Probability and Statistics I

19. A Primer on Double Integration

Method[1]

Suppose that you wish to integrate the function $f(x, y)$ over some domain in $A \subset \mathbb{R}^2$:

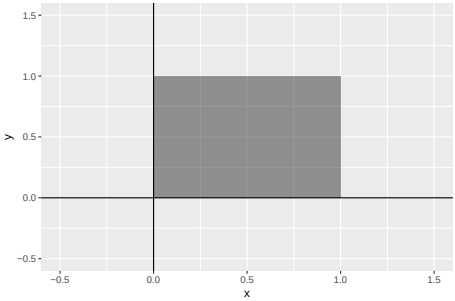
$$\int \int_A f(x, y)\ dy\ dx.$$

1) Sketch the area of integration.
2) Identify the limits of y as functions of x: $l_y(x)$ and $u_y(x)$.
3) Identify the overall limits of y: l_x and u_x.
4) Integrate $f(x, y)$ with respect to y treating x as fixed.
5) Integrate $g(x)$ with respect to x.

[1] This approach is not completely general but will work for all problems in this course. The method assumes that y is the inner variable of integration and x is the outer variable of integration. The roles would switch if you switch the order of integrations: $\int \int_A f(x, y)\ dx\ dy$.

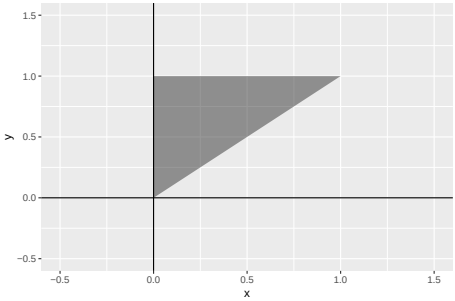
Example 19.1

Integrate $f(x, y) = xy$ over the domain $0 < x < 1$, $0 < y < 1$.



Example 19.2

Integrate $f(x, y) = xy$ over the domain $x < y < 1$, $0 < x < 1$.

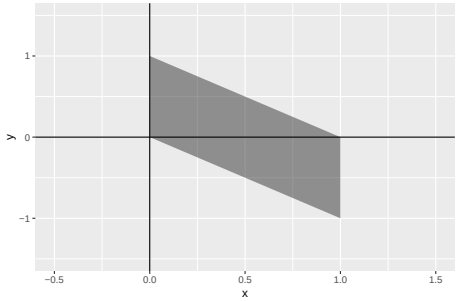


Example 19.3

Integrate $f(x, y) = xy$ over the domain $0 < x < 1$, $0 < x + y < 1$.

Example 19.4

Integrate $f(x, y) = xy$ over the domain $0 < x < 1$, $0 < x + y < 1$.



Proposition[2]

Suppose that $f(x,y) = g(x)h(y)$ and the domain of y is independent of x so that $l_y < y < u_y$ regardless of the value of x. Then

$$\int_{l_x}^{u_x} \int_{l_y}^{u_y} f(x,y)\ dy\ dx = \int_{l_x}^{u_x} g(x)\ dx \int_{l_y}^{u_y} h(y)\ dy.$$

[2] Keep in mind that this only works if the domain of y and x do not depend on each other.

Example 19.5

Integrate $f(x, y) = (1 - x^2)(1 - y^2)$ over the domain $0 < x < 1$, $0 < y < 1$.

Questions?

Exercise 19.1

1. Integrate $f(x, y) = e^{x+y} - 1$. over the region $0 < x < 1$, $0 < y < 1$.
2. Integrate $f(x, y) = e^{x+y} - 1$. over the region$y < x < 1$, $0 < y < 1$
3. Integrate $f(x, y) = xy$ over the region $x^2 + y^2 < 1$.