Probability and Statistics I

9. Expected Values of Discrete Random Variables

3.3 Expected Values of Discrete Random Variables

Expected Value or Mean

Let X be a discrete rv with set of possible values D and pmf $p(x)$.

The expected value or mean value of X is

$$E(X) = \mu_X = \sum_{x \in D} x p(x).$$

The expected value will exist provided that $\sum_{x \in D} |x| p(x) < \infty$.

Expected Value or Mean: Interpretation

The expected value is the weighted average of all possible outcomes. It tells us about the centre of the distribution.

It represents the limiting average of the random variable on repeated experiments.

If we were to repeat the same experiment many, many times and record the value of X each time then the average of these values would be very close to $E(X)$ and get closer and closer the more times we repeat the experiment.

Variance and Standard Deviation

Let x be a possible value of X. The deviation associated with x is

$$x - \mu_X = x - E(X).$$

- A positive deviation indicates that $x > E(X)$.
- A negative deviation indicates that $x < E(X)$.
- The deviation increases in magnitude the farther x is from $E(X)$.

Variance and Standard Deviation

Let X be a discrete rv with set of possible values D and pmf X. The variance of X is

$$V(X) = \sigma_X^2 = \sum_{x \in D} (x - \mu_x)^2 p(x) = E(X - \mu_X)^2.$$

The standard deviation is $\sigma_X = \sqrt{\sigma_X^2}$.

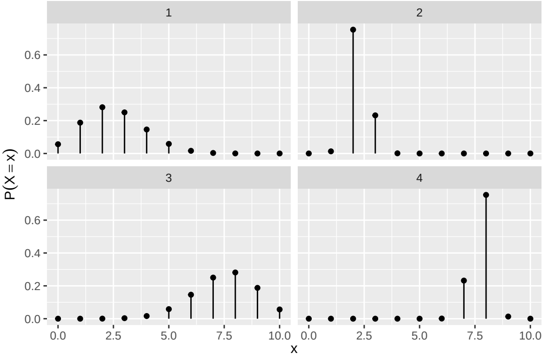
Variance and Standard Deviation: Interpretation

The variance is the weighted average of all possible *squared deviations*. It tells us about the spread of the distribution.

It represents the limiting average of the *squared deviation* on repeated experiments.

Example 9.1

Compare the mean and variance of the distributions with the following pmfs.



Example 9.2: Expected Values and Variances

Approximately 79% of the world's population has brown eyes[1].

Suppose that we sample 5 people from the population at random with replacement and record their eye-colour as brown or not brown. Let X represent the number of people in our sample with brown eyes.

a) Compute the expected value of X.
b) Compute the variance of X.
c) Compute the standard deviation of X.
d) Provide an interpretation for $E(X)$.

[1] https://www.worldatlas.com

Example 9.2 ctd: Expected Values and Variances

x	$p(x)$
0	0.00041
1	0.00768
2	0.05780
3	0.21743
4	0.40898
5	0.30771

Example 9.2 ctd: Expected Values and Variances

x	$p(x)$	$x - \mu_X$
0	0.00041	-3.95000
1	0.00768	-2.95000
2	0.05780	-1.95000
3	0.21743	-0.95000
4	0.40898	0.05000
5	0.30771	1.05000

General Functions of Random Variables

Generally, the random variable Y is a function of X if

$$Y = h(X)$$

for some function $h(\cdot)$.

If this is true then

$$E(Y) = E[h(X)] = \sum_{x \in D} h(x)p(x)$$

$$V(Y) = V[h(X)] = \sum_{x \in D} (h(x) - E[h(X)])^2 \, p(x)$$

Linear Functions of Random Variables

We say that the random variable Y is a linear function of X if

$$Y = aX + b$$

for some constants $a, b \in \mathbb{R}$.

If this is true then

$$\begin{aligned} E(Y) &= E(aX + b) = aE(X) + b \\ V(Y) &= V(aX + b) = a^2 V(X) \\ SD(Y) &= SD(aX + b) = |a| SD(X). \end{aligned}$$

Example 9.3

Approximately 79% of the world's population has brown eyes[2].

Suppose that we sample 5 people from the population at random with replacement and record their eye-colour as brown or not brown. Let Y represent the number of brown eyes in the sample plus the number of hands[3].

a) Compute the expected value of Y.
b) Compute the variance of Y.
c) Compute the standard deviation of Y.
d) Provide an interpretation for $E(Y)$.

[2] https://www.worldatlas.com
[3] We'll assume that everyone has two of each

Questions?

Exercise 9.1

A professor driving to Western must pass through 5 sets of traffic lights. There is a .75 percent chance of being stopped at each light (or so it appears to him). The time it takes him to complete the drive is 15 minutes plus 3 minutes for each light he has to stop at.
Let X be the number of lights he must stop at and Y the time it takes him in minutes.

a) Compute the expected value, variance, and standard deviation of X.
b) Provide an interpretation for the expected value.
c) Compute the expected value, variance, and standard deviation of Y.