Probability and Statistics I

15. Expected Values

4.2 Expected Values ~~and Moment Generating Functions~~

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Key Lesson

Replace

sums and pmfs (for discrete random variables)

with

integrals and pdfs (for continuous random variables).

Mean

The expected or mean value of a continuous random variable with pdf $f(x)$ is

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x)\ dx.$$

The expected values exists if $\int_{-\infty}^{\infty} |x| f(x)\ dx < \infty$.

Variance

The variance of X is

$$\sigma_X^2 = V(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\ dx.$$

The standard deviation is $\sigma_X = \sqrt{V(X)}$.
The variance and standard deviation exist if $\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\ dx < \infty$.

Variance

There is a shortcut formula for the variance. For *any* random variable X (discrete or continuous) the variance can be computed as

$$\sigma_X^2 = E(X^2) - E(X)^2 = E(X^2) - \mu_X^2.$$

Example 15.1

Consider the random variable, X, from Example 14.1. This random variable represents the distance that an object dropped from a height of 1 m falls in a randomly selected time between 0 and 1 second on the home planet of Emperor Zurg where the force of gravity is only 2 m/s^2. The pdf and cdf are

$$f(x) = \begin{cases} 0 \le 0 & \\ 2x & 0 < x < 1 \\ 0 & 1 \le x \end{cases} \quad \text{and } F(x) = \int_{-\infty}^{x} f(u)\, du = \begin{cases} 0 & x \le 0 \\ x^2 & 0 < x < 1 \\ 1 & 1 \le x \end{cases}$$

a) Find the mean of X.

b) Compute the variance of X.

c) Provide an interpretation for the mean.

Mean

Let $Y = h(X)$ for some function $h(\cdot)$. Then:

$$\mu_Y = E(Y) = \int_{-\infty}^{\infty} h(x) f(x)\, dx \text{ and } \sigma_Y^2 = V(Y) = \int_{-\infty}^{\infty} (h(y) - \mu_Y)^2 f(x)\, dx.$$

If Y is a linear function of X, $Y = aX + b$, then

$$\mu_Y = a\mu_X + b \text{ and } \sigma_Y^2 = a^2 \sigma_X^2.$$

Example 15.1 ctd

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d) Let Y be the distance traveled in inches not metres ($Y = 39.37X$). Find the mean and variance of Y.

e) Let $Z = X^2$. Find the mean and variance of Z.

Approximating the Mean and Variance

The proposition on page 174 (called the delta method) suggests that if $Y = h(X)$ for some function $h(x)$ which is differentiable and has non-zero deriviative at $\mu = E(X)$ then

$$E(Y) \approx h(\mu)$$

and

$$V(Y) \approx h'(\mu)^2 V(X)$$

if "the variance of X is small".

This is too vague! We will discuss the full result later.

Questions?

Exercise 15.1

Suppose that the random variable X has pdf

$$f(x) = \frac{3}{4}\left[2x - x^2\right], \quad 0 \leq x \leq 2.$$

a) Confirm that $f(x)$ is a valid pdf.
b) Find the mean and variance of X.
c) Find the mean and variance of $Y = 3X + 2$.
d) Find the mean and variance of $Z = X^2$.