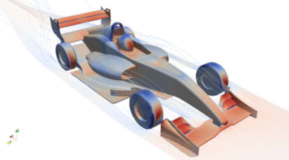


Category Theory

Cartography for the information age

Jan. 30, 2022

INCOSE IW

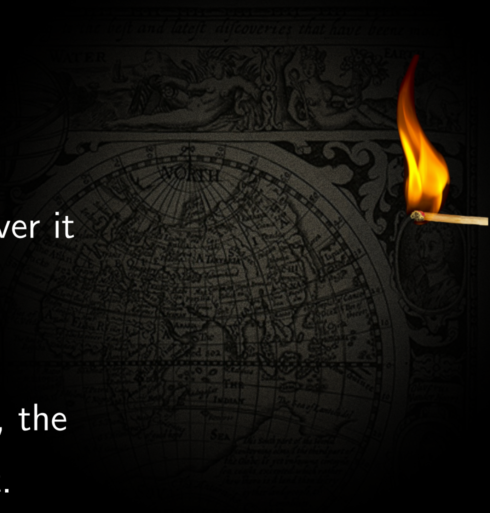
Spencer Breiner

Joint work w. E. Subrahmanian

National Institute of Standards and Technology
U.S. Department of Commerce

Chances are, the problem you’re working on (whatever it is) already has a solution.

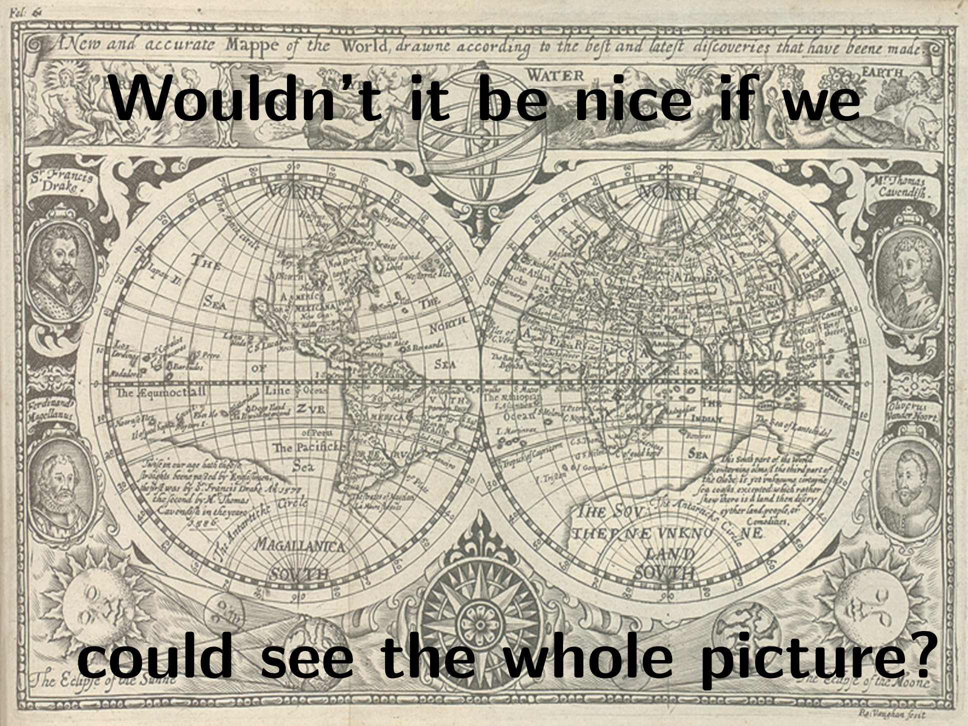
In an ocean of knowledge, the first challenge is *finding* it.



Most of the time, we have more than one problem.

The second challenge is to combine multiple insights into a single solution.

Wouldn’t it be nice if we could see the whole picture?




Too bad!
NEW YORK CITY MAP.
St. Peter's Square
St. Peter's Basilica
The Eclipse of the Sunne
The Eclipse of the Moone
EARTH

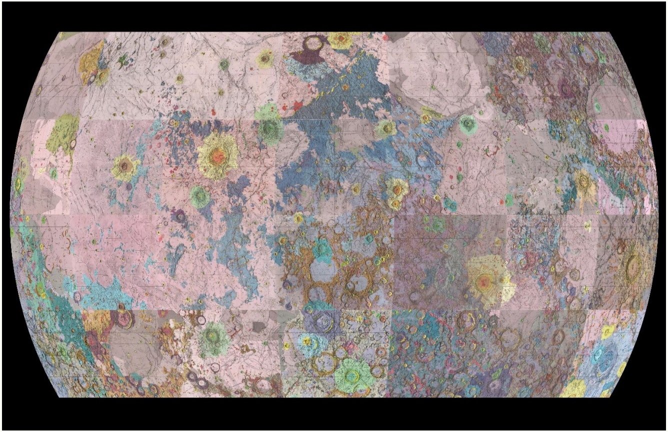
The problem of plurality:

Identify and align known methods, tools and representations to address unique, system-specific concerns.

What *can* we do?

Transform

What can we do?

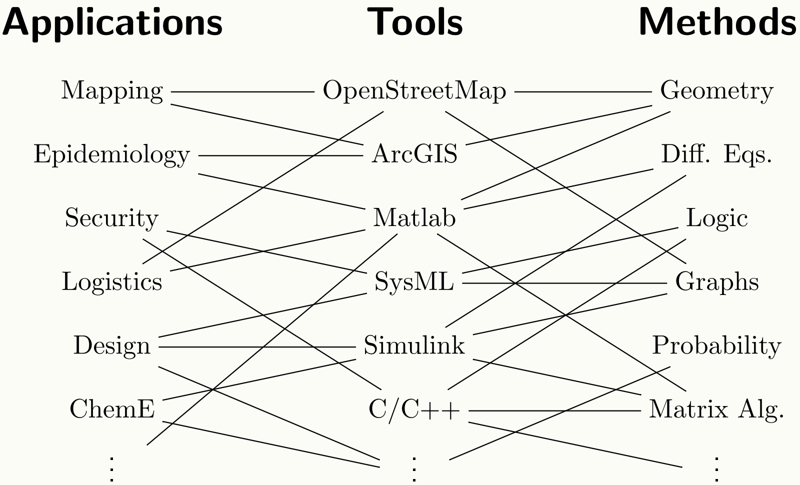


Align

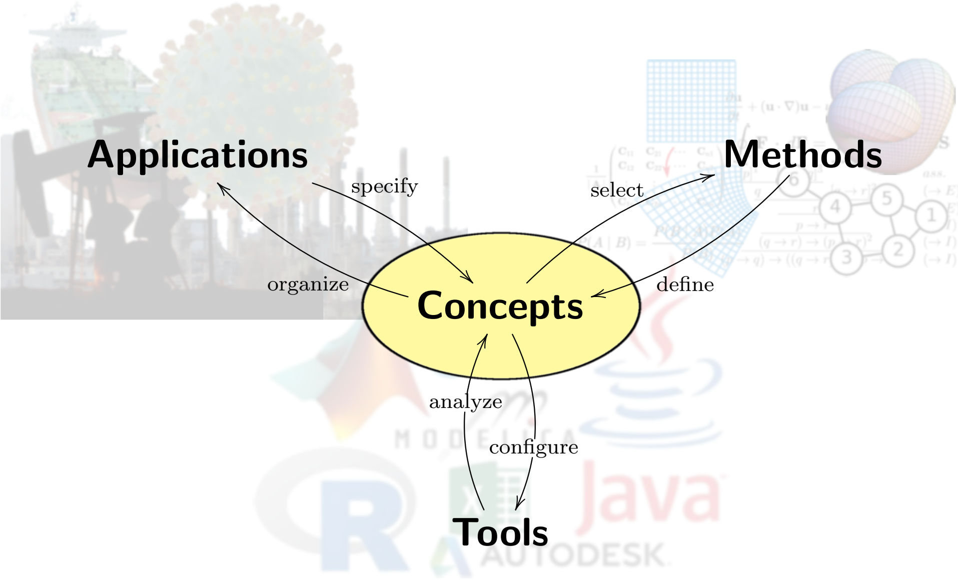
What *can* we do?

Abstract

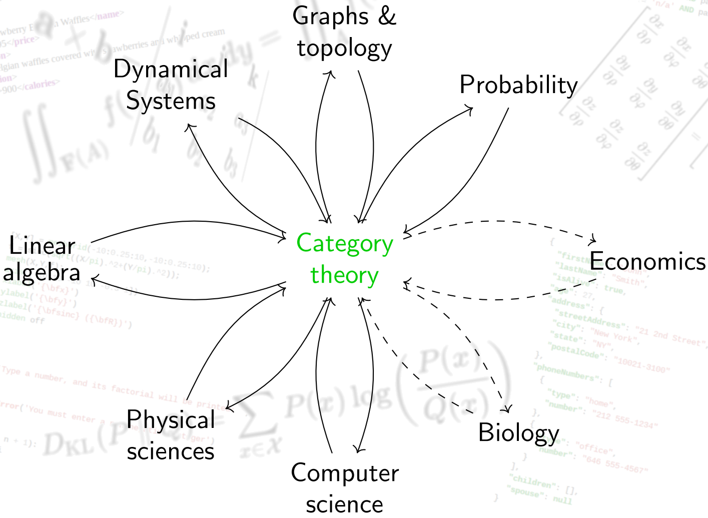
Where we are


Applications
Tools
Methods
Mapping
Epidemiology
Security
Logistics
Design
ChemE
⋮
OpenStreetMap
ArcGIS
Matlab
SysML
Simulink
C/C++
⋮
Geometry
Diff. Eqs.
Logic
Graphs
Probability
Matrix Alg.
⋮

Where we're going



How to get there?



Outline

I) Composition

II) Categories

III) Structure

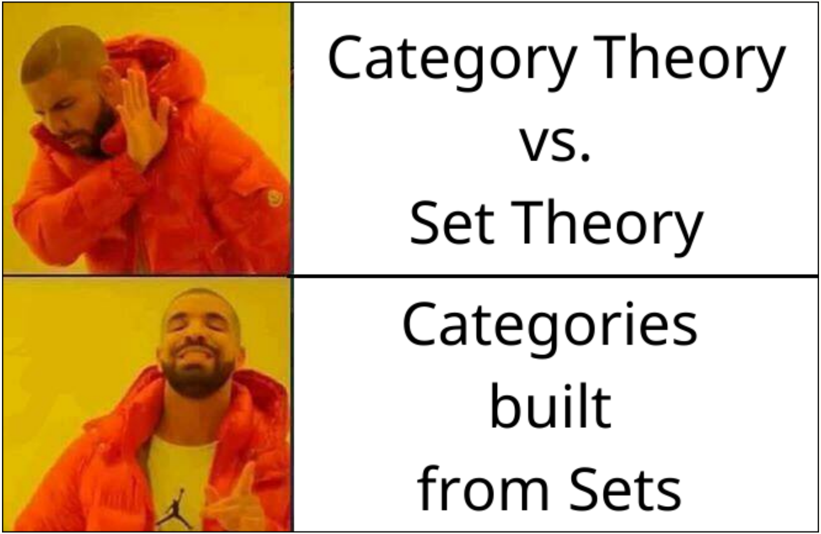
-break-

IV) Models

V) Data

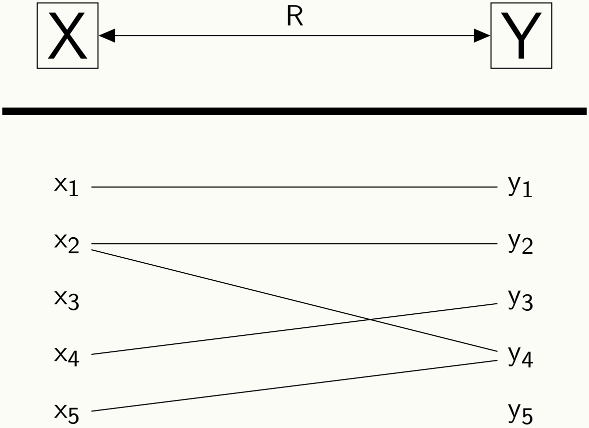
VI) Processes

Sets

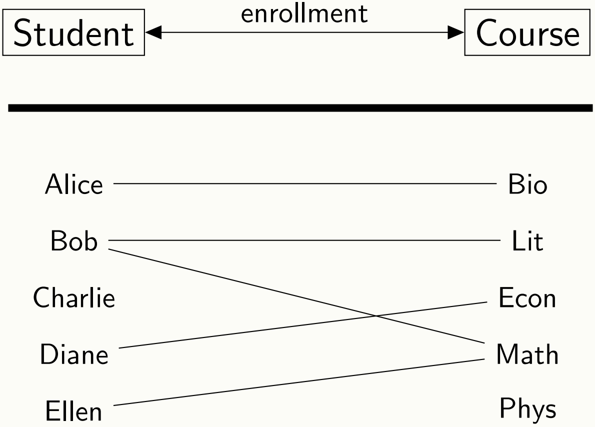

Category Theory
vs.
Set Theory
Categories
built
from Sets

Relations

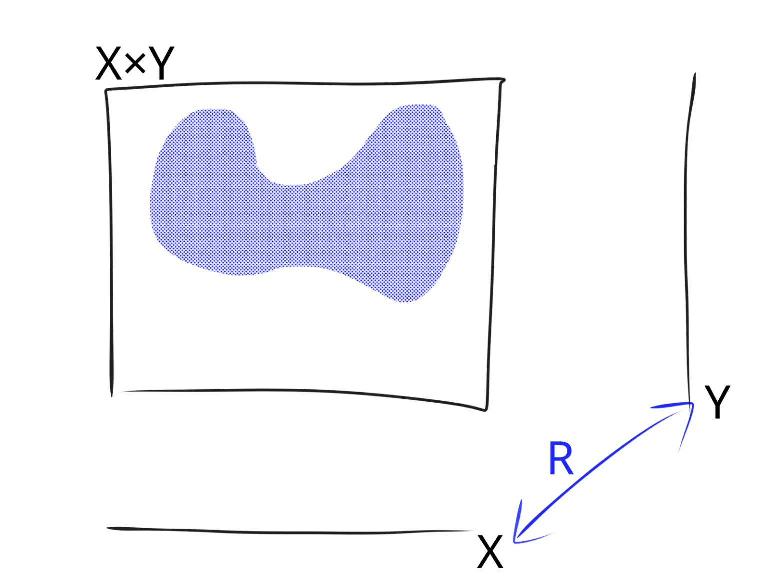
Relations



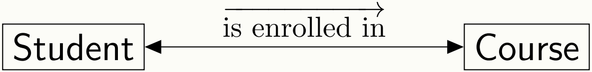
Relations

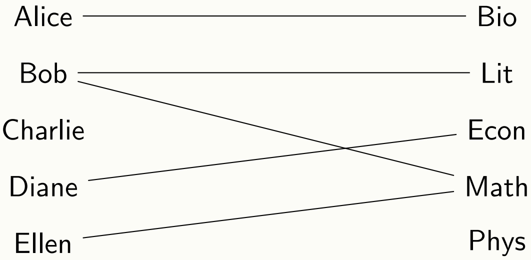

Student
enrollment
Course
Alice
Bob
Charlie
Diane
Ellen
Bio
Lit
Econ
Math
Phys

Relations, geometrically

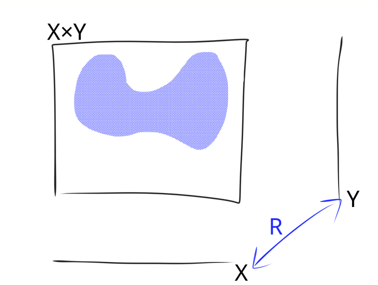


Relations are reversible

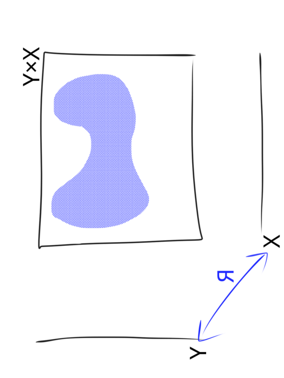




Reversal, geometrically

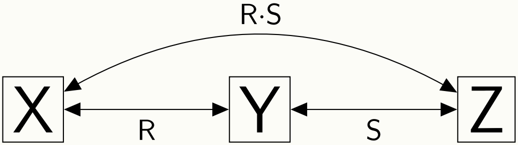


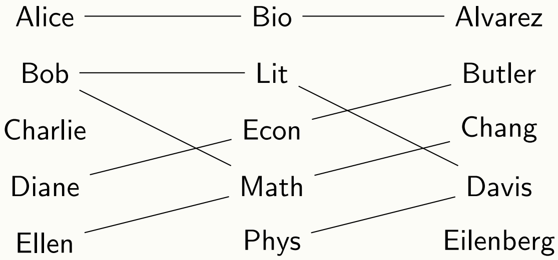
R



R^{op}

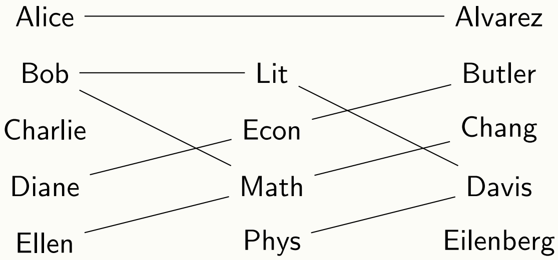
Composing Relations



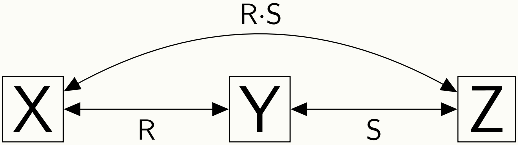


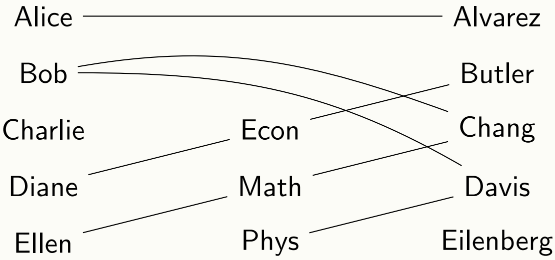
Composing Relations

$$
\begin{tikzcd}
\boxed{X} \arrow[r, leftrightarrow, "R"'] \arrow[rr, leftrightarrow, bend left, "R \cdot S"] & \boxed{Y} \arrow[r, leftrightarrow, "S"'] & \boxed{Z}
\end{tikzcd}
$$

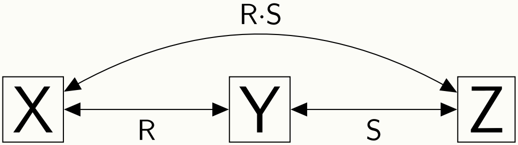


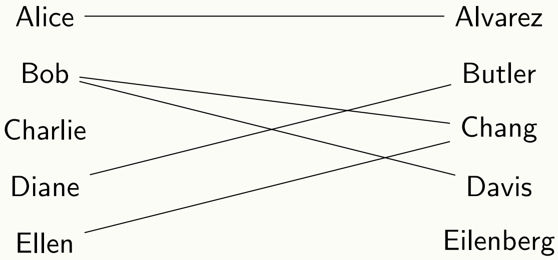
Composing Relations





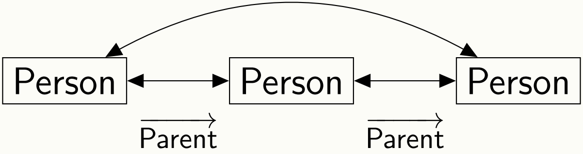
Composing Relations





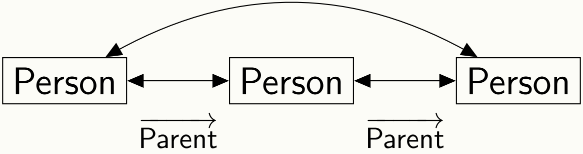
Self-Relations

$$\overrightarrow{\text{Grandparent}} = \overrightarrow{\text{Parent}} \cdot \overrightarrow{\text{Parent}}$$

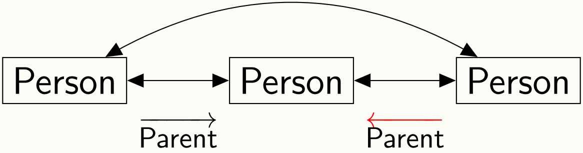


Self-Relations

$$\overrightarrow{\text{Grandparent}} = \overrightarrow{\text{Parent}} \cdot \overrightarrow{\text{Parent}}$$

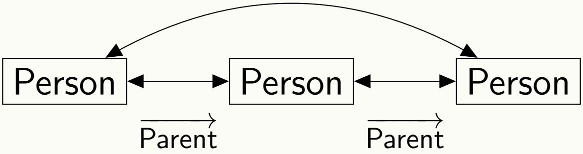


$$\overrightarrow{\text{Sibling}} = \overrightarrow{\text{Parent}} \cdot \overleftarrow{\text{Parent}}$$

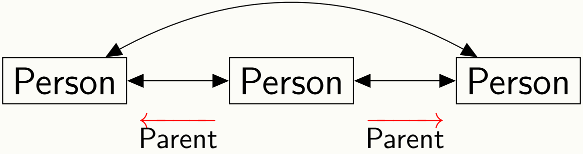


Self-Relations

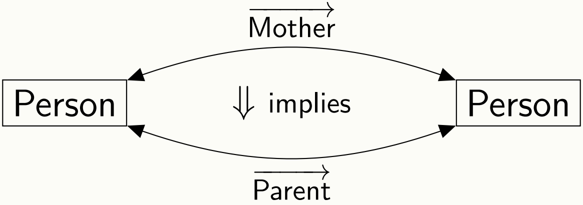
$$\overrightarrow{\text{Grandparent}} = \overrightarrow{\text{Parent}} \cdot \overrightarrow{\text{Parent}}$$



$$\overrightarrow{\text{Coparent}} = \overleftarrow{\text{Parent}} \cdot \overrightarrow{\text{Parent}}$$



Relations between relations



Relations between relations

$$
\begin{tikzcd}[column sep=huge]
\boxed{\mathsf{Person}} \arrow[r, bend left, leftrightarrow, "\overrightarrow{\mathsf{Mother}}"] \arrow[r, bend right, leftrightarrow, "\overrightarrow{\mathsf{Parent}}"'] \arrow[r, phantom, "\Downarrow"] & \boxed{\mathsf{Person}} \arrow[r, bend left, leftrightarrow, "\overrightarrow{\mathsf{Mother}}"] \arrow[r, bend right, leftrightarrow, "\overrightarrow{\mathsf{Parent}}"'] \arrow[r, phantom, "\Downarrow"] & \boxed{\mathsf{Person}}
\end{tikzcd}
$$

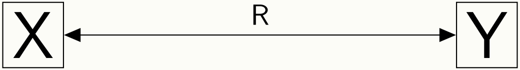
$$
\begin{tikzcd}
& \overrightarrow{\substack{\text{Maternal}\\ \text{Grandmother}}} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\overrightarrow{\mathsf{Grandmother}} \arrow[dr, Rightarrow] & & \overrightarrow{\substack{\text{Maternal}\\ \text{Grandparent}}} \arrow[dl, Rightarrow] \\
& \overrightarrow{\mathsf{Grandparent}} &
\end{tikzcd}
$$

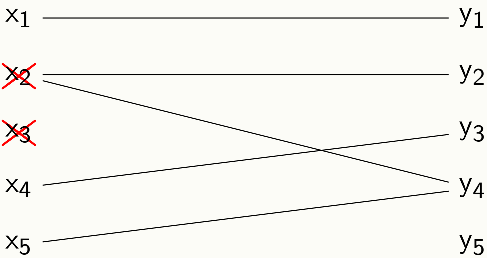
Questions?

Functions

Functions

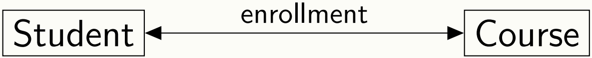
Functions are *total* and *single-valued*.

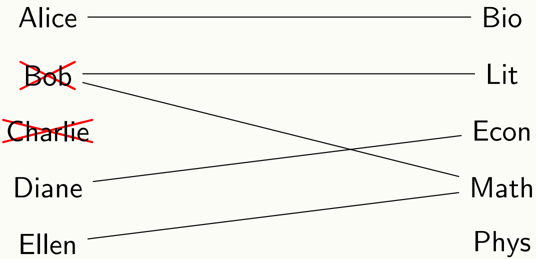




Functions

Functions are *total* and *single-valued*.





Functions

Functions are *total* and *single-valued*.

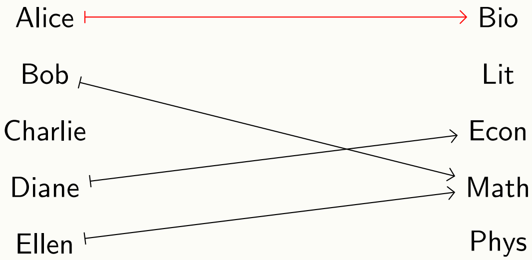
$$\boxed{\text{Student}} \xrightarrow{\text{enrollment}} \boxed{\text{Course}}$$

$$\begin{tikzcd}
\text{Alice} \arrow[r, mapsto] & \text{Bio} \\
\text{Bob} \arrow[rddd, mapsto] & \text{Lit} \\
\text{Charlie} \arrow[ru, mapsto] & \text{Econ} \\
\text{Diane} \arrow[ru, mapsto] & \text{Math} \\
\text{Ellen} \arrow[ru, mapsto] & \text{Phys}
\end{tikzcd}$$

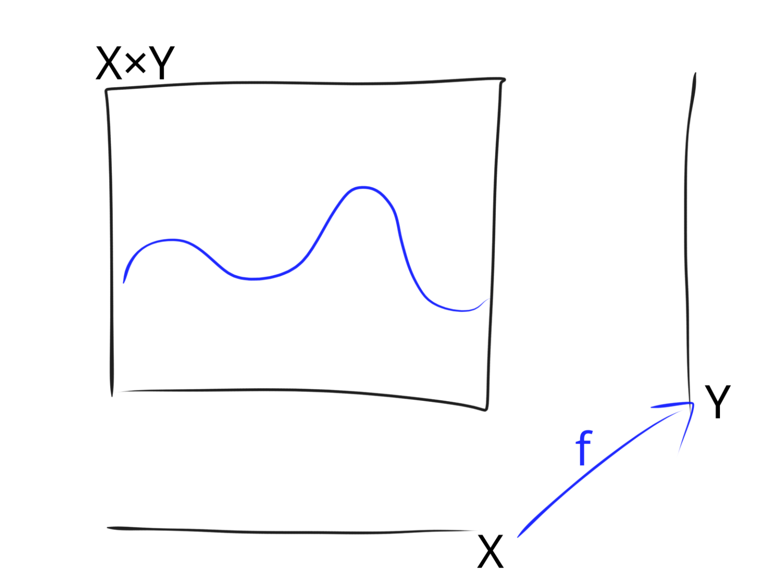
Functions

Functions are *total* and *single-valued*.

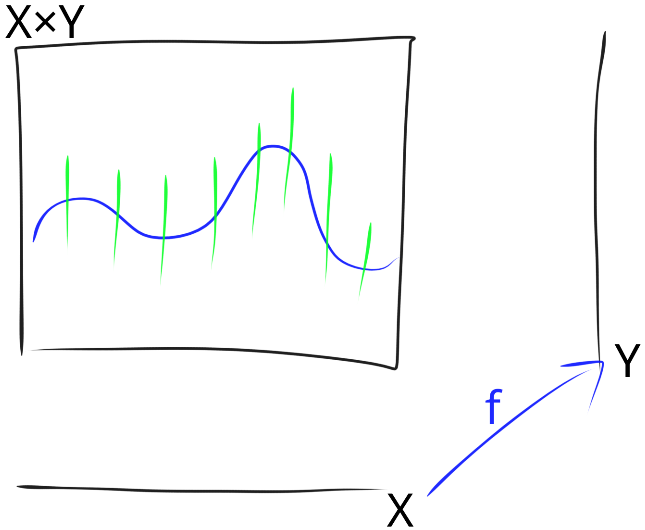
$$\boxed{\text{Student}} \xrightarrow{\text{enrollment}} \boxed{\text{Course}}$$



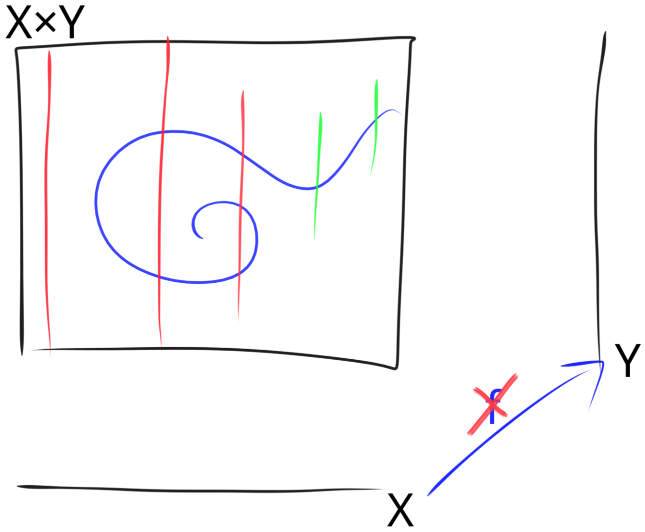
Functions, geometrically



Functions, geometrically

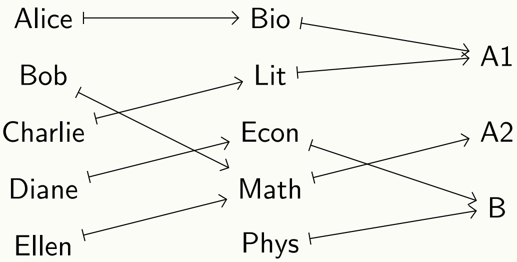


Functions, geometrically



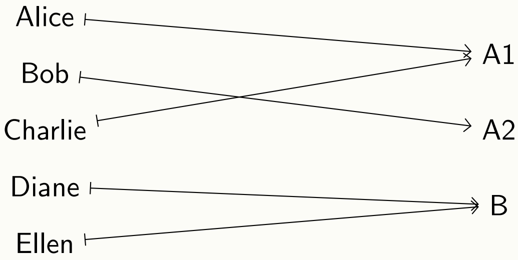
Functions compose

$$
\begin{tikzcd}
\boxed{\text{Student}} \arrow[r, "\text{enrollment}"'] \arrow[rr, bend left, "\text{enrollment}\cdot\text{classroom}"] & \boxed{\text{Course}} \arrow[r, "\text{classroom}"'] & \boxed{\text{Room}}
\end{tikzcd}
$$

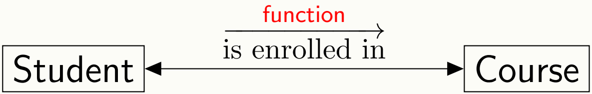


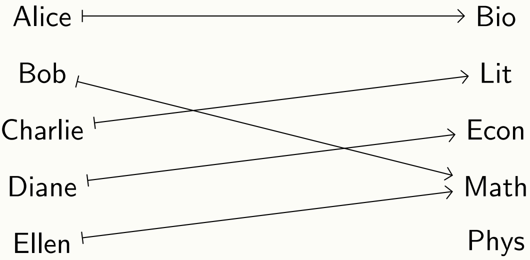
Functions compose

$$\begin{tikzcd}
\text{Student} \arrow[r, "\text{enrollment}"'] \arrow[rr, bend left, "\text{enrollment}\cdot\text{classroom}"] & \text{Course} \arrow[r, "\text{classroom}"'] & \text{Room}
\end{tikzcd}$$

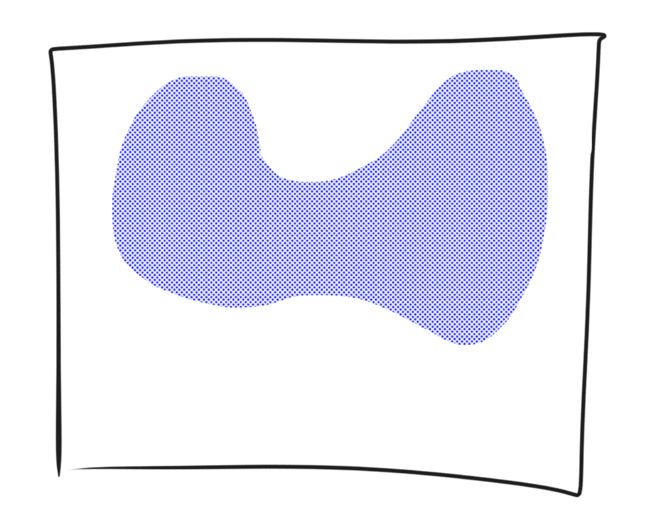


Functions do *not* reverse



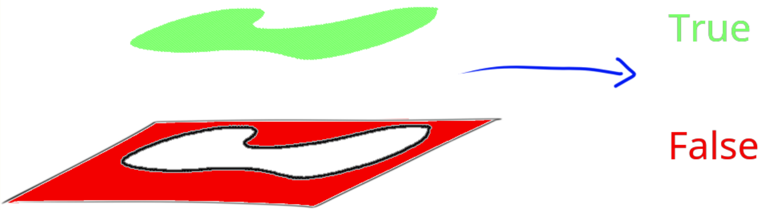


Relations as functions



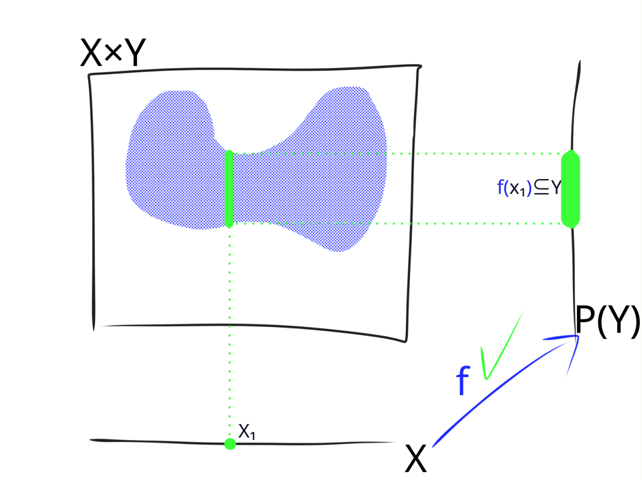
Relations as functions

Version 1: Truth functions $R : X \times Y \to \{T, F\}$



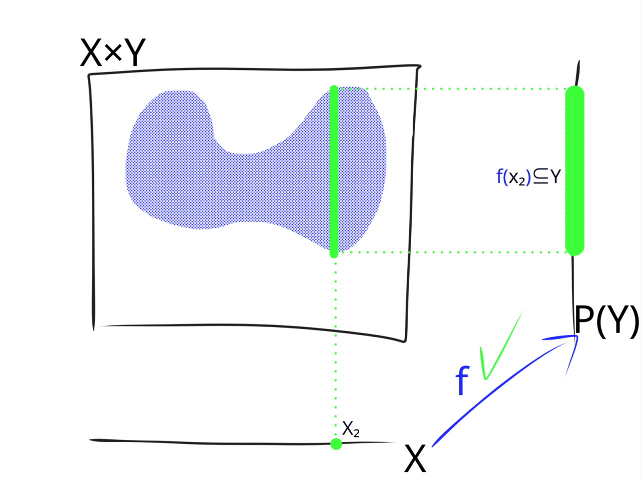
Relations as functions

Version 2: Set-valued functions $\overrightarrow{R} : X \to \mathcal{P}(Y)$



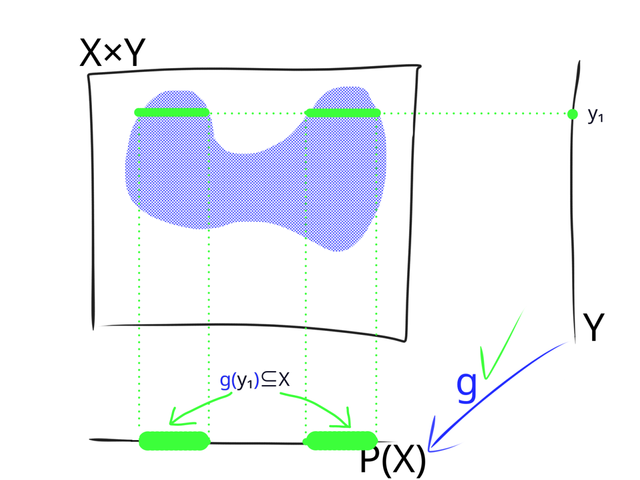
Relations as functions

Version 2: Set-valued functions $\overrightarrow{R} : X \to \mathcal{P}(Y)$



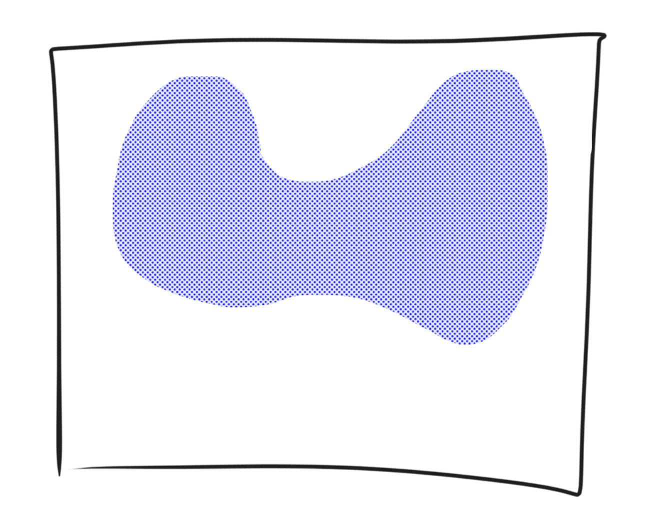
Relations as functions

Version 3: Set-valued functions $\overleftarrow{R} : Y \to \mathcal{P}(X)$



Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$



Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$

$$\begin{array}{c} R \\ \Big\downarrow{\scriptstyle \langle x, y\rangle \mapsto \langle x, y\rangle} \\ X \times Y \end{array}$$

Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$

$$
\begin{tikzcd}
 & R \arrow[dd, "\langle x, y \rangle \mapsto \langle x, y \rangle"] \\
X & \\
 & X \times Y \arrow[ul, "\mathrm{proj}_1"]
\end{tikzcd}
$$

Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$

$$\begin{tikzcd}
& R \arrow[dl, "x"'] \arrow[dd, "\langle x, y\rangle \mapsto \langle x, y\rangle" description] \\
X & \\
& X \times Y \arrow[ul, "\mathrm{proj}_1"]
\end{tikzcd}$$

Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$

$$
\begin{tikzcd}
 & R \arrow[dl, "x"'] \arrow[dd, "\langle x, y\rangle \mapsto \langle x, y\rangle"] & \\
X & & Y \\
 & X \times Y \arrow[ul, "\mathrm{proj}_1"] \arrow[ur, "\mathrm{proj}_2"'] &
\end{tikzcd}
$$

Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$

$$
\begin{tikzcd}
& R \arrow[dl, "x"'] \arrow[dr, "y"] \arrow[dd, "\langle x, y\rangle \mapsto \langle x, y\rangle" description] & \\
X & & Y \\
& X \times Y \arrow[ul, "\mathrm{proj}_1"] \arrow[ur, "\mathrm{proj}_2"'] &
\end{tikzcd}
$$

Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$

$$\begin{tikzcd} & R \arrow[dl, "x"'] \arrow[dr, "y"] & \\ X & & Y \end{tikzcd}$$

A diagram $X \leftarrow R \rightarrow Y$ is called a *span*.

