Variance Results for Simplified CJS Model

# 1 Model

I have simplified the Cormack-Jolly-Seber model by assuming:

1. That there is a single release of $n$ individuals.
2. That $\phi$ is constant.
3. That $p$ is constant *and known*.

Let $k$ denote the number of recapture occasions and $m_j$ the number of individuals recaptured on occasion $j$, $j = 1, \ldots, k$. Define

$$P_j(\phi) = \phi^j (1-p)^{(j-1)} p$$

to be the probability that an individual is recaptured on occasion $j$. The likelihood is

$$\mathrm{l}(\phi) := m_0 \cdot \log\left(1 - \sum_{j=1}^{k} P_j(\phi)\right) + \sum_{j=1}^{k} m_j \cdot \log\left(P_j(\phi)\right)$$

where $m_0 = n - \sum_{j=1}^{k} m_j$ is the number of individuals never recaptured.

# 2 Approximate Variance

Let $\hat{\phi}_k$ denote the MLE where $k$ indexes the number of recapture occasions. We can approximate $\mathrm{Var}(\hat{\phi}_k)$ through the usual arguments. The second derivative of $\mathrm{l}(\phi)$ with respect to $\phi$ is

$$\begin{aligned}\frac{d^2 l}{d\phi} = &- \frac{m_0 \cdot p \cdot \sum_{j=1}^{k} (j-1) \cdot j \cdot (1-p)^{j-1} \cdot \phi^{j-2}}{1 - p \cdot \sum_{j=1}^{k} (1-p)^{j-1} \cdot \phi^j} \\ &- \frac{m_0 \cdot p^2 \cdot \left(\sum_{j=1}^{k} j \cdot (1-p)^{j-1} \cdot \phi^{j-1}\right)^2}{\left(1 - p \cdot \sum_{j=1}^{k} (1-p)^{j-1} \cdot \phi^j\right)^2} \\ &- \frac{\sum_{j=1}^{k} j \cdot m_j}{\phi^2}.\end{aligned}$$

This is a linear function of $m_0, m_1, \ldots, m_k$ so we can compute the expected value simply by replacing these values with their expected values

$$E(m_0) = n\left(1 - \sum_{j=1}^{k} P_j(\phi)\right)$$

and

$$E(m_j) = nP_j, \quad j = 1, \ldots, k.$$

It follows that

$$\begin{aligned} I(\phi) &= n \left[ \frac{\left(\sum_{j=1}^k j(1-p)^{j-1}\phi^{j-1}p\right)^2}{1-\sum_{j=1}^k (1-p)^{j-1}\phi^j p} + \sum_{j=1}^k j^2(1-p)^{j-1}\phi^{j-2}p \right] \\ &= n \left[ \sum_{j=1}^k j^2 \frac{P_j(\phi)}{\phi^2} + \frac{\left(\sum_{j=1}^k j\frac{P_j(\phi)}{\phi}\right)^2}{1-\sum_{j=1}^k P_j(\phi)} \right] \end{aligned}$$

and we can approximate $\mathrm{Var}(\hat{\phi}_k) \approx I(\phi)^{-1}$. The file `test_simple_variance.R` assesses this through simulation, and the results seem to do quite well.

# 3 Asymptotic Behaviour

There is a very simple argument to show that $\mathrm{Var}(\hat{\phi}_k)$. Intuitively, $\mathrm{Var}(\hat{\phi}_{k+1}) < \mathrm{Var}(\hat{\phi}_k)$ because we gain more information as the number of recapture occasions increases. This should also be fairly simple to prove. Hence, $\mathrm{Var}(\hat{\phi}_k)$ is a decreasing sequence and since it is bounded below, $\mathrm{Var}(\hat{\phi}_k) \geq 0$, it must converge.

Suppose now that $\sum_{j=1}^k P_j(\phi)$ the conditions for interchange of the limit (as $k \to \infty$) and derivative so that

$$\frac{d}{d\phi}\sum_{j=1}^\infty P_j(\phi) = \sum_{j=1}^\infty \frac{d}{d\phi}P_j(\phi)$$

and

$$\frac{d^2}{d\phi^2}\sum_{j=1}^\infty P_j(\phi) = \sum_{j=1}^\infty \frac{d^2}{d\phi^2}P_j(\phi).$$

Note that

$$\frac{d}{d\phi}P_j(\phi) = j\frac{P_j(\phi)}{\phi} \text{ and } \frac{d^2}{d\phi^2}P_j(\phi) = j(j+1)\frac{P_j(\phi)}{\phi}.$$

Substituting these quantities into the expression for $I(\phi)$ above then yields

$$\lim_{k\to\infty} \mathrm{Var}(\hat{\phi}_k) = \frac{\phi(1-\phi)(1-(1-p)\phi)}{np(1-\phi^2(1-p))}.$$

Again, I have implemented this expression in `test_simple_variance.R` and it seems to work well.

## 4 Further Notes

I have just realized that there is a way to simplify $I(\phi)$ even using the fact that $\sum_{j=1}^{k} a_k = \sum_{j=1}^{\infty} a_k - \sum_{j=k+1}^{\infty} a_k$. For example

$$
\begin{aligned}
\sum_{j=1}^{k} P_j(\phi) &= \sum_{j=1}^{\infty} P_j(\phi) - \sum_{j=k+1}^{\infty} P_j(\phi) \\
&= P_1(\phi) \sum_{j=1}^{\infty} \phi^{j-1}(1-p)^{j-1} - P_{k+1}(\phi) \sum_{j=1}^{\infty} \phi^{j-1}(1-p)^{j-1} \\
&= (P_1(\phi) - P_{k+1}(\phi)) \sum_{j=0}^{\infty} \phi^{j}(1-p)^{j} \\
&= \frac{\phi p - \phi^{k+1}(1-p)^k p}{1-\phi(1-p)}
\end{aligned}
$$

using the fact that $P_{j+1}(\phi) = P_j(\phi)\phi(1-p)$ and the limit for a geometric series. The same argument can be applied along with the results for the derivatives to compute $\sum_{j=1}^{k} j^2 \frac{P_j(\phi)}{\phi^2}$ and $\sum_{j=1}^{k} j \frac{P_j(\phi)}{\phi}$. Substituting these expressions into $I(\phi)$ will remove the sums from $I(\phi)$ entirely. I'm hoping that this will then make it possible to compare $\mathrm{Var}(\hat{\phi}_k)$ and $\lim_{k\to\infty} \mathrm{Var}(\hat{\phi}_k)$ analytically to see how fast the variance decreases for given values of $n$, $p$ and $\phi$.