

Category Theory

Cartography for the information age

Jan. 30, 2022

INCOSE IW

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Joint work w. E. Subrahmanian

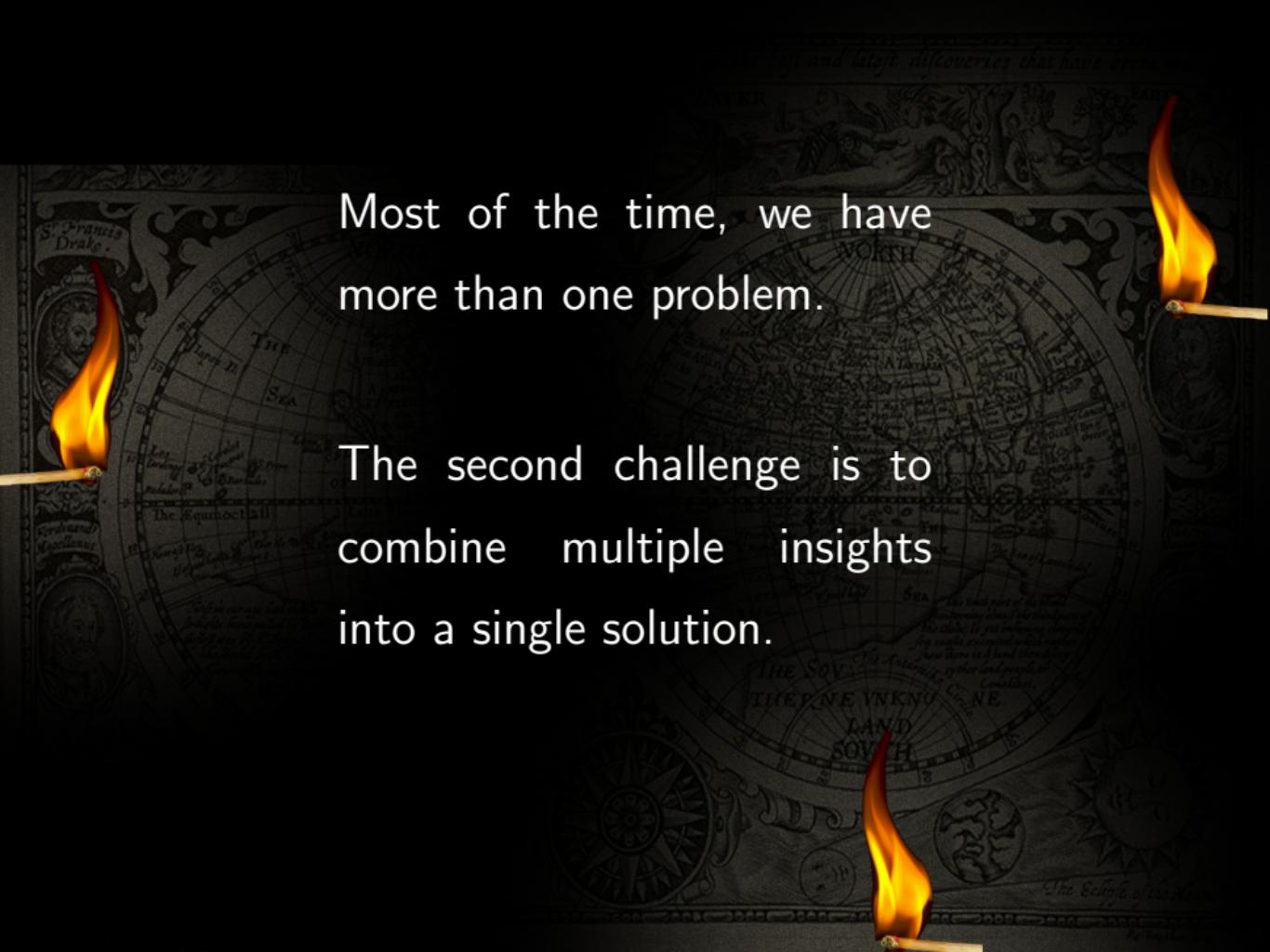
NIST

National Institute of
Standards and Technology
U.S. Department of Commerce

Chances are, the problem
you're working on (whatever it
is) already has a solution.

In an ocean of knowledge, the
first challenge is *finding* it.





Most of the time, we have
more than one problem.

The second challenge is to
combine multiple insights
into a single solution.



```

<breakfast_menu>
  <food>
    <name>Belgian Waffles</name>
    <price>$5.95</price>
    <description>
      Two of our famous Belgian Waffles with plenty of real maple syrup
    </description>
    <calories>650</calories>
  </food>
  <food>
    <name>Strawberry福地 Waffles</name>
    <price>$7.95</price>
    <description>
      Light Belgian waffles covered with strawberries and whipped cream
    </description>
    <calories>900</calories>
  </food>

```

The problem of plurality:

Identify and align known methods, tools
and representations to address unique,
system-specific concerns.

```

[X, Y] = meshgrid(-10:10, 20:10, -10:10)
r = sinc(sqrt((X.^2 + Y.^2)))
mesh(X, Y, r)
axis([10 20 -10 10])
xlabel('t\beta x')
ylabel('t\beta y')
zlabel('t\beta z \sin c')
hidden off

```

```

n = int(input('Type a number, and its factorial will be printed:'))
if n < 0:
    raise ValueError('You must enter a non-negative integer')
factorial = 1
for i in range(2, n + 1):
    factorial *= i
print(factorial)

```

$$D_{KL}(P \parallel Q) = \sum_{x \in X} P(x) \log \left(\frac{P(x)}{Q(x)} \right)$$

```

CREATE TABLE test_table (
  id          INT(10)           NOT NULL,
  part_number CHAR(500),
  part_name   DECIMAL(10, 2)    NOT NULL,
  state       DECIMAL(10, 2)    NOT NULL,
  PRIMARY KEY (id),
  CONSTRAINT test_check CHECK ((part_number > 'n/a' AND part_name IS NOT NULL) OR
                                (part_number = 'n/a' AND part_name IS NULL))

```

```

{
  "firstName": "John",
  "lastName": "Smith",
  "isAlive": true,
  "age": 25,
  "address": {
    "streetAddress": "21 2nd Street",
    "city": "New York",
    "state": "NY",
    "postalCode": "10021-3100"
  },
  "phoneNumbers": [
    {
      "type": "home",
      "number": "212 555-1234"
    },
    {
      "type": "office",
      "number": "646 555-4567"
    }
  ],
  "children": [],
  "spouse": null
}

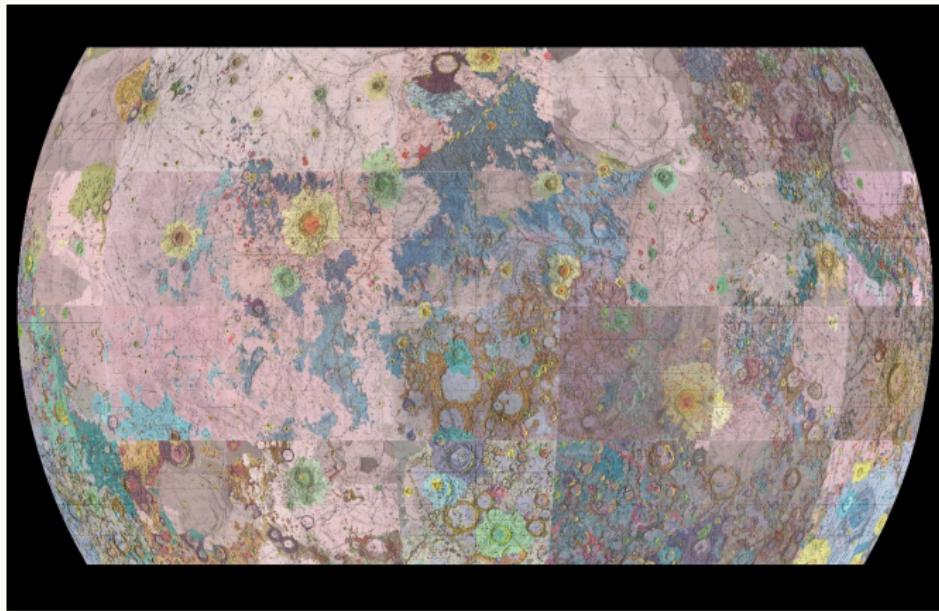
```

What *can* we do?



Transform

What *can* we do?



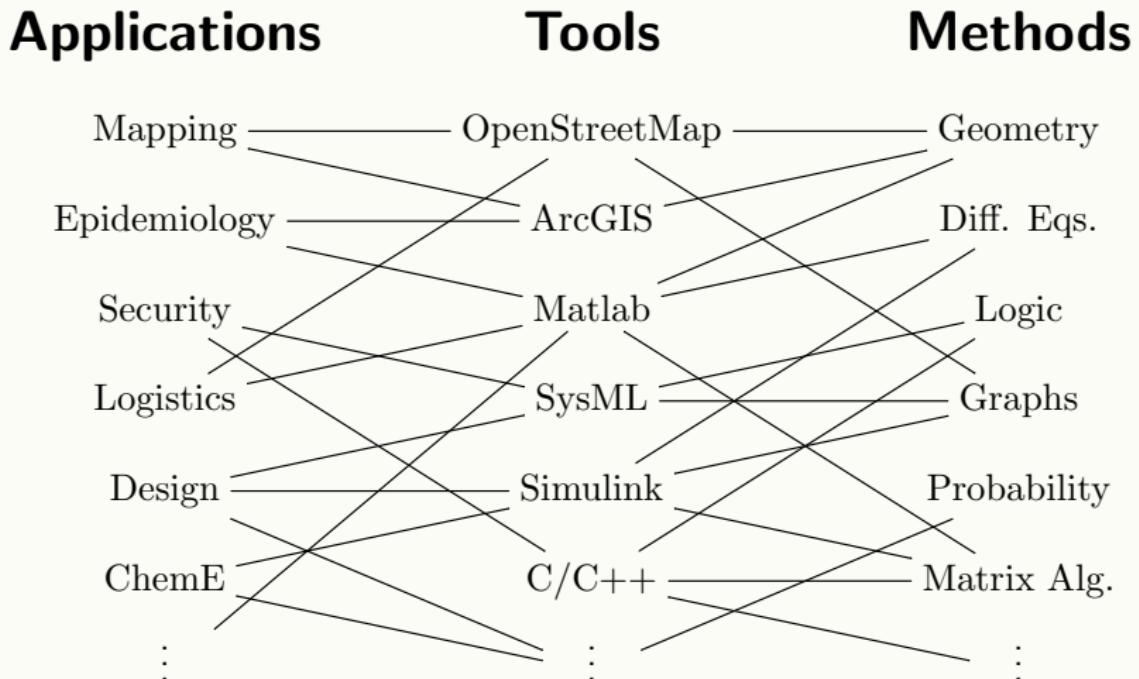
Align

What *can* we do?



Abstract

Where we are



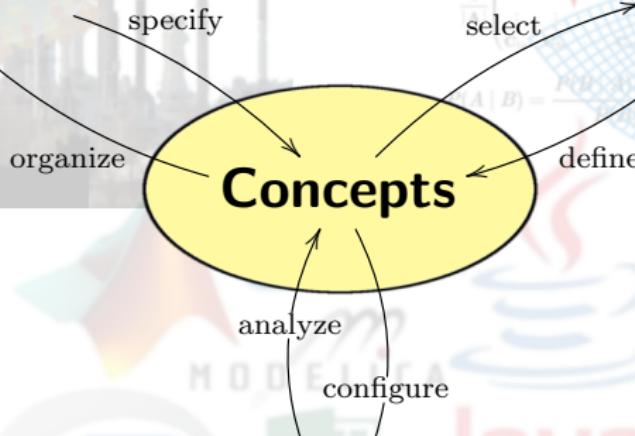
Where we're going

Applications

Methods

Concepts

Tools



How to get there?

```
<breakfast_menu>
  <food>
    <name>Belgian Waffles</name>
    <price>$5.95</price>
    <description>
      Two of our famous Belgian Waffles with plenty of real maple syrup
    </description>
    <calories>650</calories>
  </food>
  <food>
    <name>Strawberry Eggnog Waffles</name>
    <price>$7.95</price>
    <description>
      Light Belgian waffles covered with strawberries and whipped cream
    </description>
    <calories>900</calories>
  </food>
```

Linear algebra

```
def id(x,y):
    return ((x,p1),^2*(Y,p1),^2));
yLabel('(\mathbf{b}ry)')
zLabel('(\mathbf{b}simc) (\mathbf{b}r(y))')
hidden off
```

Dynamical Systems

Graphs & topology

Probability

Category theory

Physical sciences

Computer science

Economics

```
CREATE TABLE test_table
  (part_number CHAR(500),
   part_name DECIMAL,
   part_desc DEFAULT 'n/a',
   part_qty PRIMARY KEY,
   part_name NOT NULL,
   part_desc CHECK ((part_number = 'n/a' AND part_name IS NULL) OR
                    (part_number != 'n/a' AND part_name IS NOT NULL))
```

$$\begin{bmatrix} \frac{\partial z}{\partial p} & \frac{\partial z}{\partial q} & \frac{\partial z}{\partial g} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} & \frac{\partial y}{\partial g} \\ \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} & \frac{\partial x}{\partial g} \end{bmatrix} = \begin{bmatrix} \sin \varphi \cos \theta & \rho \cos \varphi \cos \theta & \rho \cos \varphi \sin \theta \\ \sin \varphi \sin \theta & \rho \sin \varphi \cos \theta & -\rho \sin \varphi \\ \cos \varphi & \rho \cos \varphi & -\rho \sin \varphi \end{bmatrix}$$

```
n = int(input('Type a number, and its factorial will be printed'))
if n < 0:
    raise ValueError('You must enter a positive integer')
factorial = 1
for i in range(2, n + 1):
    factorial *= i
print(factorial)
```

```
{
  "firstName": "John",
  "lastName": "Smith",
  "isAlive": true,
  "address": {
    "streetAddress": "21 2nd Street",
    "city": "New York",
    "state": "NY",
    "postalCode": "10021-3100"
  },
  "phoneNumbers": [
    {
      "type": "home",
      "number": "212 555-1234"
    },
    {
      "type": "office",
      "number": "646 555-4567"
    }
  ],
  "children": [],
  "spouse": null
}
```

Outline

- I) Composition
- II) Categories
- III) Structure
 - break-
- IV) Models
- V) Data
- VI) Processes

Sets



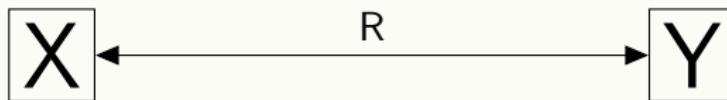
Category Theory
vs.
Set Theory



Categories
of
Sets

Relations

Relations



x_1 ————— y_1

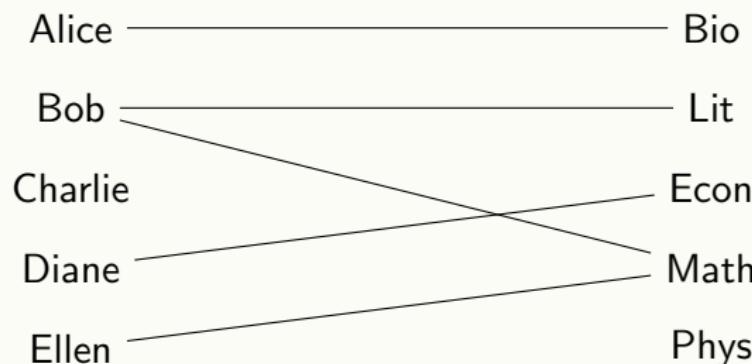
x_2 ————— y_2

x_3 ————— y_3

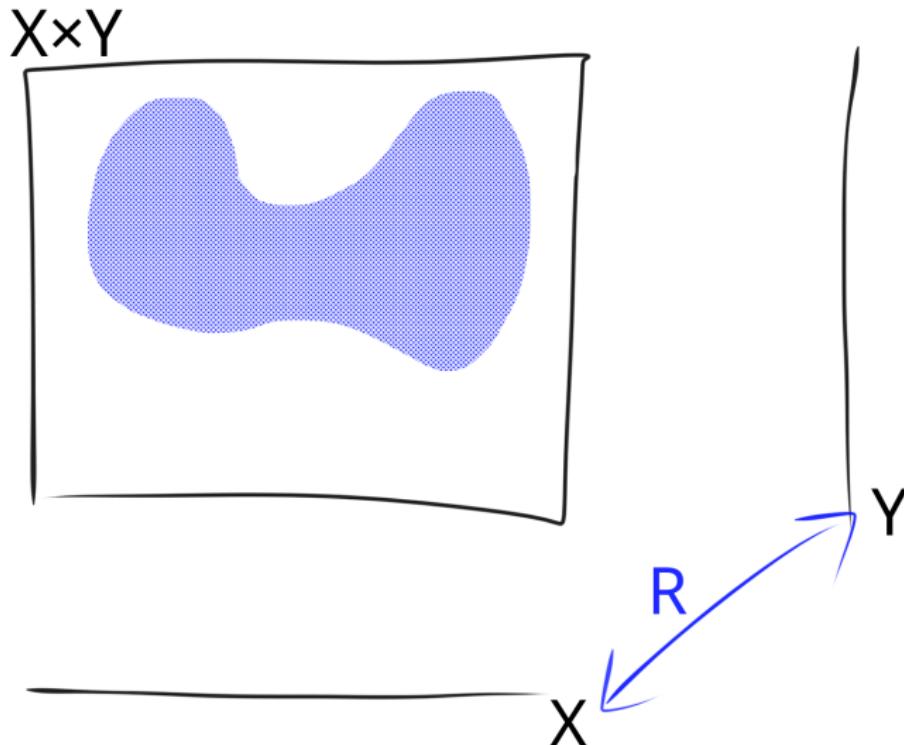
x_4 ————— y_4

x_5 ————— y_5

Relations



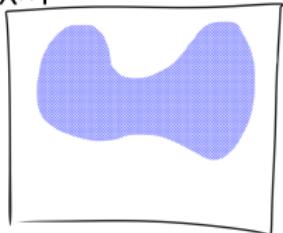
Relations, geometrically



Relations are reversible

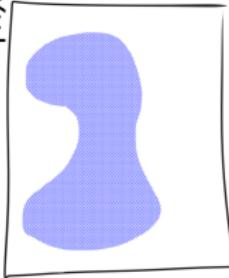
Reversal, geometrically

$X \times Y$



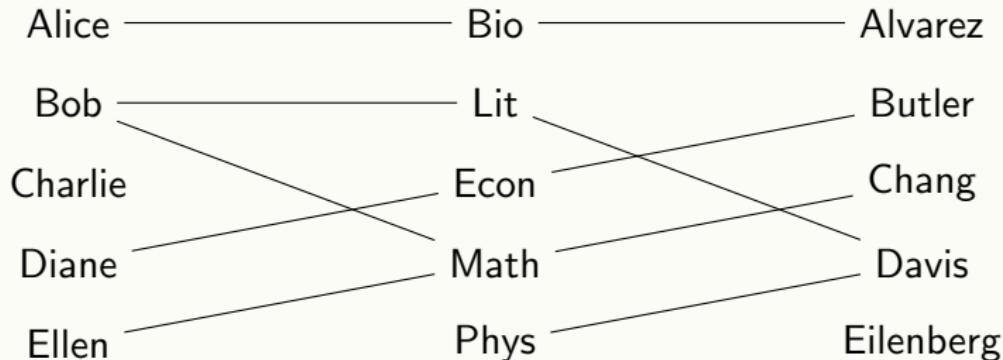
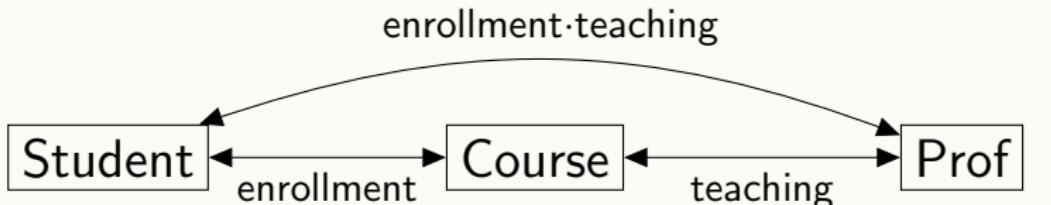
R

$Y \times X$

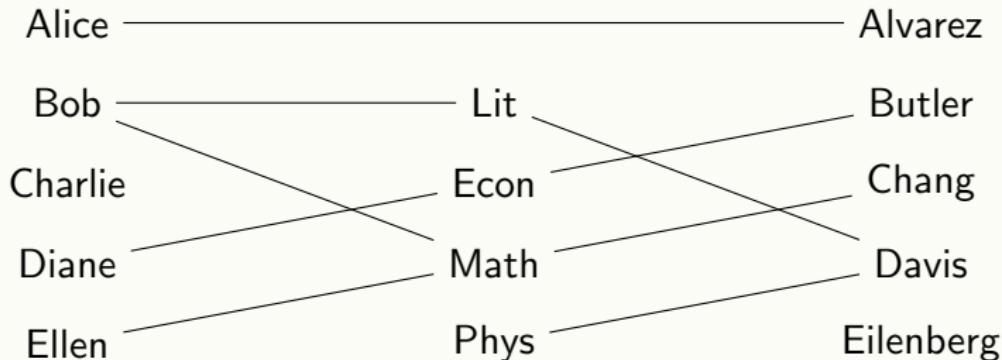
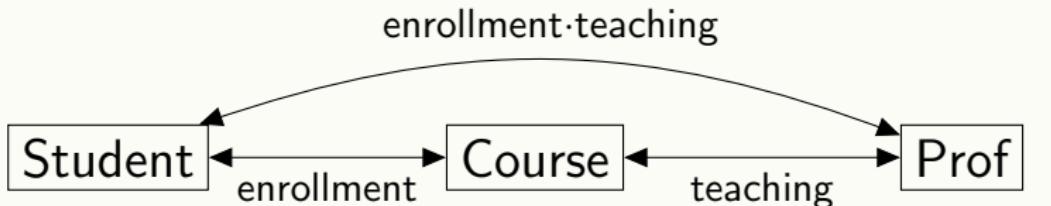


R^{op}

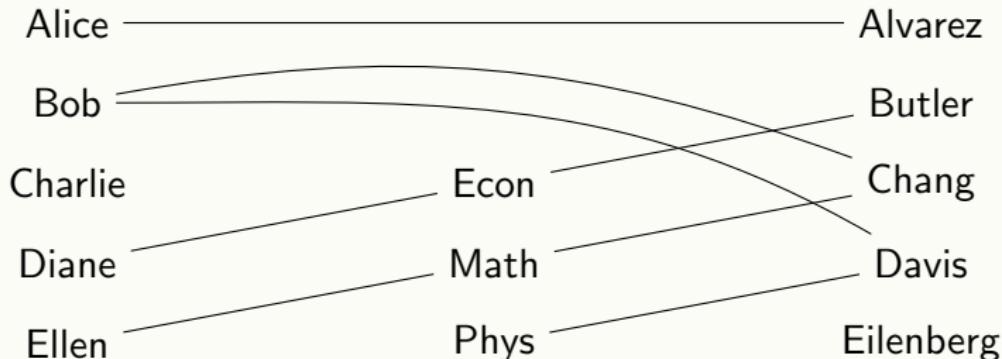
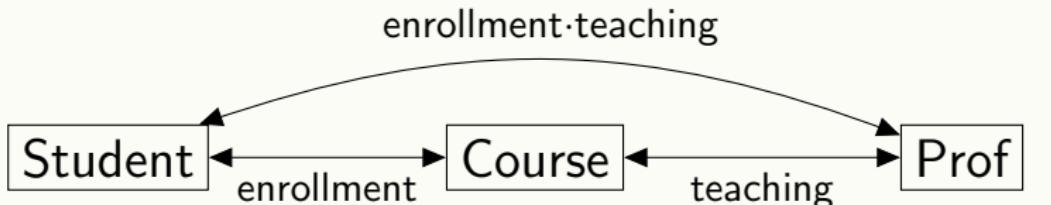
Composing Relations



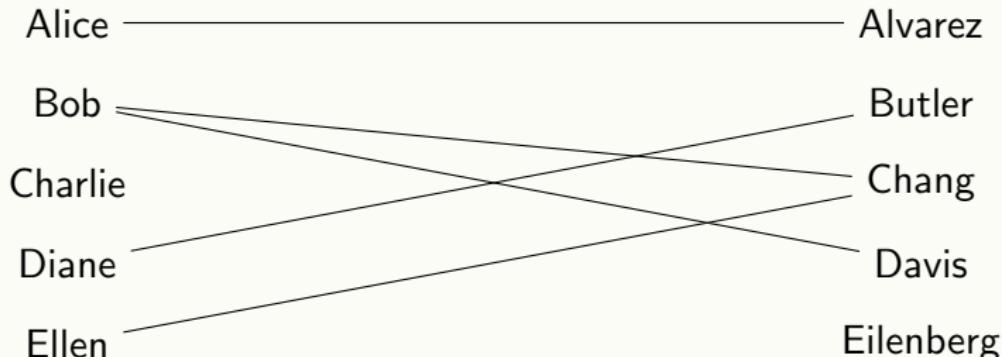
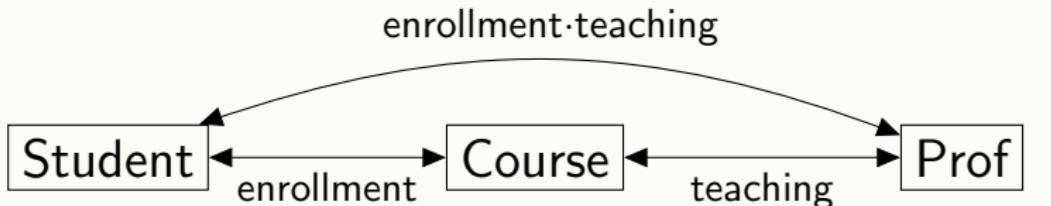
Composing Relations



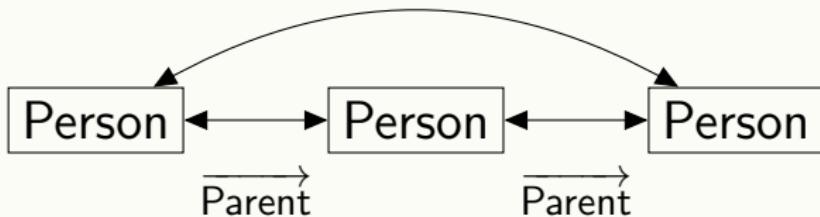
Composing Relations



Composing Relations

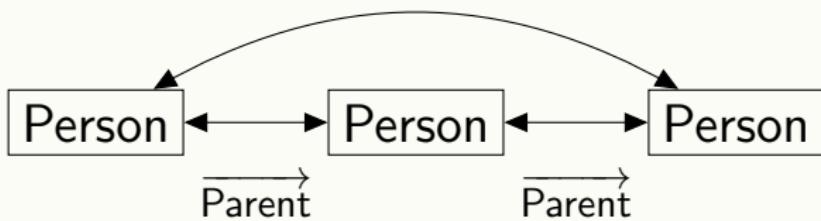


Self-Relations

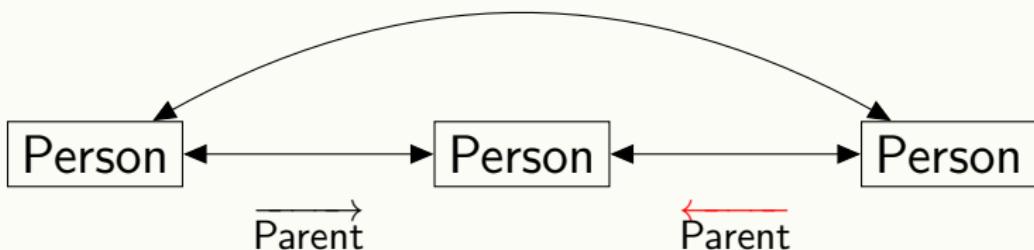
$$\text{Grandparent} = \xrightarrow{\text{Parent}} \cdot \xrightarrow{\text{Parent}}$$


Self-Relations

$$\text{Grandparent} = \text{Parent} \cdot \text{Parent}$$

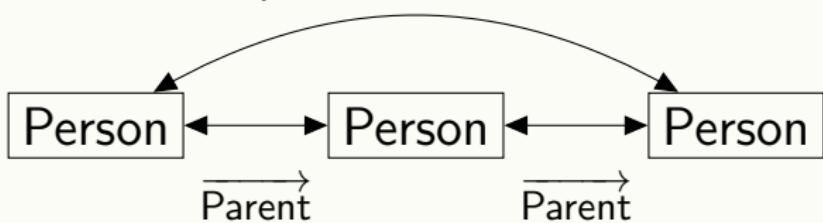


$$\text{Sibling} = \text{Parent} \cdot \text{Parent}$$

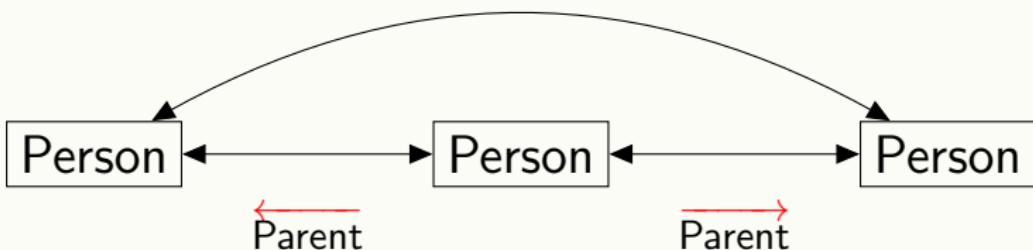


Self-Relations

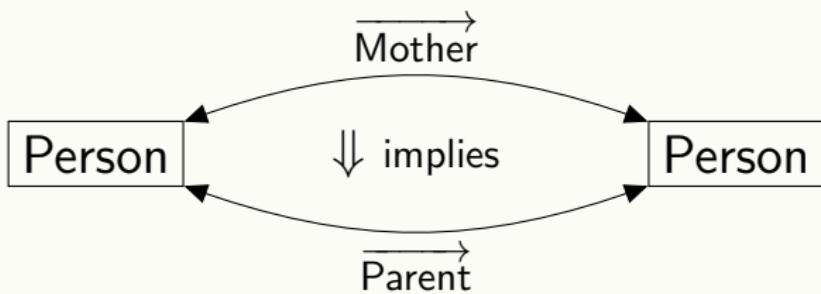
$$\text{Grandparent} = \text{Parent} \cdot \text{Parent}$$



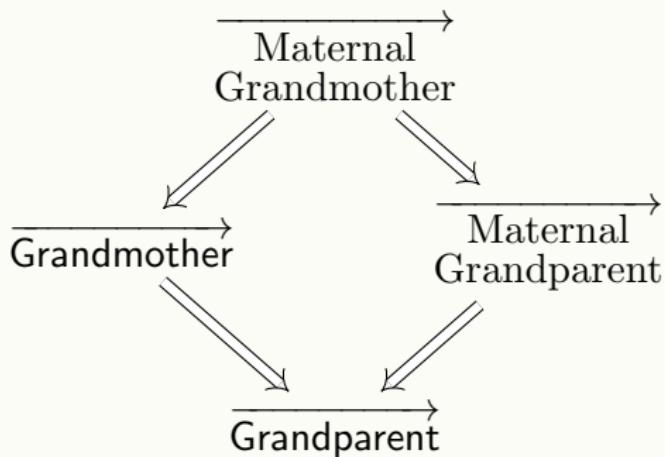
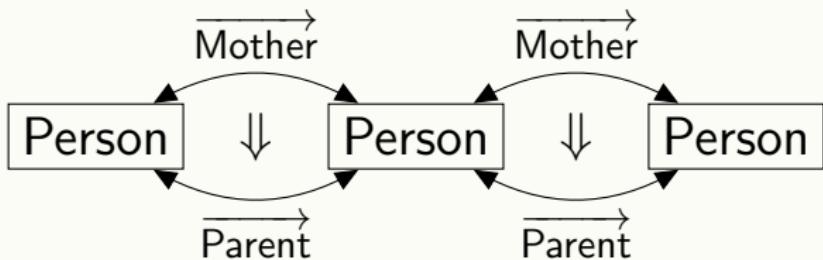
$$\text{Coparent} = \text{Parent} \cdot \text{Parent}$$



Relations between relations



Relations between relations

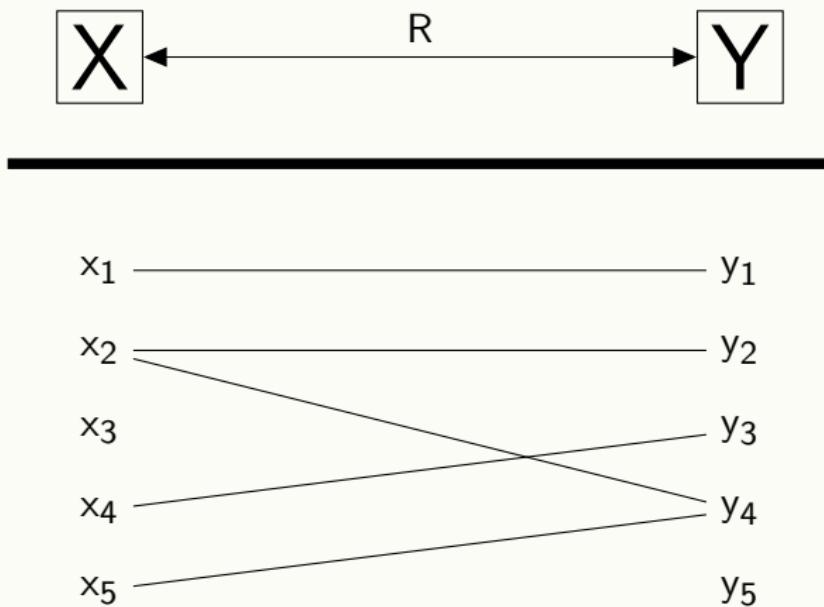


Questions?

Functions

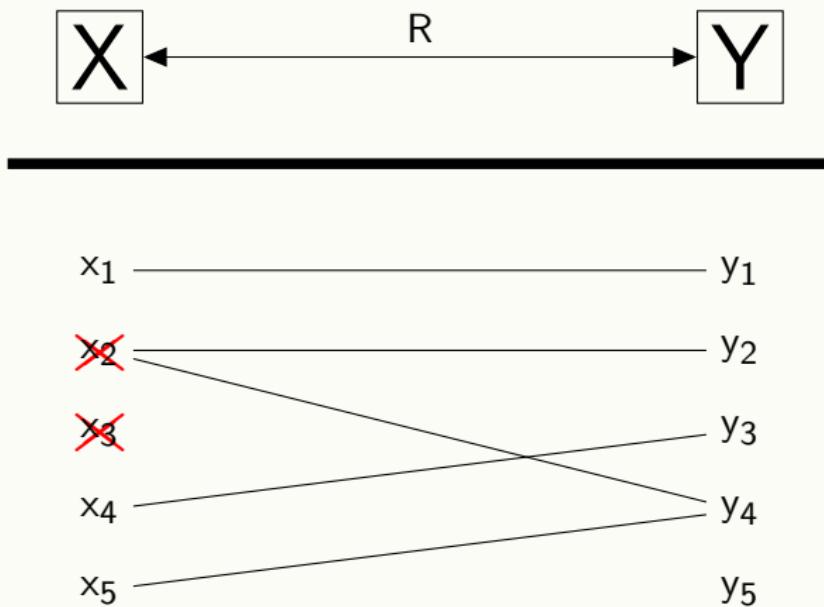
Functions

Functions are *total* and *single-valued*.



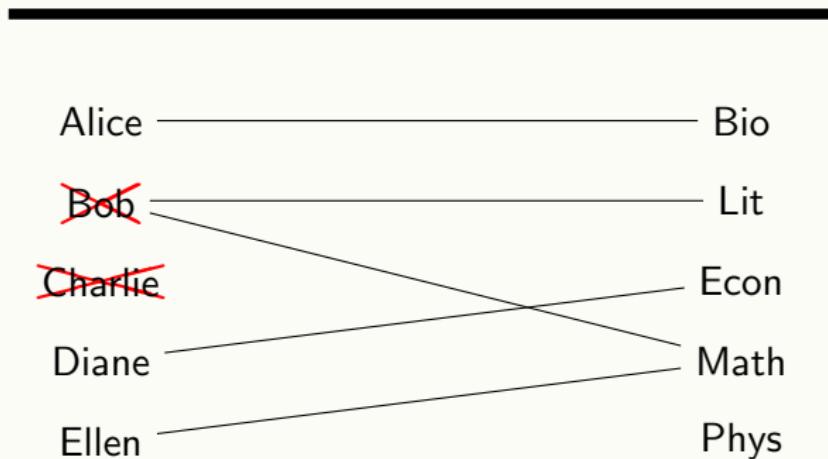
Functions

Functions are *total* and *single-valued*.



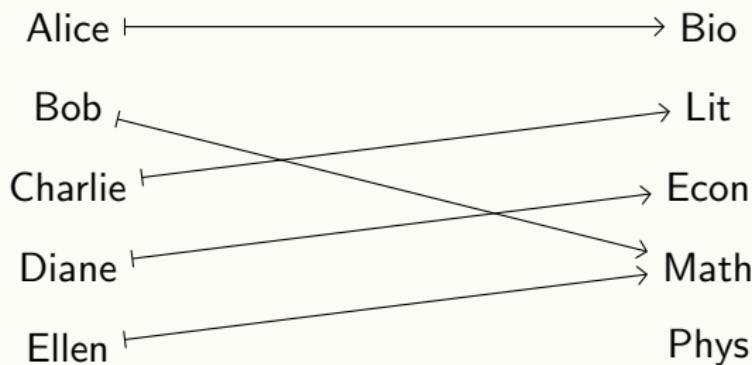
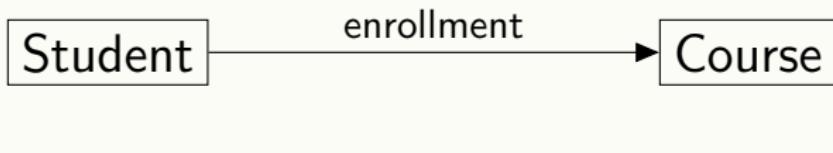
Functions

Functions are *total* and *single-valued*.



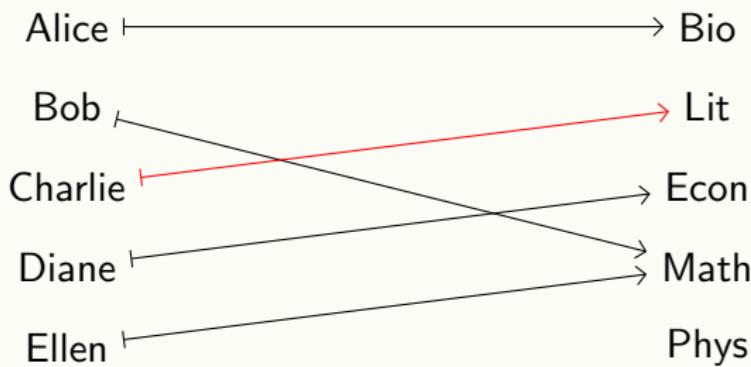
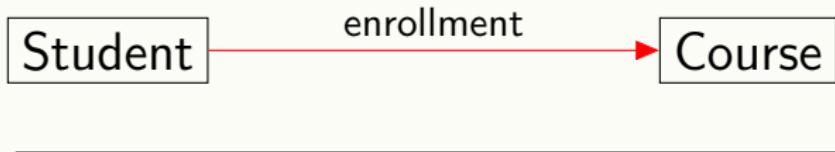
Functions

Functions are *total* and *single-valued*.

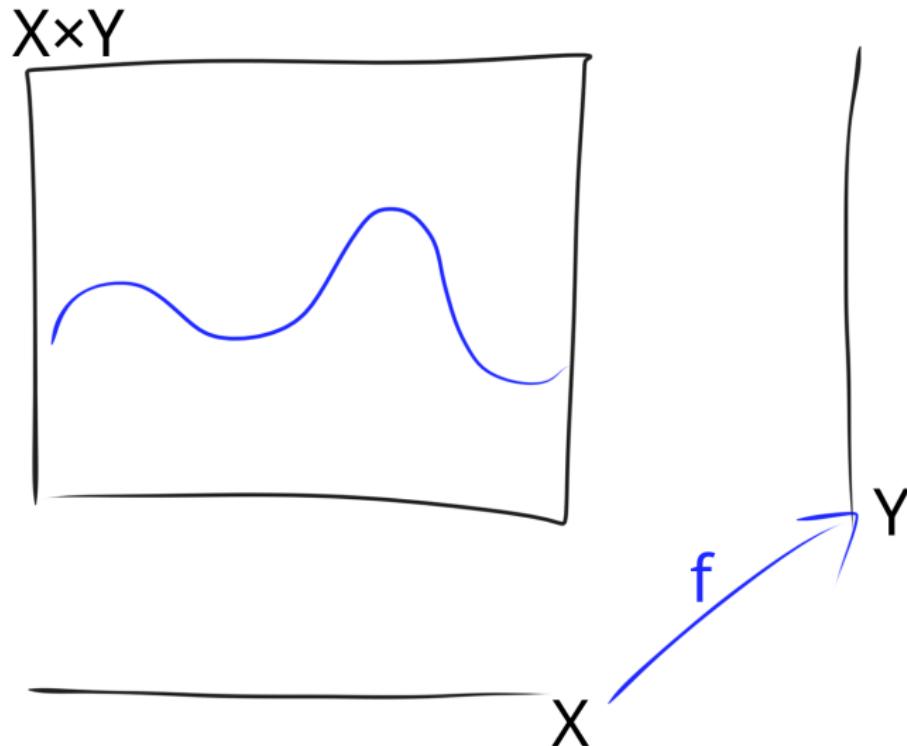


Functions

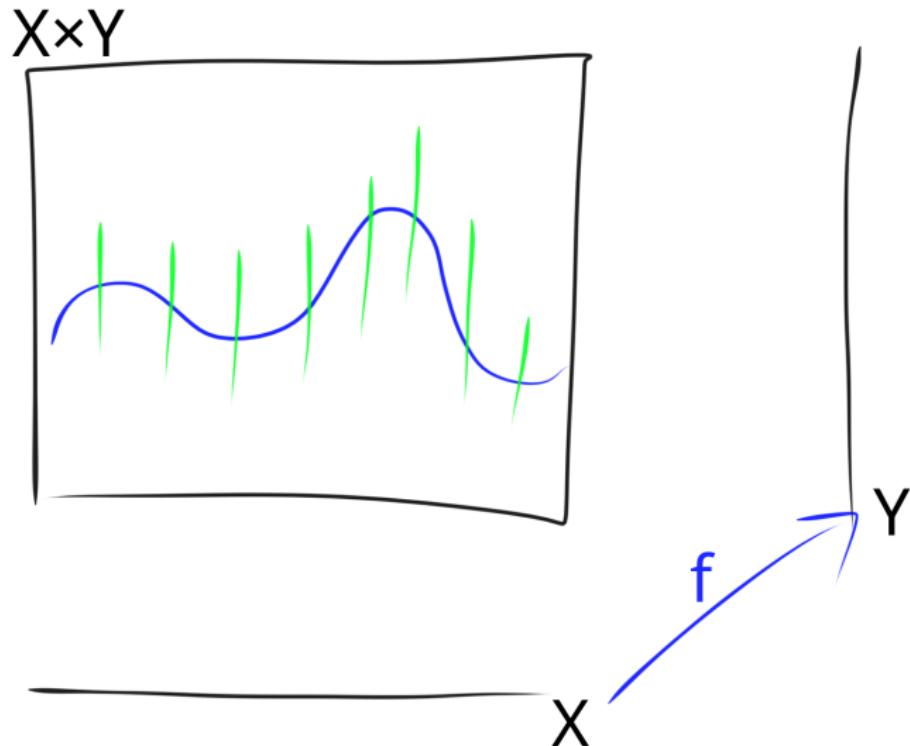
Functions are *total* and *single-valued*.



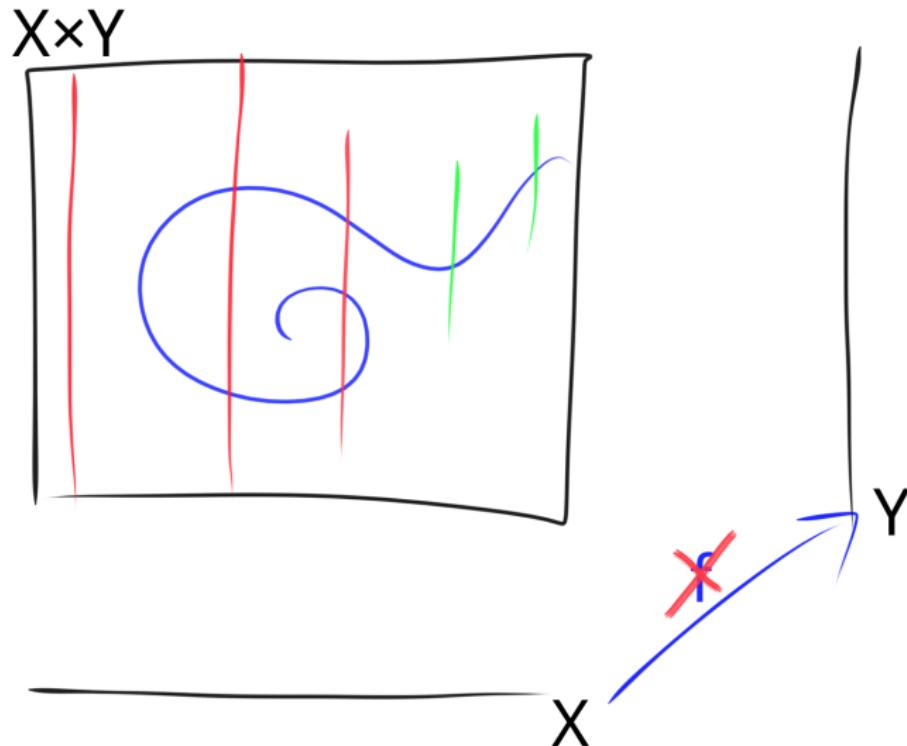
Functions, geometrically



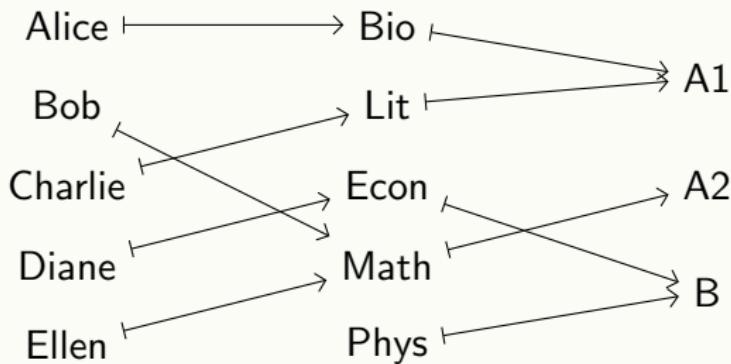
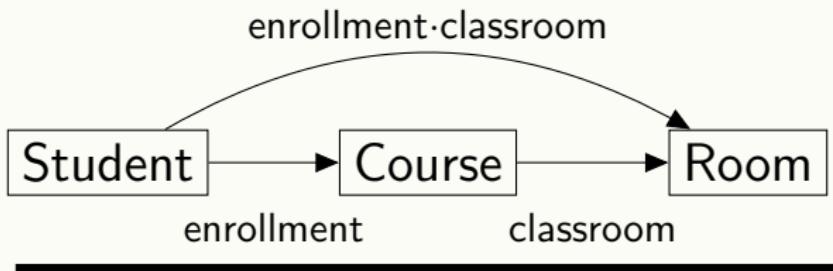
Functions, geometrically



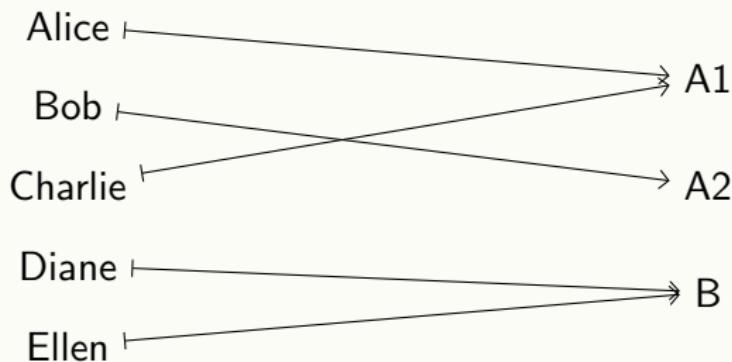
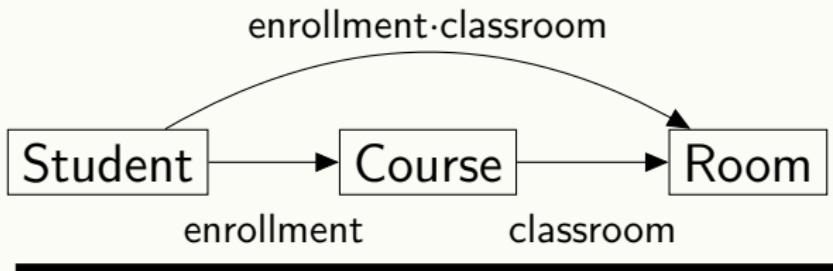
Functions, geometrically



Functions compose

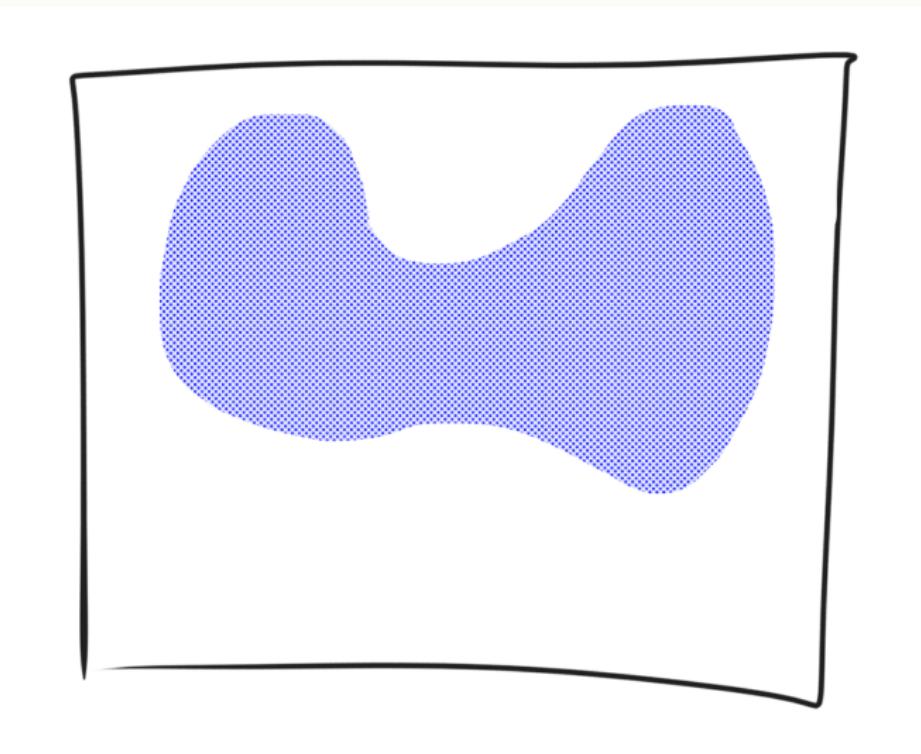


Functions compose



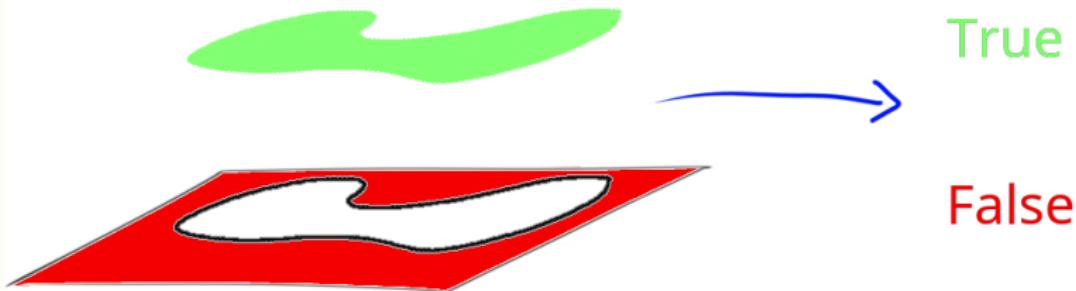
Functions do *not* reverse

Relations as functions



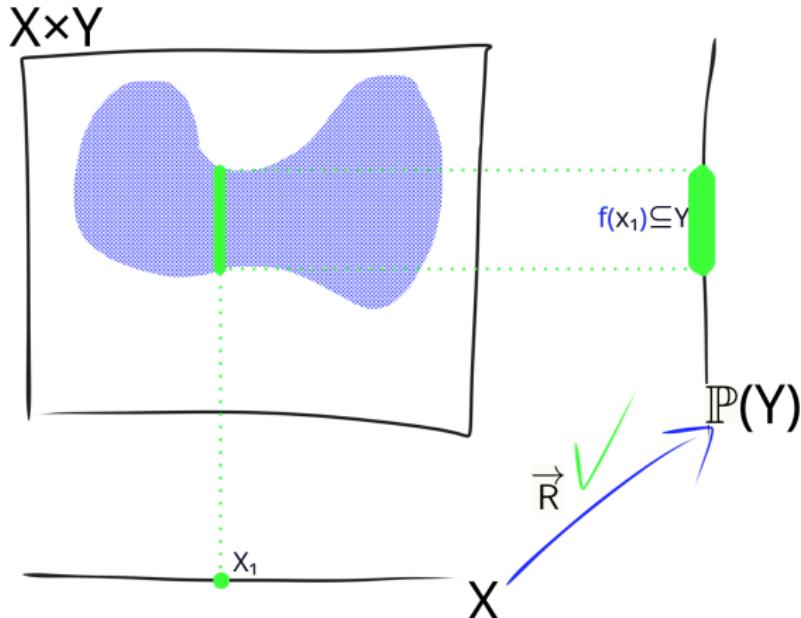
Relations as functions

Version 1: Truth functions $R : X \times Y \rightarrow \{T, F\}$



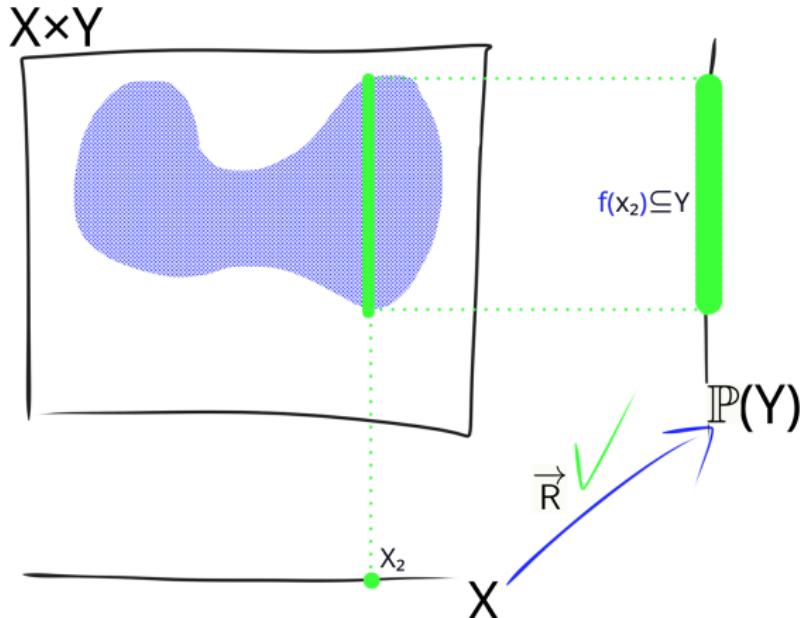
Relations as functions

Version 2: Set-valued functions $\vec{R} : X \rightarrow \mathbb{P}(Y)$



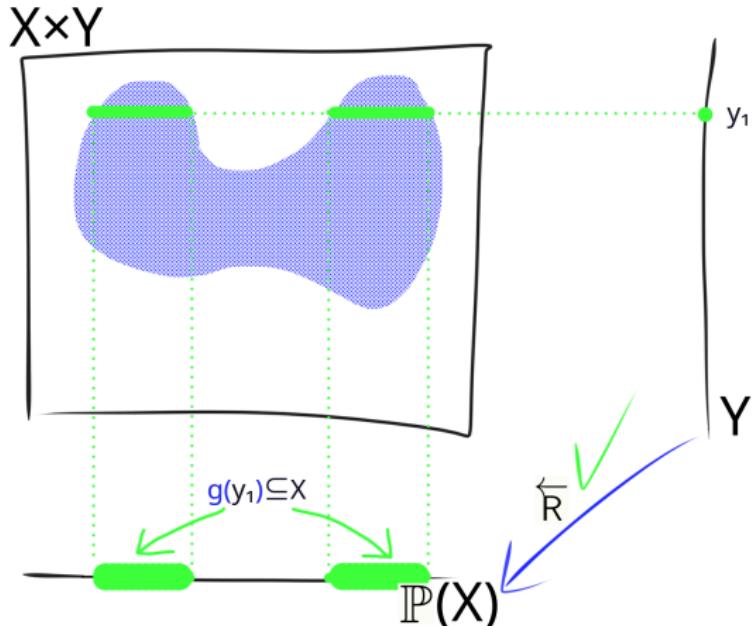
Relations as functions

Version 2: Set-valued functions $\overrightarrow{R} : X \rightarrow \mathbb{P}(Y)$



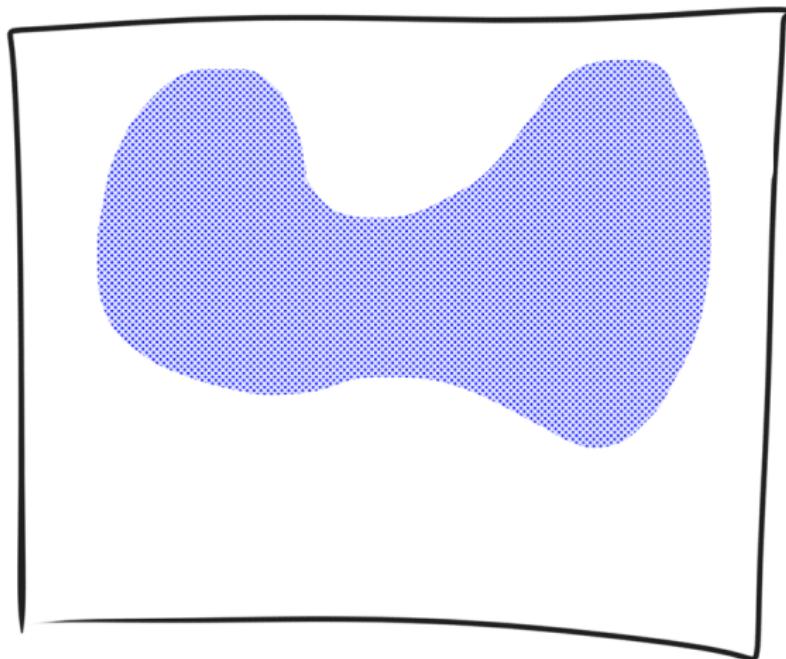
Relations as functions

Version 3: Set-valued functions $\overleftarrow{R} : Y \rightarrow \mathbb{P}(X)$



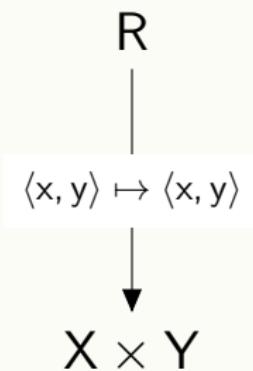
Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$



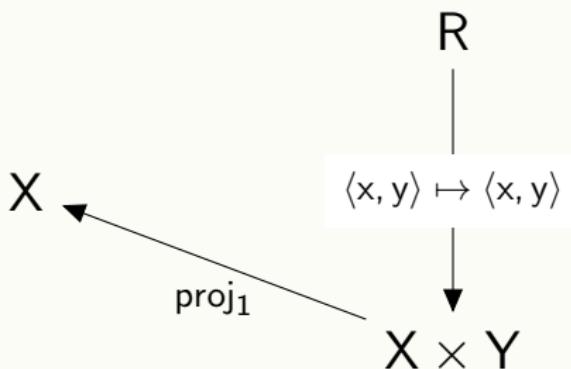
Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$



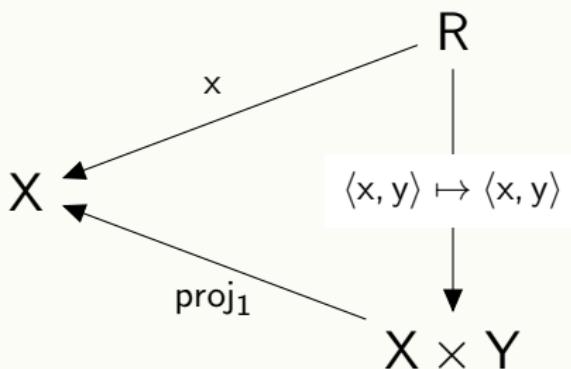
Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$



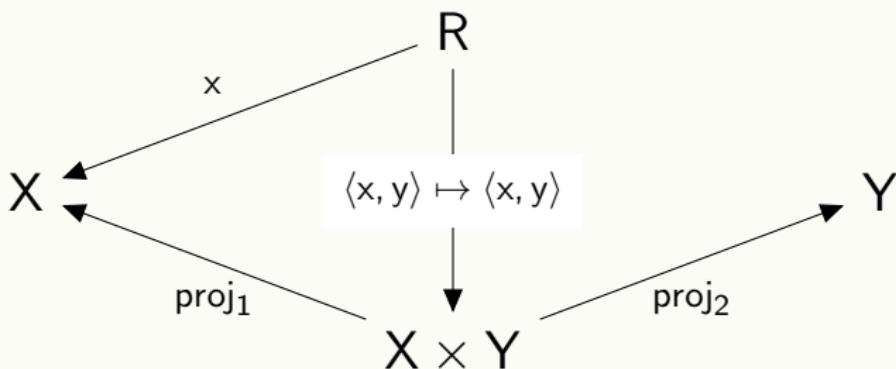
Relations as functions

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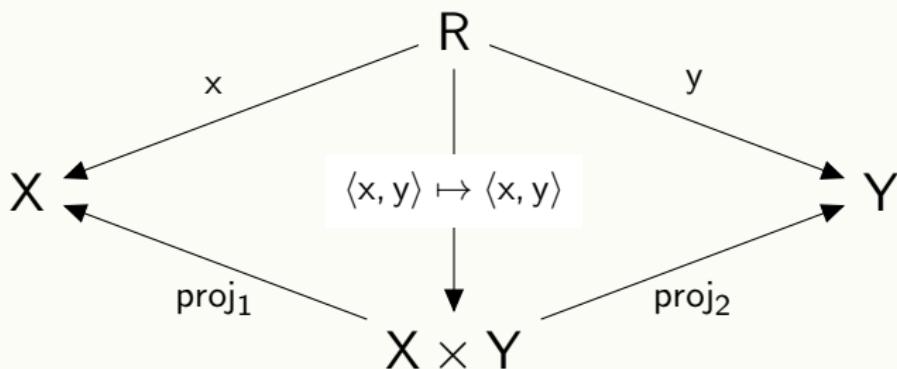
Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$



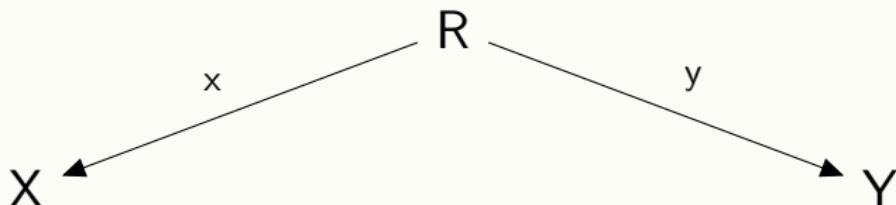
Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$



Relations as functions

Version 4: Subset inclusion $R \subseteq X \times Y$



A diagram $X \leftarrow R \rightarrow Y$ is called a *span*.