Functions, symbolically

$$y = \sqrt{2^{x^2+3}}$$

Functions, symbolically

$$y = \sqrt{2^{3^2+3}}$$

$X = \mathbb{R}$

3

Functions, symbolically

$$y = \sqrt{2^{9+3}}$$

$$\begin{array}{rcl} X = \mathbb{R} & \xrightarrow{\text{sqr}} & \mathbb{R} \\ 3 & \longmapsto & 9 \end{array}$$

Functions, symbolically

$$y = \sqrt{2^{12}}$$

$$X = \mathbb{R} \xrightarrow{\text{sqr}} \mathbb{R} \xrightarrow{+3} \mathbb{R}$$

$$3 \longmapsto 9 \longmapsto 12$$

Functions, symbolically

$$y = \sqrt{4096}$$

$$X = \mathbb{R} \xrightarrow{\text{sqr}} \mathbb{R} \xrightarrow{+3} \mathbb{R} \xrightarrow{\text{exp2}} \mathbb{R}$$

$$3 \longmapsto 9 \longmapsto 12 \longmapsto 4096$$

Functions, symbolically

$$y = 64$$

$$X = \mathbb{R} \xrightarrow{\text{sqr}} \mathbb{R} \xrightarrow{+3} \mathbb{R} \xrightarrow{\text{exp2}} \mathbb{R} \xrightarrow{\text{sqrt}} \mathbb{R} = Y$$

$$3 \longmapsto 9 \longmapsto 12 \longmapsto 4096 \longmapsto 64$$

Functions, symbolically

$$y = \sqrt{2^{x^2+3}}$$

$$X = \mathbb{R} \xrightarrow{\text{sqr}} \mathbb{R} \xrightarrow{+3} \mathbb{R} \xrightarrow{\text{exp2}} \mathbb{R} \xrightarrow{\text{sqrt}} \mathbb{R} = Y$$

$$3 \longmapsto 9 \longmapsto 12 \longmapsto 4096 \longmapsto 64$$

Functions, symbolically

$$y = \sqrt{2^{x^2+3}}$$

$$X = \mathbb{R} \xrightarrow{\text{sqr}} \mathbb{R} \xrightarrow{+3} \mathbb{R} \xrightarrow{\text{exp2}} \mathbb{R} \xrightarrow{\text{sqrt}} \mathbb{R} = Y$$

$$3 \longmapsto 9 \longmapsto 12 \longmapsto 4096 \longmapsto 64$$

Functions, symbolically

$$y = \sqrt{2^{x^2+3}}$$

$$X = \mathbb{R} \xrightarrow{\text{sqr}} \mathbb{R} \xrightarrow{+3} \mathbb{R} \xrightarrow{\text{exp2}} \mathbb{R}^{\geq 0} \xrightarrow{\text{sqrt}} \mathbb{R} = Y$$

$$3 \longmapsto 9 \longmapsto 12 \longmapsto 4096 \longmapsto 64$$

Composition, symbolically

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "f \cdot g"] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$$\underbrace{y = f(x) \qquad\qquad z = g(y)}_{z = g(f(x))}$$

Alternate notation: $f \cdot g = g \circ f$

Composition, symbolically

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "f\cdot g"] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

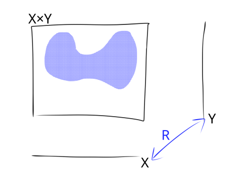
$$\underbrace{y = f(x) \qquad\qquad z = g(y)}_{z = g(f(x))}$$

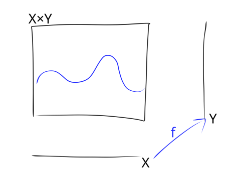
$$y = \mathtt{sqrt(exp2(plus3(sqr(x))))}$$

Alternate notation: $f \cdot g = g \circ f$

Questions?

	Relation	Function
Notation	$R : X \longleftrightarrow Y$	$f : X \rightarrow Y$
	$x \leftrightarrow y$	$x \mapsto y$
Connectivity	**Any-Any**	**Any-One**
Composes	✓	✓
Reverses	✓	X





Categories

$\mathbb{S}$et **&** $\mathbb{R}$el

The category $\mathbb{S}$et

Objects	Sets X, Y, Z,. . .	
Arrows	Functions $f : X \to Y$, $g : Y \to Z$,. . .	
Identities	$id(x) := x$	$X \to X$
Composition	$(f \cdot g)(x) := g(f(x))$	$X \to Z$
Unit	$id \cdot f(x) = f(id(x)) = f(x)$	
	$f \cdot id(x) = id(f(x)) = f(x)$	
Associativity	. . .	

The category $\mathbb{S}\mathrm{et}$

Objects	Sets $X, Y, Z, \ldots$	
Arrows	Functions $f : X \to Y$, $g : Y \to Z, \ldots$	
Identities	$id(x) := x$	$X \to X$
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	$f \cdot id(x) = id(f(x)) = f(x)$	
Associativity	$\ldots$	

The category $\mathbb{S}\mathsf{et}$

Objects	Sets X, Y, Z,...	
Arrows	Functions $f : X \to Y$, $g : Y \to Z$,...	
Identities	$id(x) := x$	$X \to X$
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Unit	$id \cdot f(x) = f(id(x)) = f(x)$	
	$f \cdot id(x) = id(f(x)) = f(x)$	
Associativity	...	

The category $\mathbb{S}\mathrm{et}$

Objects	Sets $X, Y, Z, \ldots$	
Arrows	Functions $f : X \to Y$, $g : Y \to Z, \ldots$	
Identities	$\mathsf{id}(x) := x$	$X \to X$
Composition	$(f \cdot g)(x) := g(f(x))$	$X \to Z$
Unit	$\mathsf{id} \cdot f(x) = f(\mathsf{id}(x)) = f(x)$	
	$f \cdot \mathsf{id}(x) = \mathsf{id}(f(x)) = f(x)$	
Associativity	$\ldots$	

The category $\mathbb{S}\mathrm{et}$

Objects	Sets X, Y, Z,...	
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	$f \cdot id(x) = id(f(x)) = f(x)$	
Associativity	...	

The category $\mathbb{S}\mathrm{et}$

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Associativity	...	

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Associativity	...	

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Associativity	...	

The category $\mathbb{S}\mathrm{et}$

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Associativity	$\ldots$	

The category $\mathbb{S}\mathrm{et}$

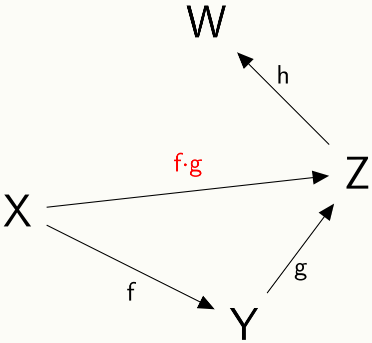
Objects	Sets X, Y, Z,...	
Arrows	Functions $f : X \to Y$, $g : Y \to Z$,...	
Identities	$\mathrm{id}(x) := x$	$X \to X$
Composition	$(f \cdot g)(x) := g(f(x))$	$X \to Z$
Unit	$\mathrm{id} \cdot f(x) = f(\mathrm{id}(x)) = f(x)$	
	$f \cdot \mathrm{id}(x) = \mathrm{id}(f(x)) = f(x)$	
Associativity	...	

Associativity

$$
\begin{tikzcd}
& W & \\
X \arrow[dr, "f"'] & & Z \arrow[ul, "h"'] \\
& Y \arrow[ur, "g"'] &
\end{tikzcd}
$$

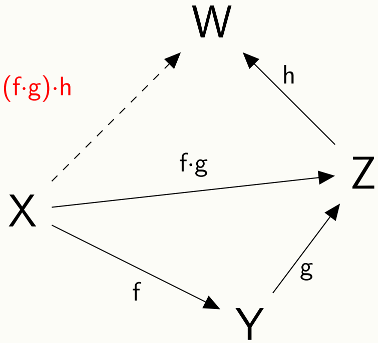
$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity



$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity



$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity

$$
\begin{tikzcd}
 & W & \\
X \arrow[rr, "f \cdot g"] \arrow[dr, "f"'] & & Z \arrow[ul, "h"'] \\
 & Y \arrow[ur, "g"'] \arrow[uu, "g \cdot h"] &
\end{tikzcd}
$$

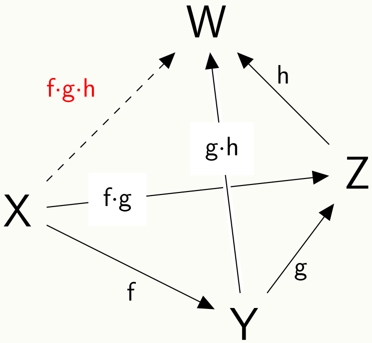
$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity

$$
\begin{tikzcd}
 & W & \\
X \arrow[ur, dashed, "f\cdot(g\cdot h)"] \arrow[rr, "f\cdot g" description] \arrow[dr, "f"'] & & Z \arrow[ul, "h"'] \\
 & Y \arrow[uu, "g\cdot h" description] \arrow[ur, "g"'] &
\end{tikzcd}
$$

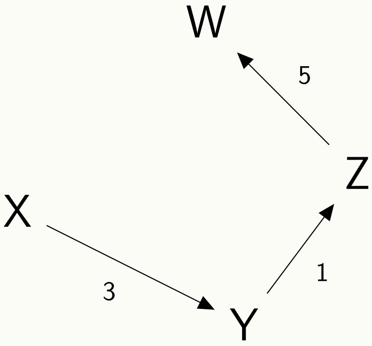
$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity



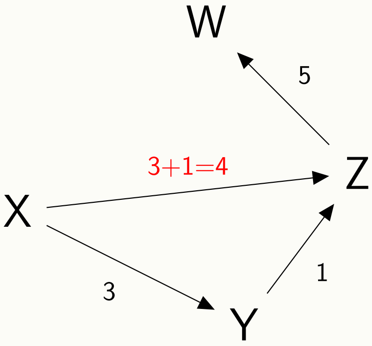
$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity



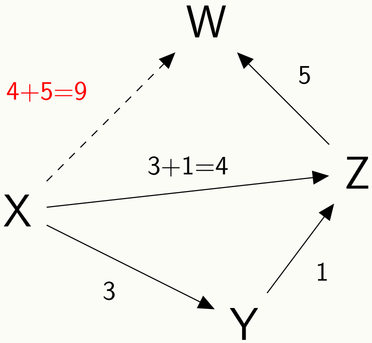
$$f \cdot g := f + g$$

Associativity



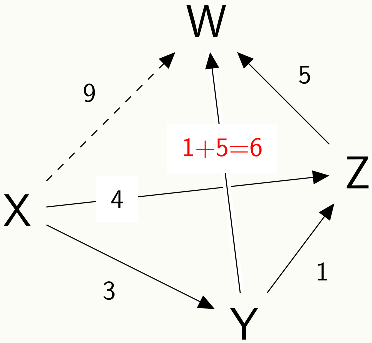
$$f \cdot g := f + g$$

Associativity



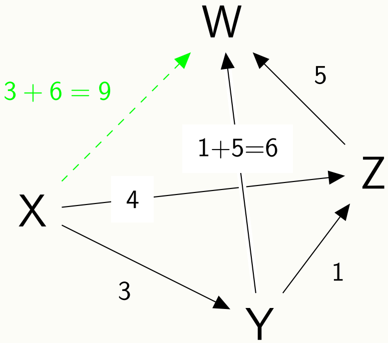
$$f \cdot g := f + g$$

Associativity



$$f \cdot g := f + g$$

Associativity



$$f \cdot g := f + g$$

(Non)Associativity

$$\begin{tikzcd}
 & & W & \\
 & & & Z \arrow[ul, "5"'] \\
X \arrow[drr, "3"'] & & & \\
 & & Y \arrow[uur, "1"'] &
\end{tikzcd}$$

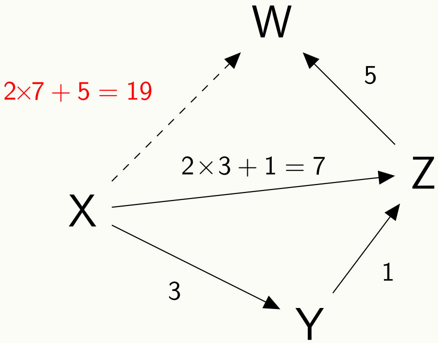
$$f \cdot g := 2 \times f + g$$

(Non)Associativity

$$\begin{tikzcd}
& W & \\
X \arrow[rr, "\color{red}{2\times 3+1=7}"] \arrow[dr, "3"'] & & Z \arrow[ul, "5"'] \\
& Y \arrow[ur, "1"'] &
\end{tikzcd}$$

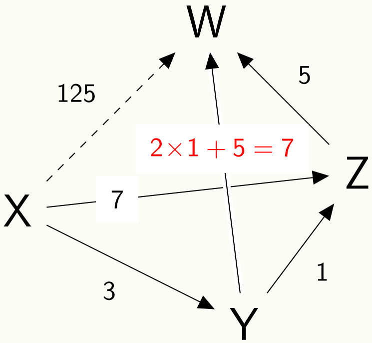
$$f \cdot g := 2\times f + g$$

(Non)Associativity



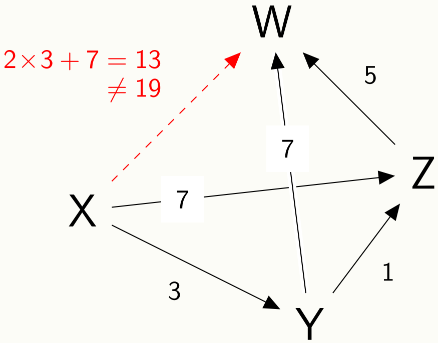
$$f \cdot g := 2\times f + g$$

(Non)Associativity



$$f \cdot g := 2 \times f + g$$

(Non)Associativity



$$f \cdot g := 2 \times f + g$$

Associativity in $\mathbb{S}$et

$$
\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rrr, bend left, "f{\cdot}g{\cdot}h"] & Y \arrow[r, "g"'] & Z \arrow[r, "g"'] & W
\end{tikzcd}
$$

$$
\begin{aligned}
w &= [(f \cdot g) \cdot h](x) \\
&= h([f \cdot g](x)) \\
&= h(g(f(x))) \\
&= [g \cdot h](f(x)) \\
&= [f \cdot (g \cdot h)](x) = w'
\end{aligned}
$$

Associativity in $\mathbb{S}$et

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rrr, bend left, "f \cdot g \cdot h"] & Y \arrow[r, "g"'] & Z \arrow[r, "g"'] & W
\end{tikzcd}$$

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w &= [(f \cdot g) \cdot h](x) \\
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Associativity in $\mathbb{S}$et

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\begin{tikzcd}
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$$
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Associativity in $\mathbb{S}\mathrm{et}$

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rrr, bend left, "f{\cdot}g{\cdot}h"] & Y \arrow[r, "g"'] & Z \arrow[r, "g"'] & W
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Associativity in $\mathbb{S}$et

$$
\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rrr, bend left, "f{\cdot}g{\cdot}h"] & Y \arrow[r, "g"'] & Z \arrow[r, "g"'] & W
\end{tikzcd}
$$

$$
\begin{aligned}
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&= h([f \cdot g](x)) \\
&= h(g(f(x))) \\
&= [g \cdot h](f(x)) \\
&= [f \cdot (g \cdot h)](x) = w'
\end{aligned}
$$

Categories

Definition: A *category* is

- A collection of (directed) relationships $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$
- closed under identities $X \xrightarrow{id} X$ and composition $X \xrightarrow{f \cdot g} Z$
- satisfying unit and associativity laws:

$$\mathrm{id} \cdot f = f = f \cdot \mathrm{id}$$

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Notation: We use a special font for categories $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{S}\mathrm{et}$, $\mathbb{R}\mathrm{el}$.

The category $\mathbb{R}\mathrm{el}$

Objects	Sets $X, Y, Z, \ldots$	
Arrows	Relations $R \subseteq X \times Y$, $S \subseteq Y \times Z, \ldots$	
Identities	$\{\langle x, x' \rangle \mid x = x'\}$	$\subseteq X \times X$
Composition	$\big\{\langle x, z \rangle \mid \exists y. R(x, y) \;\&\; S(y, z)\big\}$	$\subseteq X \times Z$
Unit	$\langle x, y \rangle \in \mathrm{id} \cdot R$	$\subseteq X \times Y$
	$\Leftrightarrow \exists x'. x = x' \;\&\; \langle x', y \rangle \in R$	
	$\Leftrightarrow \langle x, y \rangle \in R$	
Associativity	$\ldots$	$\subseteq X \times W$

The category $\mathbb{R}$el

Objects	Sets X, Y, Z,...	
Arrows	Relations $R \subseteq X \times Y$, $S \subseteq Y \times Z, \ldots$	
Identities	$\{\langle x, x' \rangle \mid x = x'\}$	$\subseteq X \times X$
Composition	$\left\{\langle x, z \rangle \mid \exists y. R(x, y)\ \&\ S(y, z)\right\}$	$\subseteq X \times Z$
Unit	$\langle x, y \rangle \in id \cdot R$	$\subseteq X \times Y$
	$\Leftrightarrow \exists x'. x = x'\ \&\ \langle x', y \rangle \in R$	
	$\Leftrightarrow \langle x, y \rangle \in R$	
Associativity	$\ldots$	$\subseteq X \times W$

The category $\mathbb{R}$el

Objects	Sets $X, Y, Z, \ldots$	
Arrows	Relations $R \subseteq X \times Y$, $S \subseteq Y \times Z, \ldots$	
Identities	$\{\langle x, x' \rangle \mid x = x'\}$	$\subseteq X \times X$
Composition	$\left\{\langle x, z\rangle \mid \exists y. R(x, y) \;\&\; S(y, z)\right\}$	$\subseteq X \times Z$
Unit	$\langle x, y\rangle \in \mathrm{id} \cdot R$	$\subseteq X \times Y$
	$\Leftrightarrow \exists x'. x = x' \;\&\; \langle x', y\rangle \in R$	
	$\Leftrightarrow \langle x, y\rangle \in R$	
Associativity	$\ldots$	$\subseteq X \times W$

The category $\mathbb{R}\mathsf{el}$

Objects	Sets $X, Y, Z, \ldots$	
Arrows	Relations $R \subseteq X \times Y$, $S \subseteq Y \times Z, \ldots$	
Identities	$\{\langle x, x'\rangle \mid x = x'\}$	$\subseteq X \times X$
Composition	$\{\langle x, z\rangle \mid \exists y. R(x, y)\ \&\ S(y, z)\}$	$\subseteq X \times Z$

Unit $\quad \langle x, y\rangle \in \mathrm{id} \cdot R \qquad \subseteq X \times Y$

$$\Leftrightarrow \exists x'. x = x'\ \&\ \langle x', y\rangle \in R$$

$$\Leftrightarrow \langle x, y\rangle \in R$$

Associativity $\quad \ldots \qquad \subseteq X \times W$

The category $\mathbb{R}$el

Objects	Sets $X, Y, Z, \ldots$	
Arrows	Relations $R \subseteq X \times Y$, $S \subseteq Y \times Z, \ldots$	
Identities	$\{\langle x, x' \rangle \mid x = x'\}$	$\subseteq X \times X$
Composition	$\big\{\langle x, z \rangle \mid \exists y. R(x, y) \;\&\; S(y, z)\big\}$	$\subseteq X \times Z$
Unit	$\langle x, y \rangle \in \mathrm{id} \cdot R$	$\subseteq X \times Y$
	$\Leftrightarrow \exists x'. x = x' \;\&\; \langle x', y \rangle \in R$	
	$\Leftrightarrow \langle x, y \rangle \in R$	
Associativity	$\ldots$	$\subseteq X \times W$

The category $\mathbb{R}\mathsf{el}$

Objects	Sets $X, Y, Z, \ldots$	
Arrows	Relations $R \subseteq X \times Y$, $S \subseteq Y \times Z, \ldots$	
Identities	$\{\langle x, x' \rangle \mid x = x'\}$	$\subseteq X \times X$
Composition	$\big\{\langle x, z \rangle \mid \exists y. R(x, y) \;\&\; S(y, z)\big\}$	$\subseteq X \times Z$
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	$\Leftrightarrow \exists x'. x = x' \;\&\; \langle x', y \rangle \in R$	
	$\Leftrightarrow \langle x, y \rangle \in R$	
Associativity	$\ldots$	$\subseteq X \times W$

The category $\mathbb{R}$el

Objects	Sets $X, Y, Z, \ldots$	
Arrows	Relations $R \subseteq X \times Y$, $S \subseteq Y \times Z, \ldots$	
Identities	$\{\langle x, x' \rangle \mid x = x'\}$	$\subseteq X \times X$
Composition	$\{\langle x, z \rangle \mid \exists y. R(x, y) \;\&\; S(y, z)\}$	$\subseteq X \times Z$
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	$\Leftrightarrow \langle x, y \rangle \in R$	
Associativity	$\ldots$	$\subseteq X \times W$

The category $\mathbb{R}$el

Objects	Sets X, Y, Z,...	
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	$\Leftrightarrow \exists x'. x = x' \;\&\; \langle x', y \rangle \in R$	
	$\Leftrightarrow \langle x, y \rangle \in R$	
Associativity	$\ldots$	$\subseteq X \times W$

Associativity in $\mathbb{R}\mathrm{el}$

$$\begin{tikzcd}
X \arrow[r, leftrightarrow, "R"'] \arrow[rrr, bend left, leftrightarrow, "R\cdot S\cdot T"] & Y \arrow[r, leftrightarrow, "S"'] & Z \arrow[r, leftrightarrow, "T"'] & W
\end{tikzcd}$$

$$\begin{aligned}
[(R\cdot S)\cdot T](x,w) &\iff \exists z.[R\cdot S](x,z)\ \&\ T(z,w)\\
&\iff \exists z.\Big(\exists y.R(x,y)\ \&\ S(y,z)\Big)\ \&\ T(z,w)\\
&\overset{y\notin T}{\iff} \exists y,z.\Big(\exists y.R(x,y)\ \&\ S(y,z)\ \&\ T(z,w)\Big)\\
&\overset{z\notin R}{\iff} \exists y.R(x,y)\ \&\ (\exists z.S(y,z)\ \&\ T(z,w))\\
[R\cdot(S\cdot T)](x,w) &\iff \exists y.R(x,y)\ \&\ [S\cdot T](y,w)
\end{aligned}$$

†-Categories

Definition: A †-*category* is a category together with a reversal operation

$$\begin{array}{ccc} X & \xrightarrow{\mathsf{r}} & Y \\ & \rightsquigarrow & \\ Y & \xrightarrow{\mathsf{r}^\dagger} & X \end{array}$$

satisfying

$$\begin{aligned} \mathsf{r}^{\dagger\dagger} &= \mathsf{r} \\ \left.\begin{aligned} \mathsf{id}^\dagger &= \mathsf{id} \\ (\mathsf{r}\cdot\mathsf{s})^\dagger &= \mathsf{s}^\dagger\cdot\mathsf{r}^\dagger \end{aligned}\right\} &\quad \text{functoriality} \end{aligned}$$

Questions

Diagrams

Diagrams

A diagram *commutes* if parallel paths (same src/tgt) are equal.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] & Y \arrow[d, "g"] \\ Y' \arrow[r, "g'"'] & Z \end{tikzcd} \qquad f \cdot g \;=\; f' \cdot g'$$

Diagrams

A diagram *commutes* if parallel paths (same src/tgt) are equal.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, "f'"'] \arrow[dr, "h"] & & Y \arrow[dd, "g"] \\
& W \arrow[ur, "i"] \arrow[dl, "j"] \arrow[dr, "k"] & \\
Y' \arrow[rr, "g'"'] & & Z
\end{tikzcd}
$$

9 equations

Diagrams

A diagram *commutes* if parallel paths (same src/tgt) are equal.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, "f'"'] \arrow[dr, "h" description] & & Y \arrow[dd, "g"] \\
& W \arrow[ur, "i" description] \arrow[dl, "j" description] \arrow[dr, "k" description] & \\
Y' \arrow[rr, "g'"'] & & Z
\end{tikzcd}
$$

$$
\begin{aligned}
f \cdot g &= h \cdot i \cdot g \\
&= h \cdot k \\
&= h \cdot j \cdot g' \\
&= f' \cdot g'
\end{aligned}
$$

Diagrams

A diagram *commutes* if parallel paths (same src/tgt) are equal.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"] \arrow[dd, "f'"'] & & Y \arrow[dd, "g"] \\
& W \arrow[ur, "i"] \arrow[dl, "j"] \arrow[dr, "k"] & \\
Y' \arrow[rr, "g'"'] & & Z
\end{tikzcd}
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\begin{aligned}
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Diagrams

A diagram *commutes* if parallel paths (same src/tgt) are equal.

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\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"] \arrow[dd, "f'"'] & & Y \arrow[dd, "g"] \\
& W \arrow[ur, "i"] \arrow[dl, "j"] \arrow[dr, "k"] & \\
Y' \arrow[rr, "g'"'] & & Z
\end{tikzcd}
$$

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f \cdot g &= h \cdot i \cdot g \\
&= h \cdot k \\
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\end{aligned}
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Diagrams

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\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"] \arrow[dd, "f'"'] & & Y \arrow[dd, "g"] \\
& W \arrow[ur, "i"] \arrow[dl, "j"] \arrow[dr, "k"] & \\
Y' \arrow[rr, "g'"'] & & Z
\end{tikzcd}
$$

$$
\begin{aligned}
f \cdot g &= h \cdot i \cdot g \\
&= h \cdot k \\
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&= f' \cdot g'
\end{aligned}
$$

Diagrams

A diagram *commutes* if parallel paths (same src/tgt) are equal.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"] \arrow[dd, "f'"'] & & Y \arrow[dd, "g"] \\
& W \arrow[ur, "i"] \arrow[dl, "j"] \arrow[dr, "k"] & \\
Y' \arrow[rr, "g'"'] & & Z
\end{tikzcd}$$

$$\begin{aligned}
f \cdot g &= h \cdot i \cdot g \\
&= h \cdot k \\
&= h \cdot j \cdot g' \\
&= f' \cdot g'
\end{aligned}$$

Diagrams

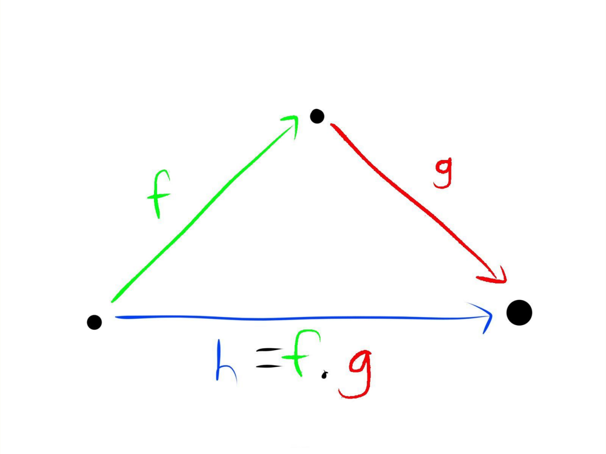
A diagram *commutes* if parallel paths (same src/tgt) are equal.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, "f'"'] \arrow[dr, "h"] & & Y \arrow[dd, "g"] \\
& W \arrow[ur, "i"] \arrow[dl, "j"] \arrow[dr, "k"] & \\
Y' \arrow[rr, "g'"'] & & Z
\end{tikzcd}$$

$$\begin{aligned}
f \cdot g &= h \cdot i \cdot g \\
&= h \cdot k \\
&= h \cdot j \cdot g' \\
&= f' \cdot g'
\end{aligned}$$

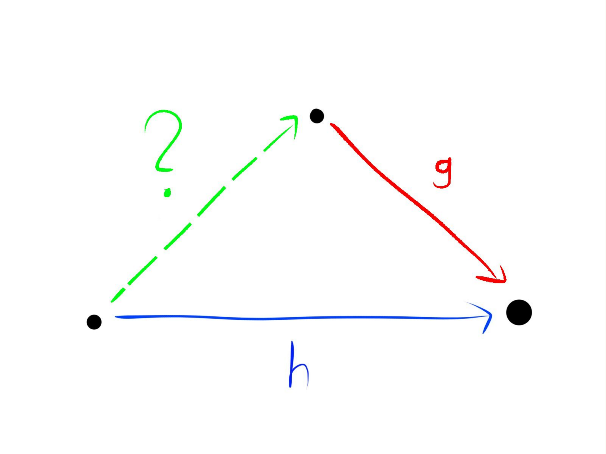
Lifting & Extension

Composition is a forward mapping $\langle f, g \rangle \mapsto h$.



Lifting & Extension

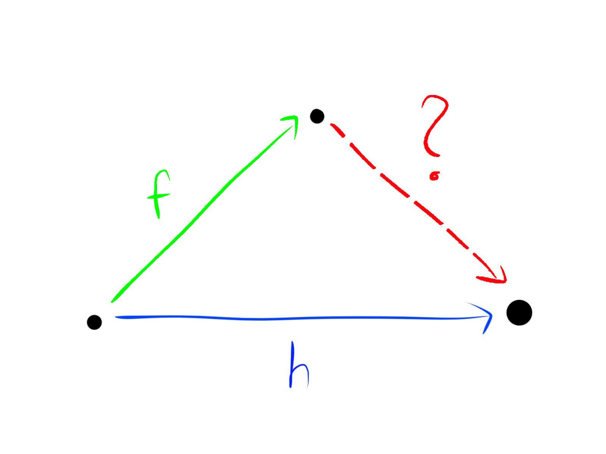
There are *two* inverse problems:



Lifting problem: given h & g, find f.

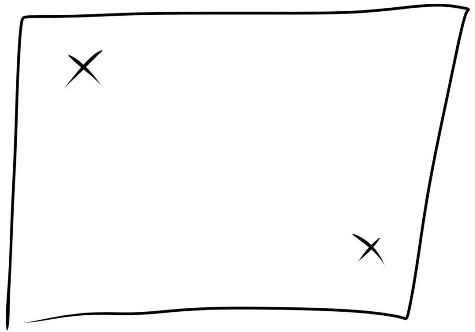
Lifting & Extension

There are *two* inverse problems:



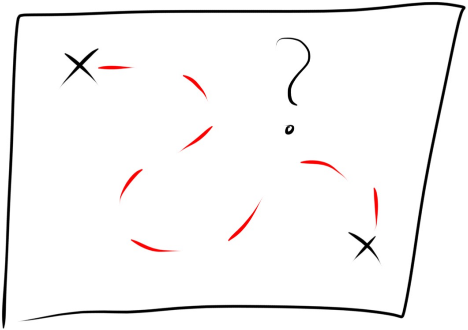
Extension problem: given h & f, find g.

Lifting & Extension



Driver's problem: given h & f, find g.

Lifting & Extension



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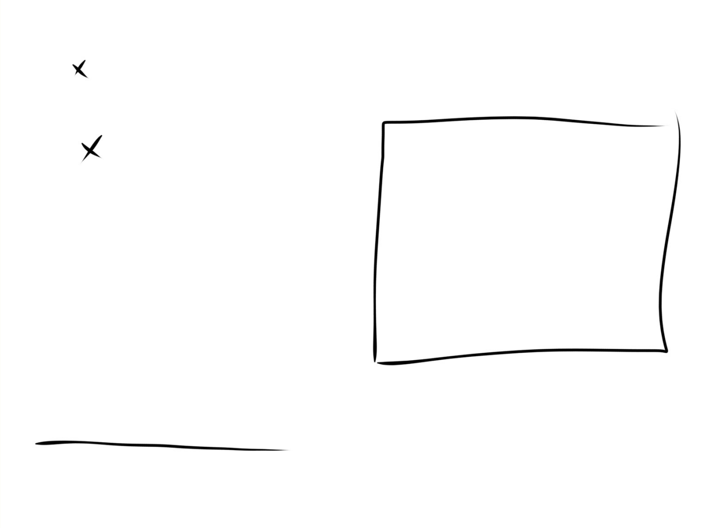

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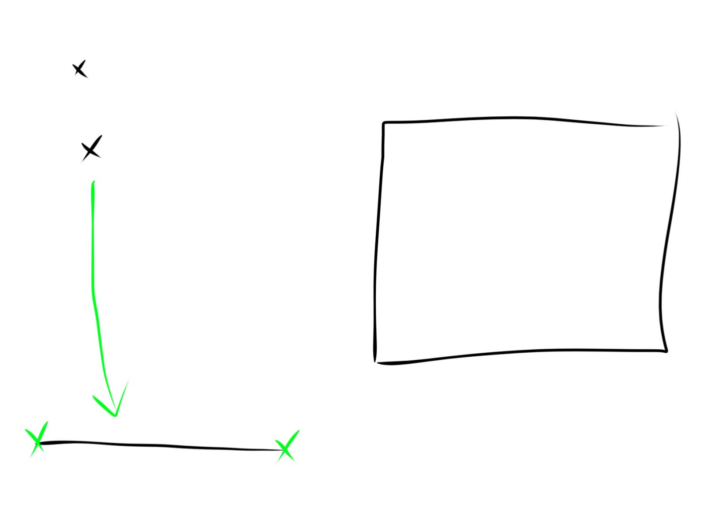

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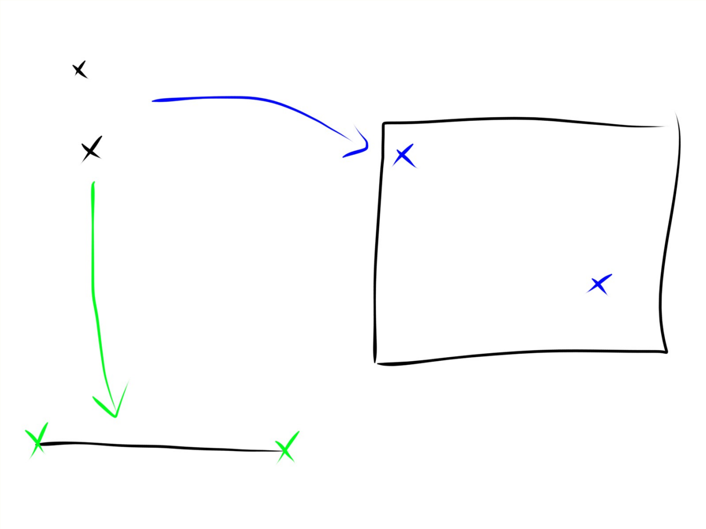

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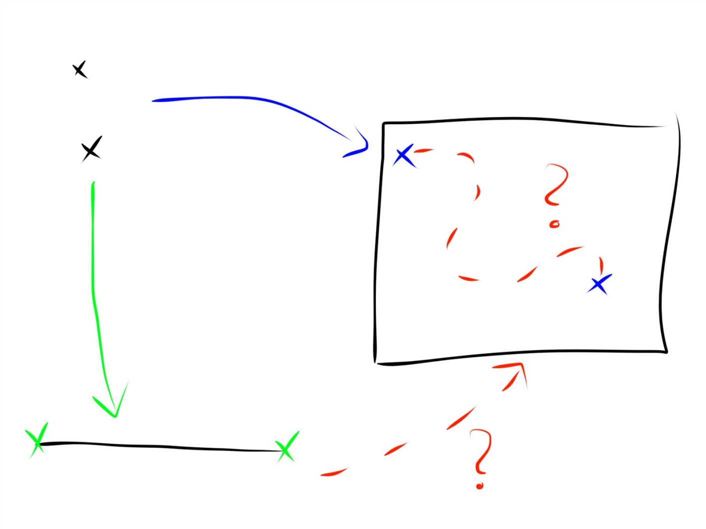

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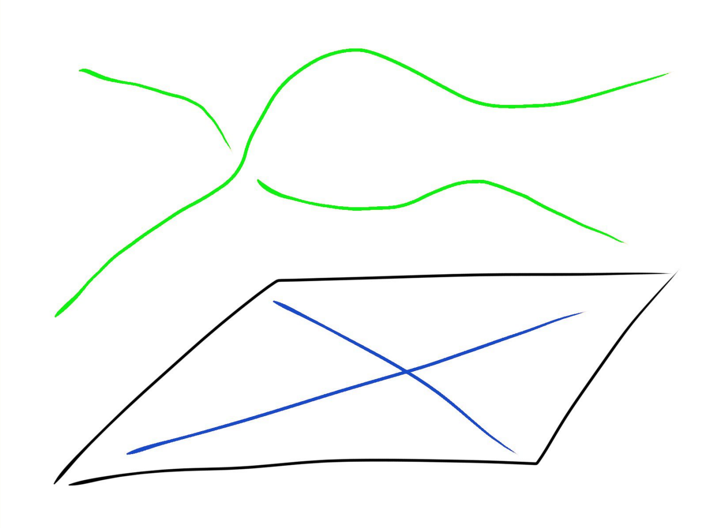
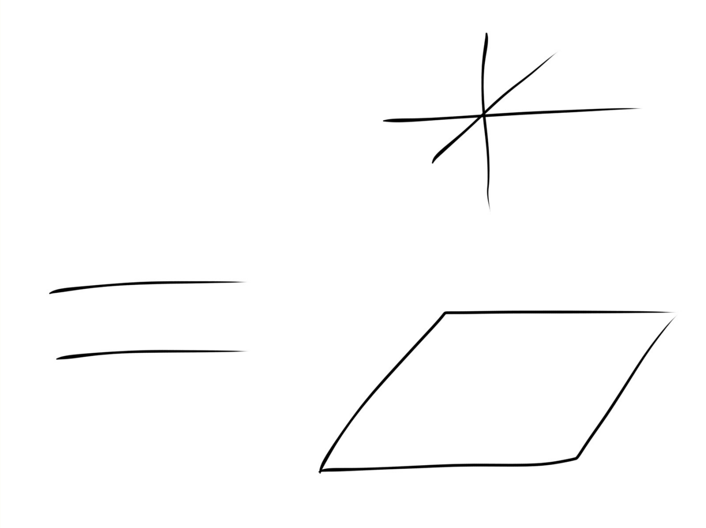


Pilot's problem: given h & g, find f.

