# STAT 2857A – Lecture 15 Examples and Exercises

**Solutions**

### Example 15.1

a) The first mean of $X$ is:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} uf(u)\ du \\ &= \int_0^1 u(2u)\ du \\ &= \left[\frac{2u^3}{3}\right]_0^1 \\ &= 2/3 \end{aligned}$$

b) There are two ways to compute the variance. First, we can compute it directly

$$\begin{aligned} \sigma_X^2 &= E[(X-\mu_X)^2] \\ &= E[(X-2/3)^2] \\ &= \int_0^1 (x-2/3)^2 \cdot 2x\ dx \\ &= \int_0^1 2x^3 - 8/3x^2 + 8/9x\ dx \\ &= \left[\frac{2x^4}{4} - \frac{8x^3}{9} + \frac{8x^2}{18}\right]_0^1 \\ &= \frac{1}{2} - \frac{8}{9} + \frac{4}{9} \\ &= 1/18. \end{aligned}$$

Alternatively, we can use the shortcut formula

$$\sigma_X^2 = E(X^2) - E(X)^2.$$

Then:

$$\begin{aligned}
E(X^2) &= \int_0^1 u^2(2u)\ du \\
&= \left[\frac{2u^4}{4}\right]_0^1 \\
&= 1/2
\end{aligned}$$

Applying the shortcut formula we get

$$\sigma_X^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}.$$

c) If we were to repeatedly drop an object from a height of 1~m on Emperor Zurg's home planet and stop it at a randomly selected time between 0 and 1 second then the average distance the objects has fallen would be very close to 2/3~m and get closer and closer the more times we drop the object.

d) In this case, $Y$ is a linear transformation of $X$. Hence

$$E(Y) = E(39.73X) = 39.73E(X) = \frac{39.73(2)}{3} = 26.4867$$

and

$$V(Y) = V(39.73X) = 39.73^2V(X) = \frac{39.73^2(1)}{18} = 87.693.$$

The mean and variance of $Y$ are 26.5 inches and 87.7 inches squared.

e) We need to apply the formula for the variance of a function directly since $Z$ is not a linear function of $X$. First, we get

$$\begin{aligned}
\mu_Z = E(Z) &= \int_0^1 x^2 \cdot 2x\ dx \\
&= \int_0^1 2x^3\ dx \\
&= \left[\frac{x^4}{2}\right]_0^1 \\
&= 1/2.
\end{aligned}$$

Then

$$\begin{aligned} E(Z^2) &= \int_0^1 x^4 \cdot 2x \ dx \\ &= \int_0^1 2x^5 \ dx \\ &= \left[\frac{x^6}{3}\right]_0^1 \\ &= 1/3. \end{aligned}$$

Applying the shortcut formula

$$\sigma_Z^2 = E(Z^2) - E(Z)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

## Exercise 15.2

a) We need to show that:

1. $f(x) \geq 0$ for all $x$, and
2. $\int_{-\infty}^{\infty} f(x) \ dx = 1$

To show the first property, note that $f(x) = 0$ for all $x < 0$ or $x > 2$. Then for $0 \leq x \leq 2$, $f(x) = 3/4[2x - x^2] > 0$. Hence, $f(x) \geq 0$ for all $x \in \mathfrak{R}$.

Then

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \ dx &= \int_0^2 \frac{3}{4}[2x - x^2] \ dx \\ &= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= \frac{3}{4}\left[4 - \frac{8}{3}\right] \\ &= 1. \end{aligned}$$

Hence, $f(x)$ is a valid pdf.

b) Note that

$$\begin{aligned}E(X) &= \int_0^2 x \cdot \frac{3}{4}[2x - x^2]\ dx \\ &= \frac{3}{4}\int_0^2 2x^2 - x^3\ dx \\ &= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 \\ &= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] \\ &= 1\end{aligned}$$

and

$$\begin{aligned}E(X^2) &= \int_0^2 x^2 \cdot \frac{3}{4}[2x - x^2]\ dx \\ &= \frac{3}{4}\int_0^2 2x^3 - x^4\ dx \\ &= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 \\ &= \frac{3}{4}\left[\frac{32}{3} - \frac{32}{5}\right] \\ &= 6/5.\end{aligned}$$

Then

$$\mu_X = E(X) = 1$$

and

$$\sigma_X^2 = E(X^2) - E(X)^2 = \frac{6}{5} - 1 = \frac{1}{5}.$$

c) Note that $Y$ is a linear function of $X$. Then

$$\mu_Y = 3\mu_X + 2 = 5$$

and

$$\sigma_Y^2 = 3^2\sigma_X^2 = \frac{9}{5} = 2.2.$$

d) Note that $Z$ is not a linear function of $X$, so we must compute the mean and variance from the definitions. We have

$$\begin{aligned}E(Z) &= E(X^2) \\ &= \frac{6}{5}\end{aligned}$$

from the result above. Further,

$$\begin{aligned}
E(Z^2) &= E(X^4) \\
&= \int_0^2 x^4 \cdot \frac{3}{4}[2x - x^2]\ dx \\
&= \frac{3}{4}\int_0^2 2x^5 - x^6\ dx \\
&= \frac{3}{4}\left[\frac{2x^6}{6} - \frac{x^7}{7}\right]_0^2 \\
&= \frac{3}{4}\left[\frac{64}{3} - \frac{128}{7}\right] \\
&= \frac{16}{7}.
\end{aligned}$$

Hence

$$\mu_Z = E(Z) = \frac{6}{5}$$

and

$$\sigma_Z^2 - E(Z^2) - E(Z)^2 = \frac{16}{7} - \left(\frac{6}{5}\right)^2 = \frac{38}{35} = 1.08571.$$