STAT 2857A – Lecture 23a Examples and Exercises

Solutions

a) We know that

$$\frac{1}{c} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x,y)}{c} \; dy \; dx$$

in order that the joint pdf integrates to 1. To compute the integral, we need to find the bounds of integration. I'll consider y as this inner variable of integration. The second constraint states that $0 < y < 1$ and the third constraint implies that $y < 1-x$. Putting these together we have that $0 < y < 1-x$. The overall bounds on x are then $0 < x < 1$ as stated. Hence:

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x,y)}{c} \; dy \; dx &= \int_0^1 \int_0^{1-x} 1-(x+y) \; dy \; dx \\
&= \int_0^1 \left[y - xy + \frac{y^2}{2} \right]_0^{1-x} \; dx \\
&= \int_0^1 \frac{1-2x+x^2}{2} \; dx \\
&= \frac{1}{2} \left[x - x^2 + \frac{x^3}{3} \right]_0^1 \\
&= \frac{1}{6}.
\end{aligned}$$

Hence, $c = 6$ and the full joint pdf is

$$f(x,y) = 6(1-(x+y)), 0 < x < 1, 0 < y < 1-x.$$

b) The marginal pdf of X is defined to be

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\ dy \\ &= \int_0^{1-x} 6(1-(x+y))\ dy \\ &= 6\left[y - xy - \frac{y^2}{2}\right]_0^{1-x} \\ &= 6\left((1-x) - x(1-x) - \frac{(1-x)^2}{2}\right) \\ &= 3(1-x)^2, \quad 0 < x < 1. \end{aligned}$$

The marginal pdf of Y is defined to be

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\ dx$$

However, we need to find the bounds on x as a function of y. The third constraint implies that $x < 1-y$. Combining this with the first constraint, we have that $0 < x < 1-y$. Then

$$f_Y(y) = \int_0^{1-y} 6(1-(x+y))\ dx$$

Note that this is exactly the same integral as above, except that we've switched the roles of x and y. We will do exactly the same operations to solve the integral. Hence, the marginal pdf will be the same except the argument (the variable in the function) will be replaced with y. We say that "by symmetry":

$$f_Y(y) = 3(1-y)^2, \quad 0 < y < 1.$$

c) By definition, the conditional pdf of $X|Y=y$ is

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{6(1-(x+y))}{3(1-x)^2} \\ &= 2\frac{1-(x+y)}{(1-x)^2}. \end{aligned}$$

However, as always, we must include the support. In this case, x must be positive and less than $1-y$ in order that $x+y<1$. Hence, the conditional pdf is

$$f_{X|Y}(x|y) = 2\frac{1-(x+y)}{(1-y)^2}, \quad 0 < x < 1-y.$$

d) The conditional mean of $X|Y=y$ is

$$
\begin{aligned}
E(X|Y=y) &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\ dx \\
&= \int_0^{1-y} \frac{2x(1-(x+y))}{(1-y)^2}\ dx \\
&= \frac{2}{(1-y)^2}\int_0^{1-y} x(1-y)-x^2\ dx \\
&= \frac{2}{(1-y)^2}\left[\frac{x^2}{2}(1-y)-\frac{x^3}{3}\right]_0^{1-y} \\
&= \frac{2}{(1-y)^2}\left[\frac{(1-y)^3}{2}-\frac{(1-y)^3}{3}\right] \\
&= \frac{1-y}{3}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
E(X^2|Y=y) &= \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\ dx \\
&= \int_0^{1-y} \frac{2x^2(1-(x+y))}{(1-y)^2}\ dx \\
&= \frac{2}{(1-y)^2}\int_0^{1-y} x^2(1-y)-x^3\ dx \\
&= \frac{2}{(1-y)^2}\left[\frac{x^3}{3}(1-y)-\frac{x^4}{4}\right]_0^{1-y} \\
&= \frac{2}{(1-y)^2}\left[\frac{(1-y)^4}{3}-\frac{(1-y)^4}{4}\right] \\
&= \frac{(1-y)^2}{6}.
\end{aligned}
$$

Applying the shortcut formula the variance is

$$
\begin{aligned}
V(X) &= E(X^2)-E(X)^2 \\
&= \frac{(1-y)^2}{6}-\frac{(1-y)^2}{9} \\
&= \frac{(1-y)^2}{18}.
\end{aligned}
$$

e) The final piece of information we need to compute the covariance and correlation is

$$\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\ dy\ dx \\
&= \int_0^1\int_0^{1-x} 6xy(1-(x+y))\ dy\ dx \\
&= 6\int_0^1\int_0^{1-x} x(1-x)y - xy^2\ dy\ dx \\
&= 6\int_0^1 \left[\frac{x(1-x)y^2}{2} - \frac{xy^3}{3}\right]_0^{1-x}\ dx \\
&= 6\int_0^1 \left(\frac{x(1-x)^3}{2} - \frac{x(1-x)^3}{3}\right)\ dx \\
&= 6\int_0^1 \frac{x(1-x)^3}{6}\ dx \\
&= \int_0^1 \frac{x(1-x)^3}{} dx \\
&= \frac{1}{20}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\
&= \frac{1}{20} - \frac{1}{16} \\
&= \frac{-1}{80}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Corr}(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} \\
&= \frac{-1/80}{3/80} \\
&= \frac{-1}{3}.
\end{aligned}$$

The fact that the correlation is negative makes sense based on the contour plot of the joint pdf. The range of Y decreases as X increases. If $X = .01$, then Y can take any value between 0 and .99. However, if $X = .5$, then Y can only take values between 0 and .5, and if $X = .99$ then Y can only take values between 0 and .01. Hence there is a negative association between the two variables.