Lifting & Extension



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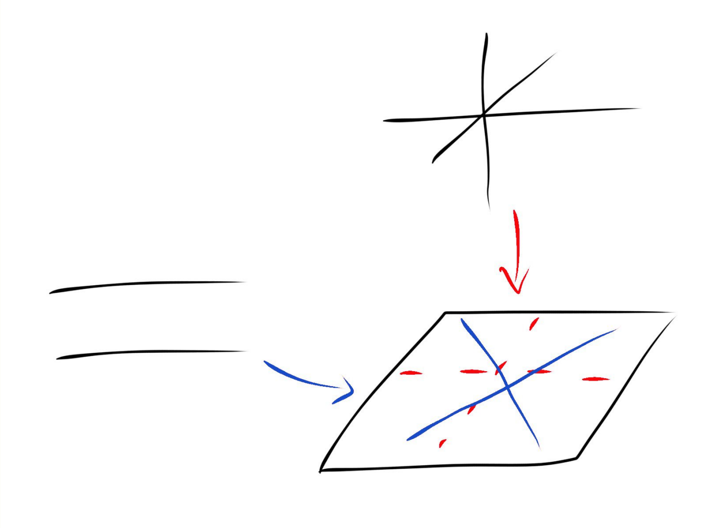

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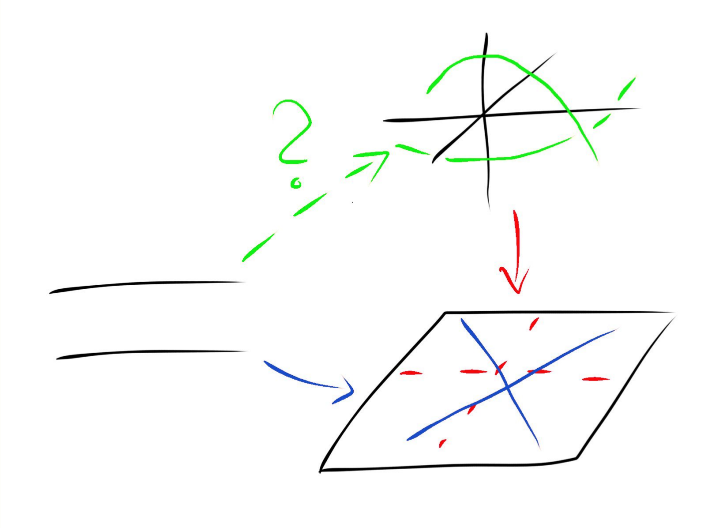

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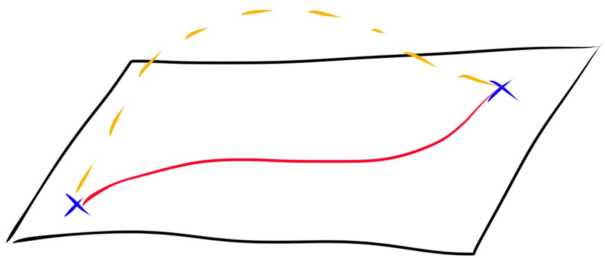
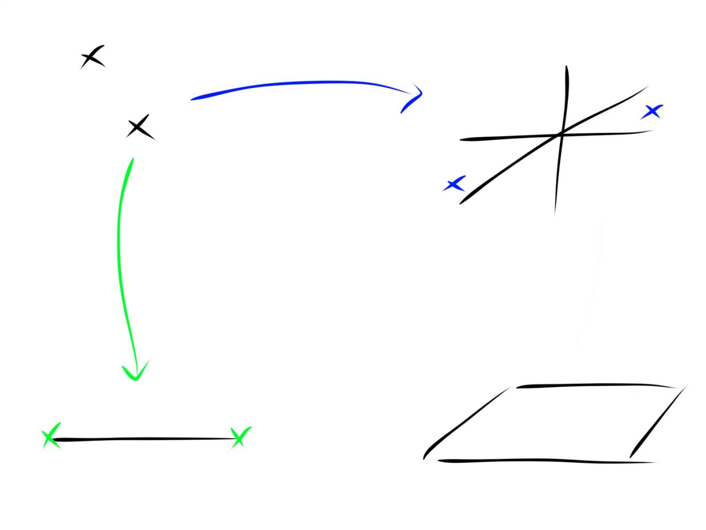


Pilot's problem (now with landing!): fill the diagonal.

Lifting & Extension



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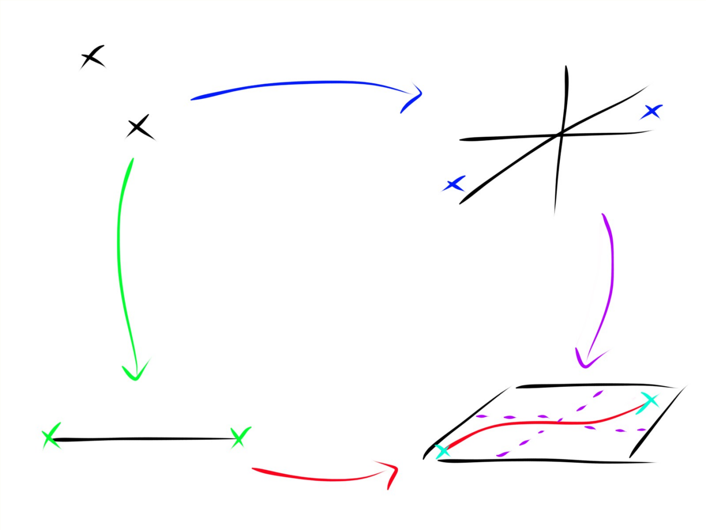

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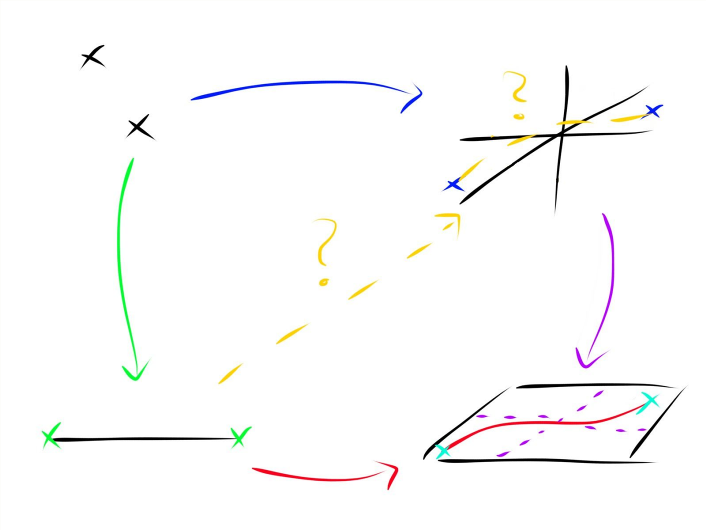
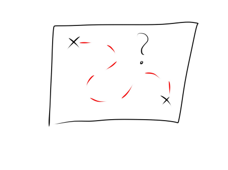
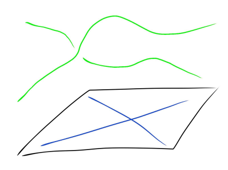
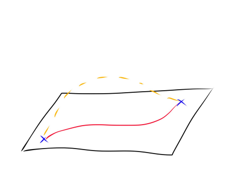


Pilot's problem (now with landing!): fill the diagonal.

Questions?

	$\mathbb{S}\mathrm{et}$	$\mathbb{R}\mathrm{el}$
Notation	$R : X \longleftrightarrow Y$	$f : X \to Y$
	$x \leftrightarrow y$	$x \mapsto y$
Category	$\checkmark$	$\checkmark$
$\dagger$-Category	$\checkmark$	X

Extension Problem	**Lifting Problem**	**Joint Constraint**
		

Structure

Structure in categories

Structure	Notation	Generalizes
Isomorphism	$X \cong Y$	Permutation, symmetry
Products	$X \times Y$	Pairs, tuples
Coproducts	$X + Y$	Conditionals (if/then/else)
Subobject	$S \rightarrowtail X$	Subset, formula
Exponential	Y^X	Function space $Y \to X$
Pullback	$Y \underset{X}{\times} Z$	Intersection, substitution
Pushout	$Y \underset{X}{+} Z$	Union
Span	$X \leftarrow R \to Y$	Relation
Cospan	$X \to C \leftarrow Y$	Connectivity diagram
NNO	$1 \xrightarrow{0} N \xrightarrow{s} N$	Nat. numbers
Subobject classifier	Ω	Truth values

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Isomorphism

Isomorphism

Definition: An *isomorphism* is an arrow $i : X \to Y$ that has an *inverse* j satisfying two equations

$$\begin{tikzcd} X \arrow[r, bend left, "i"] & Y \arrow[l, dashed, bend left, "j"] \end{tikzcd} \qquad \begin{aligned} i \cdot j &= id_X \\ j \cdot i &= id_Y \end{aligned}$$

We say X and Y are *isomorphic*: $X \cong Y$.

Identity of Indiscernibles

Two objects bearing the same properties are equal.

- Leibniz, $\sim$1715

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic.

- Yoneda, 1954

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic.

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Y

Y′

Identity of Indiscernibles

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$$Y$$

$$\forall X$$

$$Y'$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic.

- Yoneda, 1954

$$\begin{tikzcd} & Y \\ \forall X \arrow[ur, "\forall f"] & \\ & Y' \end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic*.*

- Yoneda, 1954

$$\begin{tikzcd}
 & \mathsf{Y} \\
\forall\ \mathsf{X} \arrow[ur, "\forall\ \mathsf{f}"] \arrow[dr, "\exists\ \mathsf{f}'"'] & \\
 & \mathsf{Y}'
\end{tikzcd}$$

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 & Y'
\end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~*equal*~~ isomorphic*.*

- Yoneda, 1954

$$\begin{tikzcd}
 & Y \arrow[dr, "\forall\ g"] & \\
\forall\ X \arrow[ur, "\forall\ f"] \arrow[dr, "\exists\ f'"'] & & \forall\ Z \\
 & Y' \arrow[ur, "\exists\ g'"'] &
\end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are isomorphic ~~*equal*~~.

- Yoneda, 1954

$$
\begin{tikzcd}
& Y \arrow[dr, "\forall\ g"] & \\
\forall\ X \arrow[ur, "\forall\ f"] \arrow[dr, "\exists\ f'"'] \arrow[rr, phantom, "f \cdot g = f' \cdot g'"] & & \forall\ Z \\
& Y' \arrow[ur, "\exists\ g'"'] &
\end{tikzcd}
$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic .

- Yoneda, 1954

$$\begin{tikzcd}
 & Y \arrow[dr, "\mathrm{id}"] & \\
Y \arrow[ur, "\mathrm{id}"] & & Y \\
 & Y' &
\end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic *.*

- Yoneda, 1954

$$\begin{tikzcd}
& Y \arrow[dr, "\mathrm{id}"] & \\
Y \arrow[ur, "\mathrm{id}"] \arrow[dr, "i"'] & & Y \\
& Y' \arrow[ur, "j"'] &
\end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic .

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$i \cdot j = \mathrm{id}_Y$

$$
\begin{tikzcd}
& Y \arrow[dr, "\mathrm{id}"] & \\
Y \arrow[ur, "\mathrm{id}"] \arrow[dr, "i"'] \arrow[rr, phantom, "i \cdot j = \mathrm{id}"] & & Y \\
& Y' \arrow[ur, "j"'] &
\end{tikzcd}
$$

Identity of Indiscernibles

isomorphic

Two objects bearing the same properties are ~~equal~~ .

- Yoneda, 1954

$i \cdot j = id_Y$

$$
\begin{tikzcd}
 & Y & \\
Y' \arrow[r, "id"'] & Y' \arrow[r, "id"'] & Y'
\end{tikzcd}
$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic .

- Yoneda, 1954

$$i \cdot j = \mathrm{id}_Y$$

$$\begin{tikzcd}
 & Y \arrow[dr, "j'"] & \\
Y' \arrow[ur, "i'"] \arrow[dr, "\mathrm{id}"'] & & Y' \\
 & Y' \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic *.*

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$$i \cdot j = \mathrm{id}_Y$$

$$i' \cdot j' = \mathrm{id}_{Y'}$$

$$\begin{tikzcd}
 & Y \arrow[dr, "j'"] & \\
Y' \arrow[ur, "i'"] \arrow[dr, "\mathrm{id}"'] \arrow[rr, phantom, "i' \cdot j' = \mathrm{id}"] & & Y' \\
 & Y' \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic *.*

- Yoneda, 1954

$$i \cdot j = id_Y$$

$$i' \cdot j' = id_{Y'}$$

$$Y \xrightarrow{id} Y$$

$$Y' \xrightarrow{id} Y'$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic.

- Yoneda, 1954

$$i \cdot j = id_Y$$

$$i' \cdot j' = id_{Y'}$$

$$\begin{tikzcd}
 & Y \arrow[dr, "j'"] & \\
Y \arrow[ur, "id"] \arrow[dr, "i"'] & & Y' \\
 & Y' \arrow[ur, "id"'] &
\end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~*equal*~~ isomorphic .

- Yoneda, 1954

$$i \cdot j = id_Y$$

$$i' \cdot i = id_{Y'}$$

$$\begin{tikzcd}
 & Y \arrow[dr, "j'"] & \\
Y' \arrow[ur, "id"] \arrow[dr, "i"'] \arrow[rr, phantom, "i = j'"] & & Y \\
 & Y' \arrow[ur, "id"'] &
\end{tikzcd}$$

Identity of Indiscernibles

isomorphic

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$$\begin{tikzcd}
 & Y \arrow[dr, "\mathrm{id}"] & \\
Y' \arrow[ur, "i'"] \arrow[dr, "\mathrm{id}"'] \arrow[rr, phantom, "j = i'"] & & Y \\
 & Y' \arrow[ur, "j"'] &
\end{tikzcd}$$

Identity of Indiscernibles

Two objects bearing the same properties are ~~equal~~ isomorphic .

- Yoneda, 1954

$$\dfrac{\begin{gathered} i \cdot j = \mathrm{id}_Y \\ j \cdot i = \mathrm{id}_{Y'} \end{gathered}}{Y \cong Y'}$$

$$\begin{tikzcd}
 & Y \arrow[dr, "\mathrm{id}"] & \\
Y' \arrow[ur, "i'"] \arrow[dr, "\mathrm{id}"'] & & Y \\
 & Y' \arrow[ur, "j"'] &
\end{tikzcd}$$

Questions?

Products

Cartesian products

Definition: The *Cartesian product* of X and Y is a diagram $X \xleftarrow{\mathrm{proj}_1} P \xrightarrow{\mathrm{proj}_2} Y$ such that for any object T and any arrows $x : T \to X$ and $y : T \to Y$, there is a unique arrow $p : T \to P$ satisfying $p \cdot \mathrm{proj}_1 = x$ and $p \cdot \mathrm{proj}_2 = y$.

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X Y

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$$X \xleftarrow[\mathrm{proj}_1]{} P \xrightarrow[\mathrm{proj}_2]{} Y$$

Cartesian products

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$\forall\ T$

$$\begin{tikzcd} X & & P \arrow[ll, "proj_1"'] \arrow[rr, "proj_2"'] & & Y \end{tikzcd}$$

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$$\begin{tikzcd}
& & \forall\, T \arrow[dll, "\forall\, x"'] \arrow[drr, "\forall\, y"] & & \\
X & & P \arrow[ll, "\mathrm{proj}_1"] \arrow[rr, "\mathrm{proj}_2"'] & & Y
\end{tikzcd}$$

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$$\begin{tikzcd}
 & \forall\ T \arrow[dl, "\forall\ x"'] \arrow[d, dashed, "\exists!\ p"'] \arrow[dr, "\forall\ y"] & \\
X & P \arrow[l, "\mathrm{proj}_1"] \arrow[r, "\mathrm{proj}_2"'] & Y
\end{tikzcd}$$

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\end{tikzcd}$$

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 & \forall\, T \arrow[dl, "\forall\, x"'] \arrow[d, dashed, "\exists!\, p"'] \arrow[dr, "\forall\, y"] & \\
X & P \arrow[l, "\mathsf{proj}_1"] \arrow[r, "\mathsf{proj}_2"'] & Y
\end{tikzcd}$$

$$\underbrace{\mathrm{Hom}(T, X) \times \mathrm{Hom}(T, Y)}_{\text{In } \mathbb{S}\mathrm{et}} \cong \mathrm{Hom}(T, \underbrace{X \times Y}_{\text{In } \mathbb{C}})$$

Cartesian product

$$\frac{x_0 \in X \qquad\qquad y_0 \in Y}{\langle x_0, y_0 \rangle \in X \times Y}$$

Cartesian product

$$\frac{x_0 \in X \qquad\qquad y_0 \in Y}{\langle x_0, y_0 \rangle \in X \times Y}$$

$$\frac{\vec{x} : \{*\} \xrightarrow{* \mapsto x_0} X \qquad\qquad \vec{y} : \{*\} \xrightarrow{* \mapsto y_0} Y}{\vec{p} : \{*\} \xrightarrow[* \mapsto \langle x_0, y_0 \rangle]{} X \times Y}$$

Cartesian product

$$\frac{x_0 \in X \qquad\qquad y_0 \in Y}{\langle x_0, y_0 \rangle \in X \times Y}$$

$$\frac{\vec{x} : \{*\} \longrightarrow X \qquad\qquad \vec{y} : \{*\} \longrightarrow Y}{\vec{p} : \{*\} \longrightarrow X \times Y}$$

$$\begin{tikzcd}
\{*\} \arrow[r, "\vec{p}"] \arrow[dr, dashed, "\vec{x} = \vec{p} \cdot \mathrm{proj}_1"'] & X \times Y \arrow[d, "\mathrm{proj}_1"] \\
& X
\end{tikzcd}$$

Cartesian product

$$\frac{x_0 \in X \qquad\qquad y_0 \in Y}{\langle x_0, {\color{red} y_0} \rangle \in X \times Y}$$

$$\frac{\vec{x} : \{*\} \longrightarrow X \qquad\qquad \vec{y} : \{*\} \longrightarrow Y}{\vec{p} : \{*\} \longrightarrow X \times Y}$$

$$\begin{tikzcd}
\vec{p} : \{*\} \arrow[r] \arrow[dr, dashed, "\vec{y} = \vec{p} \cdot \mathrm{proj}_2"'] & X \times Y \arrow[d, red, "{\color{red}\mathrm{proj}_2}"'] \arrow[dr, "\mathrm{proj}_1"] & \\
 & {\color{red} Y} & X
\end{tikzcd}$$

Cartesian product

$$\frac{x(t) \in X \qquad\qquad y(t) \in Y}{\langle x(t), y(t) \rangle \in X \times Y}$$

$$\frac{\vec{x} : T \longrightarrow X \qquad\qquad \vec{y} : T \longrightarrow Y}{\vec{p} : T \longrightarrow X \times Y}$$

$$\begin{tikzcd}
\vec{p} : T \arrow[r] \arrow[dr, dashed, "\vec{y} = \vec{p} \cdot \mathrm{proj}_2"'] & X \times Y \arrow[d, "\mathrm{proj}_2"'] \arrow[dr, "\mathrm{proj}_1"] & \\
& Y & X
\end{tikzcd}$$

Test objects

In $\mathbb{S}$et, the singleton $1 = \{*\}$ “sees” everything.

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$$\begin{array}{ccc} \mathrm{Hom}(1, X) & \cong & \mathrm{Hom}(1, Y) \\ & \Downarrow & \\ X & \cong & Y \end{array}$$

Test objects

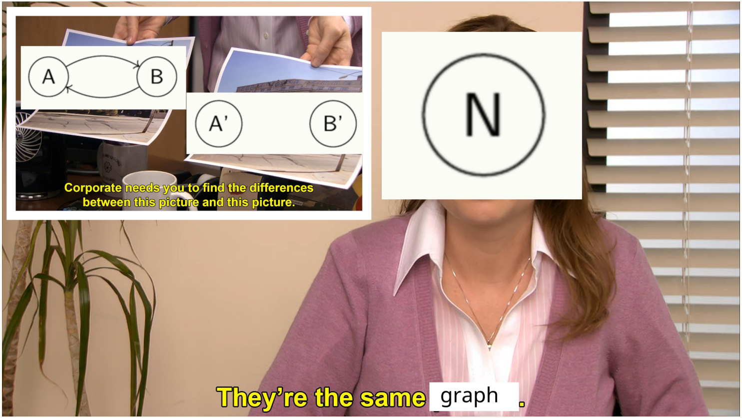
In $\mathbb{S}$et, the singleton $1 = \{*\}$ "sees" everything.

$$\begin{array}{ccc} X & \cong & \mathsf{Hom}(1, X) \\ x_0 & \longleftrightarrow & \vec{x} \end{array}$$

$$\begin{array}{rcl} \mathsf{Hom}(1, X \times Y) & \cong & X \times Y \\ & \cong & \mathsf{Hom}(1, X) \times \mathsf{Hom}(1, Y) \end{array}$$

Test objects

In $\mathbb{G}$raph, a singleton node can't "see" edges.



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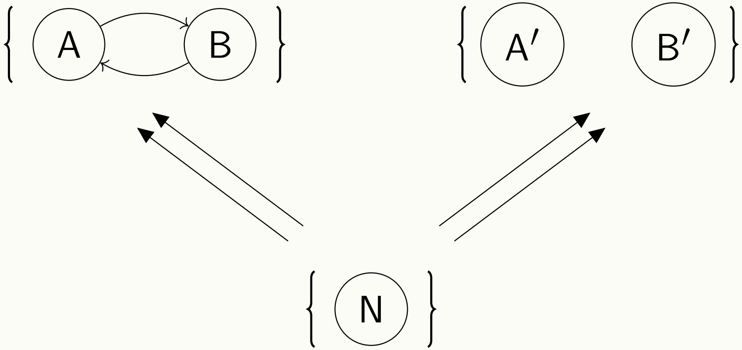

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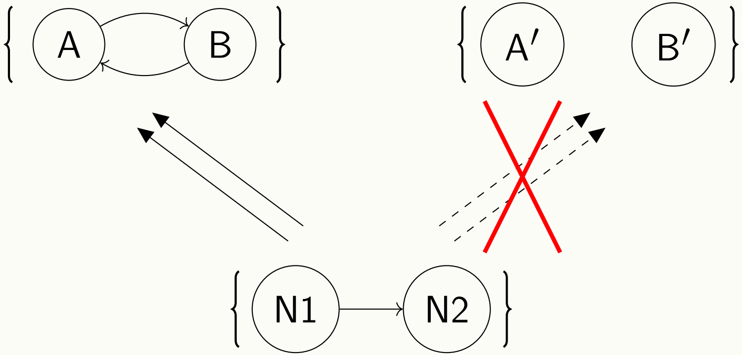


But an edge can.

Test objects

$\mathsf{P} = \mathsf{G} \times \mathsf{H}$ in $\mathbb{G}$raph if

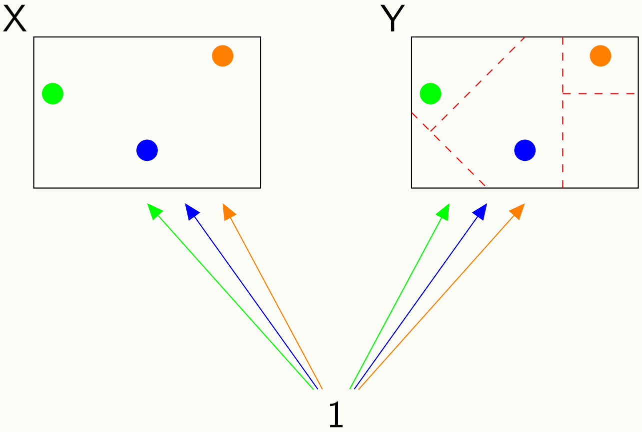
$$\frac{\mathsf{N} \longrightarrow \mathsf{G} \qquad\qquad \mathsf{N} \longrightarrow \mathsf{H}}{\mathsf{N} \longrightarrow \mathsf{P}}$$

and

$$\frac{\mathsf{E} \longrightarrow \mathsf{G} \qquad\qquad \mathsf{E} \longrightarrow \mathsf{H}}{\mathsf{E} \longrightarrow \mathsf{P}}$$

Test objects

In geometry, a point can't "see" continuity.



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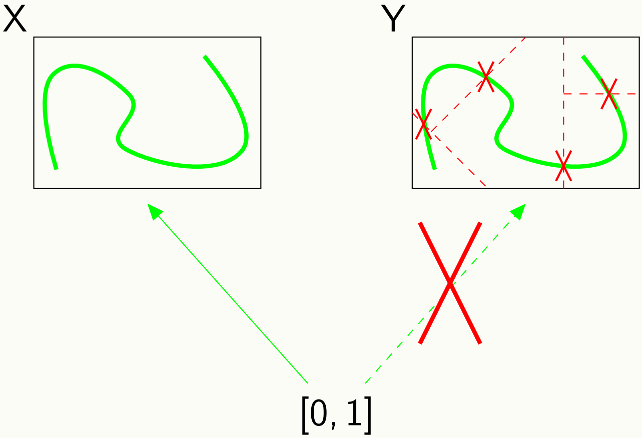
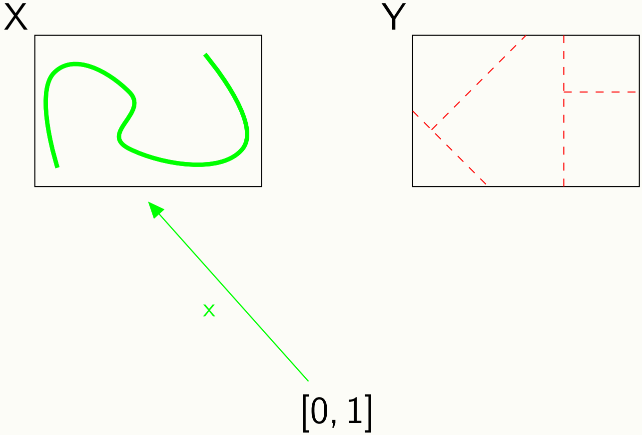


But a curve can.

Generalized elements

A curve $[0,1] \to X$ acts like an element of X.



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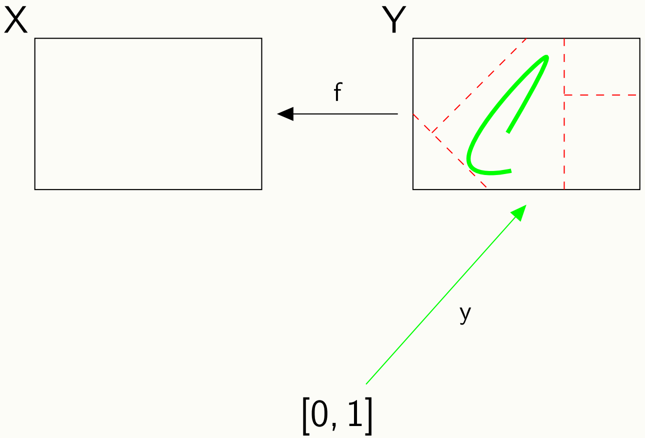

"Apply" f to an "element" by composition.

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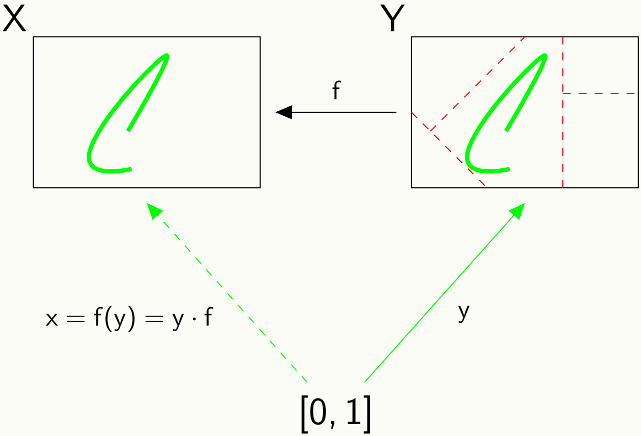

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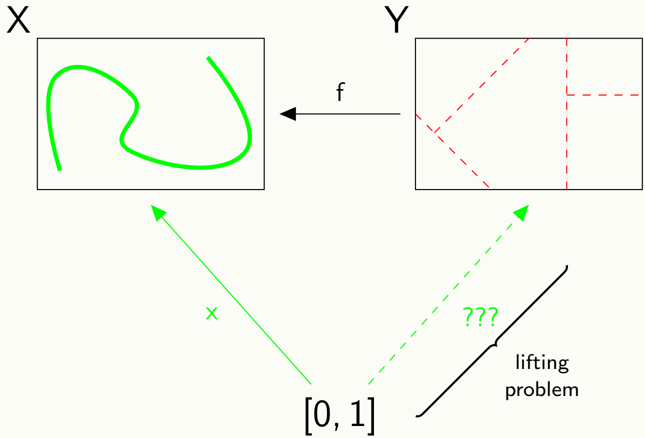

Questions?

Duality

Coproducts

Every construction in category theory has a *dual*.

Just reverse all the arrows.

$$\begin{tikzcd} X \arrow[r, "\mathsf{inj}_1"] \arrow[dr, "\forall\, x"'] & C \arrow[d, dashed, "\exists!\, c"'] & Y \arrow[l, "\mathsf{inj}_2"'] \arrow[dl, "\forall\, y"] \\ & \forall\, T & \end{tikzcd}$$

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Coproducts in $\mathbb{S}$et

In $\mathbb{S}$et, the coproduct is a sum $X + Y$ (*disjoint union*)

$$\{a, b, c, d\} + \{b, d, e\} \cong \left\{ \begin{array}{cccc} a_1, & b_1, & c_1, & d_1, \\ & b_2, & & d_2, e_2 \end{array} \right\}$$

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Given $\mathtt{f(x : X) : T}$ and $\mathtt{g(y : Y) : T}$

```
def choose(f,g)(x_or_y:X+Y):T
    if (index(x_or_y)==1)
       f(x_or_y)
    else                            # index(x_or_y)==2
       g(x_or_y)
    end
 end
```

Coproducts in $\mathbb{R}\mathsf{el}$

The same thing works in $\mathbb{R}\mathsf{el}$

$$\{a, b, c, d\} + \{b, d, e\} \cong \left\{ \begin{matrix} a_1, b_1, c_1, d_1, & \\ \quad b_2, \quad\; d_2, e_2 \end{matrix} \right\}$$

Coproducts in $\mathbb{R}$el

The same thing works in $\mathbb{R}$el

$$\{a, b, c, d\} + \{b, d, e\} \cong \left\{ \begin{matrix} a_1, b_1, c_1, d_1, & \\ \quad b_2, \quad\quad d_2, e_2 \end{matrix} \right\}$$

Given $\mathtt{R}(\mathtt{x} : \mathtt{X}, \mathtt{t} : \mathtt{T}) : \mathtt{Bool}$ and $\mathtt{S}(\mathtt{y} : \mathtt{Y}, \mathtt{t} : \mathtt{T}) : \mathtt{Bool}$

```
def choose(R,S)(x_or_y:X+Y,t:T):Bool
    if (index(x_or_y)==1)
      R(x_or_y,t)
    else                              # index(x_or_y)==2
      S(x_or_y,t)
    end
 end
```

Products in $\mathbb{R}$el

In $\mathbb{R}$el, sums and products are the same!

$$
\cfrac{\cfrac{\cfrac{\cfrac{\color{red}{\mathsf{T} \longleftrightarrow \mathsf{X} \times \mathsf{Y} \longleftrightarrow \mathsf{T}'}}{\mathsf{T} \longleftrightarrow \mathsf{X} \times \mathsf{Y} \qquad \mathsf{T}' \longleftrightarrow \mathsf{X} \times \mathsf{Y}}}{\mathsf{T} \longleftrightarrow \mathsf{X} \quad \mathsf{T} \longleftrightarrow \mathsf{Y} \quad \mathsf{T}' \longleftrightarrow \mathsf{X} \quad \mathsf{T}' \longleftrightarrow \mathsf{Y}}}{\mathsf{X} \longleftrightarrow \mathsf{T} \quad \mathsf{Y} \longleftrightarrow \mathsf{T} \quad \mathsf{X} \longleftrightarrow \mathsf{T}' \quad \mathsf{Y} \longleftrightarrow \mathsf{T}'}}{\cfrac{\mathsf{X} + \mathsf{Y} \longleftrightarrow \mathsf{T} \qquad \mathsf{X} + \mathsf{Y} \longleftrightarrow \mathsf{T}'}{\mathsf{T} \longleftrightarrow \mathsf{X} + \mathsf{Y} \longleftrightarrow \mathsf{T}'}}
$$

Products in $\mathbb{R}\mathrm{el}$

In $\mathbb{R}$el, sums and products are the same!

$$
\begin{array}{cc}
\multicolumn{2}{c}{T \longleftrightarrow X \times Y \longleftrightarrow T'} \\
\hline\hline
T \longleftrightarrow X \times Y & T' \longleftrightarrow X \times Y \\
\hline\hline
T \longleftrightarrow X \quad T \longleftrightarrow Y & T' \longleftrightarrow X \quad T' \longleftrightarrow Y \\
\hline\hline
X \longleftrightarrow T \quad Y \longleftrightarrow T & X \longleftrightarrow T' \quad Y \longleftrightarrow T' \\
\hline\hline
X + Y \longleftrightarrow T & X + Y \longleftrightarrow T' \\
\hline\hline
\multicolumn{2}{c}{T \longleftrightarrow X + Y \longleftrightarrow T'}
\end{array}
$$

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$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{T \longleftrightarrow X \times Y \longleftrightarrow T'}{T \longleftrightarrow X \times Y \qquad T' \longleftrightarrow X \times Y}}{T \longleftrightarrow X \quad T \longleftrightarrow Y \qquad T' \longleftrightarrow X \quad T' \longleftrightarrow Y}}{X \longleftrightarrow T \quad Y \longleftrightarrow T \qquad X \longleftrightarrow T' \quad Y \longleftrightarrow T'}}{X + Y \longleftrightarrow T \qquad X + Y \longleftrightarrow T'}}{T \longleftrightarrow X + Y \longleftrightarrow T'}$$

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$$
\begin{array}{c}
T \longleftrightarrow X \times Y \longleftrightarrow T' \\
\hline\hline
T \longleftrightarrow X \times Y \qquad \color{red}{T' \longleftrightarrow X \times Y} \\
\hline\hline
T \longleftrightarrow X \quad T \longleftrightarrow Y \qquad \color{red}{T' \longleftrightarrow X \quad T' \longleftrightarrow Y} \\
\hline\hline
X \longleftrightarrow T \quad Y \longleftrightarrow T \qquad X \longleftrightarrow T' \quad Y \longleftrightarrow T' \\
\hline\hline
X + Y \longleftrightarrow T \qquad X + Y \longleftrightarrow T' \\
\hline\hline
T \longleftrightarrow X + Y \longleftrightarrow T'
\end{array}
$$

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$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{T \longleftrightarrow X \times Y \longleftrightarrow T'}{T \longleftrightarrow X \times Y \qquad T' \longleftrightarrow X \times Y}}{\color{red}{T \longleftrightarrow X} \quad T \longleftrightarrow Y \quad T' \longleftrightarrow X \quad T' \longleftrightarrow Y}}{\color{red}{X \longleftrightarrow T} \quad Y \longleftrightarrow T \quad X \longleftrightarrow T' \quad Y \longleftrightarrow T'}}{X + Y \longleftrightarrow T \qquad X + Y \longleftrightarrow T'}}{T \longleftrightarrow X + Y \longleftrightarrow T'}$$

The horizontal rules between the rows are double lines.

