

Probability and Statistics I

16. The Normal Distribution

Suppose that X is a continuous random variable with pdf $f(x)$ and cdf $F(x)$.

- a) TRUE or FALSE: $f(x) \leq 1$ for all $x \in \mathbb{R}$
- b) TRUE or FALSE: $f(x)$ is continuous
- c) TRUE or FALSE: $E(X^2) \geq E(X)^2$
- d) Which of the following is not equal to all of the others?
 - A) $P(3 \leq X \leq 5)$
 - B) $P(3 < X \leq 5)$
 - C) $P(3 < X < 5)$
 - D) $P(3 < X < 4) + P(4 < X < 5)$

4.3 The Normal Distribution

The Normal Distribution

We say that a random variable X has a normal distribution with mean μ and variance σ^2 if the pdf of X is

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

for all $x \in \mathbb{R}$.

Mathematically, we write $X \sim \text{Normal}(\mu, \sigma^2)$.

THE NORMAL DISTRIBUTION

Properties

CDF: No closed form

Mean: $E(Z) = \mu$

Variance: $V(Z) = \sigma^2$

Calculator

<https://stattrek.com/online-calculator/normal>

Example 16.1

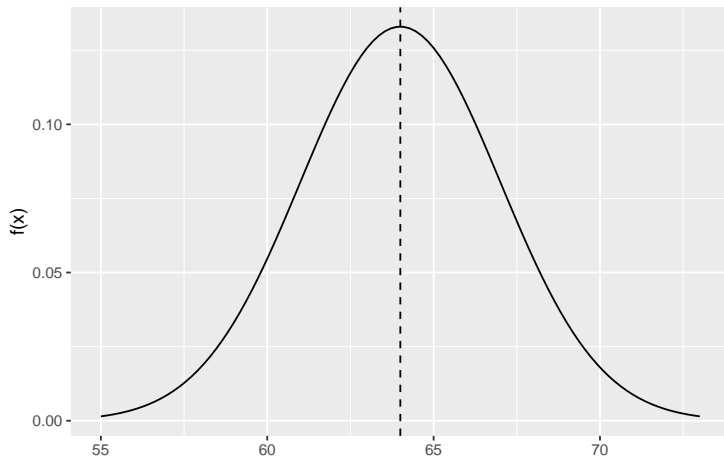
The adult heights of people assigned to be male and female at birth can be modelled amazingly well by a normal distribution. Suppose that the adult height people assigned to be female at birth is normally distributed with mean 64 inches and standard deviation 3 inches.

$$X \sim \text{Normal}(64, 9).$$

- a) What is the density of X ?
- b) What is the probability that someone assigned to be female at birth will be:
 - i) less than 5 feet tall?
 - ii) greater than 6 feet tall?
 - iii) between 5 and 6 feet tall?
- c) Find values l and u such that $P(l < X < u) \approx .95$.

THE NORMAL DISTRIBUTION

Example 16.1 ctd



Example 16.1 ctd

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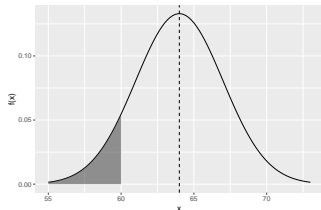
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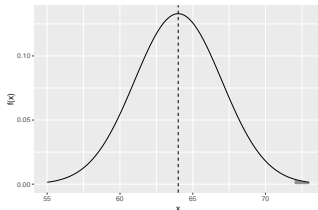
THE NORMAL DISTRIBUTION

Example 16.1 ctd

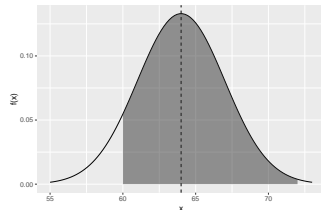
$$P(X < 60)$$



$$P(X > 72)$$



$$P(60 < X < 72)$$



The Standard Normal Distribution

We say that a random variable Z has a **standard** normal distribution if

$$Z \sim \text{Normal}(0, 1).$$

The pdf of the standard normal is

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

for all $x \in \mathbb{R}$.

Properties

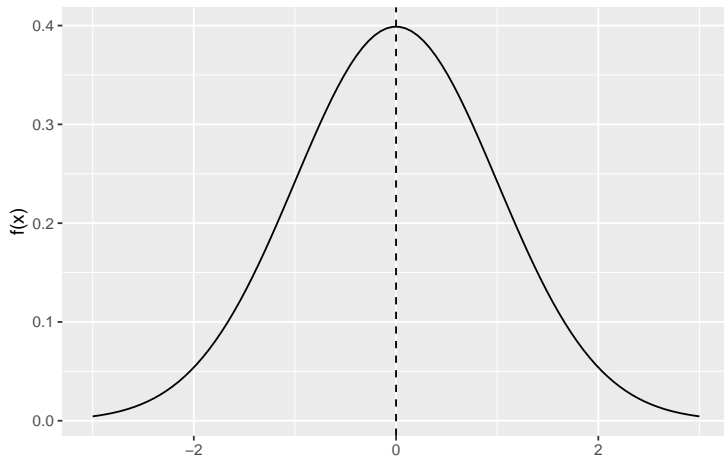
CDF: No closed form

Mean: $E(Z) = 0$

Variance: $V(Z) = 1$

THE NORMAL DISTRIBUTION

Example 16.1 ctd



THE NORMAL DISTRIBUTION

Standardization

If $X \sim \text{Normal}(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution.

