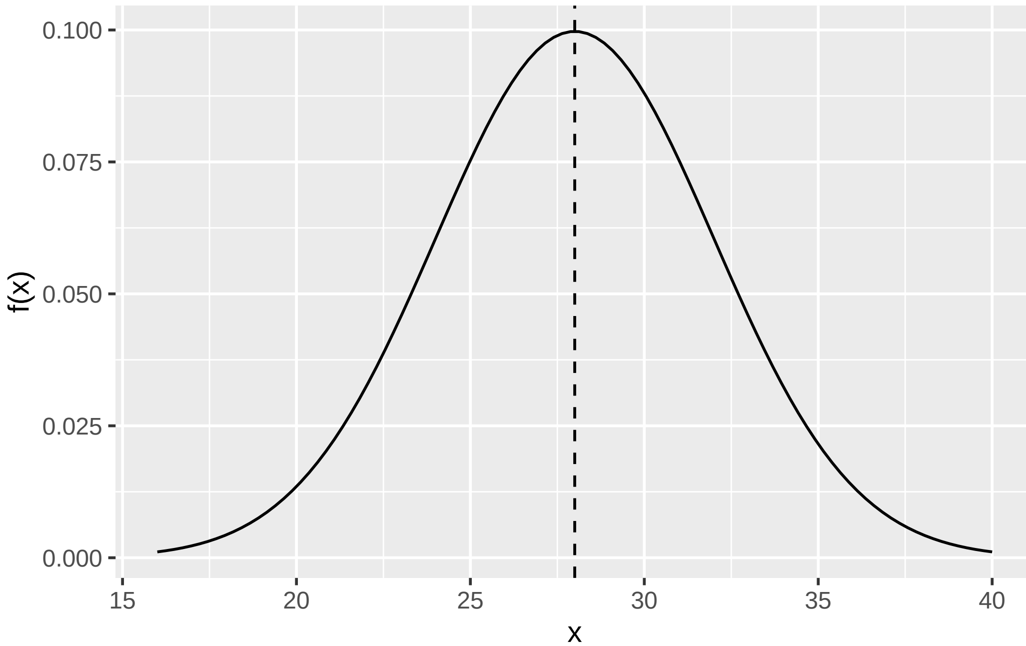
$$\begin{aligned} P(95 \leq X \leq 105) &= P(X \leq 105) - P(X \leq 94) \\ &\approx P\left(Z \leq \frac{105.5 - 200(.486)}{\sqrt{200(.486)(.514)}}\right) - P\left(Z \leq \frac{94.5 - 200(.486)}{\sqrt{200(.486)(.514)}}\right) \\ &= P(Z \leq 1.1742577) - P(Z \leq -0.3819874) \\ &= 0.8798541 - 0.3512353 \\ &= 0.5286187 \end{aligned}$$

Although the probability is not exact it is very close.

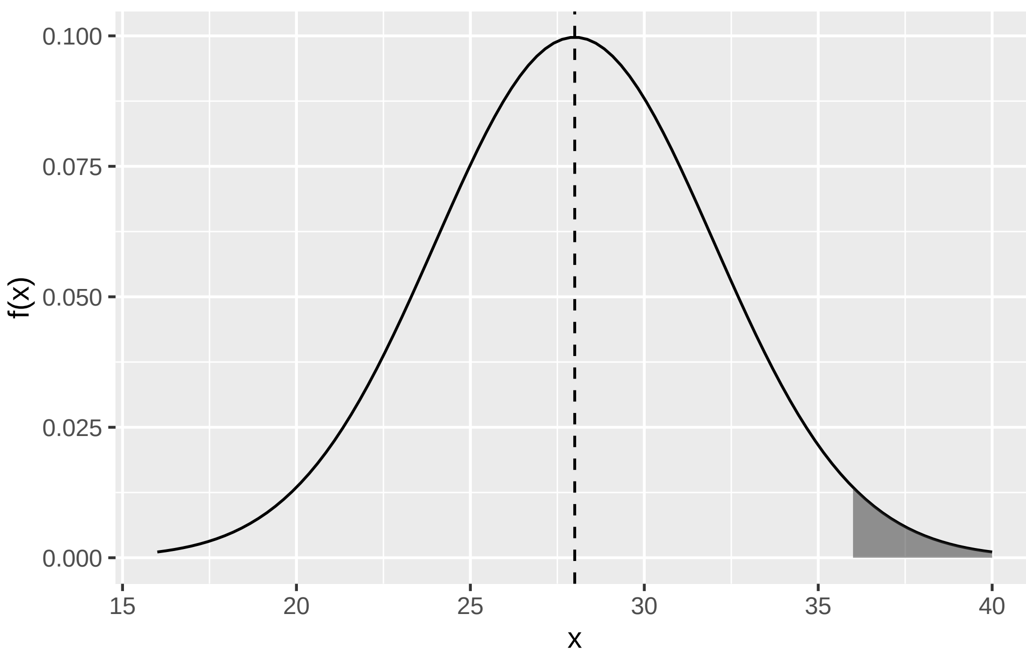
Exercise 16.1

a)

[1] The exact values are -1.960 and 1.960.



b)



c) Using the calculator

$$P(X > 34) = .06681.$$

Standardizing

$$
\begin{aligned}
P(X > 34) &= 1 - P(X \leq 34) \\
&= 1 - P\left(Z \leq \frac{34 - 28}{4}\right) \\
&= 1 - P(Z \leq 1.5) \\
&= 1 - .9332 \\
&= .0668.
\end{aligned}
$$

d) Here we can apply the empirical rule. Note that

$$36 = 28 + 8 = \mu + 2\sigma.$$

Hence

$$P(X > 36) = P(X > \mu + 2\sigma) = (1 - .95)/2 = .025.$$