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\begin{array}{c}
T \longleftrightarrow X \times Y \longleftrightarrow T' \\
\hline\hline
T \longleftrightarrow X \times Y \qquad\qquad T' \longleftrightarrow X \times Y \\
\hline\hline
T \longleftrightarrow X \quad {\color{red} T \longleftrightarrow Y} \qquad T' \longleftrightarrow X \quad T' \longleftrightarrow Y \\
\hline\hline
X \longleftrightarrow T \quad {\color{red} Y \longleftrightarrow T} \qquad X \longleftrightarrow T' \quad Y \longleftrightarrow T' \\
\hline\hline
X + Y \longleftrightarrow T \qquad\qquad X + Y \longleftrightarrow T' \\
\hline\hline
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\end{array}
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\hline\hline
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\hline\hline
T \longleftrightarrow X \quad T \longleftrightarrow Y \qquad \color{red}{T' \longleftrightarrow X} \quad T' \longleftrightarrow Y \\
\hline\hline
X \longleftrightarrow T \quad Y \longleftrightarrow T \qquad \color{red}{X \longleftrightarrow T'} \quad Y \longleftrightarrow T' \\
\hline\hline
X + Y \longleftrightarrow T \qquad\qquad X + Y \longleftrightarrow T' \\
\hline\hline
T \longleftrightarrow X + Y \longleftrightarrow T'
\end{array}
$$

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\begin{array}{cccc}
\multicolumn{4}{c}{\mathsf{T} \longleftrightarrow \mathsf{X} \times \mathsf{Y} \longleftrightarrow \mathsf{T}'} \\
\hline\hline
\multicolumn{2}{c}{\mathsf{T} \longleftrightarrow \mathsf{X} \times \mathsf{Y}} & \multicolumn{2}{c}{\mathsf{T}' \longleftrightarrow \mathsf{X} \times \mathsf{Y}} \\
\hline\hline
\mathsf{T} \longleftrightarrow \mathsf{X} & \mathsf{T} \longleftrightarrow \mathsf{Y} & \mathsf{T}' \longleftrightarrow \mathsf{X} & \color{red}{\mathsf{T}' \longleftrightarrow \mathsf{Y}} \\
\hline\hline
\mathsf{X} \longleftrightarrow \mathsf{T} & \mathsf{Y} \longleftrightarrow \mathsf{T} & \mathsf{X} \longleftrightarrow \mathsf{T}' & \color{red}{\mathsf{Y} \longleftrightarrow \mathsf{T}'} \\
\hline\hline
\multicolumn{2}{c}{\mathsf{X} + \mathsf{Y} \longleftrightarrow \mathsf{T}} & \multicolumn{2}{c}{\mathsf{X} + \mathsf{Y} \longleftrightarrow \mathsf{T}'} \\
\hline\hline
\multicolumn{4}{c}{\mathsf{T} \longleftrightarrow \mathsf{X} + \mathsf{Y} \longleftrightarrow \mathsf{T}'}
\end{array}
$$

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\begin{array}{c}
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\hline\hline
T \longleftrightarrow X \times Y \qquad\qquad T' \longleftrightarrow X \times Y \\
\hline\hline
T \longleftrightarrow X \quad T \longleftrightarrow Y \qquad T' \longleftrightarrow X \quad T' \longleftrightarrow Y \\
\hline\hline
{\color{red} X \longleftrightarrow T \quad Y \longleftrightarrow T} \qquad X \longleftrightarrow T' \quad Y \longleftrightarrow T' \\
\hline\hline
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\hline\hline
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\hline\hline
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\hline\hline
X \longleftrightarrow T \quad Y \longleftrightarrow T \qquad {\color{red} X \longleftrightarrow T' \quad Y \longleftrightarrow T'} \\
\hline\hline
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\hline\hline
T \longleftrightarrow X + Y \longleftrightarrow T'
\end{array}$$

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\hline\hline
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\hline\hline
T \longleftrightarrow X \quad T \longleftrightarrow Y \qquad T' \longleftrightarrow X \quad T' \longleftrightarrow Y \\
\hline\hline
X \longleftrightarrow T \quad Y \longleftrightarrow T \qquad X \longleftrightarrow T' \quad Y \longleftrightarrow T' \\
\hline\hline
\color{red}{X + Y \longleftrightarrow T} \qquad\qquad X + Y \longleftrightarrow T' \\
\hline\hline
\color{red}{T \longleftrightarrow X + Y \longleftrightarrow T'}
\end{array}
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\begin{array}{cccc}
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\hline\hline
T \longleftrightarrow X & T \longleftrightarrow Y & T' \longleftrightarrow X & T' \longleftrightarrow Y \\
\hline\hline
X \longleftrightarrow T & Y \longleftrightarrow T & X \longleftrightarrow T' & Y \longleftrightarrow T' \\
\hline\hline
\multicolumn{2}{c}{X + Y \longleftrightarrow T} & \multicolumn{2}{c}{X + Y \longleftrightarrow T'} \\
\hline\hline
\multicolumn{4}{c}{T \longleftrightarrow X + Y \longleftrightarrow T'}
\end{array}
$$

The category $\mathbb{X}^{\mathsf{op}}$

Objects	Same as $\mathbb{X}$
Arrows	$\mathsf{f}^{\mathsf{op}} : \mathsf{X} \to \mathsf{Y} \in \mathbb{X}^{\mathsf{op}} \longleftrightarrow \mathsf{f} : \mathsf{Y} \to \mathsf{X} \in \mathbb{X}$
Identities	$\mathsf{id}^{\mathsf{op}} ::= \mathsf{id} : \mathsf{X} \to \mathsf{X}$
Composition	$\mathsf{f}^{\mathsf{op}} \cdot \mathsf{g}^{\mathsf{op}} := (\mathsf{g} \cdot \mathsf{f})^{\mathsf{op}}$
Unit	From $\mathbb{X}$
Associativity	From $\mathbb{X}$

Questions?

Context

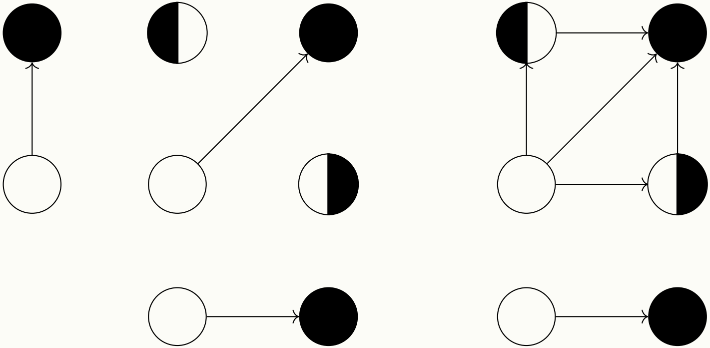
Categories are context

$\mathbb{S}$et and $\mathbb{R}$el have the *same* objects $X, Y, \ldots$, but

	Product $X \times Y$	Coproduct $X + Y$
In $\mathbb{S}$et	Set of pairs	Disj. union
	$\{\langle x, y \rangle\}$	$\{x_1\} \cup \{y_2\}$
In $\mathbb{R}$el	Disj. union	Disj. union
	$\{x_1\} \cup \{y_2\}$	$\{x_1\} \cup \{y_2\}$

Graph Products

Kronecker	Strong
$G \times H$	$G \boxtimes H$



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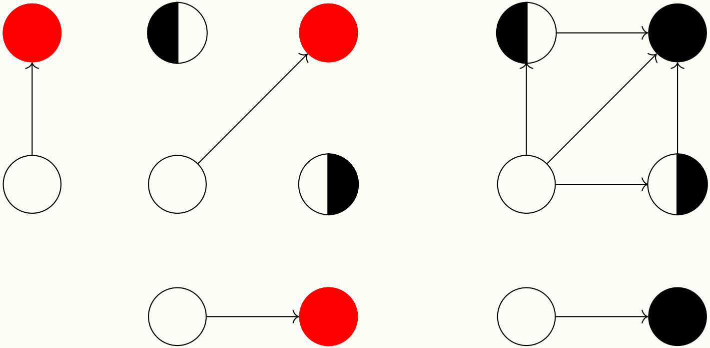

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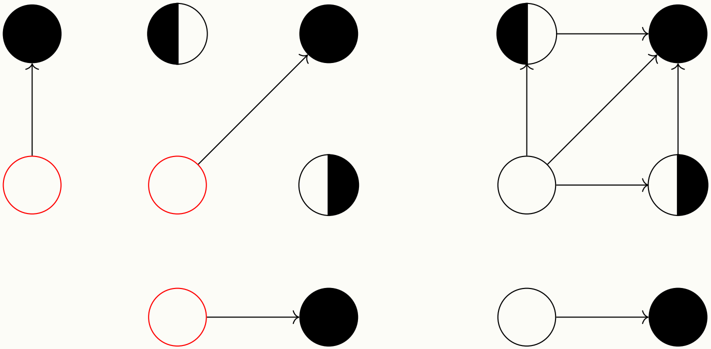

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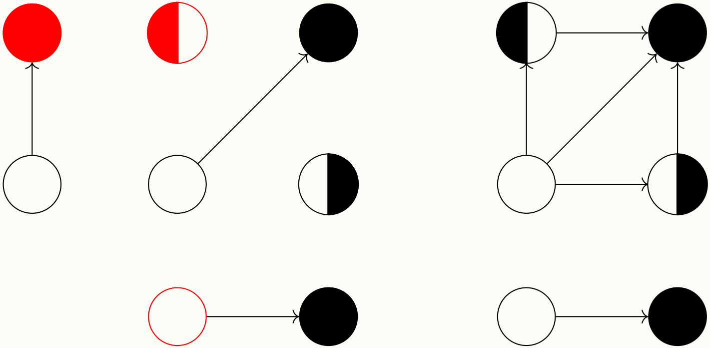

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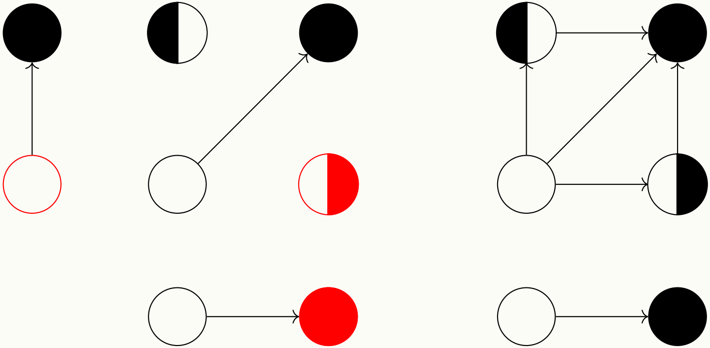

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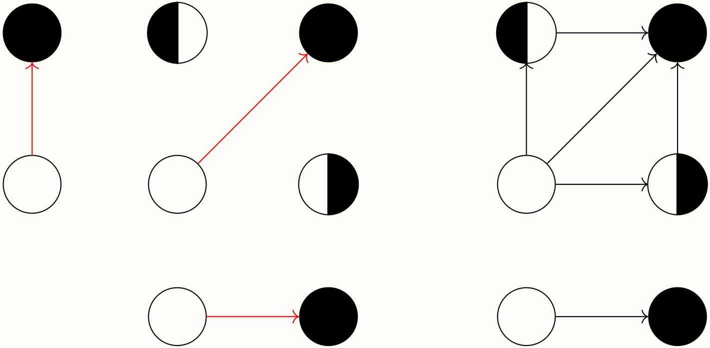

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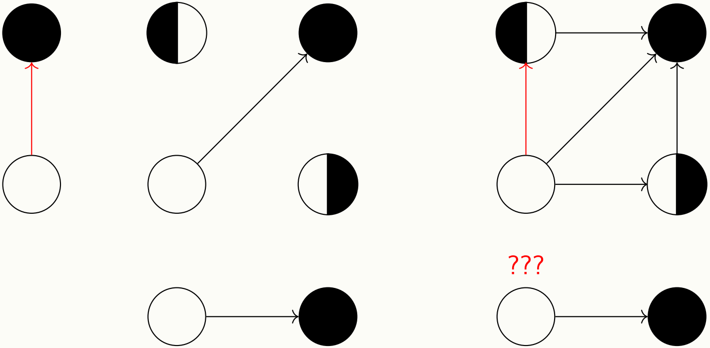

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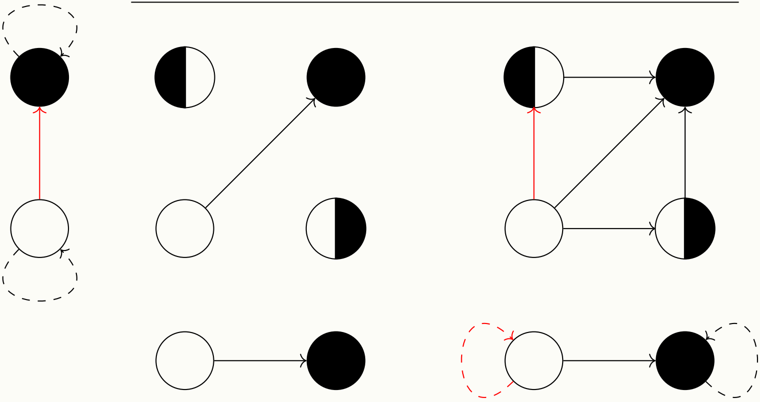
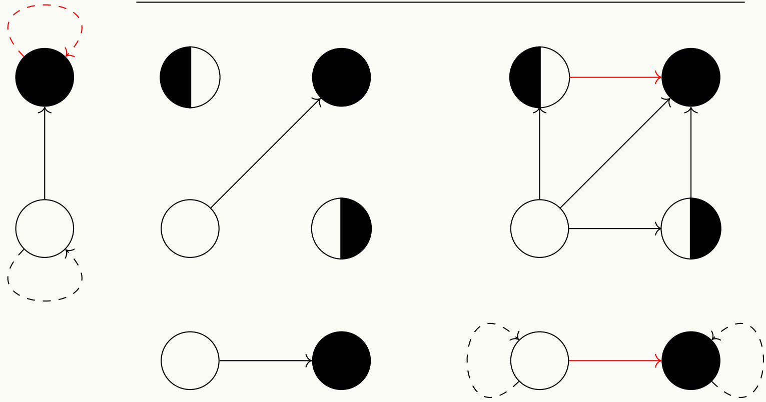
Kronecker	Strong
$G \times H$	$G \boxtimes H$



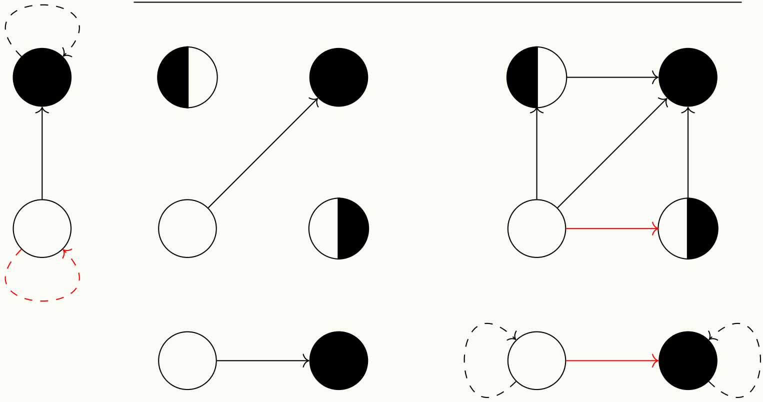
Graph Products

	Kronecker	Strong
	$G \times H$	$G \boxtimes H$



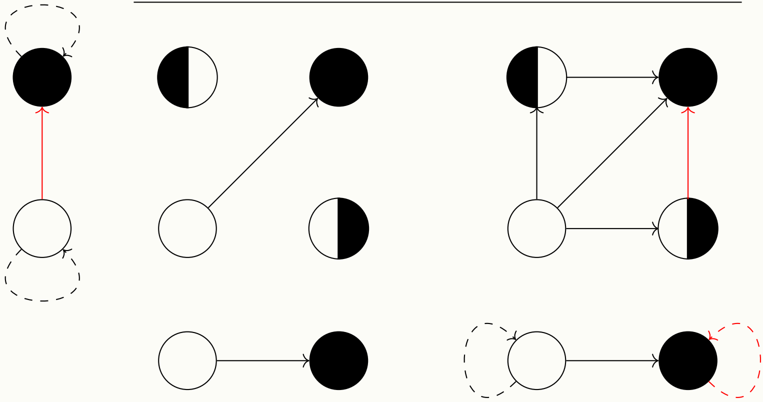
Graph Products

Kronecker	Strong
$G \times H$	$G \boxtimes H$



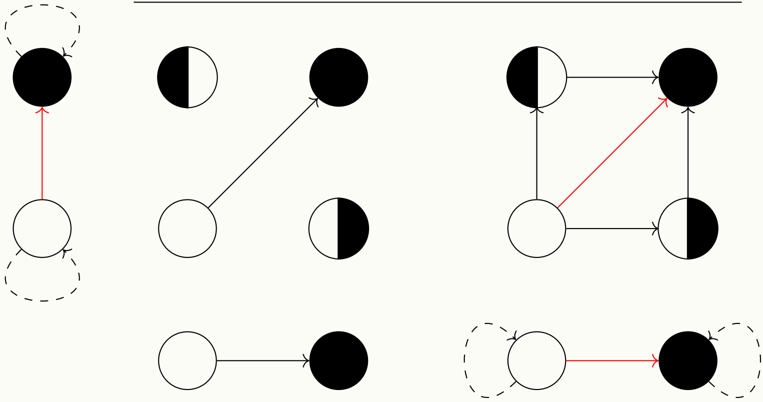
Graph Products

Kronecker	Strong
$G \times H$	$G \boxtimes H$



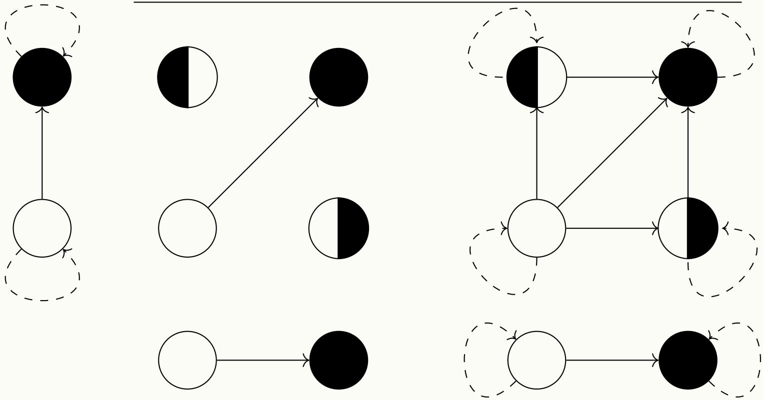
Graph Products

Kronecker | Strong
$G \times H$ | $G \boxtimes H$



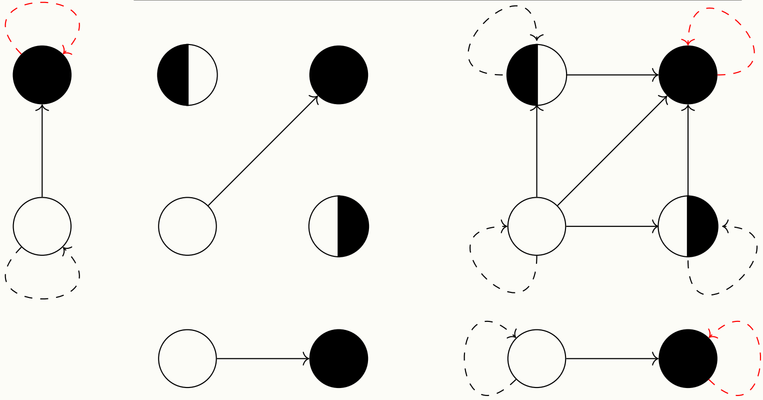
Graph Products

Kronecker	Strong
$G \times H$	$G \boxtimes H$



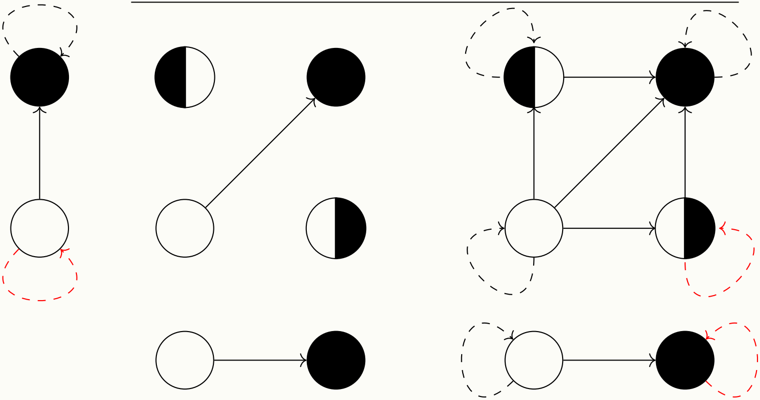
Graph Products

Kronecker	Strong
$G \times H$	$G \boxtimes H$



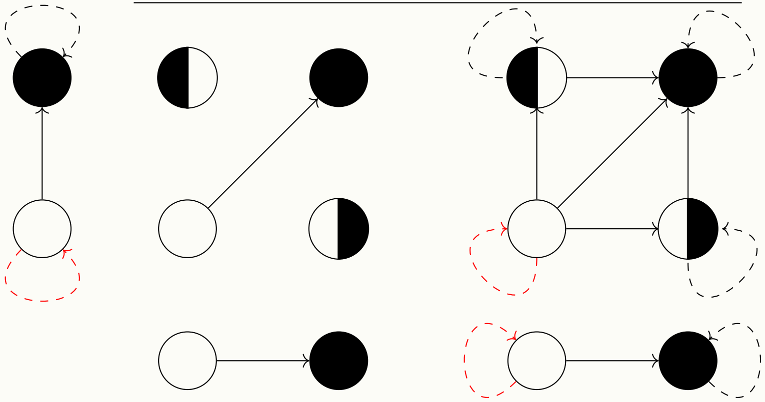
Graph Products

Kronecker	Strong
$G \times H$	$G \boxtimes H$



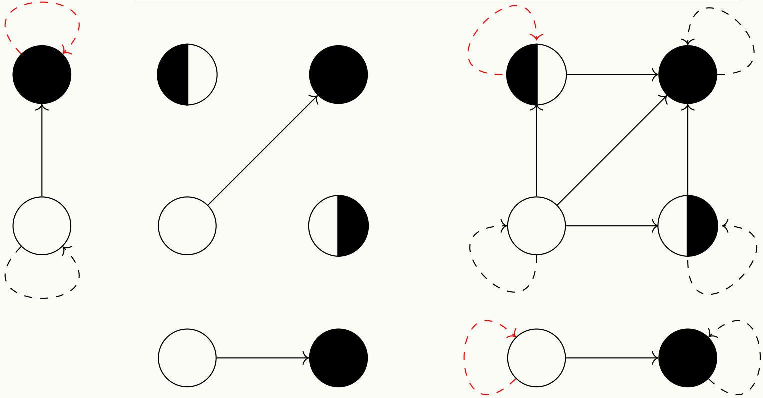
Graph Products

Kronecker	Strong
$G \times H$	$G \boxtimes H$



Graph Products

Kronecker	Strong
$G \times H$	$G \boxtimes H$



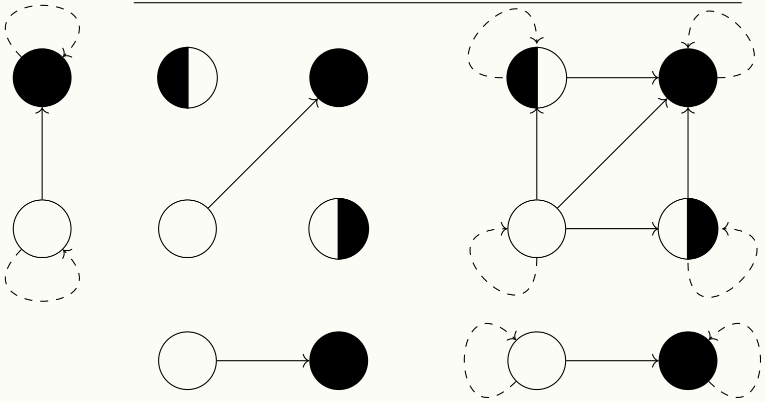
Categories are context

In $\mathbb{G}$raph

$G \times H$

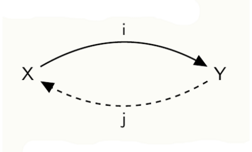
In $\mathbb{R}$Graph

$G \times H$

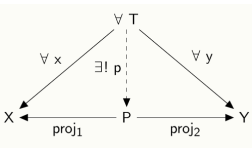


Questions?

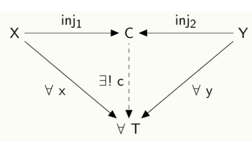
Isomorphism



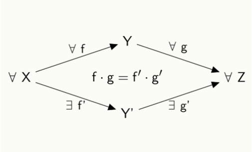
Products



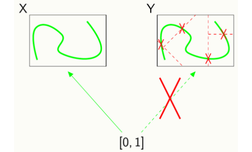
Duality



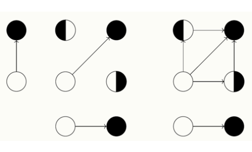
Yoneda



Test objects



Context



Todo

Monoids

Models

Functorial Semantics

A model is a mapping

$$\text{Syntax} \xrightarrow{\text{Model}} \text{Semantics}$$

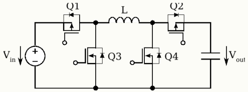
Syntax: What we *say*

What are the (types of) elements in the system?

How do they fit together?

- Easy to write down.
- Hard to compute.

Examples:

- Logical formulas: $\forall x.\exists y.R(x, y)$
- Computer code: `(x>0) ? sqrt(x) : sqrt(-x)`
- Circuit diagrams: 

Semantics: What we *mean*

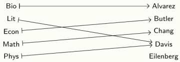
How are elements represented?

How do we model their interaction.

- Hard to write down.
- "Easy" to compute.

Examples:

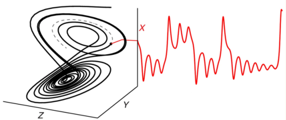
- Relations



- Matrices

$$\mathbf{K}_k\mathbf{S}_k = \left(\mathbf{H}_k\mathbf{P}_{k|k-1}\right)^\mathsf{T} = \mathbf{P}_{k|k-1}\mathbf{H}_k^\mathsf{T}$$
$$\Rightarrow \mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^\mathsf{T}\mathbf{S}_k^{-1}$$

- Dynamics



Functorial Semantics

A model is a mapping

$$\mathbb{S}\text{yntax} \xrightarrow{\mathbb{M}\text{odel}} \mathbb{S}\text{emantics}$$

Functors


Set of Elements
Category of Sets
Category of Categories

Functors

Definition: A *functor* $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ is

- a function from objects of $\mathbb{X}$ to objects of $\mathbb{Y}$
- a function from arrows of $\mathbb{X}$ to arrows of $\mathbb{Y}$
- preserving source and target, identities and composition:

$$\mathbb{F}(\overbrace{\mathsf{g} : \mathsf{X} \to \mathsf{X}'}^{\text{in } \mathbb{X}}) = \mathbb{F}(\mathsf{g}) : \overbrace{\mathbb{F}(\mathsf{X}) \to \mathbb{F}(\mathsf{X}')}^{\text{in } \mathbb{Y}}$$

$$\mathbb{F}(\mathsf{id}_{\mathbb{X}}) = \mathsf{id}_{\mathbb{F}(\mathbb{X})}$$

$$\mathbb{F}(\underbrace{\mathsf{f} \cdot \mathsf{g}}_{\text{in } \mathbb{X}}) = \underbrace{\mathbb{F}(\mathsf{f}) \cdot \mathbb{F}(\mathsf{g})}_{\text{in } \mathbb{Y}}$$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\mathsf{X} \longmapsto \mathbb{F}(\mathsf{X})$$

Object Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd} \mathsf{X} \arrow[loop above, dashed, "\mathsf{id}"] \arrow[r, mapsto] & \mathbb{F}(\mathsf{X}) \end{tikzcd}$$

Object Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\quad\mathbb{F}\quad} \mathbb{Y}$$

$$\begin{tikzcd} \mathsf{X} \arrow[loop above, dashed, "\mathrm{id}"] \arrow[rr, mapsto] & & \mathbb{F}(\mathsf{X}) \arrow[loop above, dashed, "\mathrm{id}"] \end{tikzcd}$$

Object Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{id}_{\mathsf{X}} \arrow[r, mapsto, bend left] & \mathbb{F}(\mathsf{id}_{\mathsf{X}}) = \mathsf{id}_{\mathbb{F}(\mathsf{X})} \\
{\color{red}\mathsf{X}} \arrow[r, mapsto] \arrow[loop above, dashed] & \mathbb{F}(\mathsf{X}) \arrow[loop above, dashed]
\end{tikzcd}$$

Identity Eq. $\mathbb{F}(\mathsf{id}_{\color{red}\mathsf{X}}) = \mathsf{id}_{\mathbb{F}(\mathsf{X})}$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{id}_{X} \arrow[r, mapsto, bend left] & \mathbb{F}(\mathsf{id}_{X}) = \mathsf{id}_{\mathbb{F}(X)} \\
X \arrow[r, mapsto] \arrow[loop above, dashed] & \mathbb{F}(X) \arrow[loop above, dashed]
\end{tikzcd}$$

Identity Eq. $\mathbb{F}(\mathsf{id}_{X}) = \mathsf{id}_{\mathbb{F}(X)}$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{id}_X \arrow[rr, mapsto, bend left] & & \mathbb{F}(\mathsf{id}_X) = \mathsf{id}_{\mathbb{F}(X)} \\
X \arrow[loop above, dashed] \arrow[rr, mapsto] & & \mathbb{F}(X) \arrow[loop above, dashed]
\end{tikzcd}$$

Identity Eq. $\mathbb{F}(\mathsf{id}_X) = \mathsf{id}_{\mathbb{F}(X)}$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{id}_{\mathsf{X}} \arrow[r, mapsto, bend left] & \mathbb{F}(\mathsf{id}_{\mathsf{X}}) = \mathsf{id}_{\mathbb{F}(\mathsf{X})} \\
\color{red}{\mathsf{X}} \arrow[loop above, dashed] \arrow[r, mapsto] & \mathbb{F}(\mathsf{X}) \arrow[loop above, dashed]
\end{tikzcd}$$

Identity Eq. $\mathbb{F}(\mathsf{id}_{\mathsf{X}}) = \mathsf{id}_{\mathbb{F}(\color{red}{\mathsf{X}})}$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{id}_{\mathsf{X}} \arrow[rr, mapsto, bend left] & & \mathbb{F}(\mathsf{id}_{\mathsf{X}}) = \mathsf{id}_{\mathbb{F}(\mathsf{X})} \\
\color{red}{\mathsf{X}} \arrow[loop above, dashed] \arrow[rr, mapsto, red] & & \color{red}{\mathbb{F}(\mathsf{X})} \arrow[loop above, dashed]
\end{tikzcd}$$

Identity Eq. $\mathbb{F}(\mathsf{id}_{\mathsf{X}}) = \mathsf{id}_{\color{red}{\mathbb{F}(\mathsf{X})}}$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{id}_{\mathsf{X}} \arrow[rr, bend left, mapsto] & & \mathbb{F}(\mathsf{id}_{\mathsf{X}}) = \mathsf{id}_{\mathbb{F}(\mathsf{X})} \\
\mathsf{X} \arrow[rr, mapsto] \arrow[loop above, dashed] & & \mathbb{F}(\mathsf{X}) \arrow[loop above, dashed]
\end{tikzcd}$$

Identity Eq. $\mathbb{F}(\mathsf{id}_{\mathsf{X}}) = \mathsf{id}_{\mathbb{F}(\mathsf{X})}$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{array}{ccc} X' & & \\ \downarrow{\scriptstyle h} & & \\ X & \longmapsto & \mathbb{F}(X) \end{array}$$

Arrow Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
{\color{red}X'} \arrow[r, mapsto, red] \arrow[d, "h"] & {\color{red}\mathbb{F}(X')} \\
X \arrow[r, mapsto] & \mathbb{F}(X)
\end{tikzcd}$$

Arrow Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{X}' \arrow[rrr, mapsto] \arrow[d, "\mathsf{h}"'] & & & \mathbb{F}(\mathsf{X}') \arrow[d, "\mathbb{F}(\mathsf{h})"] \\
\mathsf{X} \arrow[rrr, mapsto] & & & \mathbb{F}(\mathsf{X})
\end{tikzcd}$$

$$\mathsf{h} \longmapsto \mathbb{F}(\mathsf{h})$$

Arrow Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
X' \arrow[r, mapsto] \arrow[d, "h"] & \mathbb{F}(X') \arrow[d, "\mathbb{F}(h)"] \\
X \arrow[r, mapsto] \arrow[d, "k"] & \mathbb{F}(X) \\
X''
\end{tikzcd}$$

Composition Eq. $\mathbb{F}(h \cdot k) = \mathbb{F}(h) \cdot \mathbb{F}(k)$

Picturing a Functor

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "\mathbb{F}"] & & \mathbb{Y} \end{tikzcd}$$

$$\begin{tikzcd}
X' \arrow[r, mapsto] \arrow[d, "h"] \arrow[dd, bend right, dashed, red, "h \cdot k"'] & \mathbb{F}(X') \arrow[d, "\mathbb{F}(h)"] \\
X \arrow[r, mapsto] \arrow[d, "k"] & \mathbb{F}(X) \\
X'' &
\end{tikzcd}$$

Composition Eq. $\mathbb{F}({\color{red}h \cdot k}) = \mathbb{F}(h) \cdot \mathbb{F}(k)$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$
\begin{tikzcd}
 & X' \arrow[d, "h"] \arrow[dd, bend right, dashed] \arrow[rr, mapsto] & & \mathbb{F}(X') \arrow[d, "\mathbb{F}(h)"] \arrow[dd, bend left, dashed] & \\
h \cdot k \arrow[rrrr, mapsto, bend left] & X \arrow[d, "k"] \arrow[rr, mapsto] & & \mathbb{F}(X) & \mathbb{F}(h \cdot k) \\
 & X'' \arrow[rr, mapsto] & & \mathbb{F}(X'') &
\end{tikzcd}
$$

Composition Eq. $\mathbb{F}(h \cdot k) = \mathbb{F}(h) \cdot \mathbb{F}(k)$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{X}' \arrow[r, mapsto] \arrow[d, "\mathsf{h}"] \arrow[dd, bend right, dashed, "\mathsf{h} \cdot \mathsf{k}"'] & \mathbb{F}(\mathsf{X}') \arrow[d, "\mathbb{F}(\mathsf{h})"] \arrow[dd, bend left, dashed, "\mathbb{F}(\mathsf{h} \cdot \mathsf{k})"] \\
\mathsf{X} \arrow[r, mapsto] \arrow[d, "\mathsf{k}"] & \mathbb{F}(\mathsf{X}) \\
\mathsf{X}'' \arrow[r, mapsto] & \mathbb{F}(\mathsf{X}'')
\end{tikzcd}$$

Composition Eq. $\mathbb{F}(\mathsf{h} \cdot \mathsf{k}) = \mathbb{F}(\mathsf{h}) \cdot \mathbb{F}(\mathsf{k})$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$
\begin{tikzcd}
X' \arrow[r, mapsto] \arrow[d, no head] \arrow[dddd, bend right, dashed, "h \cdot k"'] & \mathbb{F}(X') \arrow[d, no head] \arrow[dddd, bend left, dashed, "\mathbb{F}(h \cdot k)"] \\
h \arrow[r, mapsto] \arrow[d] & \mathbb{F}(h) \arrow[d] \\
X \arrow[r, mapsto] \arrow[d, no head] & \mathbb{F}(X) \arrow[d, no head] \\
k \arrow[r, mapsto] \arrow[d] & \mathbb{F}(k) \arrow[d] \\
X'' \arrow[r, mapsto] & \mathbb{F}(X'')
\end{tikzcd}
$$

Composition Eq. $\mathbb{F}(h \cdot k) = \mathbb{F}(h) \cdot \mathbb{F}(k)$

Picturing a Functor

$$\mathbb{X} \xrightarrow{\quad\mathbb{F}\quad} \mathbb{Y}$$

$$\begin{tikzcd}
\mathsf{X}' \arrow[rrr, mapsto] \arrow[d, "\mathsf{h}"] \arrow[dd, bend right, dashed, "\mathsf{h}\cdot\mathsf{k}"'] & & & \mathbb{F}(\mathsf{X}') \arrow[d, "\mathbb{F}(\mathsf{h})"] \arrow[dd, bend left, dashed, red, "{\mathbb{F}(\mathsf{h}\cdot\mathsf{k}) = \mathbb{F}(\mathsf{h})\cdot\mathbb{F}(\mathsf{k})}"] \\
\mathsf{X} \arrow[rrr, mapsto] \arrow[d, "\mathsf{k}"] & & & \mathbb{F}(\mathsf{X}) \arrow[d, "\mathbb{F}(\mathsf{k})"] \\
\mathsf{X}'' \arrow[rrr, mapsto] & & & \mathbb{F}(\mathsf{X}'')
\end{tikzcd}$$

