SS2857 Probability and Statistics 1
Fall 2021

Lecture 5

2.5 Independence

Persi Diaconis, Susan Holmes, and Richard Montgomery (2007) *Dynamical Bias in the Coin Toss*. SIAM Review 2007 49:2, 211-235

Independence

Two events, A and B, are independent if the conditional probability of A given B is equal to the probability of A, or vice versa:

$$P(A|B) = P(A) \text{ or } P(B|A) = P(B).$$

Two events that are not independent are dependent.

Example 5.1

Suppose that A and B are disjoint events with positive probability ($P(A) > 0$ and $P(B) > 0$).

Can they be independent?

Example 5.2

Which pairs of events do you think are independent? Explain.

a) A: It rains in London, Ontario, on October 1.
   B: It rains in London, Ontario, on October 2.

b) A: It rains in London, Ontario, on October 1, 2022.
   B: It rains in London, England, on October 1, 2023.

c) A: Erin scores $> 80\%$ on an exam.
   B: Jonah scores $> 80\%$ on the same exam.

d) A: The Yankees win the baseball World Series.
   B: The Royals win the baseball World series.

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Example 5.3

Let A and B be two events such that

i) $P(A \cap B') = .15$
ii) $P(A' \cap B') = .35$
iii) $P(A' \cap B) = .35$

Are A and B independent?

Example 5.4

Show that if A' and B' are independent then A and B are also independent.

Mutually Independent

Several events are mutually independent if the probability of the intersection of any collection of the events is the product of the probabilities of the individual events.

Mathematically, $A_1, \ldots, A_n$ are mutually independent if

$$P(\cap_{i \in \mathcal{I}} A_i) = \prod_{i \in \mathcal{I}} P(A_i)$$

for any subset $\mathcal{I} \subset \{1, \ldots, n\}$.

Questions?

Exercise 5.1

Suppose that you toss a fair coin n times and count the number of heads.

a) Let H_i be the event that the coin lands heads side up on the i-th toss. What does it mean for H_1 and H_2 to be independent?
b) Does independence necessarily mean that the coin is fair?
c) What does it mean for the events $H_1, \ldots, H_n$ to be mutually independent?
d) What is the probability that the coin lands heads-side up on every one of $n = 10$ tosses?
e) What is the probability that the tosses alternate between landing heads-side up first then tails-side up etc if $n = 10$?
f) What is the probability that the coin lands heads-side up 5 times if $n = 10$?