Functions, symbolically

$$y = \sqrt{2x^2 + 3}$$

Functions, symbolically

$$y = \sqrt{2^{3^2} + 3}$$

$X = \mathbb{R}$

3

Functions, symbolically

$$y = \sqrt{2^{\textcolor{red}{9}+3}}$$

$$X = \mathbb{R} \xrightarrow{\text{sqr}} \mathbb{R}$$

$$3 \longmapsto 9$$

Functions, symbolically

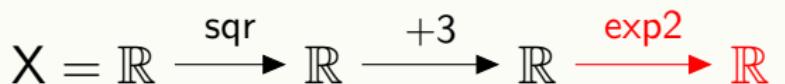
$$y = \sqrt{2^{12}}$$

$$X = \mathbb{R} \xrightarrow{\text{sqr}} \mathbb{R} \xrightarrow{+3} \mathbb{R}$$

$$3 \longmapsto 9 \xrightarrow{+3} 12$$

Functions, symbolically

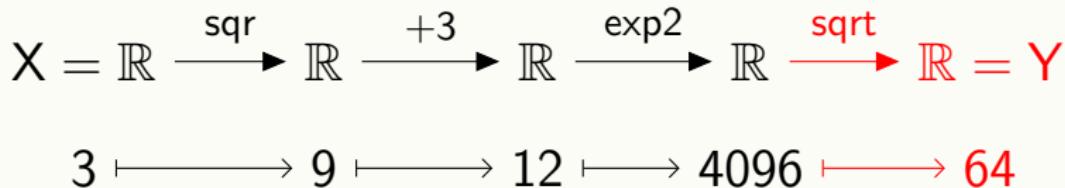
$$y = \sqrt{4096}$$



$$3 \longmapsto 9 \longmapsto 12 \longmapsto 4096$$

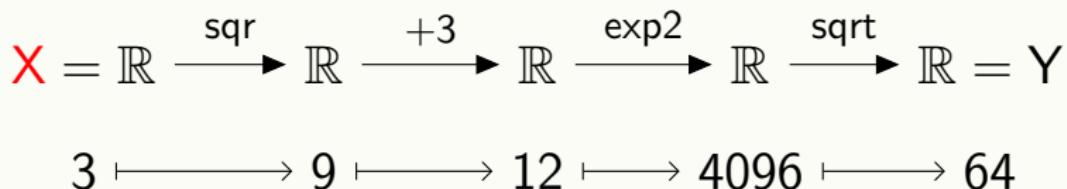
Functions, symbolically

$$y = 64$$



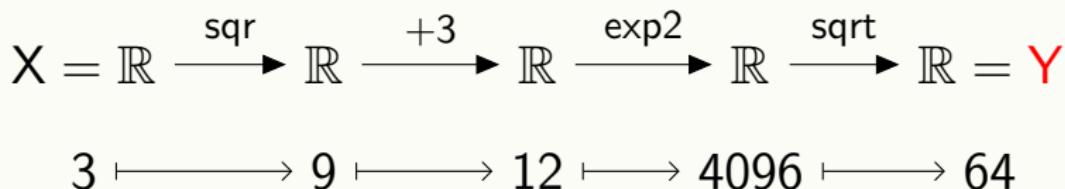
Functions, symbolically

$$y = \sqrt{2x^2 + 3}$$



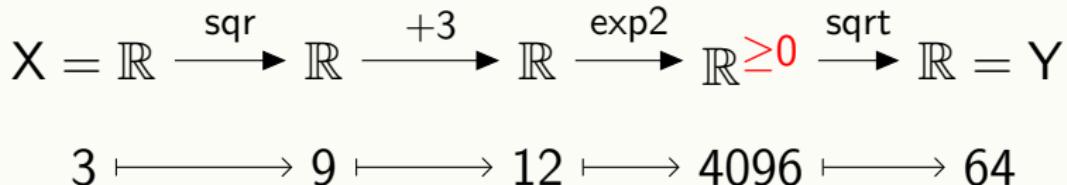
Functions, symbolically

$$y = \sqrt{2x^2 + 3}$$

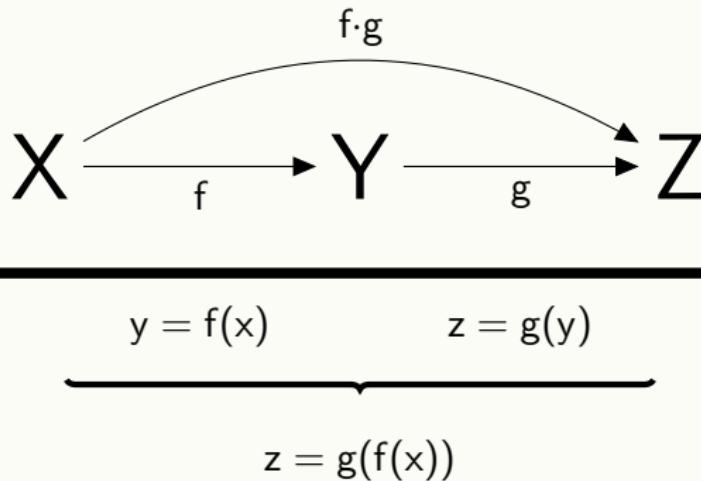


Functions, symbolically

$$y = \sqrt{2x^2 + 3}$$

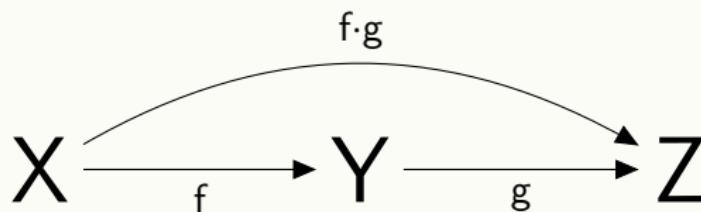


Composition, symbolically



Alternate notation: $f \cdot g = g \circ f$

Composition, symbolically



$$y = f(x)$$

$$z = g(y)$$

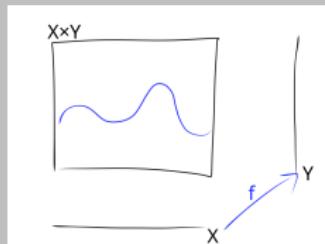
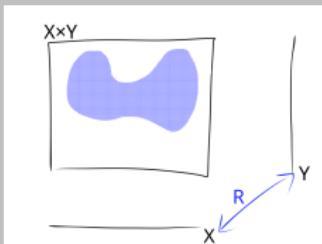
$$z = g(f(x))$$

$$y = \text{sqrt}(\text{exp2}(\text{plus3}(\text{sqr}(x))))$$

Alternate notation: $f \cdot g = g \circ f$

Questions?

| | Relation | Function |
|--------------|--|--|
| Notation | $R : X \longleftrightarrow Y$ $x \leftrightarrow y$ | $f : X \rightarrow Y$ $x \mapsto y$ |
| Connectivity | Any-Any | Any-One |
| Composes | ✓ | ✓ |
| Reverses | ✓ | ✗ |



Categories

Set & Rel

The category Set

Objects Sets X, Y, Z, \dots

Arrows Functions $f : X \rightarrow Y, g : Y \rightarrow Z, \dots$

Identities $\text{id}(x) := x$ $X \rightarrow X$

Composition $(f \cdot g)(x) := g(f(x))$ $X \rightarrow Z$

Unit $\text{id} \cdot f(x) = f(\text{id}(x)) = f(x)$

$f \cdot \text{id}(x) = \text{id}(f(x)) = f(x)$

Associativity \dots

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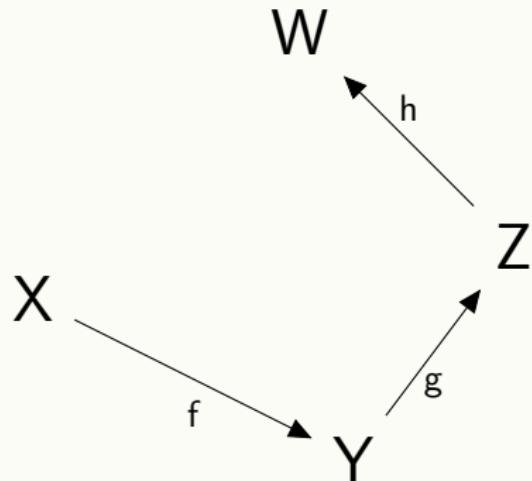
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Associativity

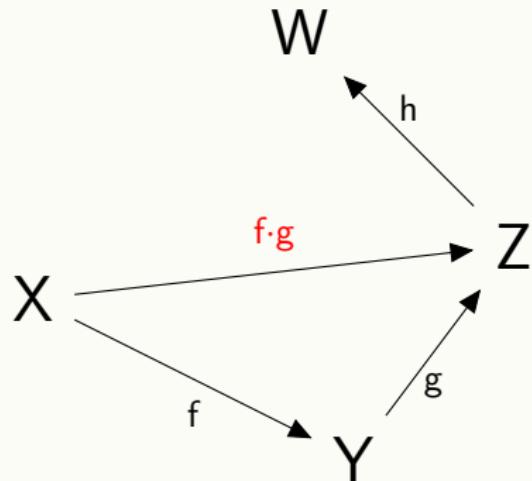
...

Associativity



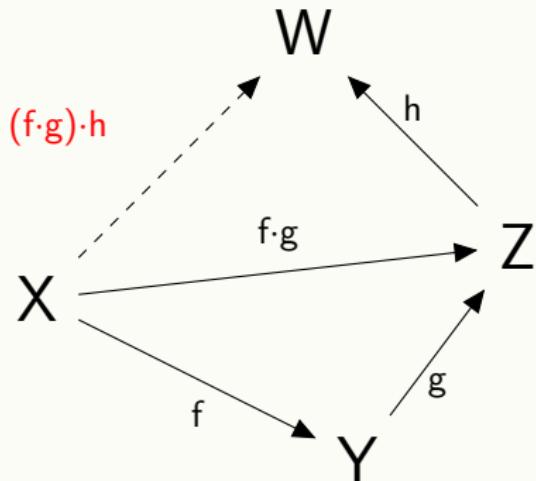
$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Associativity



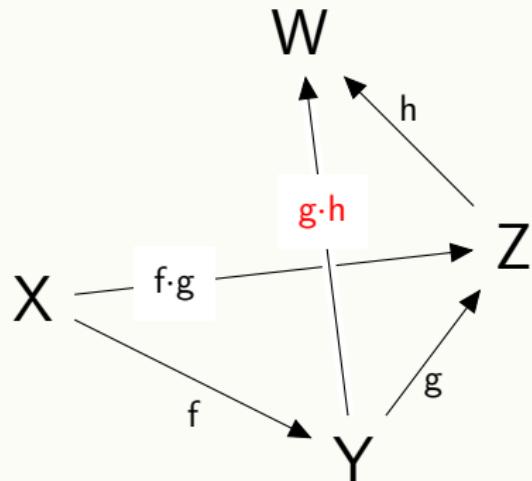
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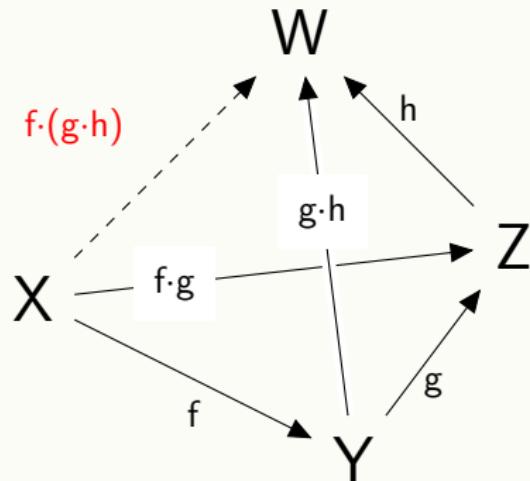
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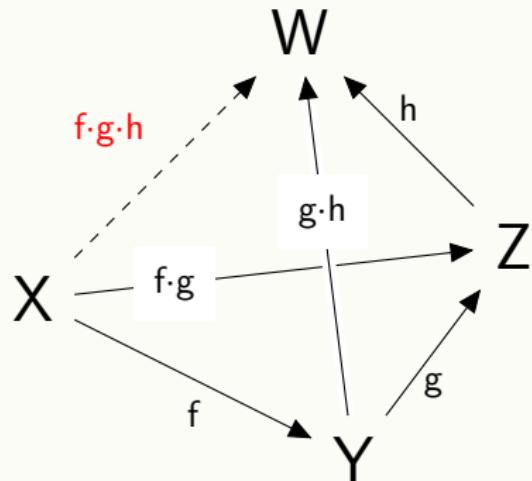
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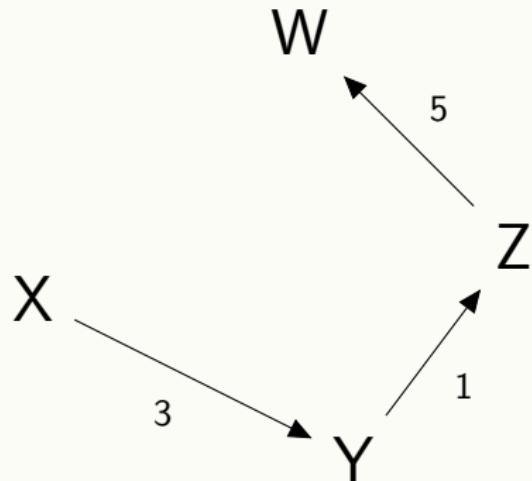
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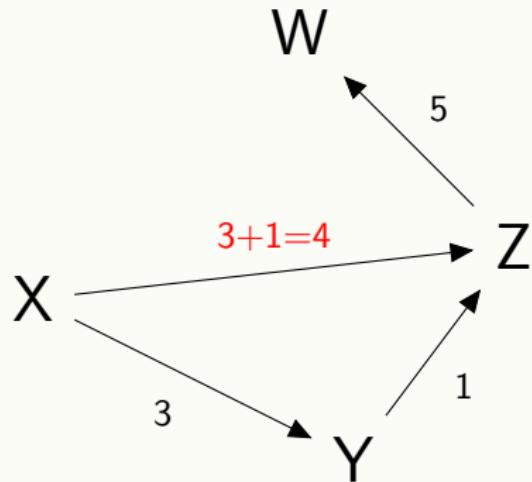
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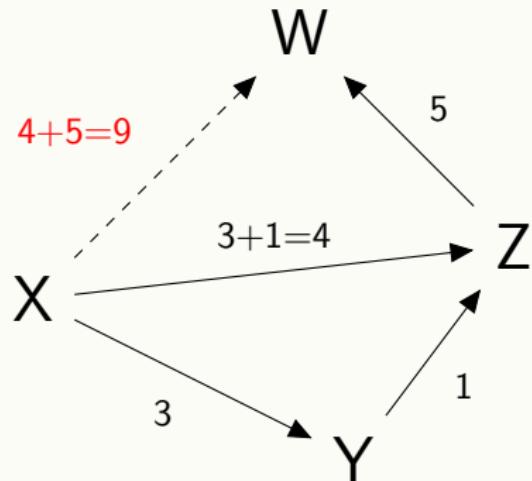
$$f \cdot g := f + g$$

Associativity



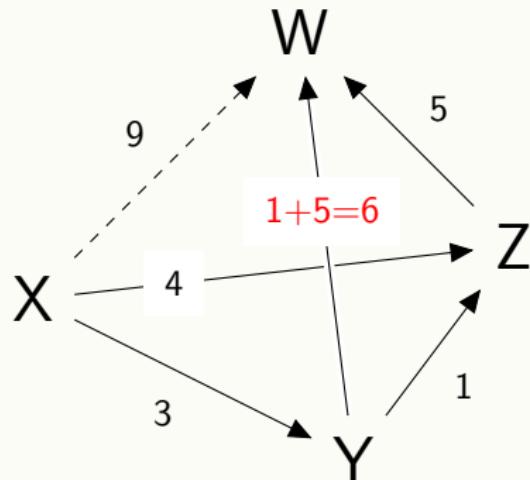
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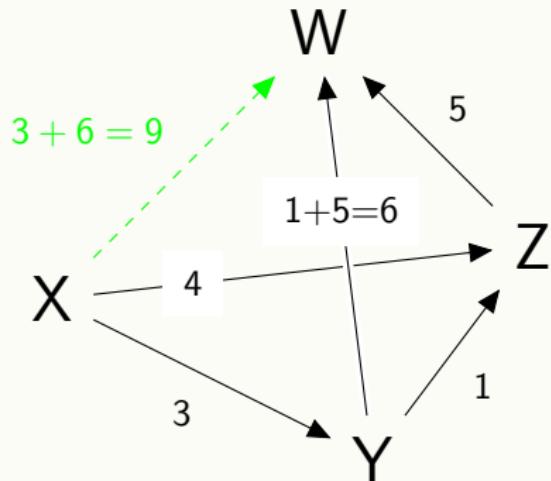
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Associativity



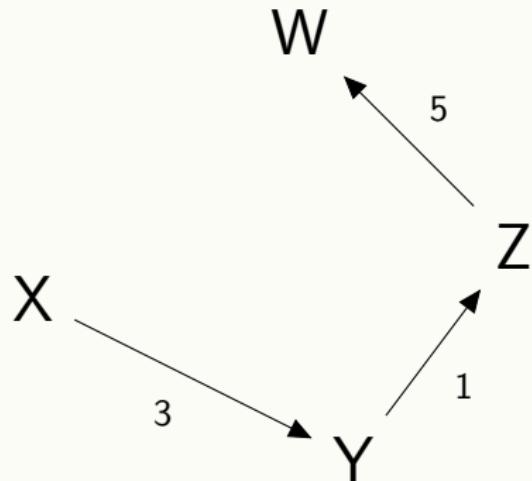
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Associativity



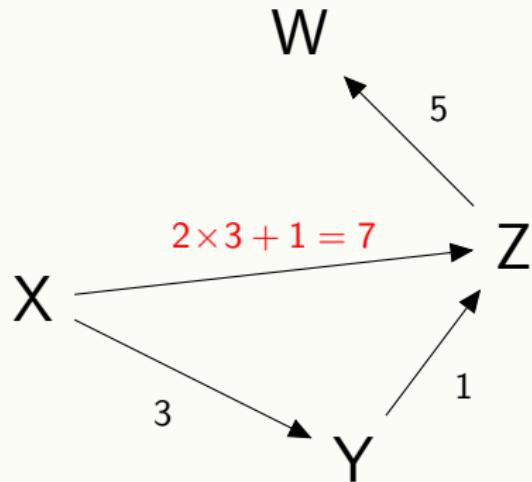
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(Non)Associativity



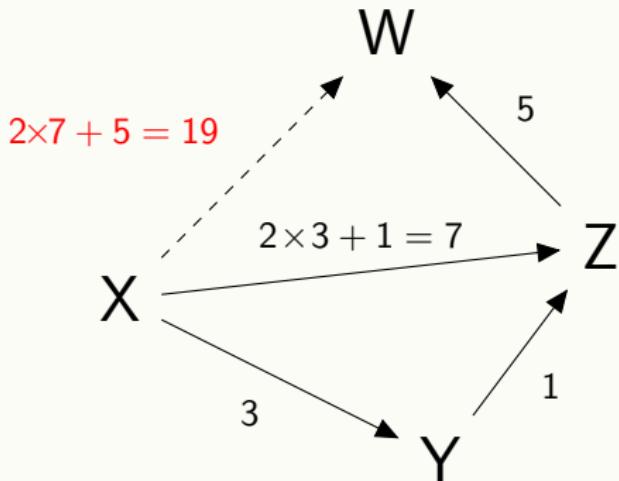
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(Non)Associativity



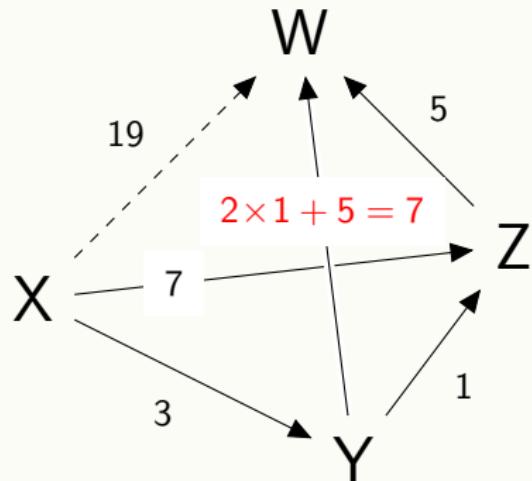
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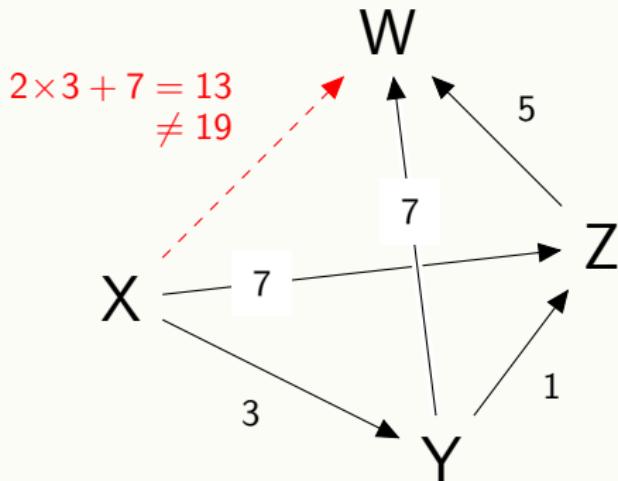
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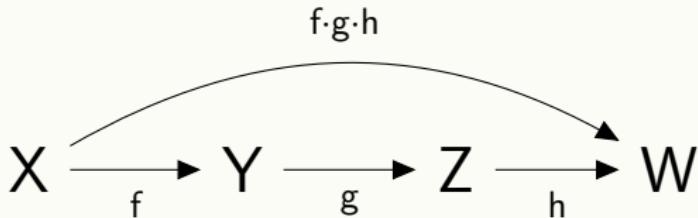
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(Non)Associativity



$$f \cdot g := 2 \times f + g$$

Associativity in Set



$$w = [(f \cdot g) \cdot h](x)$$

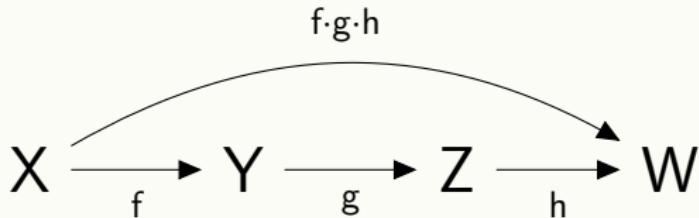
$$= h([f \cdot g](x))$$

$$= h(g(f(x)))$$

$$= [g \cdot h](f(x))$$

$$= [f \cdot (g \cdot h)](x) = w'$$

Associativity in Set



$$w = [(f \cdot g) \cdot h](x)$$

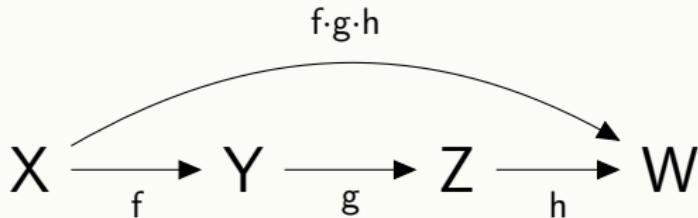
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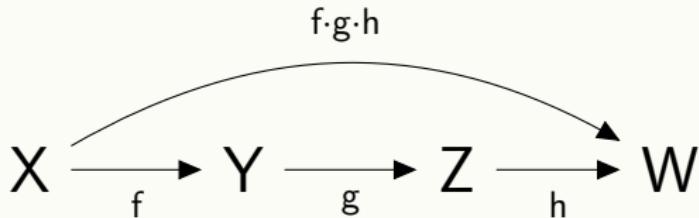
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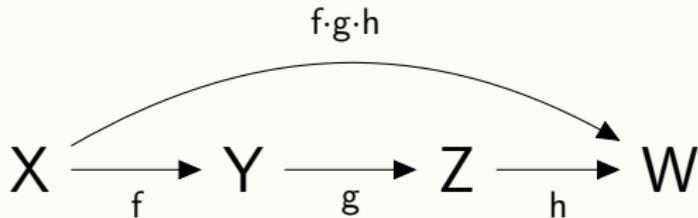
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Categories

Definition: A *category* is

- A collection of directed relationships $X \xrightarrow{f} Y, Y \xrightarrow{g} Z$
- closed under identities $X \xrightarrow{\text{id}} X$ and composition $X \xrightarrow{f \cdot g} Z$
- satisfying unit and associativity laws:

$$\text{id} \cdot f = f = f \cdot \text{id}$$

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Notation: We use a special font for categories $\mathbb{X}, \mathbb{Y}, \mathbb{S}\text{et}, \mathbb{R}\text{el}, \dots$

The category $\mathbb{R}\text{el}$

Objects Sets X, Y, Z, \dots

Arrows Relations $R \subseteq X \times Y, S \subseteq Y \times Z, \dots$

Identities $\{\langle x, x' \rangle \mid x = x'\} \subseteq X \times X$

Composition $\{\langle x, z \rangle \mid \exists y. R(x, y) \ \& \ S(y, z)\} \subseteq X \times Z$

Unit $\langle x, y \rangle \in id \cdot R \subseteq X \times Y$

$$\Leftrightarrow \exists x'. x = x' \ \& \ \langle x', y \rangle \in R$$

$$\Leftrightarrow \langle x, y \rangle \in R$$

Associativity $\dots \subseteq X \times W$

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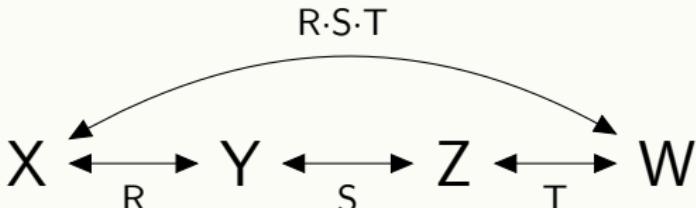
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$$\Leftrightarrow \exists x'. x = x' \ \& \ \langle x', y \rangle \in R$$

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Associativity $\dots \subseteq X \times W$

Associativity in $\mathbb{R}\text{el}$



$$\begin{aligned} [(R \cdot S) \cdot T](x, w) &\iff \exists z. [R \cdot S](x, z) \ \& \ T(z, w) \\ &\iff \exists z. (\exists y. R(x, y) \ \& \ S(y, z)) \ \& \ T(z, w) \\ &\stackrel{y \notin T}{\iff} \exists y, z. (\exists y. R(x, y) \ \& \ S(y, z) \ \& \ T(z, w)) \\ &\stackrel{z \notin R}{\iff} \exists y. R(x, y) \ \& \ (\exists z. S(y, z) \ \& \ T(z, w)) \\ [R \cdot (S \cdot T)](x, w) &\iff \exists y. R(x, y) \ \& \ [S \cdot T](y, w) \end{aligned}$$

\dagger -Categories

Definition: A \dagger -category is a category together with a reversal operation

$$\begin{array}{ccc} X & \xrightarrow{r} & Y \\ & \downarrow & \\ Y & \xrightarrow{r^\dagger} & X \end{array}$$

satisfying

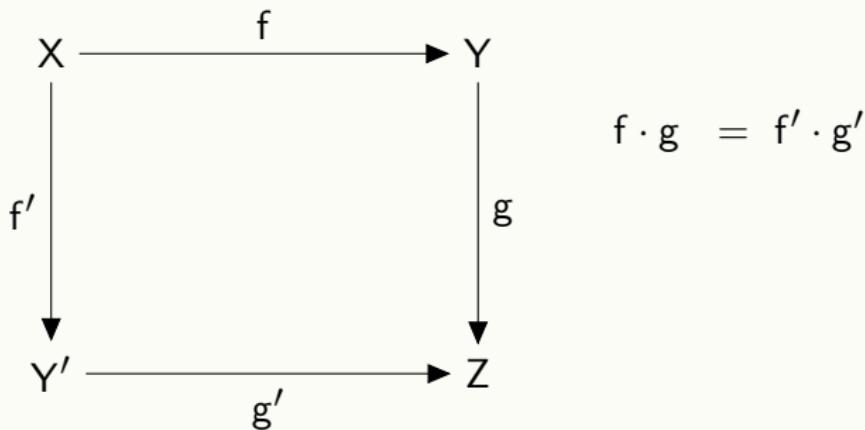
$$\left. \begin{aligned} r^{\dagger\dagger} &= r \\ id^\dagger &= id \\ (r \cdot s)^\dagger &= s^\dagger \cdot r^\dagger \end{aligned} \right\} \text{functoriality}$$

Questions?

Diagrams

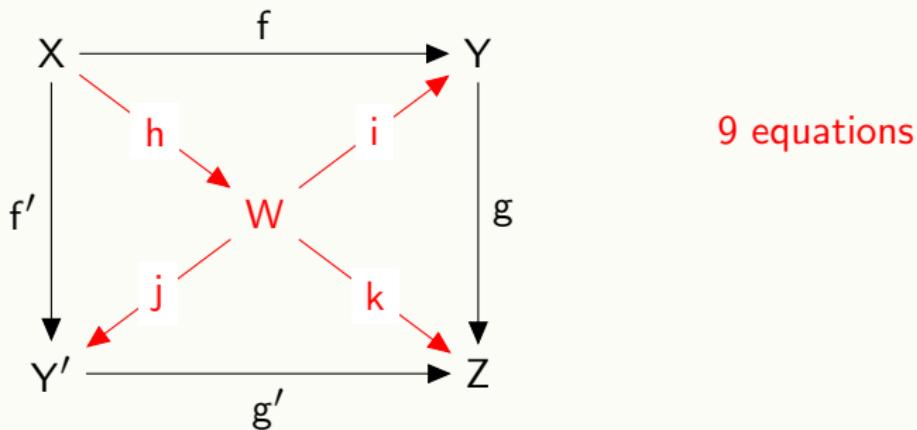
Diagrams

A diagram *commutes* if parallel paths (same src/tgt) are equal.



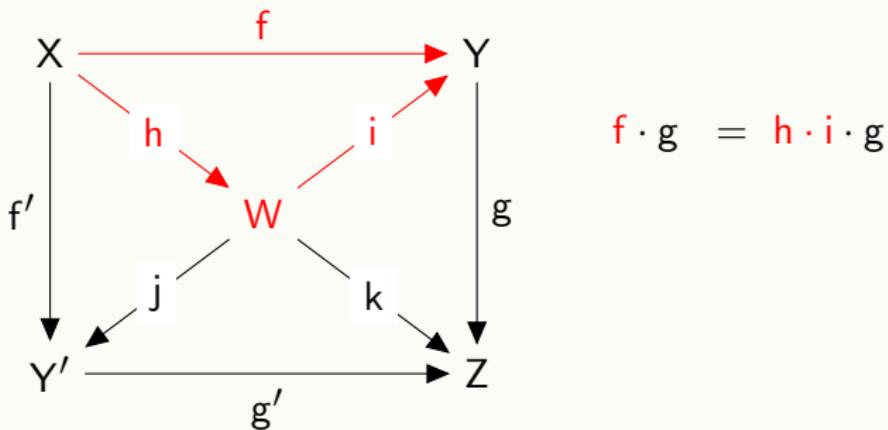
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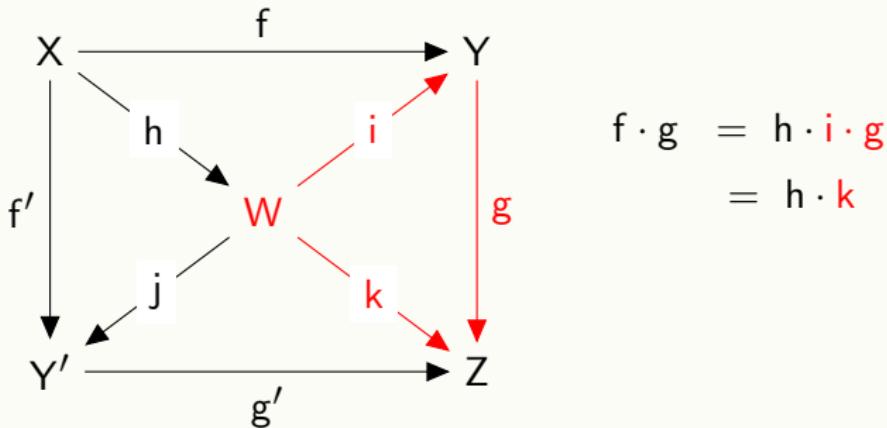
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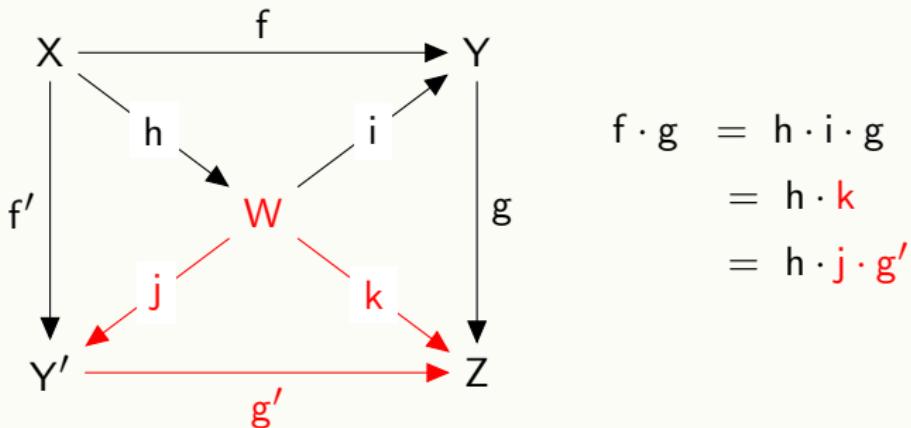
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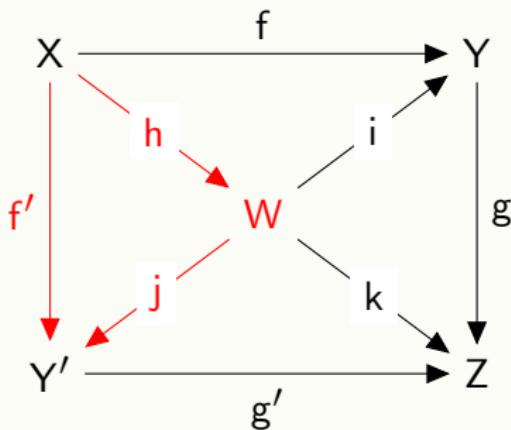
Diagrams

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Diagrams

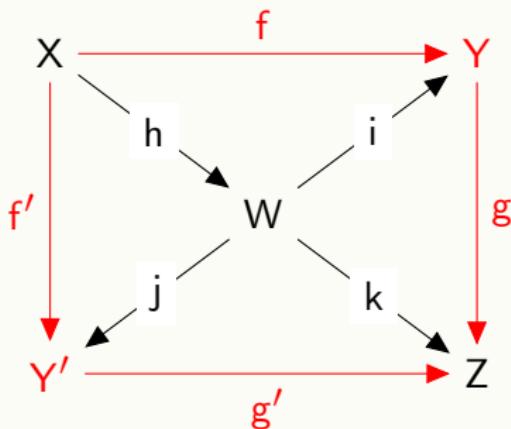
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$$\begin{aligned} f \cdot g &= h \cdot i \cdot g \\ &= h \cdot k \\ &= h \cdot j \cdot g' \\ &= f' \cdot g' \end{aligned}$$

Diagrams

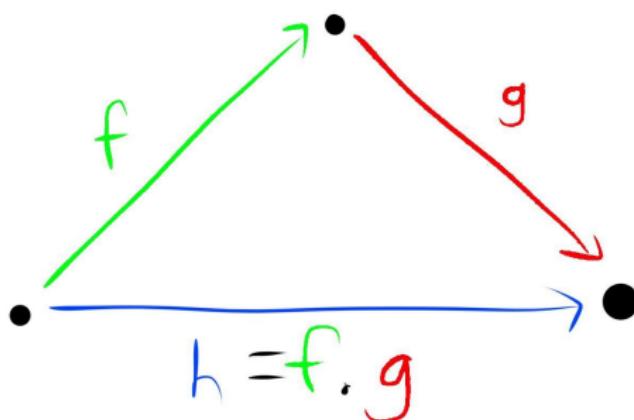
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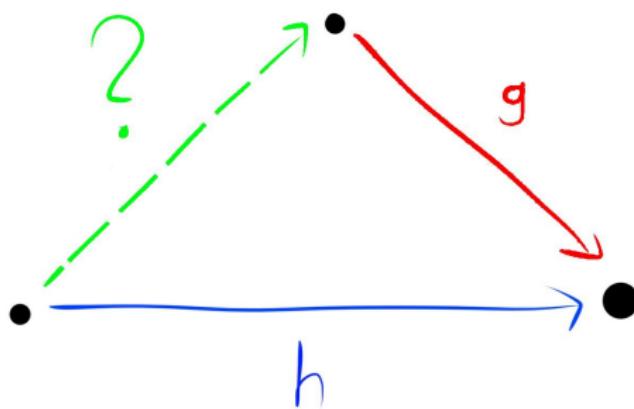
Lifting & Extension

Composition is a forward mapping $\langle f, g \rangle \mapsto h$.



Lifting & Extension

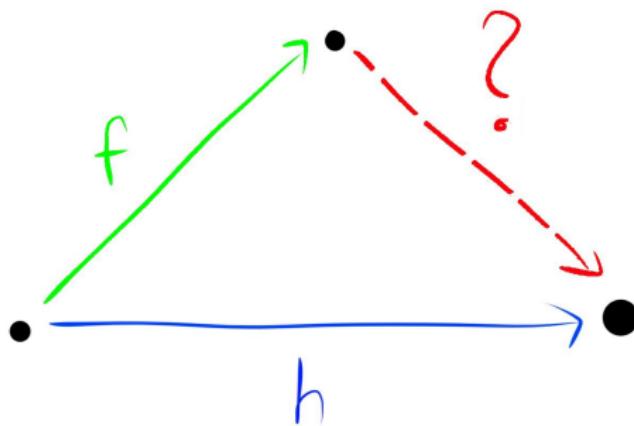
There are *two* inverse problems:



Lifting problem: given h & g , find f .

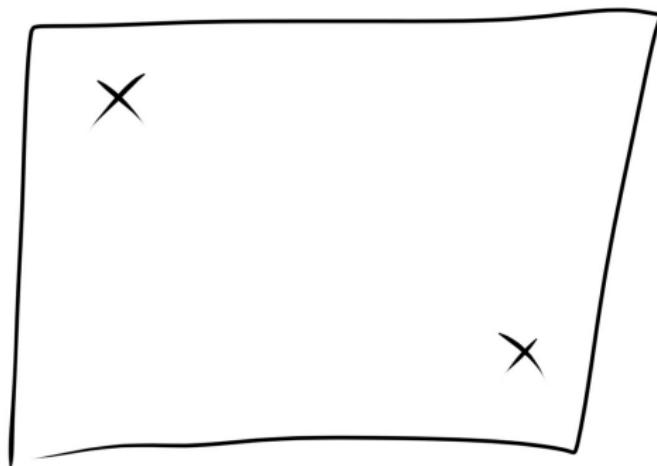
Lifting & Extension

There are *two* inverse problems:



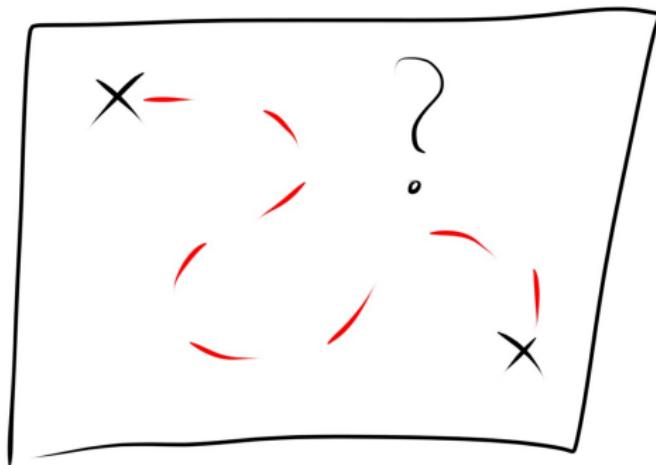
Extension problem: given h & f , find g .

Lifting & Extension



Driver's problem: given h & f , find g .

Lifting & Extension

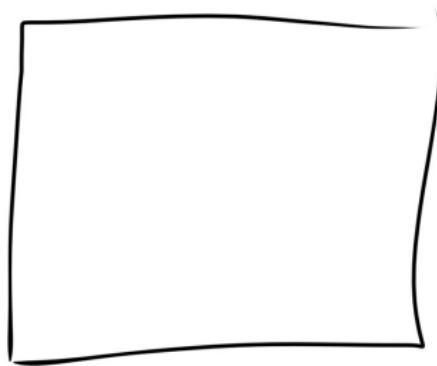


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Lifting & Extension

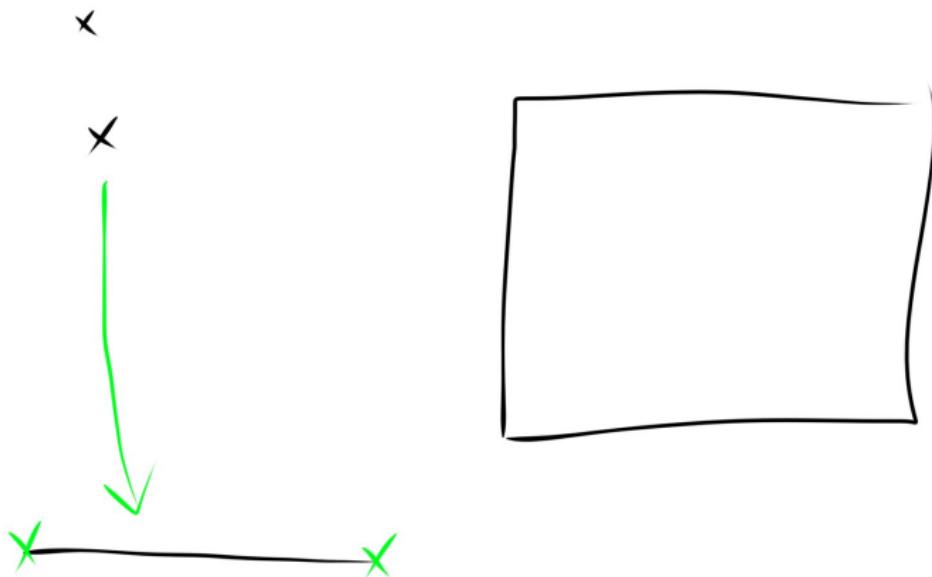
x

x



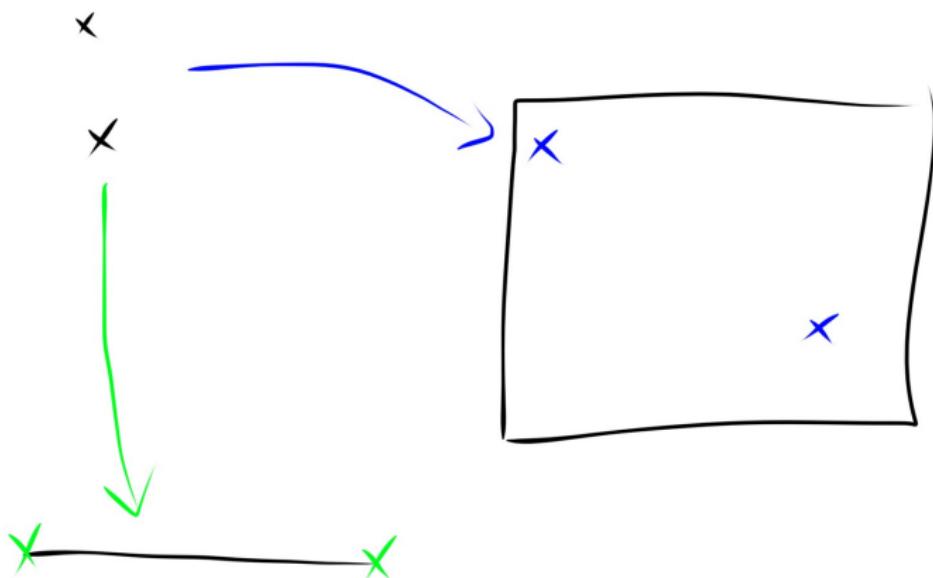
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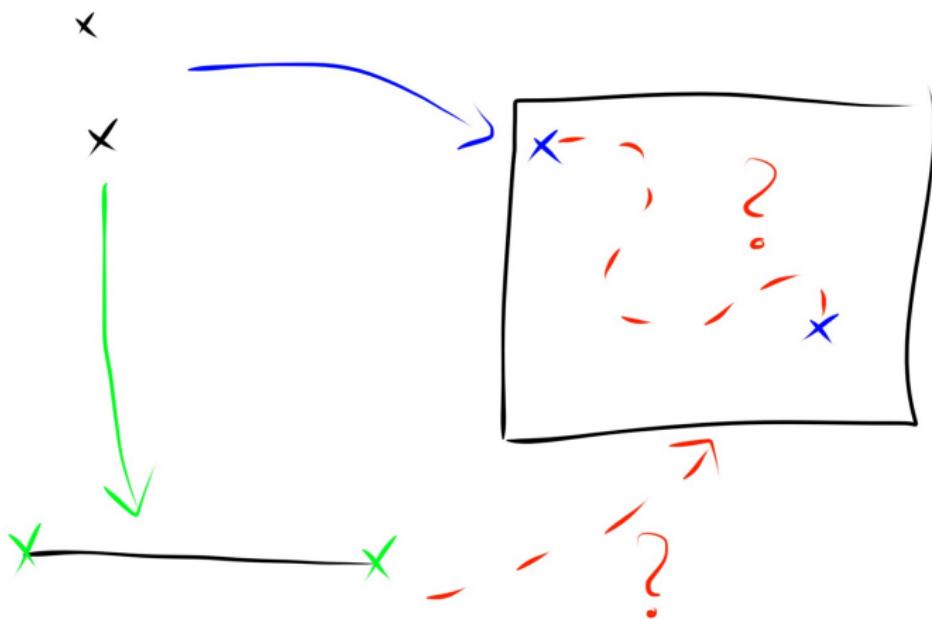
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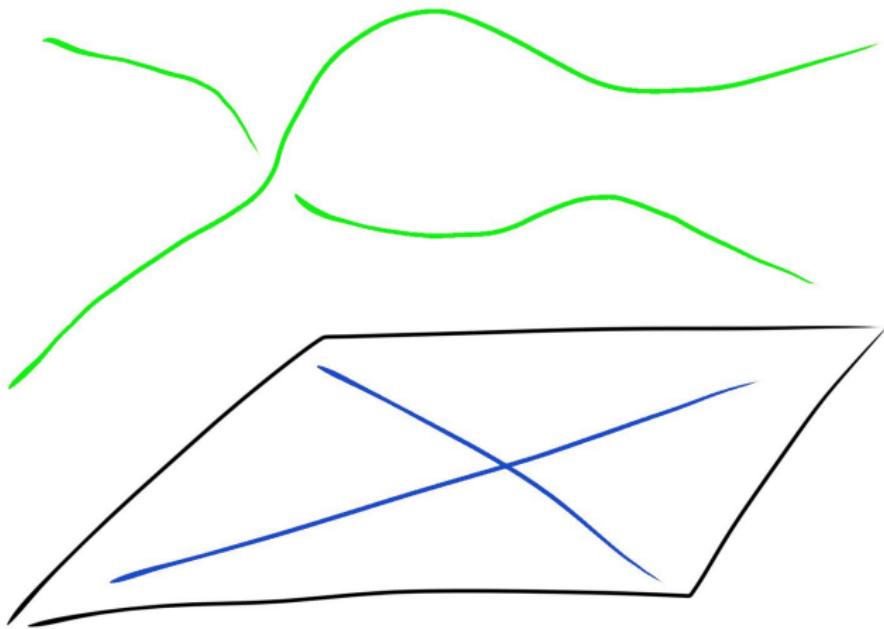
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Lifting & Extension



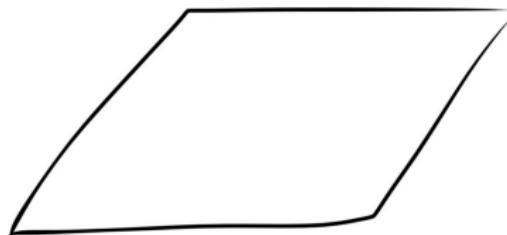
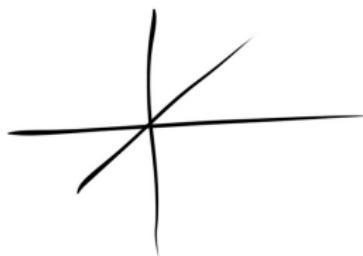
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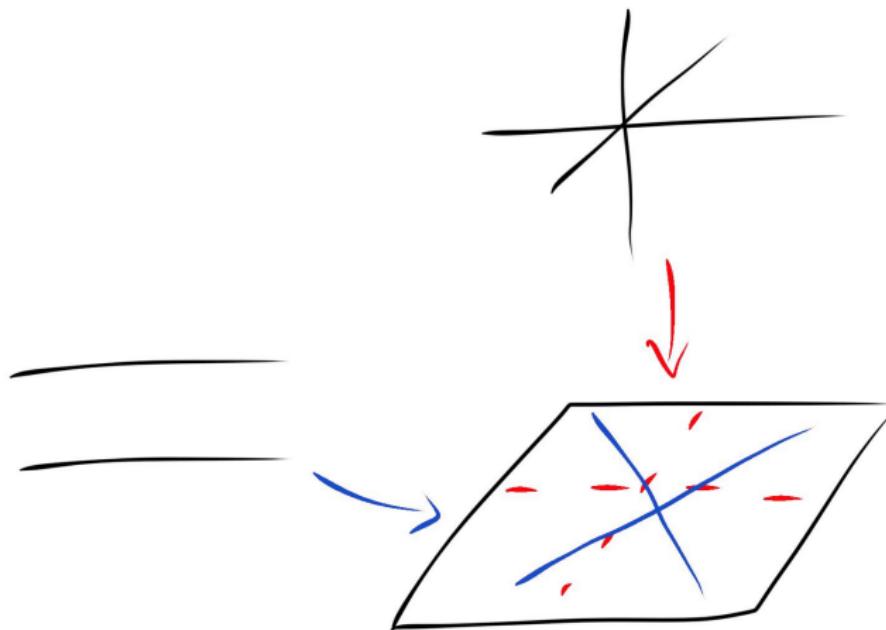
Pilots' problem: given h & g , find f .

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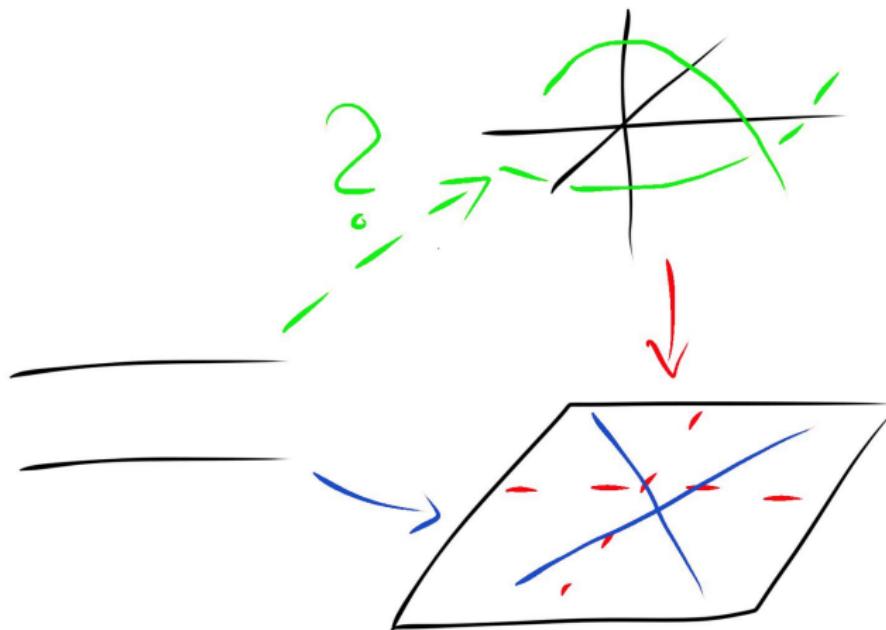
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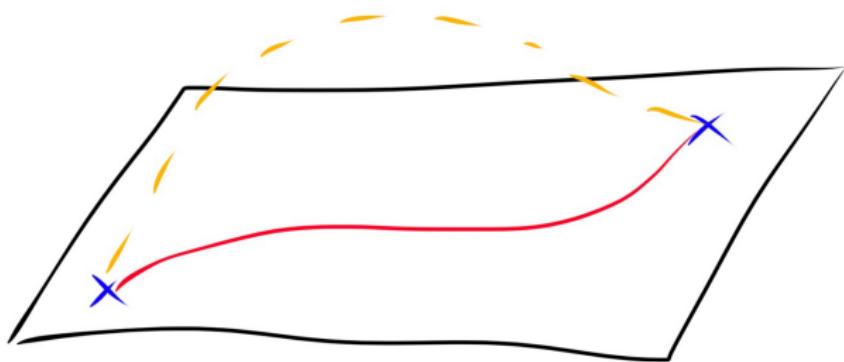
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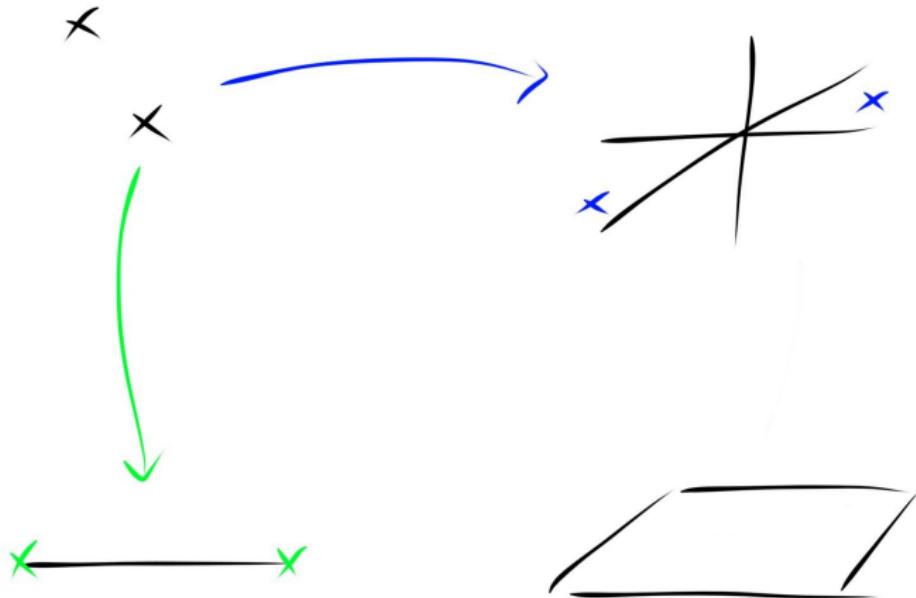
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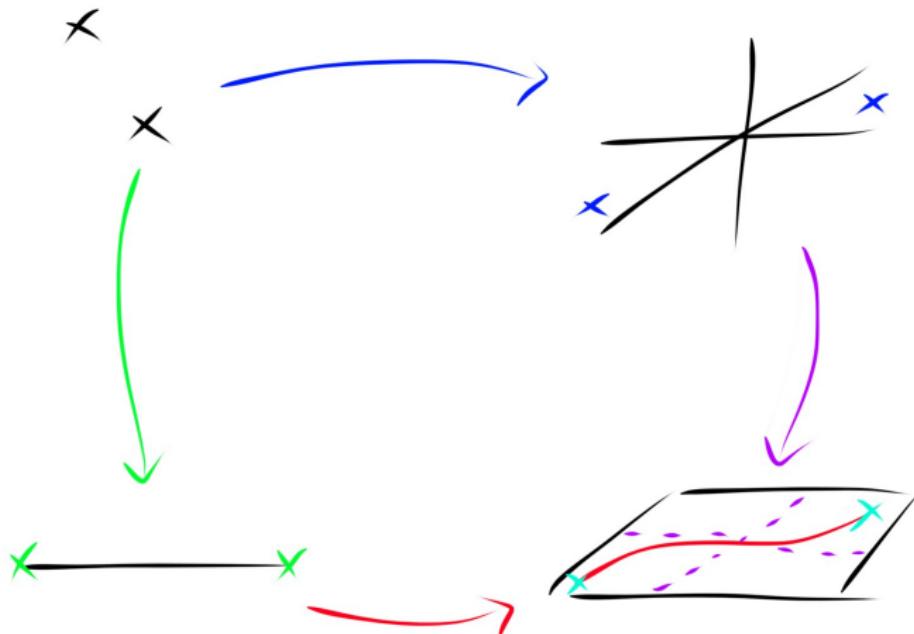
Pilot's problem (now with landing!): fill the diagonal.

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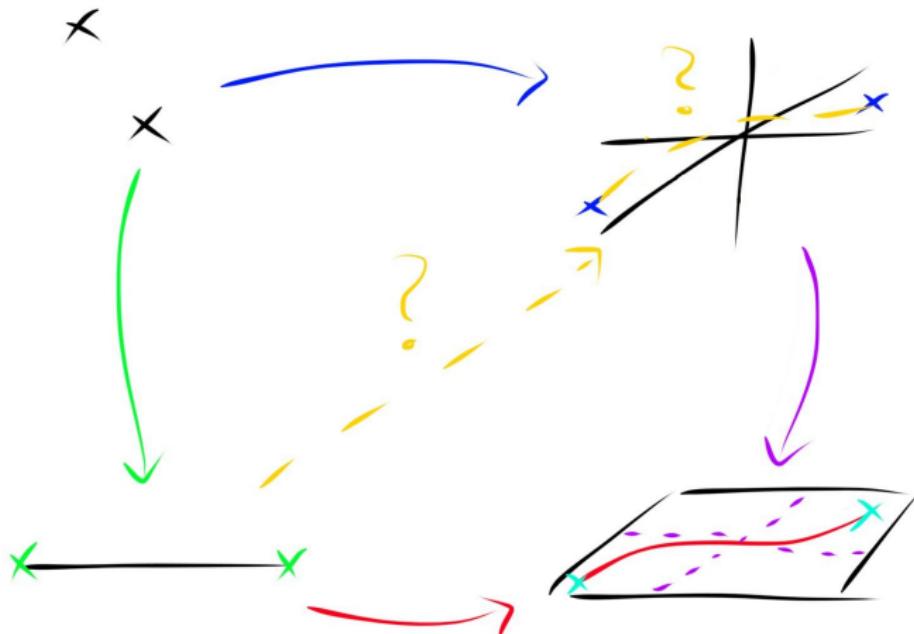
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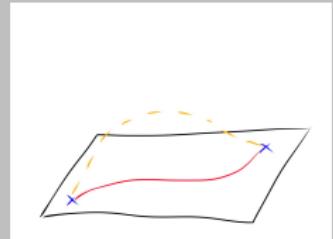
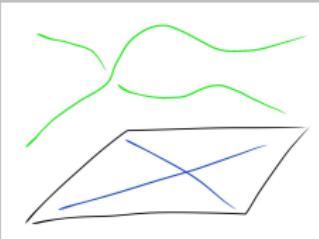
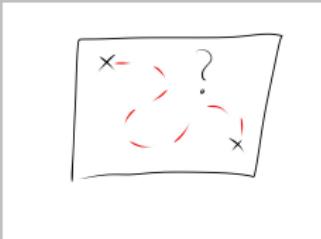


Pilot's problem (now with landing!): fill the diagonal.

Questions?

| | Set | Rel |
|-------------------|-------------------------------|-----------------------|
| Notation | $R : X \longleftrightarrow Y$ | $f : X \rightarrow Y$ |
| | $x \leftrightarrow y$ | $x \mapsto y$ |
| Category | ✓ | ✓ |
| †-Category | ✓ | X |

Extension Problem Lifting Problem Joint Constraint



Structure

Structure in categories

| Structure | Notation | Generalizes |
|----------------------|--|----------------------------------|
| Isomorphism | $X \cong Y$ | Permutation, symmetry |
| Products | $X \times Y$ | Pairs, tuples |
| Coproducts | $X + Y$ | Conditionals (if/then/else) |
| Subobject | $S \rightarrowtail X$ | Subset, formula |
| Exponential | Y^X | Function space $Y \rightarrow X$ |
| Pullback | $\begin{matrix} Y \times Z \\ \downarrow X \end{matrix}$ | Intersection, substitution |
| Pushout | $\begin{matrix} Y + Z \\ \downarrow X \end{matrix}$ | Union |
| Span | $X \leftarrow R \rightarrow Y$ | Relation |
| Cospan | $X \rightarrow C \leftarrow Y$ | Connectivity diagram |
| NNO | $1 \xrightarrow{0} N \xrightarrow{s} N$ | Nat. numbers |
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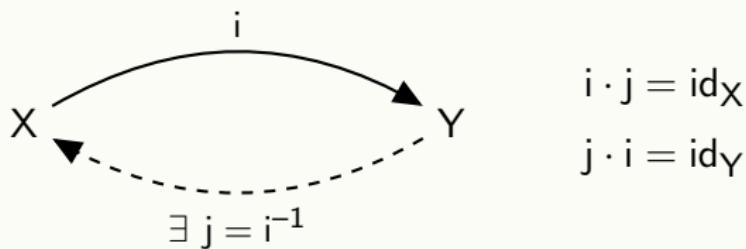
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Isomorphism

Isomorphism

Definition: An *isomorphism* is an arrow $i : X \rightarrow Y$ that has an inverse j satisfying two equations



We say X and Y are *isomorphic*: $X \cong Y$.

Identity of Indiscernibles

Two objects bearing the same properties are equal.

- Leibniz, ~1715

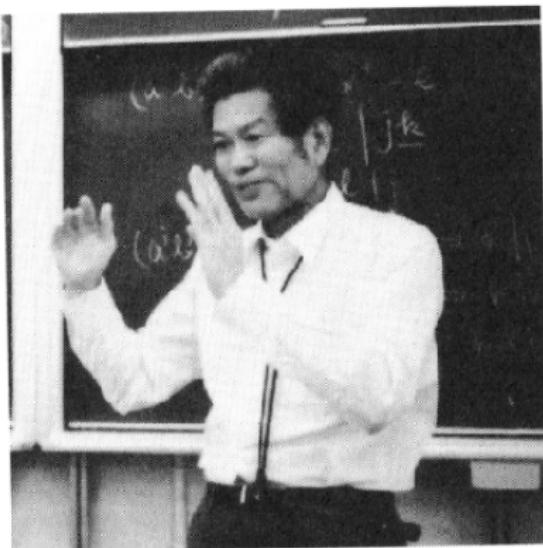


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米田信夫

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Y

Y'

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Y

$\forall X$

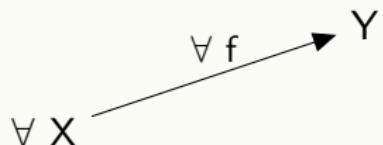
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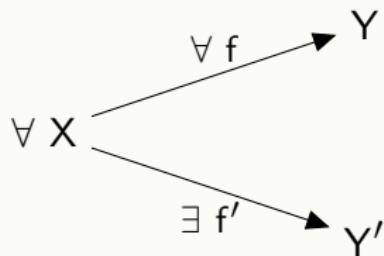


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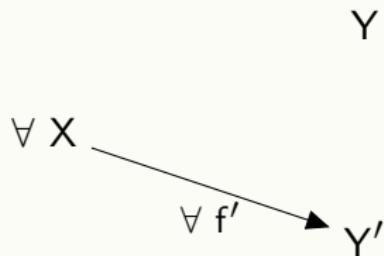
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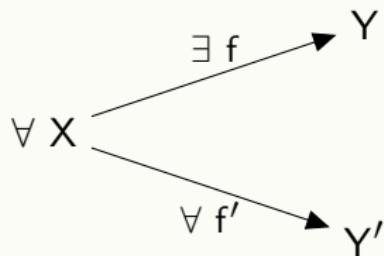


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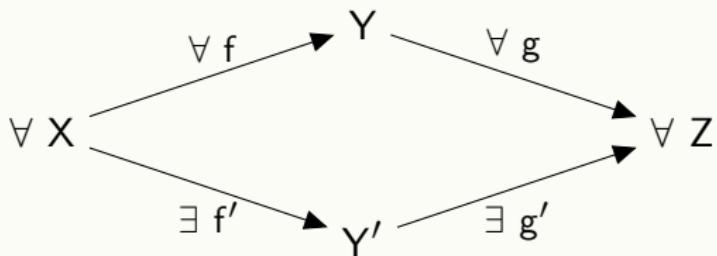


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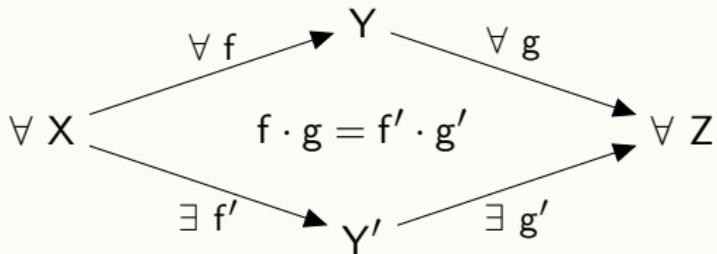
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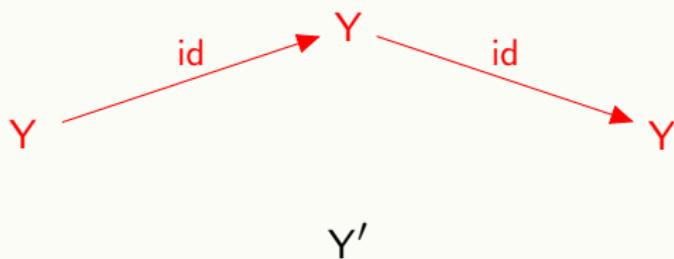


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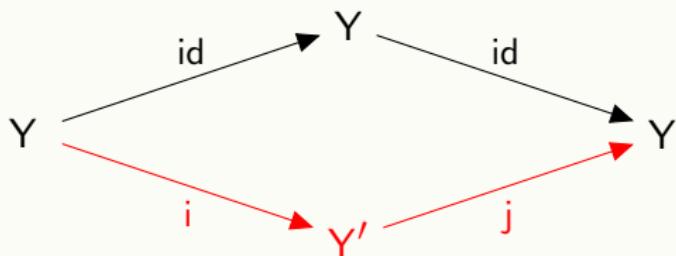
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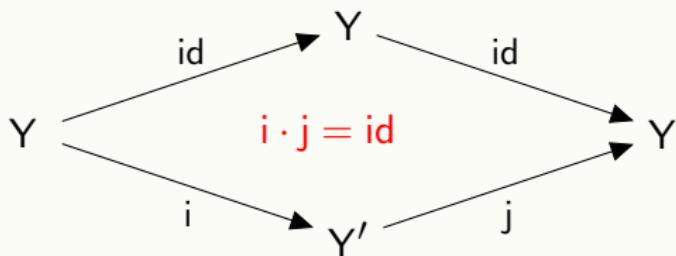
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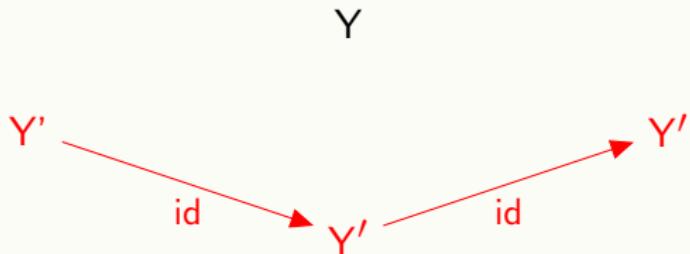


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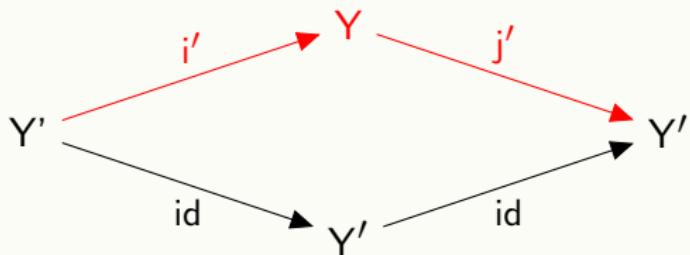


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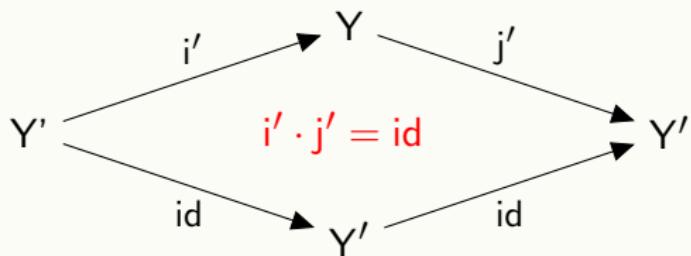
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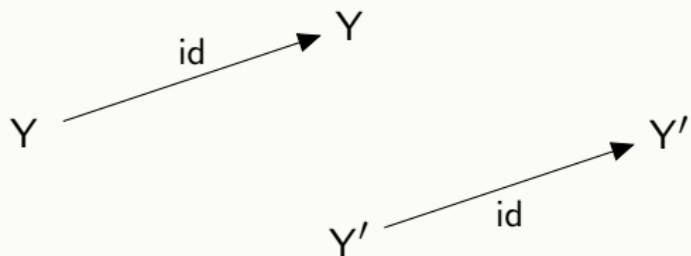
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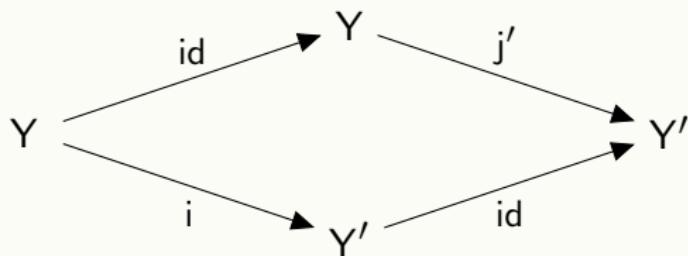
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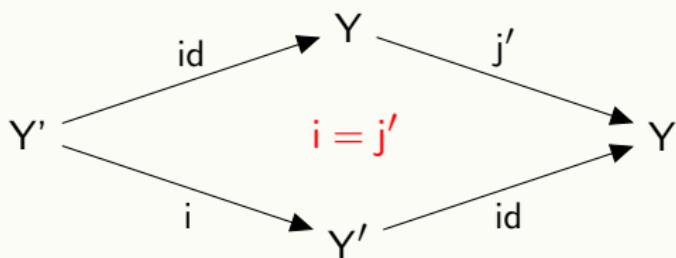
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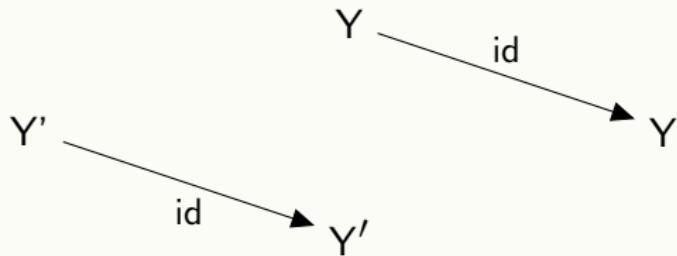
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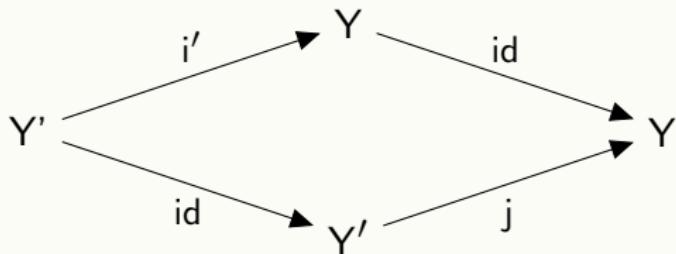
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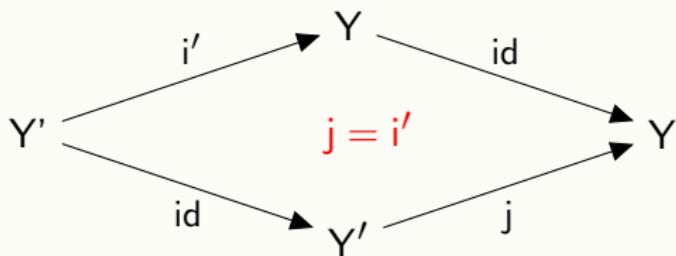
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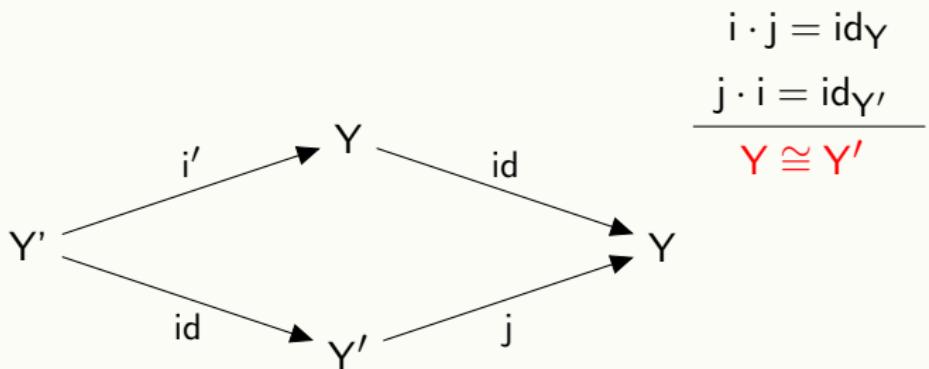
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Questions?