Composition Eq. $\mathbb{F}(\mathsf{h}\cdot\mathsf{k}) = \mathbb{F}(\mathsf{h})\cdot\mathbb{F}(\mathsf{k})$

Some functors: Products

$$\mathbb{S}\mathrm{et} \xrightarrow{- \times A} \mathbb{S}\mathrm{et}$$

$$\begin{array}{ccccc} X' & \longmapsto & X' \times A & & \langle x, a \rangle \\ \downarrow h & & \downarrow h \times A & & \downarrow \\ X & \longmapsto & X \times A & & \langle h(x), a \rangle \end{array}$$

Some functors: Products

$$\mathbb{S}\mathrm{et} \xrightarrow{- \times A} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
X' \arrow[r, mapsto] \arrow[d, "h"] & X' \times A \arrow[d, "h \times A"] & \langle x, a \rangle \arrow[d, mapsto] \\
X \arrow[r, mapsto] & X \times A & \langle h(x), a \rangle
\end{tikzcd}$$

Object function: $X \longmapsto X \times A$

Some functors: Products

$$\mathbb{S}\mathrm{et} \xrightarrow{- \times A} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
X' \arrow[r, mapsto] \arrow[d, "h"] & X' \times A \arrow[d, "h \times A"] & \langle x, a \rangle \arrow[d, mapsto] \\
X \arrow[r, mapsto] & X \times A & \langle h(x), a \rangle
\end{tikzcd}$$

Arrow function: $h \times A(x, a) := \langle h(x), a \rangle$

Some functors: Products

$$\mathbb{S}\mathrm{et} \xrightarrow{- \times A} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd} X' \arrow[r, mapsto] \arrow[d, "h"] & X' \times A \arrow[d, "h \times A"] & \langle x, a\rangle \arrow[d, mapsto] \\ X \arrow[r, mapsto] & X \times A & \langle h(x), a\rangle \end{tikzcd}$$

Identity Eq $\quad \begin{aligned} \mathrm{id}_{X\times A}(x,a) &= \langle x, a\rangle = \langle \mathrm{id}_X(x), a\rangle \\ &= \mathrm{id}_X \times A(x,a) \end{aligned}$

Some functors: Products

$$\begin{tikzcd} \mathbb{S}\mathrm{et} \arrow[r, "- \times A"] & \mathbb{S}\mathrm{et} \end{tikzcd}$$

$$\begin{tikzcd} X' \arrow[r, mapsto] \arrow[d, "h"] & X' \times A \arrow[d, "h \times A"] & \langle x, a \rangle \arrow[d, mapsto] \\ X \arrow[r, mapsto] & X \times A & \langle h(x), a \rangle \end{tikzcd}$$

$$\begin{aligned} \text{Comp. Eq} \quad f \cdot g \times A(x, a) &= \langle f \cdot g(x), a \rangle = \langle g(f(x)), a \rangle \\ &= g \times A(f(x), a) = g \times A(f \times A(x, a)) \\ &= (f \times A) \cdot (g \times A)(x, a) \end{aligned}$$

Some functors: Squaring

$$\mathbb{S}\mathrm{et} \xrightarrow{(-)^2} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
X' \arrow[r, mapsto] \arrow[d, "h"] & X' \times X' \arrow[d, "h^2"] & \langle x_1, x_2 \rangle \arrow[d, mapsto] \\
X \arrow[r, mapsto] & X \times X & \langle h(x_1), h(x_2) \rangle
\end{tikzcd}$$

Some functors: Power set

$$\mathbb{S}\mathrm{et} \xrightarrow{\color{red}{\mathbb{P} = 2^{(-)}}} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
X' \arrow[rrr, mapsto] \arrow[d, "h"] & & & \mathbb{P}(X') \arrow[d, "\mathbb{P}(h)"] & \{x_i\} \arrow[d, mapsto] \\
X \arrow[rrr, mapsto] & & & \mathbb{P}(X) & \{h(x_i)\}
\end{tikzcd}$$

Some functors: Inclusion

$$\mathbb{S}\mathrm{et} \xrightarrow{\;\mathbb{I}\;} \mathbb{R}\mathrm{el}$$

$$
\begin{tikzcd}
X' \arrow[r, mapsto] \arrow[d, "h"] & X' \arrow[d, leftrightarrow, "\{\langle x, h(x)\rangle\}"] & \{x\} \arrow[d, leftrightarrow] \\
X \arrow[r, mapsto] & X & \{h(x)\}
\end{tikzcd}
$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\quad\quad\quad\mathbb{P}\quad\quad\quad} \mathbb{S}\mathrm{et}$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\mathbb{P}} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
& \mathsf{Room} \arrow[rr, mapsto] & & \mathbb{P}(\mathsf{Room}) \\
\mathsf{Course} \arrow[ur] \arrow[rr, mapsto] & & \mathbb{P}(\mathsf{Course}) \arrow[ur] &
\end{tikzcd}$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\mathbb{P}} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
 & \text{Room} \arrow[rrr, mapsto] & & & \mathbb{P}(\text{Room}) \\
\text{Course} \arrow[ur] \arrow[rrr, mapsto] & & & \mathbb{P}(\text{Course}) \arrow[ur] &
\end{tikzcd}$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\mathbb{P}} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
 & \text{Room} \arrow[rr, mapsto] & & \mathbb{P}(\text{Room}) \\
\text{Course} \arrow[ur] \arrow[rr, mapsto] & & \mathbb{P}(\text{Course}) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \text{A1} & & & \\
\text{Bio} \arrow[ur, mapsto] & \text{A2} & & & \{\text{A1, B}\} \\
\text{Lit} \arrow[uur, mapsto] & \text{B} & & & \\
\text{Econ} \arrow[ur, mapsto] & & & \{\text{Bio, Lit, Econ}\} \arrow[uur, mapsto] & \\
\text{Math} \arrow[uuur, mapsto] & & & & \\
\text{Phys} \arrow[uuur, mapsto] & & & &
\end{tikzcd}$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\ \mathbb{P}\ } \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
& \mathrm{Room} \arrow[rr, mapsto] & & \mathbb{P}(\mathrm{Room}) \\
\mathrm{Course} \arrow[ur] \arrow[rr, mapsto] & & \mathbb{P}(\mathrm{Course}) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{A1} & & \\
\mathrm{Bio} \arrow[ur, mapsto] & \mathrm{A2} & & \{\mathrm{A1}, \mathrm{B}\} \\
\mathrm{Lit} \arrow[uur, mapsto] & \mathrm{B} & \{\mathrm{Bio}, \mathrm{Lit}, \mathrm{Econ}\} \arrow[ur, mapsto] & \\
\mathrm{Econ} \arrow[ur, mapsto] & & & \\
\mathrm{Math} \arrow[uuur, mapsto] & & & \\
\mathrm{Phys} \arrow[uuur, mapsto] & & &
\end{tikzcd}$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\ \mathbb{P}\ } \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
& \mathrm{Room} \arrow[rr, mapsto] & & \mathbb{P}(\mathrm{Room}) \\
\mathrm{Course} \arrow[ur] \arrow[rr, mapsto] & & \mathbb{P}(\mathrm{Course}) \arrow[ur] &
\end{tikzcd}$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\mathbb{P}} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
& \mathsf{Room} \arrow[rr, mapsto] & & \mathbb{P}(\mathsf{Room}) \\
\mathsf{Course} \arrow[ur] \arrow[rr, mapsto] & & \mathbb{P}(\mathsf{Course}) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathsf{A1} & & \\
\mathsf{Bio} \arrow[ur, mapsto] & \mathsf{A2} & & \{\mathsf{A1}, \mathsf{B}\} \\
\mathsf{Lit} \arrow[uur, mapsto] & \mathsf{B} & & \\
\mathsf{Econ} \arrow[ur, mapsto] & & \{\mathsf{Bio}, \mathsf{Lit}, \mathsf{Econ}\} \arrow[uur, mapsto] & \\
\mathsf{Math} \arrow[uuur, mapsto] & & & \\
\mathsf{Phys} \arrow[uuur, mapsto] & & &
\end{tikzcd}$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\mathbb{P}} \mathbb{S}\mathrm{et}$$

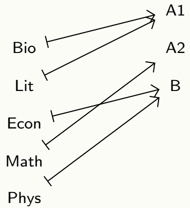
$$\begin{tikzcd}
 & \mathrm{Room} \arrow[rr, mapsto] & & \mathbb{P}(\mathrm{Room}) \\
\mathrm{Course} \arrow[ur] \arrow[rr, mapsto] & & \mathbb{P}(\mathrm{Course}) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{A1} \\
\mathrm{Bio} \arrow[ur, mapsto] & \mathrm{A2} \\
\mathrm{Lit} \arrow[uur, mapsto] & \mathrm{B} \\
\mathrm{Econ} \arrow[ur, mapsto] & \\
\mathrm{Math} \arrow[uuur, mapsto] & \\
\mathrm{Phys} \arrow[uuur, mapsto] &
\end{tikzcd}
\qquad
\{\mathrm{Bio}, \mathrm{Lit}, \mathrm{Econ}\} \mapsto \{\mathrm{A1}, \mathrm{B}\}$$

Three levels of abstraction

$$\mathbb{S}\mathrm{et} \xrightarrow{\mathbb{P}} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
& \mathsf{Room} \arrow[rr, mapsto] & & \mathbb{P}(\mathsf{Room}) \\
\mathsf{Course} \arrow[ur, red] \arrow[rr, mapsto] & & \mathbb{P}(\mathsf{Course}) \arrow[ur] &
\end{tikzcd}$$



$$\{\mathsf{Bio}, \mathsf{Lit}, \mathsf{Econ}\} \mapsto \{\mathsf{A1}, \mathsf{B}\}$$

Questions?

$\mathbb{C}$at

The category $\mathbb{C}\mathsf{at}$

Objects	Categories $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$,...	
Arrows	Functors $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$, $\mathbb{G} : \mathbb{Y} \to \mathbb{Z}$,...	
Identities	$\mathbb{id}(\mathsf{X}) = \mathsf{X}$, $\mathbb{id}(\mathsf{h}) = \mathsf{h}$	$\mathbb{X} \to \mathbb{X}$
Composition	$\mathbb{F} \cdot \mathbb{G}(-) = \mathbb{G}(\mathbb{F}(-))$	$\mathbb{X} \to \mathbb{Z}$
Unit	From $\mathbb{S}\mathsf{et}$	
Associativity	From $\mathbb{S}\mathsf{et}$	

The category $\mathbb{C}$at

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Unit	From $\mathbb{S}$et	
Associativity	From $\mathbb{S}$et	

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Composition	$\mathbb{F} \cdot \mathbb{G}(-) = \mathbb{G}(\mathbb{F}(-))$	$\mathbb{X} \to \mathbb{Z}$

Unit	From $\mathbb{S}$et
Associativity	From $\mathbb{S}$et

The category $\mathbb{C}$at

Objects	Categories $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$,...	
Arrows	Functors $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$, $\mathbb{G} : \mathbb{Y} \to \mathbb{Z}$,...	
Identities	$\mathbb{id}(X) = X$, $\mathbb{id}(h) = h$	$\mathbb{X} \to \mathbb{X}$
Composition	$\mathbb{F} \cdot \mathbb{G}(-) = \mathbb{G}(\mathbb{F}(-))$	$\mathbb{X} \to \mathbb{Z}$
Unit	From $\mathbb{S}$et	
Associativity	From $\mathbb{S}$et	

The category $\mathbb{C}$at

Objects	Categories $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$,...	
Arrows	Functors $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$, $\mathbb{G} : \mathbb{Y} \to \mathbb{Z}$,...	
Identities	$\mathbb{id}(\mathsf{X}) = \mathsf{X}$, $\mathbb{id}(\mathsf{h}) = \mathsf{h}$	$\mathbb{X} \to \mathbb{X}$
Composition	$\mathbb{F} \cdot \mathbb{G}(-) = \mathbb{G}(\mathbb{F}(-))$	$\mathbb{X} \to \mathbb{Z}$
Unit	From $\mathbb{S}$et	
Associativity	From $\mathbb{S}$et	

Lifting problems

$$\begin{tikzcd}
 & \mathbb{R}\mathsf{el} \arrow[dr, "???"] & \\
\mathbb{S}\mathsf{et} \arrow[ur, "\mathbb{I}"] \arrow[rr, "\mathbb{P}"'] & & \mathbb{S}\mathsf{et}
\end{tikzcd}$$

The category $\mathbb{X} \times \mathbb{Y}$

Objects Pairs $\langle \mathsf{X} \in \mathbb{X}, \mathsf{Y} \in \mathbb{Y} \rangle$

Arrows Pairs $\langle \mathsf{h} : \mathsf{X} \to \mathsf{X}', \mathsf{k} : \mathsf{Y} \to \mathsf{Y}' \rangle$

Identities $\mathsf{id}_{\langle \mathsf{X}, \mathsf{Y} \rangle} := \langle \mathsf{id}_{\mathsf{X}}, \mathsf{id}_{\mathsf{Y}} \rangle$ $\qquad \langle \mathsf{X}, \mathsf{Y} \rangle \to \langle \mathsf{X}, \mathsf{Y} \rangle$

Composition $\langle \mathsf{h}, \mathsf{k} \rangle \cdot \langle \mathsf{h}', \mathsf{k}' \rangle$ $\qquad \langle \mathsf{X}, \mathsf{Y} \rangle \to \langle \mathsf{X}'', \mathsf{Y}'' \rangle$

$:= \langle \mathsf{h} \cdot \mathsf{h}', \mathsf{k} \cdot \mathsf{k}' \rangle$

Unit From $\mathbb{X}$ & $\mathbb{Y}$

Associativity From $\mathbb{X}$ & $\mathbb{Y}$

Test categories

$$\mathbb{1} := \left\{ \begin{tikzcd} * \arrow[loop above, dashed, "\mathsf{id}"] \end{tikzcd} \right\}$$

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$$\mathbb{1} := \left\{ \begin{tikzcd} * \arrow[loop above, dashed, "\mathsf{id}"] \end{tikzcd} \right\}$$

$$\mathbb{1} \xrightarrow{\quad \vec{\mathsf{X}} \quad} \mathbb{X}$$

Object $\mathsf{X} \in \mathbb{X}$

Test categories

$$\mathbb{1} := \left\{ \begin{tikzcd} * \arrow[loop above, dashed, "\mathrm{id}"] \end{tikzcd} \right\}$$

$$\frac{\mathbb{1} \xrightarrow{\vec{\mathsf{X}}} \mathbb{S}\mathsf{et}}{\mathsf{Set}\ \mathsf{X}}$$

Test categories

$$\mathbb{1} := \left\{ \begin{tikzcd} * \arrow[loop above, dashed, "\mathrm{id}"] \end{tikzcd} \right\}$$

$$\frac{\mathbb{1} \xrightarrow{\quad \vec{\mathsf{X}} \quad} \mathbb{R}\mathsf{el}}{\mathsf{Set}\ \mathsf{X}}$$

Test categories

$$\mathbb{2} := \left\{ 0 \xrightarrow{\mathsf{s}} 1 \right\}$$

Test categories

$$\mathbb{2} := \left\{ 0 \xrightarrow{\mathsf{s}} 1 \right\}$$

$$\frac{\mathbb{2} \xrightarrow{\vec{\mathsf{f}}} \mathbb{X}}{}$$

2 Objects: $X_0 = \vec{f}(0)$, $X_1 = \vec{f}(1)$

1 arrow: $f : X_0 \to X_1$

Test categories

$$\mathbb{2} := \left\{ 0 \xrightarrow{\mathsf{s}} 1 \right\}$$

$$\frac{\mathbb{2} \xrightarrow{\vec{\mathsf{f}}} \mathbb{S}\mathsf{et}}{\text{Function } \mathsf{f} : \mathsf{X}_0 \to \mathsf{X}_1}$$

Test categories

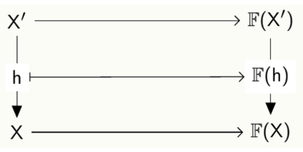
$$\mathbb{2} := \left\{ 0 \xrightarrow{s} 1 \right\}$$

$$\mathbb{2} \xrightarrow{\vec{f}} \mathbb{R}\mathrm{el}$$

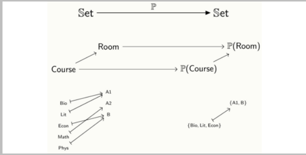
$$\text{Relation } R : X \leftrightarrow X_1$$

Questions?

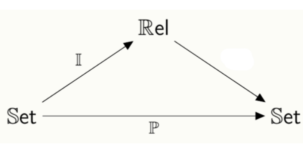
Functors



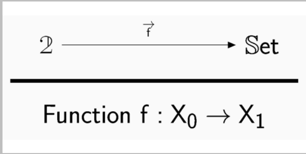
Abstraction



ℂat



Test categories



Data

Categorical Data Model

Fixed **Semantics**: $\mathbb{S}$et

Choose **Syntax**: $\mathbb{S}$chema

$$\mathbb{S}\mathrm{ch} \xrightarrow{\mathbb{D}\mathrm{ata}} \mathbb{S}\mathrm{et}$$

Categorical Data Model

Fixed **Semantics**: $\mathbb{S}$et

Choose **Syntax**: $\mathbb{S}$chema

$$\mathbb{S}\text{ch} \xrightarrow{\mathbb{D}\text{ata}} \mathbb{S}\text{et}$$

$$\overbrace{\text{Datatype} \quad \text{Entity}}^{\text{Two object types}}$$

Categorical Data Model

Fixed **Semantics**: $\mathbb{S}$et

Choose **Syntax**: $\mathbb{S}$chema

$$\mathbb{S}\mathrm{ch} \xrightarrow{\mathbb{D}\mathrm{ata}} \mathbb{S}\mathrm{et}$$

$$\begin{tikzcd}
& \text{Entity} \arrow[dl, "\text{attribute}"'] \arrow[d, "\text{map}"] \\
\text{Datatype} & \text{Entity}
\end{tikzcd}$$

Two arrow types

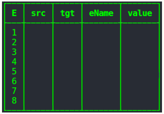
Categorical Data Model

Fixed **Semantics**: $\mathbb{S}\mathrm{et}$

Choose **Syntax**: $\mathbb{S}\mathrm{chema}$

$$\mathbb{S}\mathrm{ch} \xrightarrow{\mathbb{D}\mathrm{ata}} \mathbb{S}\mathrm{et}$$

$$\mathrm{Entity} \longmapsto$$



E	src	tgt	eName	value
1				
2				
3				
4				
5				
6				
7				
8				

Entities represent (empty) tables

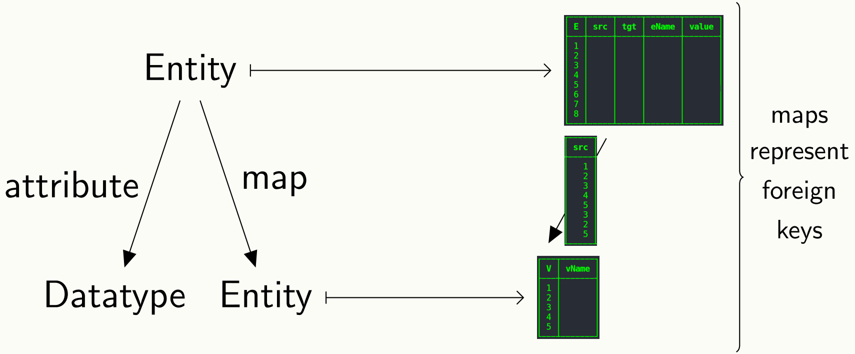
$$\begin{tikzcd} & \mathrm{Entity} \arrow[dl, "\mathrm{attribute}"'] \arrow[dr, "\mathrm{map}"] & \\ \mathrm{Datatype} & & \mathrm{Entity} \end{tikzcd}$$

Categorical Data Model

Fixed **Semantics**: $\mathbb{S}\mathrm{et}$

Choose **Syntax**: $\mathbb{S}\mathrm{chema}$

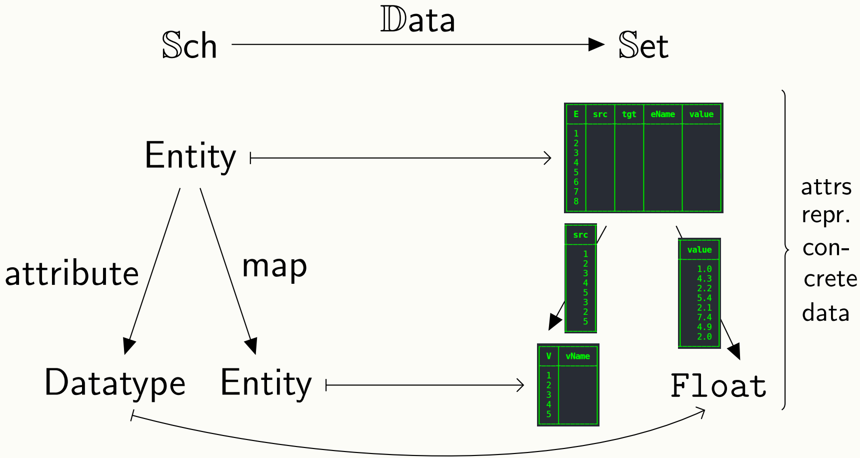
$$\mathbb{S}\mathrm{ch} \xrightarrow{\mathbb{D}\mathrm{ata}} \mathbb{S}\mathrm{et}$$



Categorical Data Model

Fixed **Semantics**: $\mathbb{S}$et

Choose **Syntax**: $\mathbb{S}$chema



Graphs

The Graph Schema

Schema $\mathbb{S}\mathsf{ch}_G$:

$$\begin{tikzcd} \texttt{Number} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left, "\mathsf{src}"] \arrow[r, shift right, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[r, "\mathsf{label}"] & \texttt{Label} \end{tikzcd}$$

The Graph Schema

Schema $\mathbb{S}\mathrm{ch}_G$**:**

$$\begin{tikzcd} \overbrace{\texttt{Number}}^{\text{Datatype}} & \overbrace{\text{Edge}}^{\text{Entity}} \arrow[l, "\text{weight}"'] \arrow[r, shift left, "\text{src}"] \arrow[r, shift right, "\text{tgt}"'] & \overbrace{\text{Vertex}}^{\text{Entity}} \arrow[r, "\text{label}"] & \overbrace{\texttt{Label}}^{\text{Datatype}} \end{tikzcd}$$

The Graph Schema

Schema $\mathbb{S}\mathrm{ch}_G$:

$$\begin{tikzcd} \texttt{Number} & \mathsf{Edge} \arrow[l, "\overbrace{\mathsf{weight}}^{\mathsf{attribute}}"'] \arrow[r, shift left=2, "\overbrace{\mathsf{src}}^{\mathsf{map}}"] \arrow[r, shift right=2, "\underbrace{\mathsf{tgt}}_{\mathsf{map}}"'] & \mathsf{Vertex} \arrow[r, "\overbrace{\mathsf{label}}^{\mathsf{attribute}}"] & \texttt{Label} \end{tikzcd}$$

The Graph Schema

Schema $\mathbb{S}\mathsf{ch}_G$:

$$
\begin{tikzcd}
\mathtt{Number} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left, "\mathsf{src}"] \arrow[r, shift right, "\mathsf{tgt}"'] \arrow[rr, bend left, dashed, "\mathsf{src}\cdot\mathsf{label}"] \arrow[rr, bend right, dashed, "\mathsf{tgt}\cdot\mathsf{label}"'] & \mathsf{Vertex} \arrow[r, "\mathsf{label}"] & \mathtt{Label}
\end{tikzcd}
$$

The Graph Schema

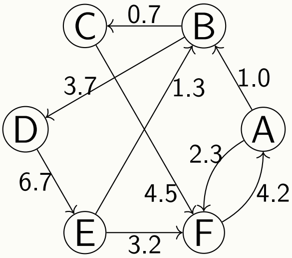
Schema $\mathbb{S}\mathsf{ch}_{\mathsf{G}}$:

$$
\begin{tikzcd}
\texttt{Number} \arrow[loop above, dashed, "\mathsf{id}"] & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[loop above, dashed, "\mathsf{id}"] \arrow[r, shift left, "\mathsf{src}"] \arrow[r, shift right, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[loop above, dashed, "\mathsf{id}"] \arrow[r, "\mathsf{label}"] & \texttt{Label} \arrow[loop above, dashed, "\mathsf{id}"]
\end{tikzcd}
$$

Graph Semantics

Schema $\mathbb{S}\mathrm{ch}_G$:

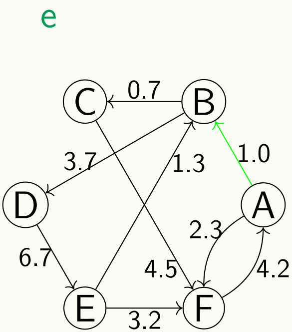
$$\texttt{Number} \xleftarrow{\text{weight}} \text{Edge} \underset{\text{tgt}}{\overset{\text{src}}{\rightrightarrows}} \text{Vertex} \xrightarrow{\text{label}} \texttt{Label}$$



Graph Semantics

Schema $\mathbb{S}\mathrm{ch}_G$:

$$
\begin{tikzcd}
\texttt{Number} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left, "\mathsf{src}"] \arrow[r, shift right, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[r, "\mathsf{label}"] & \texttt{Label}
\end{tikzcd}
$$

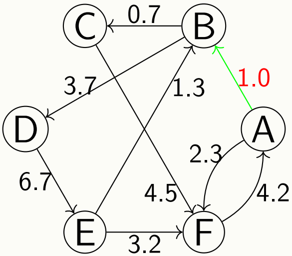


Graph Semantics

Schema $\mathbb{S}\mathrm{ch}_G$**:**

$$\begin{tikzcd} \mathtt{Number} & \mathrm{Edge} \arrow[l, "\color{red}{weight}"'] \arrow[r, shift left, "src"] \arrow[r, shift right, "tgt"'] & \mathrm{Vertex} \arrow[r, "label"] & \mathtt{Label} \end{tikzcd}$$

$$\color{red}{1.0} \longleftarrow\!\!\!| \; \color{green}{e}$$

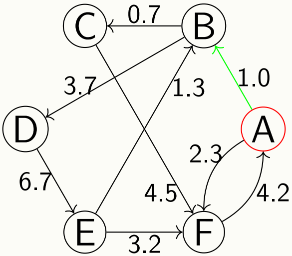


Graph Semantics

Schema $\mathbb{S}\mathrm{ch}_G$**:**

$$\begin{tikzcd} \mathtt{Number} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left, color=red, "\color{red}\mathsf{src}"] \arrow[r, shift right, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[r, "\mathsf{label}"] & \mathtt{Label} \end{tikzcd}$$

$$1.0 \longleftarrow\!\!\!\shortmid \ {\color{green}\mathsf{e}} \ \longmapsto \ {\color{red}\mathsf{v}}$$

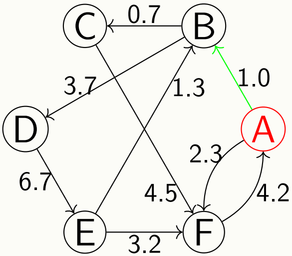


Graph Semantics

Schema $\mathbb{S}\mathrm{ch}_G$**:**

$$\begin{tikzcd} \mathtt{Number} & \mathrm{Edge} \arrow[l, "\mathrm{weight}"'] \arrow[r, shift left, "\mathrm{src}"] \arrow[r, shift right, "\mathrm{tgt}"'] & \mathrm{Vertex} \arrow[r, "\mathrm{label}"] & \mathtt{Label} \end{tikzcd}$$

$$1.0 \longleftarrow\!\!\shortmid\; e \;\shortmid\!\!\longrightarrow v \;\shortmid\!\!\longrightarrow A$$

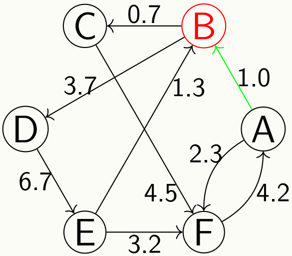


Graph Semantics

Schema $\mathbb{S}\mathrm{ch}_G$:

$$\begin{tikzcd} \mathtt{Number} & \mathrm{Edge} \arrow[l, "\mathrm{weight}"'] \arrow[r, shift left, "\mathrm{src}"] \arrow[r, shift right, "\mathrm{tgt}"'] & \mathrm{Vertex} \arrow[r, "\mathrm{label}"] & \mathtt{Label} \end{tikzcd}$$

$$\begin{tikzcd} 1.0 & e \arrow[l, mapsto] \arrow[r, mapsto] \arrow[dr, mapsto] & v \arrow[r, mapsto] & A \\ & & v' \arrow[r, mapsto] & B \end{tikzcd}$$

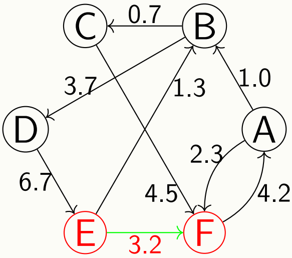


Graph Semantics

Schema $\mathbb{S}\mathrm{ch}_G$**:**

$$\begin{tikzcd} \mathtt{Number} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left, "\mathsf{src}"] \arrow[r, shift right, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[r, "\mathsf{label}"] & \mathtt{Label} \end{tikzcd}$$

$$\begin{tikzcd} & & v'' \arrow[r, mapsto] & \color{red}{E} \\ \color{red}{3.2} & \color{green}{e'} \arrow[l, mapsto] \arrow[ur, mapsto] \arrow[r, mapsto] & v''' \arrow[r, mapsto] & \color{red}{F} \end{tikzcd}$$



Graphs

Schema:

$$\begin{tikzcd} \texttt{Number} & \text{Edge} \arrow[l, "\text{weight}"'] \arrow[r, shift left, "\text{src}"] \arrow[r, shift right, "\text{tgt}"'] & \text{Vertex} \arrow[r, "\text{label}"] & \texttt{label} \end{tikzcd}$$

$$\begin{tikzcd} & & \texttt{v} = 1 \arrow[r, mapsto] & \text{A} \\ 1.0 & \texttt{e} = 1 \arrow[l, mapsto] \arrow[ur, mapsto] \arrow[dr, mapsto] & & \\ & & \texttt{v} = 2 \arrow[r, mapsto] & \text{B} \end{tikzcd}$$

E	src	tgt	weight
1	1	2	1.0
2	1	6	2.3

V	label
1	A
2	B

Reflexive Graphs

Schema $\mathbb{S}\mathrm{ch}_{\mathsf{R}}$:

$$
\begin{tikzcd}
\texttt{Float} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left=2, "\mathsf{src}"] \arrow[r, shift right=2, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[l, "\mathsf{loop}" description, red] \arrow[r, "\mathsf{label}"] & \texttt{String}
\end{tikzcd}
$$

$$\mathsf{loop} \cdot \mathsf{src} = \mathsf{id} = \mathsf{loop} \cdot \mathsf{tgt}$$

$$\mathsf{loop} \cdot \mathsf{weight} = 0.0$$

Reflexive Graphs

Schema $\mathbb{S}\mathsf{ch}_{\mathsf{R}}$**:**

$$\begin{tikzcd} \texttt{Float} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left=2, "\mathsf{src}"] \arrow[r, shift right=2, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[l, "\mathsf{loop}" description] \arrow[r, "\mathsf{label}"] & \texttt{String} \end{tikzcd}$$

$$\mathsf{loop} \cdot \mathsf{src} = \mathsf{id} = \mathsf{loop} \cdot \mathsf{tgt}$$

$$\mathsf{loop} \cdot \mathsf{weight} = 0.0$$

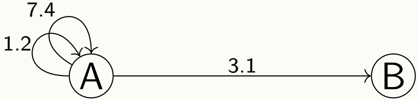
Reflexive Graphs

Schema $\mathbb{S}\mathsf{ch}_{\mathsf{R}}$:

$$
\begin{tikzcd}
\texttt{Float} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left=2, "\mathsf{src}"] \arrow[r, shift right=2, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[l, "\mathsf{loop}" description] \arrow[r, "\mathsf{label}"] & \texttt{String}
\end{tikzcd}
$$

$$\mathsf{loop} \cdot \mathsf{src} = \mathsf{id} = \mathsf{loop} \cdot \mathsf{tgt}$$

$$\mathsf{loop} \cdot \mathsf{weight} = 0.0$$



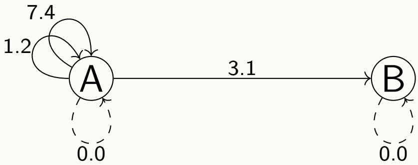
Reflexive Graphs

Schema $\mathbb{S}\mathsf{ch}_{\mathsf{R}}$:

$$
\begin{tikzcd}
\texttt{Float} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left=2, "\mathsf{src}"] \arrow[r, shift right=2, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[l, "\mathsf{loop}" description] \arrow[r, "\mathsf{label}"] & \texttt{String}
\end{tikzcd}
$$

$$\mathsf{loop} \cdot \mathsf{src} = \mathsf{id} = \mathsf{loop} \cdot \mathsf{tgt}$$

$$\mathsf{loop} \cdot \mathsf{weight} = 0.0$$



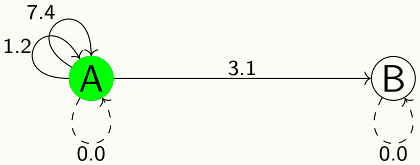
Reflexive Graphs

Schema $\mathbb{S}\mathrm{ch}_{\mathsf{R}}$:

$$\begin{tikzcd} \texttt{Float} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left=2, "\mathsf{src}"] \arrow[r, shift right=2, "\mathsf{tgt}"'] & \color{green}{\mathsf{Vertex}} \arrow[l, "\mathsf{loop}" description] \arrow[r, "\mathsf{label}"] & \texttt{String} \end{tikzcd}$$

$$\mathsf{loop} \cdot \mathsf{src} = \color{green}{\mathsf{id}} = \mathsf{loop} \cdot \mathsf{tgt}$$

$$\mathsf{loop} \cdot \mathsf{weight} = 0.0$$



Reflexive Graphs

Schema $\mathbb{S}\mathrm{ch}_{\mathsf{R}}$:

$$\begin{tikzcd}
\texttt{Float} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left=2, "\mathsf{src}"] \arrow[r, shift right=2, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[l, "\mathsf{loop}" description] \arrow[r, "\mathsf{label}"] & \texttt{String}
\end{tikzcd}$$

$$\mathsf{loop} \cdot \mathsf{src} = \mathsf{id} = \mathsf{loop} \cdot \mathsf{tgt}$$

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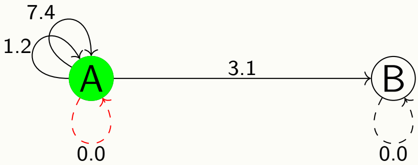


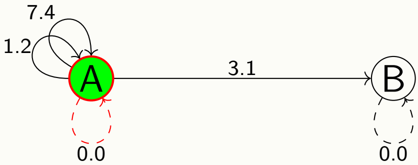
Reflexive Graphs

Schema $\mathbb{S}\mathrm{ch}_{\mathsf{R}}$:

$$\begin{tikzcd} \texttt{Float} & \mathsf{Edge} \arrow[l, "\mathsf{weight}"'] \arrow[r, shift left=2, "\mathsf{src}"] \arrow[r, shift right=2, "\mathsf{tgt}"'] & \mathsf{Vertex} \arrow[l, "\mathsf{loop}" description] \arrow[r, "\mathsf{label}"] & \texttt{String} \end{tikzcd}$$

$$\mathsf{loop} \cdot \mathsf{src} = \mathsf{id} = \mathsf{loop} \cdot \mathsf{tgt}$$

$$\mathsf{loop} \cdot \mathsf{weight} = 0.0$$

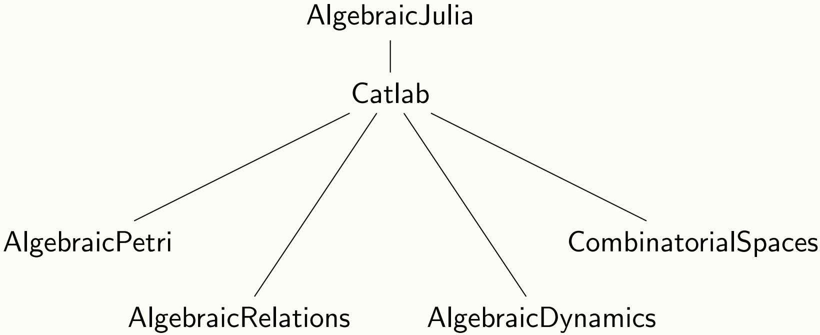


Questions?

