STAT 2857A – Lecture 23 Examples and Exercises

Solutions – Revised 27/11/24

a) First consider the number of white marbles you draw from the bag. We know that W will follow a hypergeometric distribution. Specifically

$$W \sim \text{Hypergeometric}(3, 18, 27).$$

Then

$$P(W = w) = \frac{\binom{18}{w}\binom{9}{3-w}}{\binom{27}{3}}.$$

Hence,

$$P(W = 2) = \frac{\binom{18}{w}\binom{9}{3-w}}{\binom{27}{3}} = 0.471.$$

Now consider the number of red cards that you draw from the deck. Given $W = 2$, number of cards you choose is $N = \max(2, 1) = 2$. Then

$$R|W = 2 \sim \text{Hypergeometric}(2, 26, 52).$$

So

$$P(R = 1|W = 2) = \frac{\binom{26}{1}\binom{26}{1}}{\binom{52}{2}} = 0.51.$$

Finally,

$$\begin{aligned} P(W = 2, R = 1) &= P(W = 2)P(R = 1|W = 2) \\ &= 0.471(0.51) \\ &= 0.24. \end{aligned}$$

b) Again, we start with the number of white marbles you draw from the bag. As above,

$$P(W = w) = \frac{\binom{18}{w}\binom{9}{3-w}}{\binom{27}{3}}.$$

Then, given $W = w$, $N = \max(w, 3-w)$ and

$$R|W = w \sim \text{Hypergeometric}(\max(w, 3-w), 26, 52).$$

So

$$P(R = r|W = w) = \frac{\binom{26}{r}\binom{26}{\max(w,3-w)-r}}{\binom{52}{\max(w,3-w)}}.$$

Finally, we consider the set of possible values (support) of W and R. The possible values of W are 0,1,2, and 3, and the possible values of R are 0, 1, 2, and 3 if $W = 0$ or $W = 3$ and 0,1, or 2 if $W = 1$ or $W = 2$. Hence,

$$P(W, R) = P(W = w)P(R = r|W = w) = \frac{\binom{18}{w}\binom{9}{3-w}\binom{26}{r}\binom{52}{\max(w,3-w)-r}}{\binom{27}{3}\binom{52}{\max(w,3-w)}}.$$

The following table provides the values in the intermediate calculations used to compute the table provided in the exercises:

w	N	r	P(W=w)	P(R=r|W=w)	P(W=w,R=r)
0	3	0	0.029	0.118	0.003
0	3	1	0.029	0.382	0.011
0	3	2	0.029	0.382	0.011
0	3	3	0.029	0.118	0.003
1	2	0	0.222	0.245	0.054
1	2	1	0.222	0.510	0.113
1	2	2	0.222	0.245	0.054
2	2	0	0.471	0.245	0.115
2	2	1	0.471	0.510	0.240
2	2	2	0.471	0.245	0.115
3	3	0	0.279	0.118	0.033
3	3	1	0.279	0.382	0.107
3	3	2	0.279	0.382	0.107
3	3	3	0.279	0.118	0.033

b) To answer this question, we first need to find the marginal pmf for R. This is computed by summing the joint pdf over w for each value of r. Equivalently, we sum the columns in the table above. The values are

r	P(R=r)
0	0.206
1	0.471

r	P(R=r)
2	0.287
3	0.036

Then

$$E(R) = 0(0.206) + 1(0.471) + 2(0.287) + 3(0.036) = 1.154,$$
$$E(R^2) = 0(0.206) + 1(0.471) + 4(0.287) + 9(0.036) = 1.946, \text{and}$$
$$V(R) = E(R^2) - E(R)^2 = 0.614.$$

c) You win if $R = 0$ or $R = N$. Note that these are mutually exclusive. The probability that you win is

$$\begin{aligned} P(R=0) + P(R=N) =& (0.003 + 0.054 + 0.115 + 0.033) + (0.003 + 0.113 + 0.115 + 0.033) \\ =& 0.47 \end{aligned}$$

d) For any w and r, the conditional probability of $W = w|R = r$ is

$$P(W = w|R = r) = \frac{P(W = w, R = r)}{P(R = r)}.$$

For the specific case of interest, we have

$$P(R = 2) = 0.011 + 0.054 + 0.115 + 0.107 = 0.011, 0.054, 0.115, 0.107.$$

Then

$$P(W = w|R = 2) = \frac{P(W = w, R = 2)}{0.287}$$

The entries in the conditional pdf are shown in the following table:

w	r	P(W=w,R=r)	P(W=w|R=2)
0	2	0.011	0.038
1	2	0.054	0.189
2	2	0.115	0.402
3	2	0.107	0.371

e) Using the values from the previous parts

$$E(W|R = 2) = 0(0.038) + 1(0.189) + 2(0.402) + 3(0.371) = 2.106$$

f) Using the values from the previous parts

$$E(W^2|R=2) = 0(0.038) + 1(0.189) + 4(0.402) + 9(0.371) = 5.136$$

and

$$V(W|R=2) = E(W^2|R=r) - E(W|R=r)^2 = 0.702.$$

g) The shortcut formula implies that

$$\mathrm{Cov}(W,R) = E(WR) - E(W)E(R).$$

Since W is hypergeometric, we know that

$$E(W) = \frac{3(18)}{27} = 2.$$

and

$$V(W) = \frac{27-3}{27-1} \cdot \left(\frac{3(18)}{27}\right)\left(1 - \frac{18}{27}\right) = 0.615$$

We also have that

$$E(R) = 1.154.$$

Then

$$\begin{aligned} E(RW) &= 1(1)(0.113) + 1(2)(0.054) + 2(1)(0.24) + 2(2)(0.115) + 3(1)(0.107) + 3(2)(0.107) + 3(3)(0.033) \\ &= 2.418. \end{aligned}$$

Hence

$$\mathrm{Cov}(W,R) = 2.418 - (1.154)(2) = 0.111.$$

Finally

$$\begin{aligned} \mathrm{Corr}(W,R) &= \frac{\mathrm{Cov}(W,R)}{\sqrt{V(W)V(R)}} \\ &= \frac{0.111}{\sqrt{(0.615)(0.614)}} \\ &= 0.18 \end{aligned}$$