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If $X \sim \text{Normal}(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution.

We can use this fact to compute probabilities for any normal random variable from the probabilities of a standard normal random variable:

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

where $Z \sim \text{Normal}(0, 1)$.

Example 16.1 ctd

The adult heights of people assigned to be male and female at birth can be modelled amazingly well by a normal distribution. Suppose that the adult height people assigned to be female at birth is normally distributed with mean 64 inches and standard deviation 3 inches:

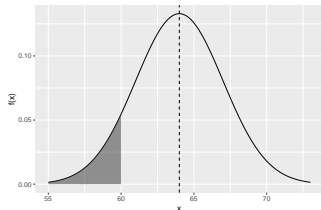
$$X \sim \text{Normal}(64, 9).$$

- d) Repeat part b) using standardization

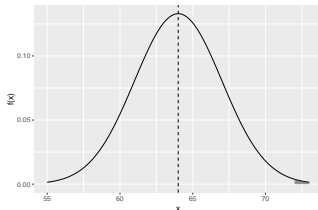
THE NORMAL DISTRIBUTION

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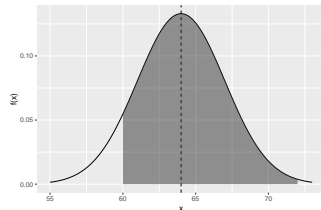
$$P(X < 60)$$



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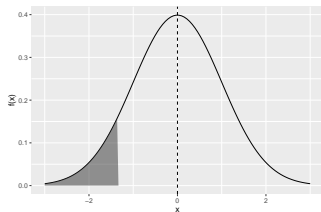
$$P(60 < X < 72)$$



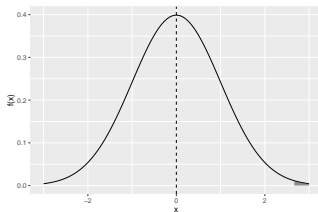
THE NORMAL DISTRIBUTION

Example 16.1 ctd

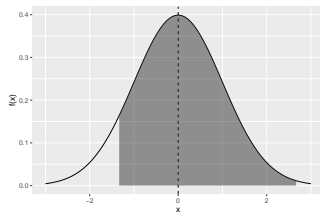
$$\begin{aligned} P\left(Z < \frac{(60-64)}{3}\right) \\ = P(Z < -4/3) \end{aligned}$$



$$\begin{aligned} P\left(Z > \frac{(72-64)}{3}\right) \\ = P(Z > 8/3) \end{aligned}$$



$$\begin{aligned} P\left(\frac{(60-64)}{3} < Z < \frac{(72-64)}{3}\right) \\ = P(-4/3 < Z < 8/3) \end{aligned}$$



The Empirical Rule

If $X \sim \text{Normal}(\mu, \sigma^2)$ then

$$P(\mu - \sigma < X < \mu + \sigma) \approx .68$$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx .95$$

$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx .997$$

Example 16.1 ctd

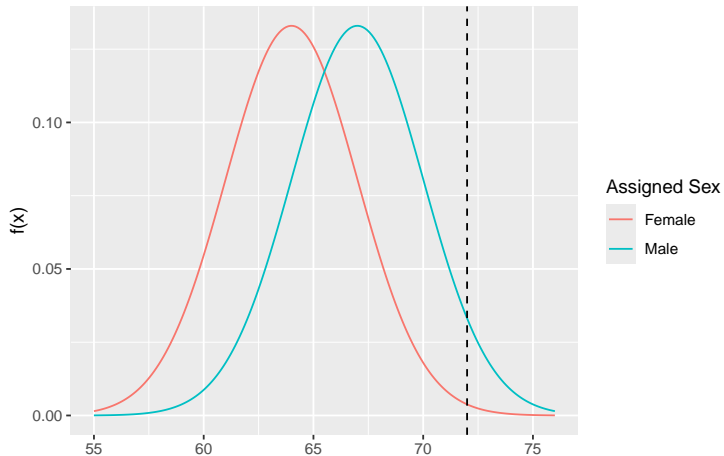
The adult heights of people assigned to be male and female at birth can be modelled amazingly well by a normal distribution. Suppose that the adult height people assigned to be female at birth is normally distributed with mean 64 inches and standard deviation 3 inches.

Suppose that the adult height of people assigned to be male at birth is normally distributed with mean 67 inches with standard deviation 3 inches.

What is the probability that someone's assigned sex was female if they are over 6 feet tall?

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Example 16.1 ctd



Example 16.2

A standard roulette wheel has 37 pockets in which the ball may land. Of these, 18 pockets are red, 18 are black, and 1 is green.

Suppose that you place \$1 bets that the ball will land in a black pocket on **200** consecutive games. Let X be the number of times you win.

- a) What is the exact probability that you win between 95 and 105 games inclusive?
- b) Approximate this probability with the normal distribution?

Normal Approximation to the Binomial

Suppose that $X \sim \text{Binomial}(n, p)$ with $np \geq 10$ and $n(1 - p) \geq 10$. Then

$$P(X \leq x) \approx P\left(Z \leq \frac{x + .5 - np}{\sqrt{np(1 - p)}}\right)$$

where $Z \sim \text{Normal}(0, 1)$.

Questions?

Exercise 16.1