Catlab

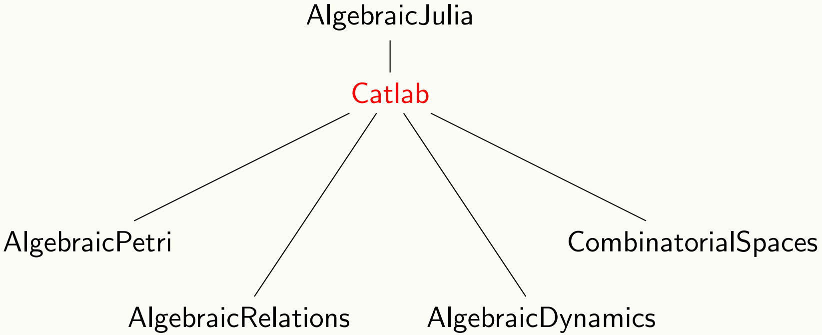
AlgebraicJulia

"Scientific computing based on applied category theory"



AlgebraicJulia

“Scientific computing based on applied category theory”



Data Migration

Schema functors

Schema functors create data transformations

$$\mathbb{S}\mathsf{ch}_\mathsf{G} \xrightarrow{\quad\quad\quad \mathbb{L} \quad\quad\quad} \mathbb{S}\mathsf{ch}_\mathsf{R}$$

Schema functors

Schema functors create data transformations

$$\begin{tikzcd} \mathbb{G}\mathsf{raph} & & \mathbb{R}\mathsf{Graph} \arrow[ll, "\Delta_{\mathbb{L}}"'] \\ \mathbb{S}\mathsf{ch}_{\mathsf{G}} \arrow[rr, "\mathbb{L}"] & & \mathbb{S}\mathsf{ch}_{\mathsf{R}} \end{tikzcd}$$

Schema functors

Schema functors create data transformations

$$\begin{tikzcd}
\mathbb{G}\mathsf{raph} & & \mathbb{R}\mathsf{Graph} \arrow[ll, "\Delta_{\mathbb{L}}"'] \\
\mathbb{S}\mathsf{ch}_{\mathsf{G}} \arrow[rr, "\mathbb{L}"] & & \mathbb{S}\mathsf{ch}_{\mathsf{R}} \arrow[d, "\mathbb{R}"] \\
& & \mathbb{S}\mathsf{et}
\end{tikzcd}$$

Schema functors

Schema functors create data transformations

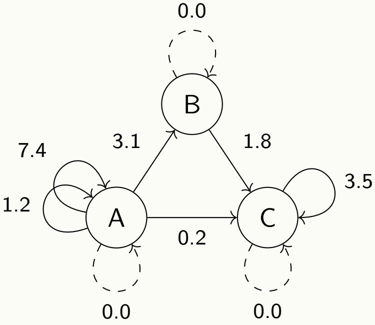
$$\mathbb{G}\mathsf{raph} \xleftarrow{\Delta_{\mathbb{L}}} \mathbb{R}\mathsf{Graph}$$

$$\begin{tikzcd}
\mathbb{S}\mathsf{ch}_{\mathsf{G}} \arrow[r, "\mathbb{L}"] \arrow[dr, dashed, "\Delta_{\mathbb{L}}(\mathbb{R}) := \mathbb{L} \cdot \mathbb{R}"'] & \mathbb{S}\mathsf{ch}_{\mathsf{R}} \arrow[d, "\mathbb{R}"] \\
 & \mathbb{S}\mathsf{et}
\end{tikzcd}$$

Schema functors

Schema functors create data transformations

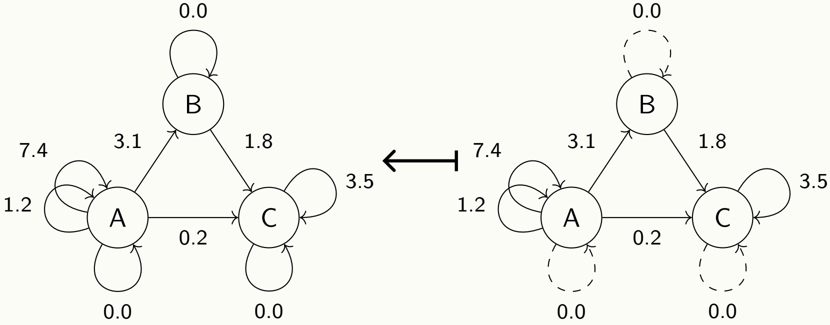
$$\mathbb{G}\text{raph} \xleftarrow{\Delta_{\mathbb{L}}} \mathbb{R}\text{Graph}$$



Schema functors

Schema functors create data transformations

$$\mathbb{G}\mathsf{raph} \xleftarrow{\;\Delta_{\mathbb{L}}\;} \mathbb{R}\mathsf{Graph}$$



Schema functors

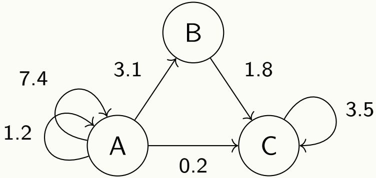
Schema functors create data transformations

$$\begin{tikzcd} \mathbb{G}\mathrm{raph} \arrow[rr, bend left, red, "\Sigma_{\mathbb{L}}"] & & \mathbb{R}\mathrm{Graph} \arrow[ll, "\Delta_{\mathbb{L}}"] \end{tikzcd}$$

Schema functors

Schema functors create data transformations

$$\begin{tikzcd} \mathbb{G}\mathsf{raph} \arrow[rr, bend left, "\Sigma_{\mathbb{L}}"] & & \mathbb{R}\mathsf{Graph} \arrow[ll, "\Delta_{\mathbb{L}}"] \end{tikzcd}$$



Schema functors

Schema functors create data transformations

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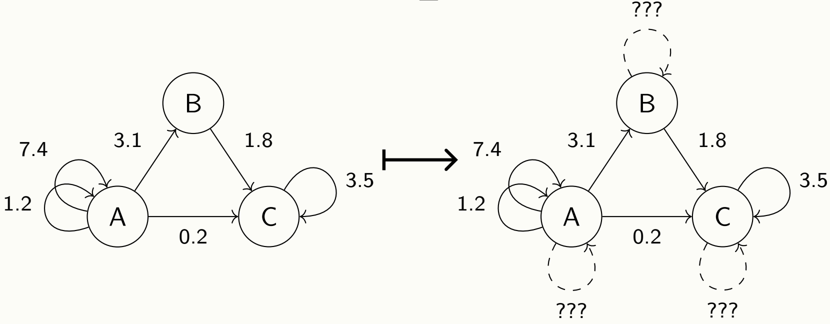
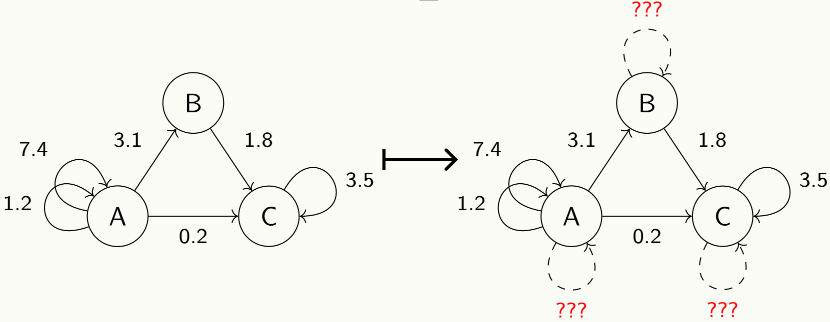


Schema functors

Schema functors create data transformations

$$\begin{tikzcd} \mathbb{G}\mathrm{raph} \arrow[rr, bend left, "\Sigma_{\mathbb{L}}"] & & \mathbb{R}\mathrm{Graph} \arrow[ll, "\Delta_{\mathbb{L}}"] \end{tikzcd}$$



Schema functors

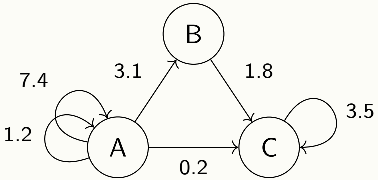
Schema functors create data transformations

$$\begin{tikzcd}
\mathbb{G}\mathsf{raph} \arrow[rr, bend left, "\Sigma_{\mathbb{L}}"] \arrow[rr, bend right, "\Pi_{\mathbb{L}}"'] & & \mathbb{R}\mathsf{Graph} \arrow[ll, "\Delta_{\mathbb{L}}" description]
\end{tikzcd}$$

Schema functors

Schema functors create data transformations

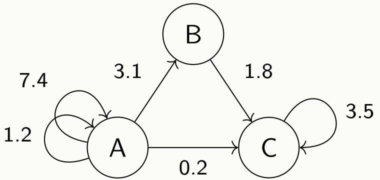
$$\begin{tikzcd}[column sep=huge] \mathbb{G}\mathrm{raph} \arrow[r, bend left, "\Sigma_{\mathbb{L}}"] \arrow[r, bend right, "\Pi_{\mathbb{L}}"'] & \mathbb{R}\mathrm{Graph} \arrow[l, "\Delta_{\mathbb{L}}" description] \end{tikzcd}$$



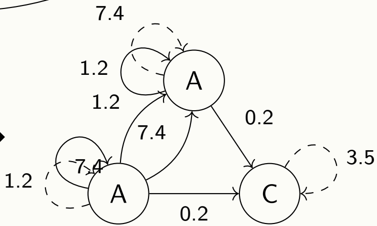
Schema functors

Schema functors create data transformations

$$\begin{tikzcd}
\mathbb{G}\mathrm{raph} \arrow[rr, bend left, "\Sigma_{\mathbb{L}}"] \arrow[rr, bend right, "\Pi_{\mathbb{L}}"'] & & \mathbb{R}\mathrm{Graph} \arrow[ll, "\Delta_{\mathbb{L}}" description]
\end{tikzcd}$$

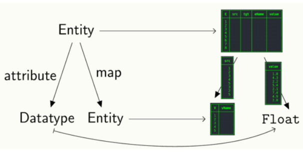


$\longmapsto$

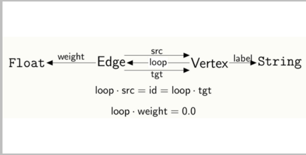


Questions?

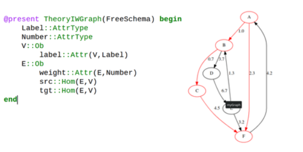
Data Model



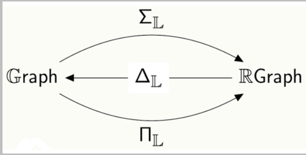
Graphs



Catlab



Migration



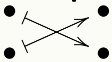
Processes

Building with blocks

The category $\mathbb{F}\mathrm{Set}$

All finite functions are built from four atomic components.

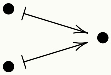
Swap



Id

Create

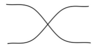
Merge



The category $\mathbb{F}\mathrm{Set}$

All finite functions are built from four atomic components.

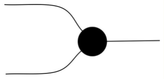
Swap



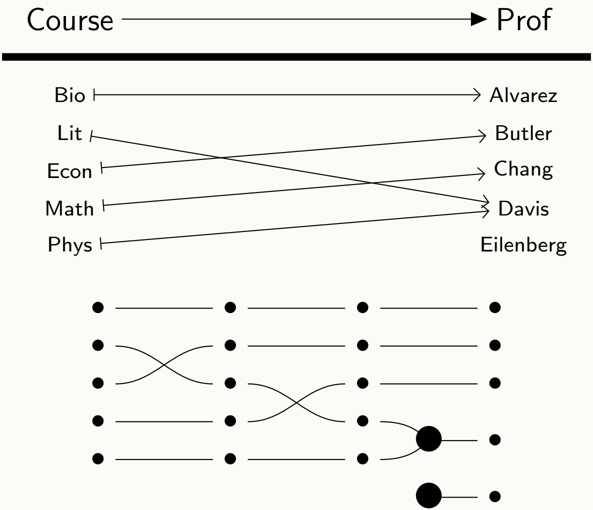
Id

Create

Merge

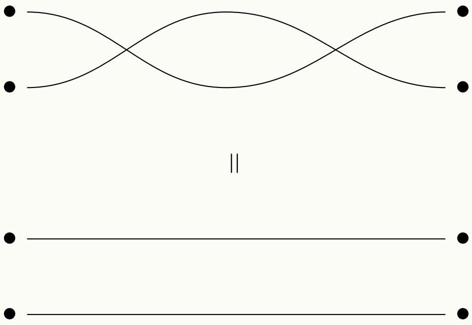


Building with bricks



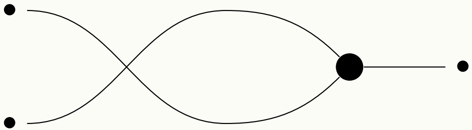
Rules of the game

$$\begin{tikzcd}
 & 2 \arrow[dr, "\mathsf{swap}"] & \\
2 \arrow[ur, "\mathsf{swap}"] \arrow[rr, "\mathsf{id}+\mathsf{id}"'] & & 2
\end{tikzcd}$$

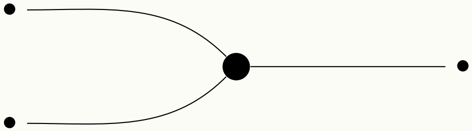


Rules of the game

$$
\begin{tikzcd}
 & 2 \arrow[dr, "\text{merge}"] & \\
2 \arrow[ur, "\text{swap}"] \arrow[rr, "\text{merge}"'] & & 1
\end{tikzcd}
$$

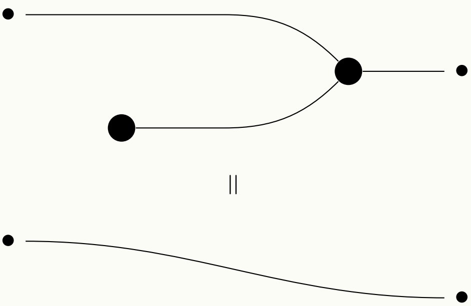


$\|$



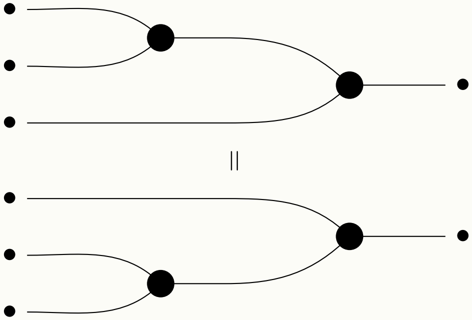
Rules of the game

$$\begin{tikzcd}
 & 2 \arrow[dr, "\text{merge}"] & \\
1 \arrow[ur, "\text{id+create}"] \arrow[rr, "\text{id}"'] & & 1
\end{tikzcd}$$



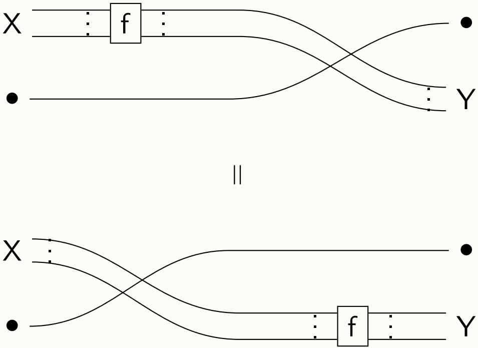
Rules of the game

$$\begin{tikzcd}[column sep=large]
 & 2 \arrow[dr, "\text{merge}"] & \\
3 \arrow[ur, "\text{merge+id}"] \arrow[dr, "\text{id+merge}"'] & & 1 \\
 & 2 \arrow[ur, "\text{merge}"'] &
\end{tikzcd}$$



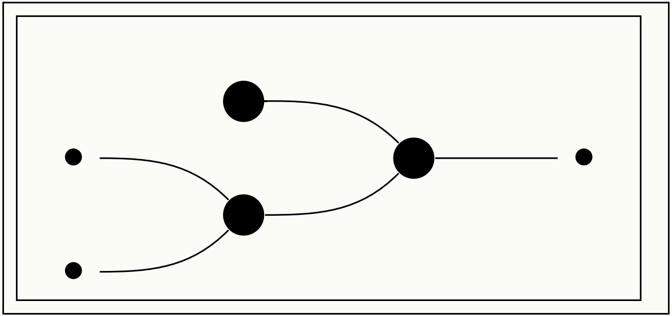
Rules of the game

$$
\begin{tikzcd}
 & Y+1 \arrow[dr, "\mathsf{swap}_X"] & \\
X+1 \arrow[ur, "f+id"] \arrow[dr, "\mathsf{swap}_X"'] & \forall\ f & 1+Y \\
 & 1+X \arrow[ur, "id+f"'] &
\end{tikzcd}
$$



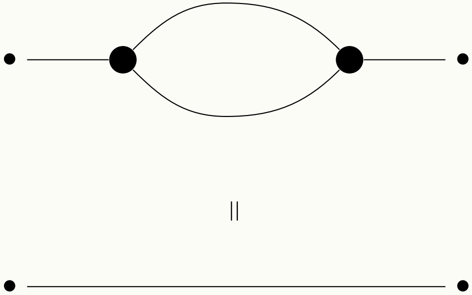
Picture Proofs

$$(\text{create}+\text{merge})\cdot\text{merge} = \text{merge}$$



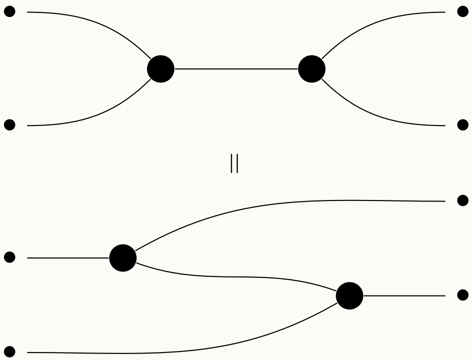
Put it in backwards?

$$
\begin{tikzcd}
 & 2 \arrow[dl, "\mathrm{merge}^{\mathrm{op}}"'] \arrow[dr, "\mathrm{merge}"] & \\
1 \arrow[rr, "\mathrm{id}"'] & & 1
\end{tikzcd}
$$

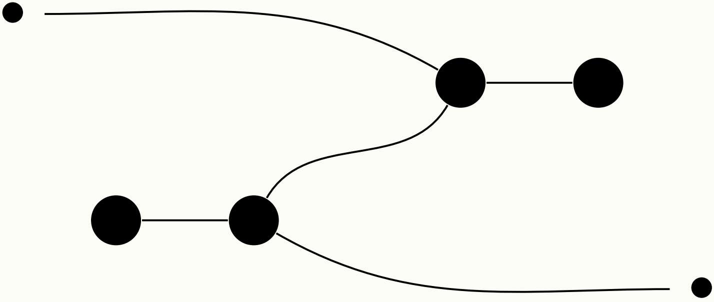


Put it in backwards?

$$\begin{tikzcd}
 & 1 \arrow[dr, leftrightarrow, "\mathrm{merge}^{\mathrm{op}}"] & \\
2 \arrow[ur, "\mathrm{merge}"] \arrow[dr, leftrightarrow, "\mathrm{merge}^{\mathrm{op}}+\mathrm{id}"'] & & 2 \\
 & 3 \arrow[ur, "\mathrm{id}+\mathrm{merge}"'] &
\end{tikzcd}$$

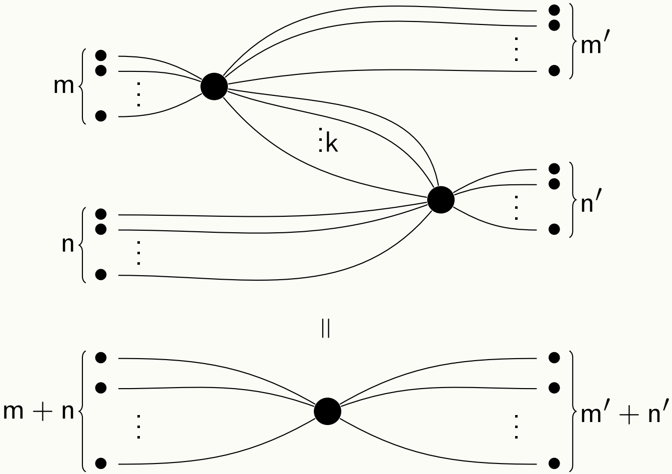


Yanking Lemma



The Spider Theorem

Connected diagrams with identical inputs/outputs are equal.



Questions?

Monoidal Categories

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$

$$\dfrac{X \xrightarrow{f} Y \qquad X' \xrightarrow{f'} Y'}{X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y'}$$

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$$

$$X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y' \xrightarrow{g \otimes g'} Z \otimes Z'$$

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

$$\begin{tikzcd} X \otimes X' \arrow[r, "f \otimes f'"] \arrow[rr, bend right, dashed, red, "(f \otimes f') \cdot (g \otimes g')"'] & Y \otimes Y' \arrow[r, "g \otimes g'"] & Z \otimes Z' \end{tikzcd}$$

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, dashed, red, "f \cdot g"] & Y \arrow[r, "g"] & Z \\
X' \arrow[r, "f'"] \arrow[rr, bend right, dashed, red, "f' \cdot g'"'] & Y' \arrow[r, "g'"] & Z'
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \otimes X' \arrow[r, "f \otimes f'"] \arrow[rr, bend right, dashed, "(f \otimes f') \cdot (g \otimes g')"'] & Y \otimes Y' \arrow[r, "g \otimes g'"] & Z \otimes Z'
\end{tikzcd}
$$

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$

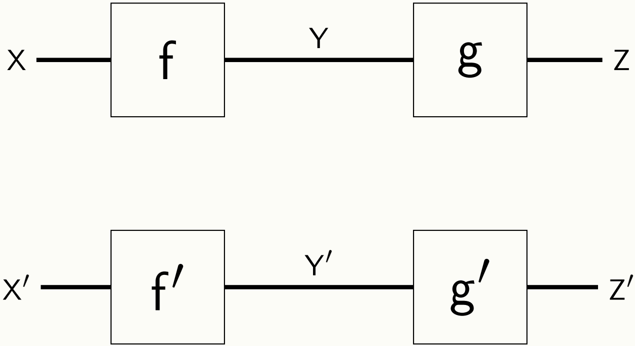
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, dashed, "f \cdot g"] & Y \arrow[r, "g"] & Z \\ X' \arrow[r, "f'"] \arrow[rr, bend right, dashed, "f' \cdot g'"'] & Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

$$\begin{tikzcd} X \otimes X' \arrow[r, "f \otimes f'"] \arrow[rr, bend right, dashed, red, "\substack{(f \otimes f') \cdot (g \otimes g') \\ = \\ (f \cdot g) \otimes (f' \cdot g')}"'] & Y \otimes Y' \arrow[r, "g \otimes g'"] & Z \otimes Z' \end{tikzcd}$$

String diagrams

The interchange law:

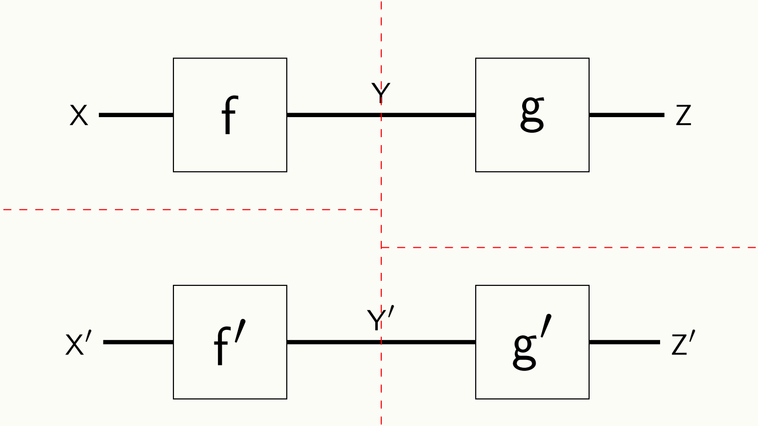
$$(f \otimes f') \cdot (g \otimes g') = (f \cdot g) \otimes (f' \cdot g')$$



String diagrams

The interchange law:

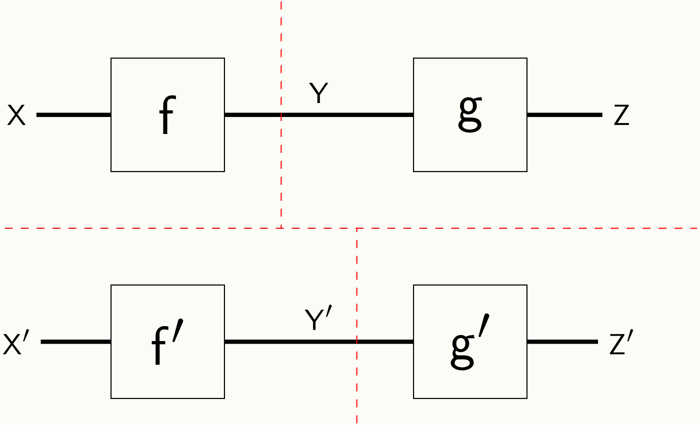
$$(f \otimes f') \cdot (g \otimes g') = (f \cdot g) \otimes (f' \cdot g')$$



String diagrams

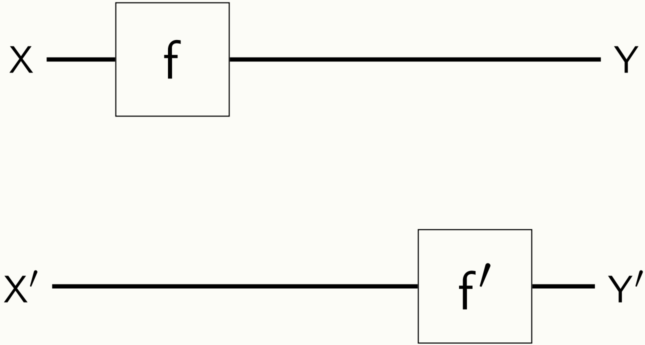
The interchange law:

$$(f \otimes f') \cdot (g \otimes g') = \color{red}{(f \cdot g) \otimes (f' \cdot g')}$$



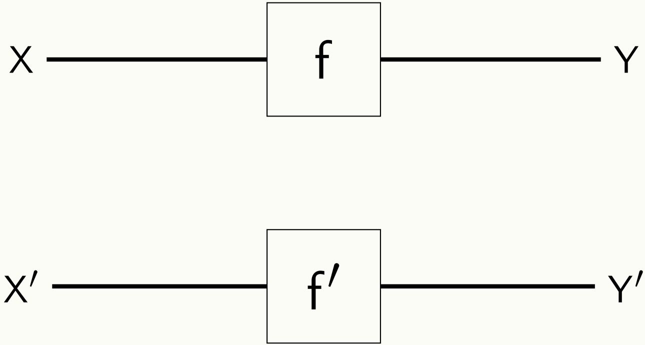
String diagrams

Logical time only!



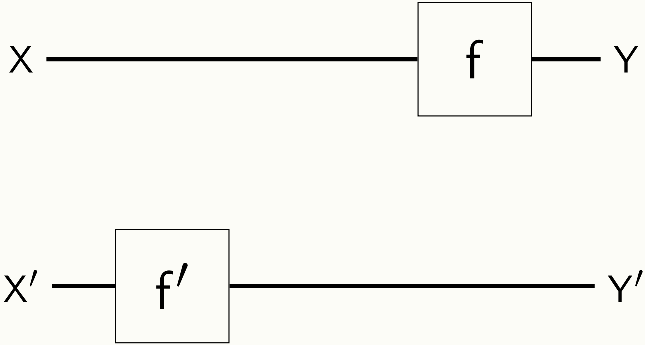
String diagrams

Logical time only!



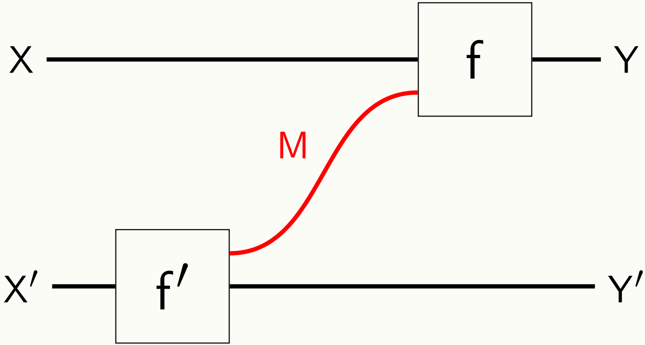
String diagrams

Logical time only!

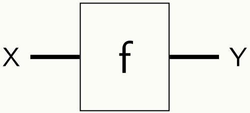
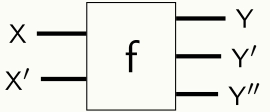
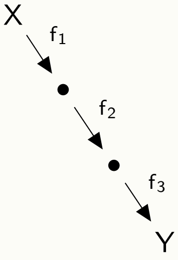
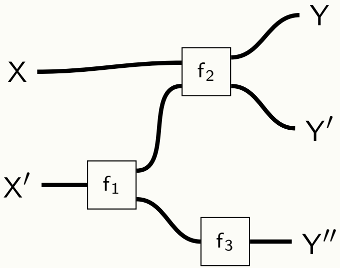


String diagrams

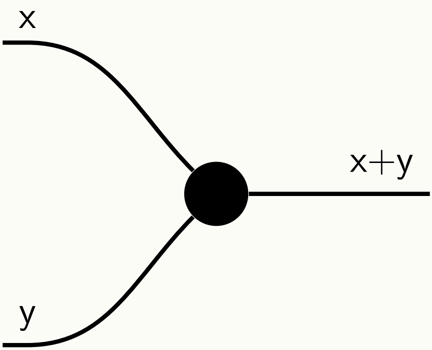
Logical time only! Sequence by messaging



Many in, many out

Context	Vanilla categories	Monoidal categories
Basic unit	$X \xrightarrow{f} Y$	$X \otimes X' \xrightarrow{f} Y \otimes Y' \otimes Y''$
		
Composite		

Values on the wire

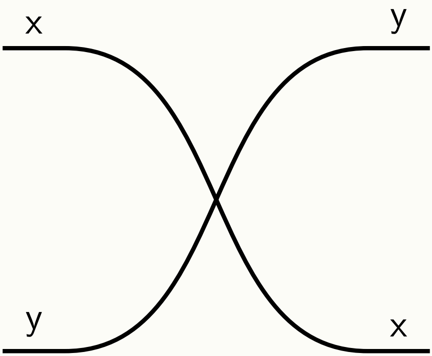


Merge

Values on the wire

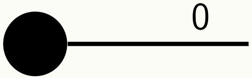
Identity

Values on the wire



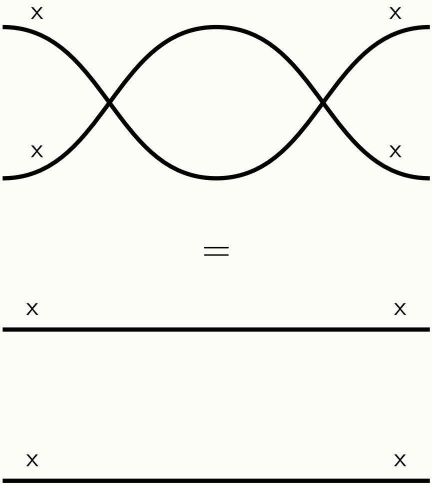
Swap

Values on the wire



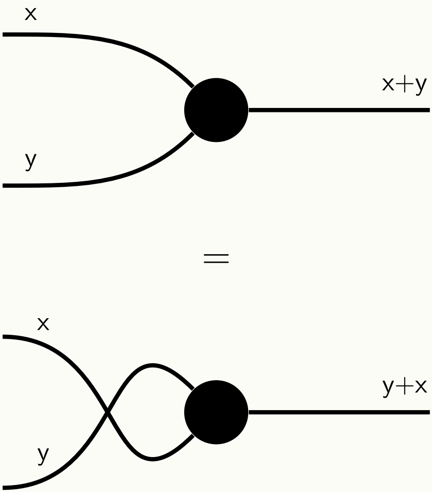
Create

Following the rules



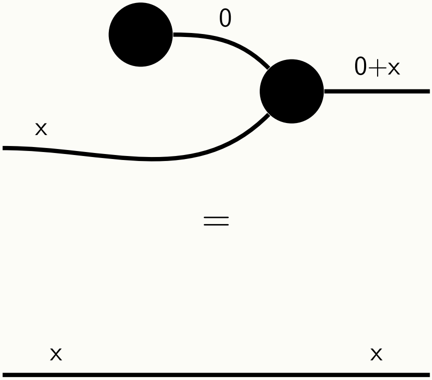
Double swap

Following the rules



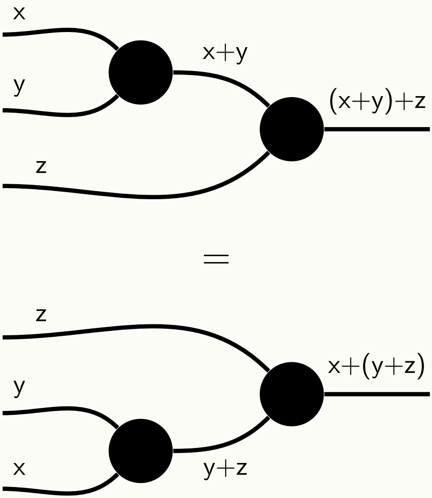
Commutativity

Following the rules



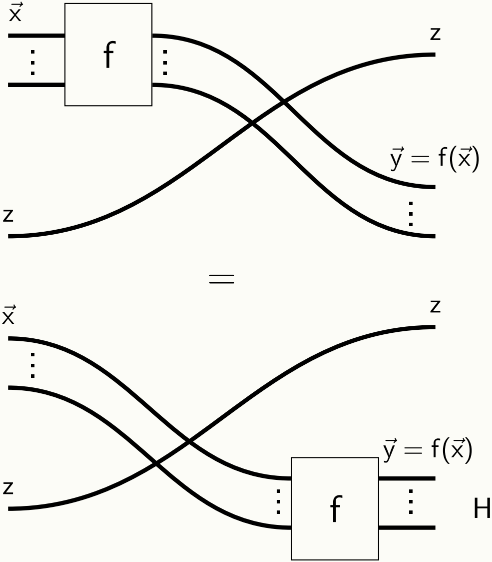
Unit

Following the rules



Associativity

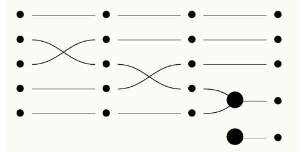
Following the rules



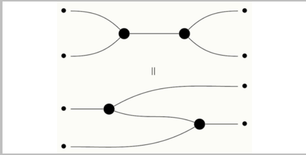
Hop the wire

Questions?

