STAT 2857A – Lecture 17 Examples and Exercises

Solutions

Review

a) FALSE (I've divided by the variance not the standard deviation.)
b) d
c) a
d) a

Example 17.1

a) The pdf of X is:

$$f(x) = \frac{1}{8.662^{0.628}\Gamma(0.628)} x^{0.628-1} e^{-x/8.662}, \quad x \geq 0.$$

b) The mean and variance are:

$$\begin{aligned} E(X) &= \alpha\beta = 0.628(8.662) = 5.439736 \\ V(X) &= \alpha\beta^2 = 0.628(8.662^2) = 47.1189932 \end{aligned}$$

c) The probability that the total precipitation is more than 10 mm given that it rains at all is:

$$\begin{aligned} P(X \geq 10) &= 1 - P(X \leq 10) \\ &= 1 - 0.8271421 \\ &= 0.1728579 \end{aligned}$$

Example 17.2

a) To write down the pdf we need first to understand what the half-life means. The half-life means that the probability of decay within the given time is .5 (i.e., the median). If we let $t_{1/2}$ denote the half-life then

$$\begin{aligned} F(t_{1/2}) &= .5 \\ 1 - e^{-\lambda t_{1/2}} &= .5 \\ \lambda t_{1/2} &= -\log(.5) \\ \lambda &= 0.9848639 \end{aligned}$$

The pdf is

$$\begin{aligned} f(t) &= e^{-\lambda t} \\ &= e^{-0.9848639t}, \quad t > 0. \end{aligned}$$

b) The mean and variance of T are

$$\begin{aligned} E(T) &= \frac{1}{\lambda} = \frac{1}{0.9848639} = 1.0153688 \\ V(T) &= \frac{1}{\lambda^2} = \frac{1}{0.9699568} = 1.0309737 \end{aligned}$$

c) The probability that $T > 1$ is

$$\begin{aligned} P(T > 1) &= 1 - P(T \leq 1) \\ &= 1 - (1 - e^{-0.9848639}) \\ &= e^{-0.9848639} \\ &= 0.3734901 \end{aligned}$$

d) The probability that $T > 2$ given $T > 1$ is

$$\begin{aligned} P(T > 2 | T > 1) &= \frac{P(T > 2, T > 1)}{P(T > 1)} \\ &= \frac{P(T > 2)}{P(T > 1)} \\ &= \frac{e^{-2(0.9848639)}}{e^{-(0.9848639)}} \\ &= e^{-(0.9848639)} \\ &= 0.3734901 \\ &= P(T > 1) \end{aligned}$$

e) The probability that $T > 100,001$ given $T > 100,000$ is

$$\begin{aligned} P(T > 100,001 | T > 100,000) &= \frac{P(T > 100,001)}{P(T > 100,000)} \\ &= \frac{e^{-100,001(0.9848639)}}{e^{-100,000(0.9848639)}} \\ &= e^{-(0.9848639)} \\ &= 0.3734901 \\ &= P(T > 1) \end{aligned}$$

In fact, for any $t_0, t > 0$

$$P(T > t + t_0 | T > t_0) = P(T > t).$$

Example 17.3

Suppose that $Z \sim N(0, 1)$ then

$$\begin{aligned} P(Z^2 \leq 2) &= P(-sqrt2 < Z < \sqrt{2}) \\ &= P(-1.4142 < Z < 1.4142) \\ &= P(Z < 1.4142) - P(Z < -1.4142) \\ &= .9213 - .0787 \\ &= .8426. \end{aligned}$$

Using the online calculator at stattrek we find that

$$P(X < 2) = .8426$$

exactly as required. In fact, this would hold for any value, not just 2.

Exercise 17.1

a) According to the question,

$$T_5 \sim \text{Gamma}(5, 10).$$

Then $\mu_{T_5} = E(T_5) = 50$ and $\sigma^2_{T_5} = V(T_5) = 500$.

b) Using the online calculator

$$P(T_5 > 60) = 0.28506.$$

c) Using the calculator, the 5-th, 50-th (median), and 95-th percentiles of T_5 and the normal distribution with mean 50 and variance 500 are

d) The following plot compares the ratio of the 5-th, 50-th, and 95-th percentiles for the gamma distribution with $\alpha = k$ and $\beta = 10$ and the normal distribution with matching mean and variance ($\mu = 10k$ and $\sigma^2 = 100k$). For all three percentiles, the ratio gets closer to 1.00 as k increases. This implies that the percentiles of the gamma and the percentiles of the normal distribution are becoming closer and closer, relatively speaking.