STAT 2857A – Lecture 14 Examples and Exercises

Solutions – Revised 31/10/24

Example 14.1

a) To show that $f(x)$ is a proper density function we need to show that:

i. $f(x) \geq 0$ for all $x \in \mathfrak{R}$
ii. $\int_{\infty}^{\infty} f(u)\ du = 1$

The first criterion requires that $c \geq 0$. Then we need

$$\begin{aligned}
\int_{\infty}^{\infty} f(u)\ du &= 1 \\
\int_{-\infty}^{0} 0\ du + \int_{0}^{1} cu^2\ du + \int_{1}^{\infty} 0\ du &= 1 \\
\left[\frac{cu^2}{2}\right]_0^1 &= 1 \\
c &= 2
\end{aligned}$$

Hence, the pdf is proper if $c = 2$. This give us

$$f(x) = \begin{cases} 0 \leq 0 & \\ 2x & 0 < x < 1 \\ 0 & 1 \leq x \end{cases}$$

b) By definition, the cdf is

$$F(x) = \int_{-\infty}^{x} f(u)\ du = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 < x < 1 \\ 1 & 1 \leq x \end{cases}$$

c)

d) By definition:

$$\begin{aligned} P(X \le .5) &= F(.5) \\ &= .5^2 \\ &= .25 \end{aligned}$$

ii) For any continuous random variable $P(X = x) = 0$ for any $x \in \mathfrak{R}$. Hence, $P(X = .5) = 0$.

iii) Consider that:

$$\begin{aligned} P(X < .5) &= P(X \le .5) - P(X = .5) \\ &= F(.5) - 0 \\ &= .5^2 \\ &= .25 \end{aligned}$$

Note that $P(X < .5) = P(X \le .5)$. In general, if X is a continuous random variable then $P(X < x) = P(X \le x) = F(x)$.

iv)

$$\begin{aligned} P(.25 \le X \le .75) &= P(X \le .75) - P(X < .25) \\ &= P(X \le .75) - P(X \le .25) \\ &= .75^2 - .25^2 \\ &= .50 \end{aligned}$$

v)

$$\begin{aligned} P((X < .25) \cup (X > .75) &= 1 - P(.25 \le X \le .75) \\ &= 1 - [F(.75) - F(.25)] \\ &= 1 - F(.75) + F(.25) \\ &= 1 - .75^2 + .25^2 \\ &= 1 - .5625 + .0625 \end{aligned}$$

vi) The random variable X is continuous, according to Devore and Berk, if:

i) the possible values (support) of X is a union of (possibly one) disjoint interval in $\mathfrak{R}$, and
ii) $P(X = x) = 0$ for any $x \in \mathfrak{R}$.

The first criterion is satisfied because the support of X is a single interval, $(0, 1)$.

To show that the second criterion is true consider that

$$\begin{aligned}
P(X=x) &= P(X \le x) - P(X < x) \\
&= P(X \le x) - \lim_{x^- \nearrow} P(X \le x^-) \\
&= F(x) - \lim_{x^- \nearrow x} F(x^-) \\
&= x^2 - \lim_{x^- \nearrow x} (x^-)^2 \\
&= 0.
\end{aligned}$$

What the second criterion really implies is that $F(x)$ is continuous. Hence a continuous random variable!

Exercise 14.2

We know from Example 14.1. that the cdf is

$$F(x) = \int_{-\infty}^{x} f(u)\, du = \begin{cases} 0 & x \le 0 \\ x^2 & 0 < x < 1 \\ 1 & 1 \le x \end{cases}$$

Inverting the cdf we have that

$$\eta_p = \sqrt{p}$$

for any $0 \le p \le 1$ where we are considering the positive square root.

a) The median is

$$\eta_{.5} = \sqrt{.5} = .7071.$$

b) The 5th and 95th percentiles are

$$\begin{aligned}
\eta_{.05} &= \sqrt{.05} = .2236 \\
\eta_{.95} &= \sqrt{.95} = .9747.
\end{aligned}$$

c) Since $f(x)$ is monotonically increasing where it is positive, $(0,1)$, the shortest interval will occur when $x_2 = 1$. Then x_1 must satisfy