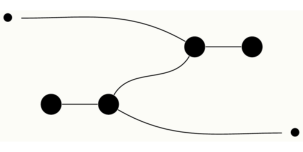
Generators



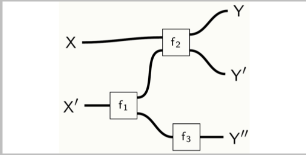
Rules



Proof



String diagrams



Linear
Algebra

Monoidal Functors

$$(\mathbb{F}\mathsf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathsf{et}, \times, 1)$$

Monoidal Functors

$$(\mathbb{F}\mathsf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathsf{et}, \times, 1)$$

1

Monoidal Functors

$$\begin{tikzcd}
(\mathbb{F}\mathsf{Set}, +, \emptyset) \arrow[r, "\mathbb{M}"] & (\mathbb{S}\mathsf{et}, \times, 1) \\
1 \arrow[r, mapsto] & \mathsf{X} := \mathbb{M}(1)
\end{tikzcd}$$

Monoidal Functors

$$(\mathbb{F}\mathsf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathsf{et}, \times, 1)$$

$$1 \longmapsto \mathsf{X} := \mathbb{M}(1)$$

$$2 \cong 1 + 1$$

Monoidal Functors

$$(\mathbb{F}\mathsf{Set}, +, \emptyset) \xrightarrow{\quad\quad\mathbb{M}\quad\quad} (\mathbb{S}\mathsf{et}, \times, 1)$$

$$1 \longmapsto \mathsf{X} := \mathbb{M}(1)$$

$$2 \cong 1 + 1 \longmapsto \mathsf{X} \times \mathsf{X} \cong \mathsf{X}^2$$

Monoidal Functors

$$(\mathbb{F}\mathsf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathsf{et}, \times, 1)$$

$$1 \longmapsto \mathsf{X} := \mathbb{M}(1)$$

$$\mathsf{I} \cong 1 + \ldots + 1 \longmapsto \mathsf{X} \times \ldots \times \mathsf{X} \cong \mathsf{X}^{\mathsf{I}}$$

Monoidal Functors

$$(\mathbb{F}\mathsf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathsf{et}, \times, 1)$$

$$1 \longmapsto \mathsf{X} := \mathbb{M}(1)$$

$$\begin{array}{c} \mathsf{I} \\ \downarrow{\scriptstyle \mathsf{f}} \\ \mathsf{J} \end{array}$$

Monoidal Functors

$$
\begin{tikzcd}
(\mathbb{F}\mathsf{Set}, +, \emptyset) \arrow[rr, "\mathbb{M}"] & & (\mathbb{S}\mathsf{et}, \times, 1) \\
1 \arrow[rr, mapsto] & & X := \mathbb{M}(1) \\
I \arrow[dd] & & X^I \arrow[dd] \\
f \arrow[rr, mapsto] & & X^f \\
J & & X^J
\end{tikzcd}
$$

Monoidal Functors

$$\begin{tikzcd}
(\mathbb{F}\mathsf{Set}, +, \emptyset) \arrow[rr, "\mathbb{M}"] & & (\mathbb{S}\mathsf{et}, \times, 1)
\end{tikzcd}$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{tikzcd}
I \arrow[d, "f"] & & X^I \arrow[d, "X^f"'] & \langle x_i \rangle \arrow[d, mapsto] \\
J & & X^J & \left\langle \sum_{f(i)=j} x_i \right\rangle
\end{tikzcd}
\qquad f \longmapsto X^f$$

Addition defines a *monoidal functor*

Monoidal Functors

$$(\mathbb{F}\mathrm{Set}, +, \emptyset) \xrightarrow{\color{red}(\mathrm{Int}, +, 0)} (\mathbb{S}\mathrm{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{tikzcd}
I \arrow[d] & & X^I \arrow[d] & \langle x_i \rangle \arrow[d, mapsto] \\
J & f \longmapsto X^f & X^J & \left\langle \sum_{f(i)=j} x_i \right\rangle
\end{tikzcd}$$

Monoidal Functors

$$
\begin{tikzcd}
(\mathbb{F}\mathrm{Set}, +, \emptyset) \arrow[rr, "(\textcolor{red}{\mathrm{Float}}, +, 0)"] & & (\mathbb{S}\mathrm{et}, \times, 1)
\end{tikzcd}
$$

$$
1 \longmapsto X := \mathbb{M}(1)
$$

$$
\begin{tikzcd}
I \arrow[d] & & X^I \arrow[d] & \langle x_i \rangle \arrow[d, mapsto] \\
J & f \longmapsto X^f & X^J & \left\langle \sum_{f(i)=j} x_i \right\rangle
\end{tikzcd}
$$

Monoidal Functors

$$
\begin{tikzcd}
(\mathbb{F}\mathrm{Set}, +, \emptyset) \arrow[rr, "(\mathrm{Float}^{n}, +, 0)"] & & (\mathbb{S}\mathrm{et}, \times, 1) & \\
1 \arrow[rr, mapsto] & & X := \mathbb{M}(1) & \\
I \arrow[d, "f"] & & X^{I} \arrow[d, "X^{f}"'] & \langle x_i \rangle \arrow[d, mapsto] \\
J & & X^{J} & \left\langle \sum_{f(i)=j} x_i \right\rangle
\end{tikzcd}
$$

$$f \longmapsto X^{f}$$

Monoidal Functors

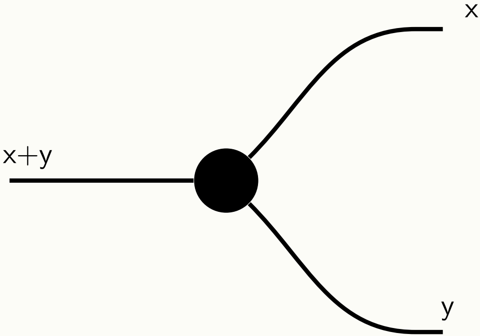
$$(\mathbb{F}\mathsf{Set}, +, \emptyset) \xrightarrow{(\mathsf{Float}, \textcolor{red}{\mathsf{min}}, 0)} (\mathbb{S}\mathsf{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{tikzcd}
I \arrow[d] & & X^I \arrow[d] & \langle x_i \rangle \arrow[d, mapsto] \\
J & f \longmapsto X^f & X^J & \left\langle \textcolor{red}{\min\limits_{f(i)=j}} \{x_i\} \right\rangle
\end{tikzcd}$$

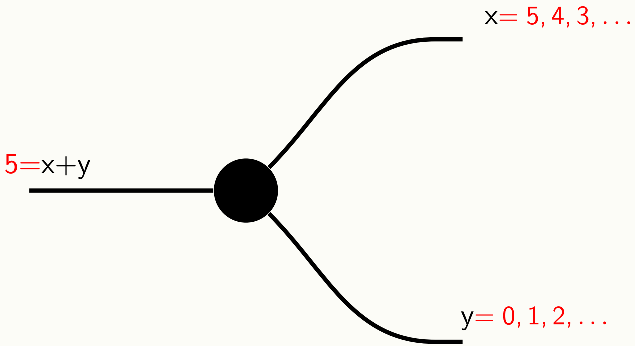
Put it in backwards?

$$(\mathbb{F}\mathsf{Set}^{\color{red}{\mathsf{op}}}, +, \emptyset) \xrightarrow{???} (\mathbb{S}\mathsf{et}, \times, 1)$$



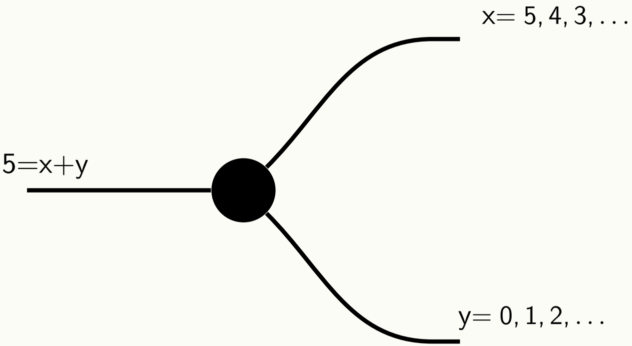
Put it in backwards?

$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{???} \cancel{(\mathbb{S}\mathsf{et}, \times, 1)}$$



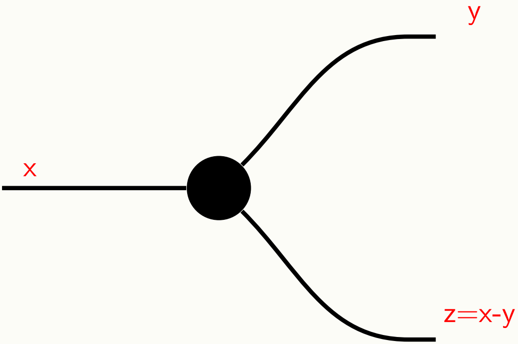
Put it in backwards?

$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{\color{red}\checkmark} {\color{red}(\mathbb{R}\mathsf{el}, \times, 1)}$$

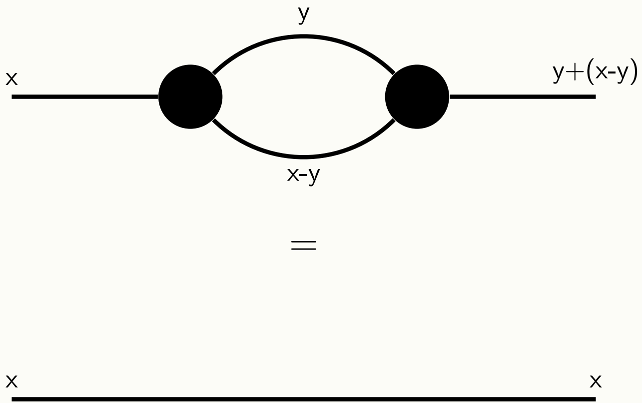


Put it in backwards?

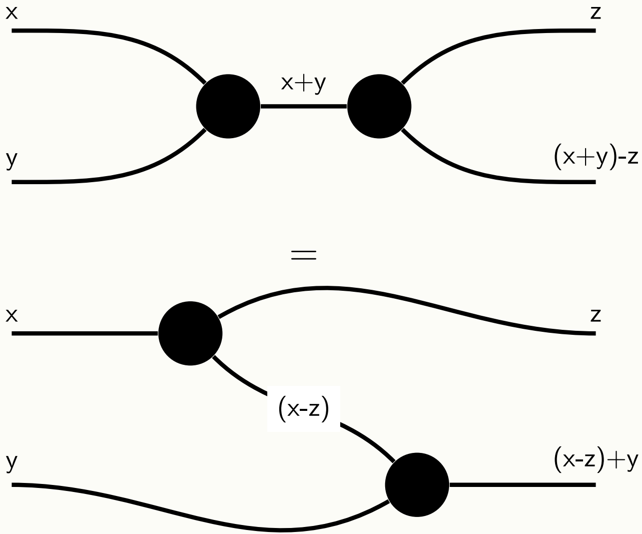
$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{\quad\checkmark\quad} (\mathbb{R}\mathsf{el}, \times, 1)$$



More rules



More rules



Semantic extension

$$(\mathbb{F}\mathsf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathsf{et}, \times, 1)$$

Semantic extension

$$\begin{tikzcd}
(\mathbb{F}\mathsf{Set}, +, \emptyset) \arrow[r, "\mathbb{M}"] \arrow[d, "\mathbb{I}"'] & (\mathbb{S}\mathsf{et}, \times, 1) \\
(\mathbb{F}\mathsf{Rel}, +, \emptyset) &
\end{tikzcd}$$

Semantic extension

$$\begin{tikzcd}
(\mathbb{F}\mathsf{Set}, +, \emptyset) \arrow[r, "\mathbb{M}"] \arrow[d, "\mathbb{I}"'] & (\mathbb{S}\mathsf{et}, \times, 1) \\
(\mathbb{F}\mathsf{Rel}, +, \emptyset) \arrow[ur, dashed, "???"] &
\end{tikzcd}$$

Semantic extension

$$\begin{tikzcd}
(\mathbb{F}\mathsf{Set}, +, \emptyset) \arrow[rr, "\mathbb{M}"] \arrow[d, "\mathbb{I}"'] & & (\mathbb{S}\mathsf{et}, \times, 1) \\
(\mathbb{F}\mathsf{Rel}, +, \emptyset) \arrow[urr, dashed, "???"] & &
\end{tikzcd}$$

$\supseteq \mathbb{F}\mathsf{Set}^{\mathsf{op}}$

Semantic extension

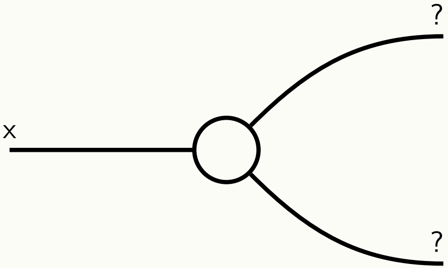
$$\begin{tikzcd}
(\mathbb{F}\mathsf{Set}, +, \emptyset) \arrow[r, "\mathbb{M}"] \arrow[d, "\mathbb{I}"'] & (\mathbb{S}\mathsf{et}, \times, 1) \arrow[d, "\mathbb{I}"] \\
(\mathbb{F}\mathsf{Rel}, +, \emptyset) & (\mathbb{R}\mathsf{el}, \times, 1)
\end{tikzcd}$$

Semantic extension

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(\mathbb{F}\mathsf{Set}, +, \emptyset) \arrow[r, "\mathbb{M}"] \arrow[d, "\mathbb{I}"'] & (\mathbb{S}\mathsf{et}, \times, 1) \arrow[d, "\mathbb{I}"] \\
(\mathbb{F}\mathsf{Rel}, +, \emptyset) \arrow[r, "\mathbb{M}^\dagger"'] & (\mathbb{R}\mathsf{el}, \times, 1)
\end{tikzcd}$$

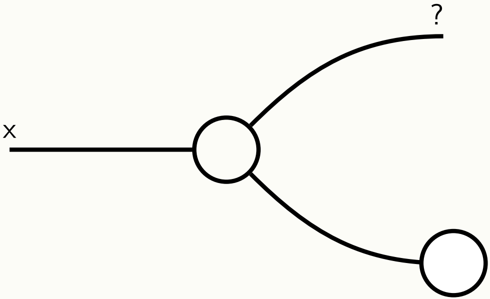
Force it in backwards?

$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{???} (\mathbb{S}\mathsf{et}, \times, 1)$$



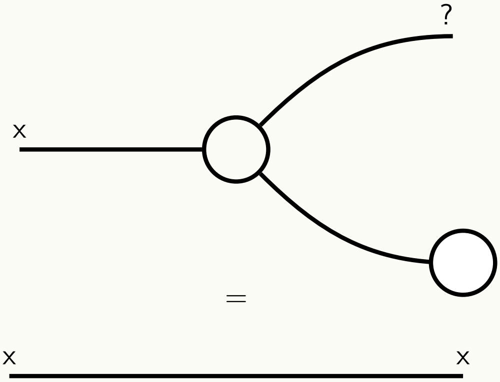
Force it in backwards?

$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{???} (\mathbb{S}\mathsf{et}, \times, 1)$$



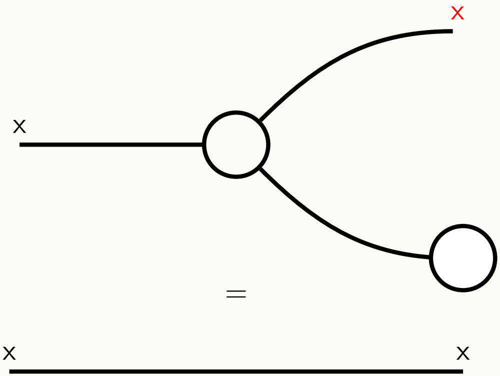
Force it in backwards?

$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{???} (\mathbb{S}\mathsf{et}, \times, 1)$$



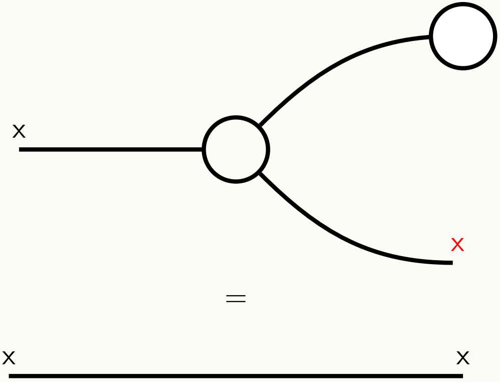
Force it in backwards?

$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{???} (\mathbb{S}\mathsf{et}, \times, 1)$$



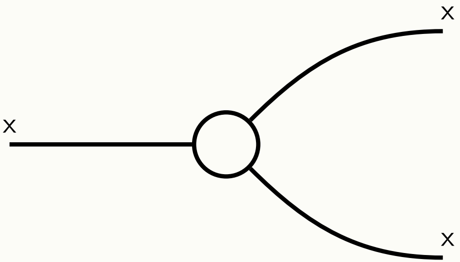
Force it in backwards?

$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{???} (\mathbb{S}\mathsf{et}, \times, 1)$$

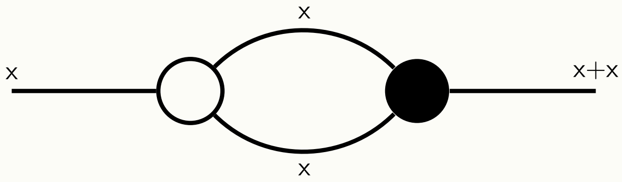


Force it in backwards?

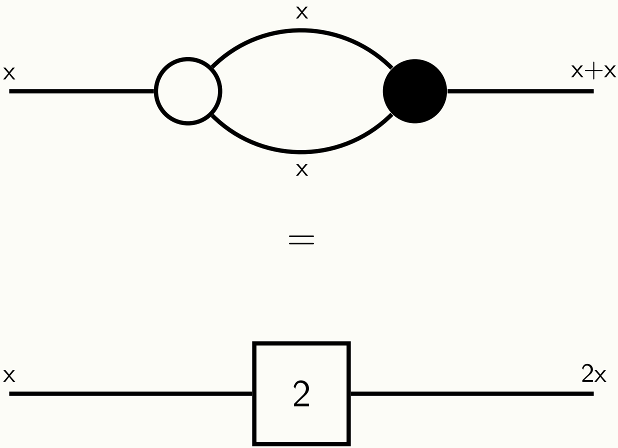
$$(\mathbb{F}\mathsf{Set}^{\mathsf{op}}, +, \emptyset) \xrightarrow{\color{red}{(\mathsf{X},\mathsf{copy})}} (\mathbb{S}\mathsf{et}, \times, 1)$$



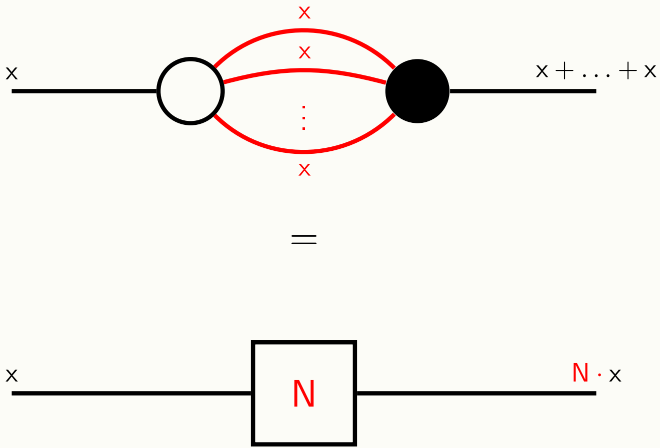
Add meets copy



Add meets copy



Add meets copy



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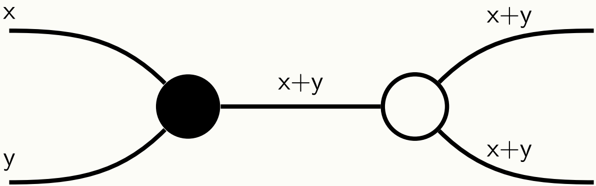

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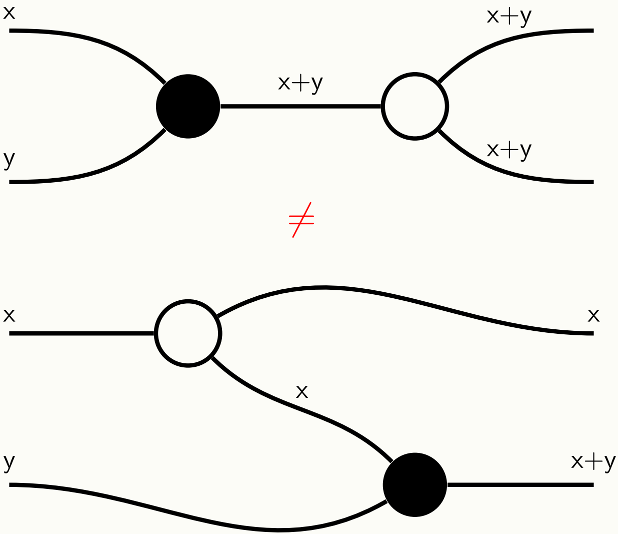
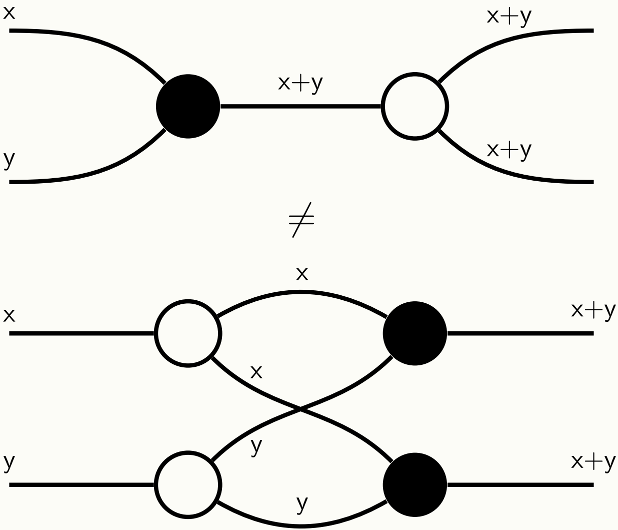
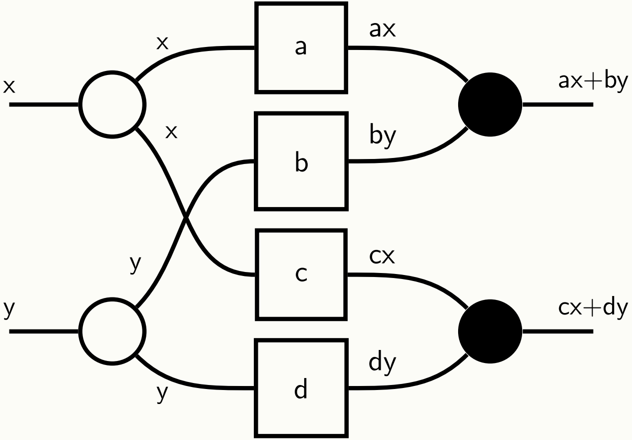


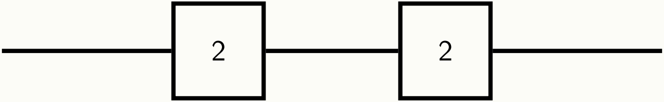
Add meets copy



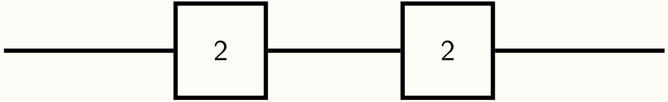
Matrices



Diagrams multiply



Diagrams multiply



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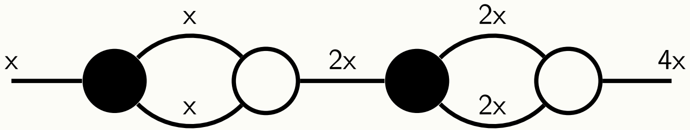
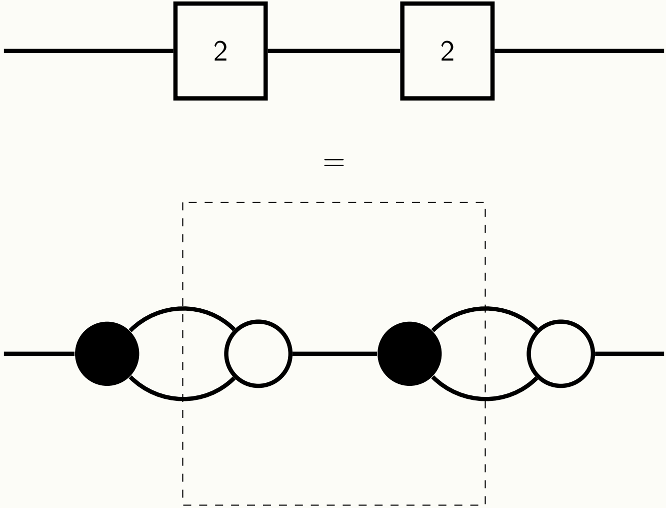
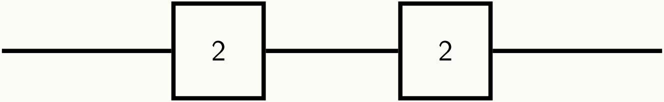


Diagrams multiply



Diagrams multiply



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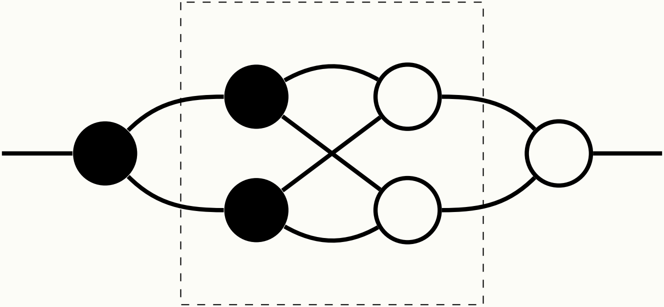
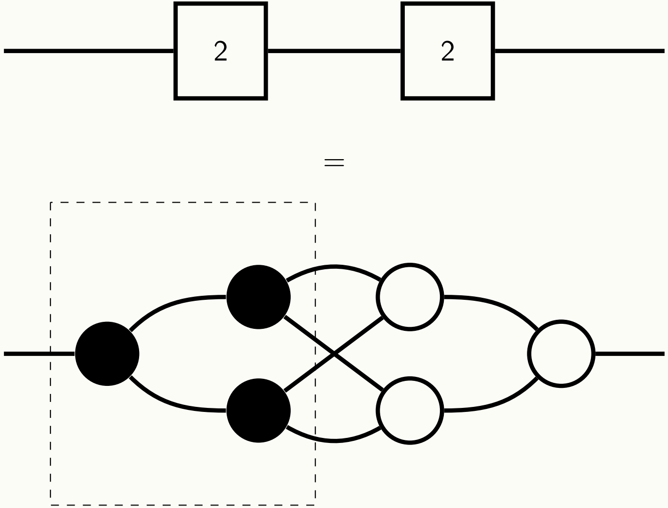
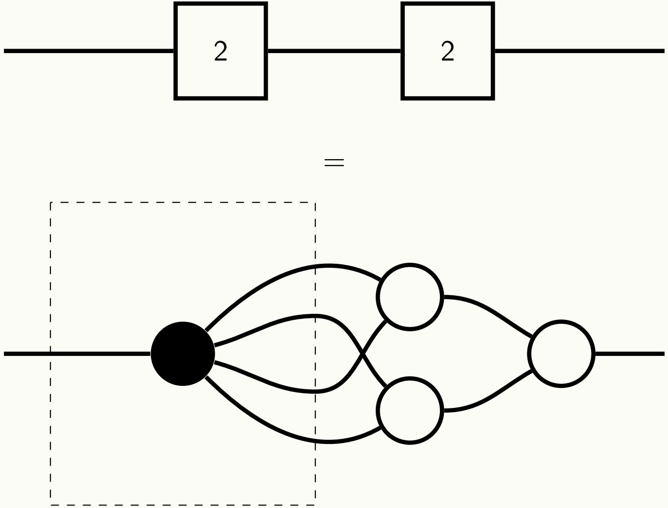


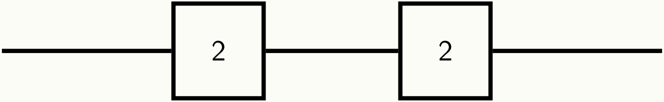
Diagrams multiply



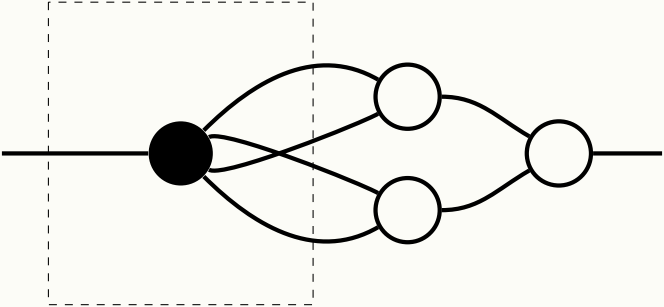
Diagrams multiply



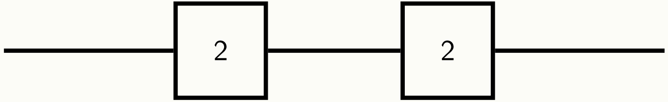
Diagrams multiply



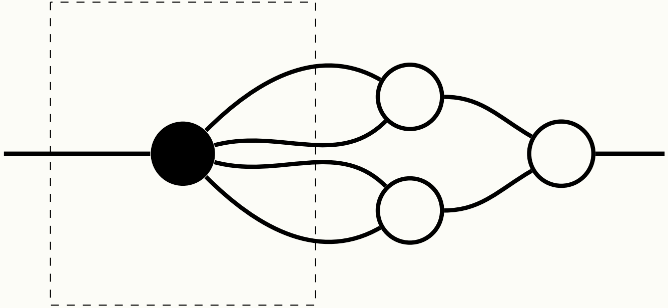
$$=$$



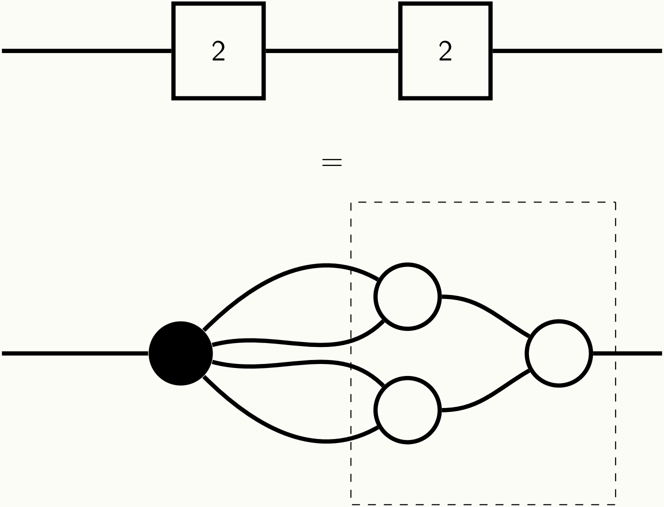
Diagrams multiply



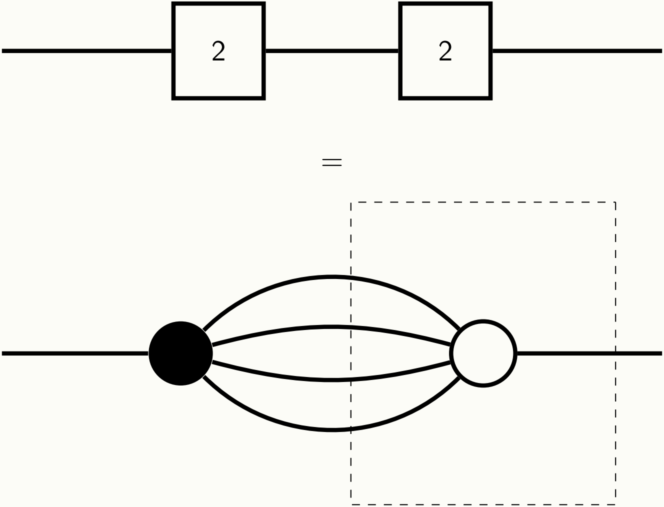
$$=$$



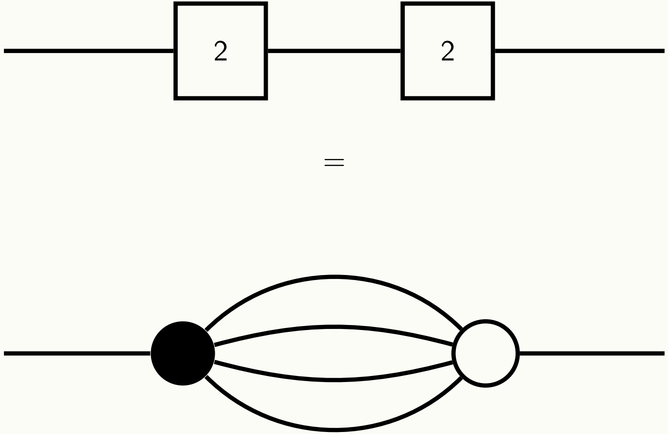
Diagrams multiply



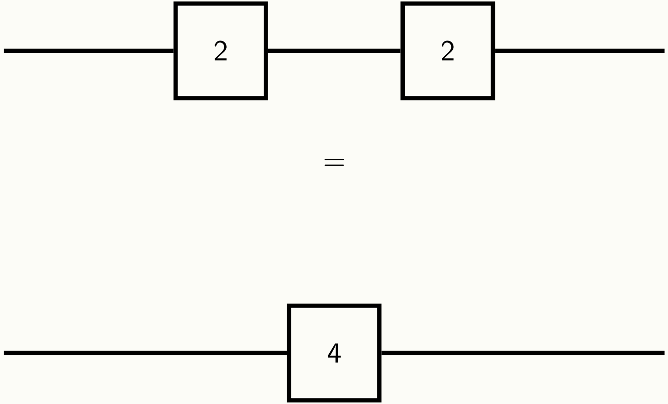
Diagrams multiply



Diagrams multiply

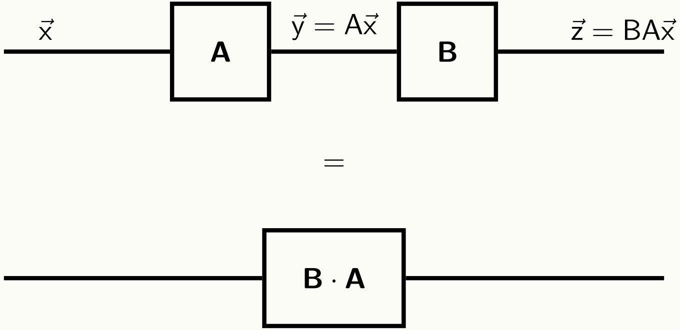


Diagrams multiply



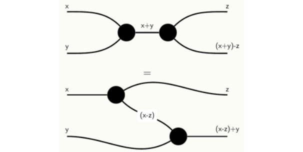
Homework

For matrices A ($k \times n$) and B ($m \times k$), show



Questions?

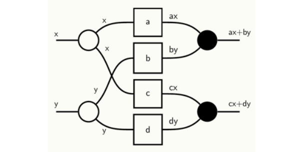
Addition^{op}



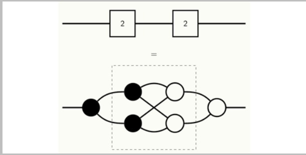
Semantic extension

$$\begin{tikzcd}
(\mathbb{F}\mathrm{Set}, +, \emptyset) \arrow[r, "\mathbb{M}"] \arrow[d, "\mathbb{I}"'] & (\mathbb{S}\mathrm{et}, \times, 1) \arrow[d, "\mathbb{I}"] \\
(\mathbb{F}\mathrm{Rel}, +, \emptyset) \arrow[r, "\mathbb{M}^\dagger"'] & (\mathbb{R}\mathrm{el}, \times, 1)
\end{tikzcd}$$

Matrices



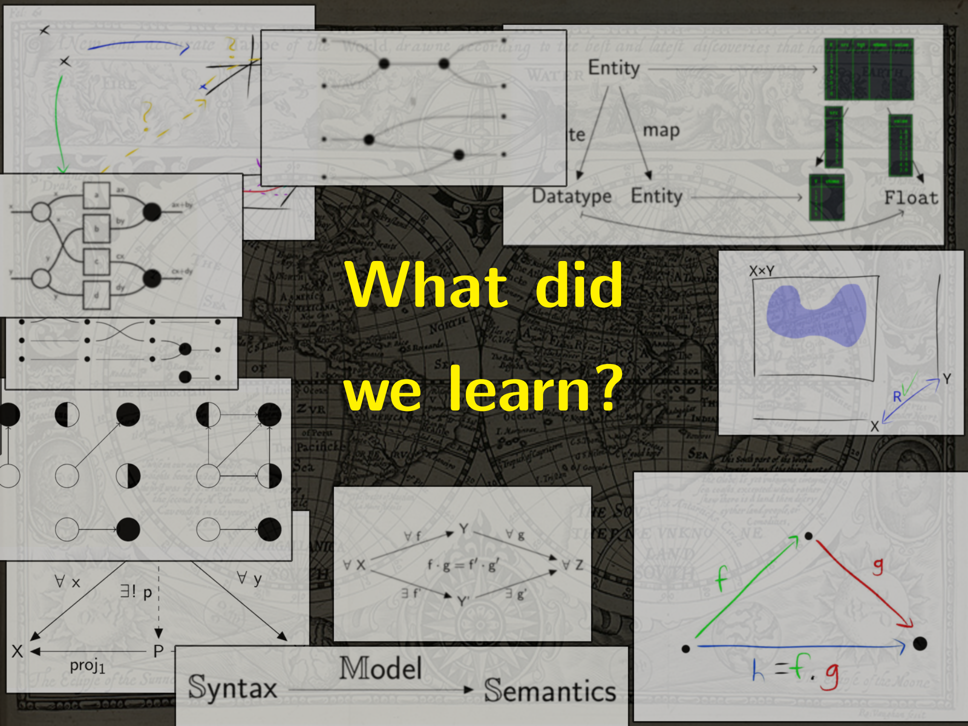
Multiplication

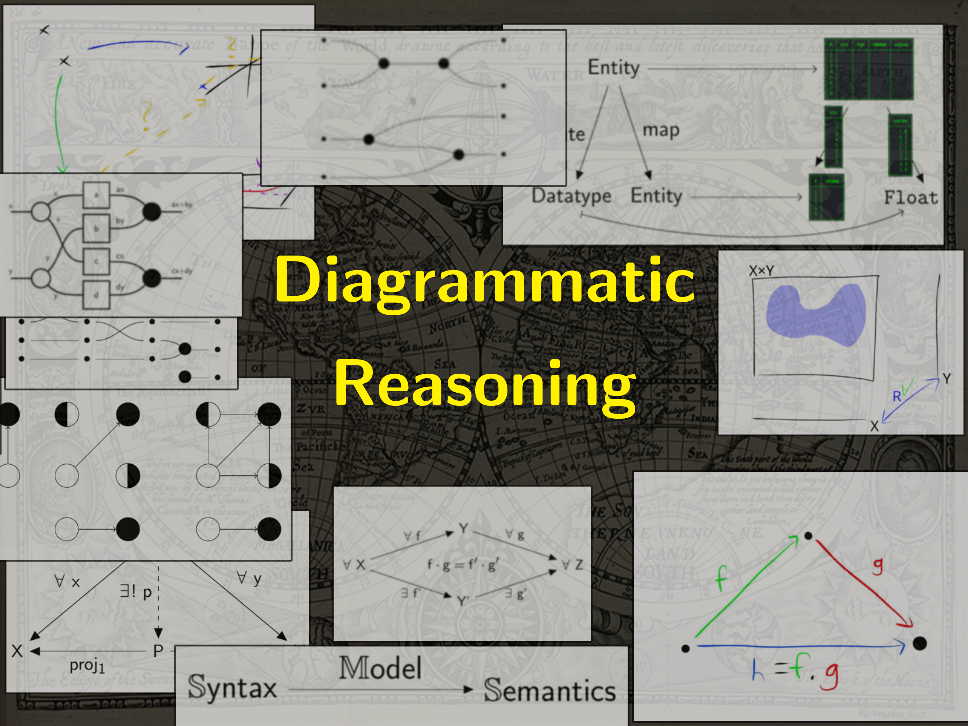


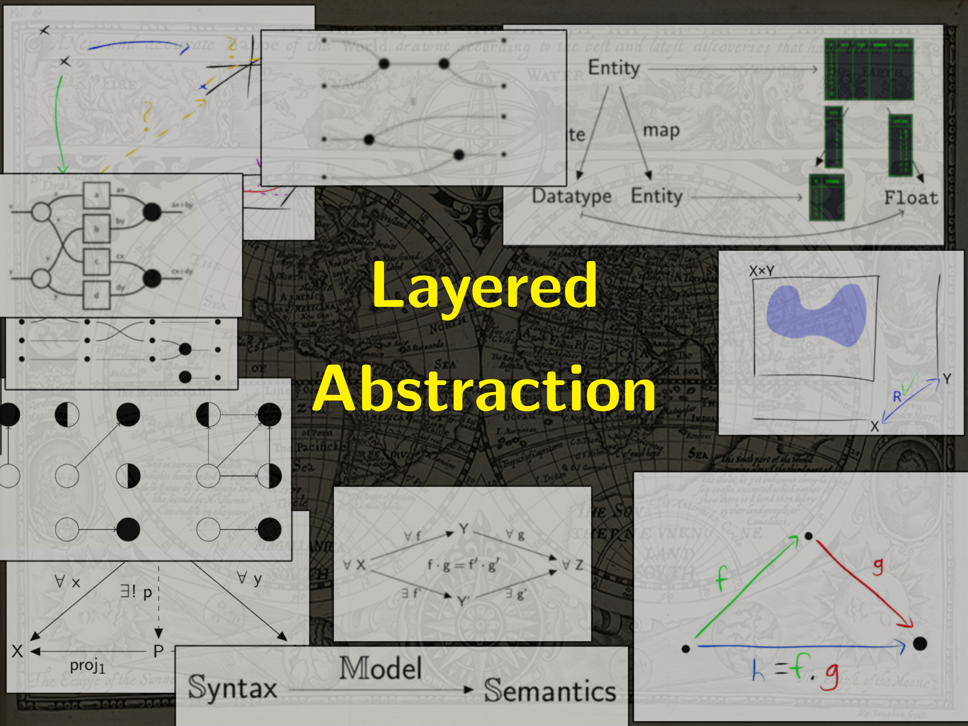
Wrapping up

What we did

Part 1	Composition	$\mathbb{S}$et, $\mathbb{R}$el
	Categories	Diagrams
	Structure	$X \cong Y$, $X \times Y$
Part 2	Models	$\mathbb{M} : \mathbb{S}\text{yn} \to \mathbb{S}\text{em}$
	Data	`Catlab`
	Processes	Add $\heartsuit$ Copy


What did we learn?
Entity
map
Datatype
Entity
Float
X×Y
R
Y
X
∀ f
Y
∀ g
∀ X
f · g = f′ · g′
∀ Z
∃ f′
Y′
∃ g′
∀ x
∃! p
∀ y
X
P
proj1
Model
Syntax
Semantics
f
g
h = f . g


Diagrammatic
Reasoning
Entity
map
Datatype
Entity
Float
X×Y
R
X
Y
∀ x
∃! p
∀ y
X
proj1
P
∀ f
Y
∀ g
∀ X
f · g = f′ · g′
∀ Z
∃ f′
Y′
∃ g′
Syntax
Model
Semantics
f
g
h = f . g


Layered Abstraction
Entity
map
te
Datatype
Entity
Float
X×Y
R
Y
X
∀ f
Y
∀ g
∀ X
f · g = f′ · g′
∀ Z
∃ f′
Y′
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∀ x
∃! p
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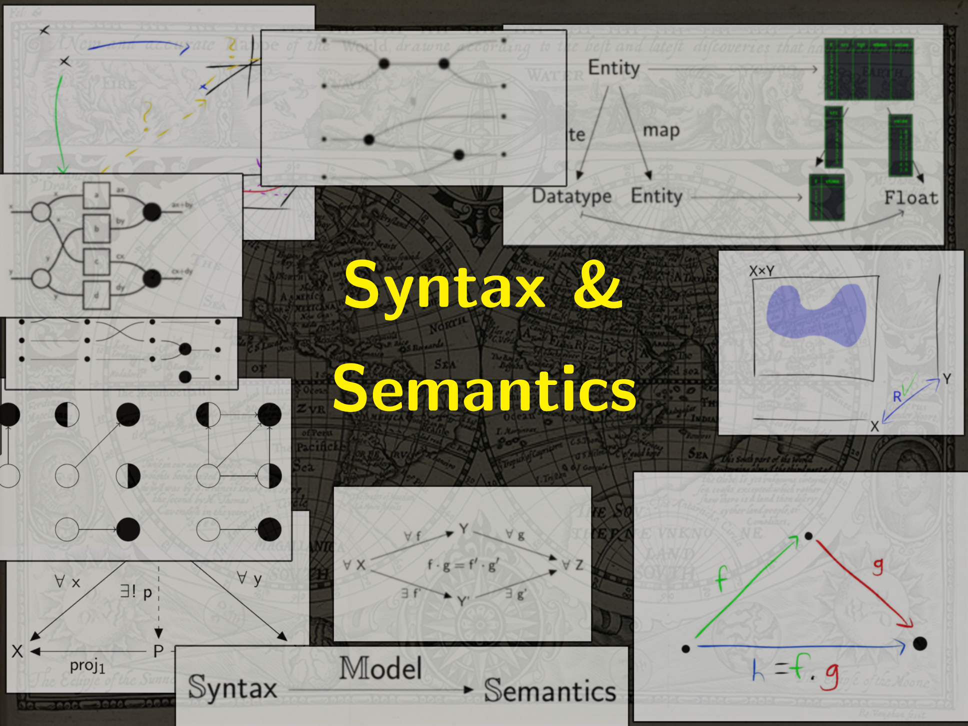
X
P
proj1
Model
Syntax
Semantics
f
g
h = f . g

Syntax & Semantics



What we didn't do

Real applications

Cyber physical systems	Robotics
Graph transformation	Design
Data Analytics	Databases
Programming	Epidemiology
Networks	Ecology
Neural Networks	Language
Molecular Biology	Probability

What we didn't do

Real proofs

Books

Spivak & Fong - Seven Sketches

Lawvere & Schanuel - Conceptual Mathematics

Spivak - CT for the sciences

Papers

Bradley - What is applied category theory?