Products

Cartesian products

Definition: The *Cartesian product* of X and Y is a diagram $X \xleftarrow{\text{proj}_1} P \xrightarrow{\text{proj}_2} Y$ such that for any object T and any arrows $x : T \rightarrow X$ and $y : T \rightarrow Y$, there is a unique arrow $p : T \rightarrow P$ satisfying $p \cdot \text{proj}_1 = x$ and $p \cdot \text{proj}_2 = y$.

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X

Y

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$$X \xleftarrow{\text{proj}_1} P \xrightarrow{\text{proj}_2} Y$$

Cartesian products

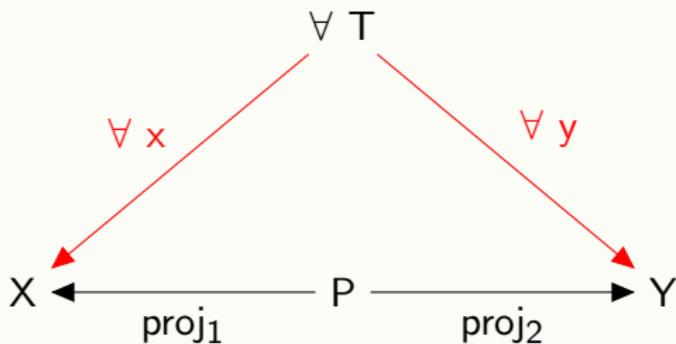
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$\forall T$

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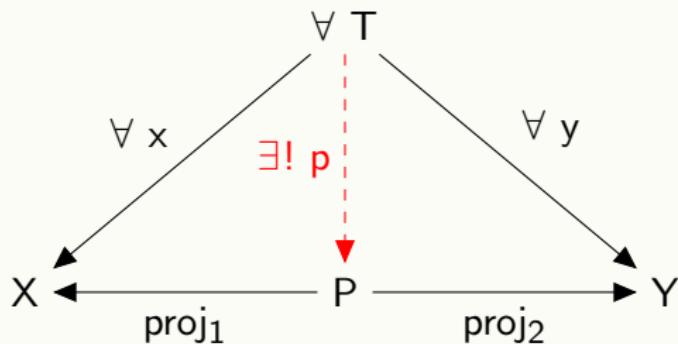
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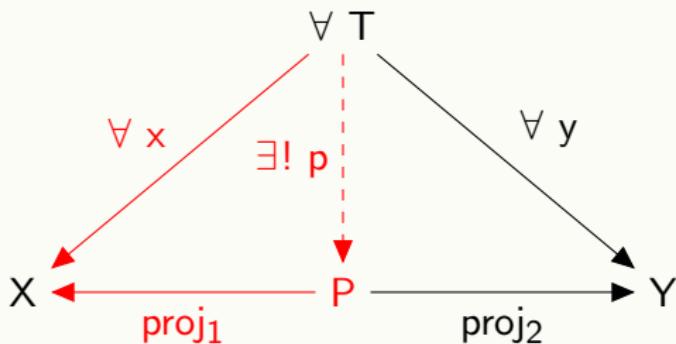
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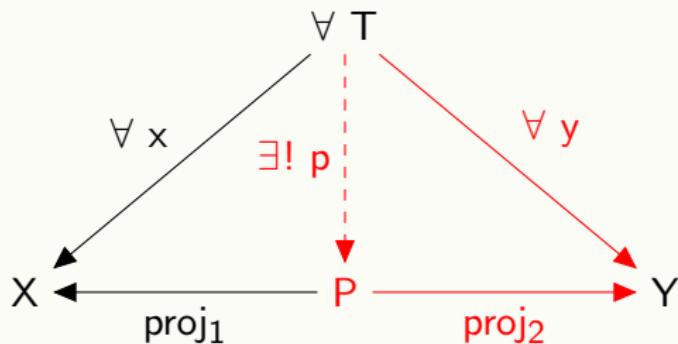
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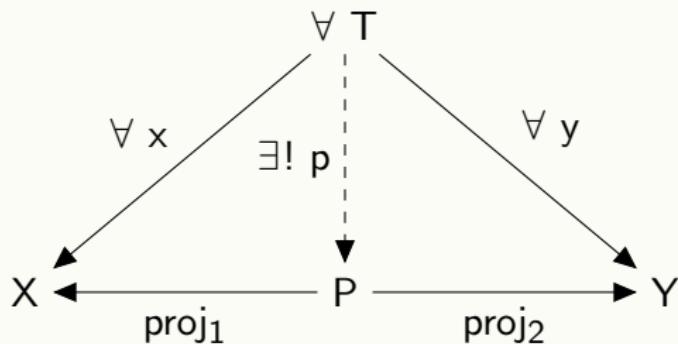
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$$\underbrace{\text{Hom}(T, X \times Y)}_{\text{In } \mathbb{C}} \cong \underbrace{\text{Hom}(T, X) \times \text{Hom}(T, Y)}_{\text{In } \mathbb{Set}}$$

A trick

$$\frac{x_0 \in X \qquad \qquad y_0 \in Y}{\langle x_0, y_0 \rangle \in X \times Y}$$

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$$\vec{x} : \{*\} \xrightarrow{*\mapsto x_0} X$$

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$$\vec{x} : \{*\} \longrightarrow X \qquad \qquad \vec{y} : \{*\} \xrightarrow{*\mapsto y_0} Y$$

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$$\langle x_0, y_0 \rangle \in X \times Y$$

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$$\vec{p} : \{*\} \longrightarrow X \times Y$$

A trick

$$x_0 \in X$$

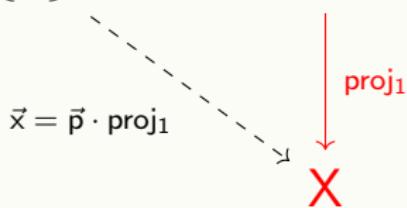
$$y_0 \in Y$$

$$\langle \textcolor{red}{x}_0, y_0 \rangle \in X \times Y$$

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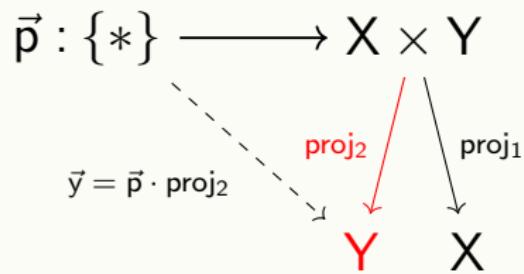


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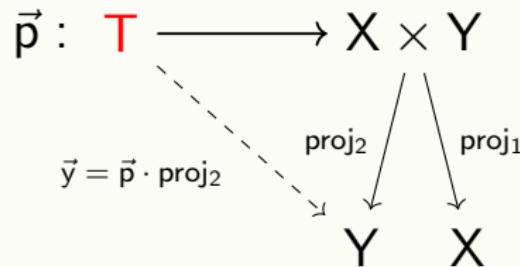
$$x_t \in X$$

$$y_t \in Y$$

$$\langle x_t, y_t \rangle \in X \times Y$$

$$\vec{x} : T \longrightarrow X$$

$$\vec{y} : T \longrightarrow Y$$



Test objects

In $\mathbb{S}et$, the singleton $1 = \{\ast\}$ “sees” everything.

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$$X \cong \text{Hom}(1, X)$$

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$$X \cong \text{Hom}(1, X)$$

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$$\text{Hom}(1, X) \cong \text{Hom}(1, Y)$$



$$X \cong Y$$

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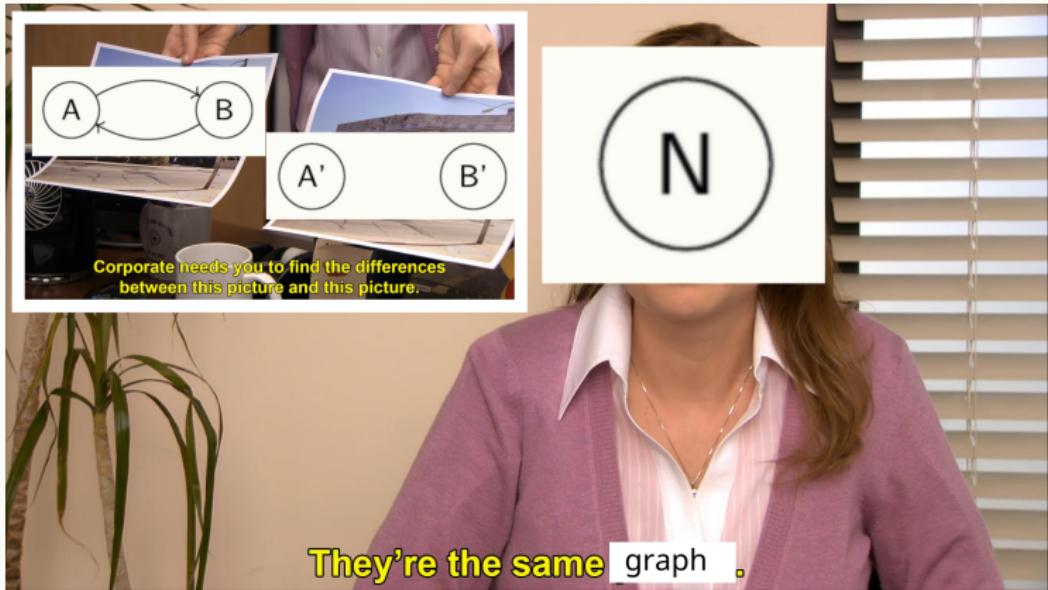
$$x_0 \longleftrightarrow \vec{x}$$

$$\text{Hom}(1, X \times Y) \cong X \times Y$$

$$\cong \text{Hom}(1, X) \times \text{Hom}(1, Y)$$

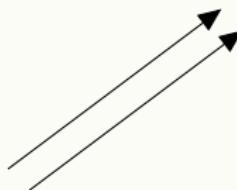
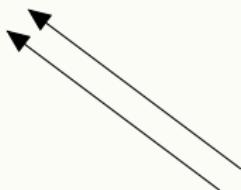
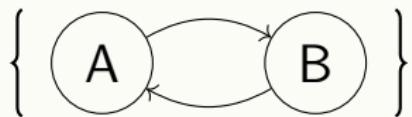
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In Graph, a singleton node can't "see" edges.



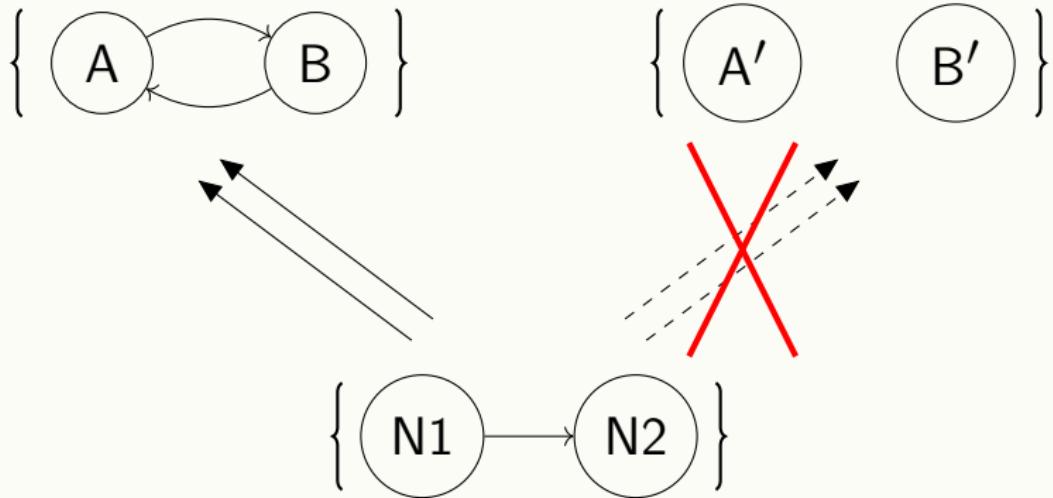
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Test objects

In Graph, a singleton node can't "see" edges.



But an edge can.

Test objects

$P = G \times H$ in $\mathbb{G}\text{raph}$ if

$$\frac{N \rightarrow G \quad N \rightarrow H}{N \rightarrow P}$$

and

$$\frac{E \rightarrow G \quad E \rightarrow H}{E \rightarrow P}$$

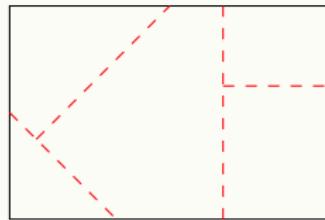
Test objects

In geometry, a point can't "see" continuity.

X

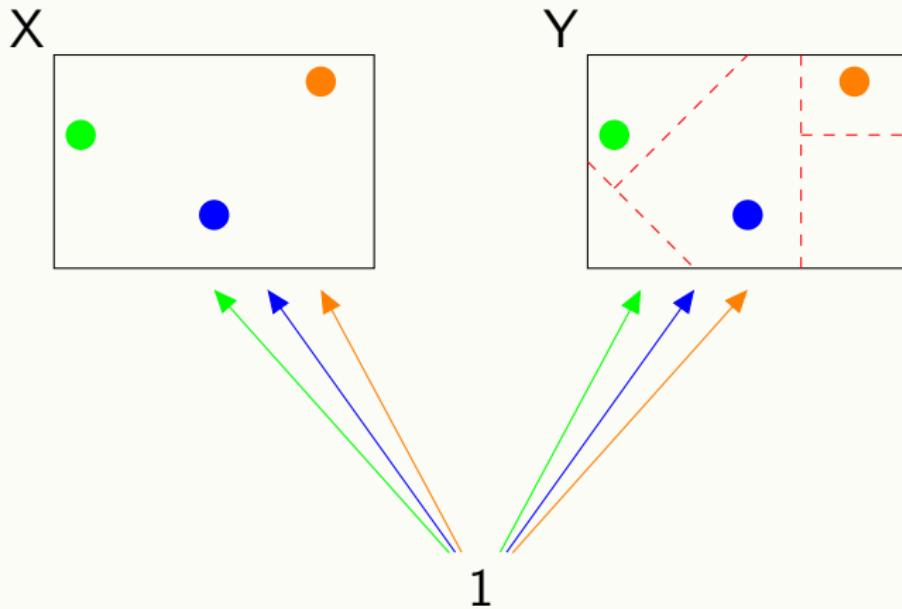


Y



Test objects

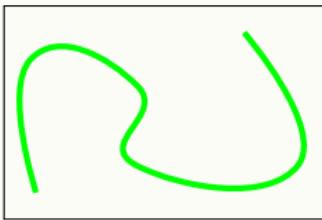
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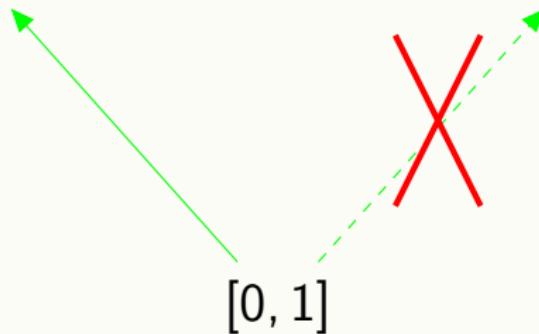
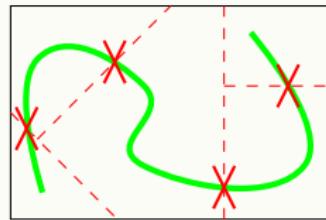
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X



Y

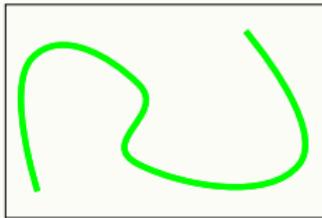


But a curve can.

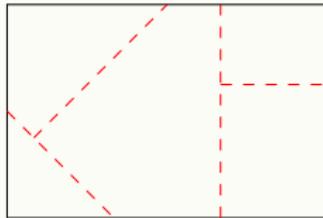
Generalized elements

A curve $[0, 1] \rightarrow X$ acts like an element of X .

X

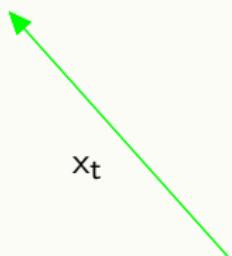


Y



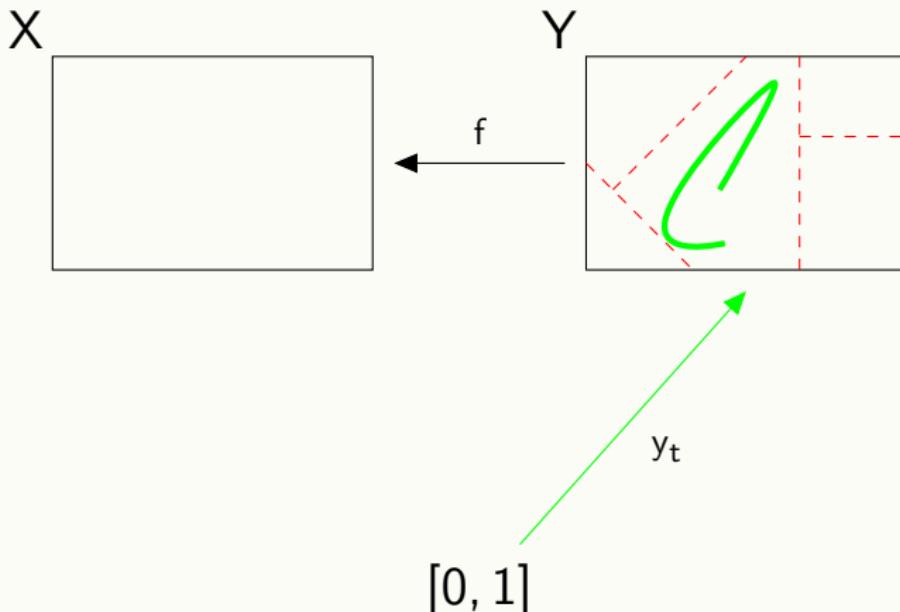
x_t

$[0, 1]$



Generalized elements

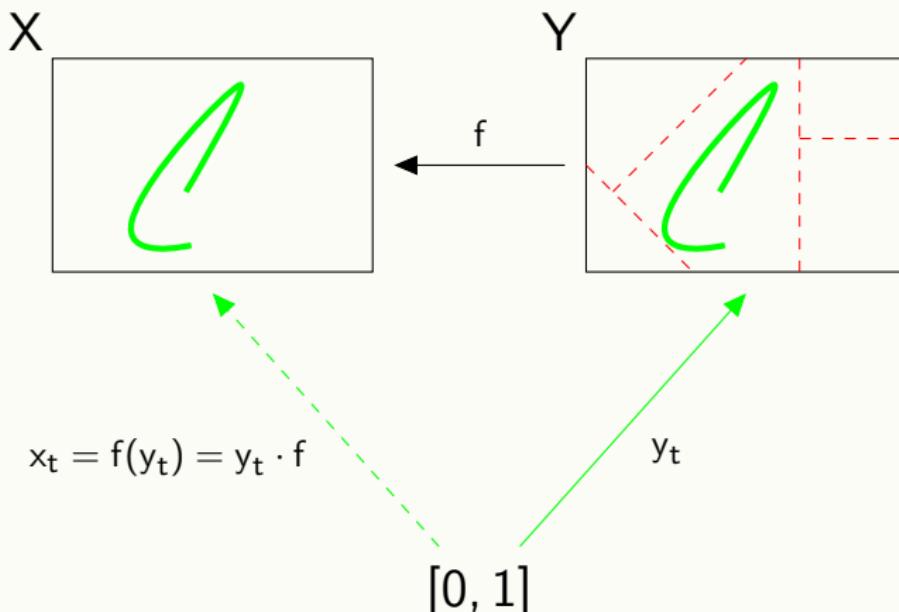
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“Apply” f to an “element” by composition.

Generalized elements

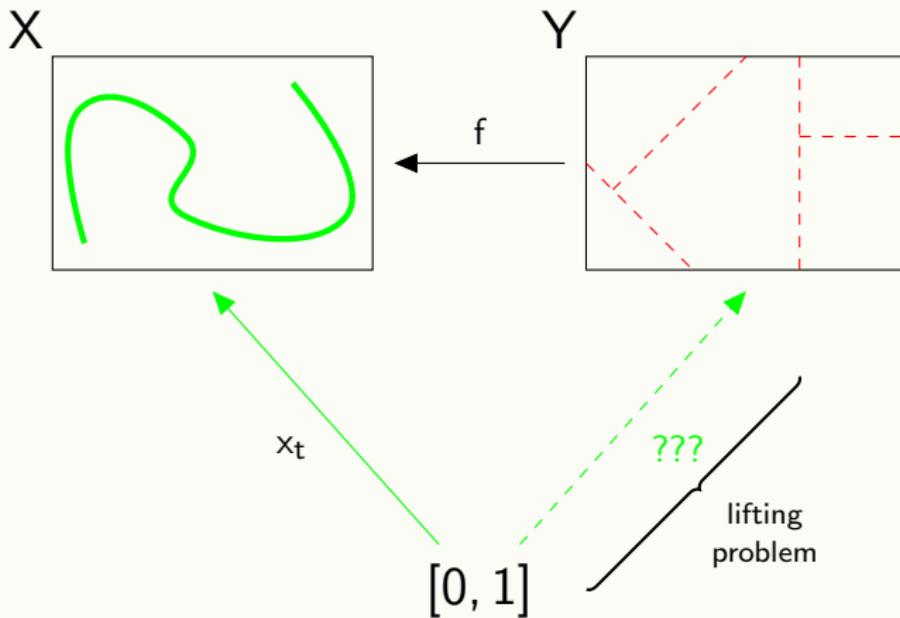
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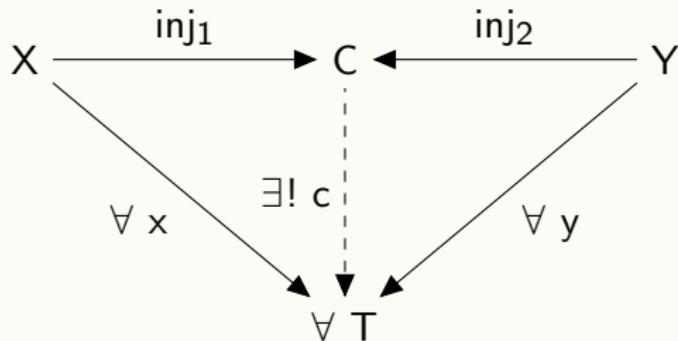
Questions?

Duality

Coproducts

Every construction in category theory has a *dual*.

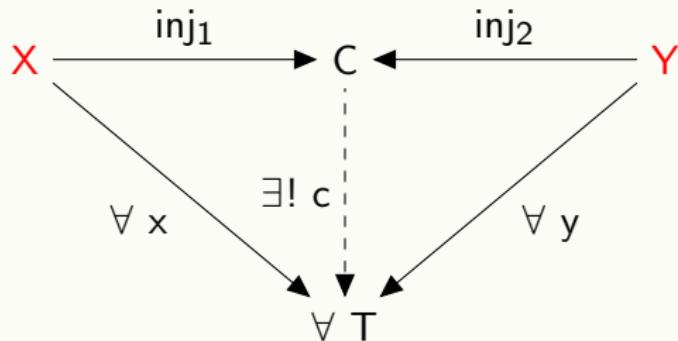
Just reverse all the arrows.



Coproducts

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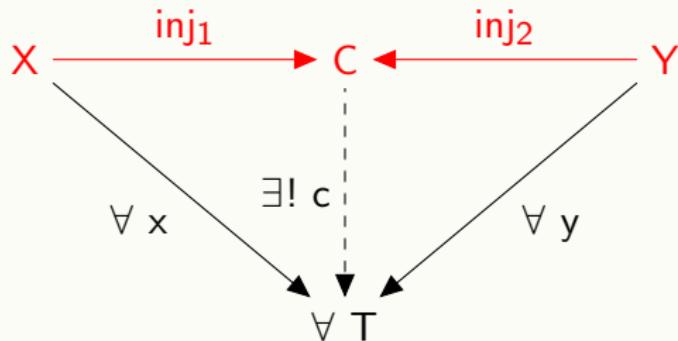


Definition: The *coproduct* of X and Y

Coproducts

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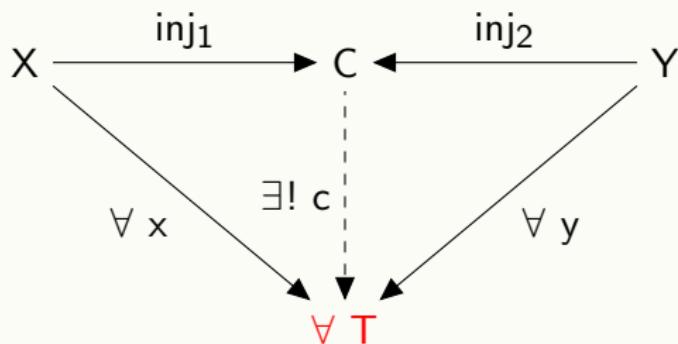
Definition: The *coproduct* of X and Y is a **diagram**

$$X \xrightarrow{\text{inj}_1} C \xleftarrow{\text{inj}_2} Y$$

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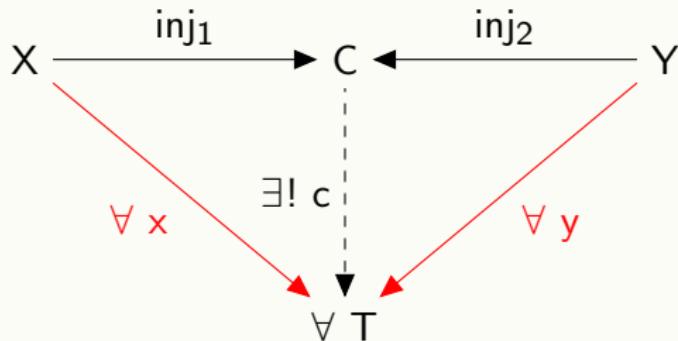
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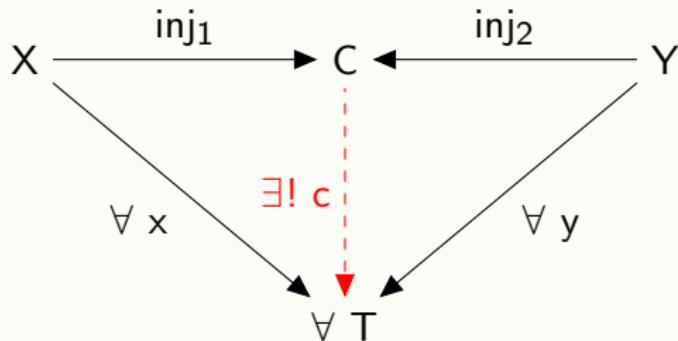
Definition: The *coproduct* of X and Y is a diagram

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 $x : X \rightarrow T$ and $y : Y \rightarrow T$,

Coproducts

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Just reverse all the arrows.



Definition: The *coproduct* of X and Y is a diagram

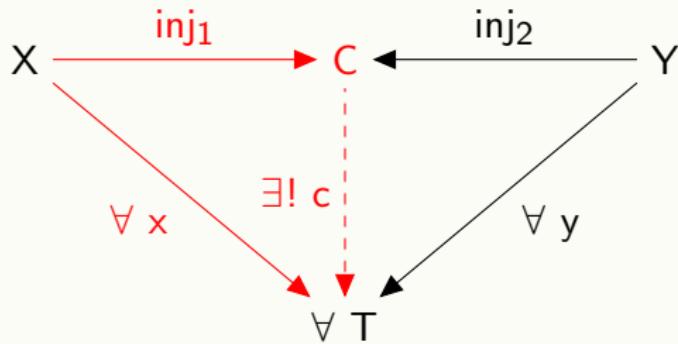
$X \xrightarrow{\text{inj}_1} C \xleftarrow{\text{inj}_2} Y$ such that for any object T and any arrows

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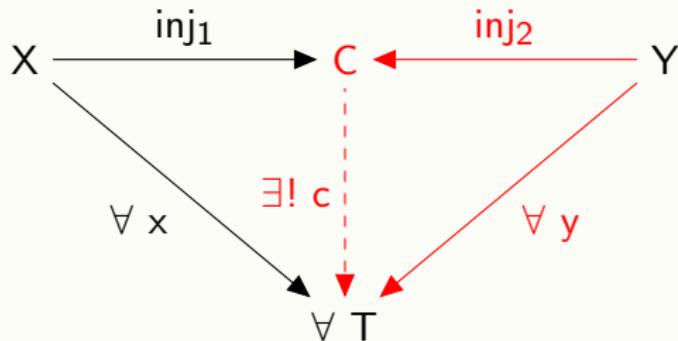
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satisfying $\text{inj}_1 \cdot c = x$

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satisfying $\text{inj}_1 \cdot c = x$ and $\text{inj}_2 \cdot c = y$.

Coproducts in Set

In Set , the coproduct is a sum $X + Y$ (*disjoint union*)

$$\{a, b, c, d\} + \{b, d, e\} \cong \left\{ \begin{array}{ll} a_1, b_1, c_1, d_1, \\ b_2, \quad d_2, e_2 \end{array} \right\}$$

Coproducts in Set

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Given $f(x : X) : T$ and $g(y : Y) : T$

```
def choose(f,g)(x_or_y:X+Y):T
  if (index(x_or_y)==1)
    f(x_or_y)
  else
    # index(x_or_y)==2
    g(x_or_y)
  end
end
```

Coproducts in $\mathbb{R}\text{el}$

The same thing works in $\mathbb{R}\text{el}$

$$\{a, b, c, d\} + \{b, d, e\} \cong \left\{ \begin{array}{ll} a_1, b_1, c_1, d_1, \\ b_2, & d_2, e_2 \end{array} \right\}$$

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Given $R(x : X, t : T) : \text{Bool}$ and $S(y : Y, t : T) : \text{Bool}$

```
def choose(R,S)(x_or_y:X+Y,t:T):Bool
    if (index(x_or_y)==1)
        R(x_or_y,t)
    else
        # index(x_or_y)==2
        S(x_or_y,t)
    end
end
```

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$T \longleftrightarrow X \times Y \longleftrightarrow T'$$

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$$T \longleftrightarrow X \times Y \qquad \qquad T' \longleftrightarrow X \times Y$$

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$\begin{array}{c} T \longleftrightarrow X \times Y \longleftrightarrow T' \\ \hline\hline \textcolor{red}{T \longleftrightarrow X \times Y} & \textcolor{red}{T' \longleftrightarrow X \times Y} \\ \hline\hline \textcolor{red}{T \longleftrightarrow X} & \textcolor{red}{T \longleftrightarrow Y} \end{array}$$

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$\frac{T \longleftrightarrow X \times Y \longleftrightarrow T' \quad T \longleftrightarrow X \times Y \quad T' \longleftrightarrow X \times Y}{T \longleftrightarrow X \quad T \longleftrightarrow Y \quad T' \longleftrightarrow X \quad T' \longleftrightarrow Y}$$

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$\begin{array}{c} T \longleftrightarrow X \times Y \longleftrightarrow T' \\ \hline\hline \\ T \longleftrightarrow X \times Y & & T' \longleftrightarrow X \times Y \\ \hline\hline \\ T \longleftrightarrow X & T \longleftrightarrow Y & T' \longleftrightarrow X & T' \longleftrightarrow Y \\ \hline\hline \\ X \longleftrightarrow T \end{array}$$

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

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The diagram illustrates the proof that $X \times Y \cong X + Y$ in $\mathbb{R}\text{el}$. It shows a sequence of equivalences between the product type $X \times Y$ and the sum type $X + Y$, mediated by types T and T' .

- The top row shows $T \leftrightarrow X \times Y \leftrightarrow T'$.
- The second row shows $T \leftrightarrow X \times Y$ and $T' \leftrightarrow X \times Y$.
- The third row shows $T \leftrightarrow X$, $T \leftrightarrow Y$, $T' \leftrightarrow X$, and $T' \leftrightarrow Y$.
- The bottom row shows $X \leftrightarrow T$ and $Y \leftrightarrow T$.

Red text highlights the diagonal equivalence $T \leftrightarrow Y$ and the bottom equivalence $Y \leftrightarrow T$.

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$\begin{array}{c} T \longleftrightarrow X \times Y \longleftrightarrow T' \\ \hline \hline T \longleftrightarrow X \times Y & T' \longleftrightarrow X \times Y \\ \hline \hline T \longleftrightarrow X & T \longleftrightarrow Y & T' \longleftrightarrow X & T' \longleftrightarrow Y \\ \hline \hline X \longleftrightarrow T & Y \longleftrightarrow T & X \longleftrightarrow T' \end{array}$$

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$\begin{array}{c} T \longleftrightarrow X \times Y \longleftrightarrow T' \\ \hline\hline \\ T \longleftrightarrow X \times Y & & T' \longleftrightarrow X \times Y \\ \hline\hline \\ T \longleftrightarrow X & T \longleftrightarrow Y & T' \longleftrightarrow X & T' \longleftrightarrow Y \\ \hline\hline \\ X \longleftrightarrow T & Y \longleftrightarrow T & X \longleftrightarrow T' & Y \longleftrightarrow T' \end{array}$$

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$\begin{array}{c} T \longleftrightarrow X \times Y \longleftrightarrow T' \\ \hline\hline \\ \begin{array}{cc} T \longleftrightarrow X \times Y & T' \longleftrightarrow X \times Y \end{array} \\ \hline\hline \\ \begin{array}{cccc} T \longleftrightarrow X & T \longleftrightarrow Y & T' \longleftrightarrow X & T' \longleftrightarrow Y \end{array} \\ \hline\hline \\ \begin{array}{cccc} X \xleftrightarrow{\textcolor{red}{T}} T & Y \xleftrightarrow{\textcolor{red}{T}} T & X \xleftrightarrow{T'} T' & Y \xleftrightarrow{T'} T' \end{array} \\ \hline\hline \\ X + Y \longleftrightarrow T \end{array}$$

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$\begin{array}{c} T \longleftrightarrow X \times Y \longleftrightarrow T' \\ \hline\hline \\ \begin{array}{cc} T \longleftrightarrow X \times Y & T' \longleftrightarrow X \times Y \end{array} \\ \hline\hline \\ \begin{array}{cccc} T \longleftrightarrow X & T \longleftrightarrow Y & T' \longleftrightarrow X & T' \longleftrightarrow Y \end{array} \\ \hline\hline \\ \begin{array}{ccccc} X \longleftrightarrow T & Y \longleftrightarrow T & X \longleftrightarrow T' & Y \longleftrightarrow T' \\ \hline\hline \\ X + Y \longleftrightarrow T & & X + Y \longleftrightarrow T' \end{array} \end{array}$$

Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

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Products in $\mathbb{R}\text{el}$

In $\mathbb{R}\text{el}$, $X \times Y \cong X + Y!$

$$T \longleftrightarrow X \times Y \longleftrightarrow T'$$

$$T \longleftrightarrow X \times Y$$

$$T' \longleftrightarrow X \times Y$$

$$T \longleftrightarrow X$$

$$T \longleftrightarrow Y$$

$$T' \longleftrightarrow X$$

$$T' \longleftrightarrow Y$$

$$X \longleftrightarrow T$$

$$Y \longleftrightarrow T$$

$$X \longleftrightarrow T'$$

$$Y \longleftrightarrow T'$$

$$X + Y \longleftrightarrow T$$

$$X + Y \longleftrightarrow T'$$

$$T \longleftrightarrow X + Y \longleftrightarrow T'$$

The category \mathbb{X}^{op}

Objects Same as \mathbb{X}

Arrows $f^{\text{op}} : X \rightarrow Y \in \mathbb{X}^{\text{op}} \longleftrightarrow f : Y \rightarrow X \in \mathbb{X}$

Identities $\text{id}^{\text{op}} := \text{id} : X \rightarrow X$

Composition $f^{\text{op}} \cdot g^{\text{op}} := (g \cdot f)^{\text{op}}$

Unit From \mathbb{X}

Associativity From \mathbb{X}

Questions?

Context

Categories are context

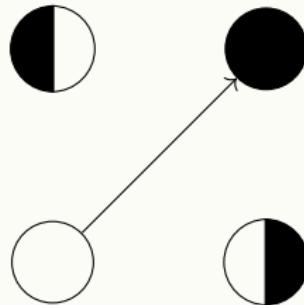
$\mathbb{S}et$ and $\mathbb{R}el$ have the *same* objects X, Y, \dots , but

| | Product $X \times Y$ | Coproduct $X + Y$ |
|-------------------|--|---------------------------------------|
| In $\mathbb{S}et$ | Set of pairs $\{\langle x, y \rangle\}$ | Disj. union $\{x_1\} \cup \{y_2\}$ |
| In $\mathbb{R}el$ | Disj. union $\{x_1\} \cup \{y_2\}$ | Disj. union $\{x_1\} \cup \{y_2\}$ |

Graph Products

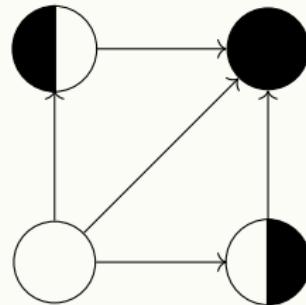
Kronecker

$G \times H$



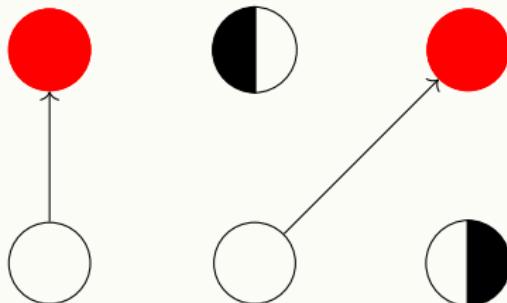
Strong

$G \boxtimes H$

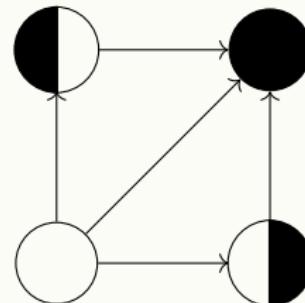


Graph Products

Kronecker

$$G \times H$$


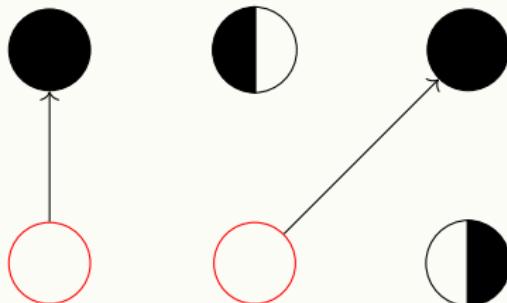
Strong

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Graph Products

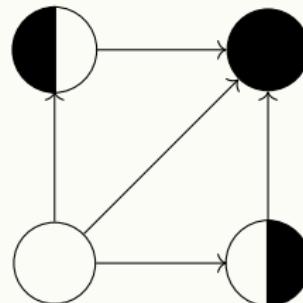
Kronecker

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Strong

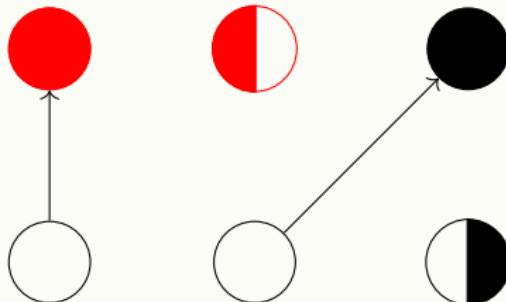
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Graph Products

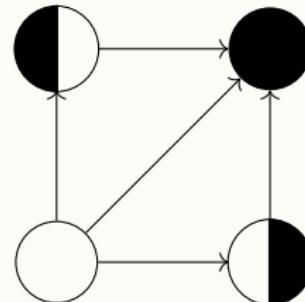
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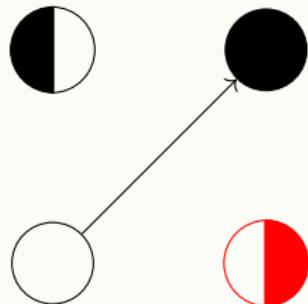
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Graph Products

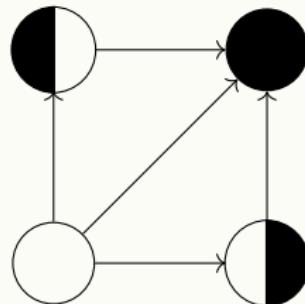
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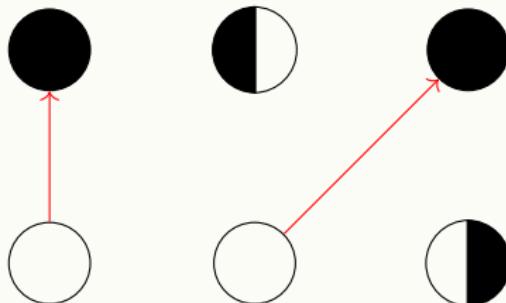
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Graph Products

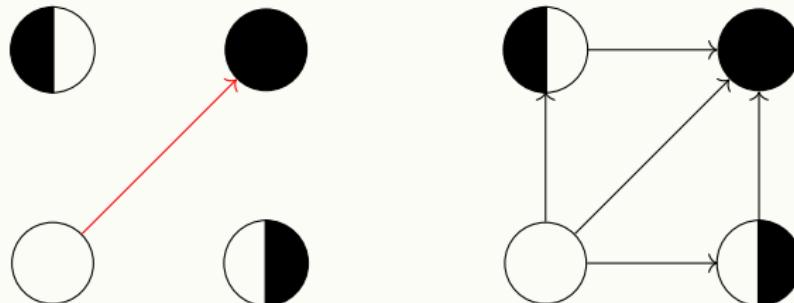
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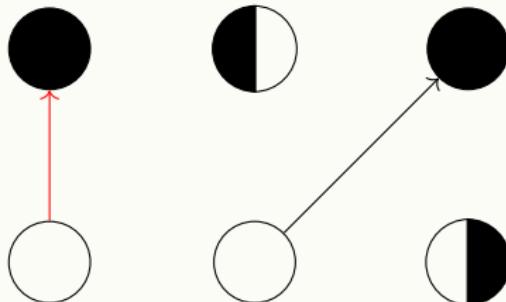
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Graph Products

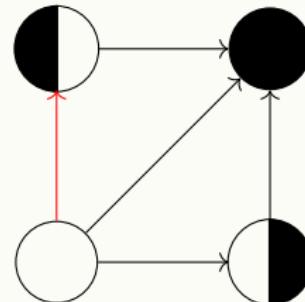
Kronecker

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???



Graph Products

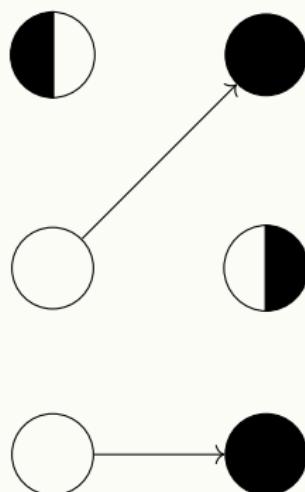
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Graph Products

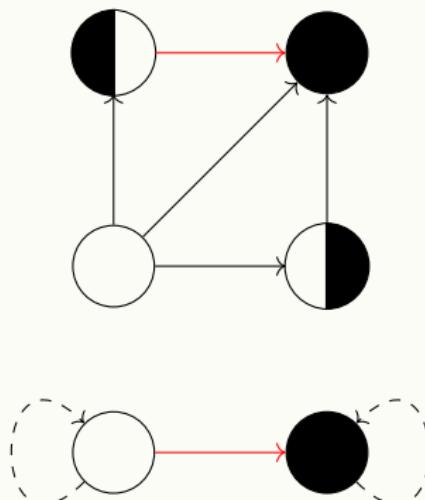
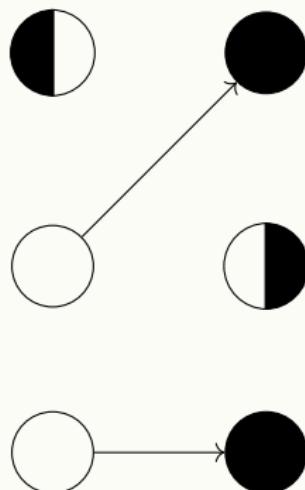
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$$G \times H$$



Strong

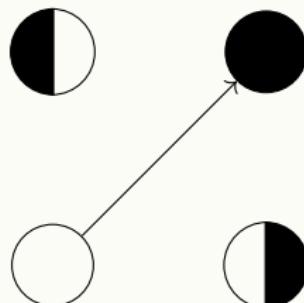
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Graph Products

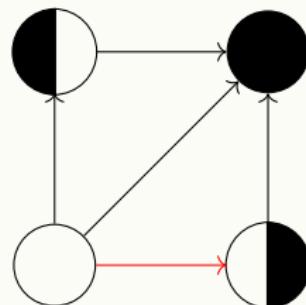
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Graph Products

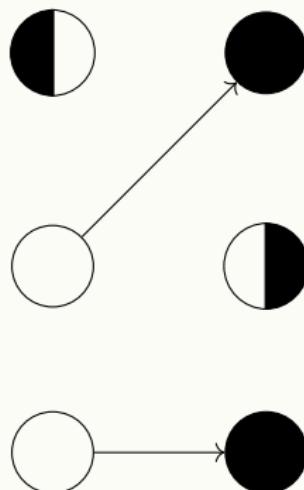
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Graph Products

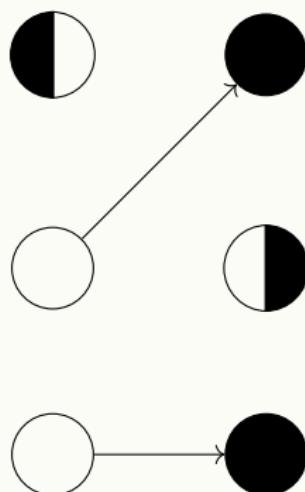
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Strong

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Graph Products

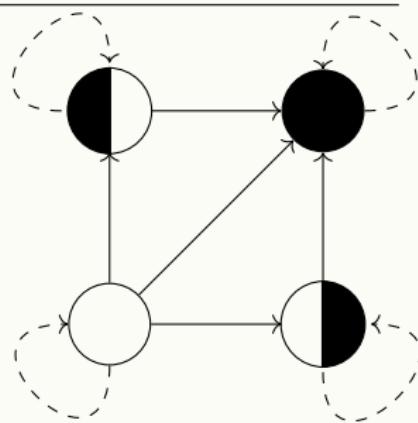
Kronecker

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Graph Products

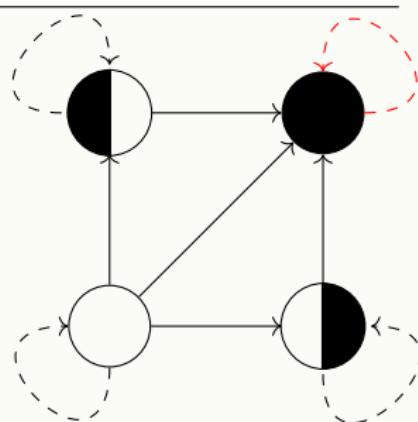
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Strong

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Graph Products

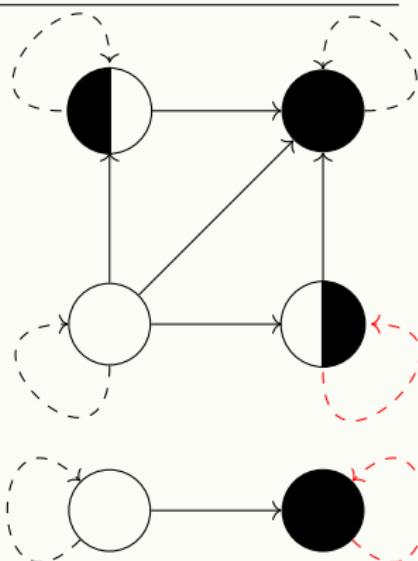
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Graph Products

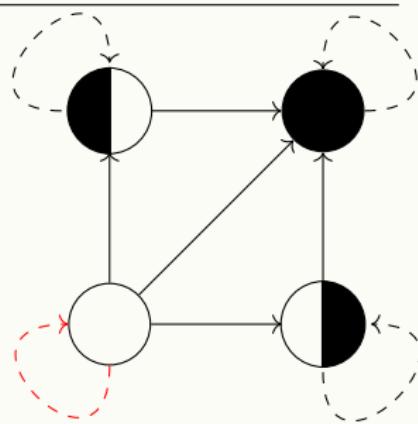
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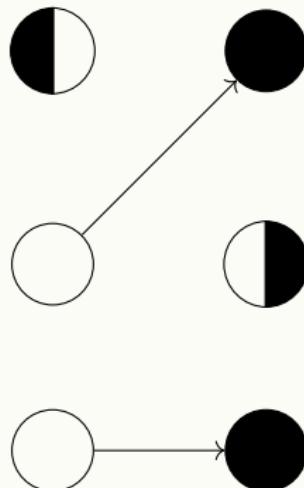
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Graph Products

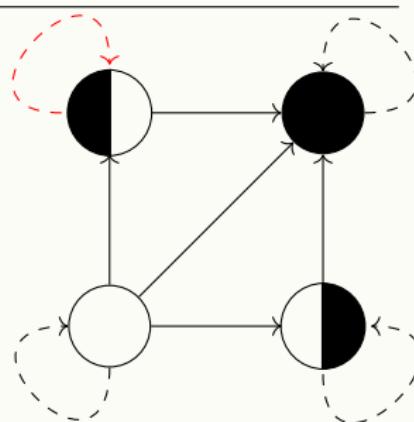
Kronecker

$G \times H$

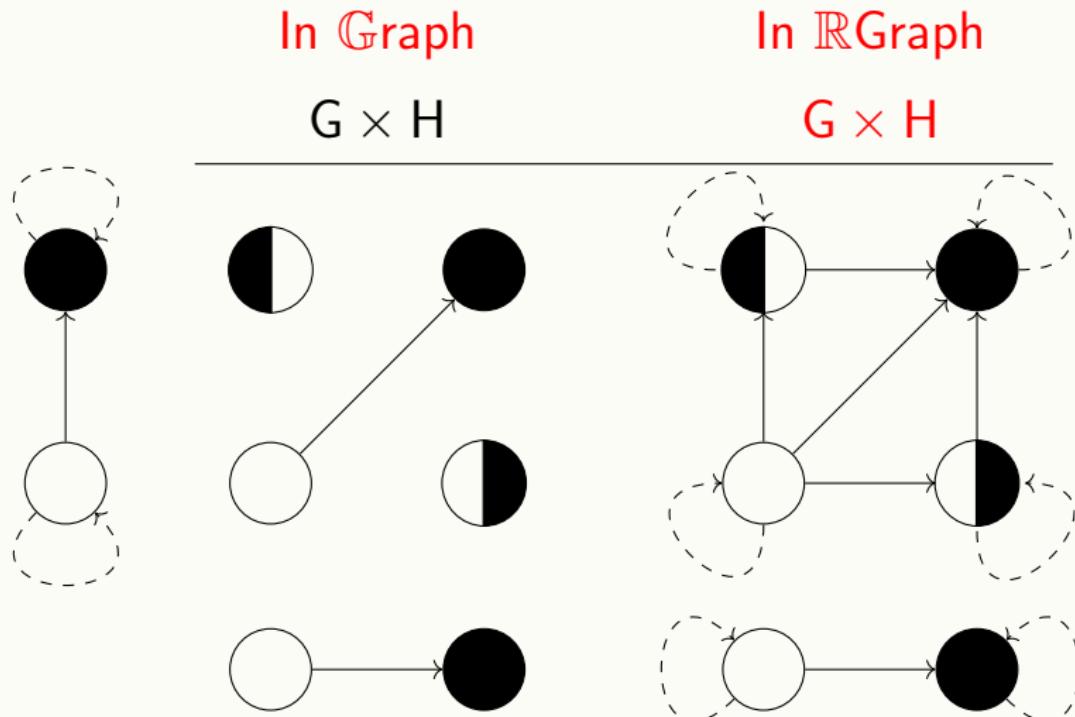


Strong

$G \boxtimes H$

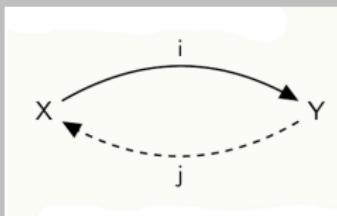


Categories are context

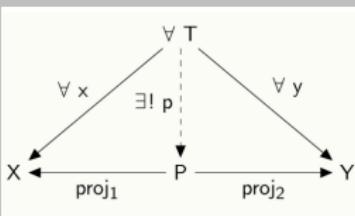


Questions?

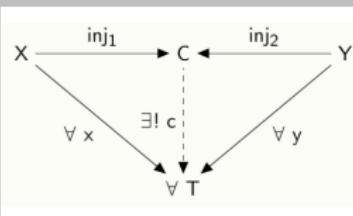
Isomorphism



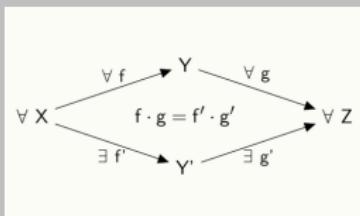
Products



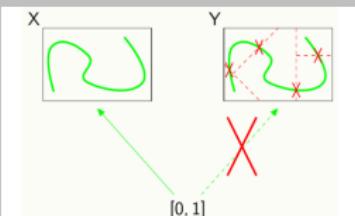
Duality



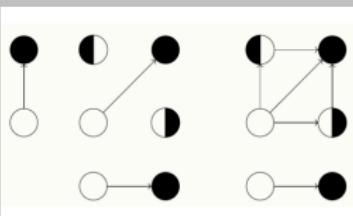
Yoneda



Test objects



Context



Models

Functorial Semantics

A model is a mapping



Syntax: What we *say*

What are the (types of) elements in the system?

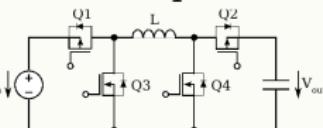
How do they fit together?

- Easy to write down.
- Hard to compute.

Examples:

- Logical formulas: $\forall x. \exists y. R(x, y)$
- Computer code: `(x>0) ? sqrt(x) : sqrt(-x)`

- Circuit diagrams:



Semantics: What we *mean*

How are elements represented?

How do we model their interaction.

- Hard to write down.
- “Easy” to compute.

Examples:

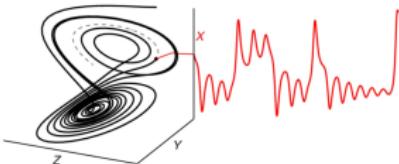
- Relations



- Matrices

$$\begin{aligned}\mathbf{K}_k \mathbf{S}_k &= (\mathbf{H}_k \mathbf{P}_{k|k-1})^\top = \mathbf{P}_{k|k-1} \mathbf{H}_k^\top \\ \Rightarrow \mathbf{K}_k &= \mathbf{P}_{k|k-1} \mathbf{H}_k^\top \mathbf{S}_k^{-1}\end{aligned}$$

- Dynamics



Functorial Semantics

A model is a mapping



Functors

Set of
Elements



Category
of Sets



Category of
Categories



Functors

Definition: A *functor* $\mathbb{F} : \mathbb{X} \rightarrow \mathbb{Y}$ is

- ▶ a function from objects of \mathbb{X} to objects of \mathbb{Y}
- ▶ a function from arrows of \mathbb{X} to arrows of \mathbb{Y}
- ▶ preserving source and target, identities and composition:

$$\mathbb{F}\underbrace{(g : X \rightarrow X')}_{\text{in } \mathbb{X}} = \mathbb{F}(g) : \underbrace{\mathbb{F}(X) \rightarrow \mathbb{F}(X')}_{\text{in } \mathbb{Y}}$$

$$\mathbb{F}(\text{id}_{\mathbb{X}}) = \text{id}_{\mathbb{F}(\mathbb{X})}$$

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Picturing a Functor

\mathbb{X} $\xrightarrow{\mathbb{F}}$ \mathbb{Y}



Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$



$$X \mapsto \mathbb{F}(X)$$

Object Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

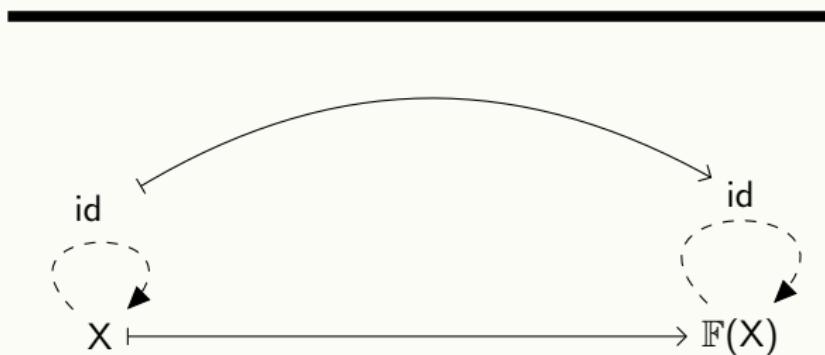


$$\begin{array}{ccc} \text{id} & & \\ \circlearrowleft & & \\ X & \xrightarrow{\quad} & \mathbb{F}(X) \end{array}$$

Object Function

Picturing a Functor

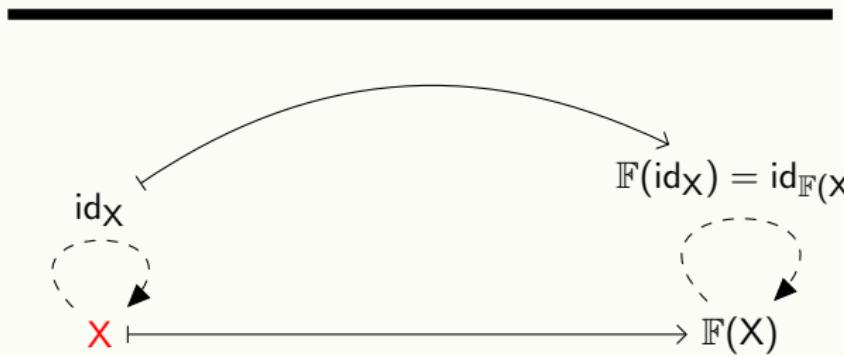
$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$



Object Function

Picturing a Functor

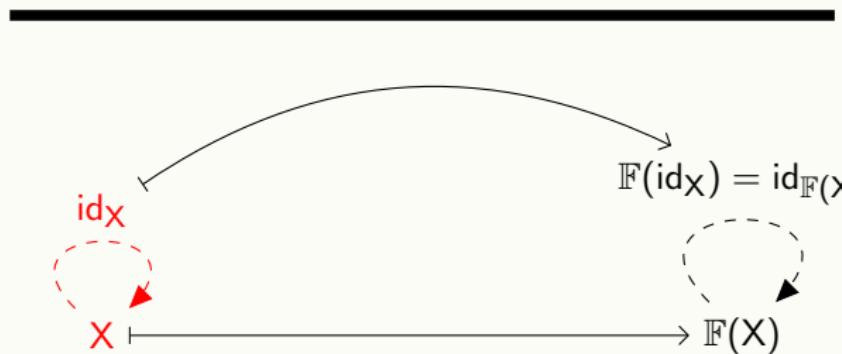
$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$



Identity Eq. $\mathbb{F}(id_X) = id_{\mathbb{F}(X)}$

Picturing a Functor

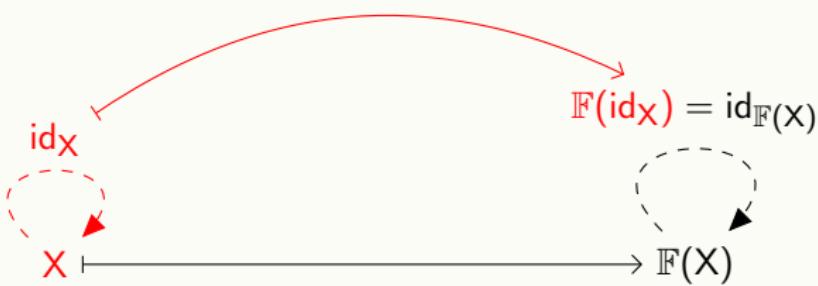
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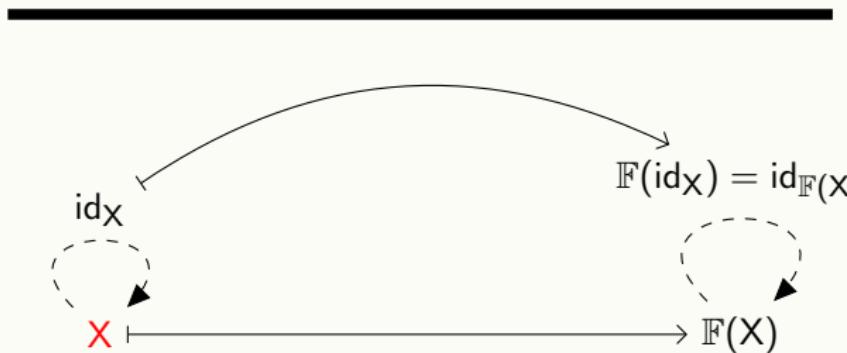
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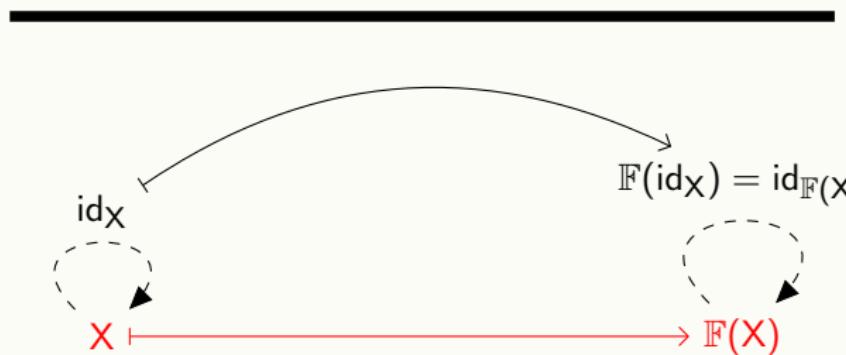
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Identity Eq. $\mathbb{F}(\text{id}_{\mathbb{X}}) = \text{id}_{\mathbb{F}(\textcolor{red}{X})}$

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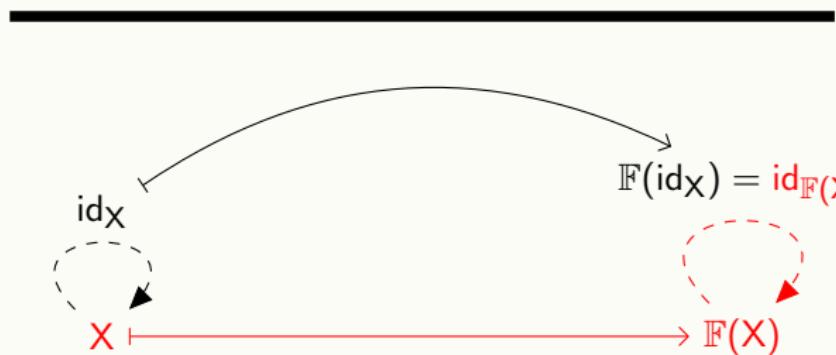
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Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$



X'

h



X

$\mathbb{F}(X)$

Arrow Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$



$$\begin{array}{ccc} \mathbb{X}' & \xrightarrow{\hspace{2cm}} & \mathbb{F}(\mathbb{X}') \\ | & & \\ h & \downarrow & \\ \mathbb{X} & \xrightarrow{\hspace{2cm}} & \mathbb{F}(\mathbb{X}) \end{array}$$

Arrow Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

$$\begin{array}{ccc} X' & \xrightarrow{\hspace{3cm}} & \mathbb{F}(X') \\ | & & | \\ h & \xrightarrow{\hspace{3cm}} & \mathbb{F}(h) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\hspace{3cm}} & \mathbb{F}(X) \end{array}$$

Arrow Function

Picturing a Functor

$$\mathbb{X} \xrightarrow{\mathbb{F}} \mathbb{Y}$$

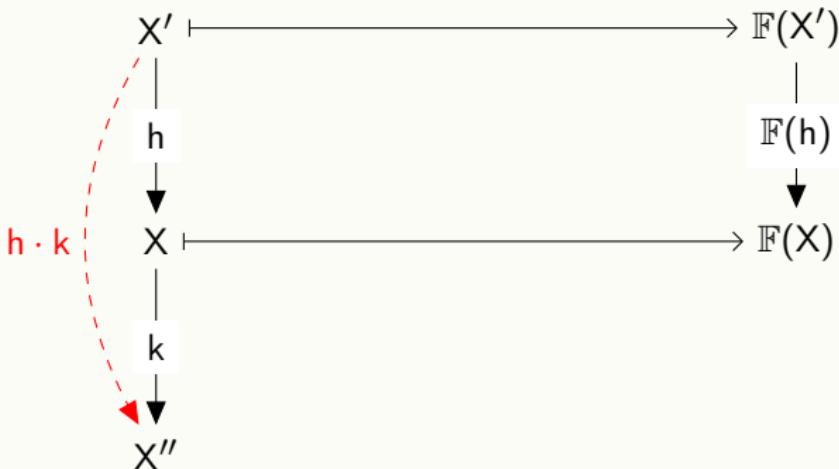


$$\begin{array}{ccc} X' & \xrightarrow{\hspace{3cm}} & \mathbb{F}(X') \\ | & & | \\ h & \downarrow & \mathbb{F}(h) \\ X & \xrightarrow{\hspace{3cm}} & \mathbb{F}(X) \\ | & & | \\ k & \downarrow & \\ X'' & & \end{array}$$

