SS2857 Probability and Statistics 1
Fall 2024

Lecture 2

2.2 Axioms, Interpretations, and Properties of Probability

Axioms of Probability

Let $\mathcal{S}$ be a samples space for some experiment. The three axioms of probability are:

1. The probability of any event is non-negative:

$$P(A) \geq 0 \text{ for any } A \subset \mathcal{S}.$$

2. The probability of the entire sample space is 1:

$$P(\mathcal{S}) = 1.$$

3. The probability of an infinite union of disjoint events is the sum of the probabilities.
   Given $A_1, A_2, A_3, \ldots \subset \mathcal{S}$ such that $A_i \cap A_j = \emptyset$ for every $i \neq j$

$$P(A_1 \cup A_2 \cup A_3 \ldots) = P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

Key Results 1

1. Complement rule:
$$P(A') = 1 - P(A).$$
2. Probabilities are less than 1:
$$P(A) \leq 1.$$
3. Finite union of disjoint events:
Given $A_1, A_2, A_3, \ldots, A_k \subset \mathcal{S}$ such that $A_i \cap A_j = \emptyset$ for every $i \neq j$ then
$$P(A_1 \cup A_2 \cup A_3 \ldots \cup A_k) = P(\cup_{i=1}^{k} A_i) = \sum_{i=1}^{k} P(A_i).$$

Key Results 2

1. Union of two events:
   For any $A, B \subset \mathcal{S}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

2. Union of three events:
   For any $A, B, C \subset \mathcal{S}$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Example 2.1

Suppose that the sample space, $\mathcal{S}$, contains $N > 1$ outcomes and we assign each event $A \subset \mathcal{S}$ probability

$$P(A) = \left(\frac{N(A)}{N}\right)^k$$

where $N(A)$ denotes the number of outcomes in A for some positive integer k, i.e. $k \in \mathbb{Z}^+$.

For what values of k is this assignment valid?

If all outcomes in the sample space are equally likely, then

1. the probability of any simple event (outcome) is $1/N$,
2. the probability of any event A is

$$P(A) = \frac{N(A)}{A}.$$

Example 2.2

Happy Birthmonth!

Consider the events E_1, E_2, and E_3 from Example 1.1 part 2.

- $E_1 = A_1 \bigcap B_1 \bigcap C_1$
- $E_2 = \bigcup_{i=1}^{12} (A_i \bigcap B_i \bigcap C_i)$
- $E_3 = \bigcup_{i=1}^{12} (A_i \bigcap B_i \bigcap C_i')$

Suppose that the probability of any outcome is equally likely.

a) What is the probability of each event?

b) What is the probability that exactly 2 of the students are born in the same month?

c) What is the probability that at least 2 of the students are born in the same month?

d) What does the probability in part c) mean?

Definition

Suppose that we repeat the same experiment n times. The probability of an event A is the *limiting relative frequency* of the event as $n \to \infty$:

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n}.$$

If we repeat the same experiment many, many times then the proportion of times we observe the event A will be very close to $P(A)$ and get closer and closer as the number of replicates increases.

Example 2.3

Provide an interpretation for the following statements:

1. The probability that a randomly selected number between 1 and 10 is prime is .5.
2. The probability that we draw a club from a well shuffled deck of cards is .25.
3. The probability that a randomly selected newborn baby is assigned to be male at birth is .503.
4. The probability that it will rain this afternoon is .70.

Exercise 2.1

You and your friend play a game of chance. They think of a number between 1 and 10, and you try to guess it. You win if you guess the number.
Suppose that your friend choose the number at random (i.e., the numbers are equally likely).

a) Compute the probability of winning and provide an interpretation.
b) What is the probability that your guess is exactly one number away from the number your friend chose?
c) What is the probability that your guess is within one number of the number your friend chose?
d) What is the probability that your guess is more than one number away from the number your friend chose?

Questions?