$$P(x_1 < X < 1) = .90$$

which implies that

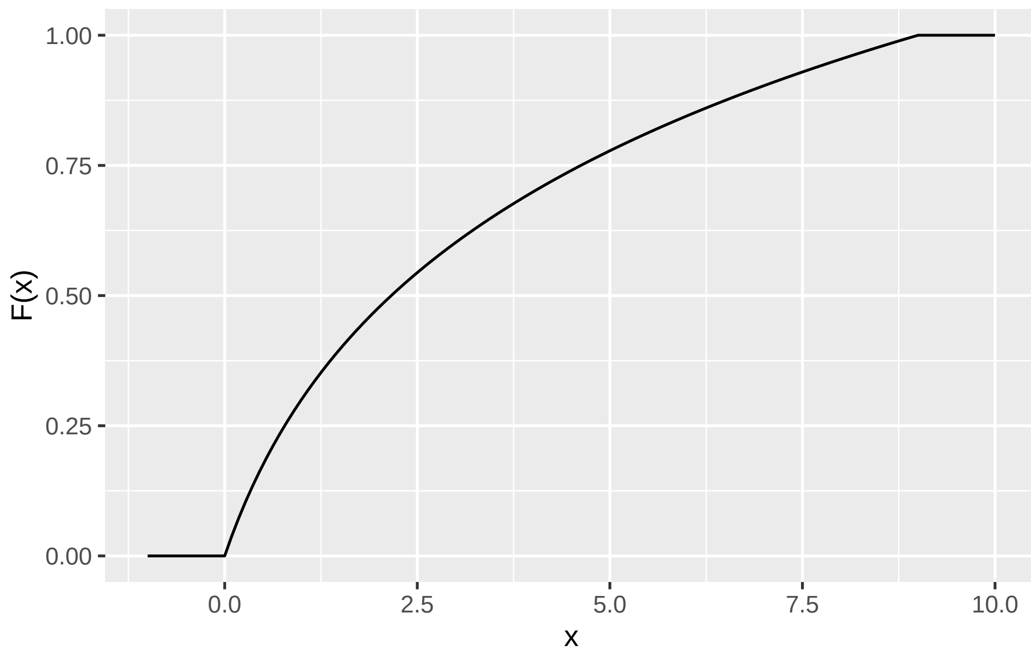
$$P(X < x_1) = .10$$

or

$$x_1 = \eta_{.10} = \sqrt{.10} = .3162.$$

Exercise 14.1

a)



b) The pdf is given by

$$f(x) = \frac{d}{dx} F(x).$$

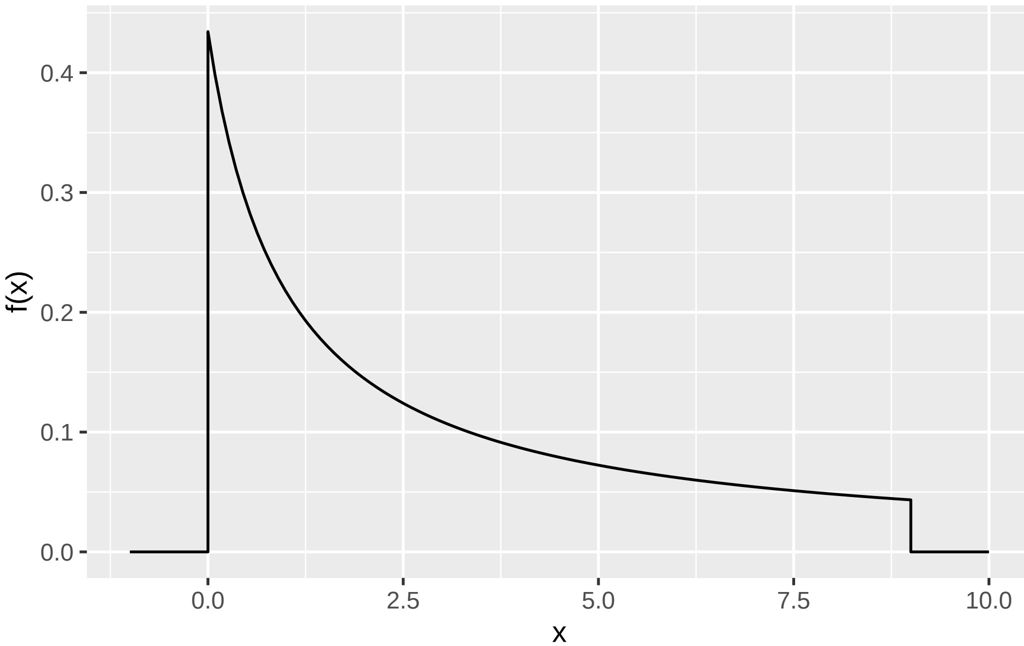
Note that $F(x)$ is flat over the regions $x < 0$ and $x > 9$ so

$$f(x) = \frac{d}{dx} F(x) = 0.$$

For $0 \leq x \leq 9$

$$\begin{aligned} f(x) &= \frac{d}{dx} \log_1 0(1 + x) \\ &= \frac{1}{(x+1)\ln(10)}. \end{aligned}$$

c)



d) i) $P(X \leq \sqrt{10}-1) = F(\sqrt{10}-1) = \log_{10}(\sqrt{10}) = .50.$

ii) Since the cdf is continuous, $P(X < \sqrt{10}-1) = P(X \leq \sqrt{10}-1) = .50.$

iii) Since the cdf is continuous, $P(X = \sqrt{10}-1) = 0.$

iv) $P(X > \sqrt{10}-1) = 1 - P(X \leq \sqrt{10}-1) = .5$ v)$P(X \geq \sqrt{10}-1) = 1 - P(X < \sqrt{10}-1) = .5$