Baez & Stay - A Rosetta Stone

Coecke, et al. - Mathematical theory of resources

More resources

Blogs

AlgebraicJulia

Math3ma

The n-Category Café

Videos

Cheng - Category Theory in Life

Compositionality (Simons Inst.)

Compositional Robotics (ICRA workshop)

What's still missing

User Interface

Use cases

Methodology

Thank You!

Please join us for an open discussion on category theory and systems engineering after the break.

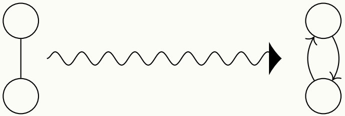
Endnotes

Symmetric Graphs

Schema:

$$\begin{tikzcd} \text{Edge} \arrow[loop left, "\text{rev}"] \arrow[r, shift left, "\text{src}"] \arrow[r, shift right, "\text{tgt}"'] & \text{Vertex} \end{tikzcd}$$

$\text{rev}\cdot\text{rev}=\text{id}$ $\qquad$ $\text{rev}\cdot\text{src}=\text{tgt}$



Symmetric Graphs

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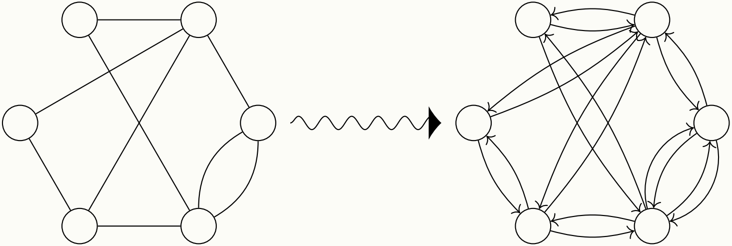

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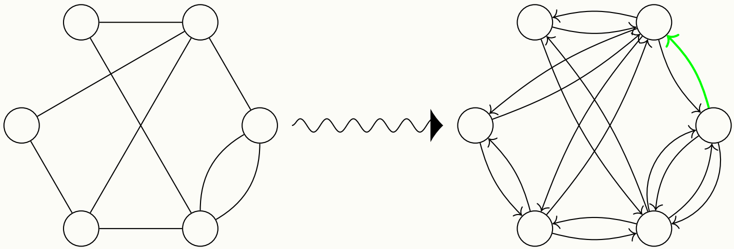

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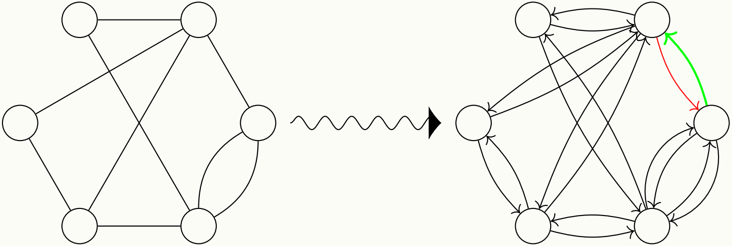

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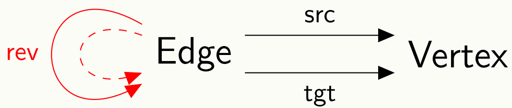

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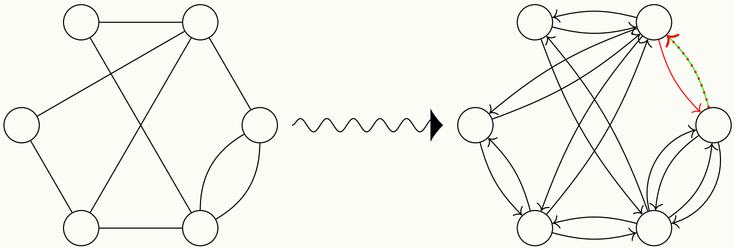

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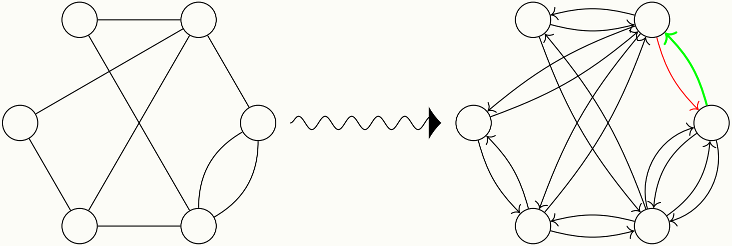

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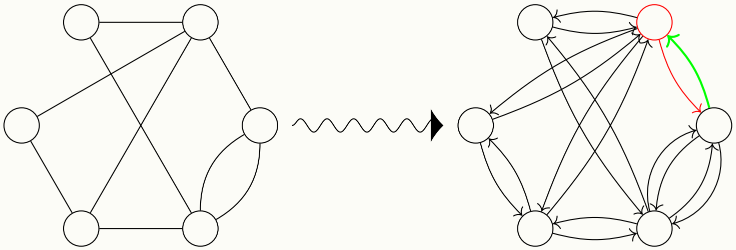

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Equational reasoning:

$\mathsf{rev} \cdot \mathsf{tgt}$

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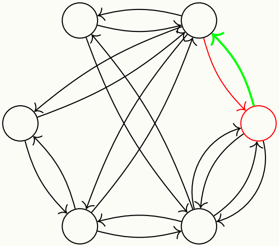

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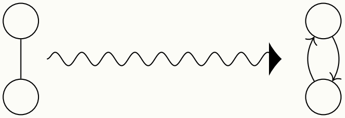

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\end{tikzcd}
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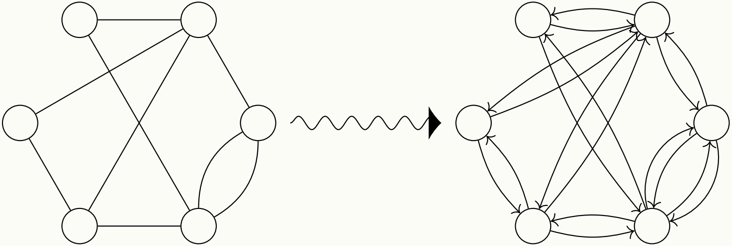

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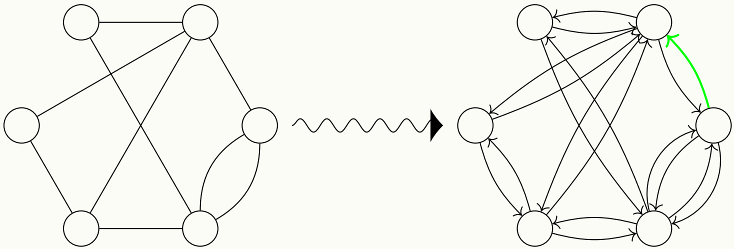

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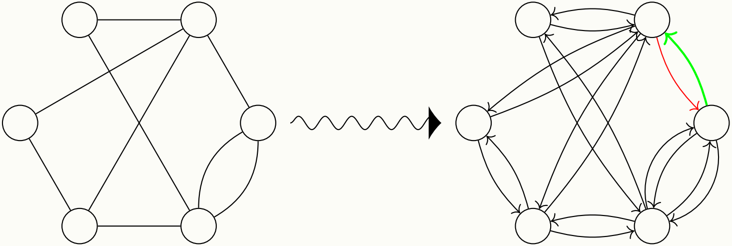

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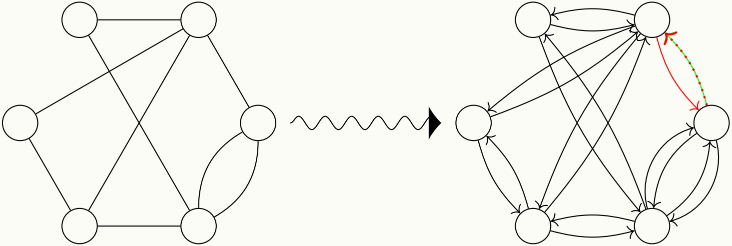

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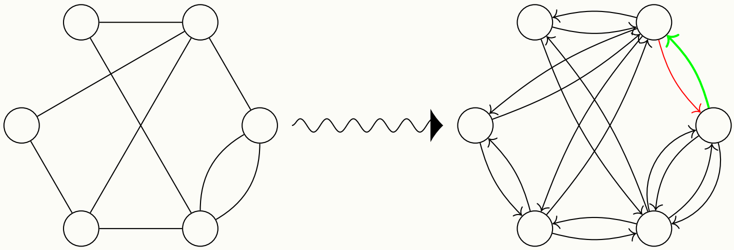

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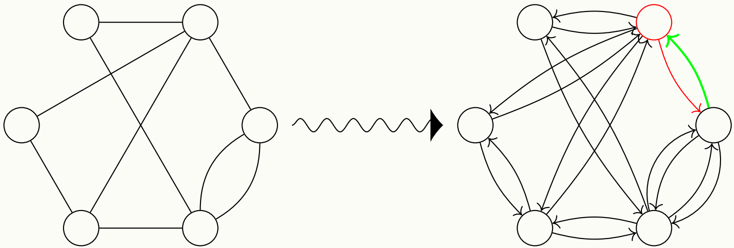

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Equational reasoning:

$\text{rev} \cdot \text{tgt}$

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Equational reasoning:

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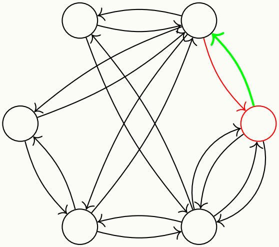
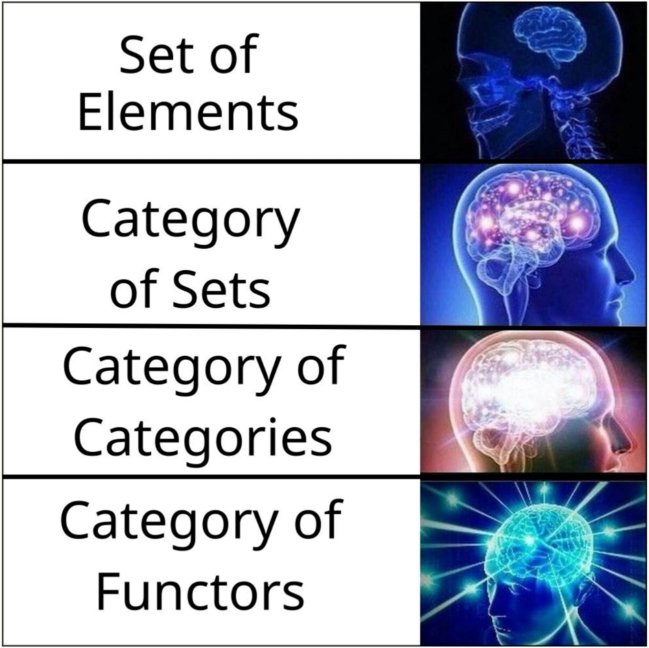


Natural Transformations


Set of Elements
Category of Sets
Category of Categories
Category of Functors

Natural Transformations

Definition: Given two functors $\mathbb{F}, \mathbb{G} : \mathbb{X} \rightrightarrows \mathbb{Y}$, a *natural transformation* $\mathbb{t} : \mathbb{F} \Rightarrow \mathbb{G}$ is

- a function from objects $\mathsf{X} \in \mathbb{X}$ to arrows $\mathbb{F}(\mathsf{X}) \to \mathbb{G}(\mathsf{X}) \in \mathbb{Y}$
- satisfying naturality $\mathbb{t}(\mathsf{X}) \cdot \mathbb{G}(\mathsf{h}) = \mathbb{F}(\mathsf{h}) \cdot \mathbb{t}(\mathsf{X}')$

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$$
\begin{tikzcd}
\mathbb{F}(\mathsf{X}) \arrow[r, dash, "\mathbb{t}(\mathsf{X})"] \arrow[d, dash, "\mathbb{F}(\mathsf{h})"'] & \mathbb{G}(\mathsf{X}) \arrow[d, dash, "\mathbb{G}(\mathsf{h})"] \\
\mathbb{F}(\mathsf{X}') \arrow[r, dash, "\mathbb{t}(\mathsf{X}')"'] & \mathbb{G}(\mathsf{X}')
\end{tikzcd}
$$

Projections are natural

$$\mathbb{p}\mathsf{roj}_1 : - \times \mathsf{A} \Rightarrow \mathbb{id}$$

Projections are natural

$$\mathbb{S}\mathrm{et} \xrightarrow{- \times \mathsf{A}} \mathbb{S}\mathrm{et}$$

$$\mathbb{p}\mathrm{roj}_1 : - \times \mathsf{A} \Rightarrow \mathbb{id}$$

Projections are natural

$$\begin{tikzcd} \mathbb{S}\mathsf{et} \arrow[r, bend left, "- \times \mathsf{A}"] \arrow[r, bend right, "\mathbb{id}"'] & \mathbb{S}\mathsf{et} \end{tikzcd}$$

$$\mathbb{p}\mathsf{roj}_1 : - \times \mathsf{A} \Rightarrow \mathbb{id}$$

Projections are natural

$$\begin{tikzcd}[column sep=huge] \mathbb{S}\mathsf{et} \arrow[r, bend left, "- \times \mathsf{A}"{name=U}] \arrow[r, bend right, "\mathbb{id}"'{name=D}] \arrow[r, phantom, "\Downarrow \mathbb{p}\mathsf{roj}_1"] & \mathbb{S}\mathsf{et} \end{tikzcd}$$

$$\color{red}{\mathbb{p}\mathsf{roj}_1} : - \times \mathsf{A} \Rightarrow \mathbb{id}$$

Projections are natural

$$\begin{tikzcd}[column sep=huge] \mathbb{S}\mathsf{et} \arrow[r, bend left, "- \times A"] \arrow[r, bend right, "\mathbb{id}"'] \arrow[r, phantom, "\Downarrow \mathbb{p}\mathsf{roj}_1"] & \mathbb{S}\mathsf{et} \end{tikzcd}$$

X

Projections are natural

$$\begin{tikzcd} \mathbb{S}\mathrm{et} \arrow[r, bend left, "- \times A"] \arrow[r, bend right, "\mathbb{id}"'] \arrow[r, phantom, "\Downarrow \mathbb{p}\mathrm{roj}_1"] & \mathbb{S}\mathrm{et} \end{tikzcd}$$

$$X \longmapsto X \times A$$

Projections are natural

$$\begin{tikzcd} \mathbb{S}\mathrm{et} \arrow[rr, bend left, "- \times A"] \arrow[rr, bend right, "\mathbb{id}"'] \arrow[rr, phantom, "\Downarrow \mathbb{p}\mathrm{roj}_1"] & & \mathbb{S}\mathrm{et} \end{tikzcd}$$

$$X \times A$$

$$X \longmapsto X$$

Projections are natural

$$\begin{tikzcd}
\mathbb{S}\mathsf{et} \arrow[rr, bend left, "- \times A"] \arrow[rr, bend right, "\mathbb{id}"'] \arrow[rr, phantom, "\Downarrow \mathbb{proj}_1"] & & \mathbb{S}\mathsf{et}
\end{tikzcd}$$

$$X \longmapsto \mathbb{proj}_1(X) \qquad \begin{tikzcd} X \times A \arrow[d] \\ X \end{tikzcd}$$

Projections are natural

$$\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[r, bend left, "- \times A"] \arrow[r, bend right, "\mathbb{id}"'] \arrow[r, phantom, "\Downarrow \mathbb{p}\mathrm{roj}_1"] & \color{red}{\mathbb{S}\mathrm{et}}
\end{tikzcd}$$

$$\begin{tikzcd}
X & \color{red}{X \times A} \arrow[dr, "\mathbb{p}\mathrm{roj}_1(X)"'] & \\
& & X
\end{tikzcd}$$

$\color{red}{\langle x, a \rangle}$

Projections are natural

$$\begin{tikzcd} \mathbb{S}\mathsf{et} \arrow[rr, bend left, "- \times \mathsf{A}"] \arrow[rr, bend right, "\mathbb{id}"'] \arrow[rr, phantom, "\Downarrow \mathbb{p}\mathsf{roj}_1"] && \color{red}{\mathbb{S}\mathsf{et}} \end{tikzcd}$$

X

$$\color{red}{\begin{tikzcd} \mathsf{X} \times \mathsf{A} \arrow[dr, "\mathbb{p}\mathsf{roj}_1(\mathsf{X})"'] & \\ & \mathsf{X} \end{tikzcd}}$$

$$\color{red}{\begin{tikzcd} \langle \mathsf{x}, \mathsf{a} \rangle \arrow[dr, mapsto] & \\ & \mathsf{x} \end{tikzcd}}$$

Projections are natural

$$\begin{tikzcd}
\color{red}{\mathbb{S}\mathrm{et}} \arrow[r, bend left, "- \times A"] \arrow[r, bend right, "\mathbb{id}"'] \arrow[r, phantom, "\Downarrow \mathbb{proj}_1"] & \mathbb{S}\mathrm{et}
\end{tikzcd}$$

$$\color{red}{X \xrightarrow{h} X'}$$

$$\begin{tikzcd}
X \times A \arrow[dr, "\mathbb{proj}_1(X)"'] & \\
& X
\end{tikzcd}$$

$$\begin{tikzcd}
\langle x, a \rangle \arrow[dr, mapsto] & \\
& x
\end{tikzcd}$$

Projections are natural

$$
\begin{tikzcd}
\mathbb{S}\mathsf{et} \arrow[rr, bend left, "- \times A"] \arrow[rr, bend right, "\mathbb{id}"'] \arrow[rr, phantom, "\Downarrow \mathbb{proj}_1"] & & \mathbb{S}\mathsf{et}
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "h"] & X'
\end{tikzcd}
\qquad X' \mapsto \mathbb{proj}_1(X')
$$

$$
\begin{tikzcd}
X \times A \arrow[dr, "\mathbb{proj}_1(X)"'] & \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \times A \arrow[dr, "\mathbb{proj}_1(X')"] & \\
& X'
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle x, a \rangle \arrow[dr, mapsto] & \\
& x
\end{tikzcd}
$$

Projections are natural

$$\begin{tikzcd}
\mathbb{S}\mathsf{et} \arrow[rr, bend left, "- \times A"] \arrow[rr, bend right, "\mathbb{id}"'] \arrow[rr, phantom, "\Downarrow \mathbb{proj}_1"] & & \mathbb{S}\mathsf{et}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "h"] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times A \arrow[r, "h \times A"] \arrow[dr, "\mathbb{proj}_1(X)"'] & X' \times A \arrow[dr, "\mathbb{proj}_1(X')"] & \\
& X & X'
\end{tikzcd}$$

$$\begin{tikzcd}
\langle x, a \rangle \arrow[dr, mapsto] & \\
& x
\end{tikzcd}$$

Projections are natural

$$
\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[rr, bend left, "- \times \mathsf{A}"] \arrow[rr, bend right, "\mathbb{id}"', red] \arrow[rr, phantom, "\Downarrow \mathbb{proj}_1"] & & \mathbb{S}\mathrm{et}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{h}", red] & \mathsf{X}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{X} \times \mathsf{A} \arrow[rr, "\mathsf{h} \times \mathsf{A}"] \arrow[dr, "\mathbb{proj}_1(\mathsf{X})"'] & & \mathsf{X}' \times \mathsf{A} \arrow[dr, "\mathbb{proj}_1(\mathsf{X}')"] & \\
& \mathsf{X} \arrow[rr, "\mathsf{h}"', red] & & \mathsf{X}'
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle \mathsf{x}, \mathsf{a} \rangle \arrow[dr, mapsto] & \\
& \mathsf{x}
\end{tikzcd}
$$

Projections are natural

$$
\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[rr, bend left, "- \times A"] \arrow[rr, bend right, "\mathbb{id}"'] \arrow[rr, phantom, "\Downarrow \mathbb{proj}_1"] & & \mathbb{S}\mathrm{et}
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "h"] & X'
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \times A \arrow[rr, "h \times A"] \arrow[dr, "\mathbb{proj}_1(X)"'] & & X' \times A \arrow[dr, "\mathbb{proj}_1(X')"] & \\
& X \arrow[rr, "h"'] & & X'
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle x, a \rangle \arrow[dr, mapsto] & & \\
& x \arrow[rr, mapsto] & & h(x)
\end{tikzcd}
$$

Projections are natural

$$
\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[rr, bend left, "- \times A"] \arrow[rr, bend right, "\mathbb{id}"'] \arrow[rr, phantom, "\Downarrow \mathbb{proj}_1"] & & \color{red}{\mathbb{S}\mathrm{et}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "h"] & X'
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \times A \arrow[r, red, "h \times A"] \arrow[dr, "\mathbb{proj}_1(X)"'] & X' \times A \arrow[dr, "\mathbb{proj}_1(X')"] & \\
 & X \arrow[r, "h"'] & X'
\end{tikzcd}
$$

$$
\begin{tikzcd}
\langle x, a \rangle \arrow[r, mapsto, red] \arrow[dr, mapsto] & \color{red}{\langle h(x), a \rangle} & \\
 & x \arrow[r, mapsto] & h(x)
\end{tikzcd}
$$

Projections are natural

$$\begin{tikzcd}[column sep=huge] \mathbb{S}\mathrm{et} \arrow[r, bend left, "- \times \mathsf{A}"] \arrow[r, bend right, "\mathbb{id}"'] \arrow[r, phantom, "\Downarrow \mathbb{proj}_1"] & \mathbb{S}\mathrm{et} \end{tikzcd}$$

$$\mathsf{X} \xrightarrow{\mathsf{h}} \mathsf{X}'$$

$$\begin{tikzcd} \mathsf{X} \times \mathsf{A} \arrow[r, "\mathsf{h} \times \mathsf{A}"] \arrow[dr, "\mathbb{proj}_1(\mathsf{X})"'] & \mathsf{X}' \times \mathsf{A} \arrow[dr, "\mathbb{proj}_1(\mathsf{X}')"] & \\ & \mathsf{X} \arrow[r, "\mathsf{h}"'] & \mathsf{X}' \end{tikzcd}$$

$$\begin{tikzcd} \langle \mathsf{x}, \mathsf{a} \rangle \arrow[r, mapsto] \arrow[dr, mapsto] & \langle \mathsf{h}(\mathsf{x}), \mathsf{a} \rangle \arrow[dr, mapsto] & \\ & \mathsf{x} \arrow[r, mapsto] & \mathsf{h}(\mathsf{x}) \end{tikzcd}$$

Singletons are natural

$$\{-\} : \mathbb{id} \Rightarrow \mathbb{P}$$

Singletons are natural

$$\begin{tikzcd}[column sep=huge] \mathbb{S}\mathrm{et} \arrow[r, bend left, "\mathbb{id}"] & \mathbb{S}\mathrm{et} \end{tikzcd}$$

$$\{-\} : \color{red}{\mathbb{id}} \color{black}{\Rightarrow \mathbb{P}}$$

Singletons are natural

$$\begin{tikzcd} \mathbb{S}\mathrm{et} \arrow[rr, bend left, "\mathbb{id}"] \arrow[rr, bend right, "\mathbb{P}"'] & & \mathbb{S}\mathrm{et} \end{tikzcd}$$

$$\{-\} : \mathbb{id} \Rightarrow \color{red}{\mathbb{P}}$$

Singletons are natural

$$\begin{tikzcd}[column sep=huge] \mathbb{S}\mathrm{et} \arrow[r, bend left, "\mathbb{id}"{name=U}] \arrow[r, bend right, "\mathbb{P}"'{name=D}] & \mathbb{S}\mathrm{et} \arrow[Rightarrow, from=U, to=D, phantom, "\Downarrow \{-\}"] \end{tikzcd}$$

$$\color{red}{\{-\}} \color{black} : \mathbb{id} \Rightarrow \mathbb{P}$$

Singletons are natural

$$\begin{tikzcd} \mathbb{S}\mathrm{et} \arrow[rr, bend left, "\mathbb{id}"] \arrow[rr, bend right, "\mathbb{P}"'] \arrow[rr, phantom, "\Downarrow \{-\}"] & & \mathbb{S}\mathrm{et} \end{tikzcd}$$

X

Singletons are natural

$$
\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[rr, bend left, "\mathbb{id}"] \arrow[rr, bend right, "\mathbb{P}"'] \arrow[rr, phantom, "\Downarrow \{-\}"] & & \mathbb{S}\mathrm{et}
\end{tikzcd}
$$

$$
x \longmapsto x
$$

Singletons are natural

$$\begin{tikzcd} \mathbb{S}\mathrm{et} \arrow[rr, bend left, "\mathbb{id}"] \arrow[rr, bend right, "\mathbb{P}"'] \arrow[rr, phantom, "\Downarrow \{-\}"] & & \mathbb{S}\mathrm{et} \end{tikzcd}$$

x

$X \longmapsto \mathbb{P}(X)$

Singletons are natural

$$\begin{tikzcd} \mathbb{S}\mathrm{et} \arrow[rr, bend left, "\mathbb{id}"] \arrow[rr, bend right, "\mathbb{P}"'] \arrow[rr, phantom, "\Downarrow \{-\}"] & & \mathbb{S}\mathrm{et} \end{tikzcd}$$

$$X \longmapsto \begin{tikzcd} X \arrow[d, "\{-\}_X"'] \\ \mathbb{P}(X) \end{tikzcd}$$

Singletons are natural

$$
\begin{tikzcd}[column sep=huge]
\mathbb{S}\mathrm{et} \arrow[r, bend left, "\mathbb{id}"{name=U}] \arrow[r, bend right, "\mathbb{P}"'{name=D}] & \mathbb{S}\mathrm{et}
\arrow[from=U, to=D, phantom, "\Downarrow\{-\}"]
\end{tikzcd}
$$

X

$$
\begin{tikzcd}
\mathsf{X} \arrow[dr, "\{-\}_{\mathsf{X}}"'] & \\
& \mathbb{P}(\mathsf{X})
\end{tikzcd}
$$

x

Singletons are natural

$$\begin{tikzcd}[column sep=huge]
\mathbb{S}\mathrm{et} \arrow[r, bend left, "\mathbb{id}"] \arrow[r, bend right, "\mathbb{P}"'] \arrow[r, phantom, "\Downarrow \{-\}"] & \color{red}{\mathbb{S}\mathrm{et}}
\end{tikzcd}$$

X

$$\color{red}{X \xrightarrow{\{-\}_X} \mathbb{P}(X)}$$

$$\color{red}{x \mapsto \{x\}}$$

Singletons are natural

$$\begin{tikzcd}
\color{red}{\mathbb{S}\mathrm{et}} \arrow[rr, bend left, "\mathbb{id}"] \arrow[rr, bend right, "\mathbb{P}"'] \arrow[rr, phantom, "\Downarrow \{-\}"] & & \mathbb{S}\mathrm{et}
\end{tikzcd}$$

$$\color{red}{X \xrightarrow{h} X'}$$

$$\begin{tikzcd}
X \arrow[dr, "\{-\}_X"'] & \\
& \mathbb{P}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[dr, mapsto] & \\
& \{x\}
\end{tikzcd}$$

Singletons are natural

$$
\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[rr, bend left, "\mathbb{id}"] \arrow[rr, bend right, "\mathbb{P}"'] \arrow[rr, phantom, "\Downarrow \{-\}"] & & \mathbb{S}\mathrm{et}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{h}"] & \mathsf{X}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{X} \arrow[dr, "\{-\}_{\mathsf{X}}"'] & \\
& \mathbb{P}(\mathsf{X})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{X}' \arrow[dr, "\{-\}_{\mathsf{X}'}"] & \\
& \mathbb{P}(\mathsf{X}')
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{x} \arrow[dr, mapsto] & \\
& \{\mathsf{x}\}
\end{tikzcd}
$$

Singletons are natural

$$
\begin{tikzcd}[column sep=huge]
\mathbb{S}\mathrm{et} \arrow[r, bend left, "\mathrm{id}"{name=U}] \arrow[r, bend right, "\mathbb{P}"'{name=D}] & \mathbb{S}\mathrm{et}
\arrow[from=U, to=D, phantom, "\Downarrow \{-\}"]
\end{tikzcd}
$$

$$
X \xrightarrow{h} X' \quad \longmapsto \quad
\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\{-\}_X"'] & & X' \arrow[dr, "\{-\}_{X'}"] & \\
& \mathbb{P}(X) & & \mathbb{P}(X')
\end{tikzcd}
$$

$$
\begin{tikzcd}
x \arrow[dr, mapsto] & \\
& \{x\}
\end{tikzcd}
$$

Singletons are natural

$$\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[rr, bend left, "\mathrm{id}"] \arrow[rr, bend right, "\mathbb{P}"', red] \arrow[rr, phantom, "\Downarrow \{-\}"] & & \mathbb{S}\mathrm{et}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "h", red] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\{-\}_X"'] & & X' \arrow[dr, "\{-\}_{X'}"] & \\
& \mathbb{P}(X) \arrow[rr, "\mathbb{P}(h)"', red] & & \mathbb{P}(X')
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[dr, mapsto] & \\
& \{x\}
\end{tikzcd}$$

Singletons are natural

$$\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[rr, bend left, "\mathbb{id}"] \arrow[rr, bend right, "\mathbb{P}"'] \arrow[rr, phantom, "\Downarrow \{-\}"] & & \mathbb{S}\mathrm{et}
\end{tikzcd}$$

$$X \xrightarrow{h} X'$$

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, "\{-\}_X"'] & X' \arrow[dr, "\{-\}_{X'}"] & \\
& \mathbb{P}(X) \arrow[r, "\mathbb{P}(h)"'] & \mathbb{P}(X')
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[dr, mapsto] & \\
& \{x\}
\end{tikzcd}$$

$$\{x_1, x_2, x_3\} \mapsto \{h(x_1), h(x_2), h(x_3)\}$$

Singletons are natural

$$\begin{tikzcd} \mathbb{S}\mathrm{et} \arrow[r, bend left, "\mathbb{id}"] \arrow[r, bend right, "\mathbb{P}"'] \arrow[r, phantom, "\Downarrow \{-\}"] & \color{red}{\mathbb{S}\mathrm{et}} \end{tikzcd}$$

$$X \xrightarrow{h} X'$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "\{-\}_X"'] & X' \arrow[dr, "\{-\}_{X'}"] & \\ & \mathbb{P}(X) \arrow[r, red, "\mathbb{P}(h)"'] & \mathbb{P}(X') \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[dr, mapsto] & & \\ & \{x\} \arrow[r, red, mapsto] & \color{red}{\{h(x)\}} \end{tikzcd}$$

$\mapsto$

Singletons are natural

$$\begin{tikzcd}[column sep=huge] \mathbb{S}\mathrm{et} \arrow[r, bend left, "\mathbb{id}"{name=U}] \arrow[r, bend right, "\mathbb{P}"'{name=D}] \arrow[Rightarrow, from=U, to=D, phantom, "\Downarrow\{-\}"] & \mathbb{S}\mathrm{et} \end{tikzcd}$$

$$X \xrightarrow{h} X'$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "\{-\}_X"'] & X' \arrow[dr, "\{-\}_{X'}"] & \\ & \mathbb{P}(X) \arrow[r, "\mathbb{P}(h)"'] & \mathbb{P}(X') \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[dr, mapsto] & h(x) & \\ & \{x\} \arrow[r, mapsto] & \{h(x)\} \end{tikzcd}$$

$\mapsto$

Singletons are natural

$$\begin{tikzcd}
\mathbb{S}\mathrm{et} \arrow[r, bend left, "\mathbb{id}"] \arrow[r, bend right, "\mathbb{P}"'] \arrow[r, phantom, "\Downarrow \{-\}"] & \mathbb{S}\mathrm{et}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "h"] & X'
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\{-\}_X"'] & & X' \arrow[dr, red, "\{-\}_{X'}"] & \\
& \mathbb{P}(X) \arrow[rr, "\mathbb{P}(h)"'] & & \mathbb{P}(X')
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[rr, mapsto] \arrow[dr, mapsto] & & h(x) \arrow[dr, mapsto, red] & \\
& \{x\} \arrow[rr, mapsto] & & \{h(x)\}
\end{tikzcd}$$

$\mapsto$

Natural Isomorphism

$$(X \times Y) \times Z \quad \neq \quad X \times (Y \times Z)$$

Natural Isomorphism

$$\begin{aligned}(X \times Y) \times Z &\neq X \times (Y \times Z) \\ \langle\langle x, y\rangle, z\rangle &\neq \langle x, \langle y, z\rangle\rangle\end{aligned}$$

Natural Isomorphism

$$\begin{array}{ccc} (X \times Y) \times Z & \cong & X \times (Y \times Z) \\ \langle\langle x, y\rangle, z\rangle & \leftrightarrow & \langle x, \langle y, z\rangle\rangle \end{array}$$