Composition Eq. $\mathbb{F}(h \cdot k) = \mathbb{F}(h) \cdot \mathbb{F}(k)$

Picturing a Functor

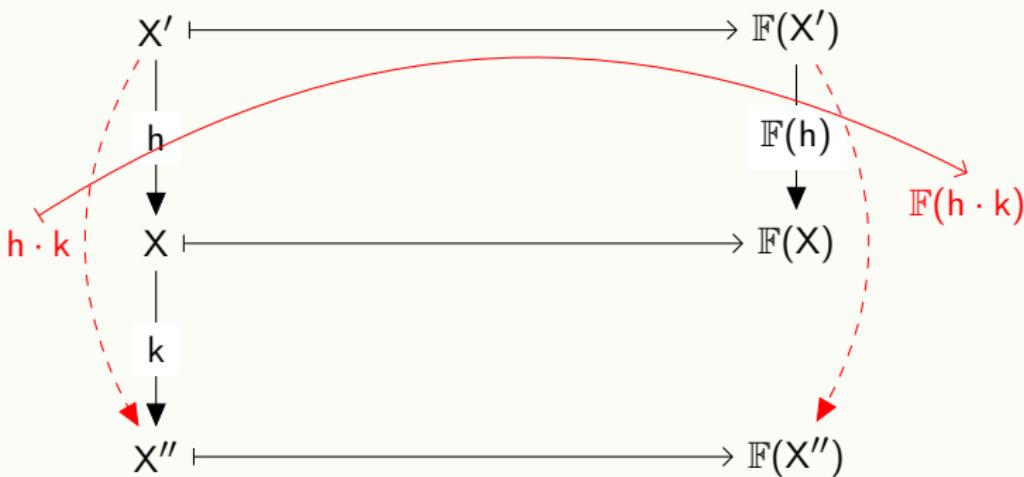
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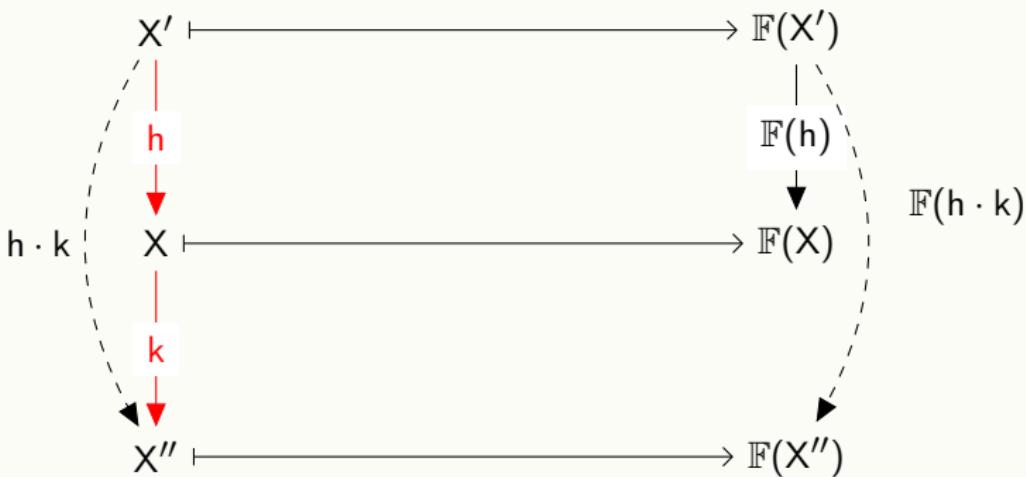
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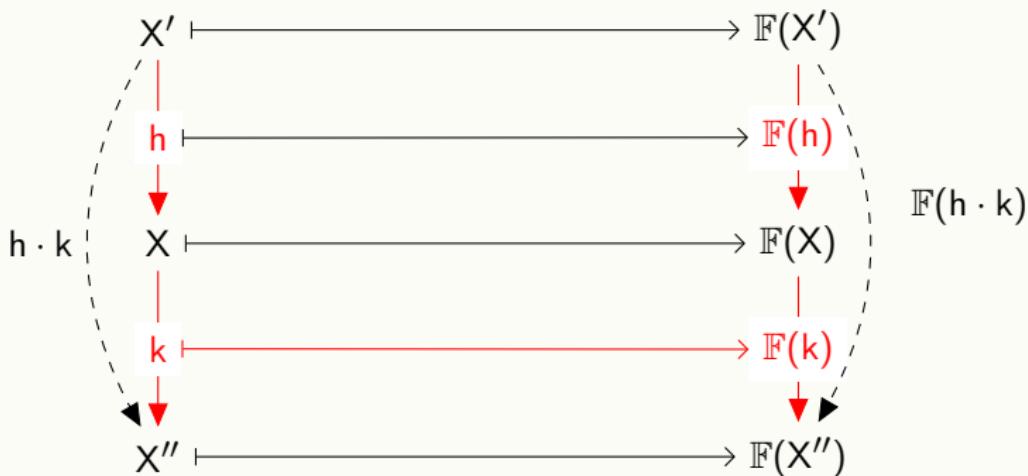
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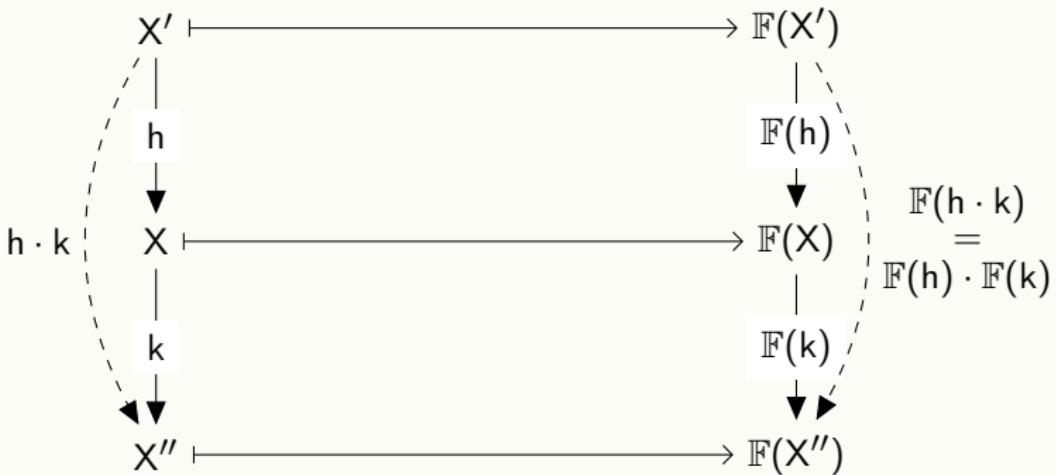
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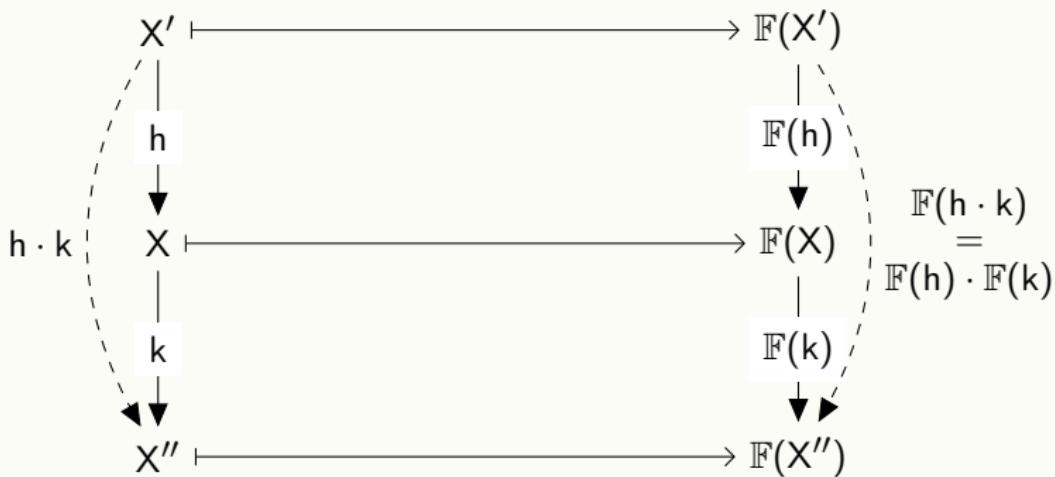
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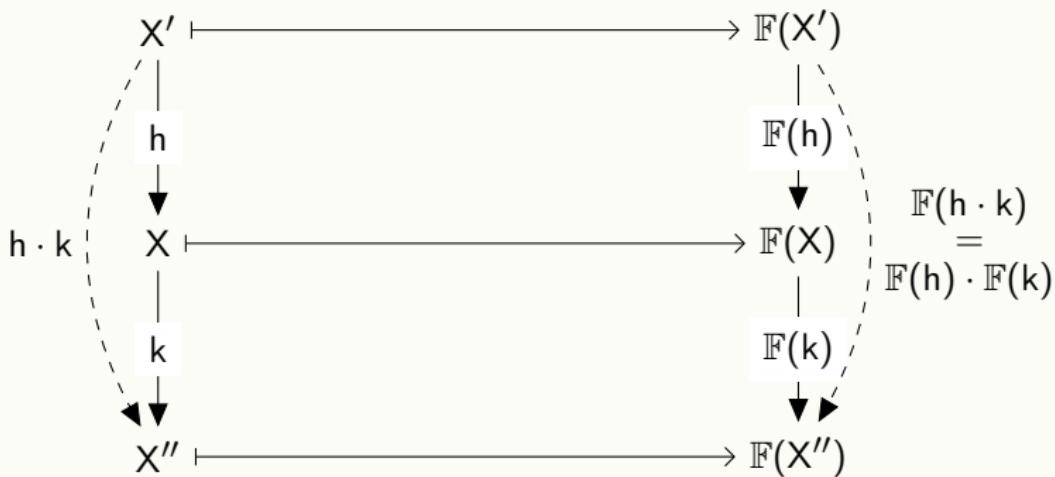
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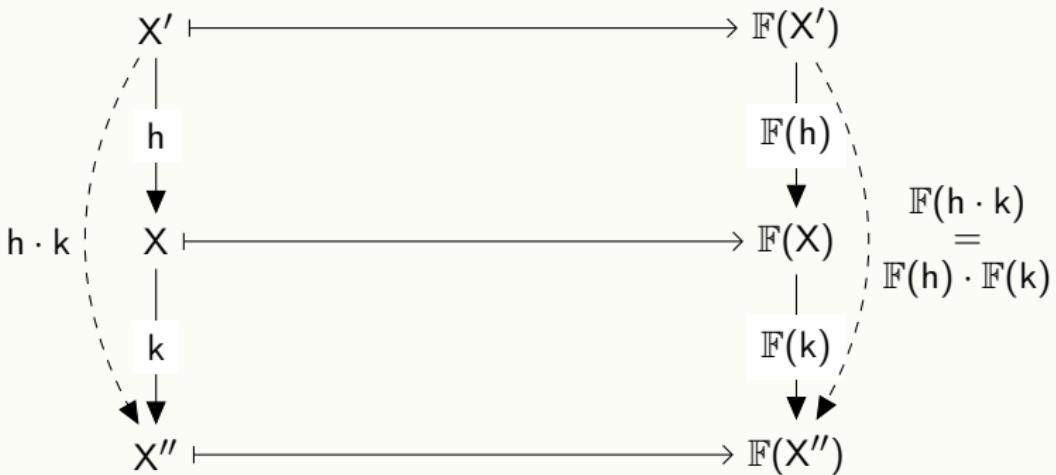
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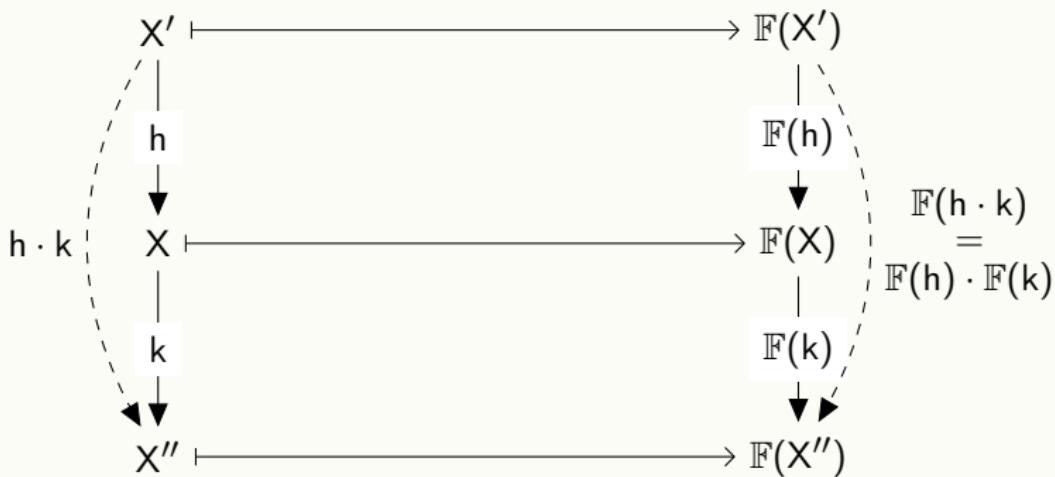
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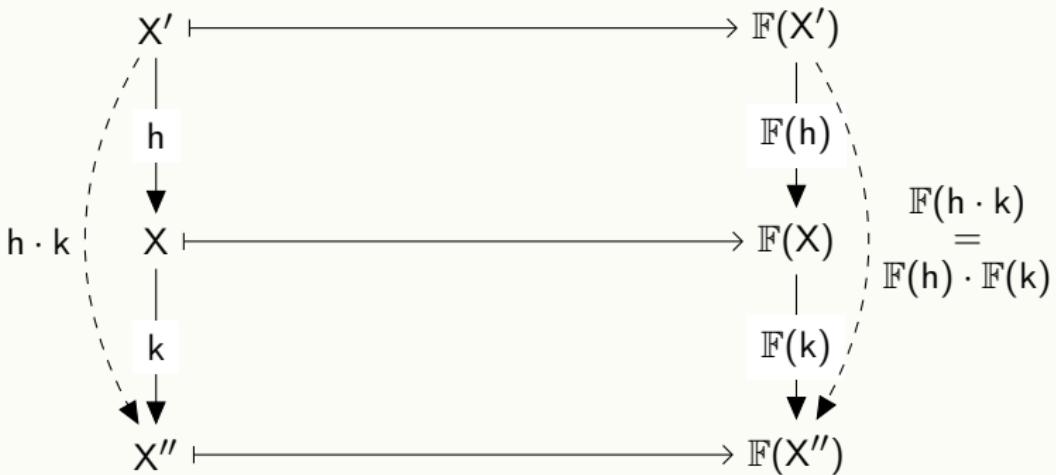
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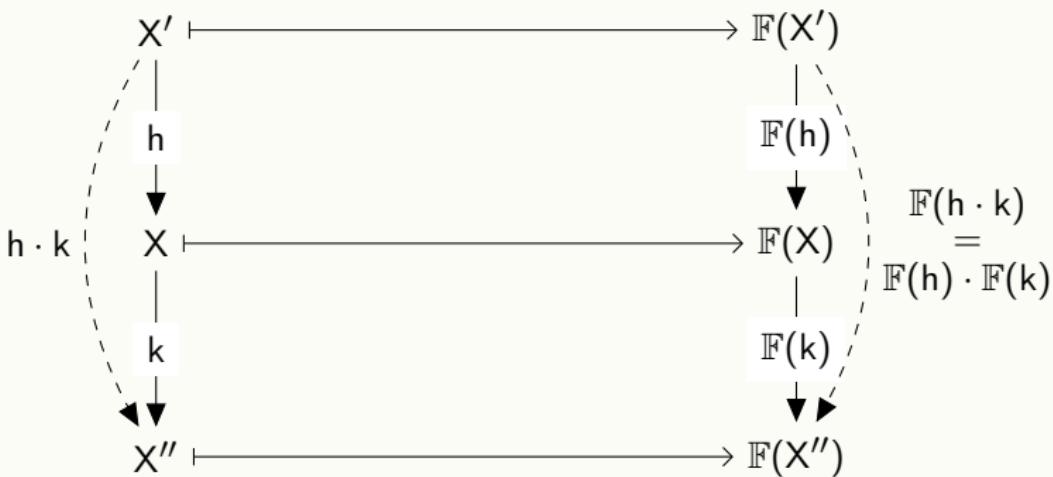
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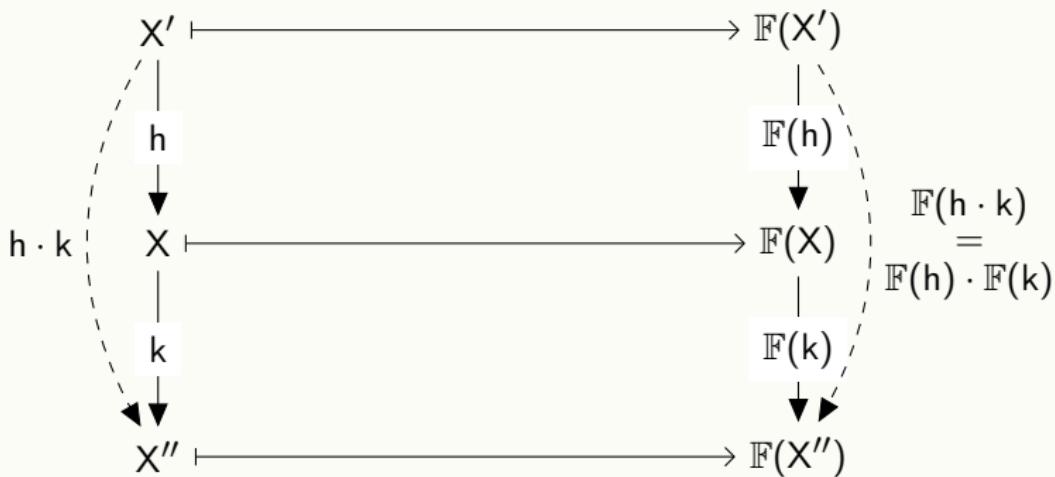
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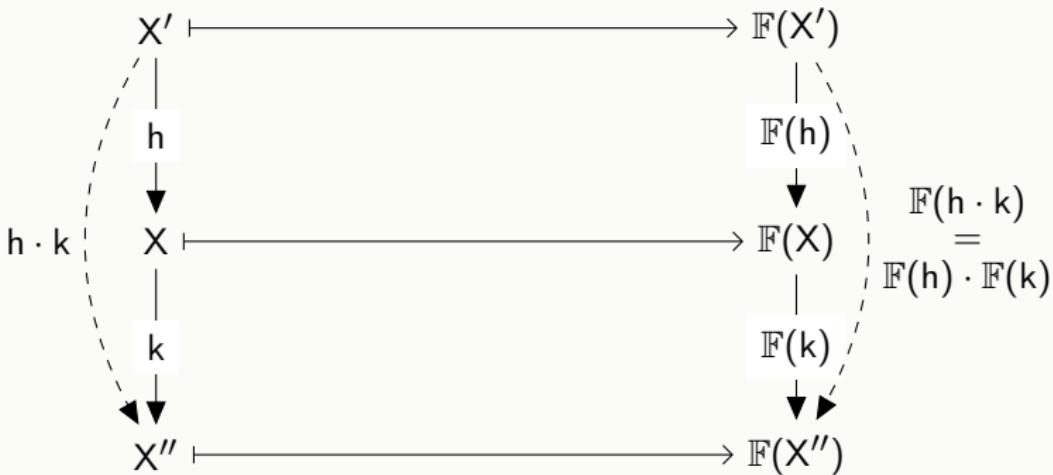
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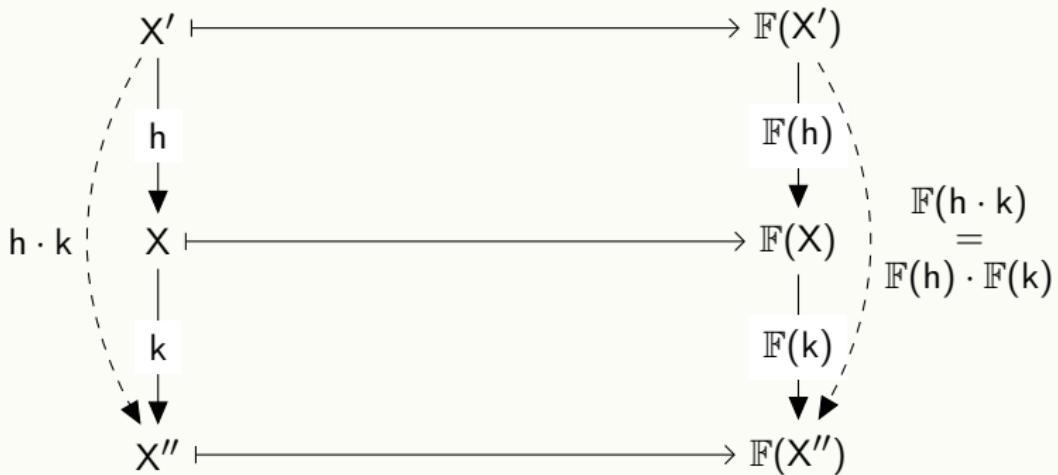
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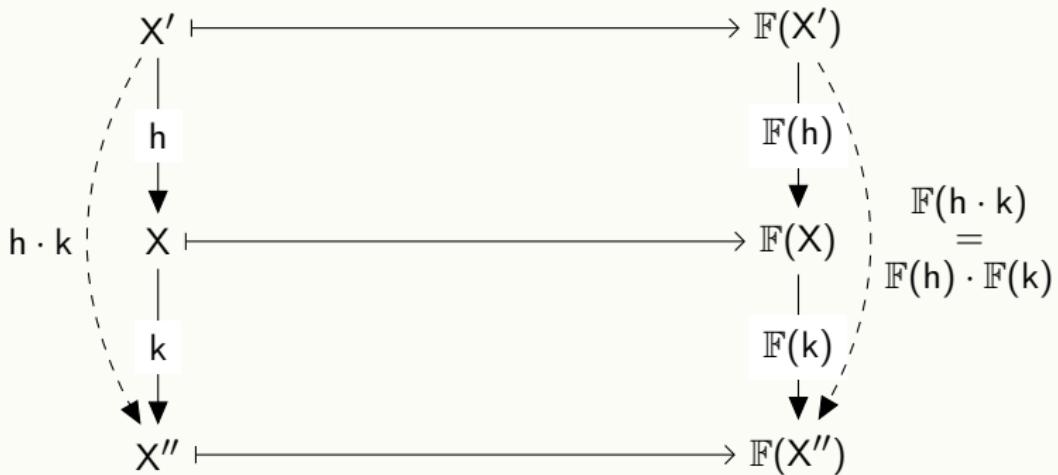
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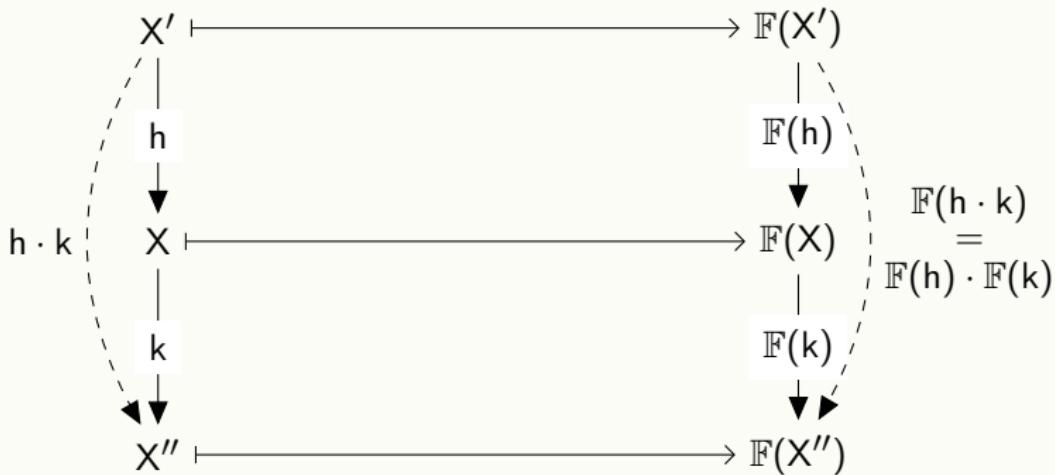
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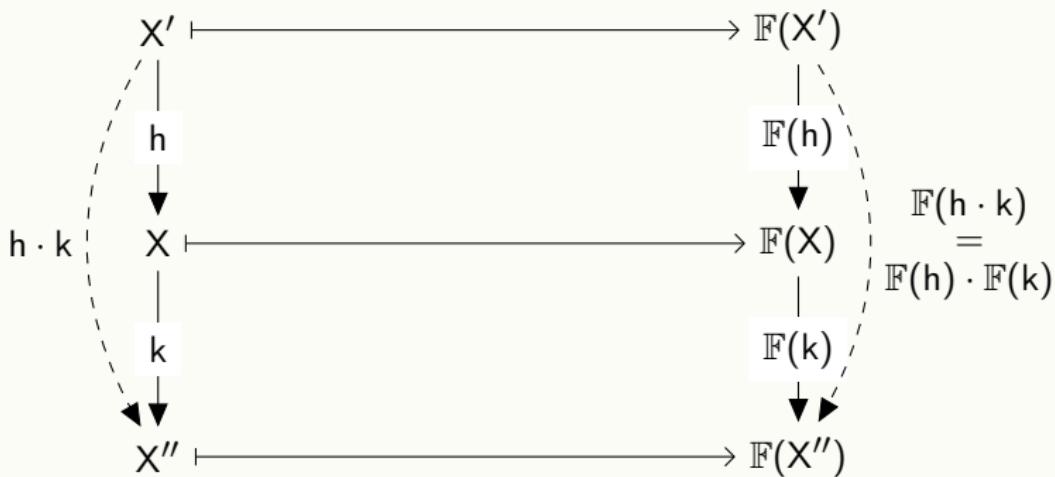
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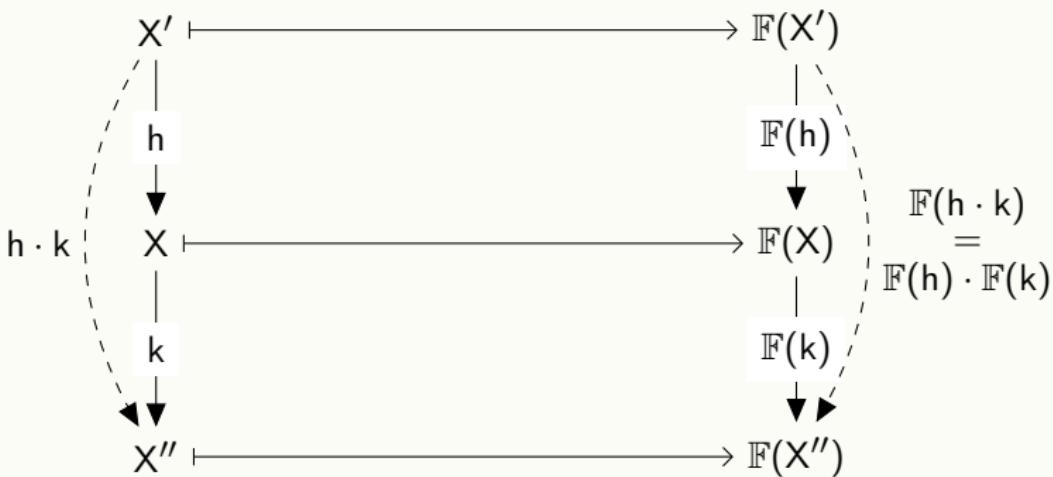
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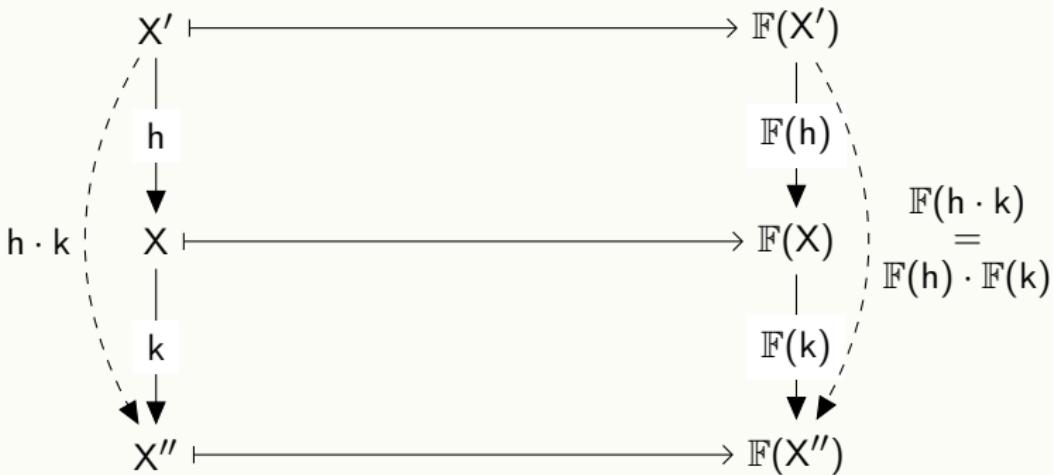
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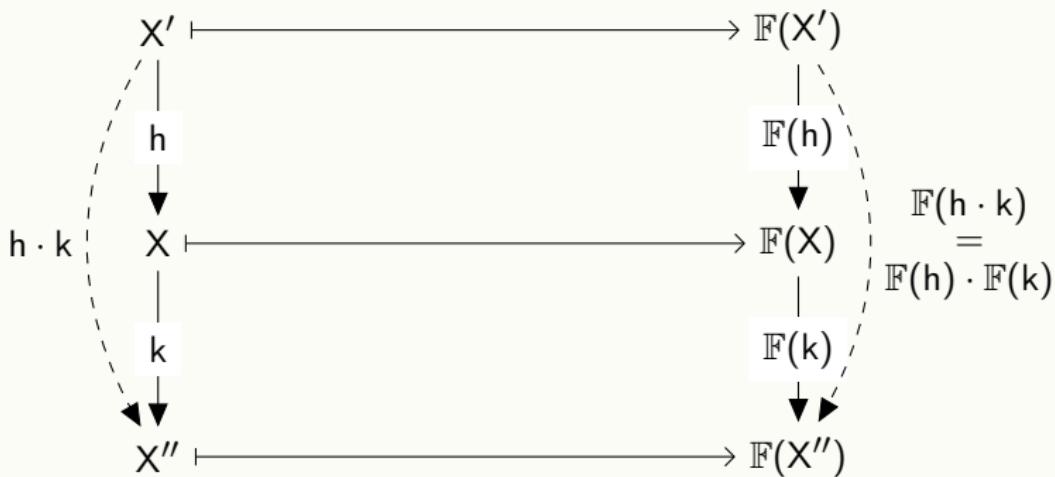
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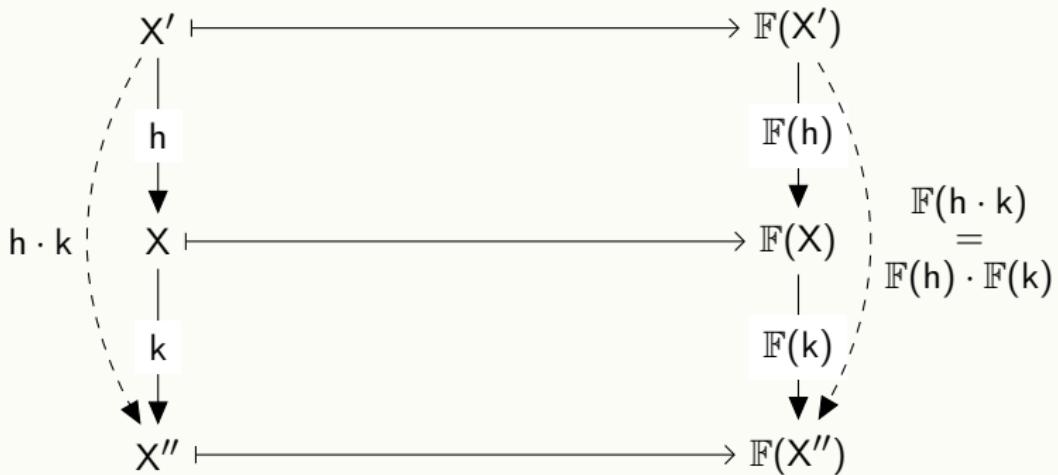
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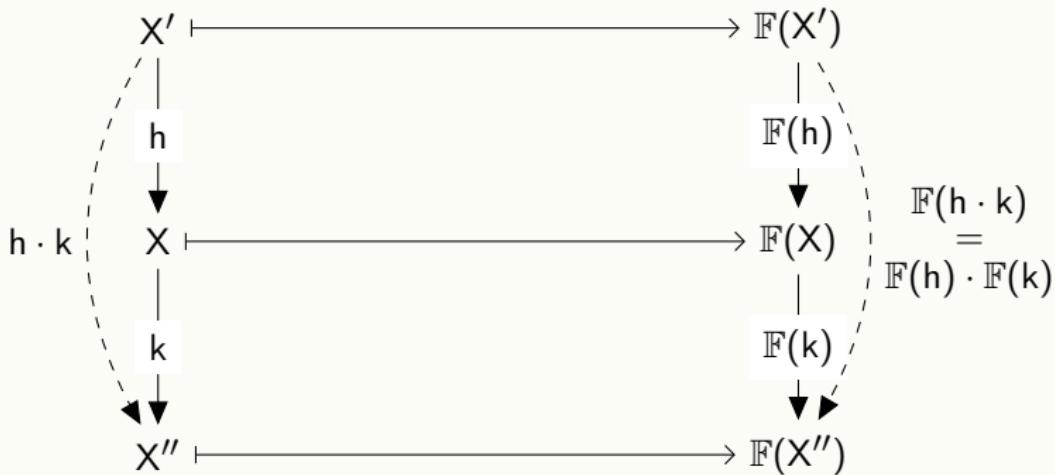
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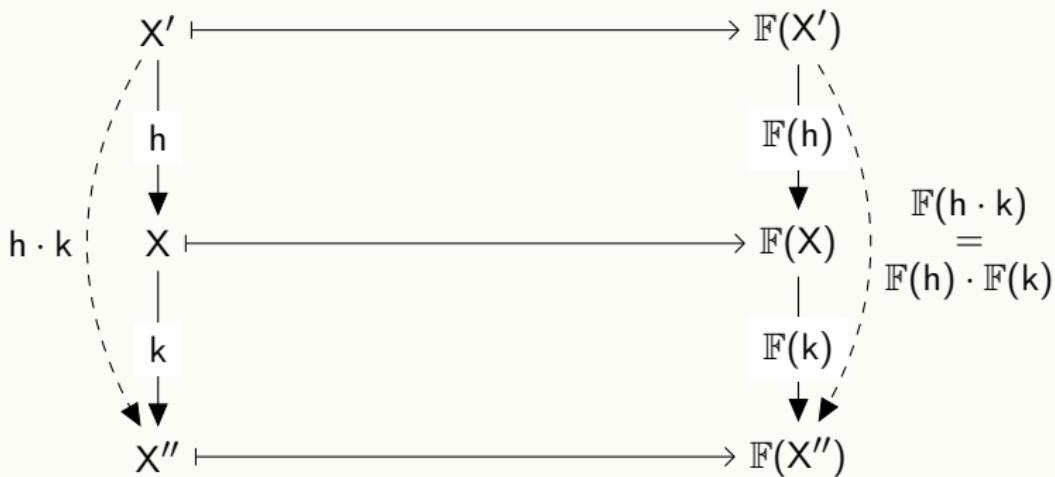
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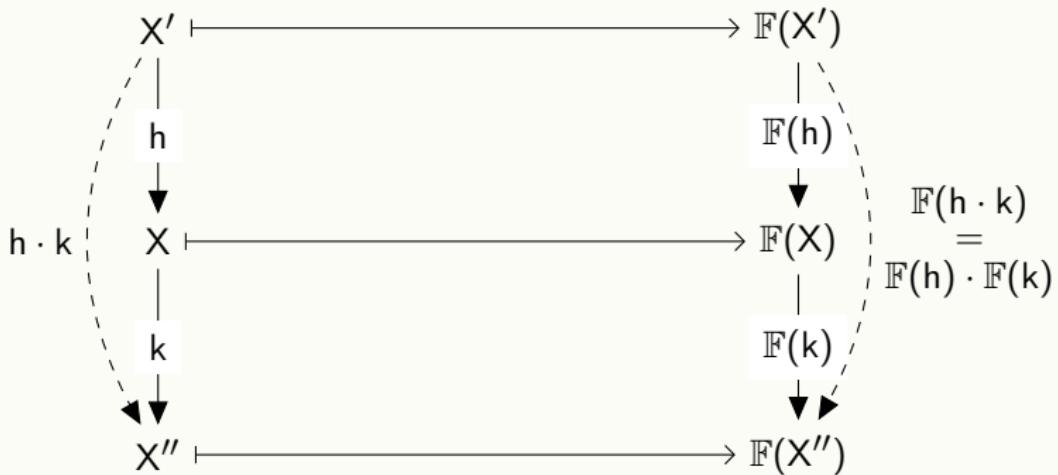
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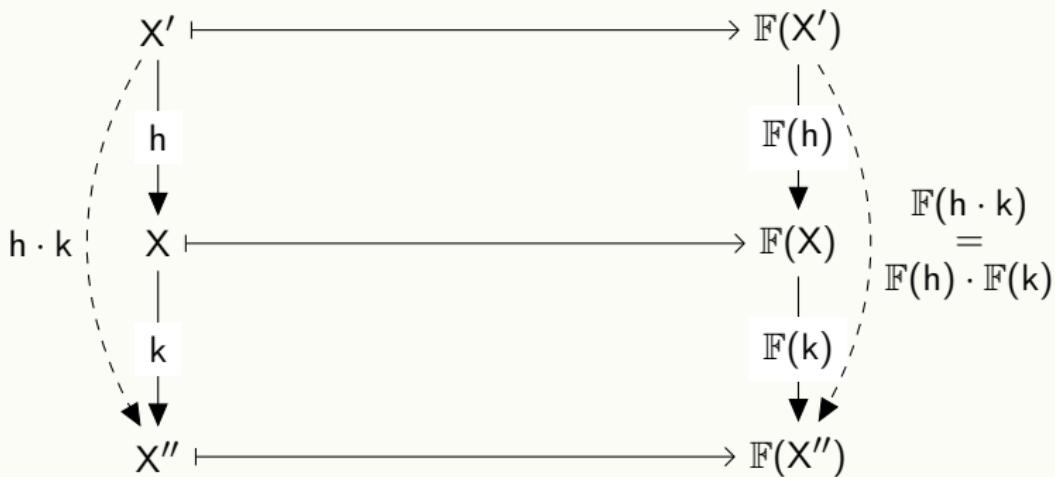
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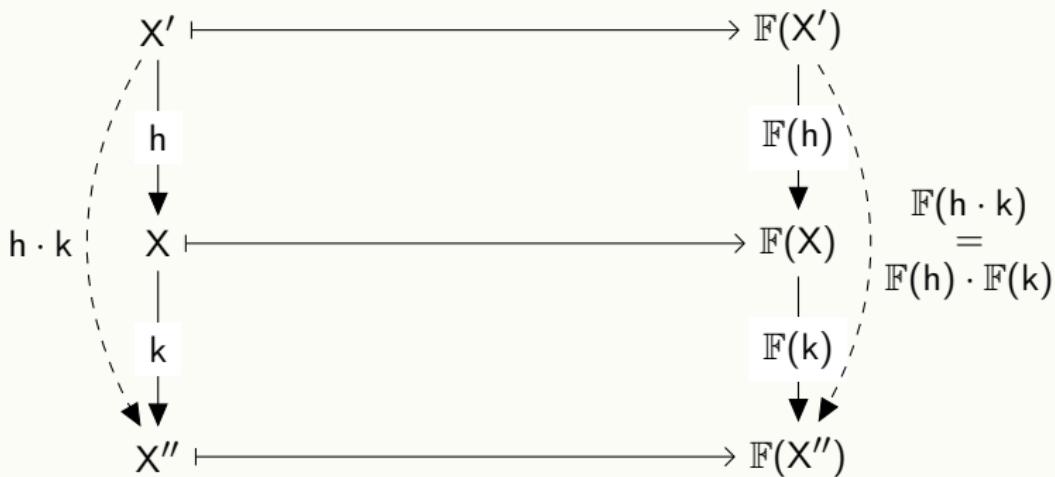
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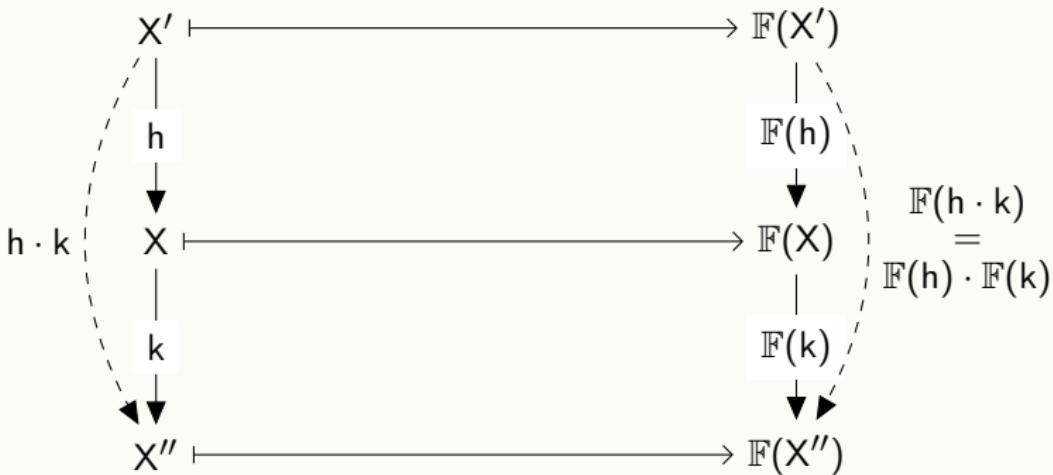
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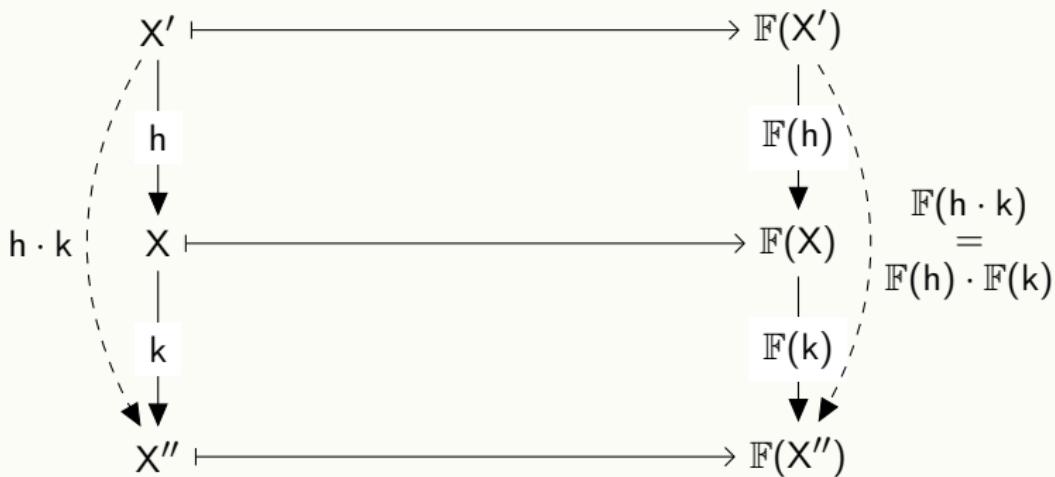
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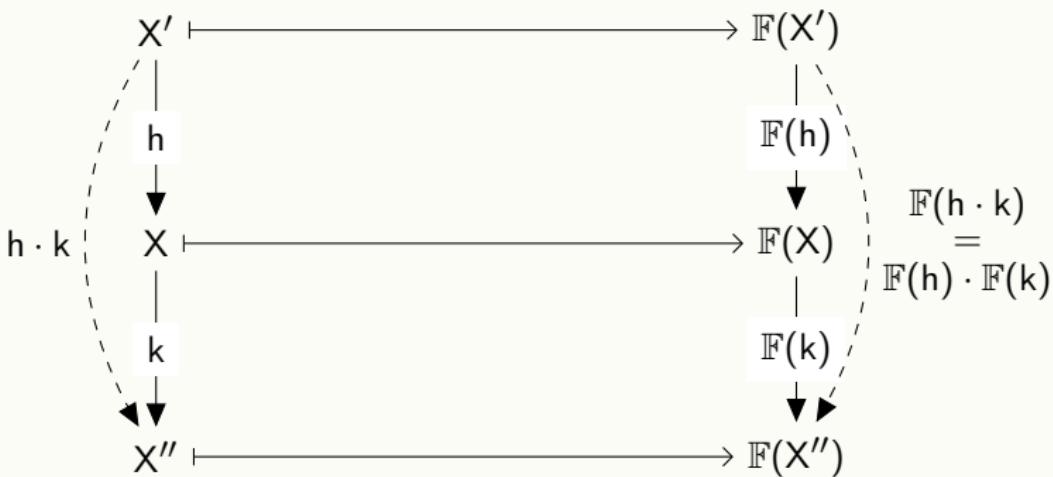
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Picturing a Functor

