Probability and Statistics I

12. The Poisson Distribution

3.7 The Poisson Distribution

The Poisson Distribution

We say that X has a Poisson distribution with mean λ if X has pmf

$$P(X = x) = p(x; \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots.$$

Mathematically

$$X \sim \text{Poisson}(\lambda).$$

The Poisson Distribution

PMF and CDF

- PMF: $p(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots.$
- CDF: requires special functions

Properties

- Mean: $E(X) = \lambda$
- Variance: $V(X) = \lambda$

The Poisson Distribution

The Poisson distribution is most commonly used to model the number of times a specific event occurs within a fixed time period or the number of items in a fixed area.

E.g.:

- The number of patients admitted to hospital in a day.
- The number of goals a soccer player scores in a season.
- The number of claims to an insurance company in a year.
- the number of students in a class.

Poisson Process

1. The probability of exactly one event in a short time interval of length Δt tends toward $\alpha\Delta t$ as Δt decreases.
2. The probability of exactly zero events in a short time interval of length Δt tends toward $1 - \alpha\Delta t$ as Δt decreases.
3. The number of events in disjoint intervals are independent.

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Points 1 and 2 imply:

- the rate of events is constant, α events per unit time on average.
- events cannot occur simultaneously.

Poisson Process

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3. The number of events in disjoint intervals are independent.

Under these conditions, the number of events in an interval of length 1 has a Poisson distribution with parameter $\lambda = \alpha$.

Mathematically

$$X \sim \text{Poisson}(\alpha).$$

Poisson Process

1. The probability of exactly one event in a short time interval of length Δt tends toward $\alpha\Delta t$ as Δt decreases.
2. The probability of exactly zero events in a short time interval of length Δt tends toward $1 - \alpha\Delta t$ as Δt decreases.
3. The number of events in disjoint intervals are independent.

Under these conditions, the number of events in an interval of length t has a Poisson distribution with parameter αt.

Mathematically

$$X \sim \text{Poisson}(\alpha t).$$

Example 12.1

According to the book "United States Water Law: An Introduction" by John W. Johnson, heavy rain falls at about 495 drops per second per metre square. Let X be the number of rain drops that falls in one metre square in t seconds.

a) What is the distribution of X?

b) What is the pmf of X?

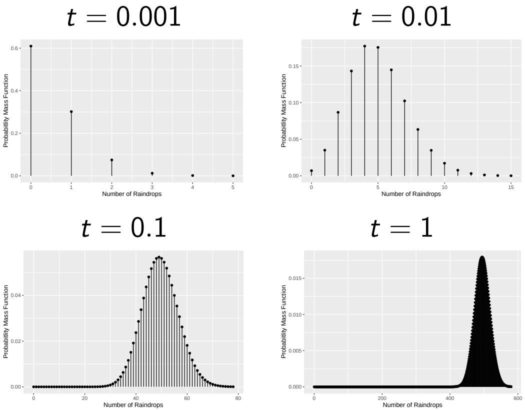
c) What are the mean and variance of X?

d) How does the shape of the distribution vary with t?

Example 12.1 ctd

t	Mean	Variance
0.001	0.495	0.495
0.010	4.950	4.950
0.100	49.500	49.500
1.000	495.000	495.000

Example 12.1 ctd



Poisson Approximation to the Binomial

Suppose that

$$X \sim \text{Binomial}(n, p)$$

such that n is large and $\mu_X = np$ is small[1].

Then we can approximate the distribution of X with a Poisson distribution

$$X \dot{\sim} \text{Poisson}(np).$$

[1] Your book uses the rule of thumb $n > 50$ and $np < 5$

Example 12.2

In lecture 4, we showed that the probability that a randomly selected person is colour blind is about .04512. Let X be the number of colour blind students in a class of 100.

a) What is the distribution of X?
b) What are the mean and variance of X?
c) What is the probability that the class contains more than 5 students who are colour blind?
d) Approximate the distribution of X by a Poisson and repeat the questions above.
e) Do you think the Poisson approximation is appropriate?

Questions?

Exercise 12.1

One gram of Uranium-235 contains 2.35×10^{21} atoms. Each atom has probability 9.85×10^{-10} of decaying in one year. Let X be the number of atoms that decay in 1 year. You may assume that atoms decay independently of one another.

a) What is the distribution of X?

b) What are the mean and variance of X?

c) What is the probability that the number of decays in one year is greater than the mean?

d) Approximate the distribution of X by a Poisson and repeat the questions above.

e) Do you think the Poisson approximation is appropriate?