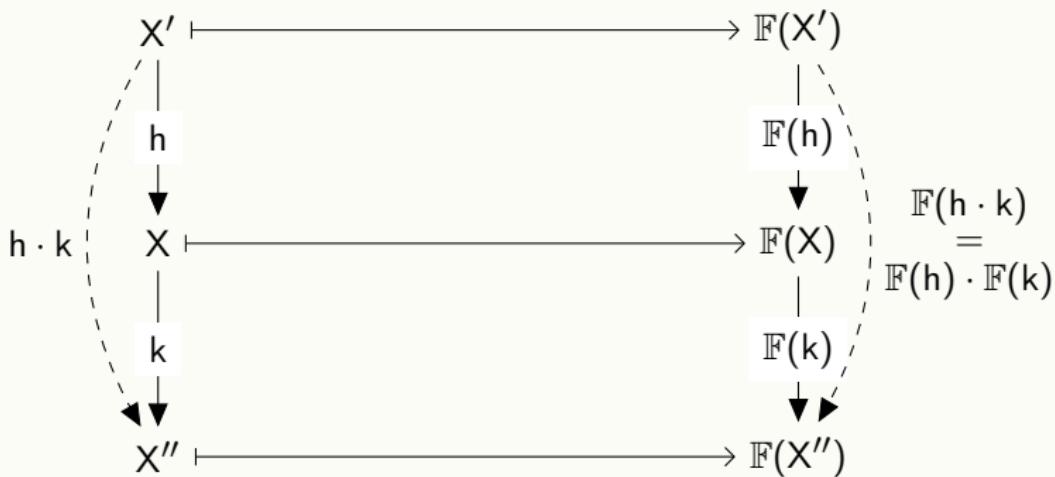
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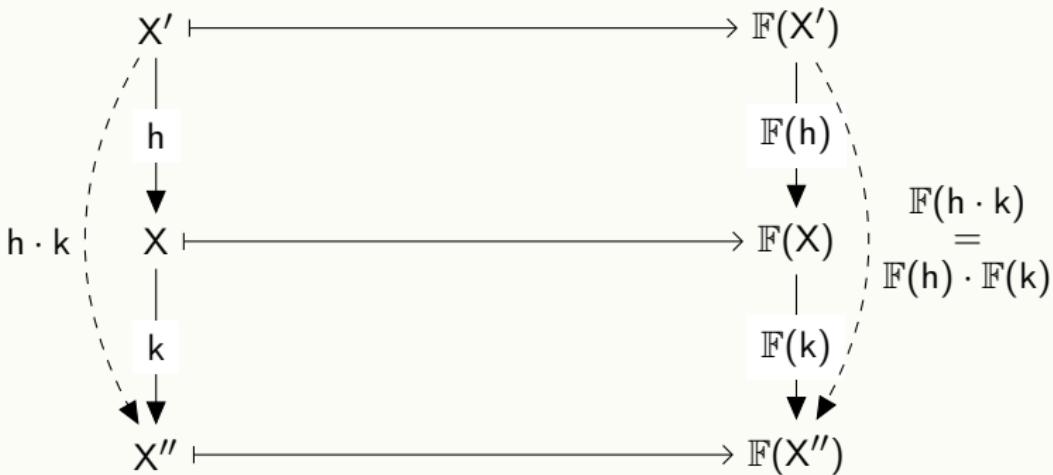
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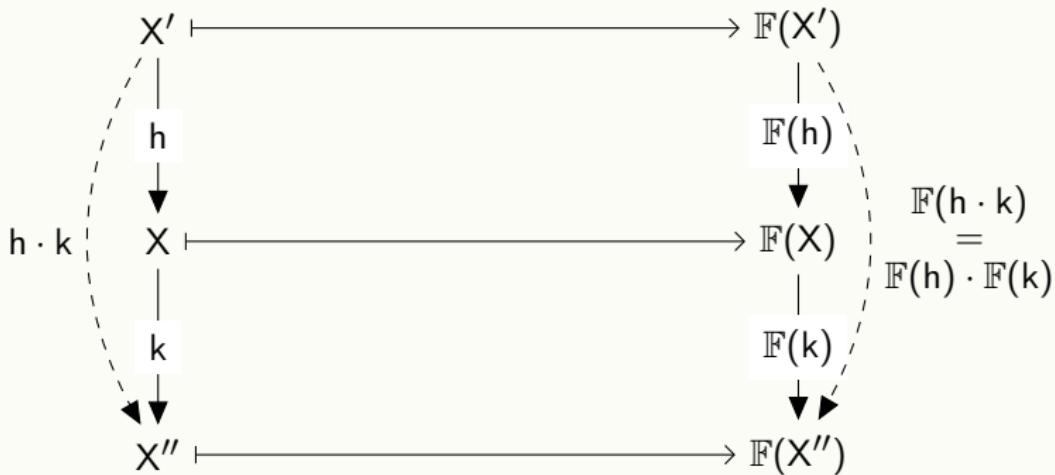
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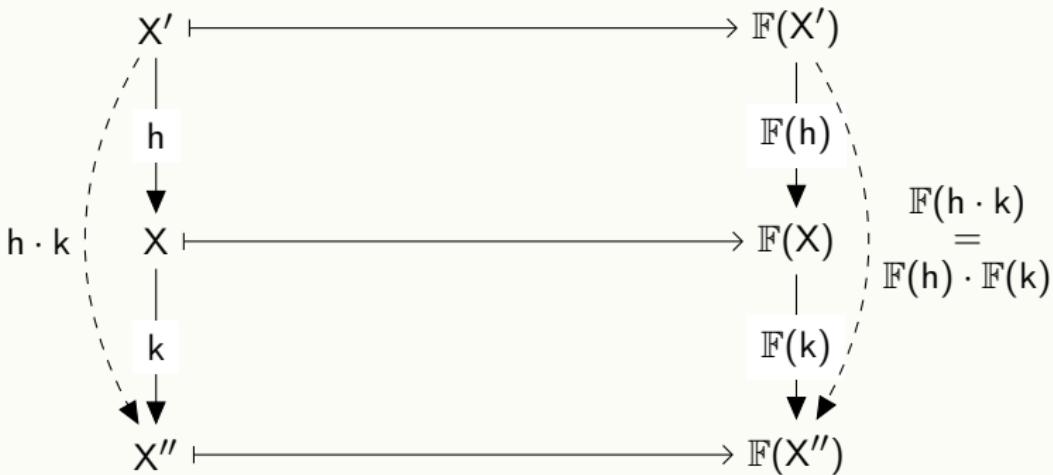
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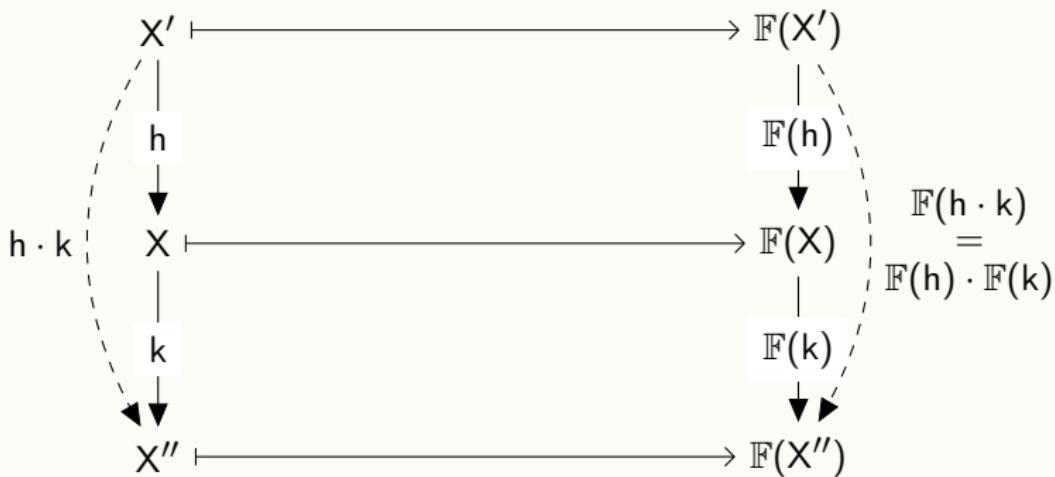
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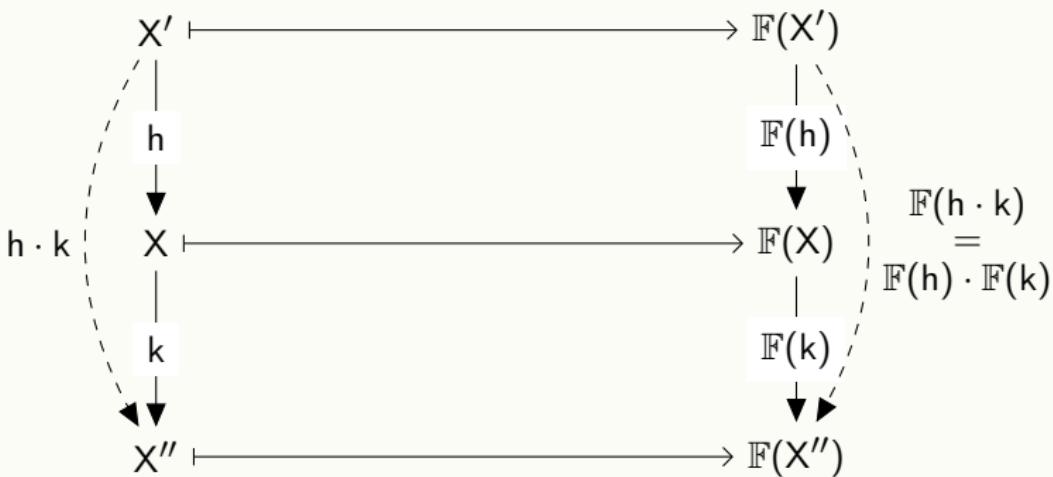
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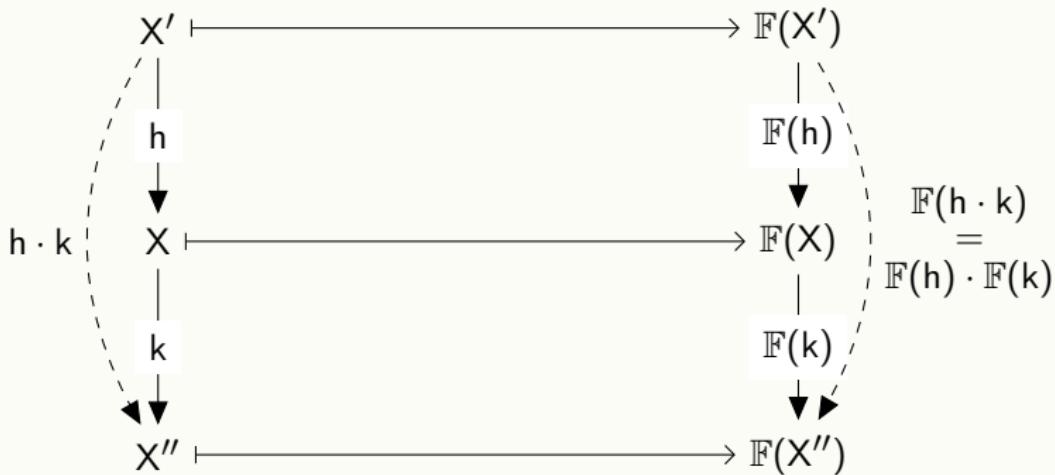
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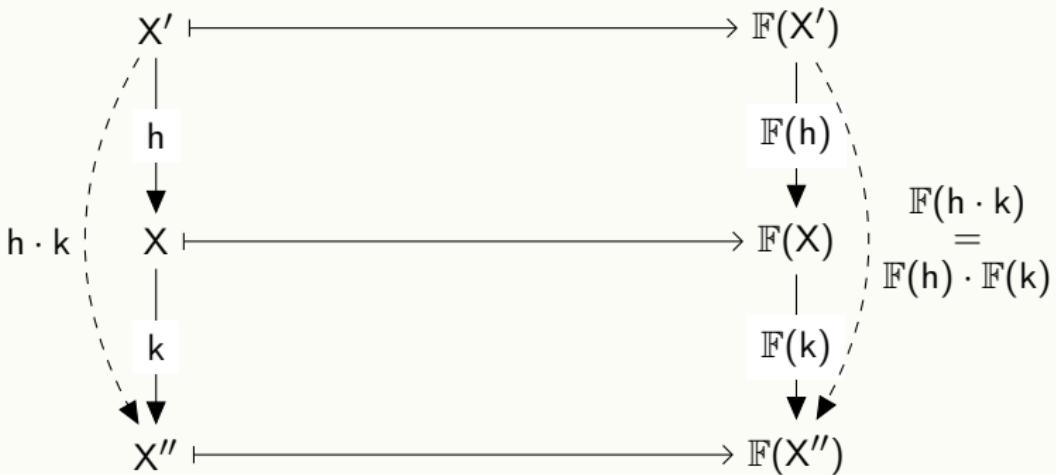
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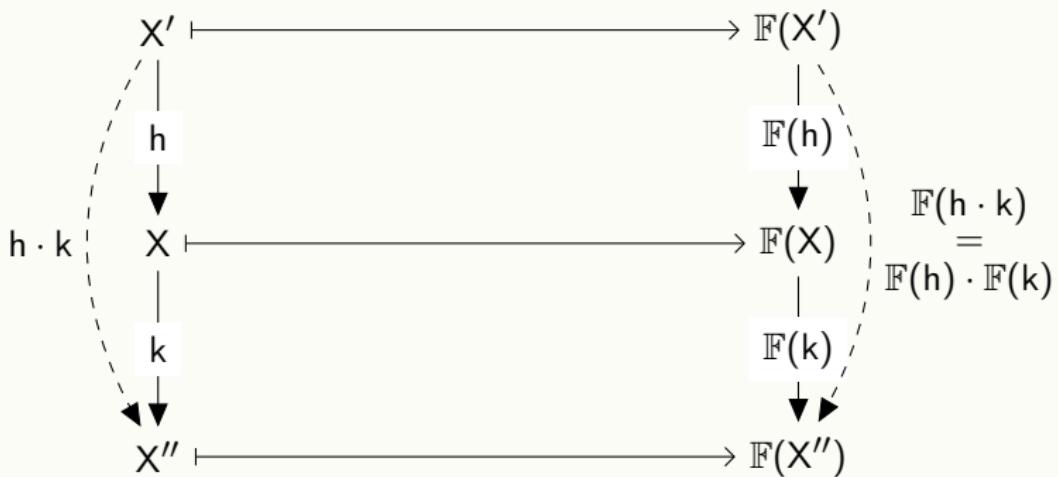
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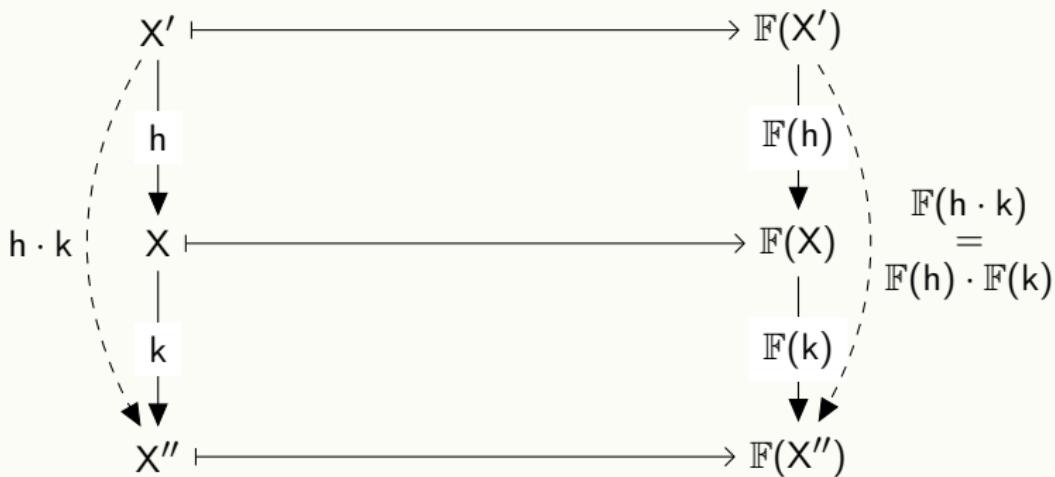
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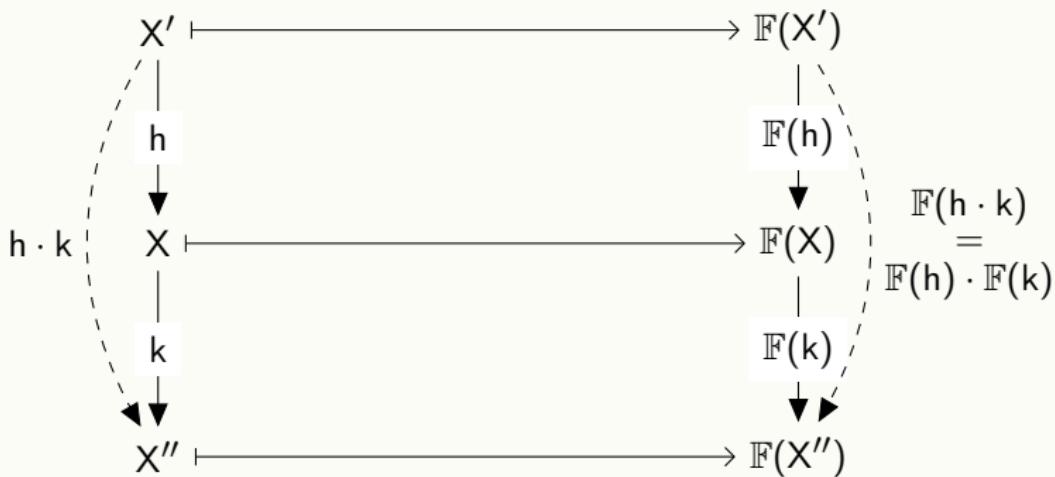
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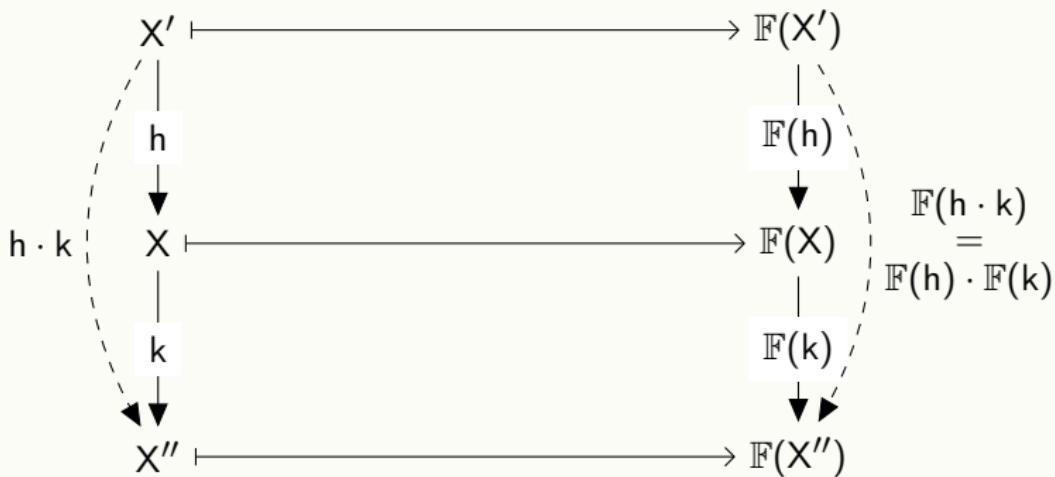
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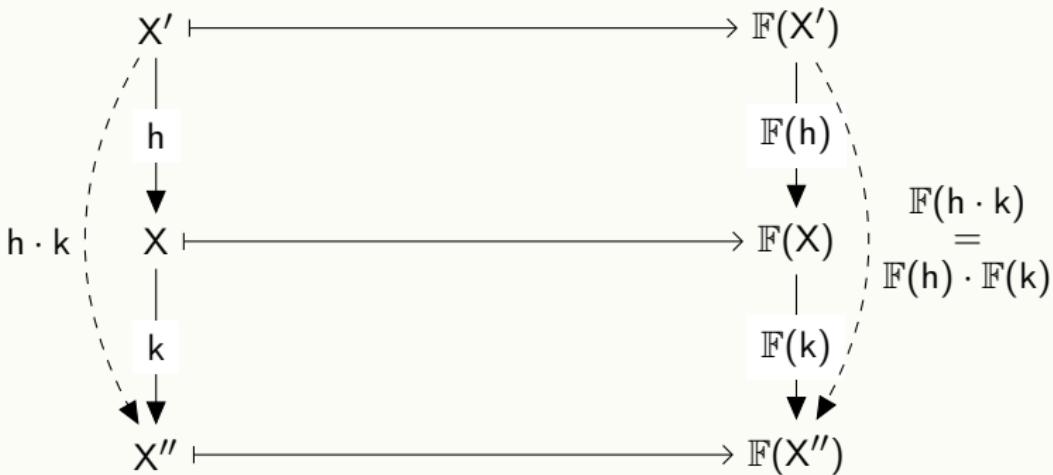
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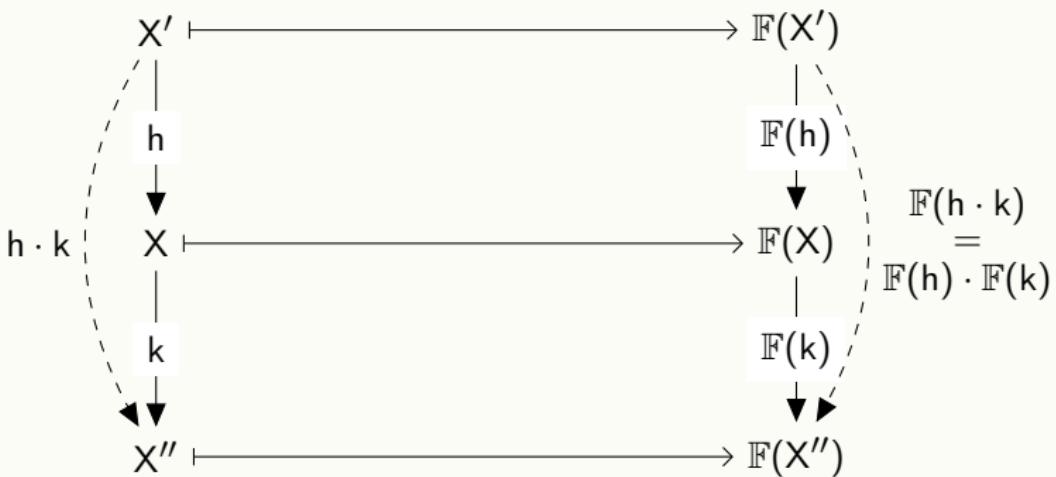
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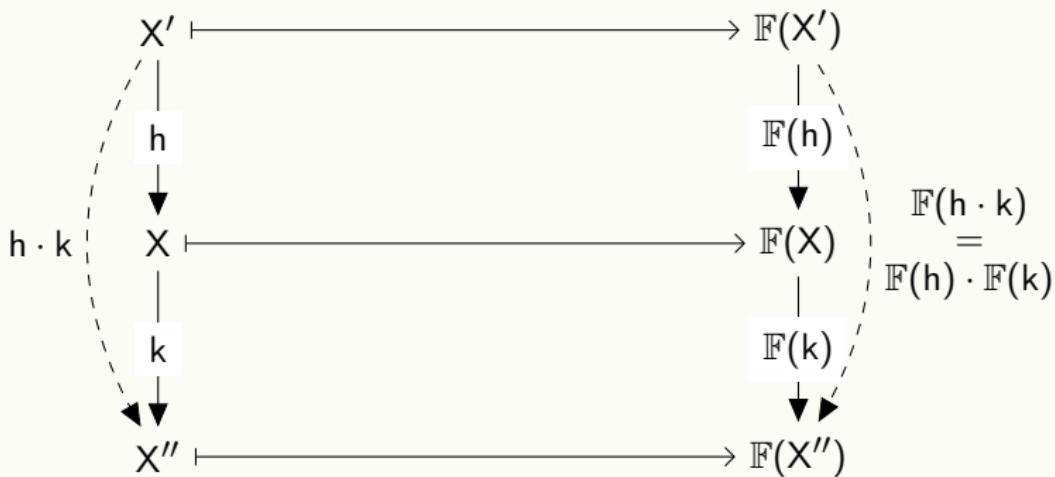
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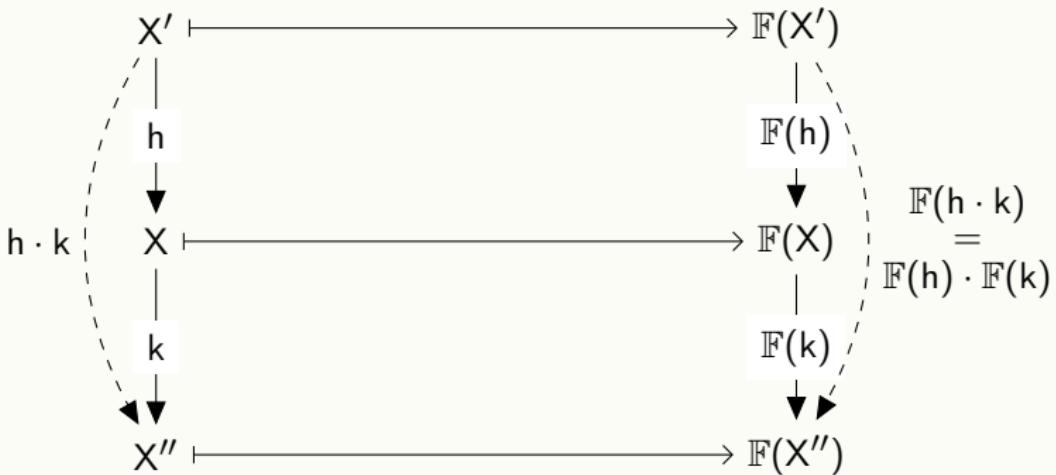
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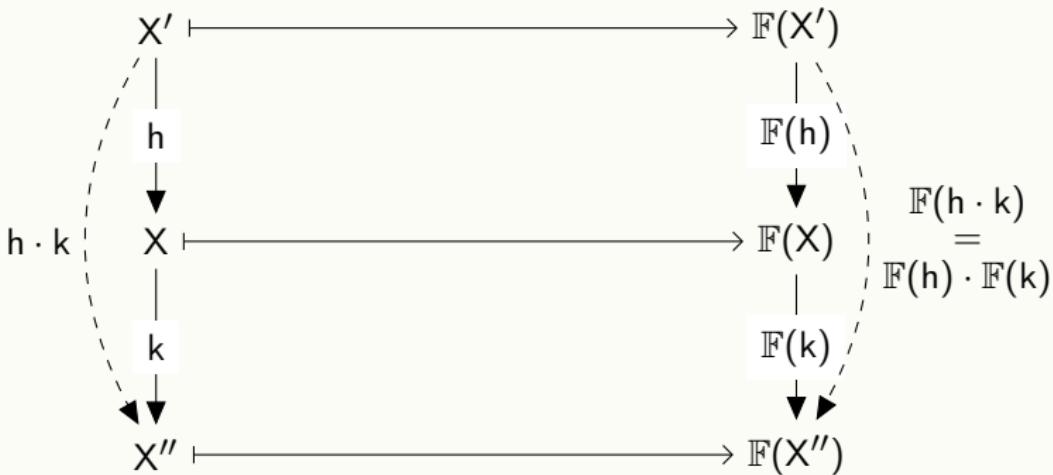
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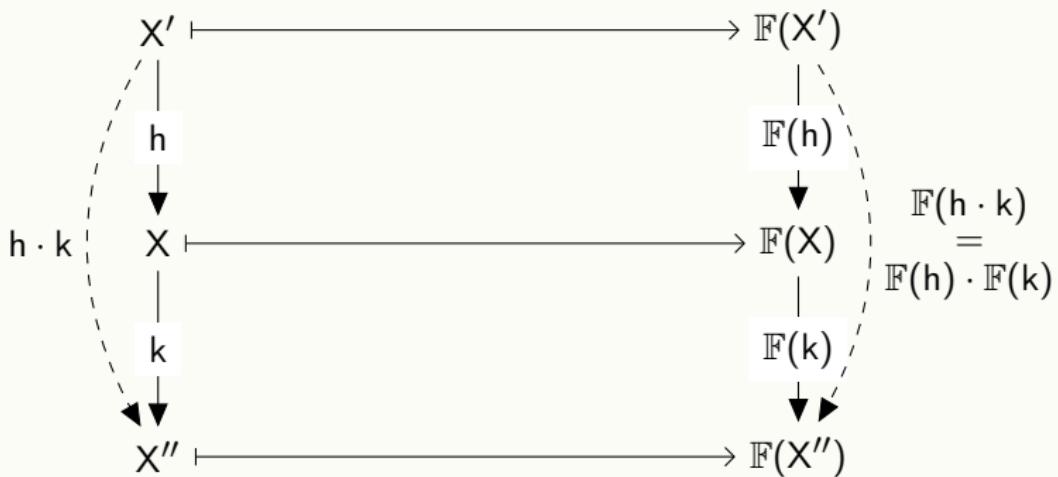
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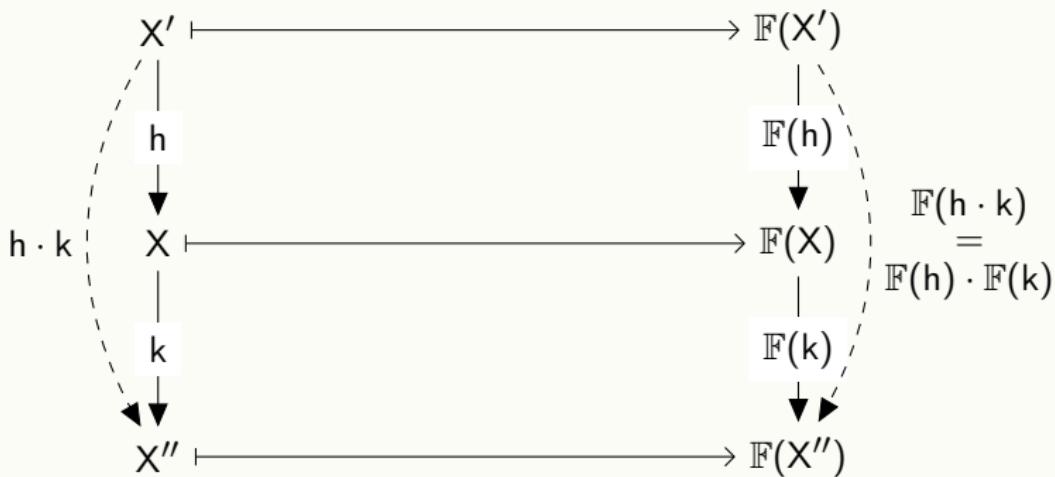
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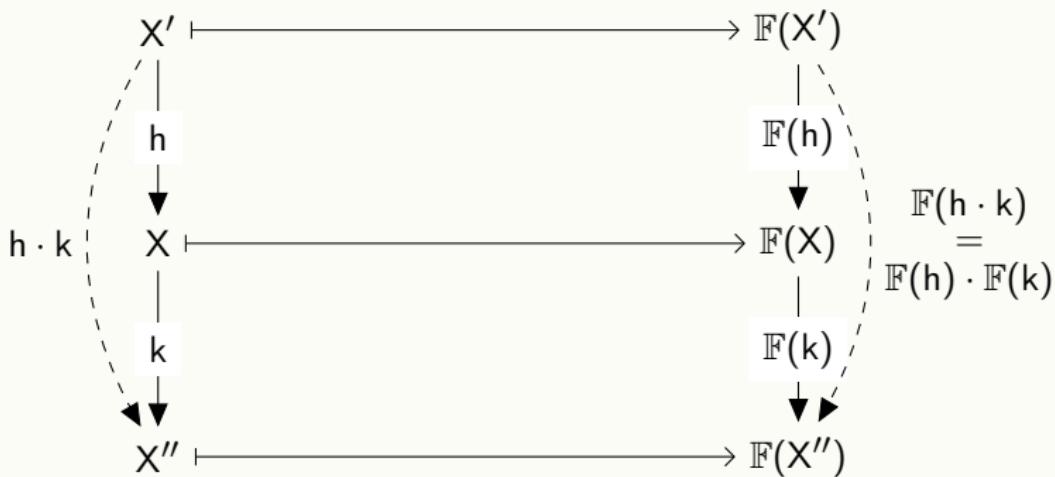
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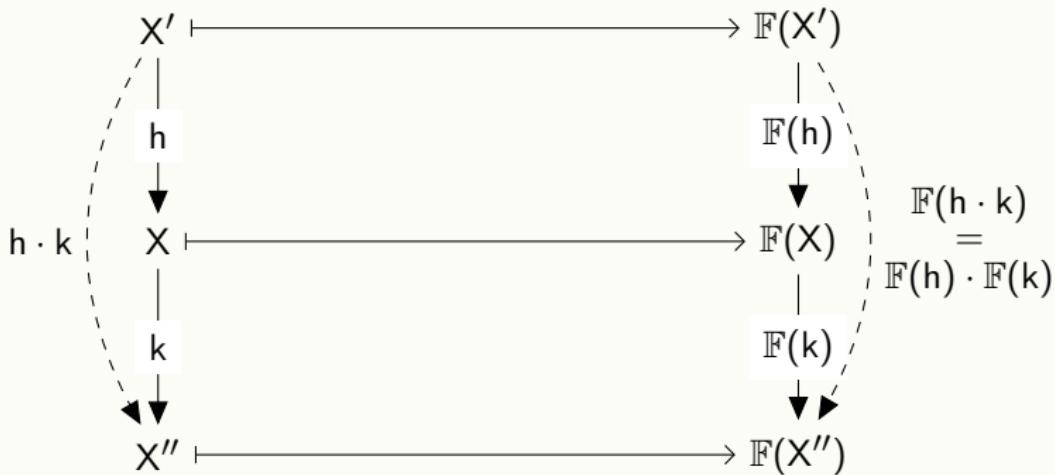
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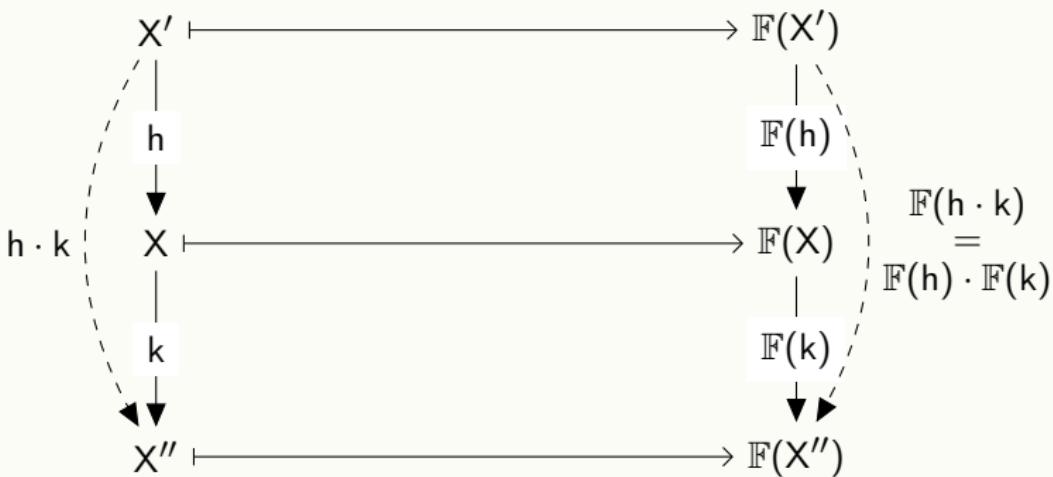
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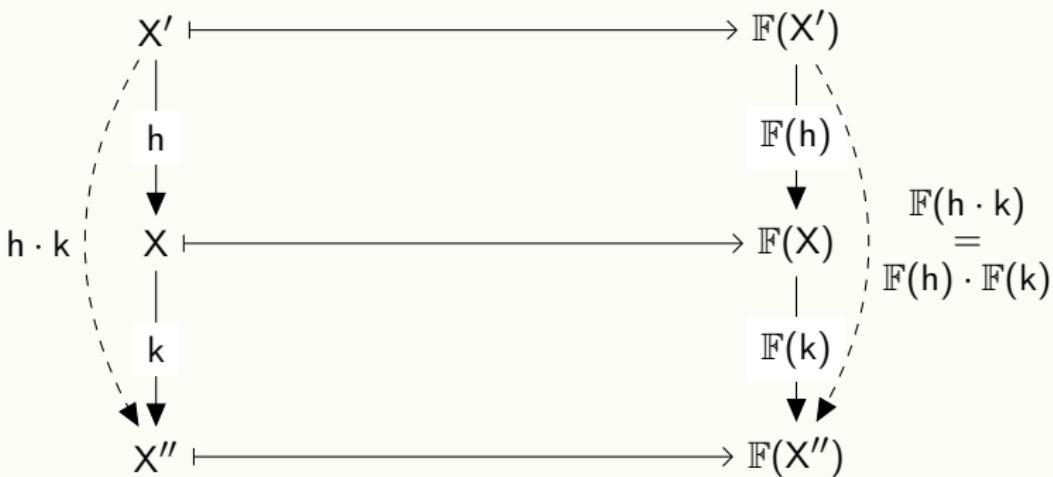
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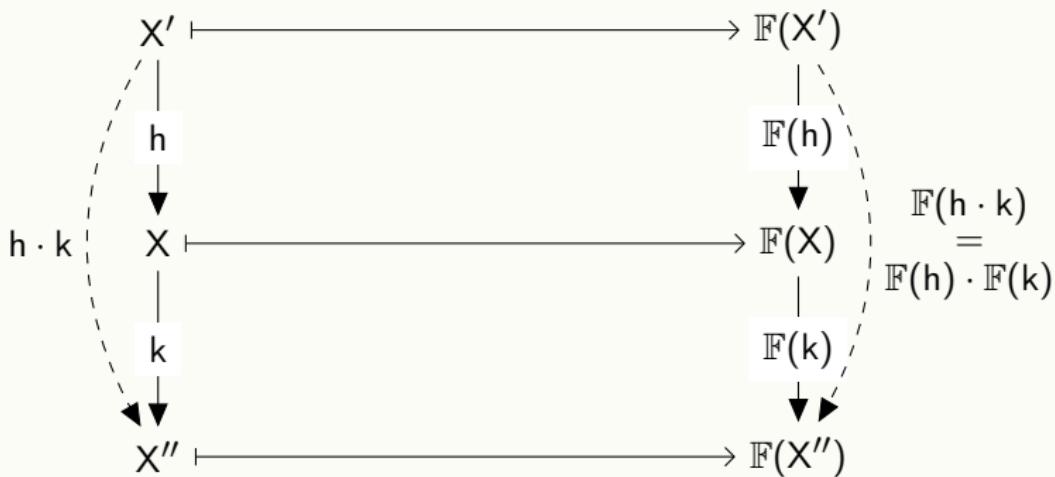
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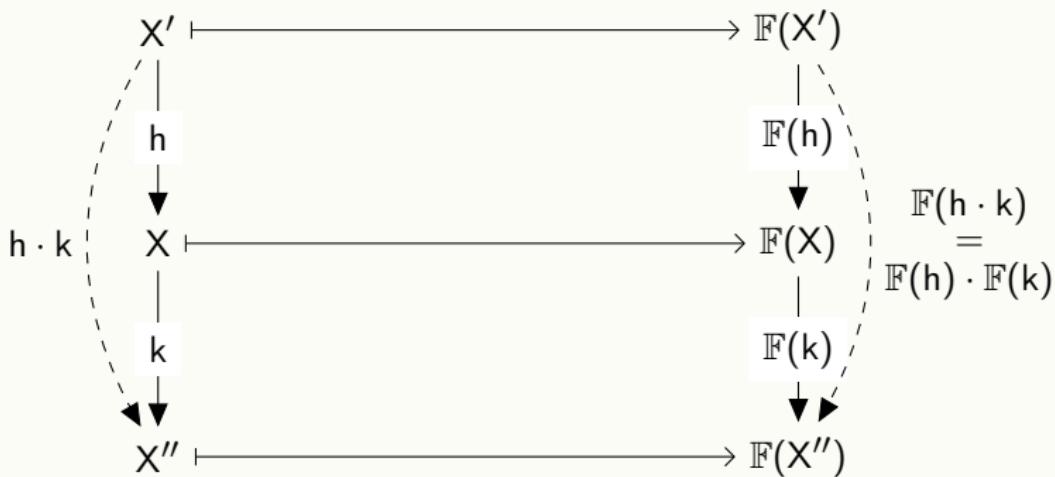
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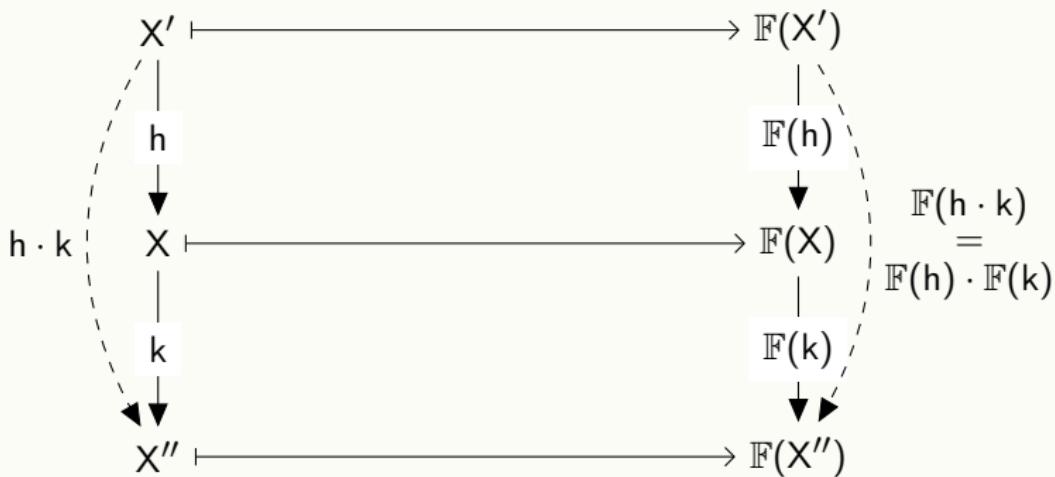
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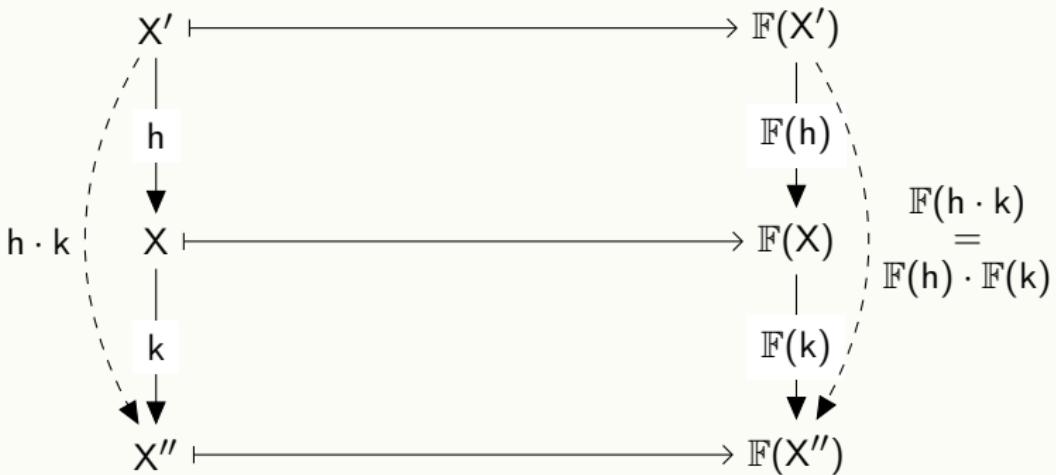
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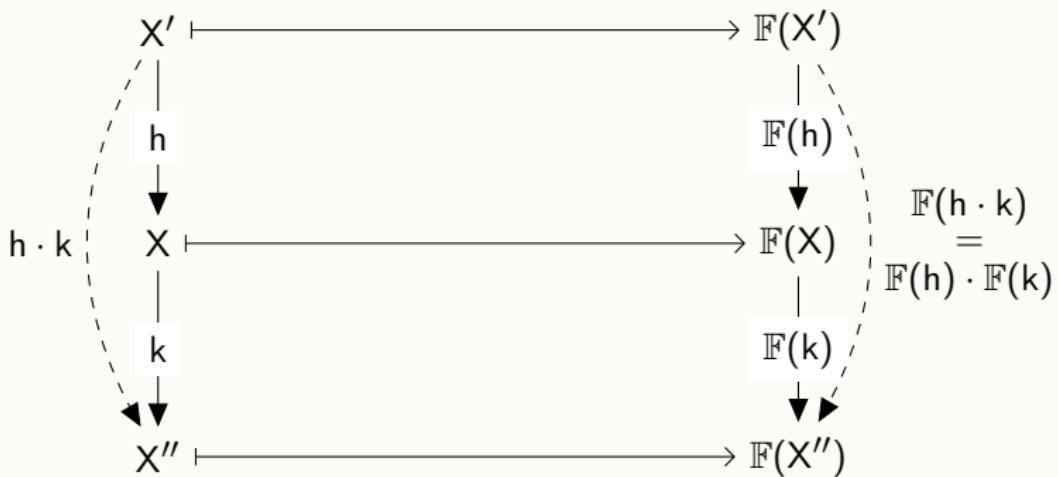
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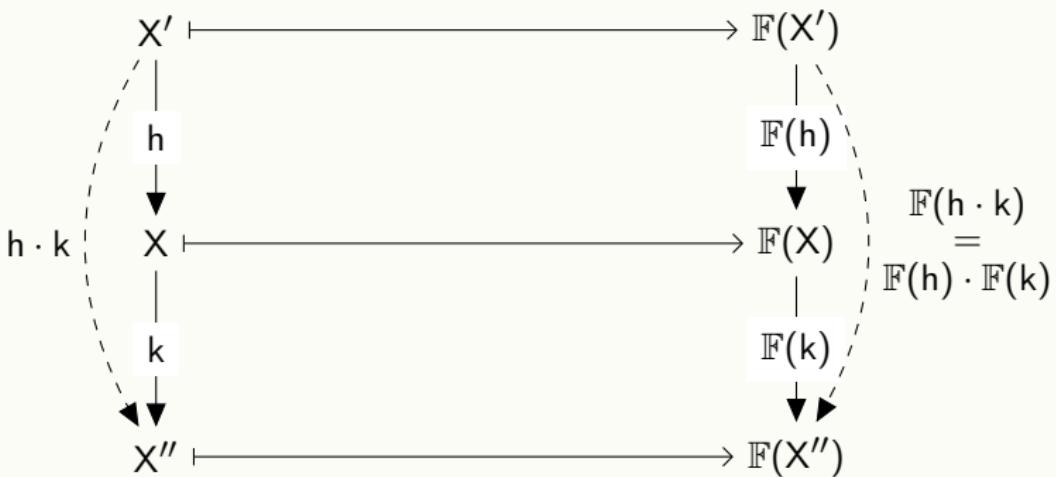
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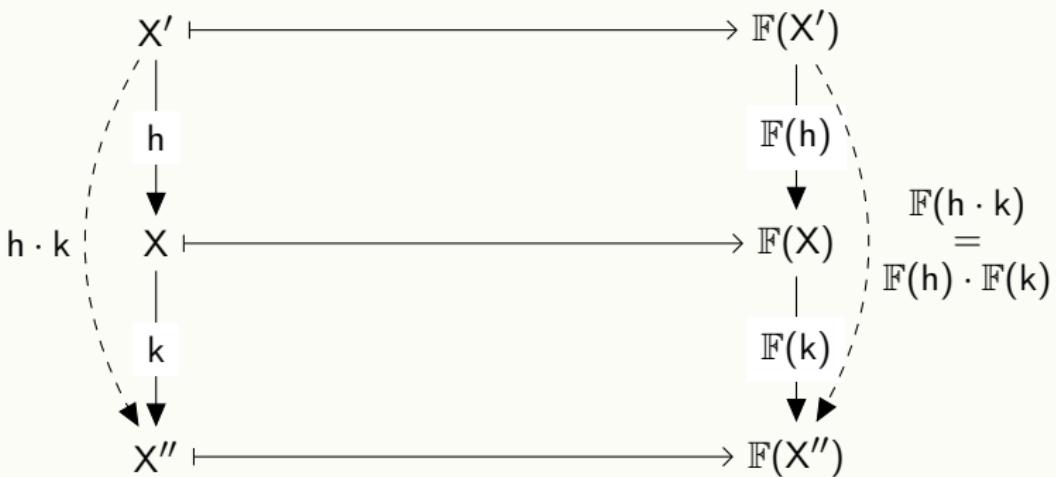
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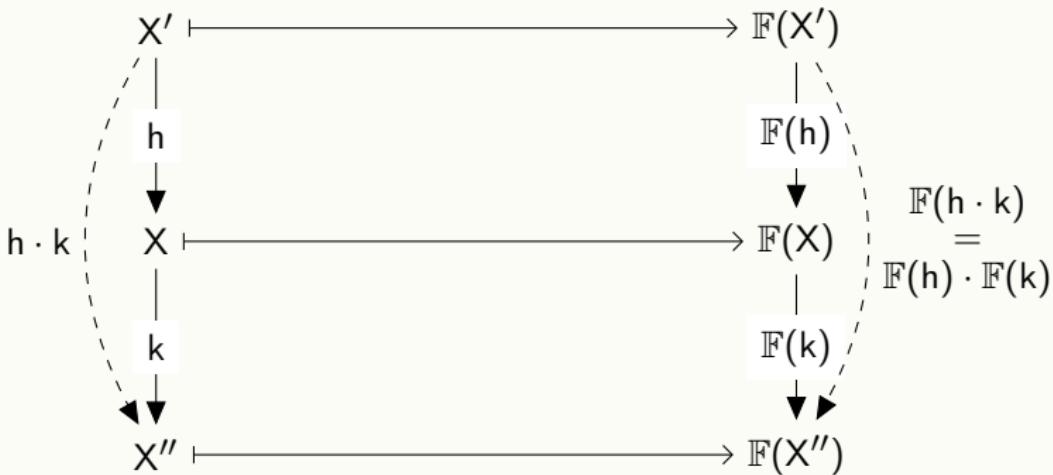
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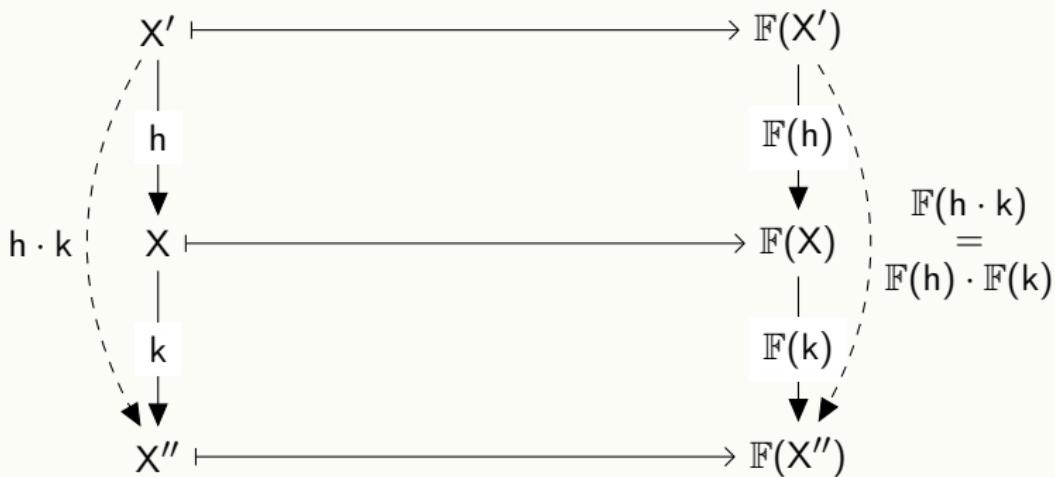
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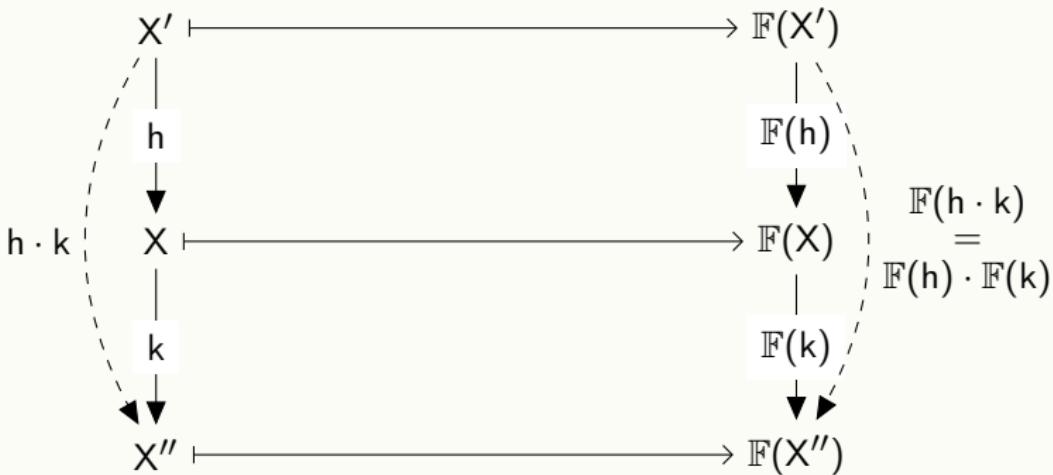
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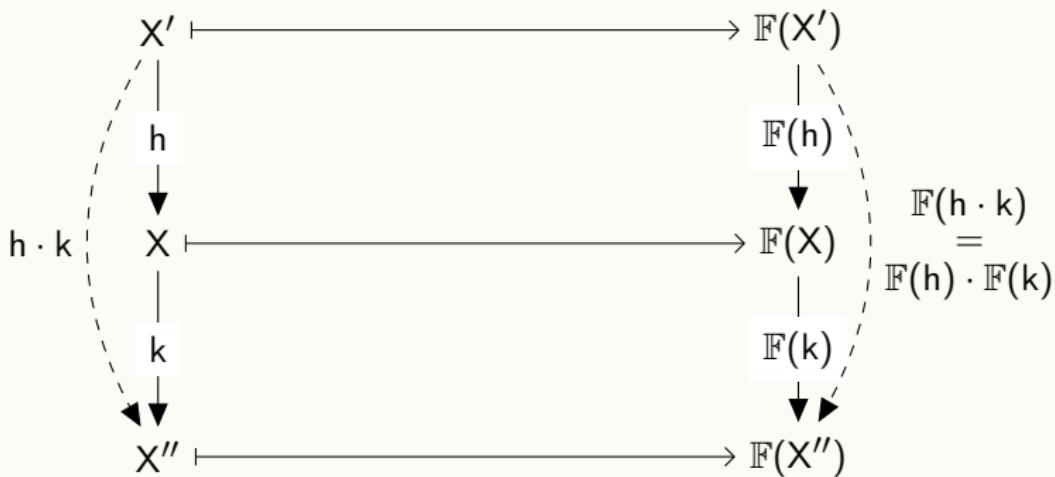
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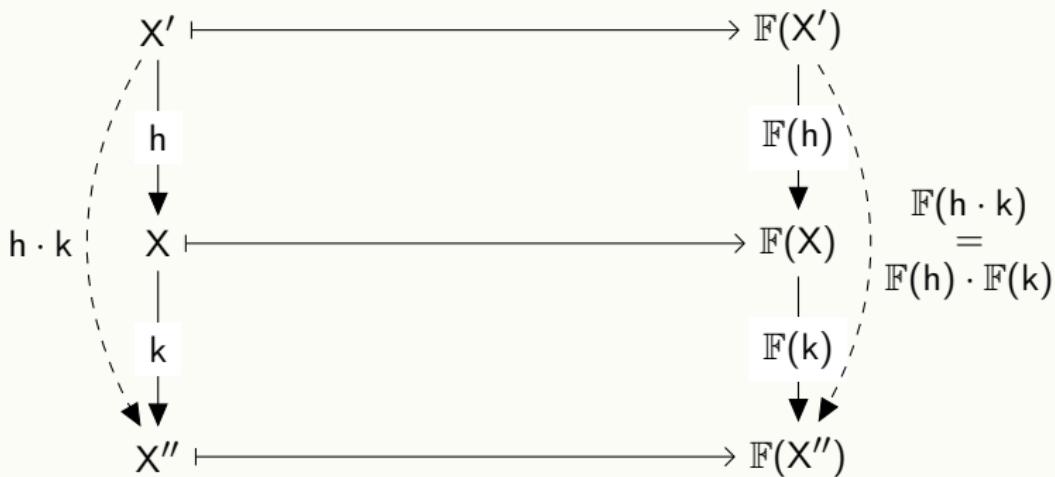
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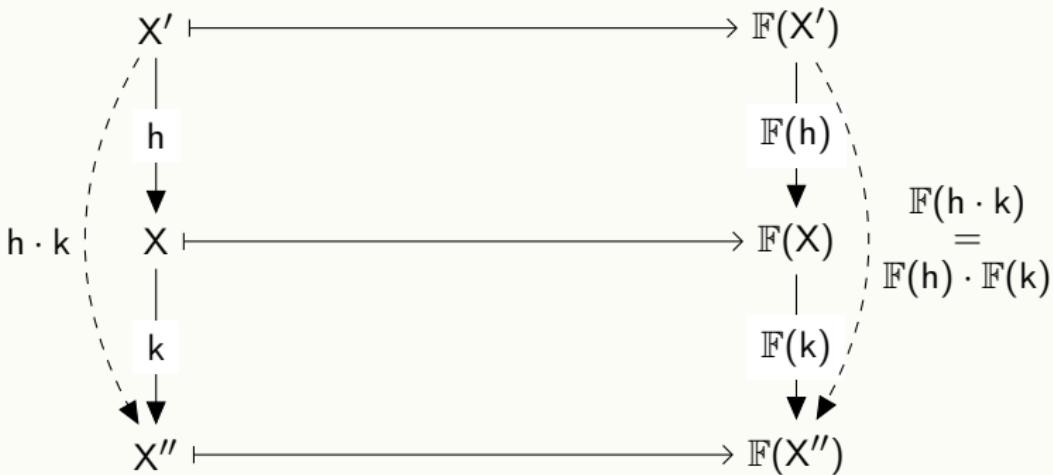
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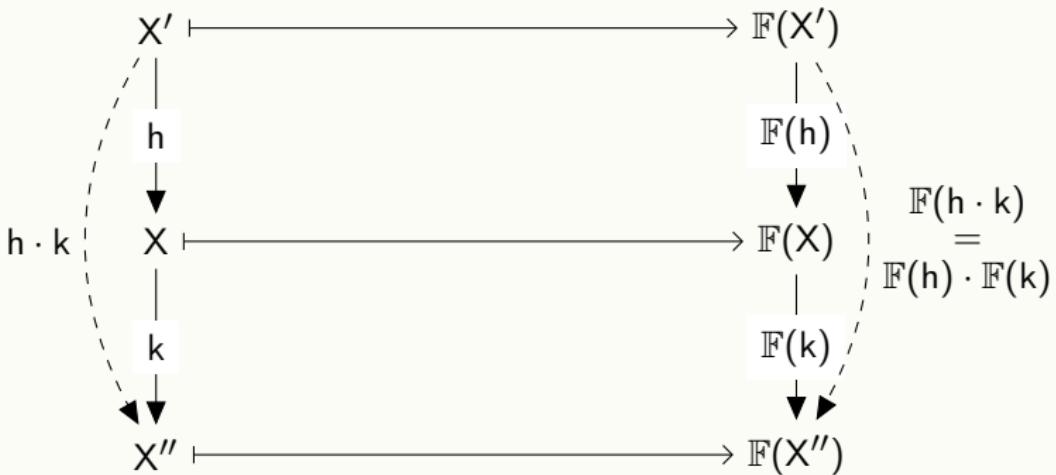
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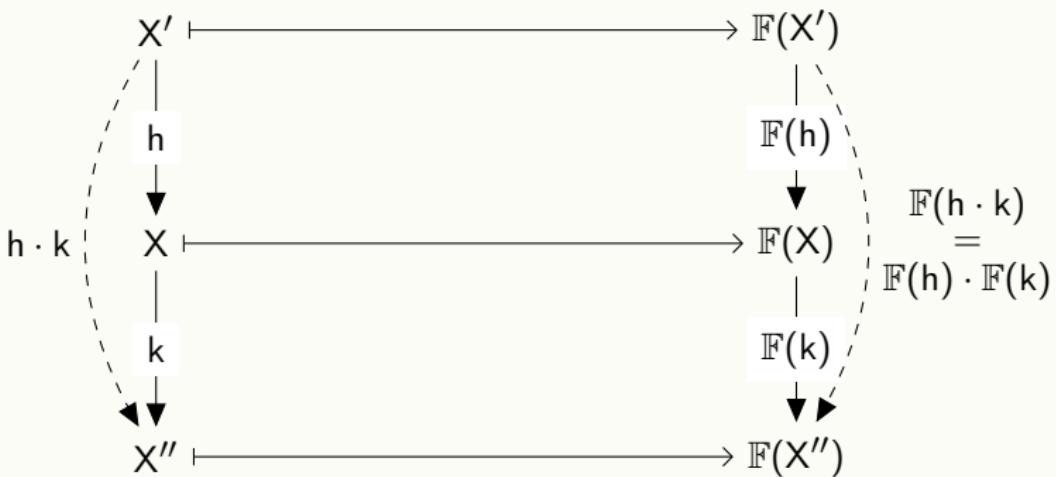
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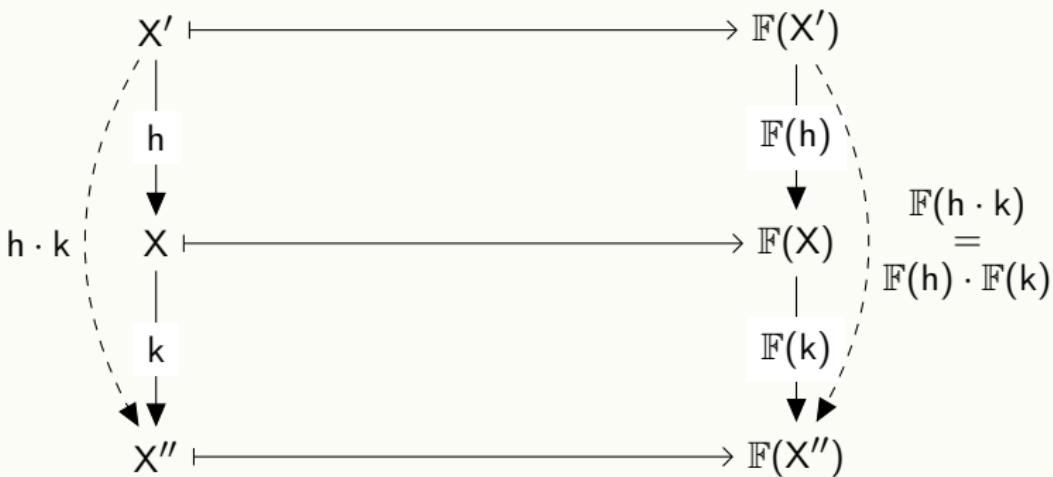
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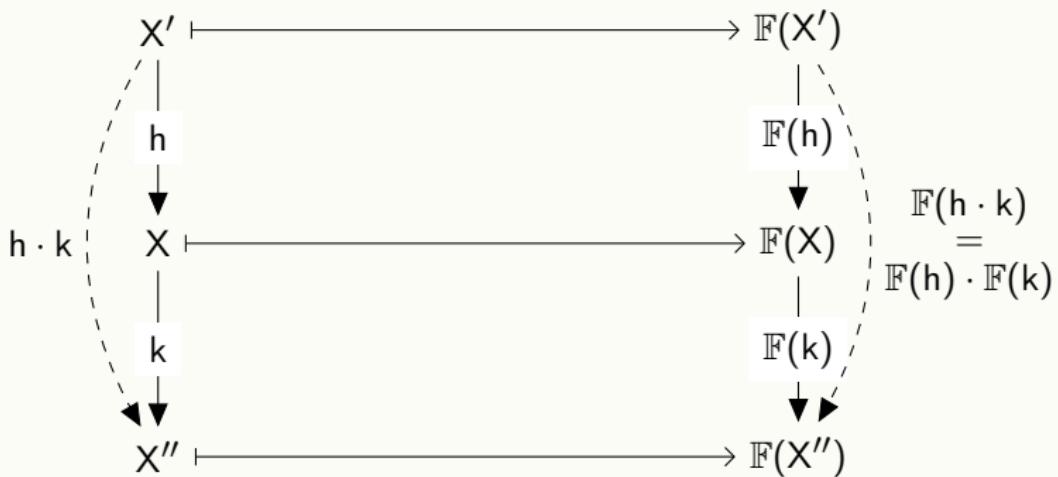
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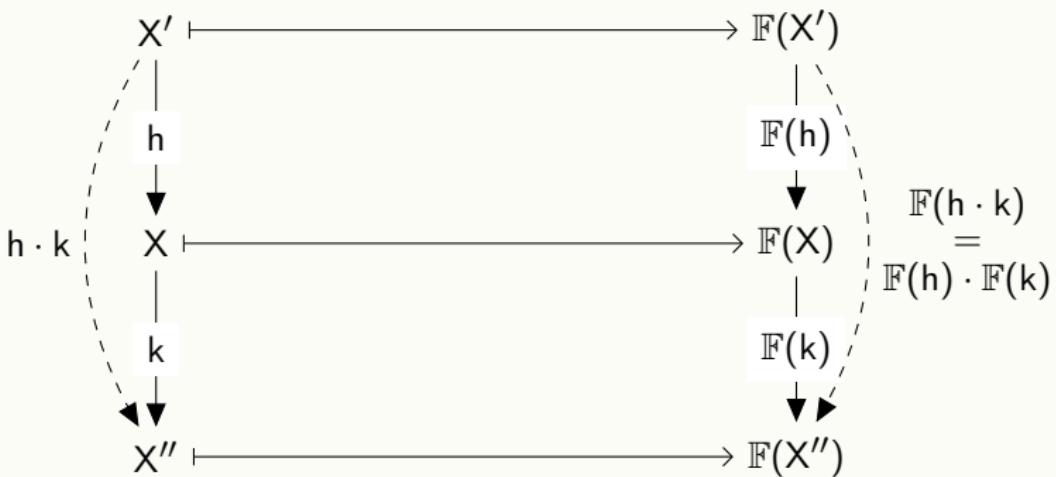
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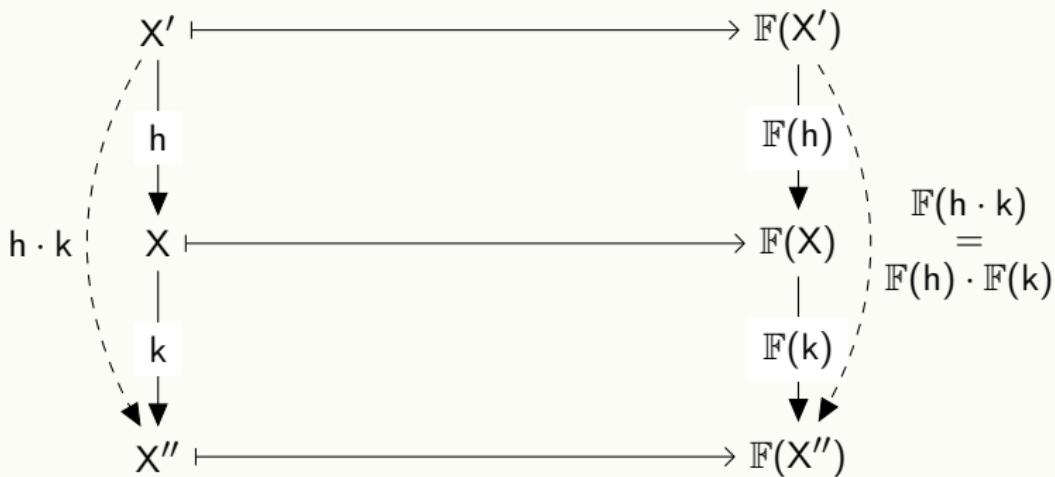
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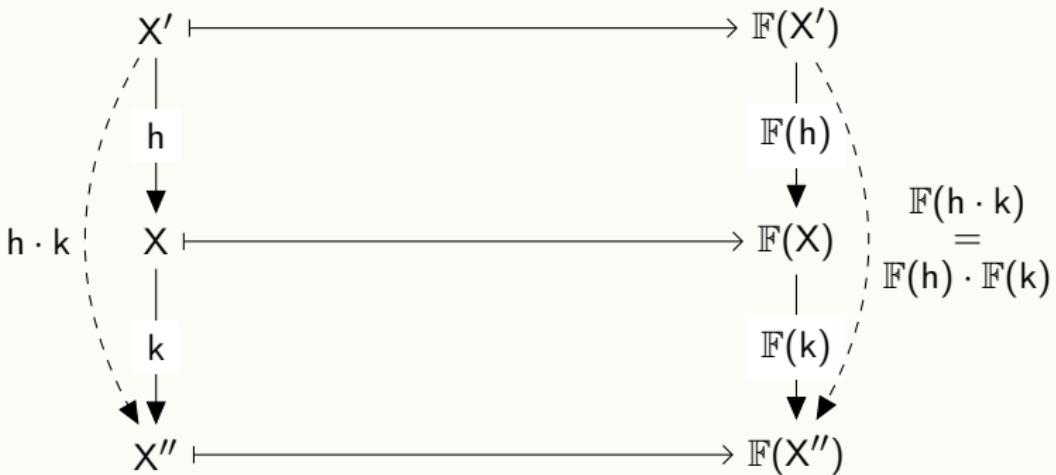
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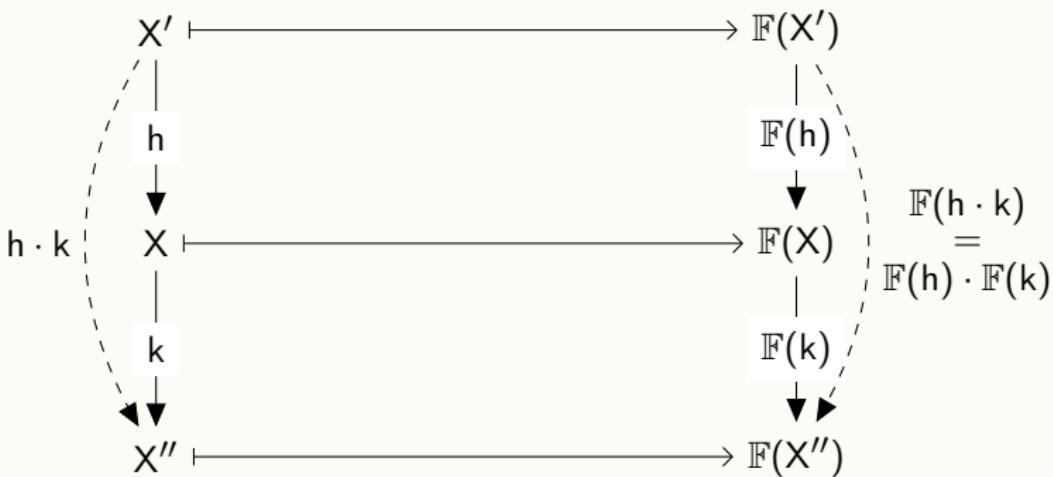
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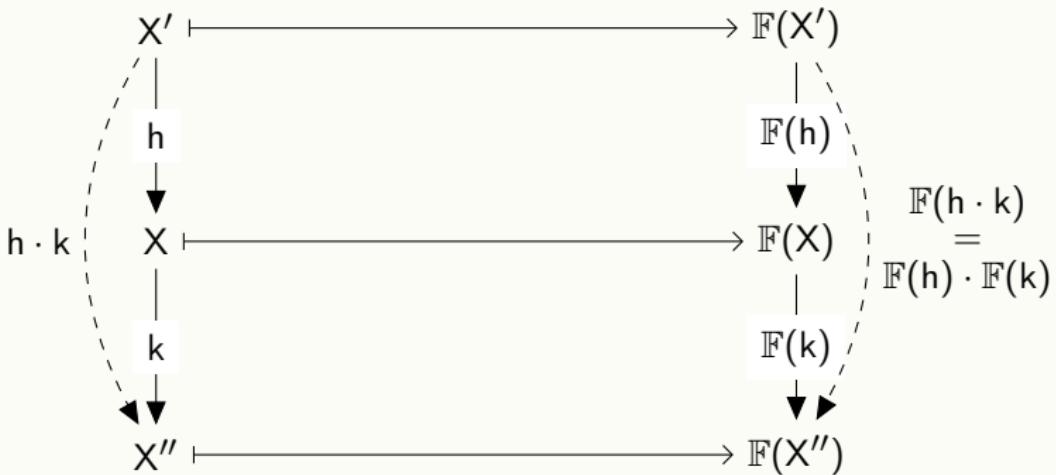
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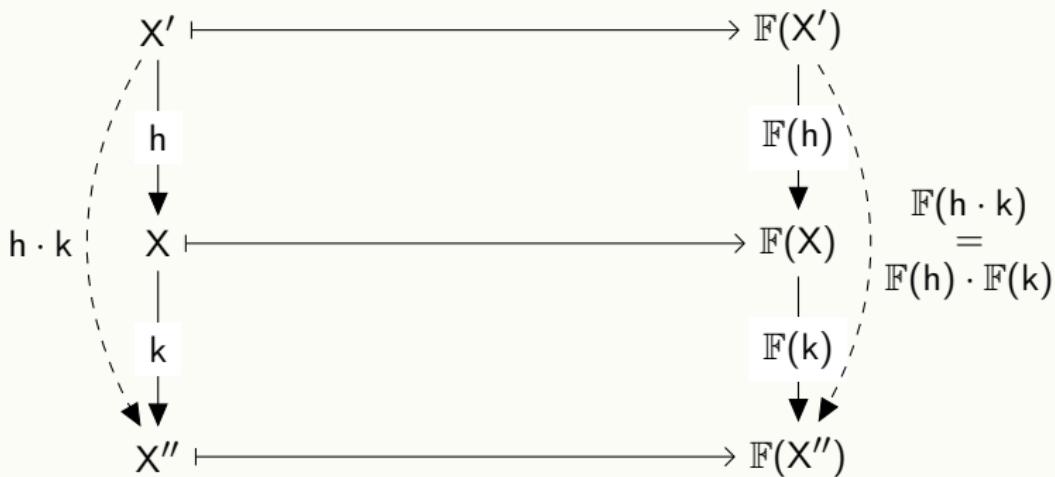
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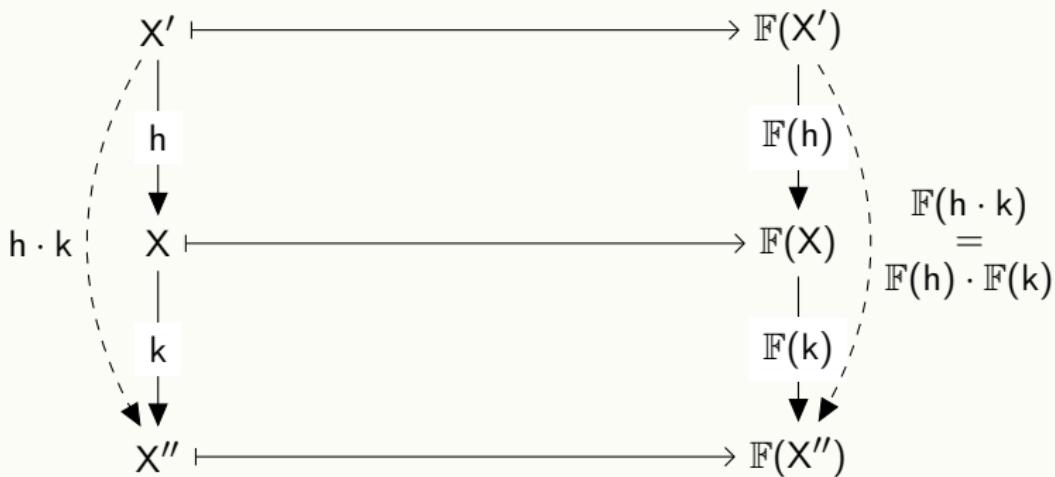
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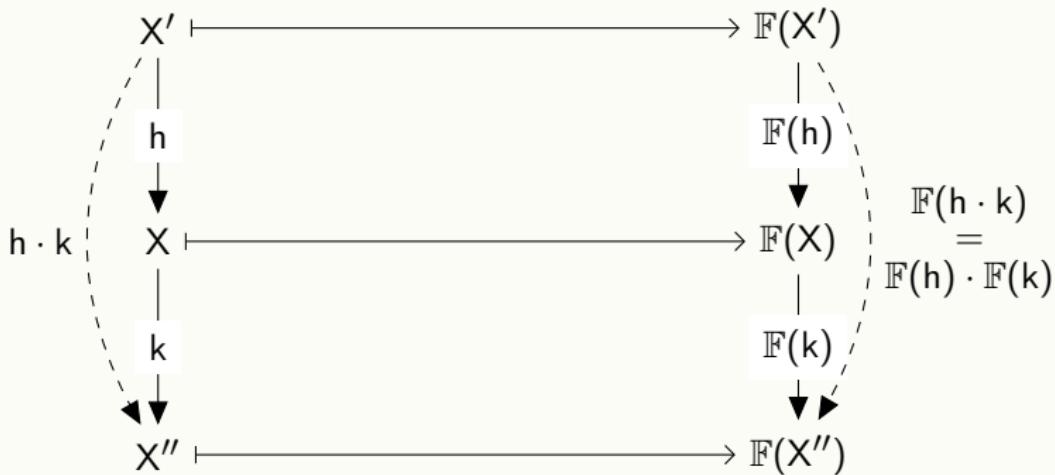
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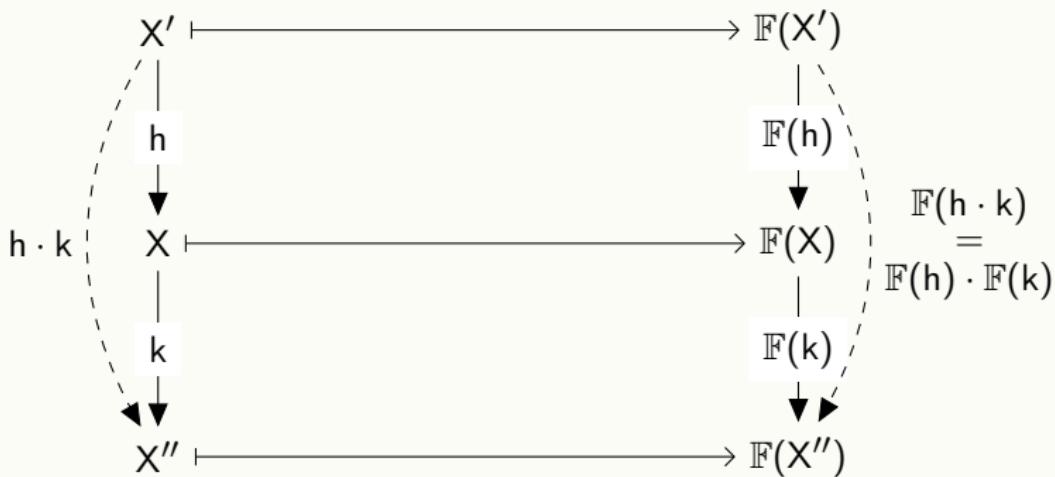
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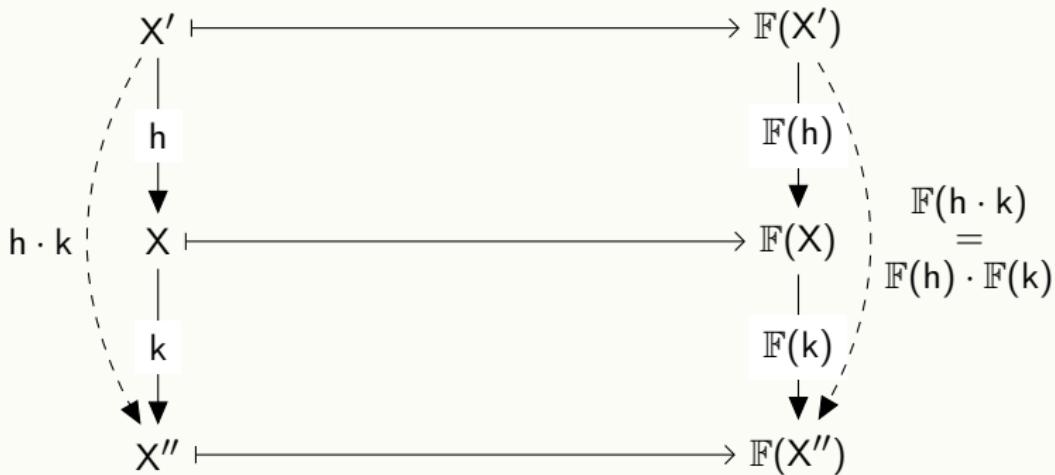
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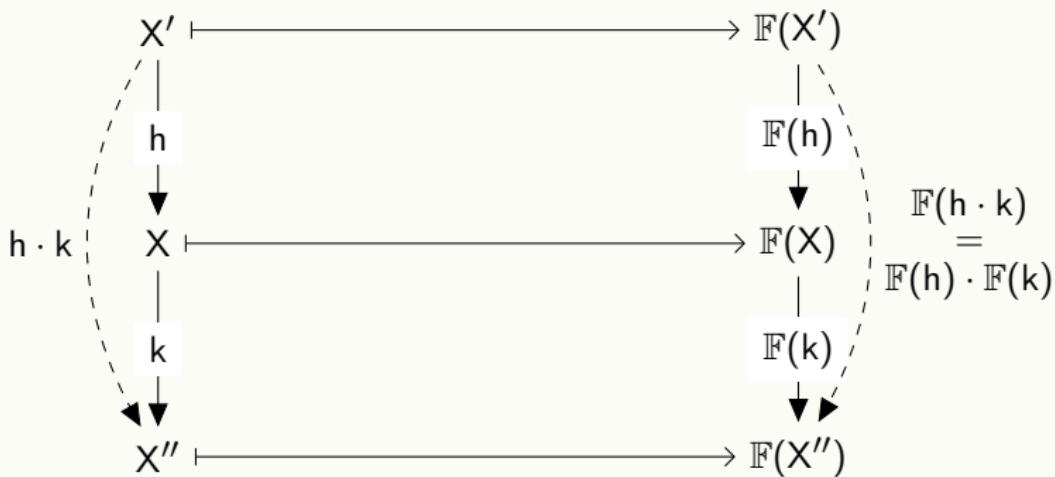
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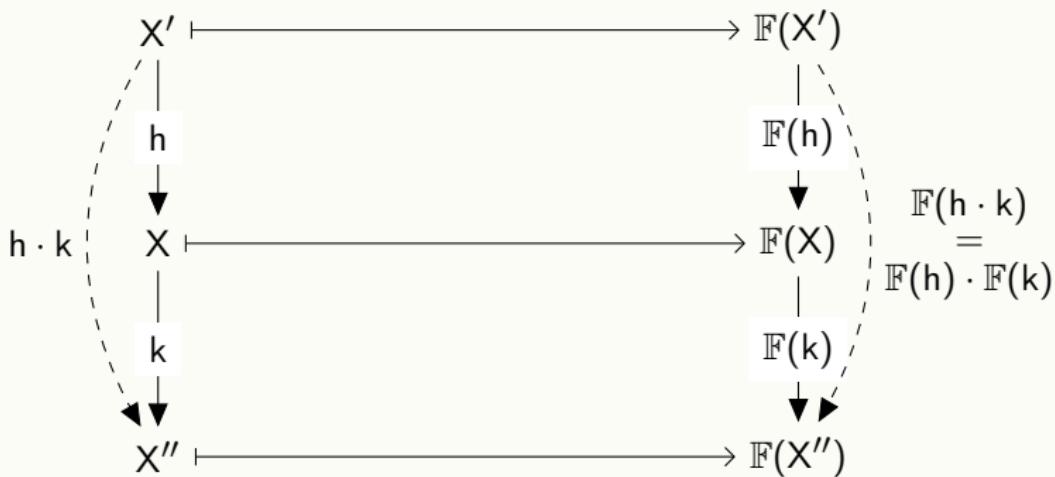
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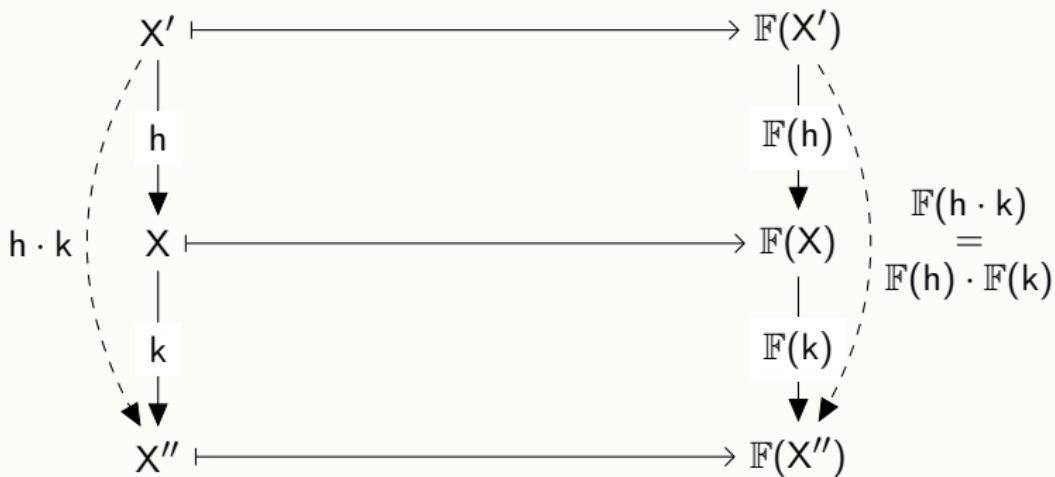
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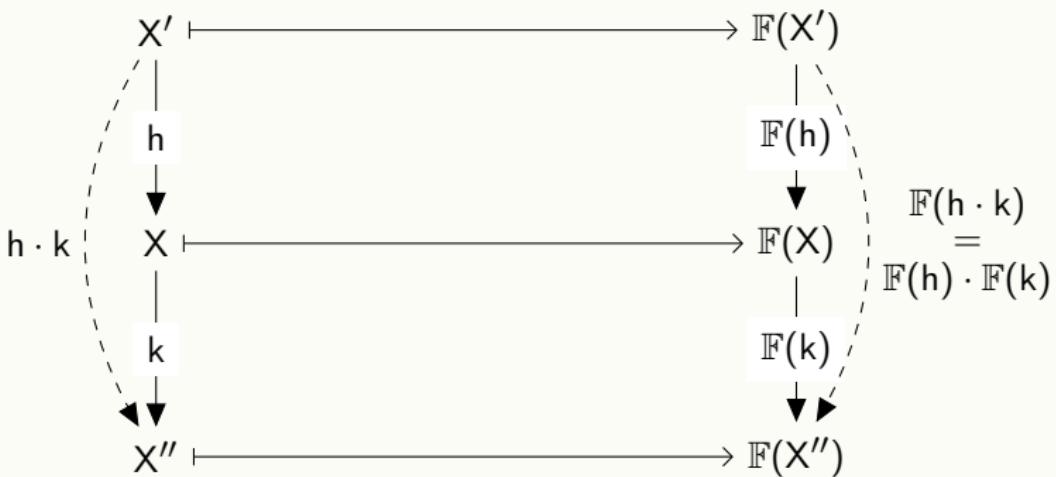
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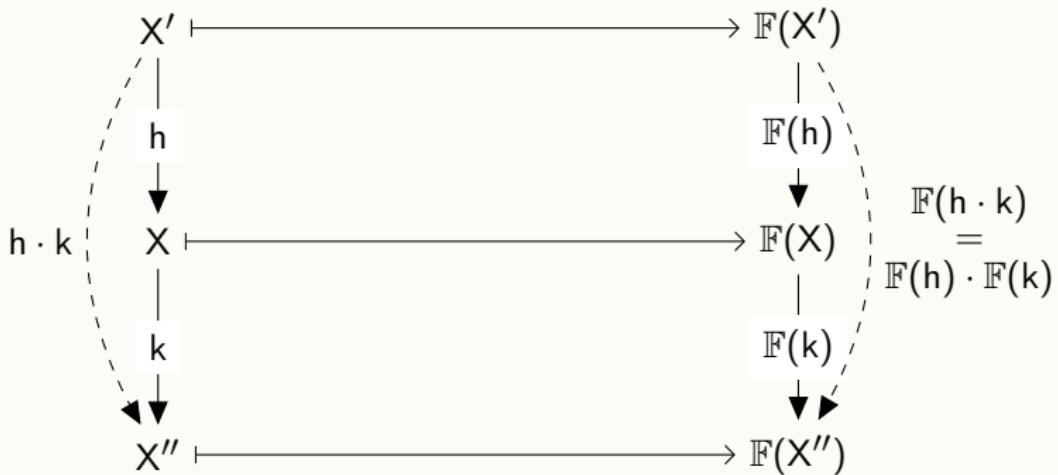
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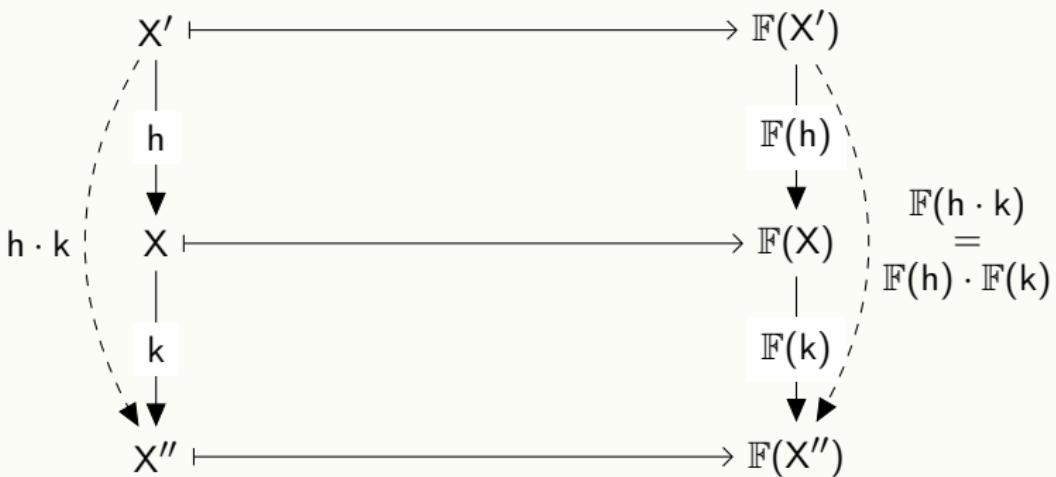
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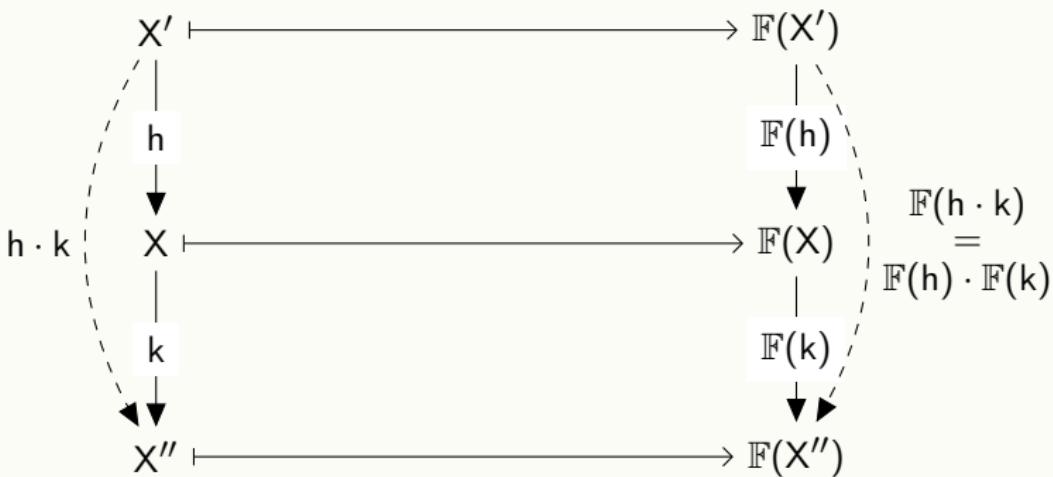
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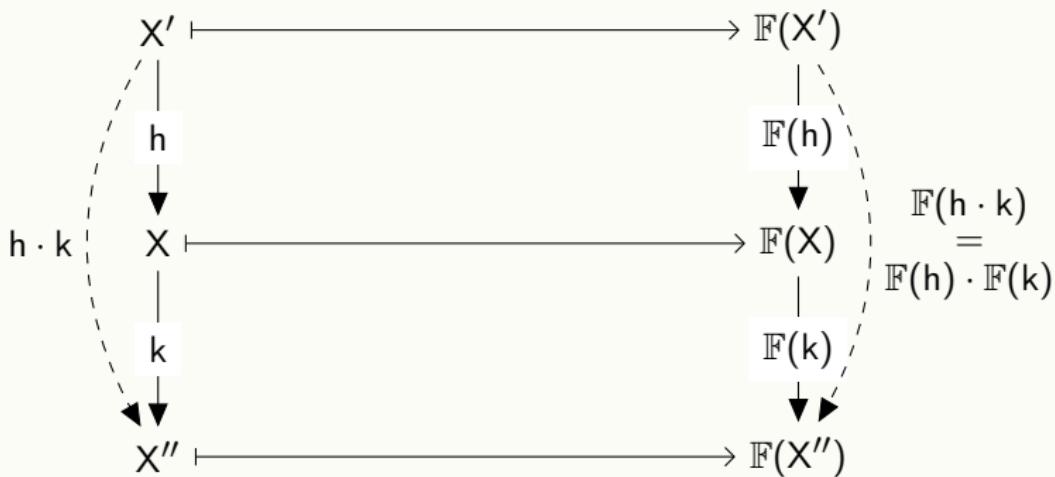
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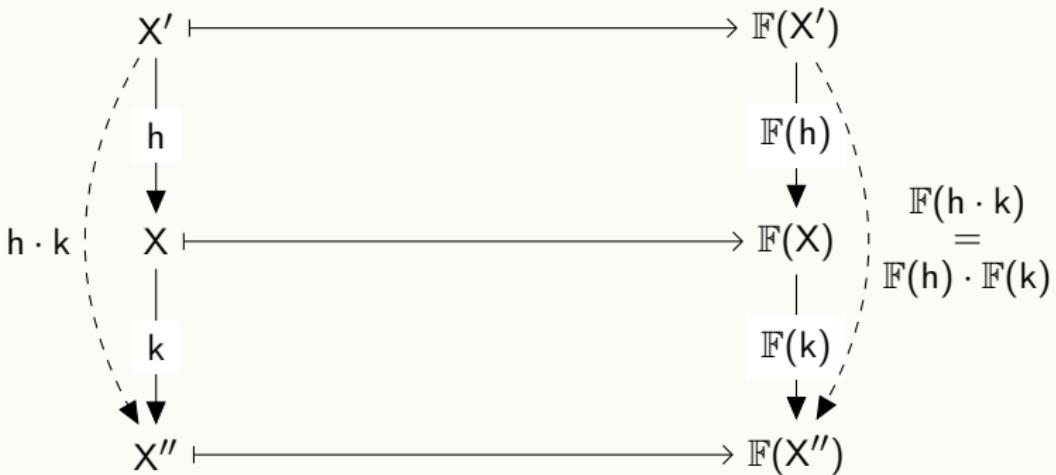
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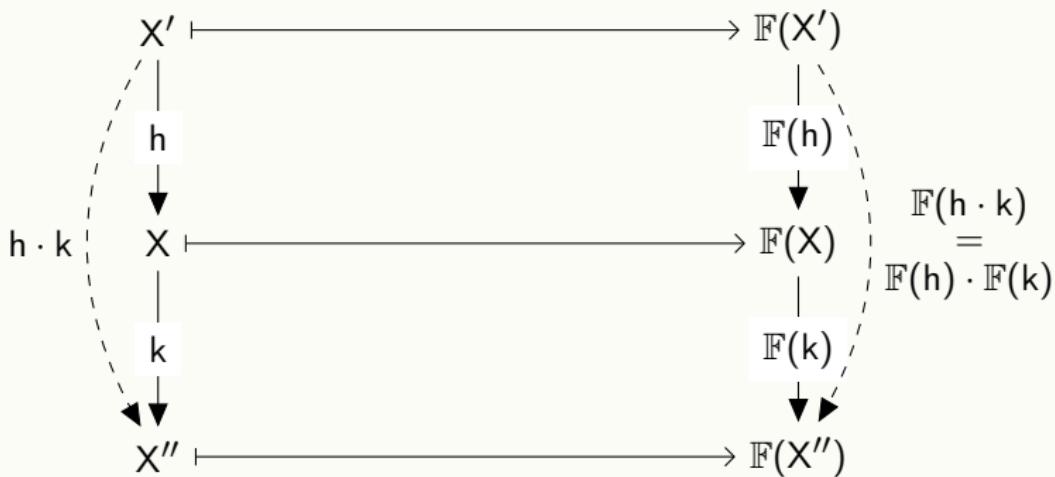
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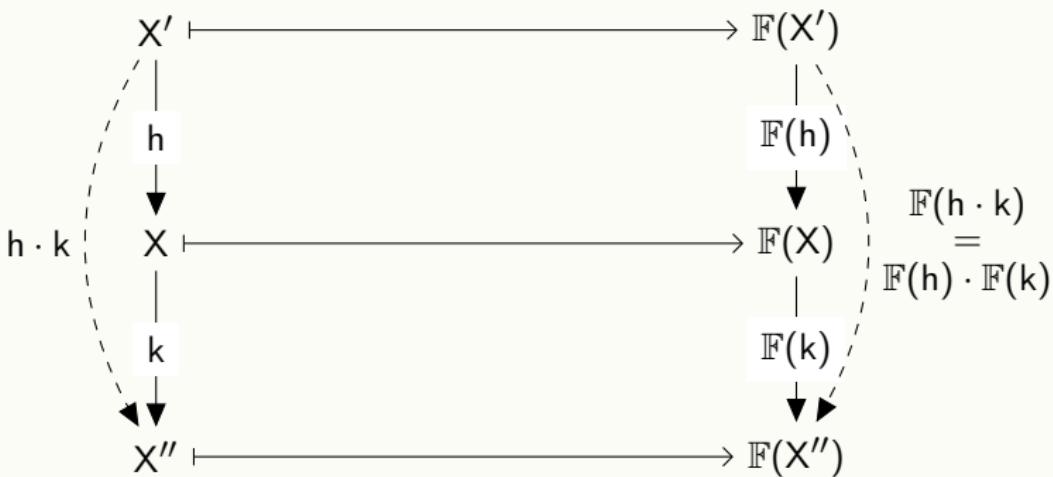
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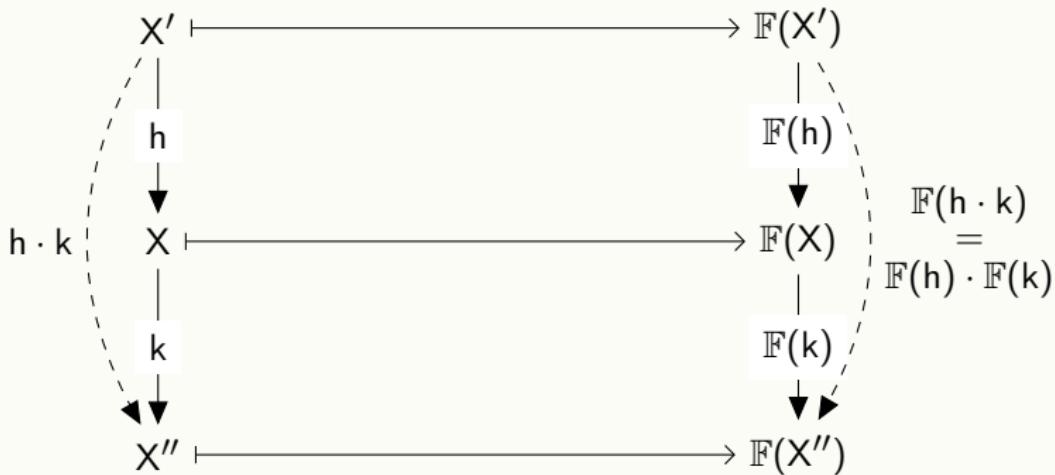
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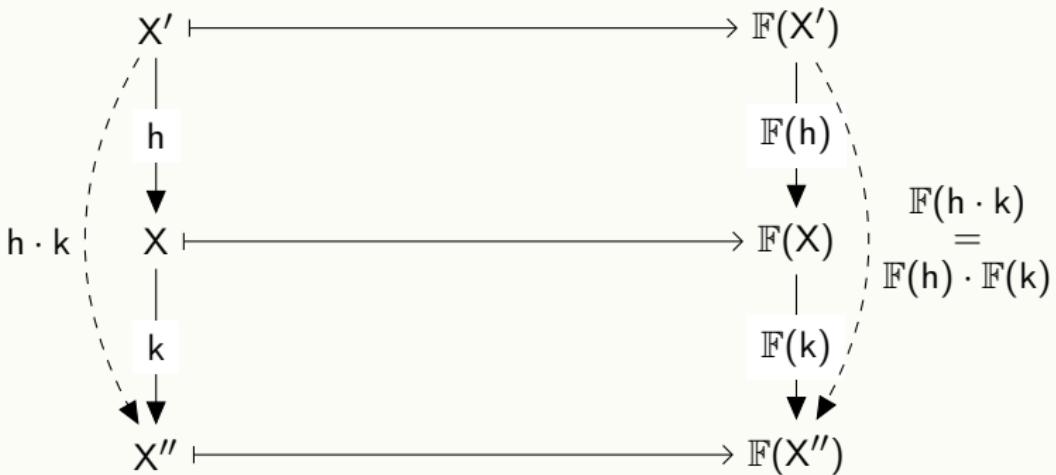
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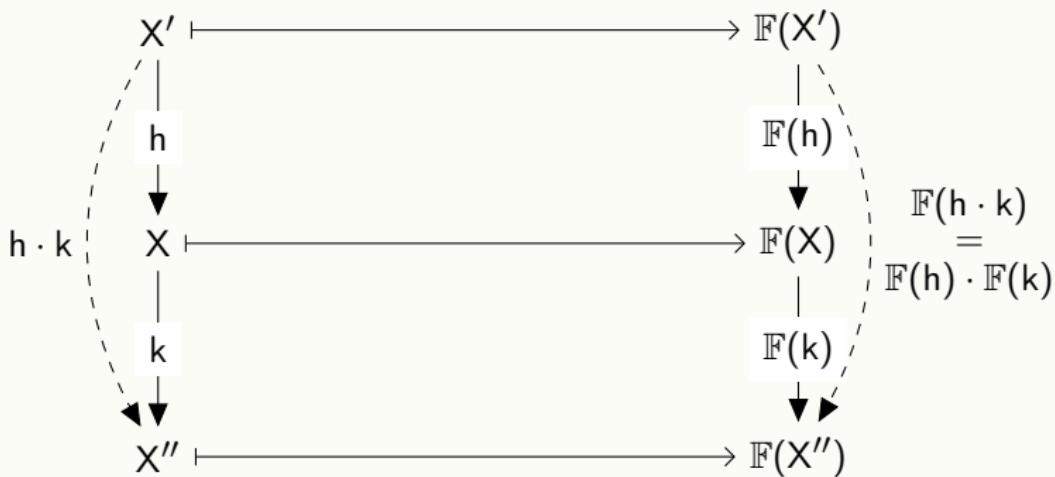
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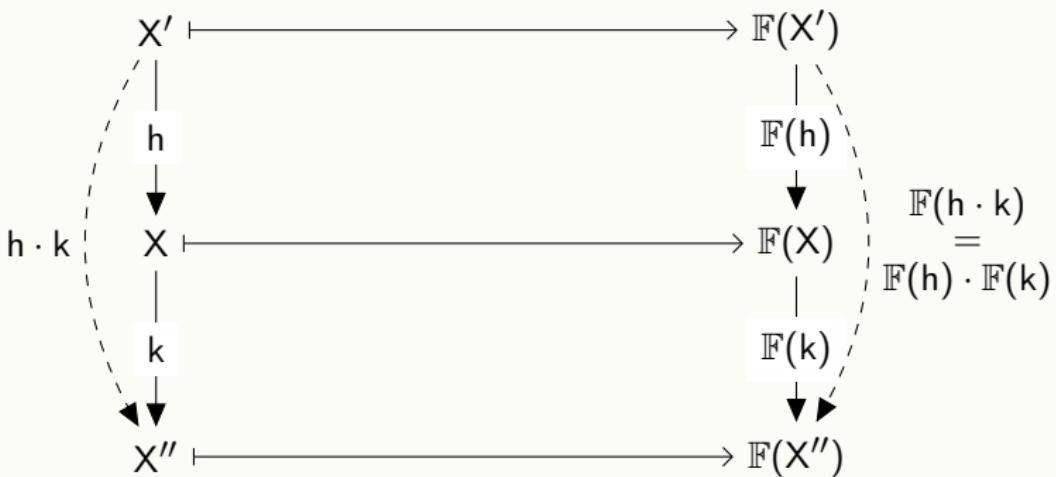
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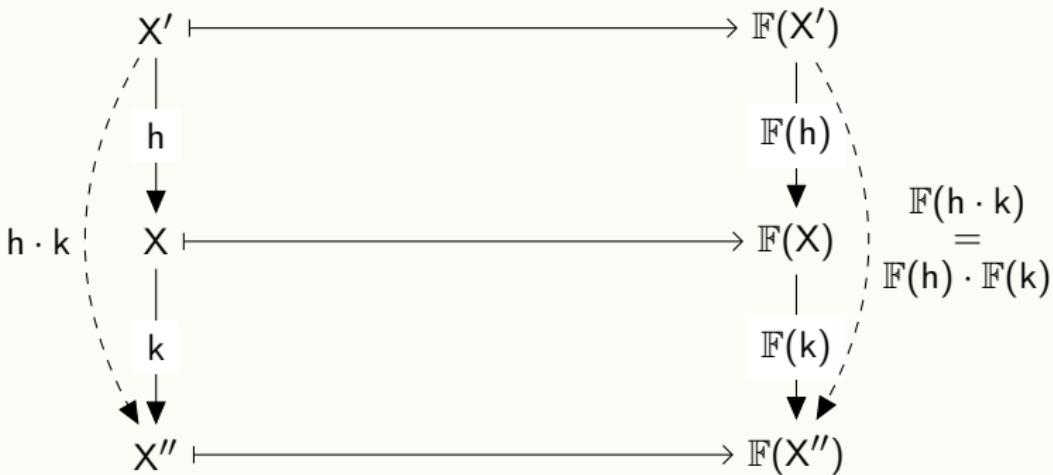
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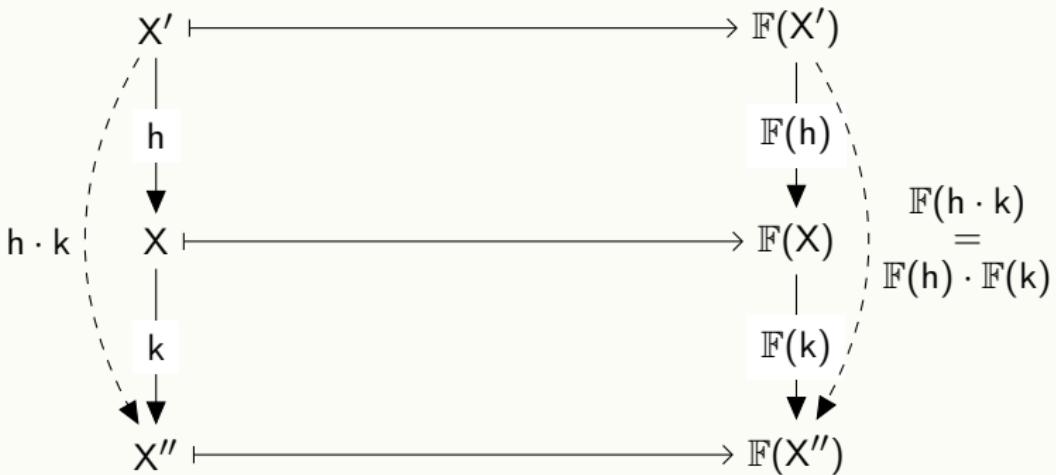
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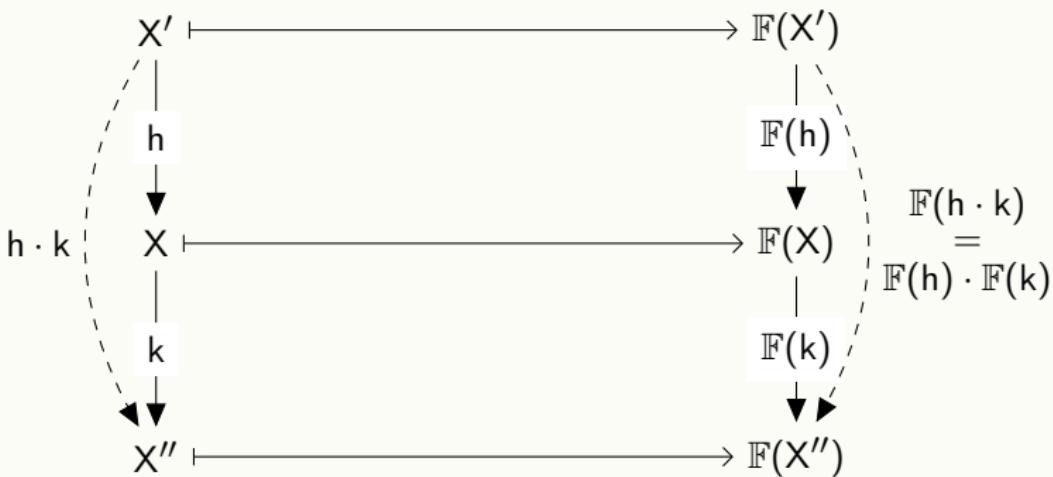
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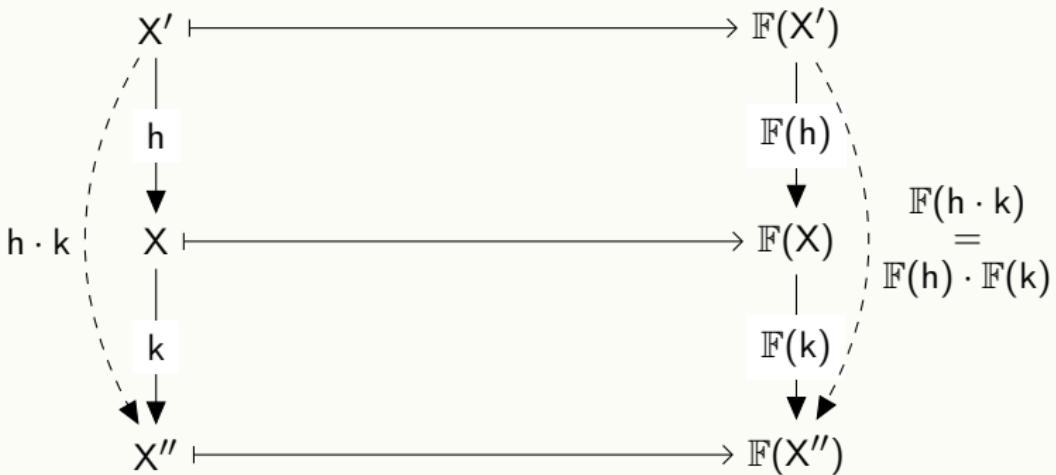
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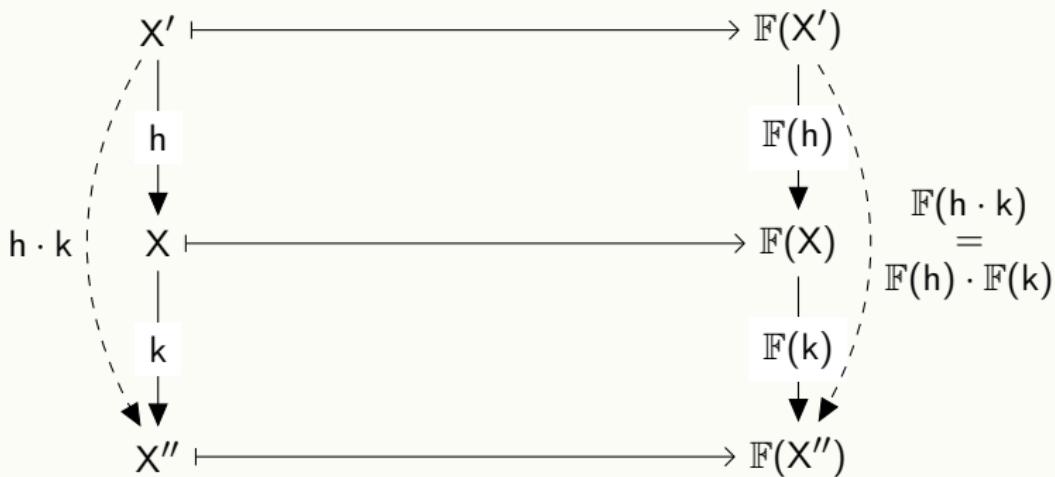
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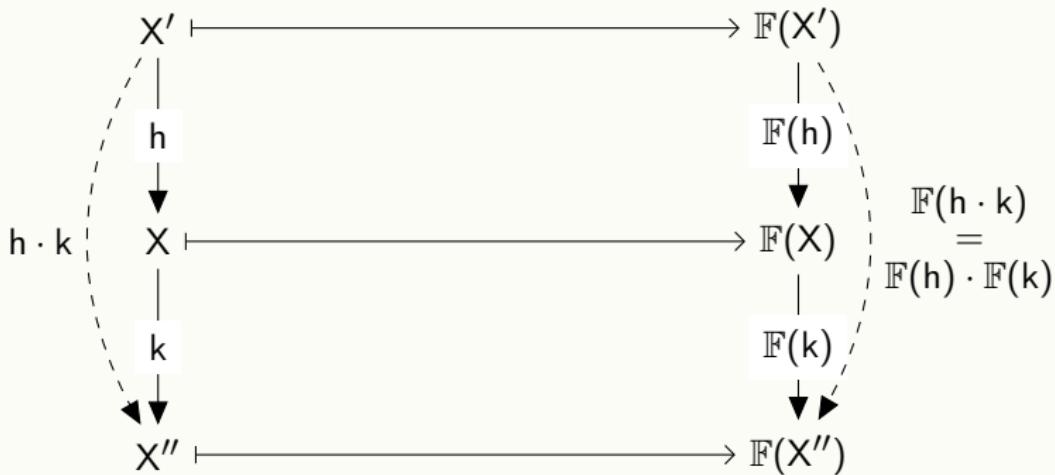
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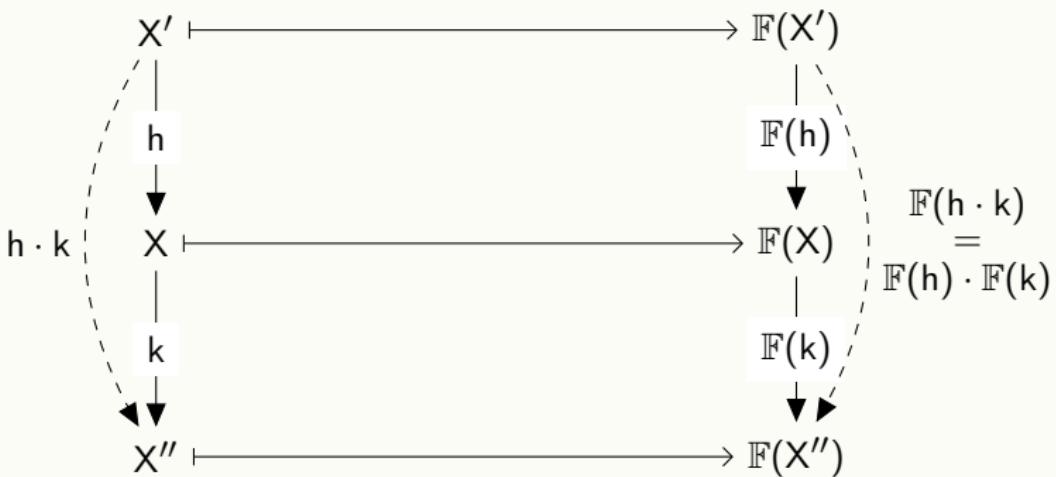
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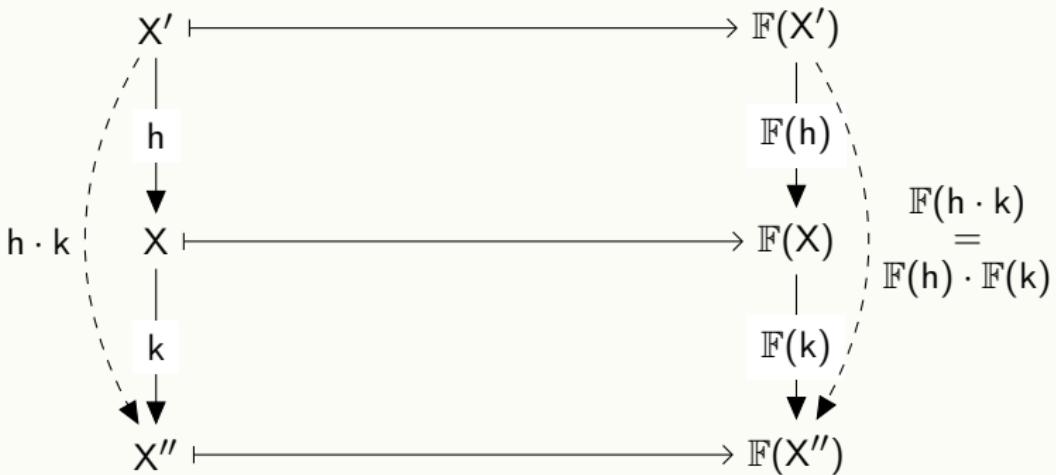
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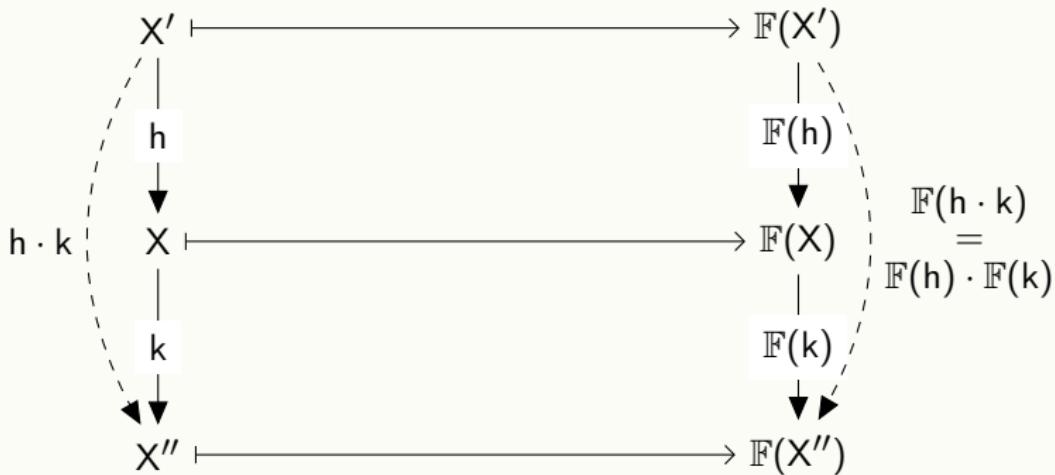
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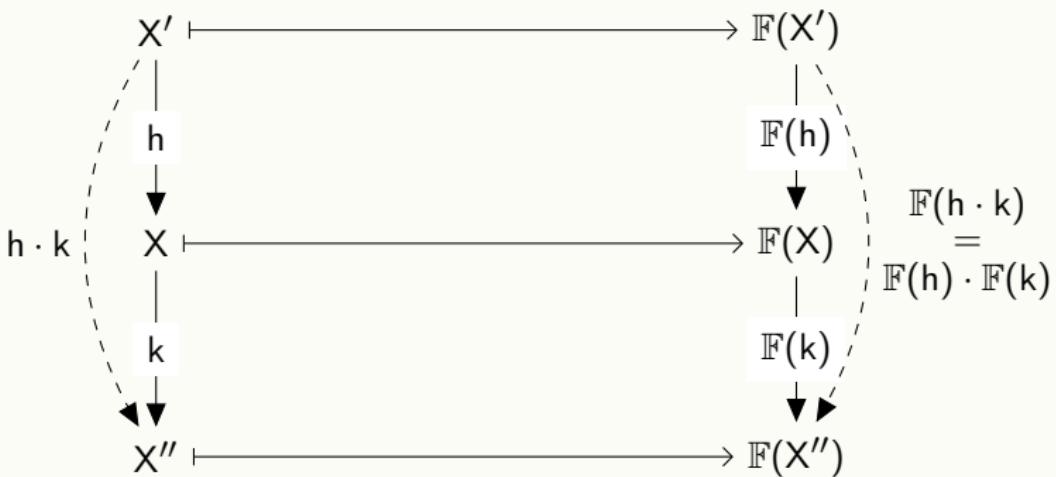
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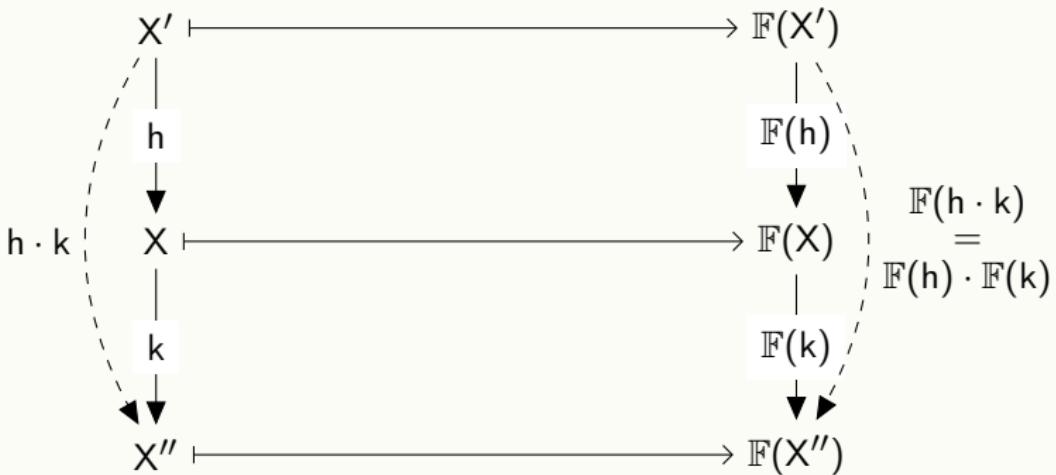
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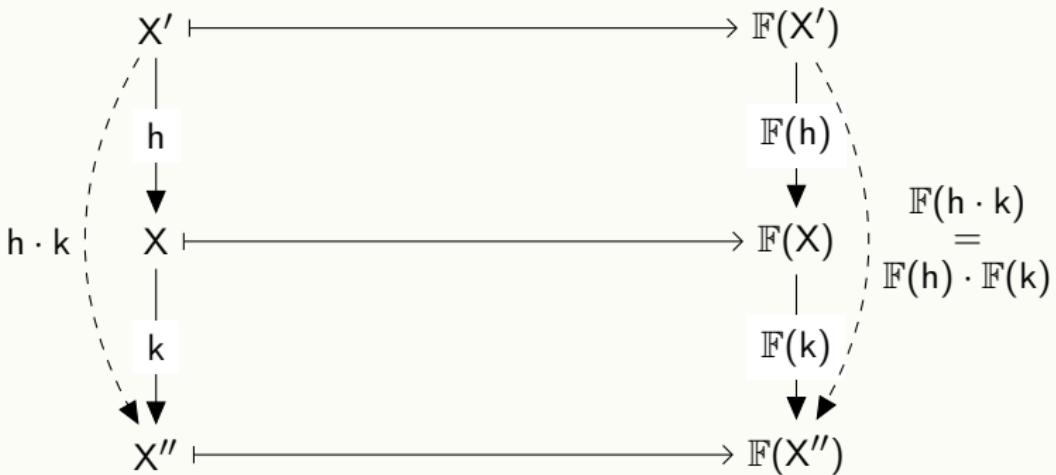
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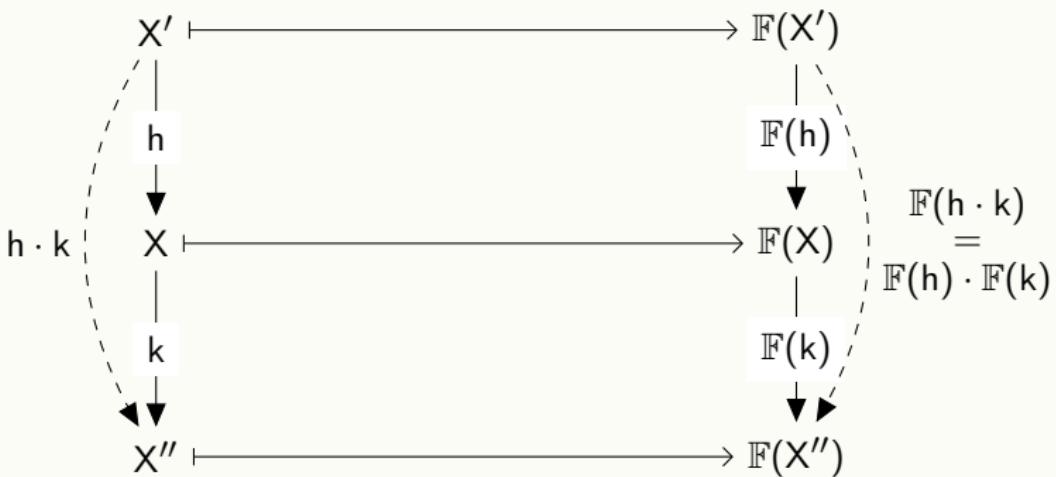
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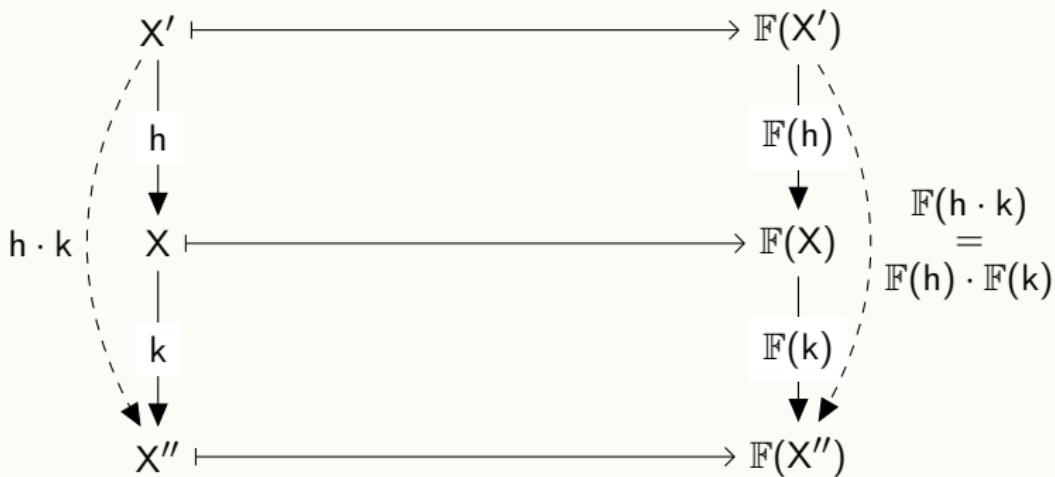
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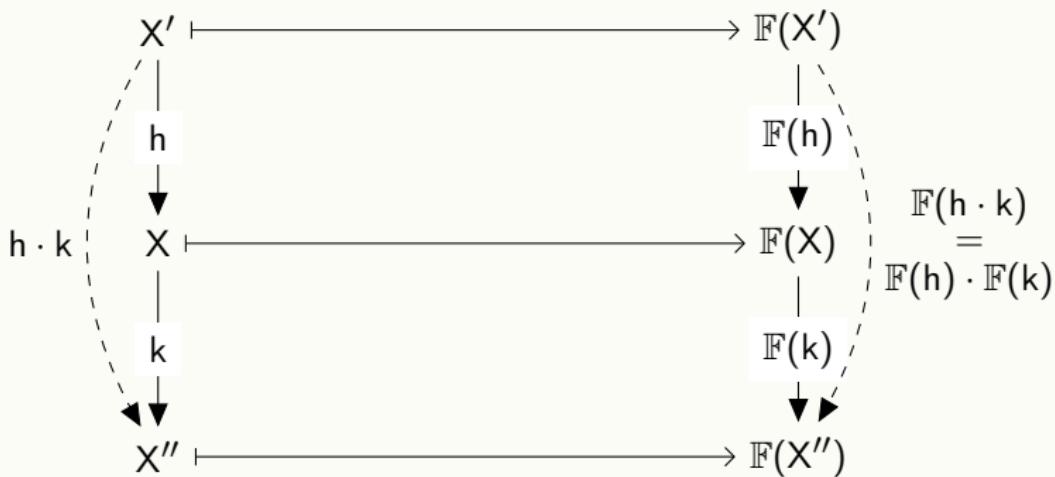
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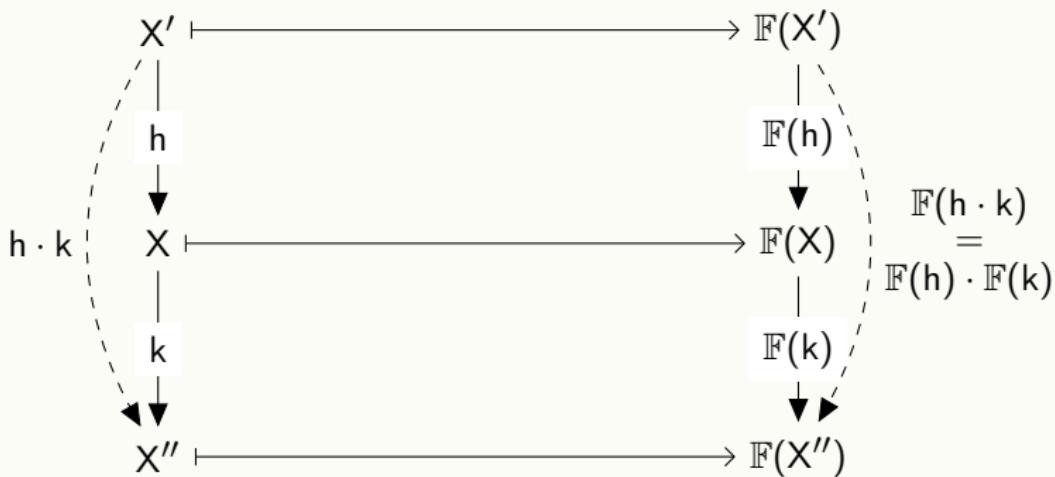
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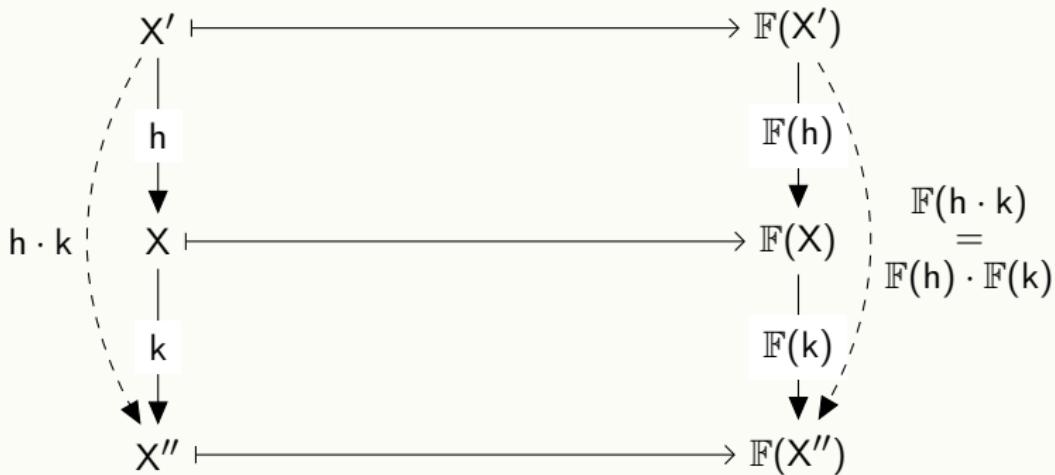
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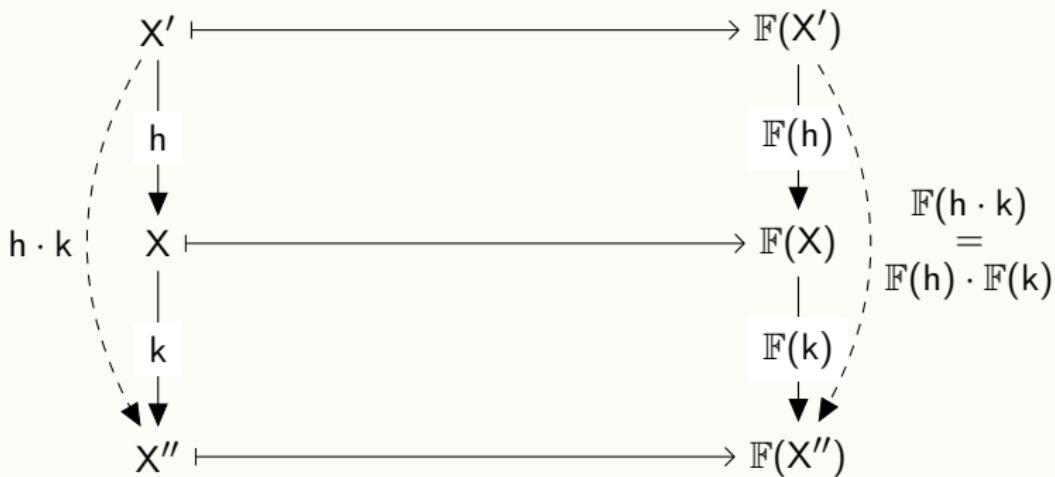
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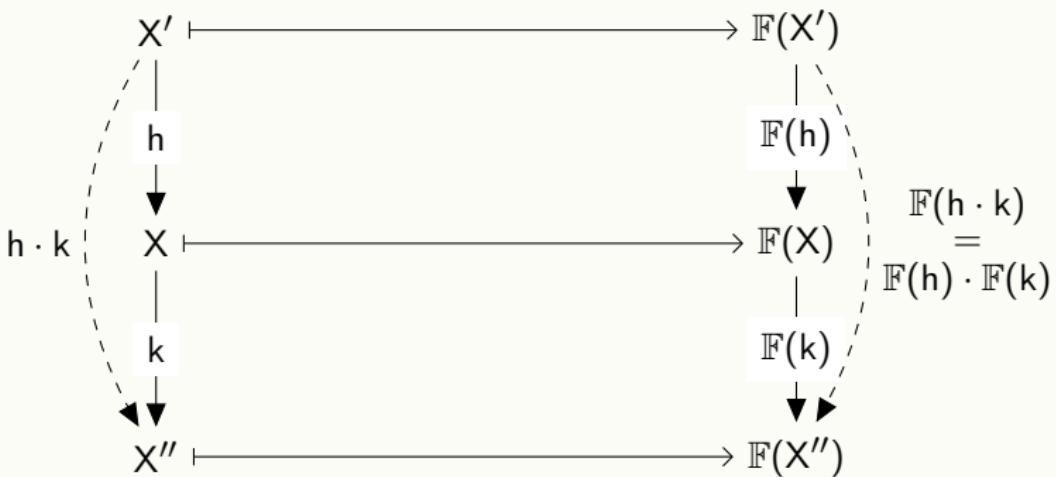
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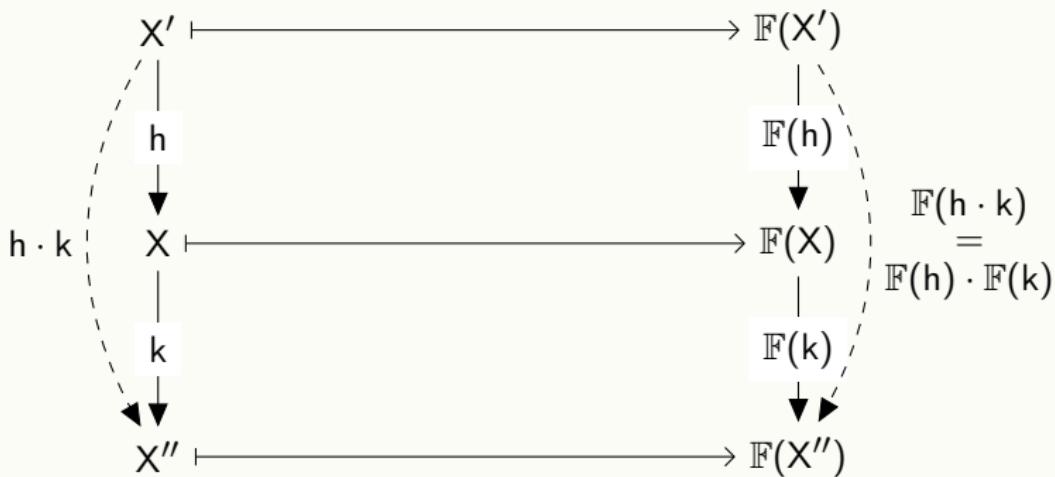
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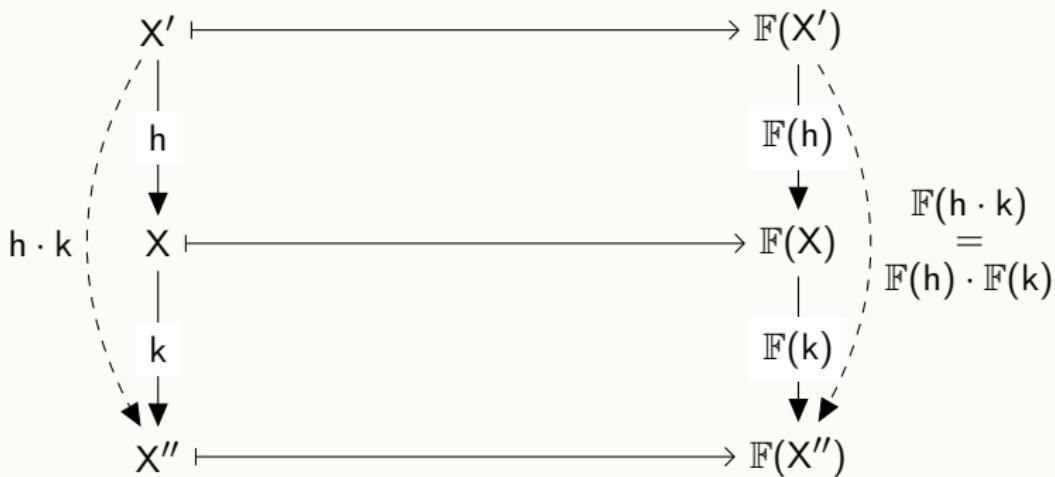
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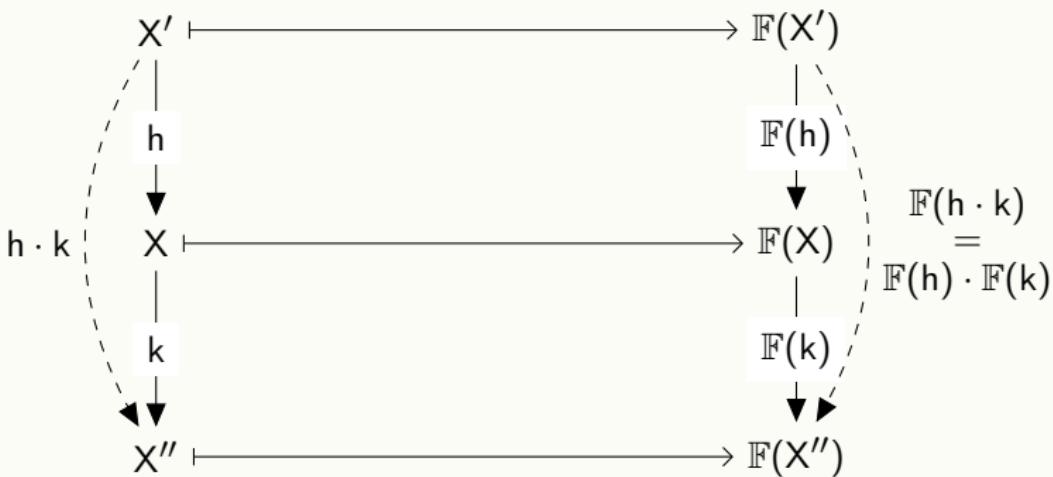
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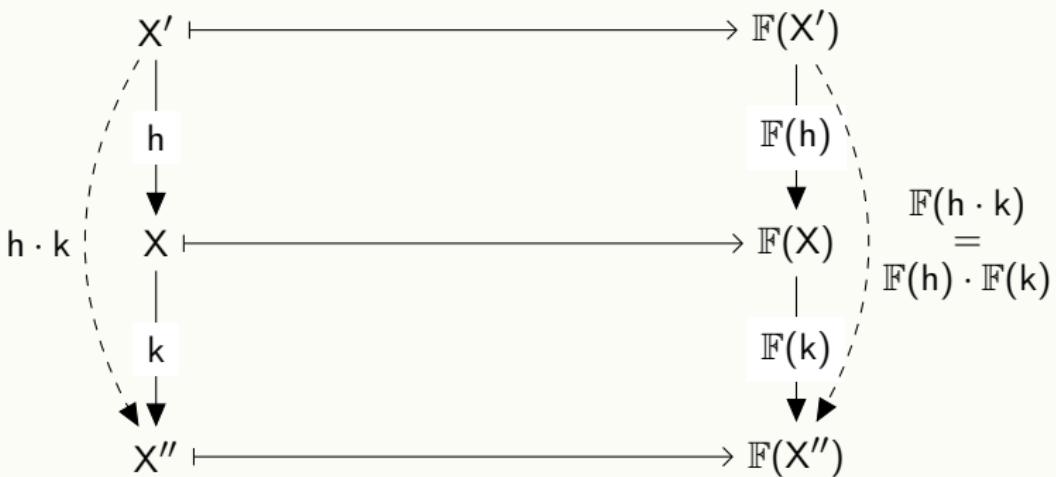
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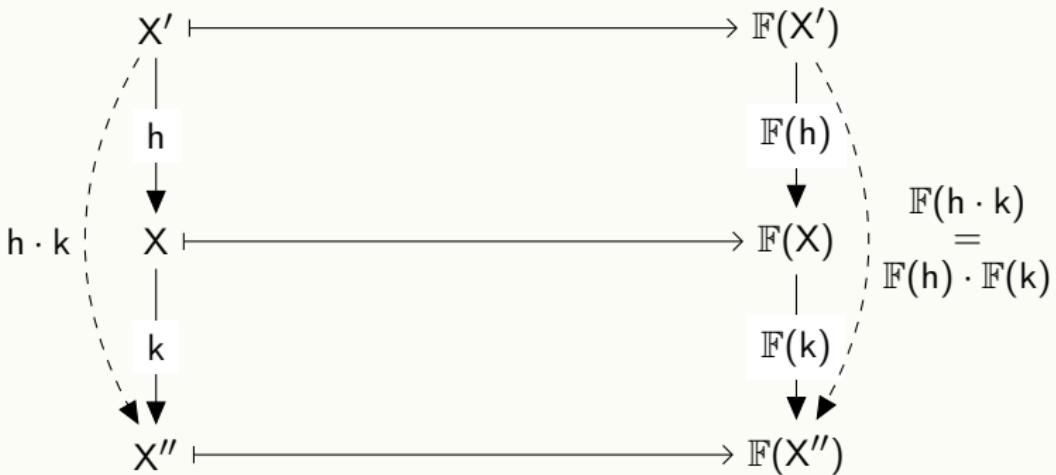
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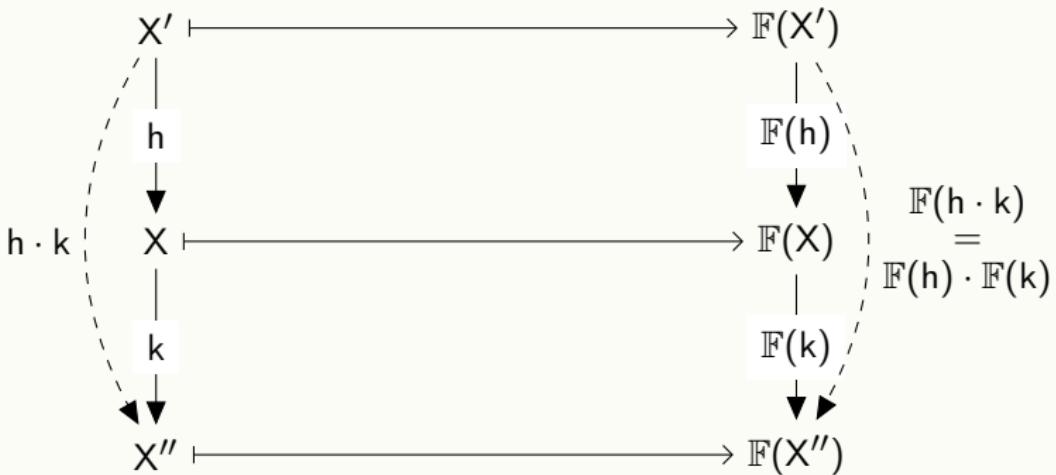
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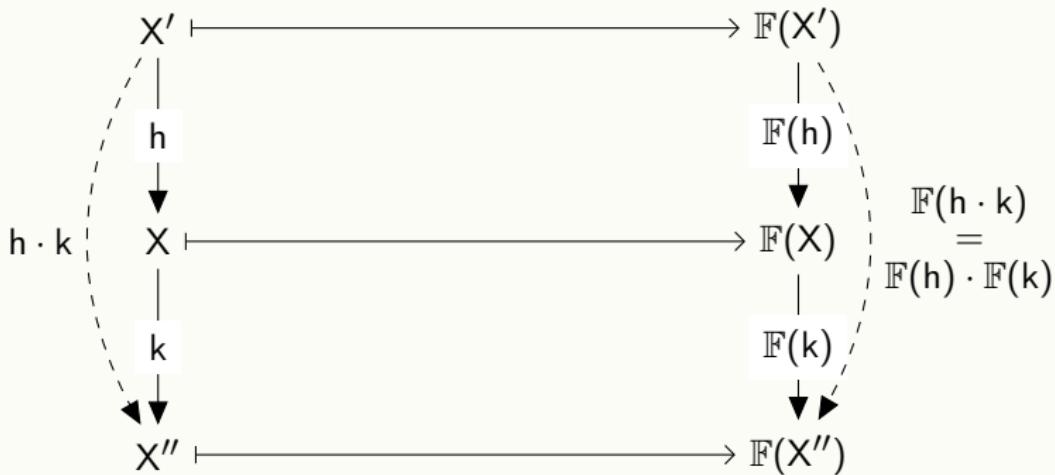
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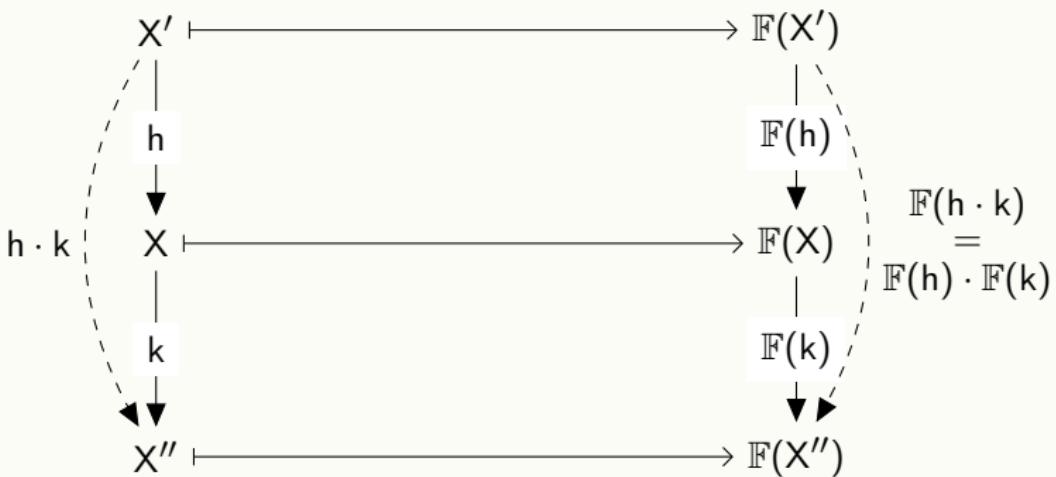
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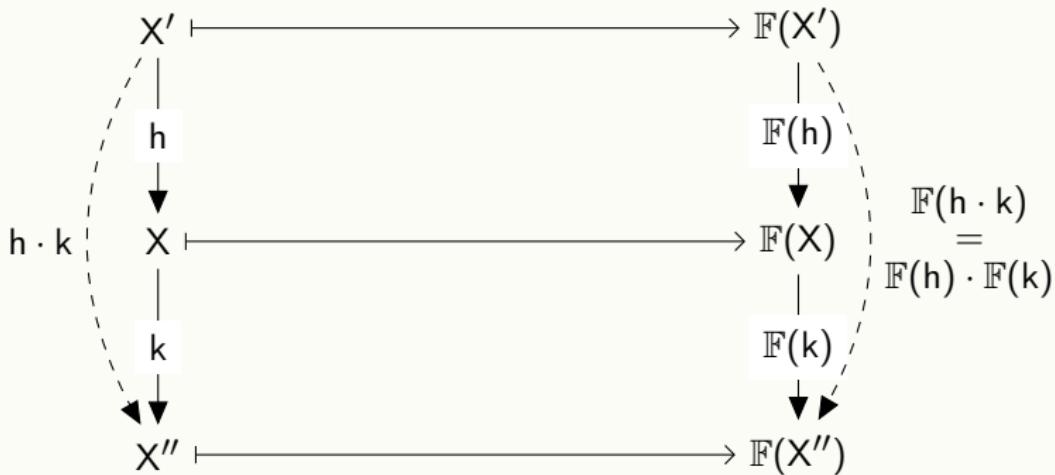
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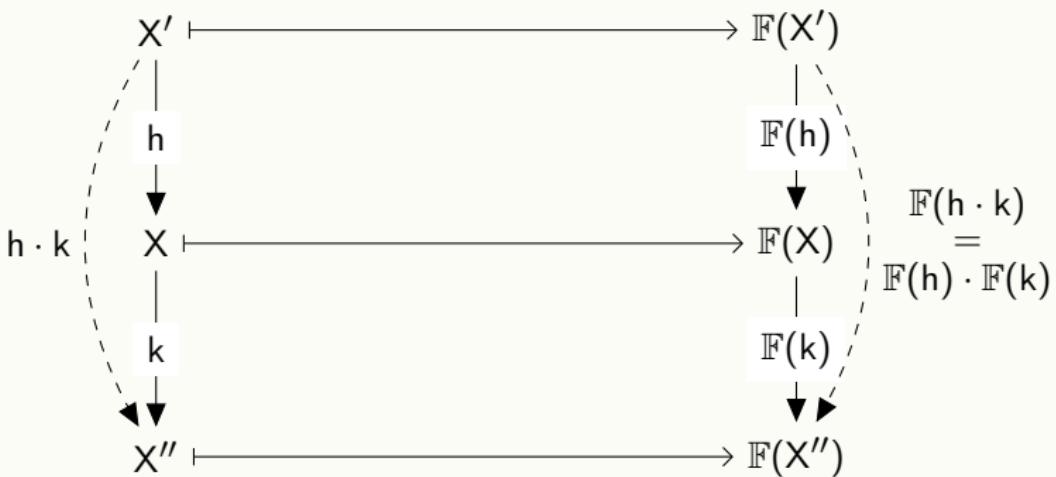
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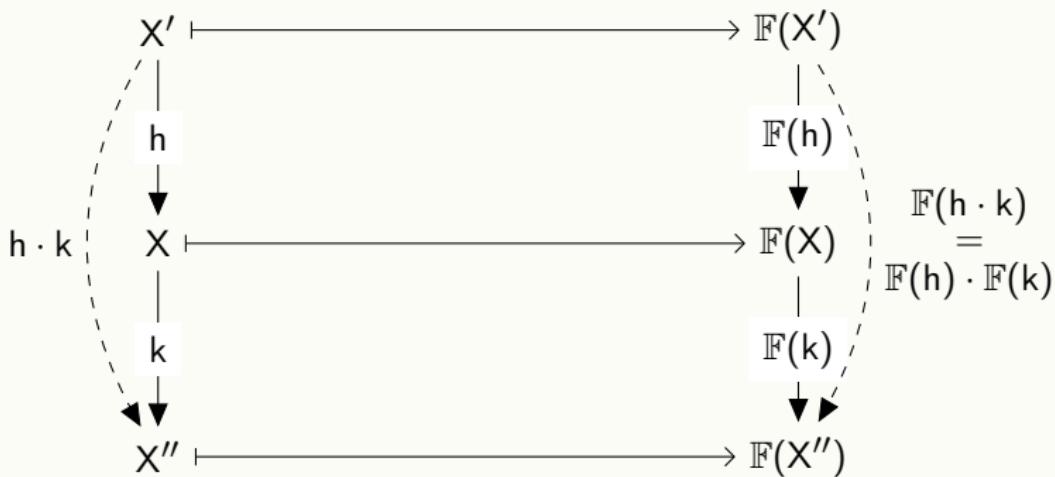
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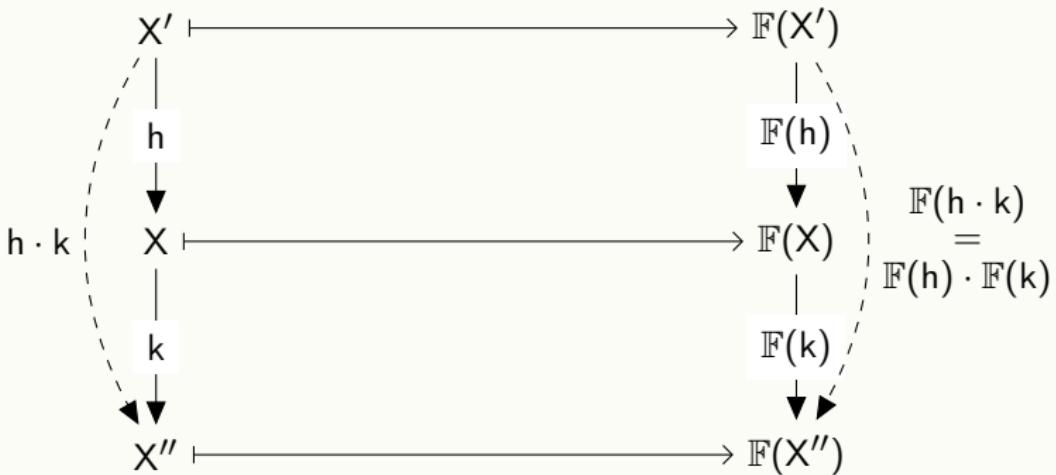
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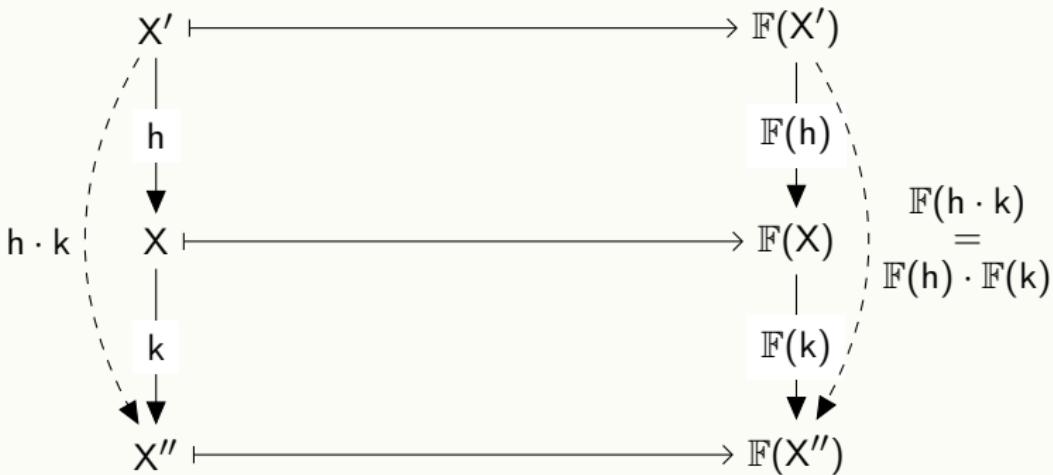
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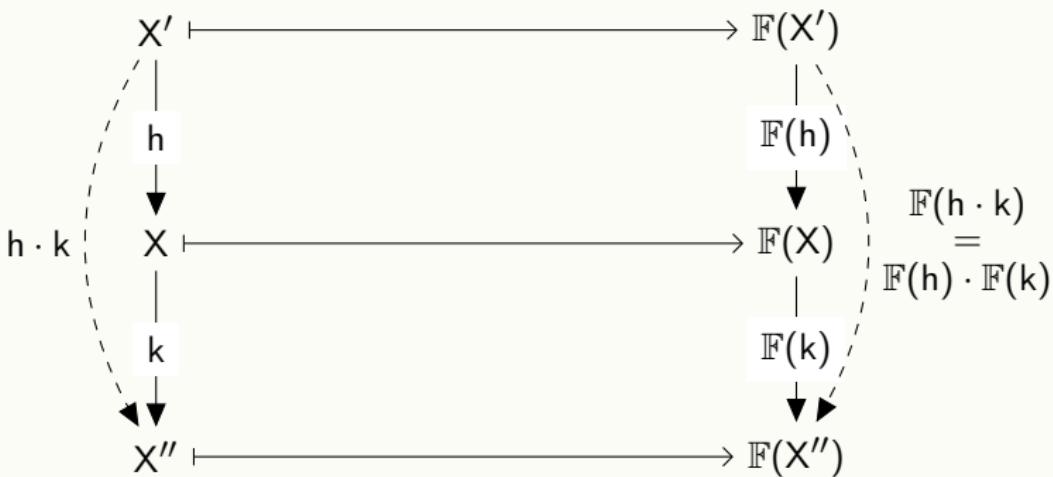
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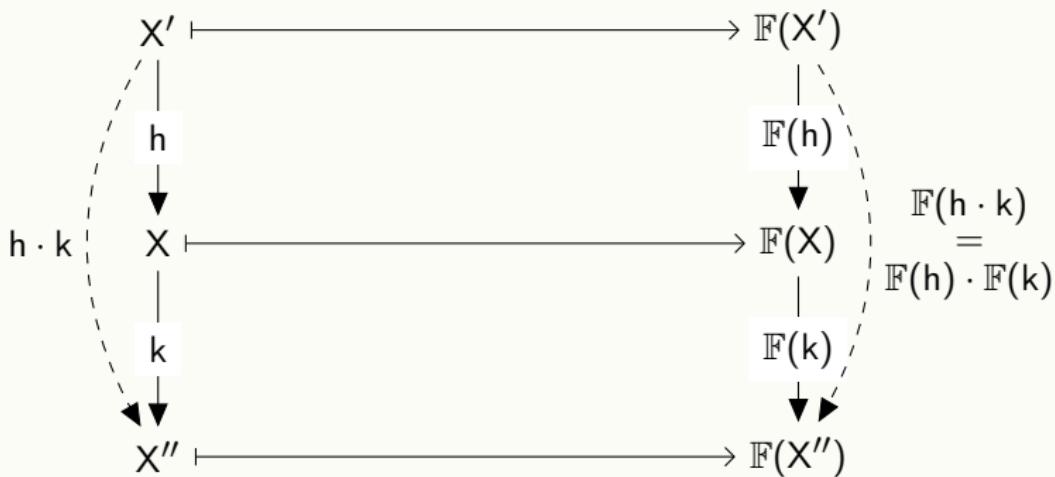
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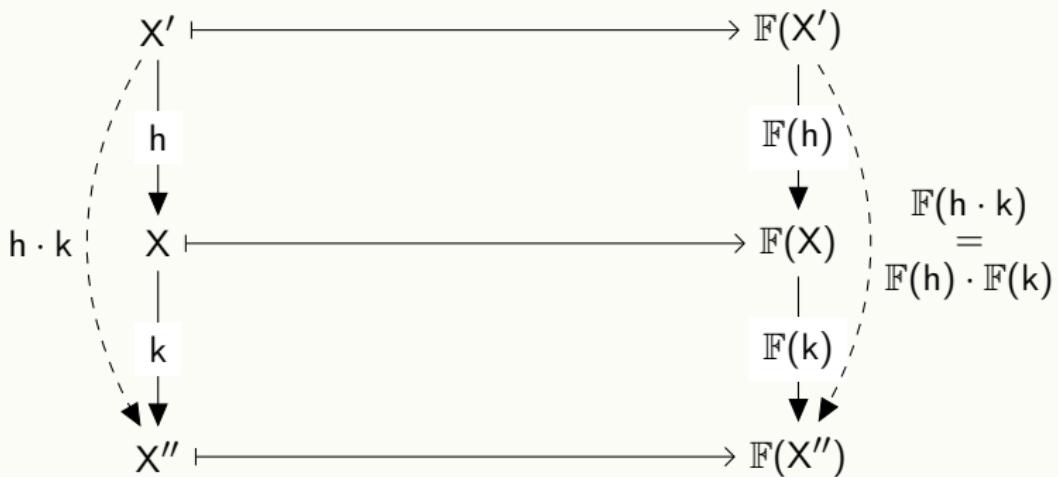
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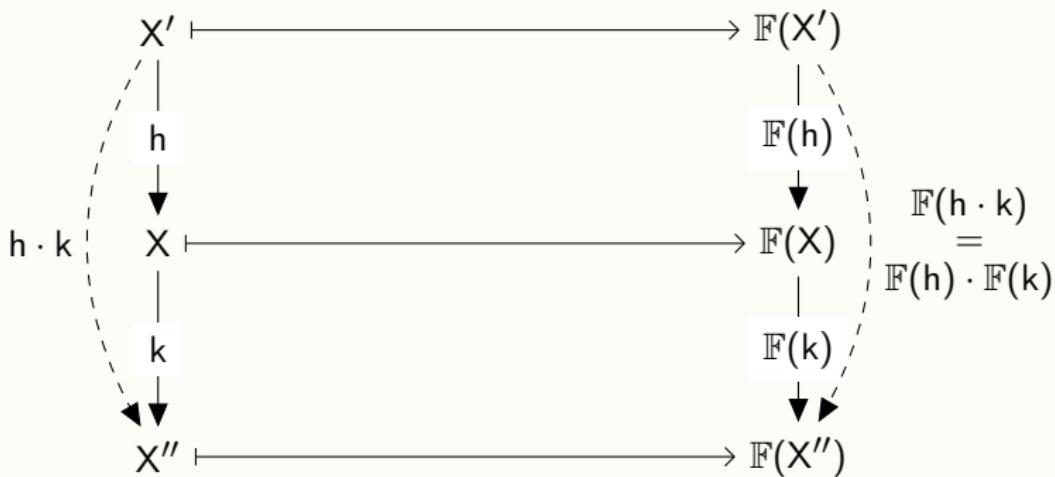
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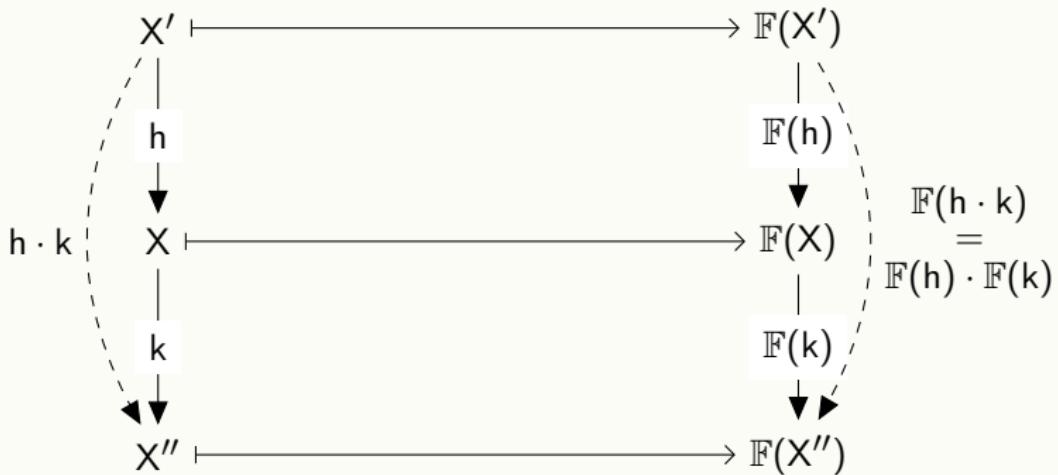
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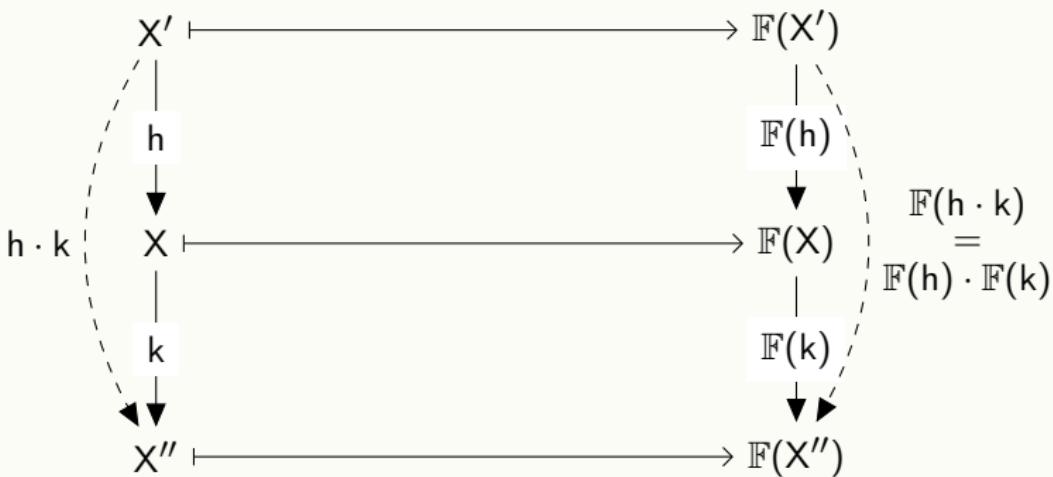
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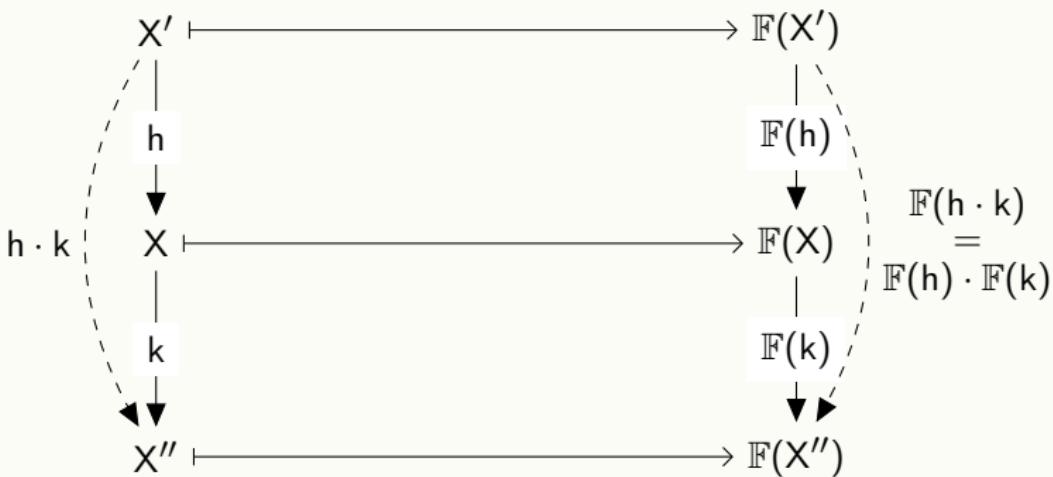
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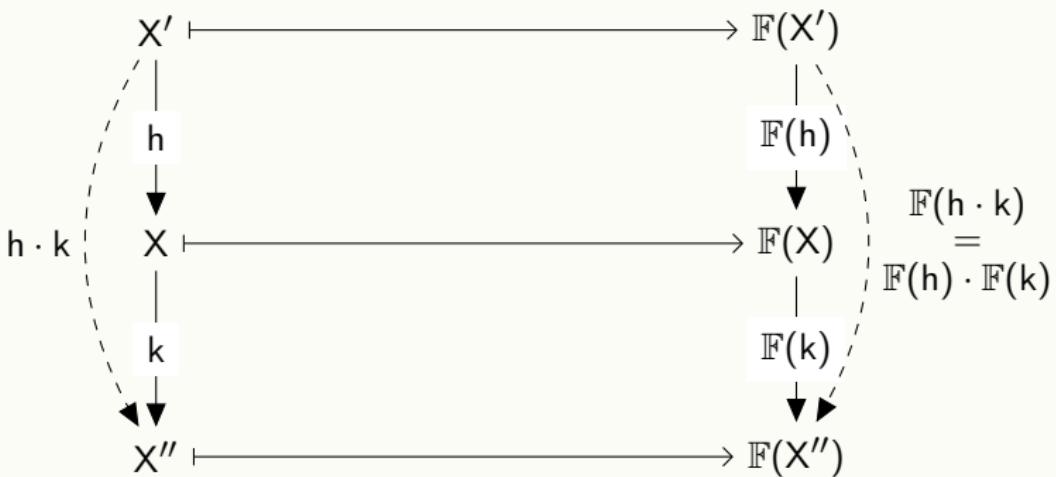
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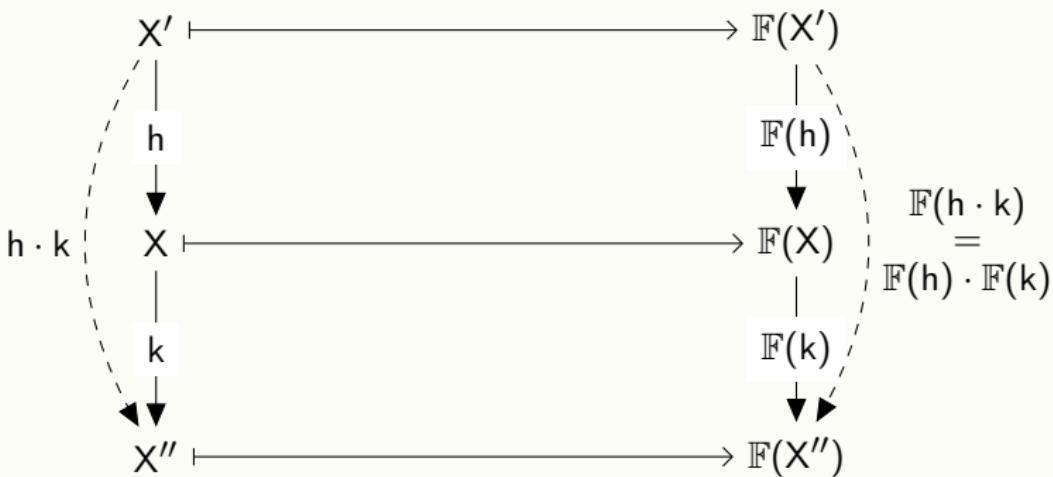
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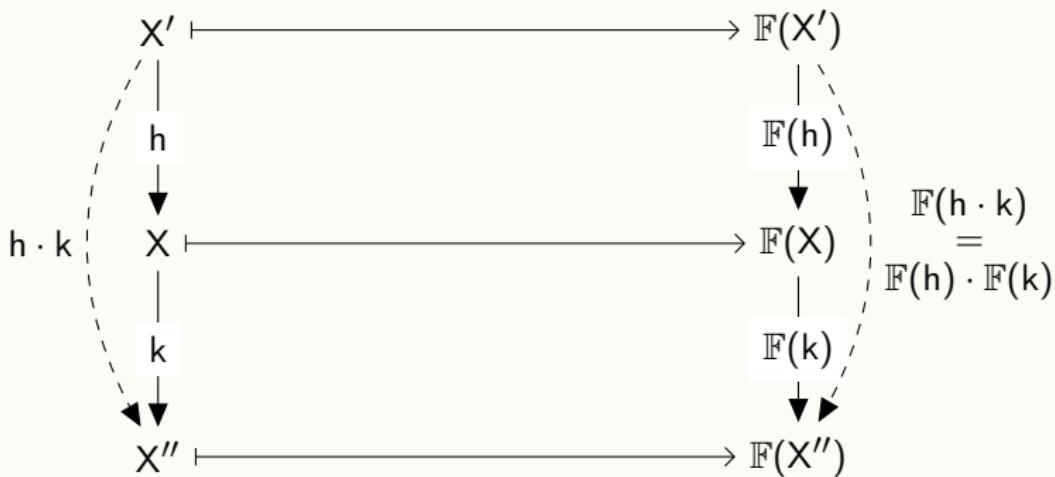
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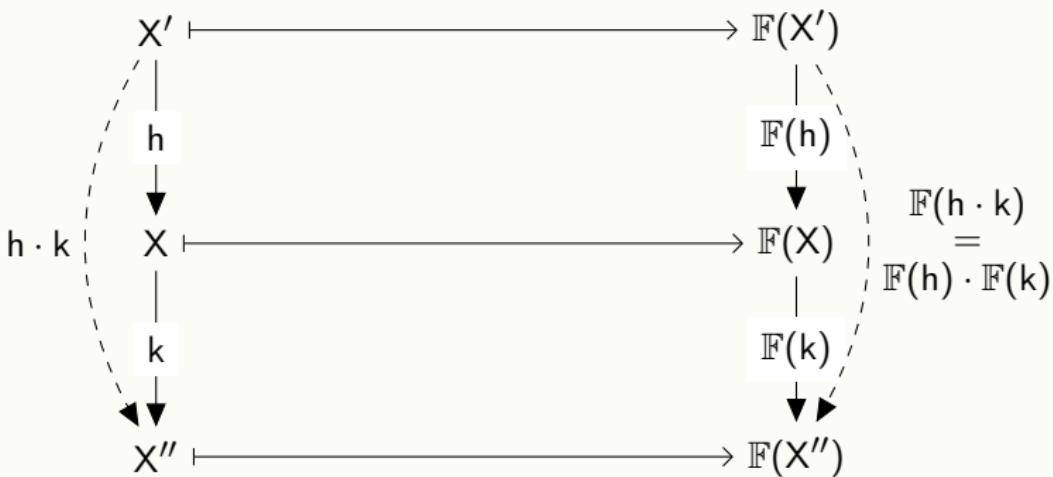
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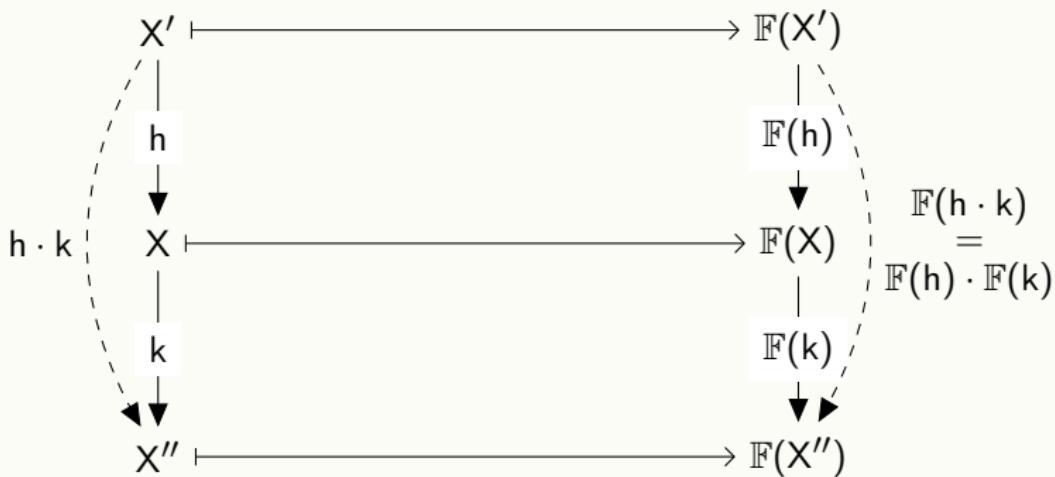
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Composition Eq. $\mathbb{F}(h \cdot k) = \mathbb{F}(h) \cdot \mathbb{F}(k)$

Picturing a Functor

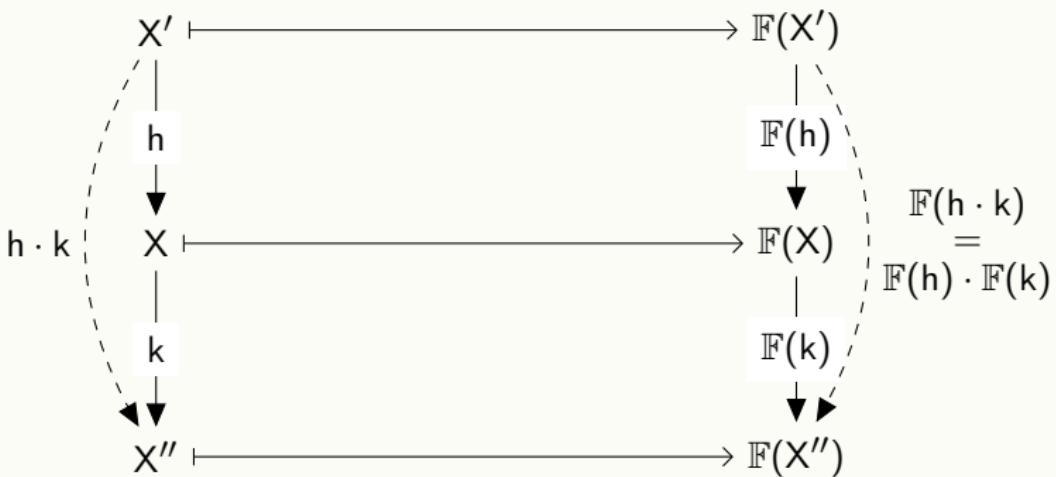
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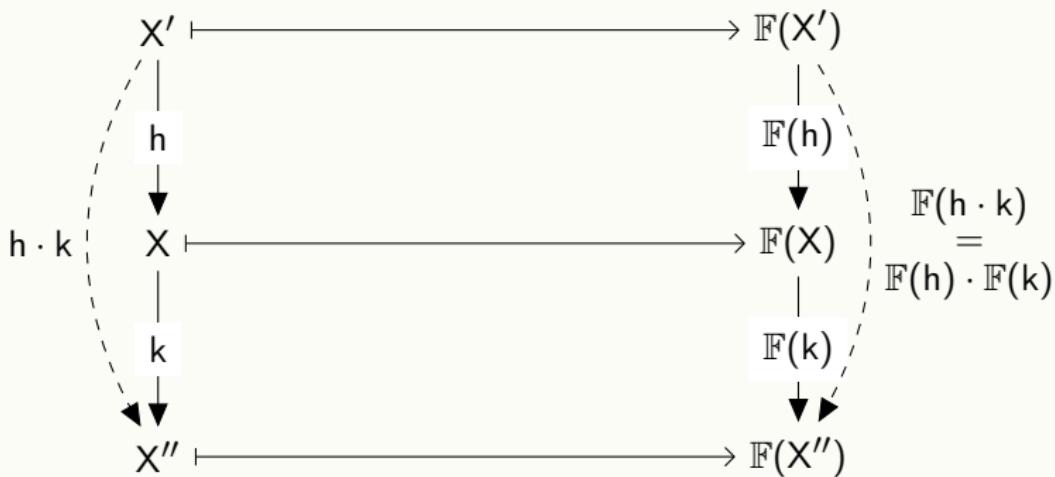
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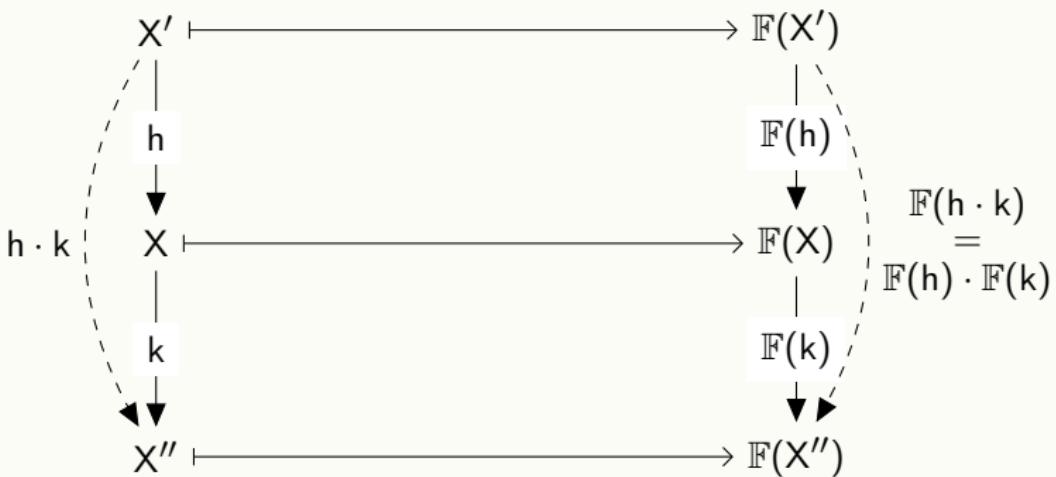
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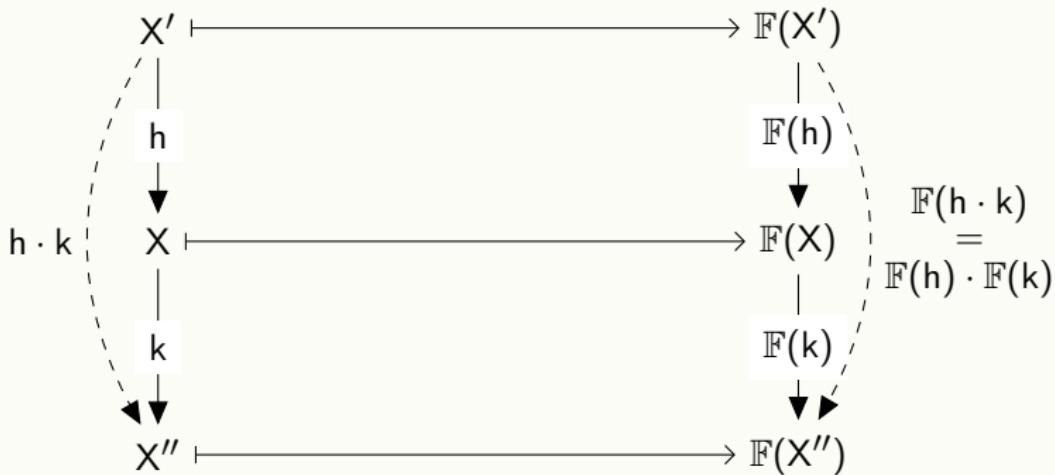
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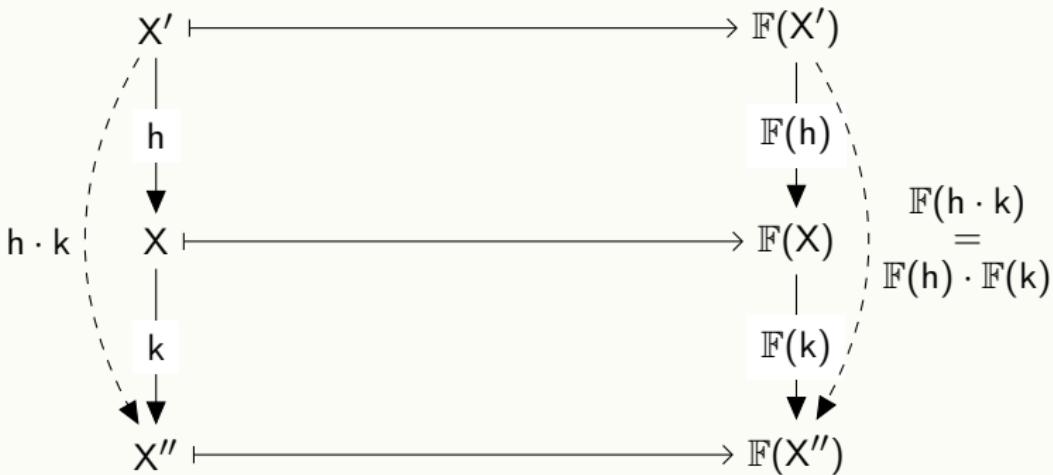
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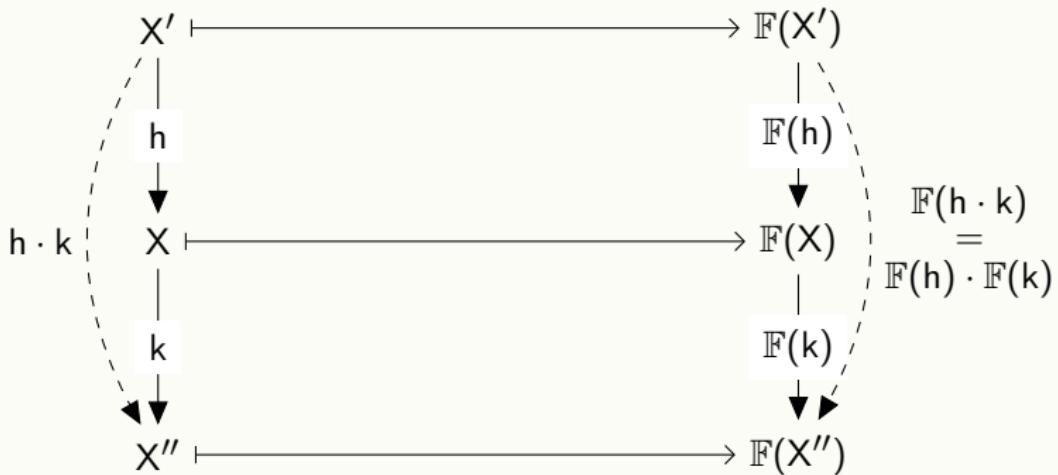
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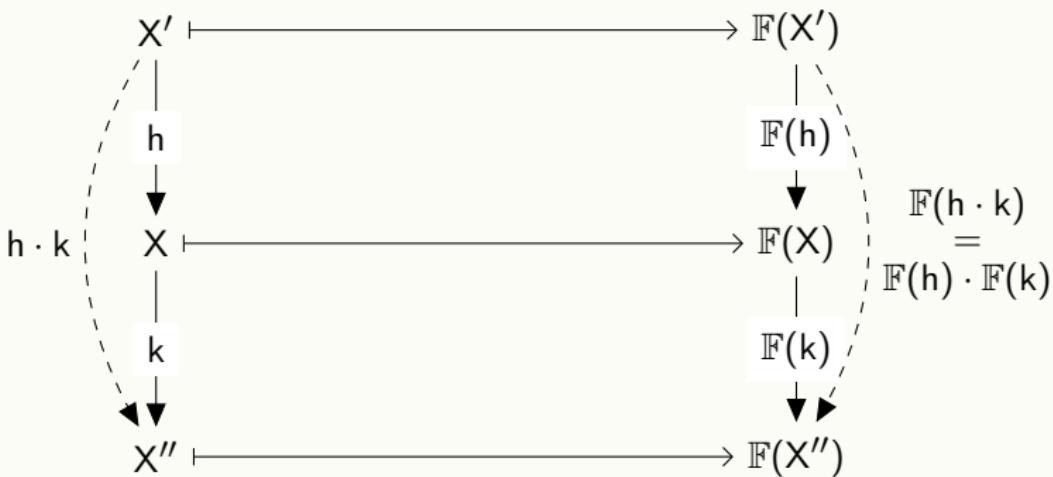
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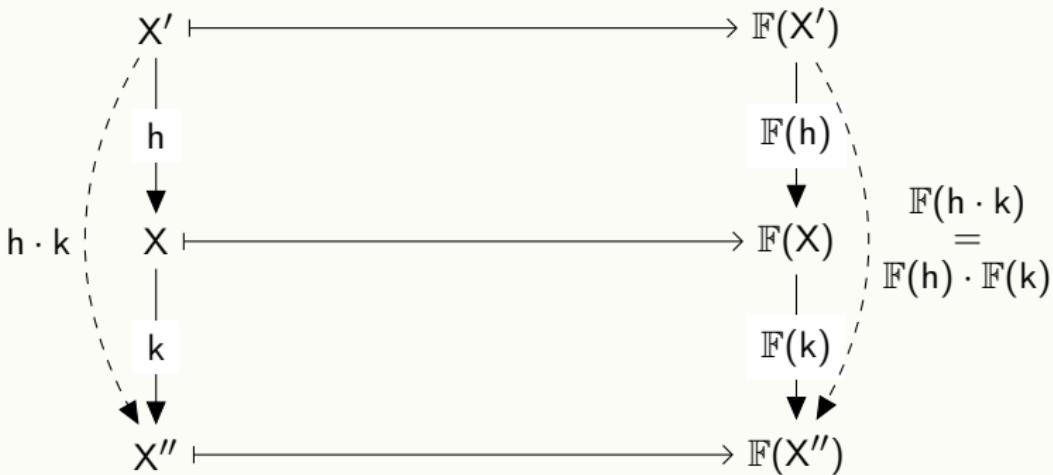
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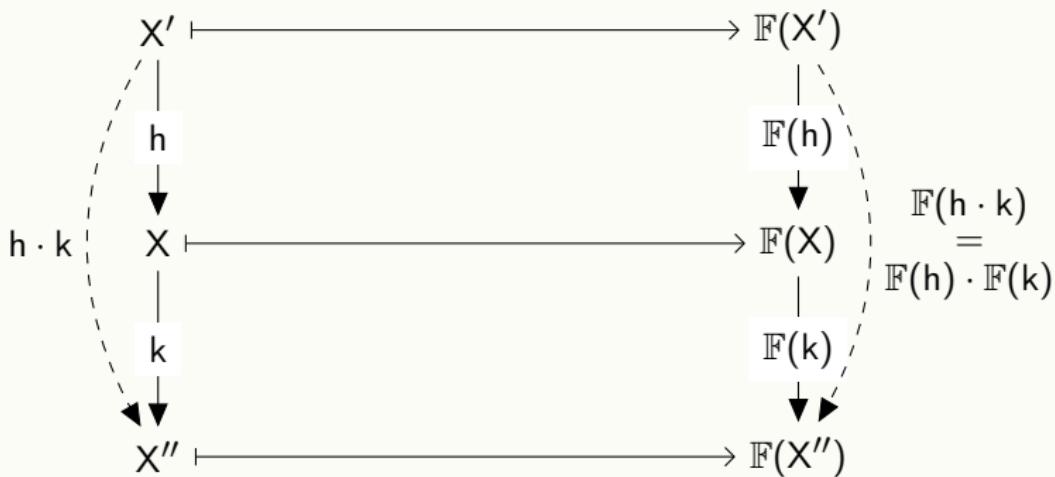
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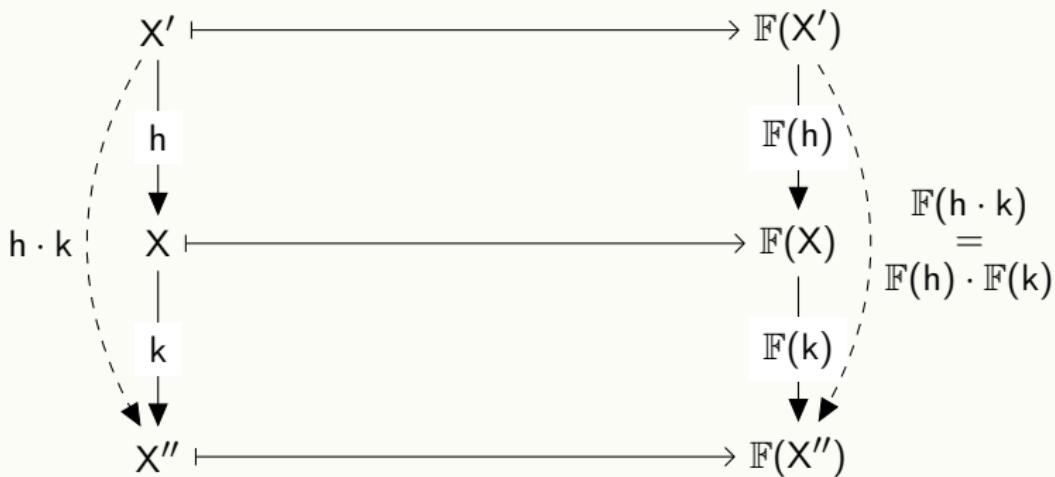
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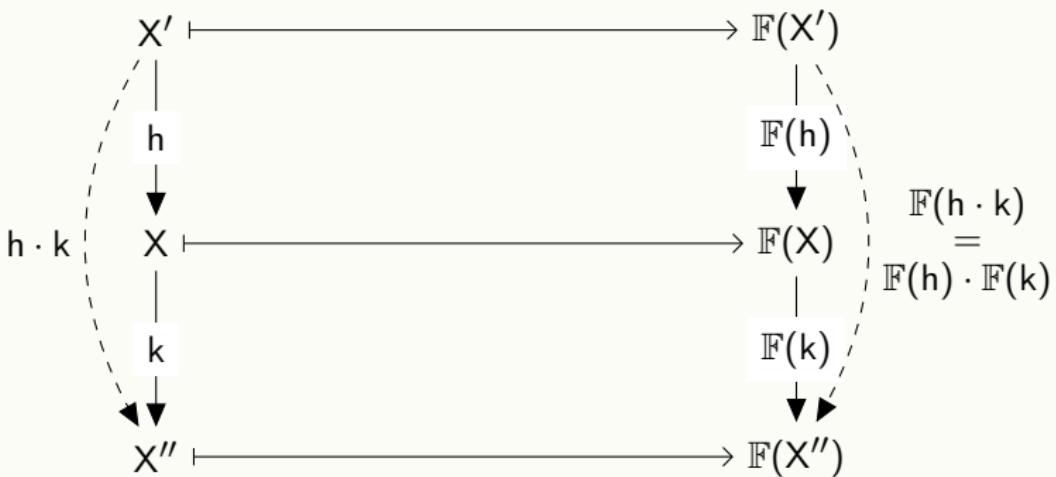
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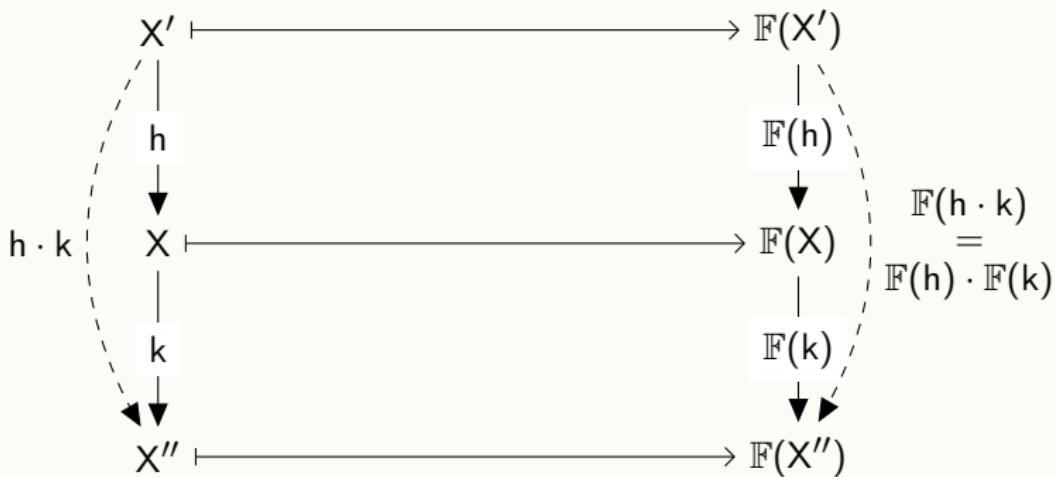
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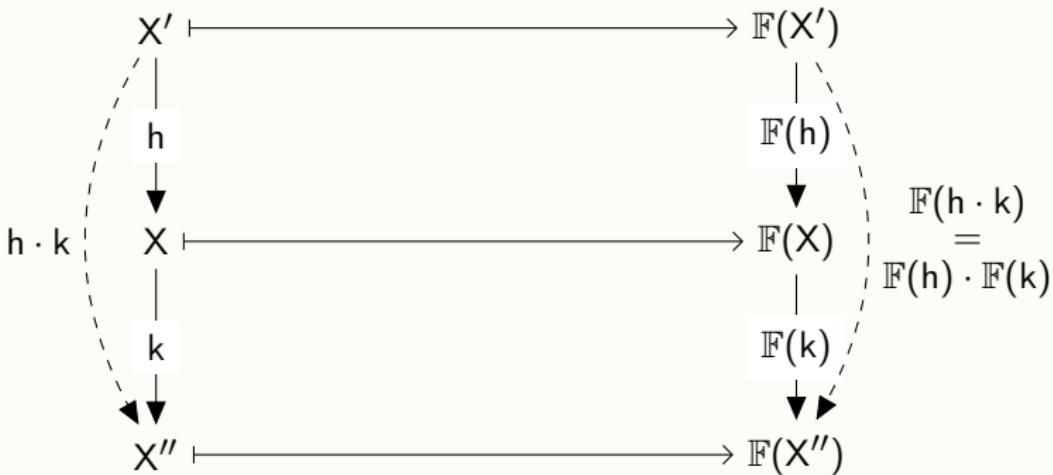
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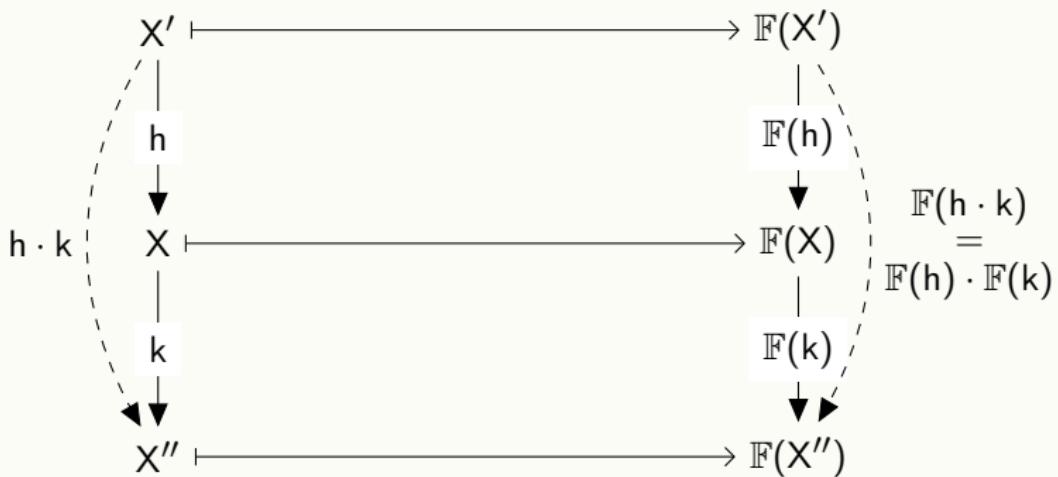
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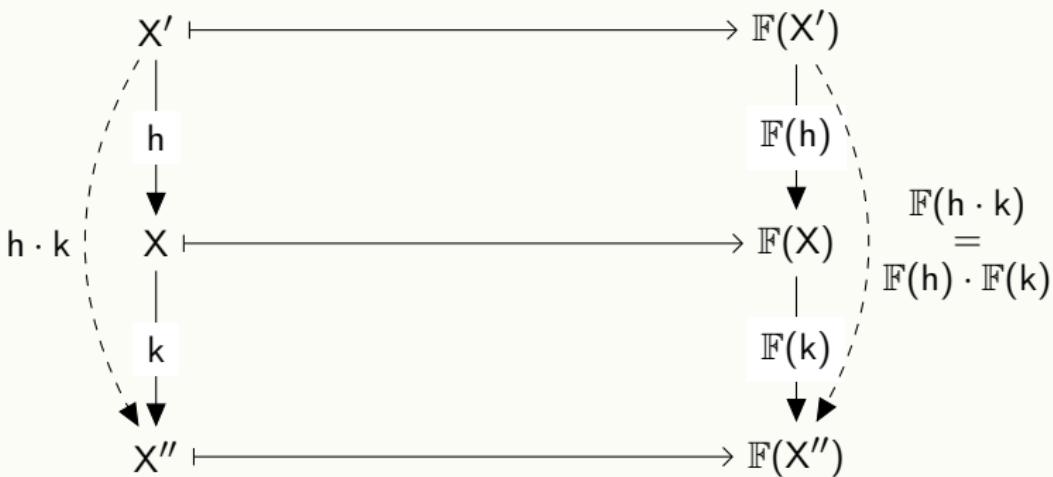
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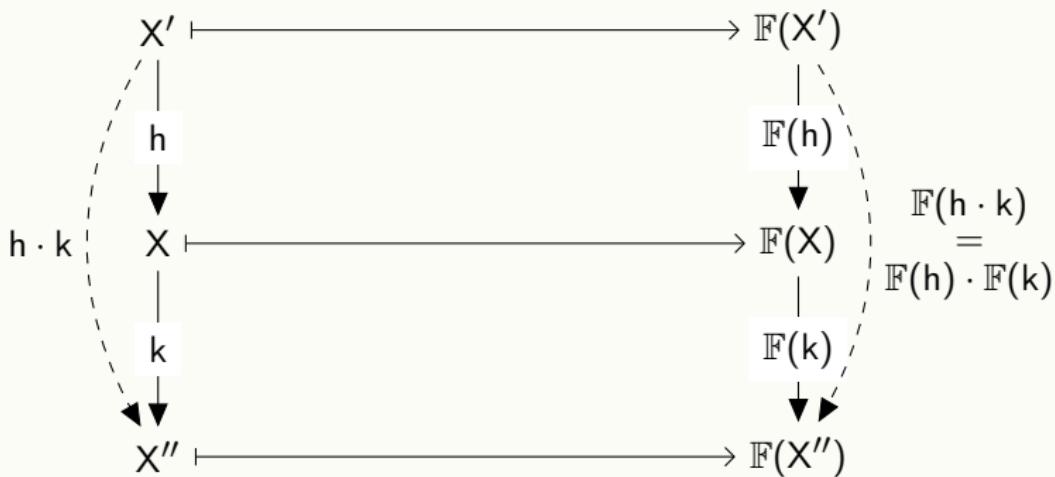
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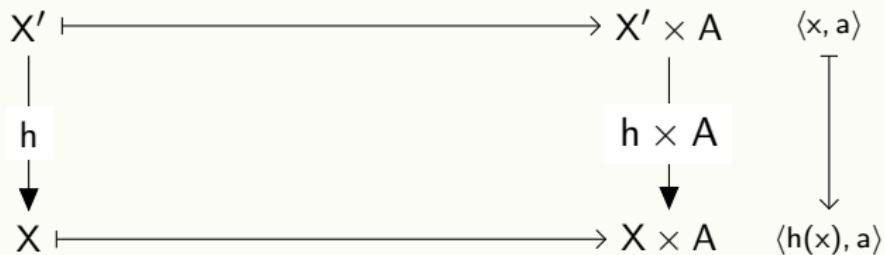
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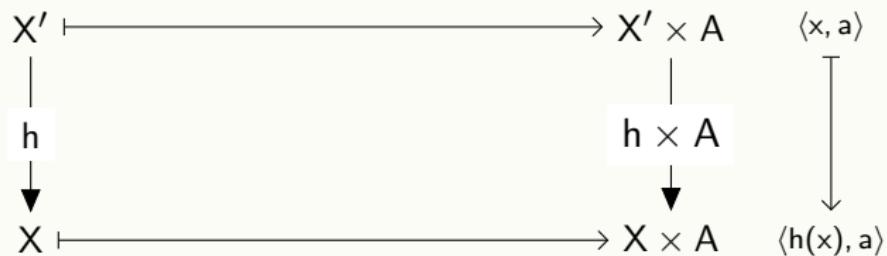
Some functors: Products

$\text{Set} \xrightarrow{- \times A} \text{Set}$



Some functors: Products

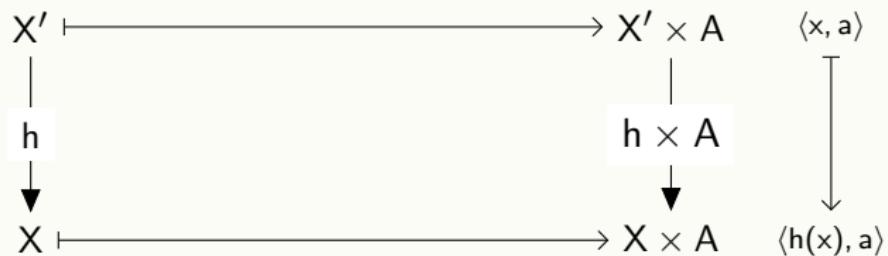
$\text{Set} \xrightarrow{- \times A} \text{Set}$



Object function: $X \mapsto X \times A$

Some functors: Products

$\text{Set} \xrightarrow{- \times A} \text{Set}$



Arrow function: $h \times A(x, a) := \langle h(x), a \rangle$

Some functors: Products

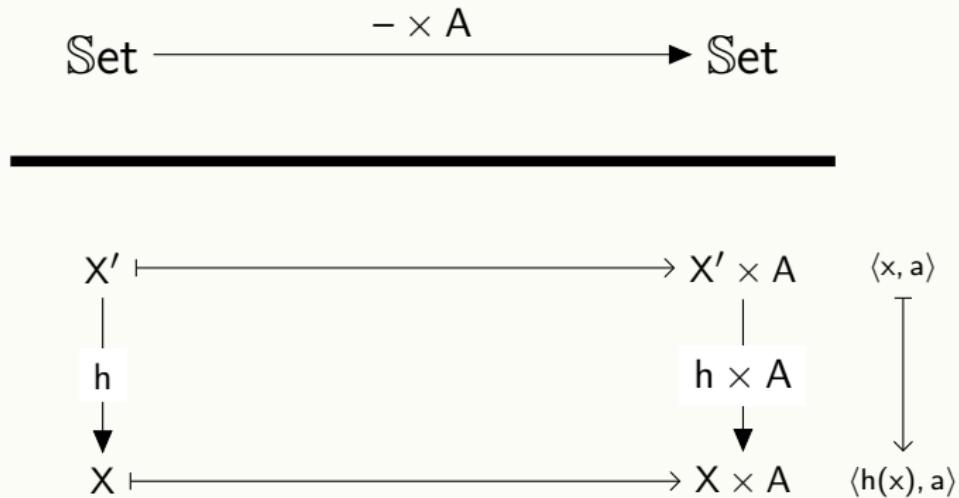
$\text{Set} \xrightarrow{- \times A} \text{Set}$



$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X' \times A & \langle x, a \rangle \\ | & & | & | \\ h & & h \times A & \downarrow \\ \downarrow & & \downarrow & \\ X & \xrightarrow{\quad} & X \times A & \langle h(x), a \rangle \end{array}$$

Identity Eq $\text{id}_{X \times A}(x, a) = \langle x, a \rangle = \langle \text{id}_X(x), a \rangle$
 $= \text{id}_X \times \text{id}_A(x, a)$

Some functors: Products



$$\begin{aligned} \text{Comp. Eq } f \cdot g \times A(x, a) &= \langle f \cdot g(x), a \rangle = \langle g(f(x)), a \rangle \\ &= g \times A(f(x), a) = g \times A(f \times A(x, a)) \\ &= (f \times A) \cdot (g \times A)(x, a) \end{aligned}$$

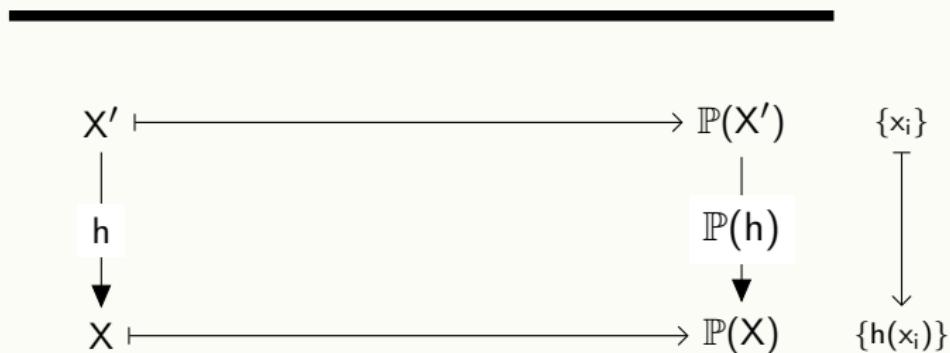
Some functors: Squaring

$$\text{Set} \xrightarrow{(-)^2} \text{Set}$$

$$\begin{array}{ccc} X' & \xrightarrow{\hspace{3cm}} & X' \times X' & \langle x_1, x_2 \rangle \\ \downarrow h & & \downarrow h^2 & \downarrow \\ X & \xrightarrow{\hspace{3cm}} & X \times X & \langle h(x_1), h(x_2) \rangle \end{array}$$

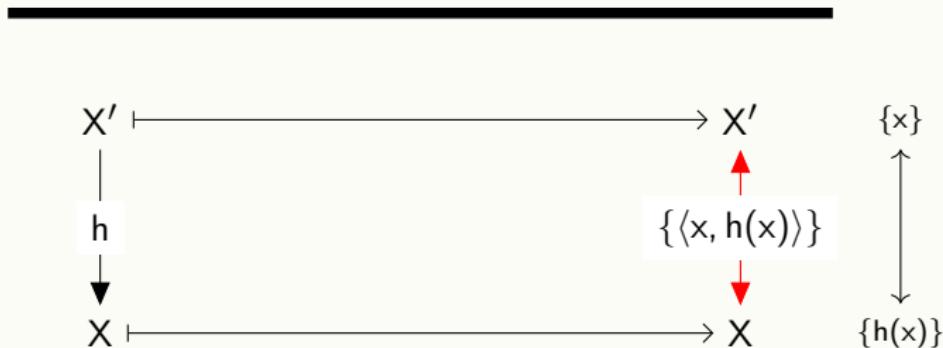
Some functors: Power set

$\text{Set} \xrightarrow{\mathbb{P} = 2^{(-)}} \text{Set}$



Some functors: Inclusion

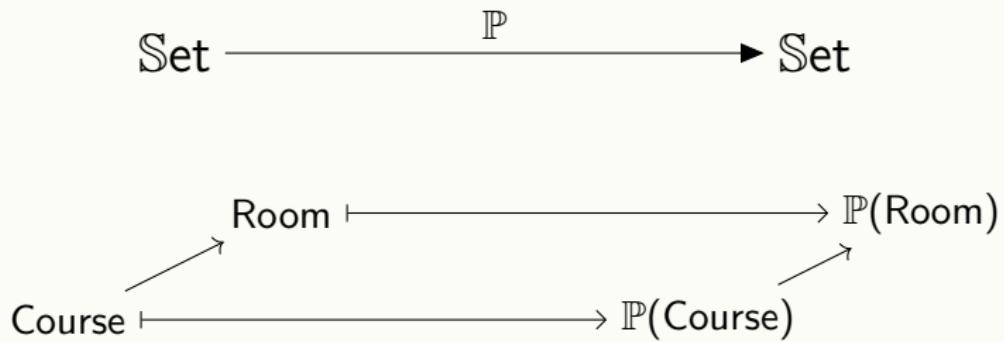
$$\text{Set} \xrightarrow{\quad \mathbb{I} \quad} \text{Rel}$$



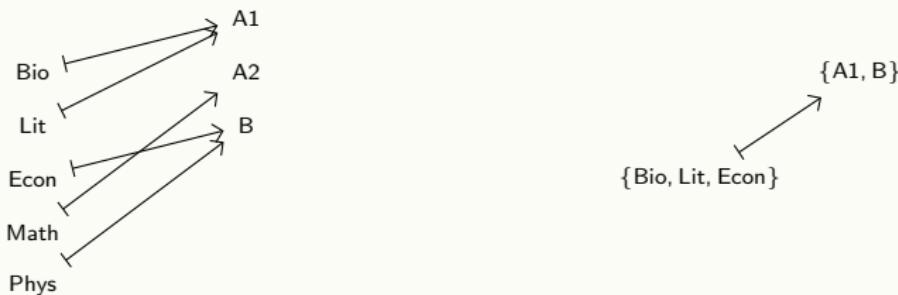
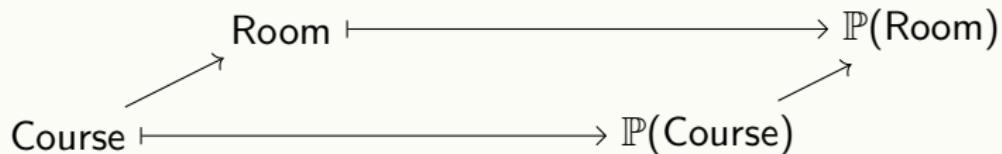
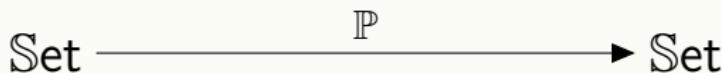
Three levels of abstraction

$$\text{Set} \xrightarrow{\mathbb{P}} \text{Set}$$

Three levels of abstraction



Three levels of abstraction



Three levels of abstraction

$$\text{Set} \xrightarrow{\mathbb{P}} \text{Set}$$

$$\begin{array}{ccc} \text{Room} & \xleftarrow{\quad} & \mathbb{P}(\text{Room}) \\ \swarrow & & \searrow \\ \text{Course} & \xrightarrow{\quad} & \mathbb{P}(\text{Course}) \end{array}$$



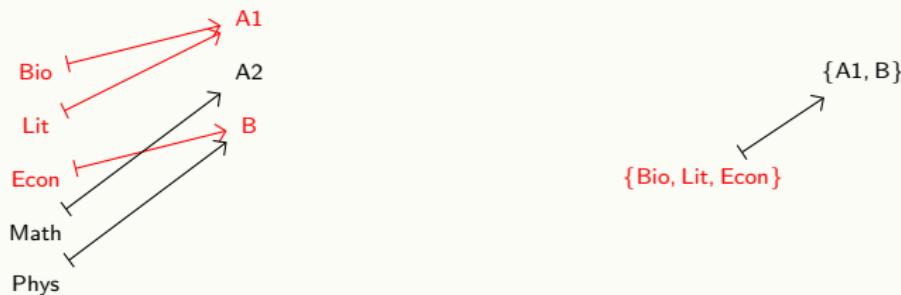
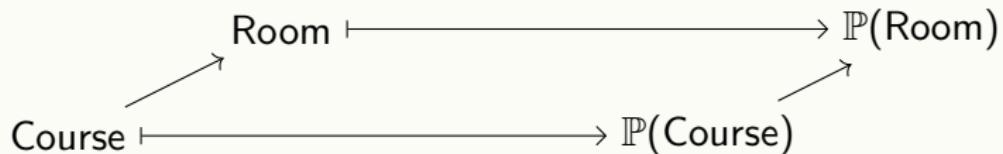
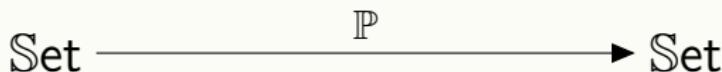
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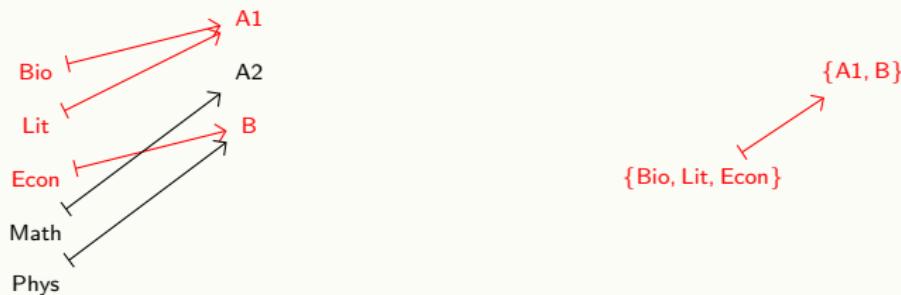
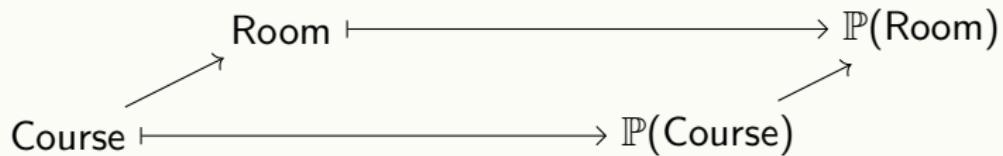
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Three levels of abstraction



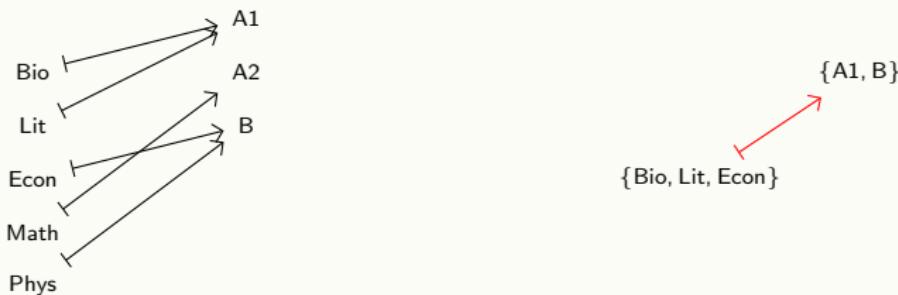
Three levels of abstraction



Three levels of abstraction

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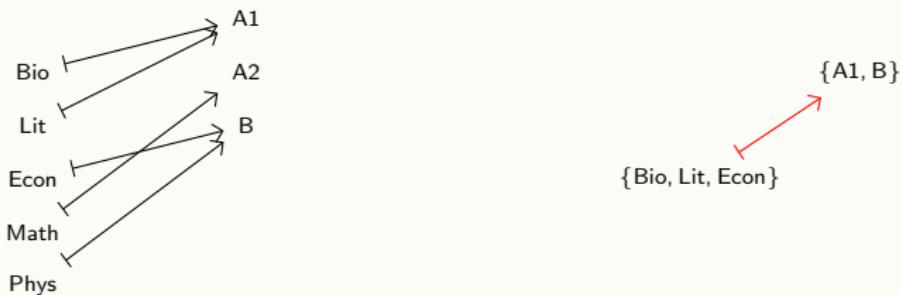
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Three levels of abstraction

$$\text{Set} \xrightarrow{\mathbb{P}} \text{Set}$$

$$\begin{array}{ccc} \text{Room} & \xleftarrow{\quad} & \mathbb{P}(\text{Room}) \\ \text{Course} & \xrightarrow{\quad} & \mathbb{P}(\text{Course}) \end{array}$$



Questions?

Cat

The category $\mathbb{C}\mathbf{at}$

Objects Categories $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \dots$

Arrows Functors $\mathbb{F} : \mathbb{X} \rightarrow \mathbb{Y}, \mathbb{G} : \mathbb{Y} \rightarrow \mathbb{Z}, \dots$

Identities $\text{id}(\mathbb{X}) = \mathbb{X}, \text{id}(h) = h : \mathbb{X} \rightarrow \mathbb{X}$

Composition $\mathbb{F} \cdot \mathbb{G}(-) = \mathbb{G}(\mathbb{F}(-)) : \mathbb{X} \rightarrow \mathbb{Z}$

Unit From $\mathbb{S}\mathbf{et}$

Associativity From $\mathbb{S}\mathbf{et}$

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Unit From Set

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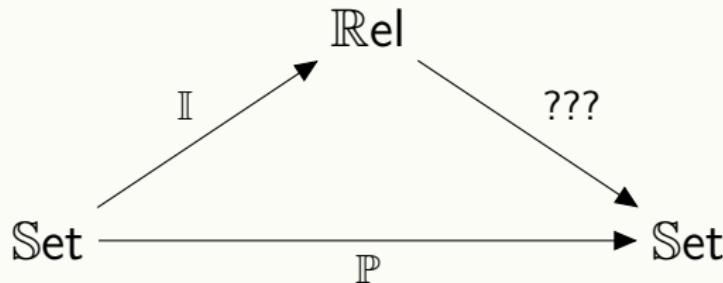
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Unit From Set

Associativity From Set

Lifting problems



The category $\mathbb{X} \times \mathbb{Y}$

Objects Pairs $\langle X \in \mathbb{X}, Y \in \mathbb{Y} \rangle$

Arrows Pairs $\langle h : X \rightarrow X', k : Y \rightarrow Y' \rangle$

Identities $\text{id}_{\langle X, Y \rangle} := \langle \text{id}_X, \text{id}_Y \rangle$ $\langle X, Y \rangle \rightarrow \langle X, Y \rangle$

Composition $\langle h, k \rangle \cdot \langle h', k' \rangle$ $\langle X, Y \rangle \rightarrow \langle X'', Y'' \rangle$
 $:= \langle h \cdot h', k \cdot k' \rangle$

Unit From \mathbb{X} & \mathbb{Y}

Associativity From \mathbb{X} & \mathbb{Y}

Test categories

$$\mathbb{1} := \left\{ \begin{array}{c} \text{id} \\ \text{*} \end{array} \right\}$$


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$$\mathbb{1} := \left\{ \begin{array}{c} \text{id} \\ \text{*} \end{array} \right\}$$

$$\mathbb{1} \xrightarrow{\quad \overrightarrow{X} \quad} X$$



Object $X \in \mathbb{X}$

Test categories

$$\mathbb{1} := \left\{ \begin{array}{c} \text{id} \\ \text{---} \\ * \end{array} \right\}$$

$$\mathbb{1} \xrightarrow{\quad \overrightarrow{x} \quad} \text{Set}$$



Set X

Test categories

$$\mathbb{1} := \left\{ \begin{array}{c} \text{id} \\ \text{---} \\ * \end{array} \right\}$$

$$\mathbb{1} \xrightarrow{\quad \overrightarrow{x} \quad} \text{Rel}$$



Set X

Test categories

$$\mathcal{Z} := \left\{ 0 \xrightarrow{s} 1 \right\}$$

Test categories

$$\mathcal{D} := \left\{ 0 \xrightarrow{s} 1 \right\}$$

$$\mathcal{D} \xrightarrow{\vec{f}} \mathbb{X}$$

2 Objects: $X_0 = \vec{f}(0)$, $X_1 = \vec{f}(1)$

Test categories

$$\mathcal{D} := \left\{ 0 \xrightarrow{s} 1 \right\}$$

$$\mathcal{D} \xrightarrow{\vec{f}}$$

1 arrow: $f = \vec{f}(s) : X_0 \rightarrow X_1$

Test categories

$$\mathcal{Z} := \left\{ 0 \xrightarrow{s} 1 \right\}$$

$$\mathcal{Z} \xrightarrow{\vec{f}} \textcolor{red}{\mathbb{S}\text{et}}$$

Function $f : X_0 \rightarrow X_1$

Test categories

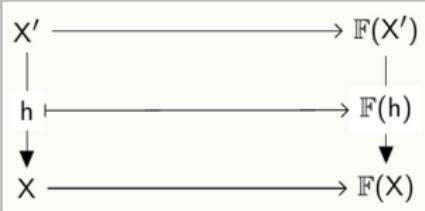
$$\mathcal{Z} := \left\{ 0 \xrightarrow{s} 1 \right\}$$

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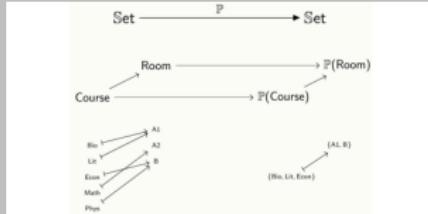
Relation $R : X \leftrightarrow X_1$

Questions?

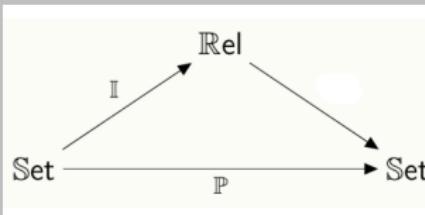
Functors



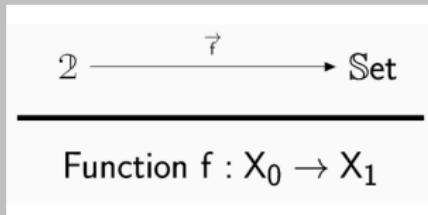
Abstraction



Cat



Test categories



Data

Categorical Data Model

Fixed **Semantics**: Set

Choose **Syntax**: Schema



Categorical Data Model

Fixed **Semantics**: Set

Choose **Syntax**: Schema

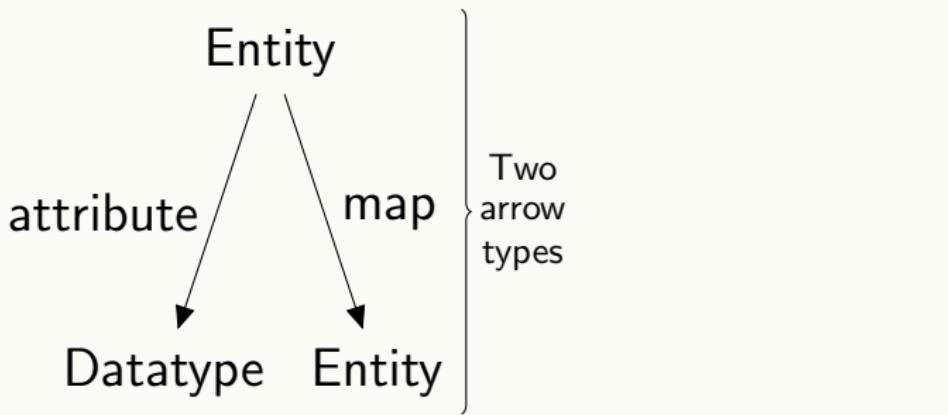


Two object types
Datatype Entity

Categorical Data Model

Fixed **Semantics**: Set

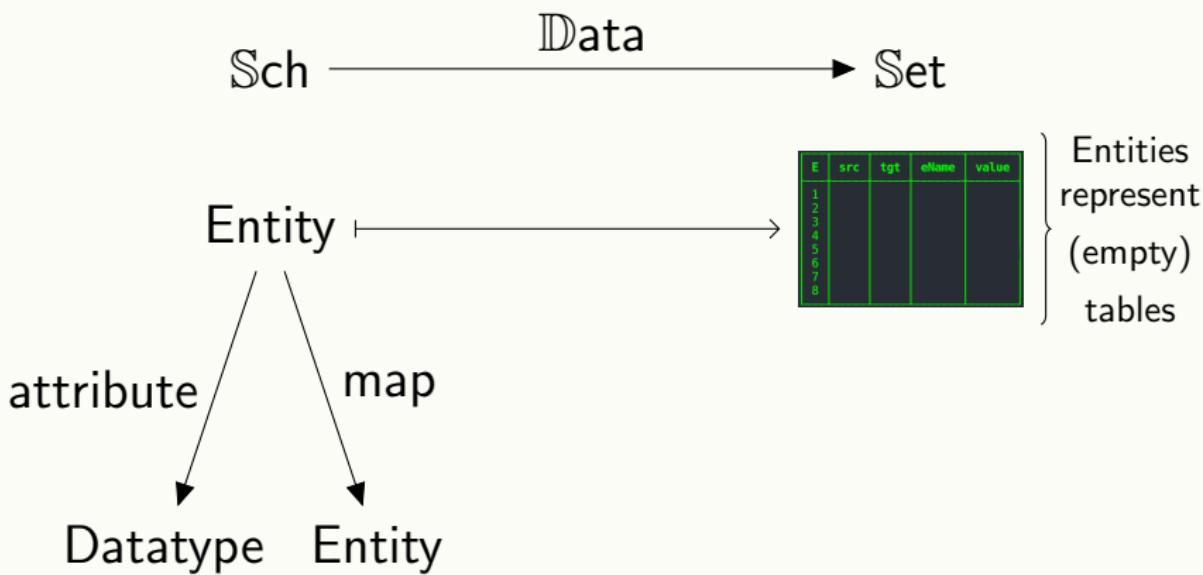
Choose **Syntax**: Schema



Categorical Data Model

Fixed **Semantics**: Set

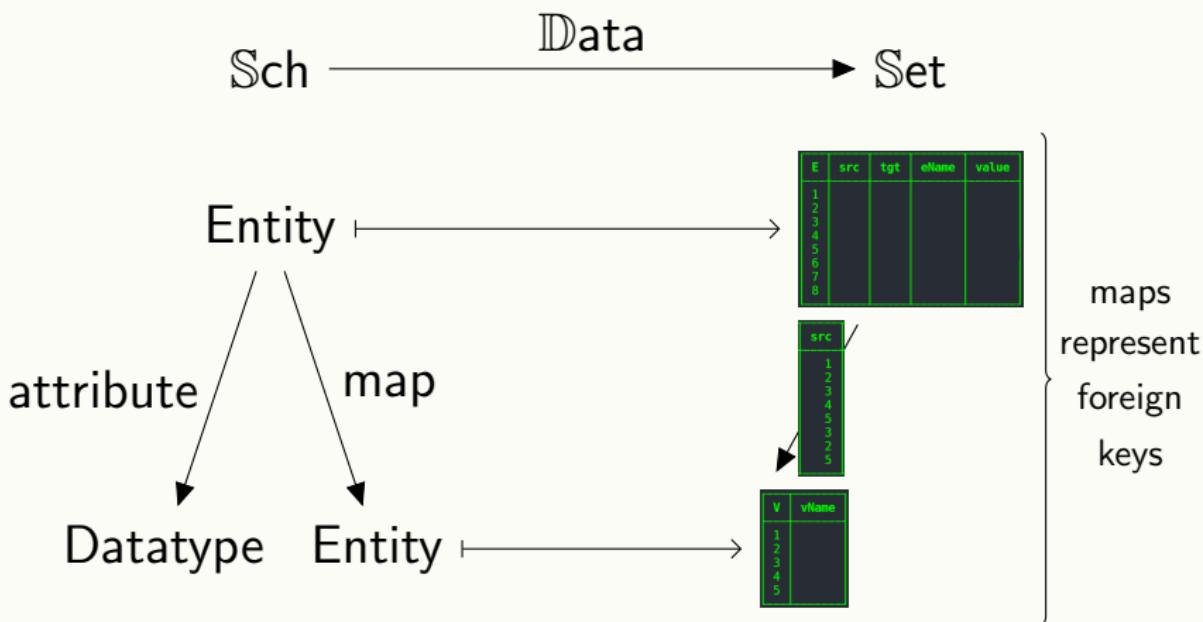
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Categorical Data Model

Fixed **Semantics**: Set

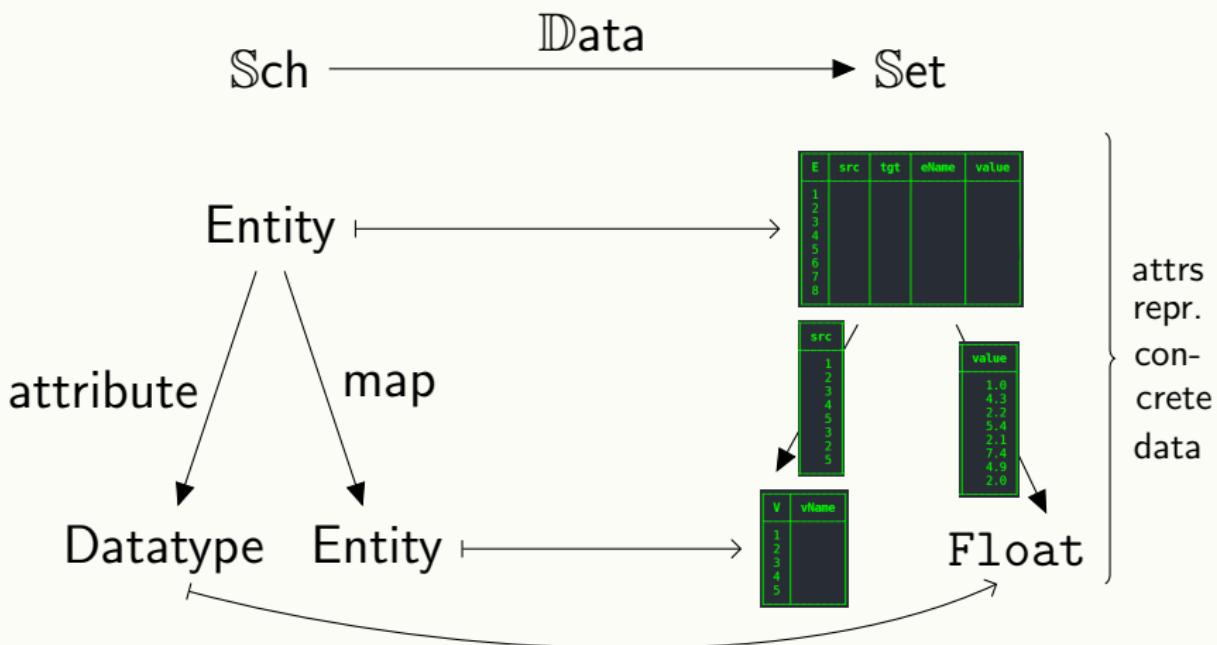
Choose **Syntax**: Schema



Categorical Data Model

Fixed **Semantics**: Set

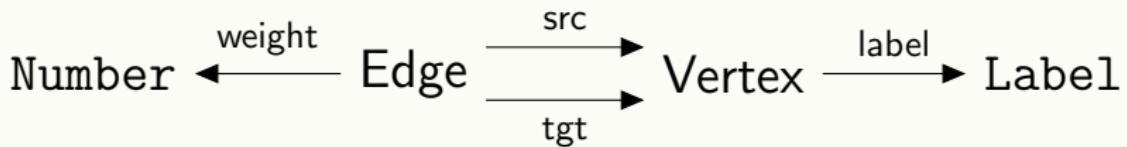
Choose **Syntax**: Schema



Graphs

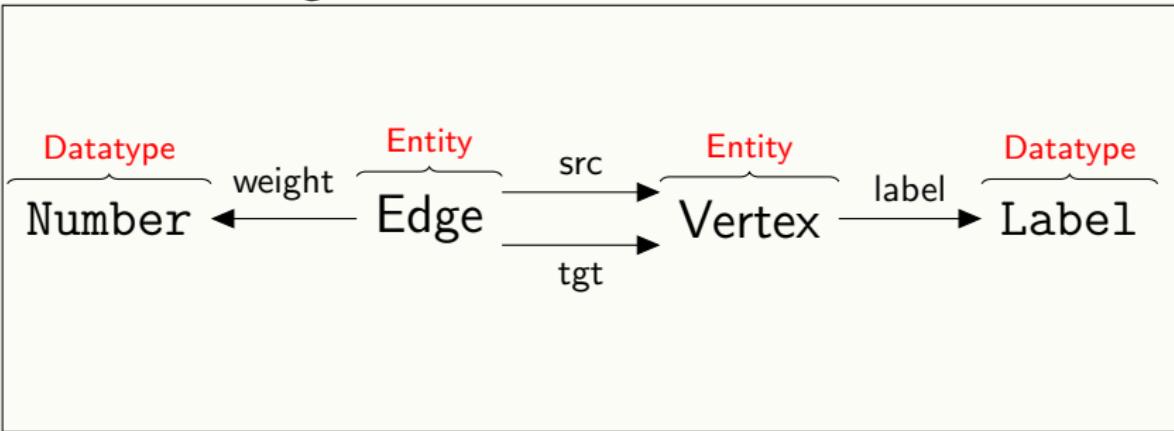
The Graph Schema

Schema Sch_G :



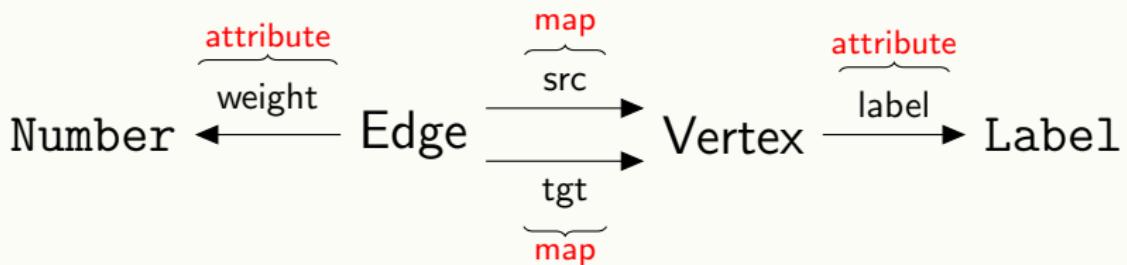
The Graph Schema

Schema Sch_G :



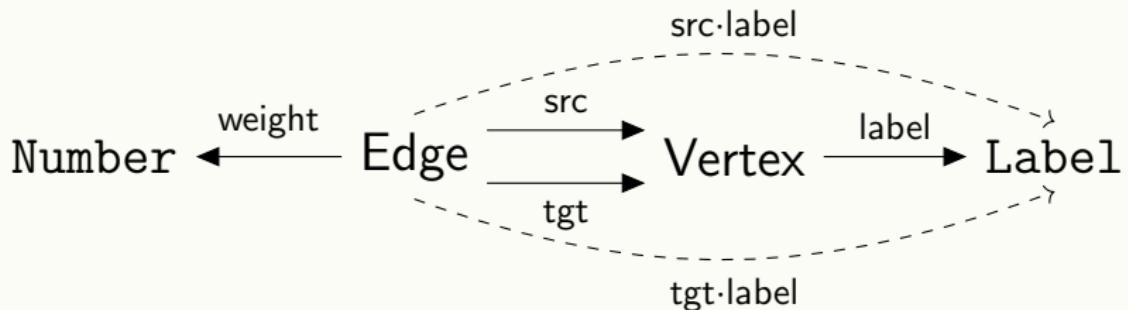
The Graph Schema

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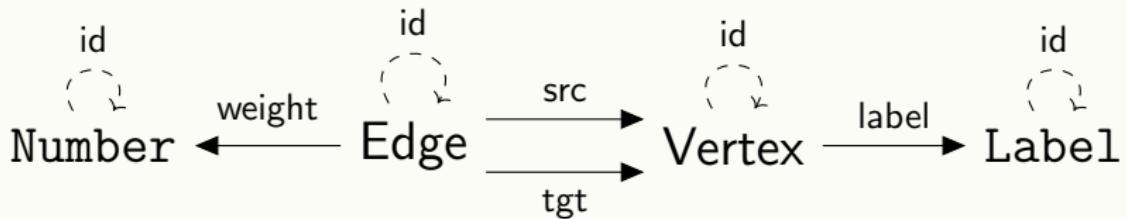
The Graph Schema

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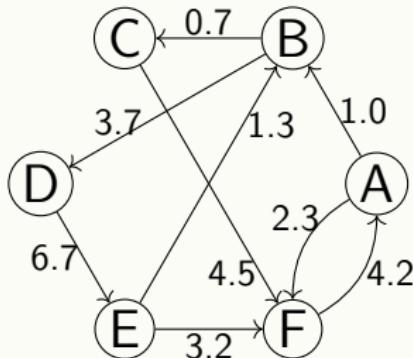
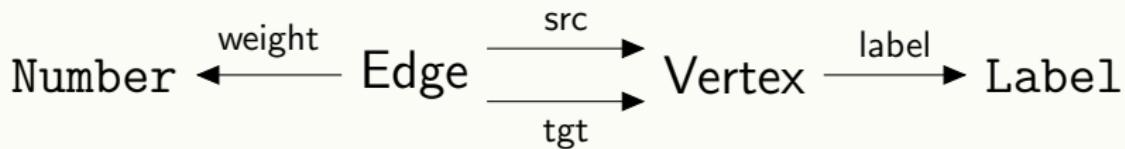
The Graph Schema

Schema Sch_G :



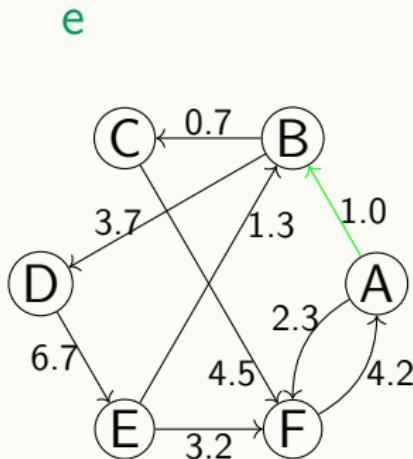
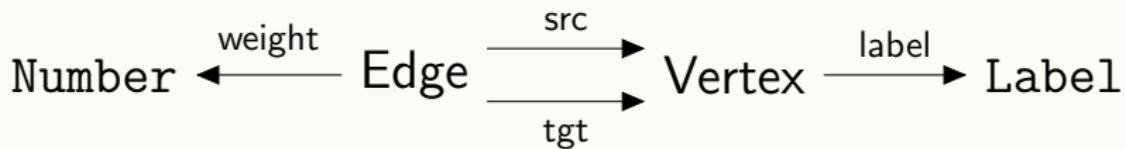
Graph Semantics

Schema Sch_G :



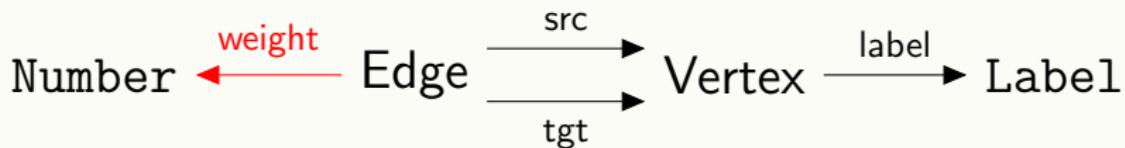
Graph Semantics

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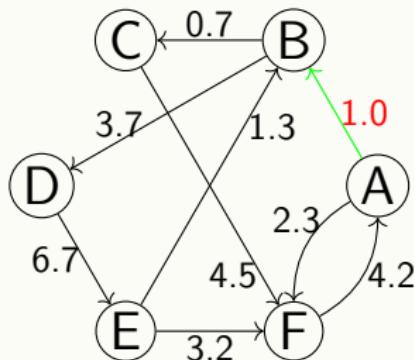


Graph Semantics

Schema Sch_G :

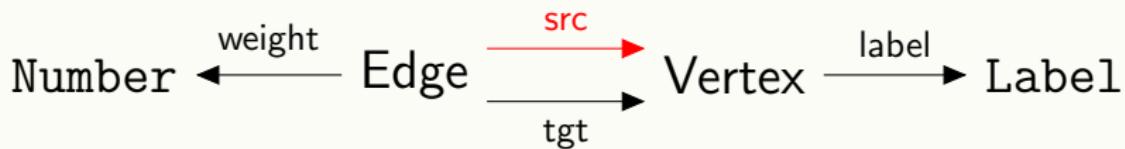


1.0 ← e

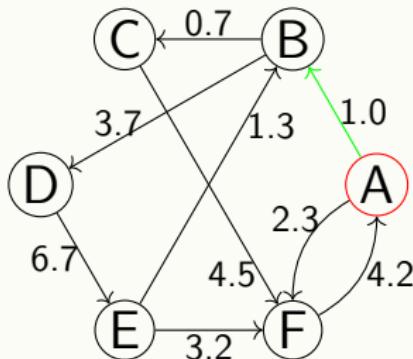


Graph Semantics

Schema Sch_G :

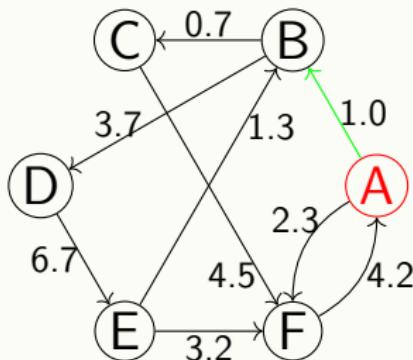
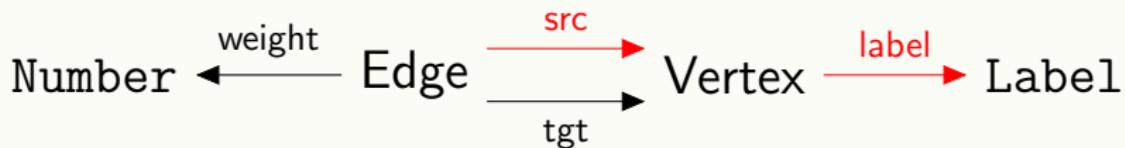


$$1.0 \leftarrow e \xrightarrow{v}$$



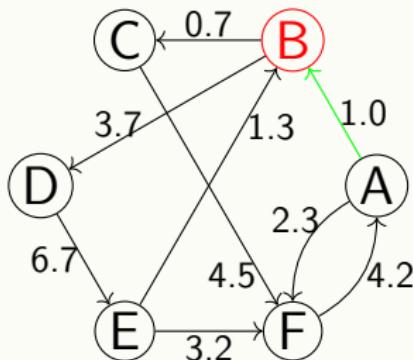
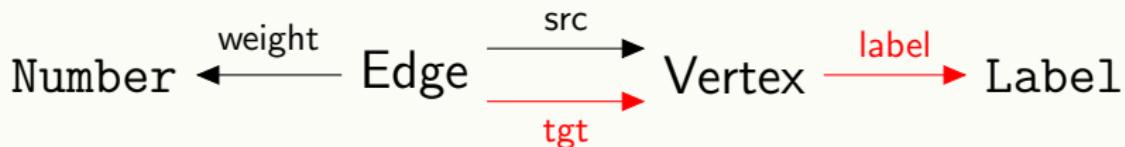
Graph Semantics

Schema Sch_G :



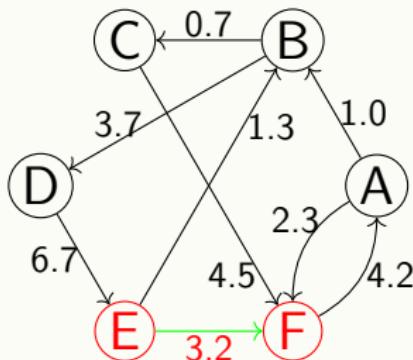
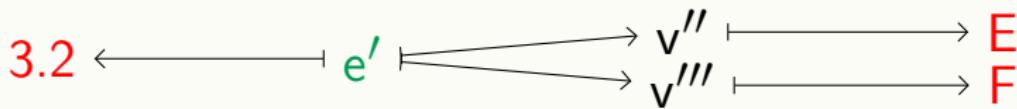
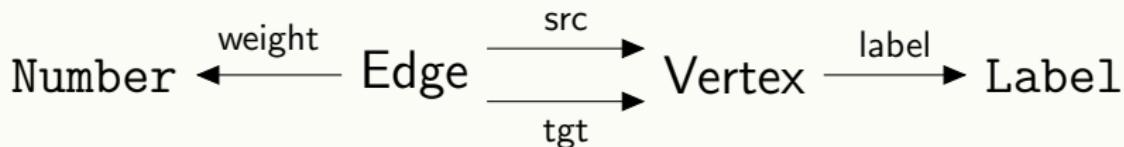
Graph Semantics

Schema Sch_G :



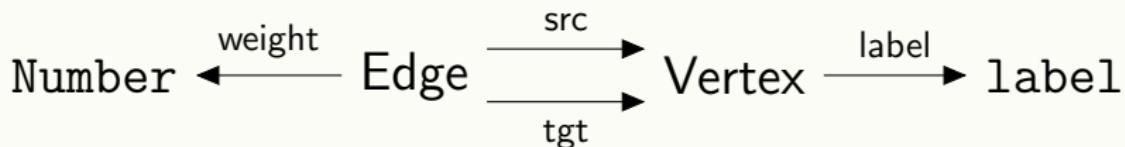
Graph Semantics

Schema Sch_G :



Graphs

Schema:

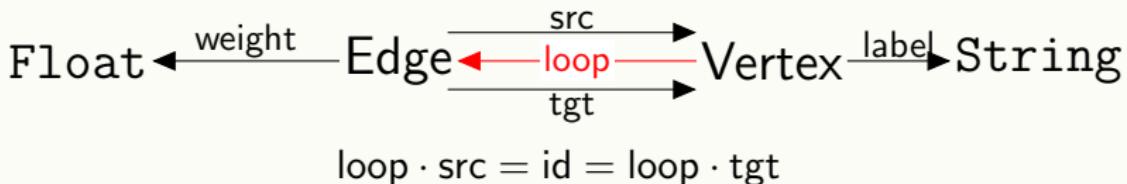


| E | src | tgt | weight |
|---|-----|-----|--------|
| 1 | 1 | 2 | 1.0 |
| 2 | 1 | 6 | 2.3 |

| V | label |
|---|-------|
| 1 | A |
| 2 | B |

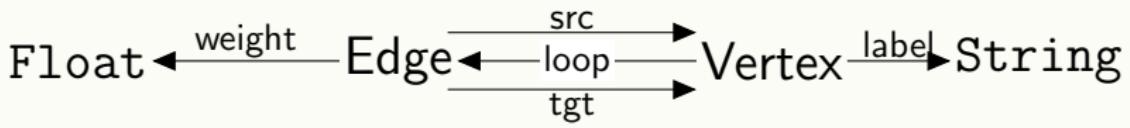
Reflexive Graphs

Schema Sch_R:



Reflexive Graphs

Schema Sch_R:

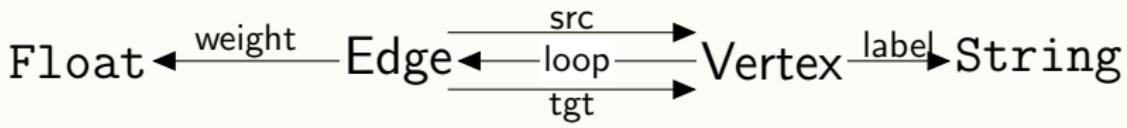


$$\text{loop} \cdot \text{src} = \text{id} = \text{loop} \cdot \text{tgt}$$

$$\text{loop} \cdot \text{weight} = 0.0$$

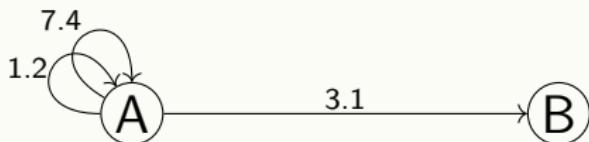
Reflexive Graphs

Schema Sch_R:



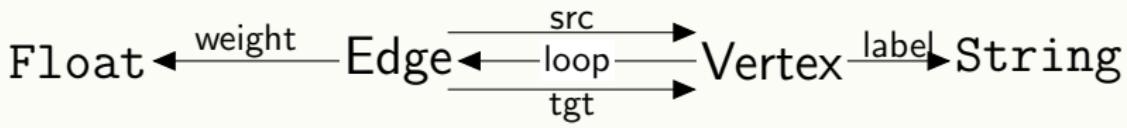
$$\text{loop} \cdot \text{src} = \text{id} = \text{loop} \cdot \text{tgt}$$

$$\text{loop} \cdot \text{weight} = 0.0$$



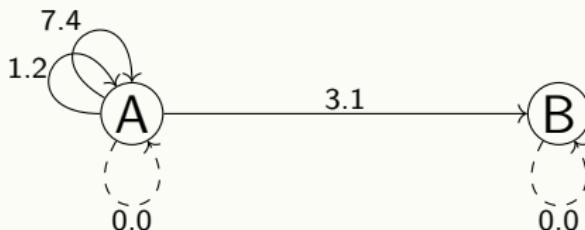
Reflexive Graphs

Schema Sch_R:



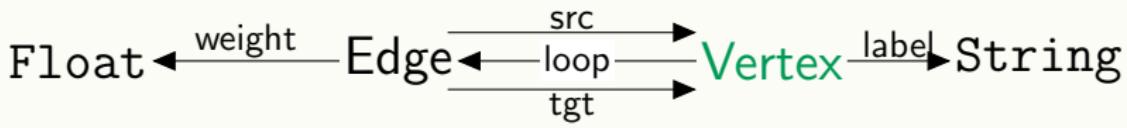
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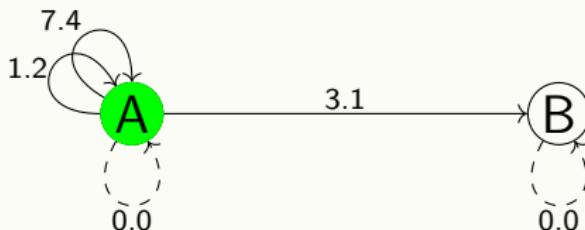
Reflexive Graphs

Schema Sch_R:



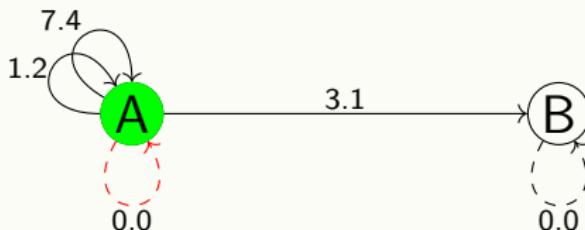
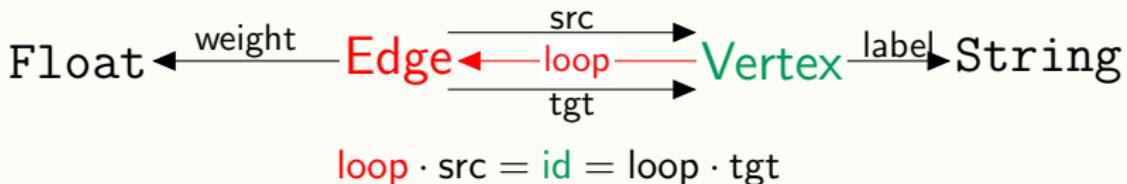
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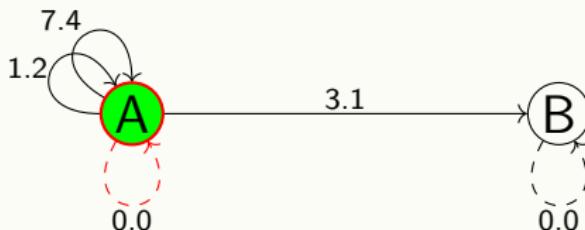
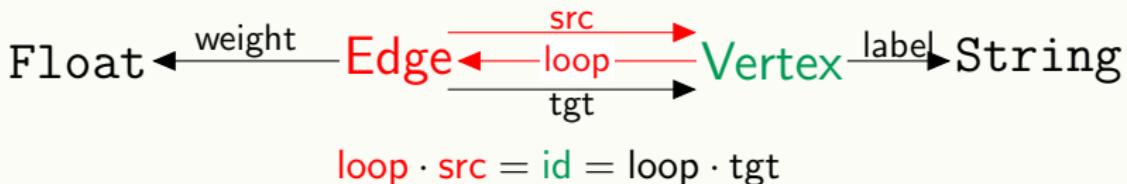
Reflexive Graphs

Schema Sch_R:



Reflexive Graphs

Schema Sch_R:



Questions?

Data Migration

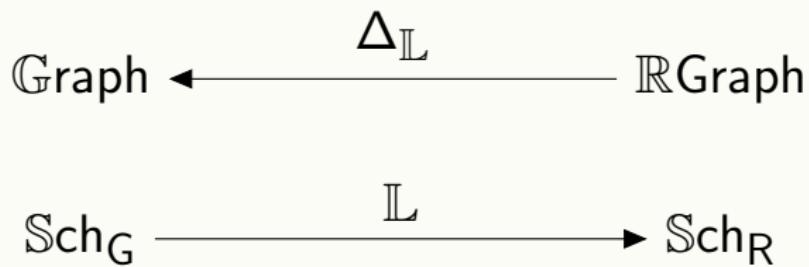
Schema functors

Schema functors create data transformations

$$\mathbb{S}\text{ch}_G \xrightarrow{\mathbb{L}} \mathbb{S}\text{ch}_R$$

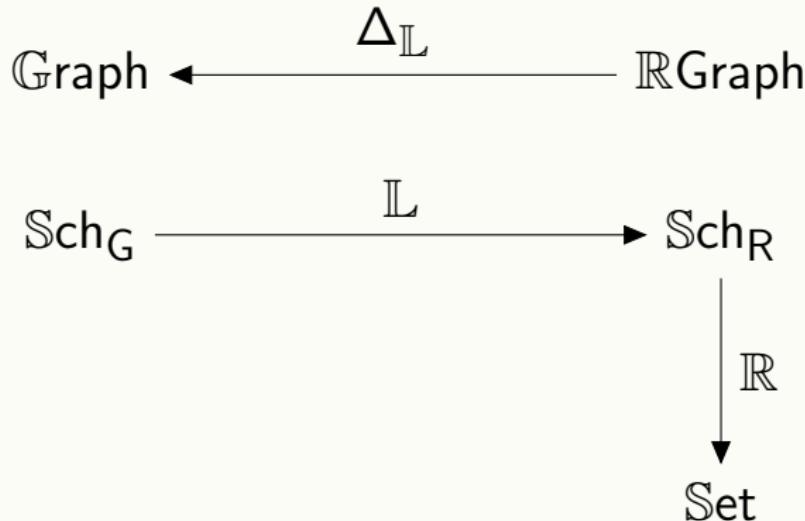
Schema functors

Schema functors create data transformations



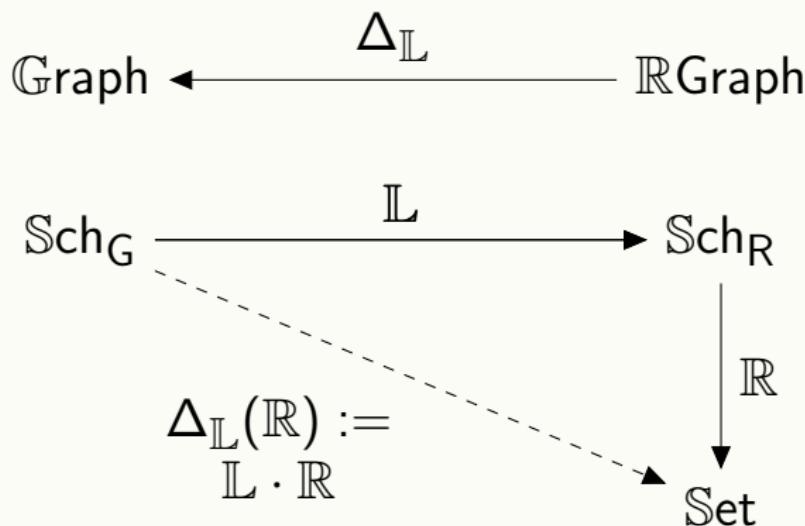
Schema functors

Schema functors create data transformations



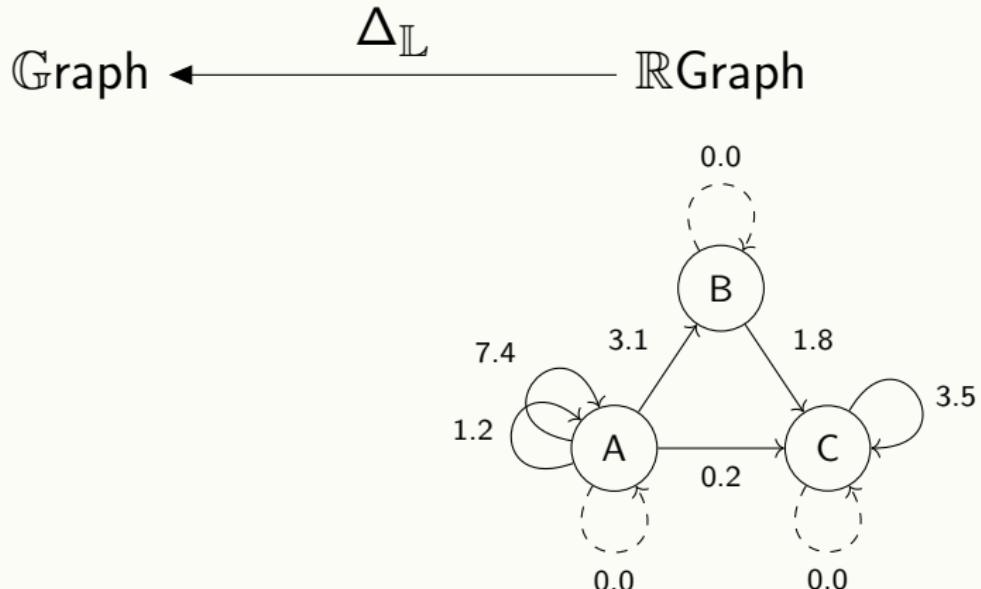
Schema functors

Schema functors create data transformations



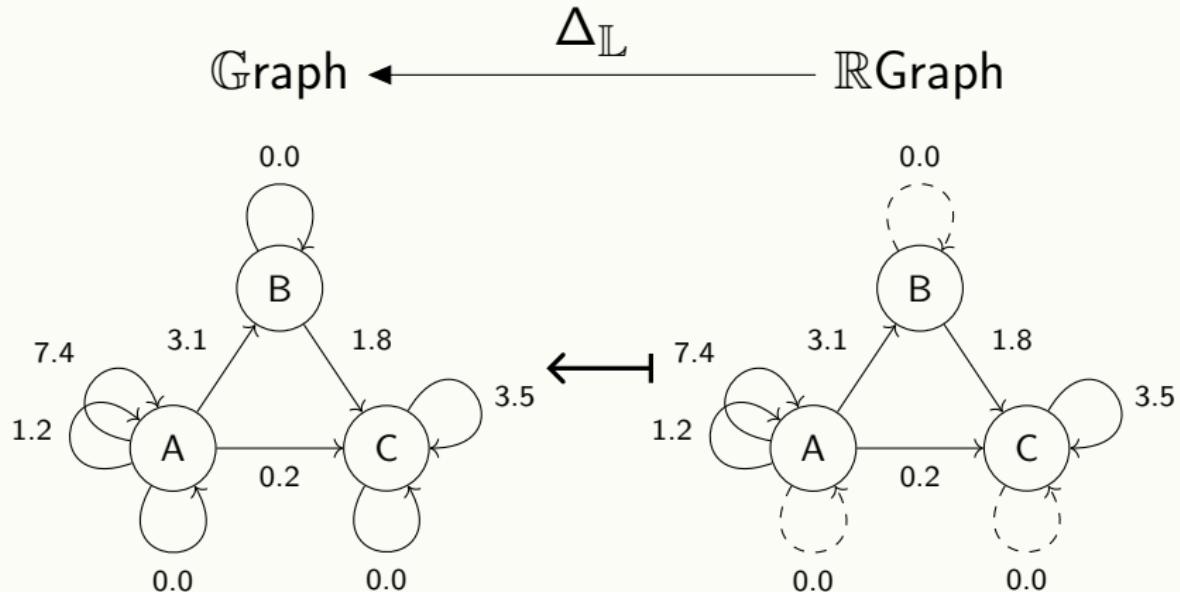
Schema functors

Schema functors create data transformations



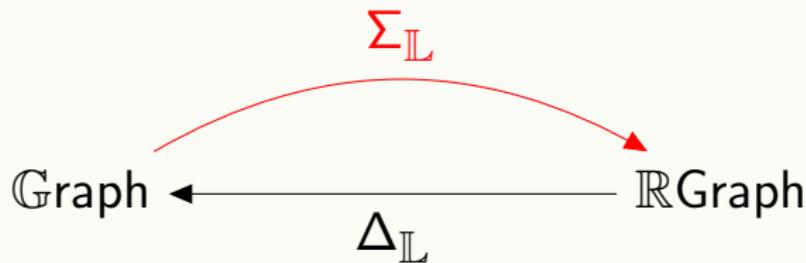
Schema functors

Schema functors create data transformations



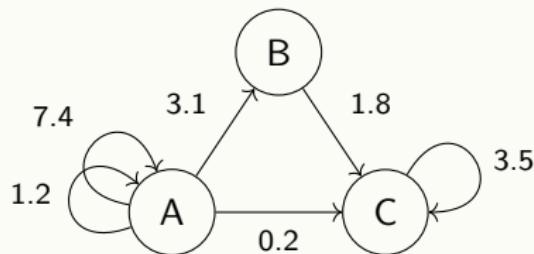
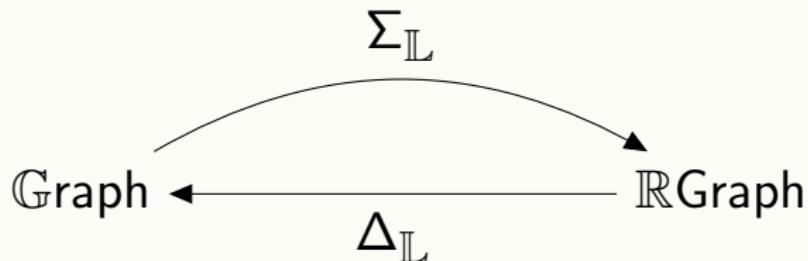
Schema functors

Schema functors create data transformations



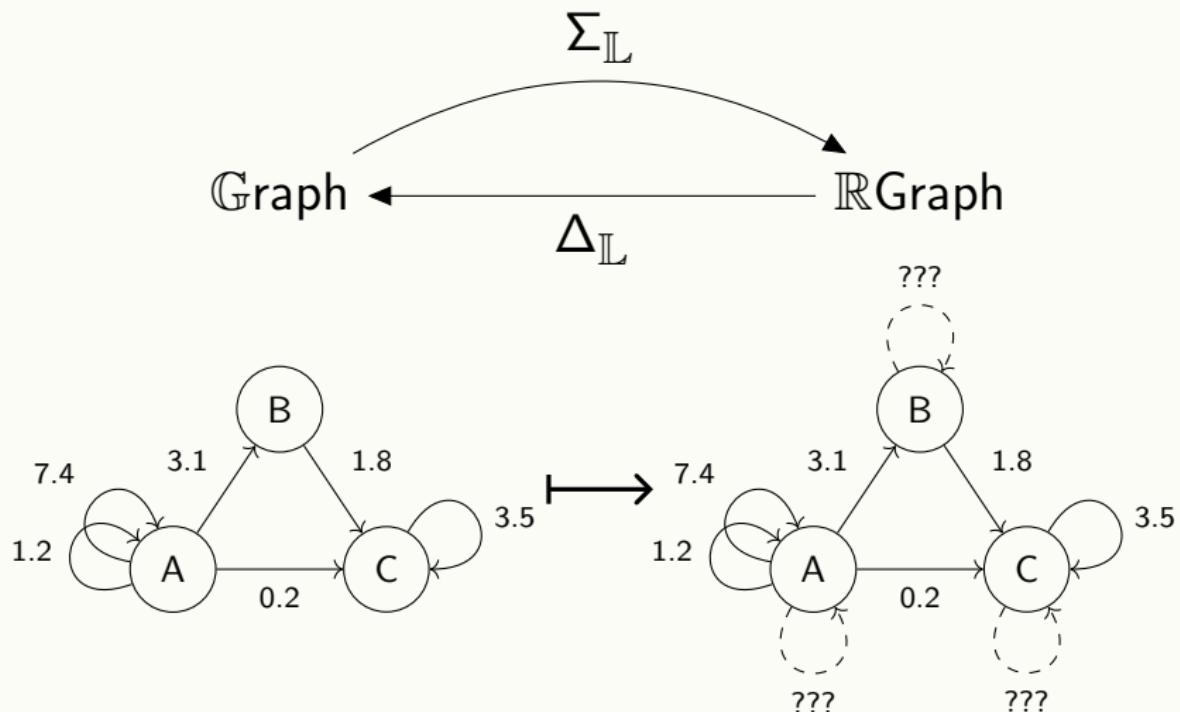
Schema functors

Schema functors create data transformations



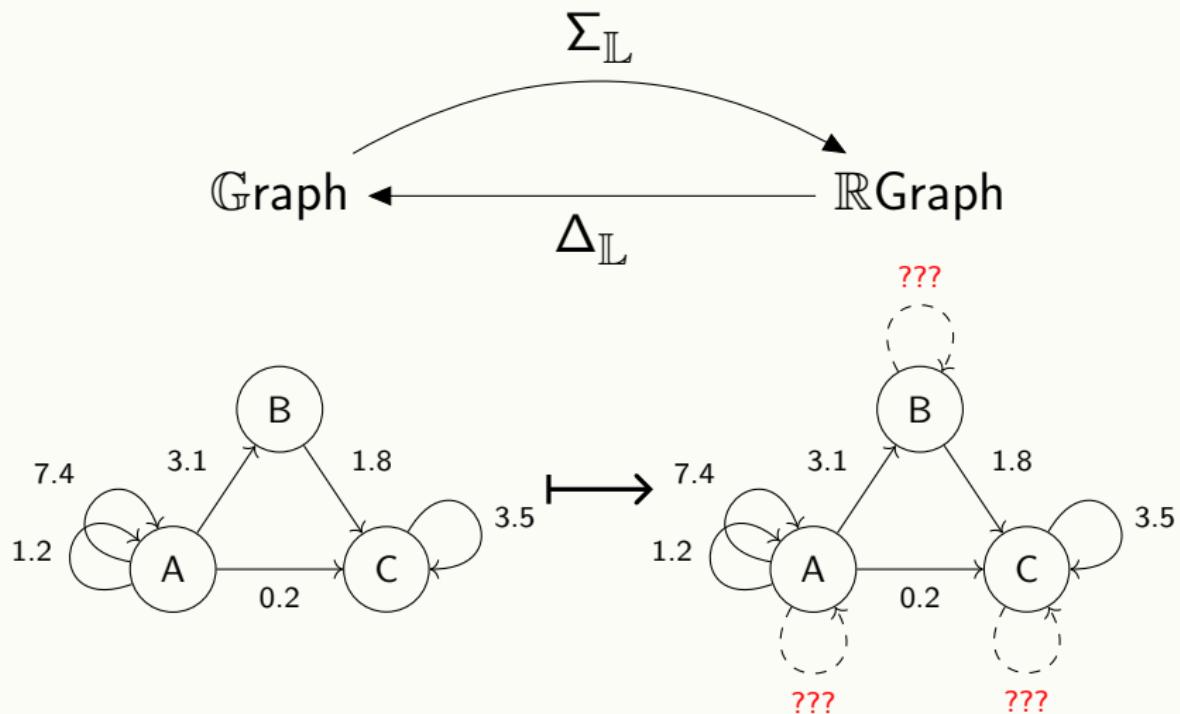
Schema functors

Schema functors create data transformations



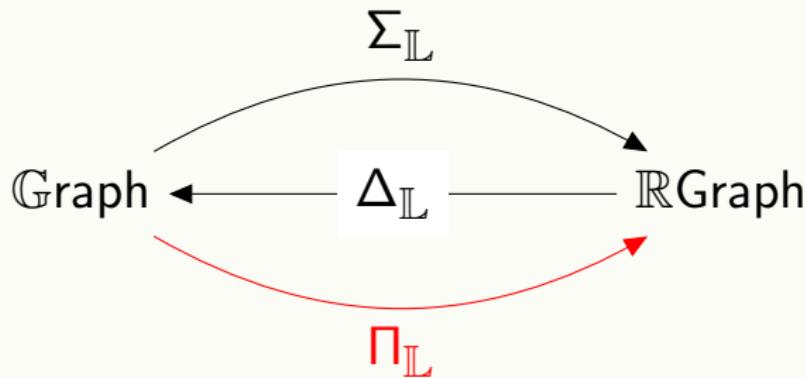
Schema functors

Schema functors create data transformations



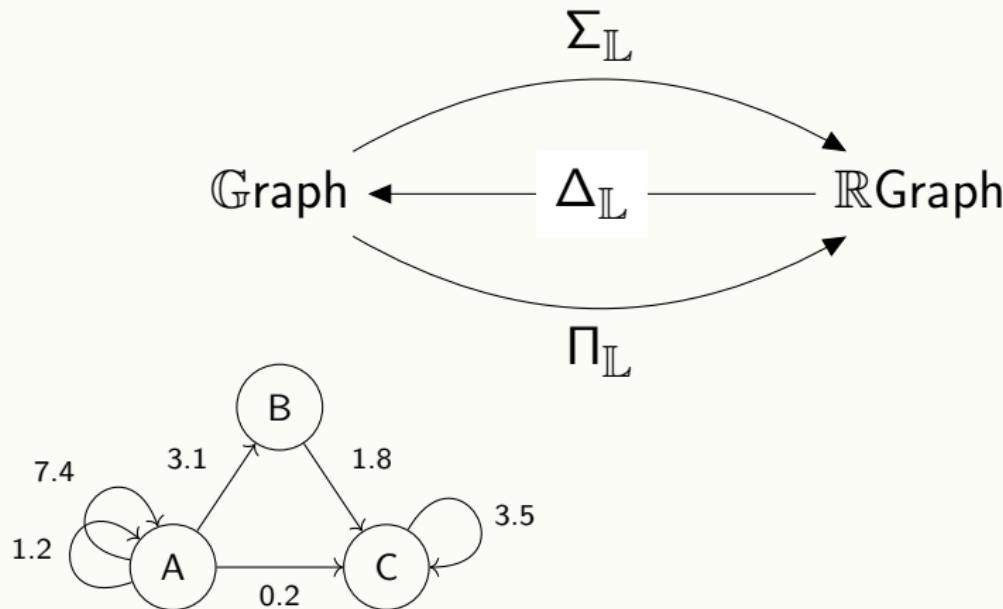
Schema functors

Schema functors create data transformations



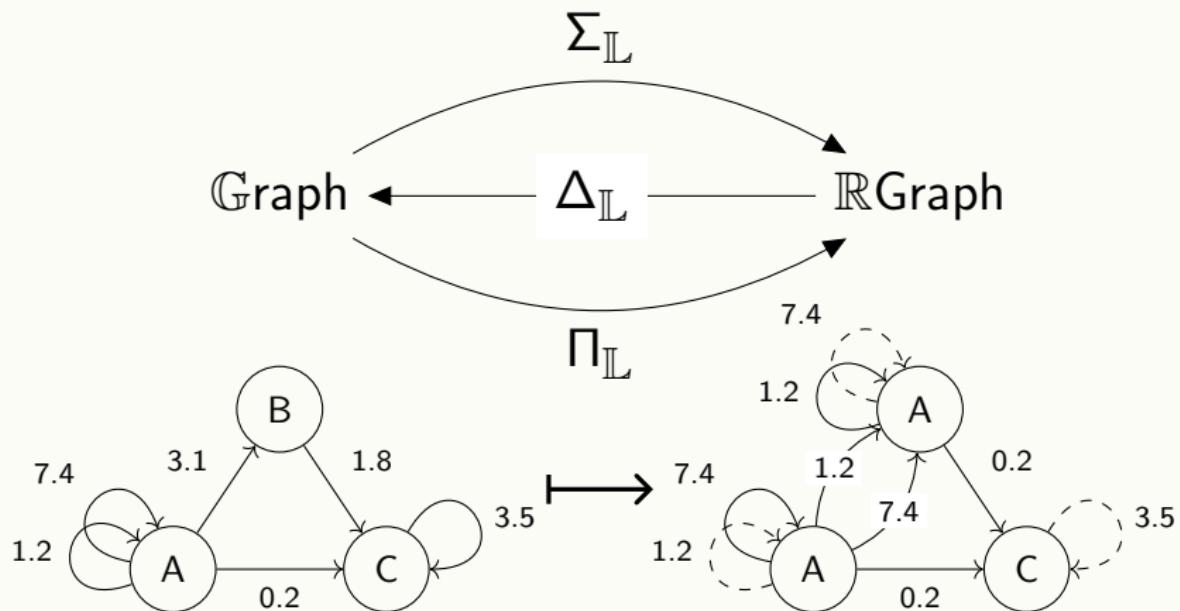
Schema functors

Schema functors create data transformations



Schema functors

Schema functors create data transformations

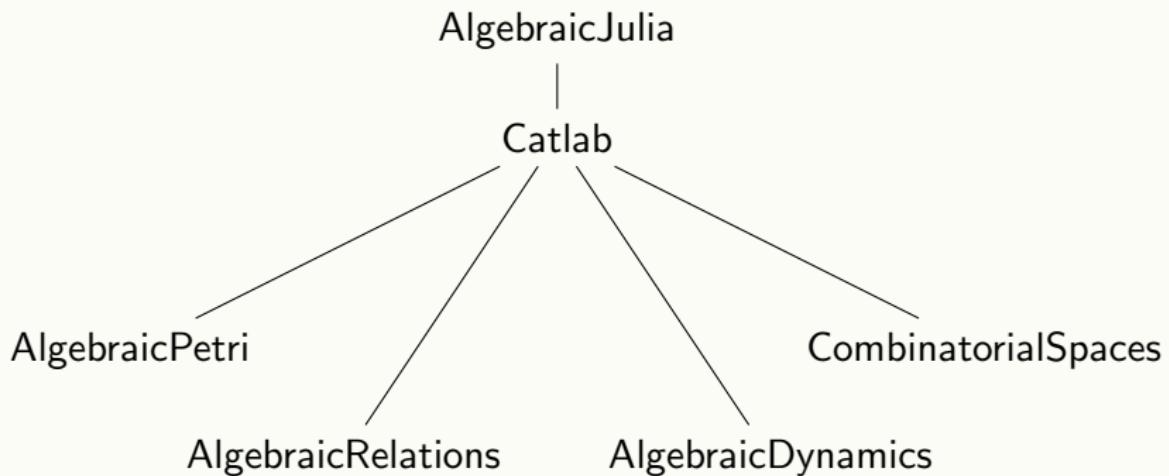


Questions?

Catlab

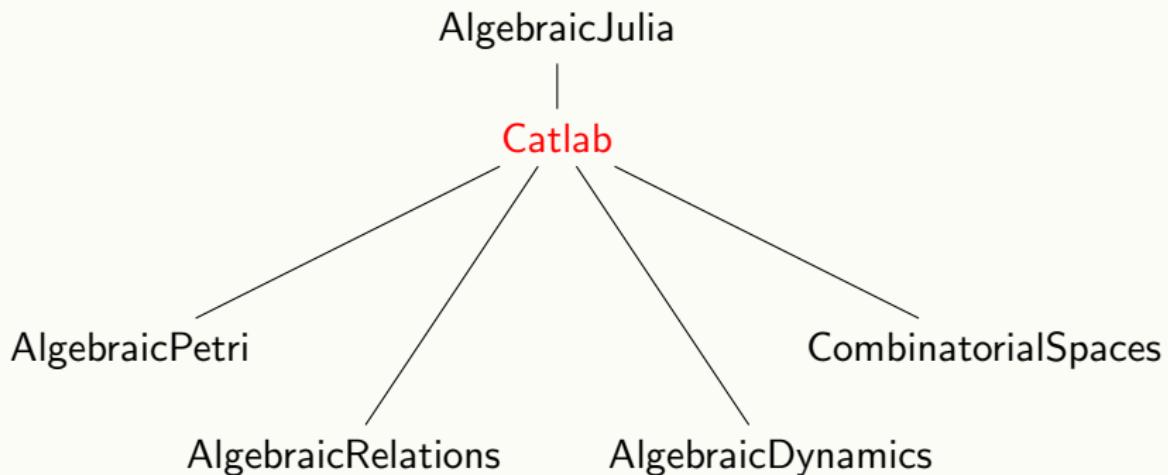
AlgebraicJulia

“Scientific computing based on applied category theory”



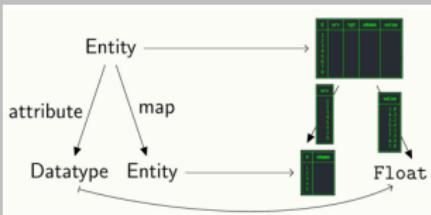
AlgebraicJulia

“Scientific computing based on applied category theory”

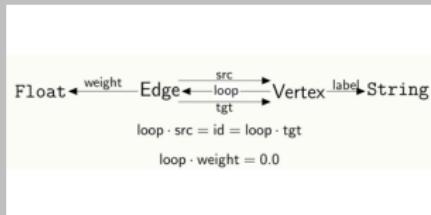


Questions?

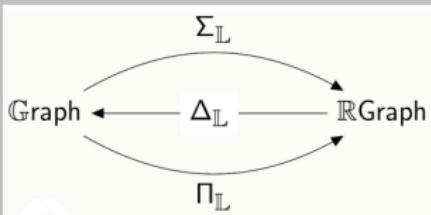
Data Model



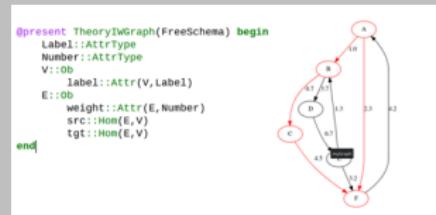
Graphs



Migration



Catlab

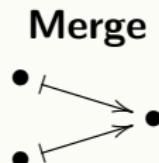
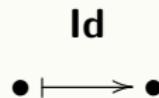
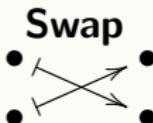


Processes

**Building
with
blocks**

The category $\mathbb{F}\text{Set}$

All **finite** functions are built from four atomic components.



The category $\mathbb{F}\text{Set}$

All **finite** functions are built from four atomic components.

Swap



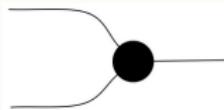
Id



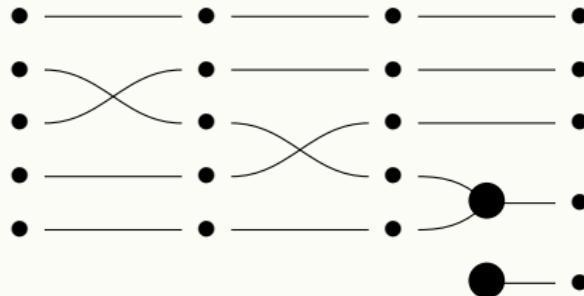
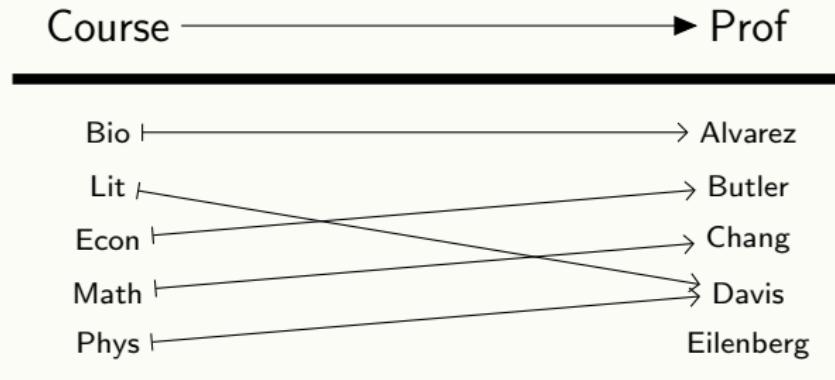
Create



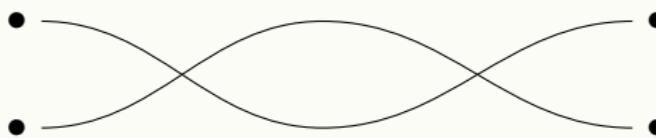
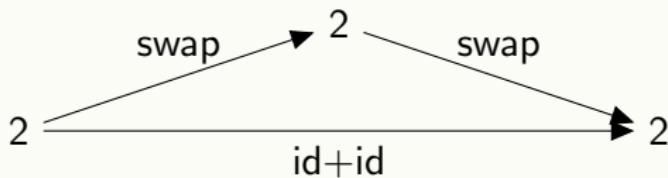
Merge



Building with bricks



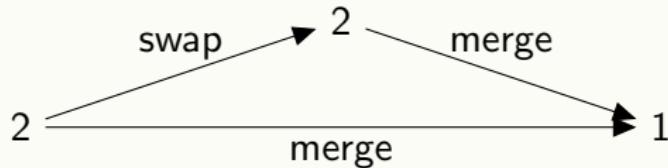
Rules of the game

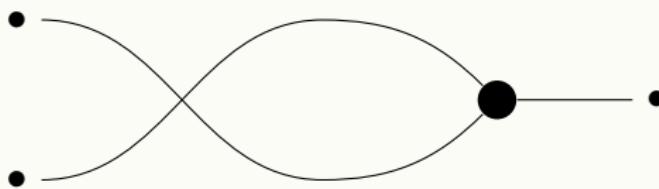


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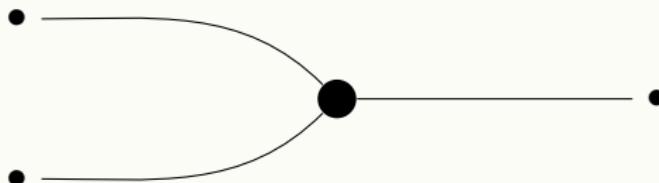


Rules of the game

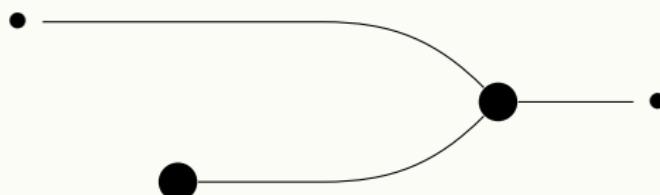
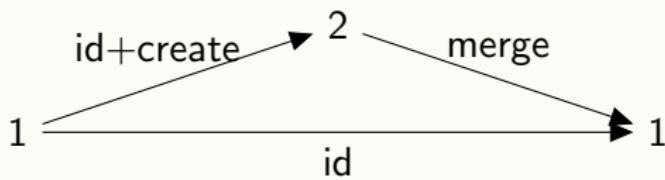




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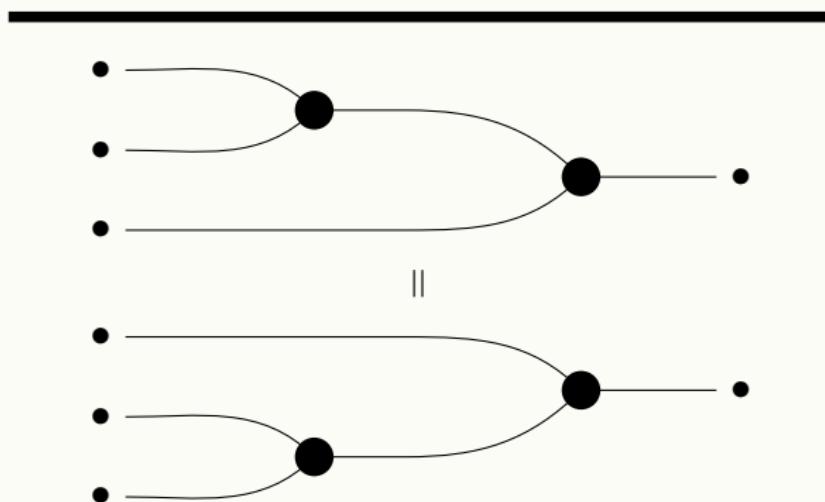
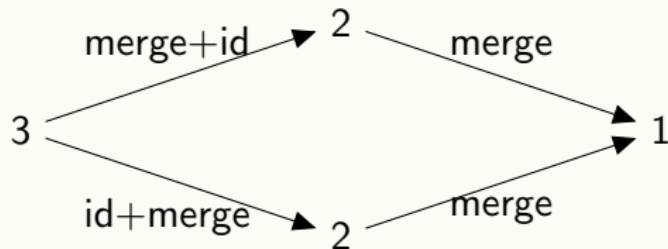
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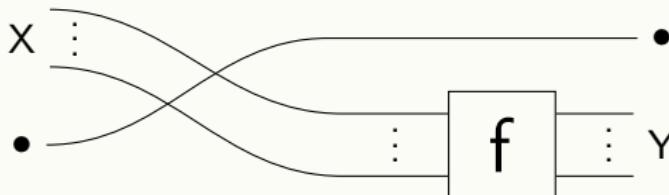
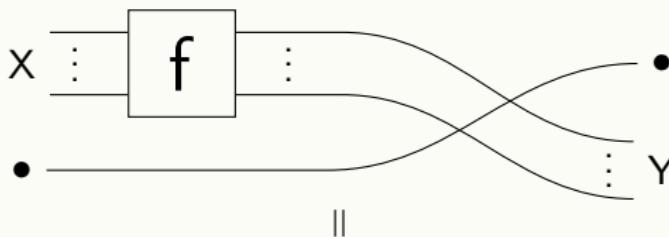
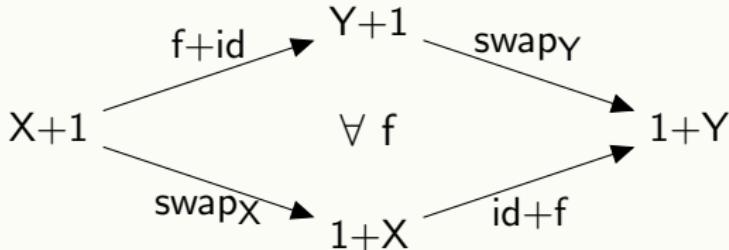
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Rules of the game



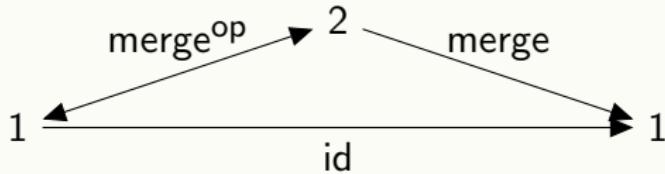
Rules of the game

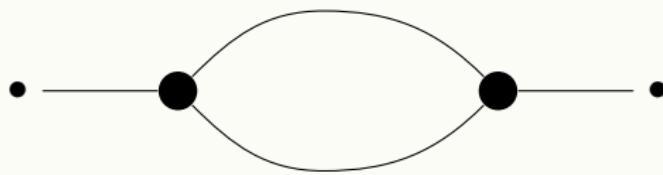


Picture Proofs

$(\text{create} + \text{merge}) \cdot \text{merge} = \text{merge}$

Put it in backwards?

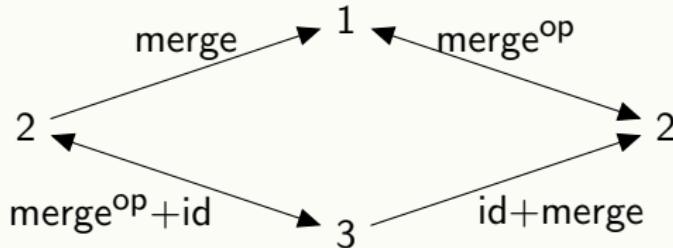


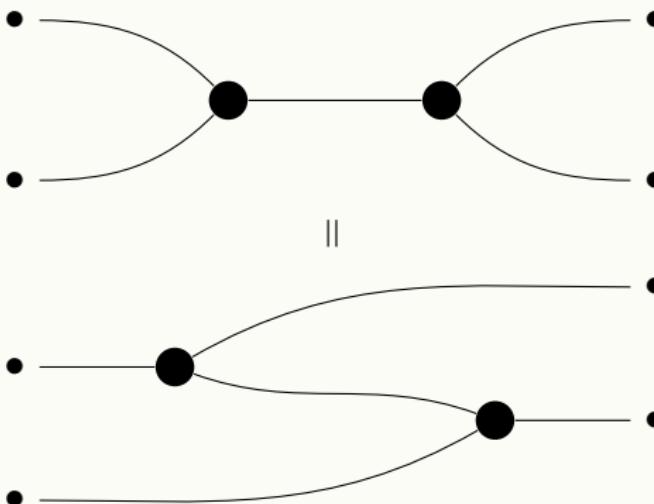


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Put it in backwards?

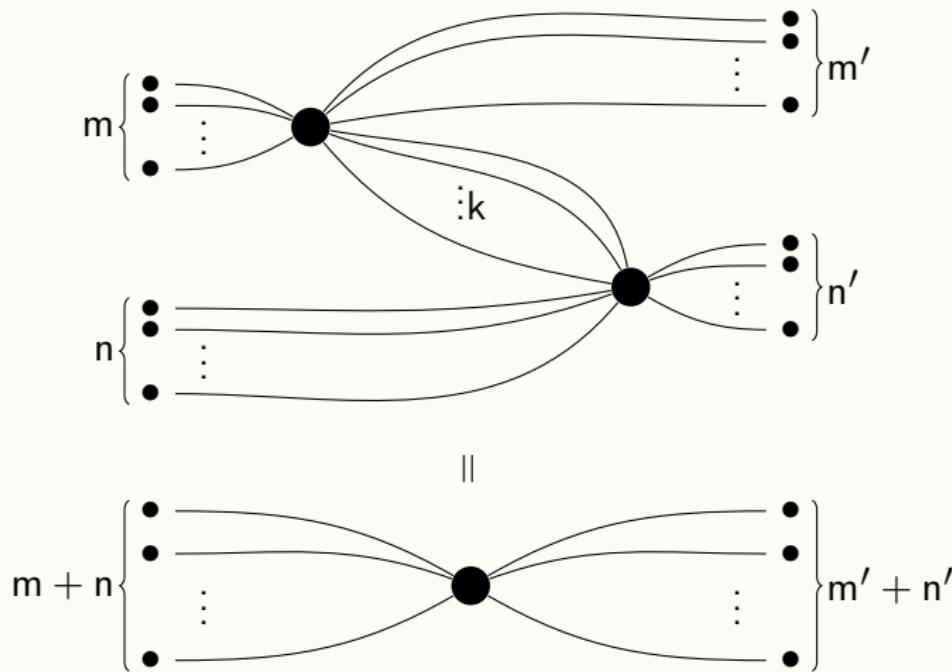




Yanking Lemma

The Spider Theorem

Connected diagrams with identical inputs/outputs are equal.



Questions?

Monoidal Categories

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X}$

$$X \xrightarrow{f} Y$$

$$X' \xrightarrow{f'} Y'$$

$$X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y'$$

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X}$

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$



$$X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y' \xrightarrow{g \otimes g'} Z \otimes Z'$$

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X}$

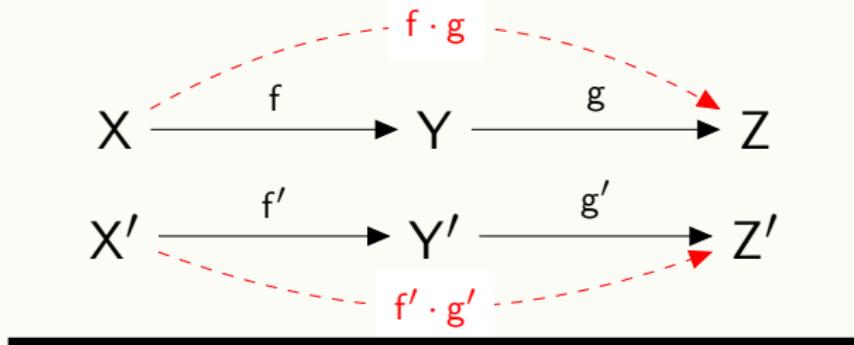
$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

$$X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y' \xrightarrow{g \otimes g'} Z \otimes Z'$$

$(f \otimes f') \cdot (g \otimes g')$

Parallel Composition

A monoidal product is a functor $\otimes : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X}$

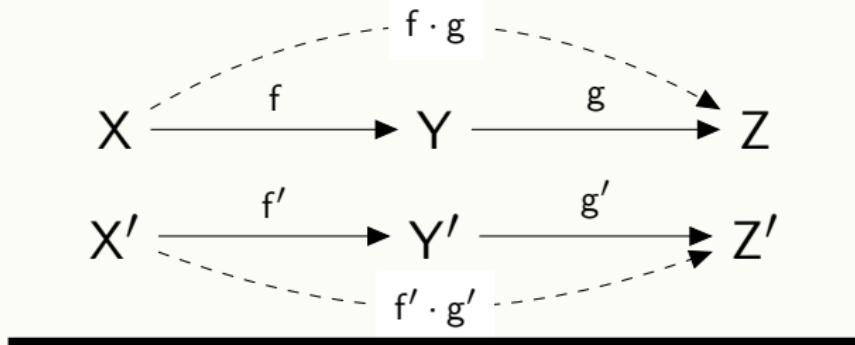


$$X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y' \xrightarrow{g \otimes g'} Z \otimes Z'$$

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Parallel Composition

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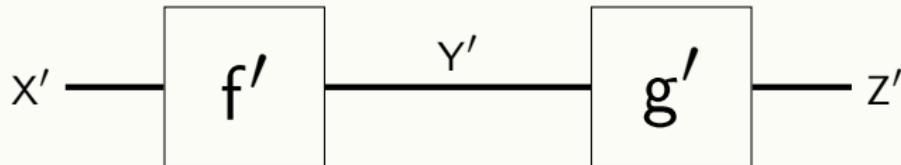
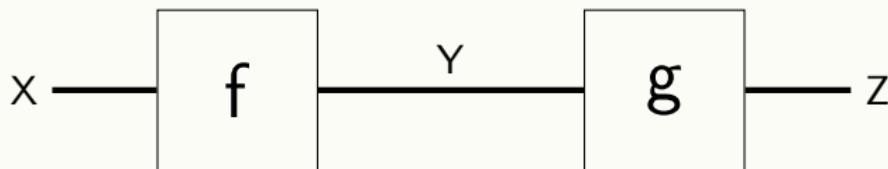
$$X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y' \xrightarrow{g \otimes g'} Z \otimes Z'$$

$$(f \otimes f') \cdot (g \otimes g') = (f \cdot g) \otimes (f' \cdot g')$$

String diagrams

The interchange law:

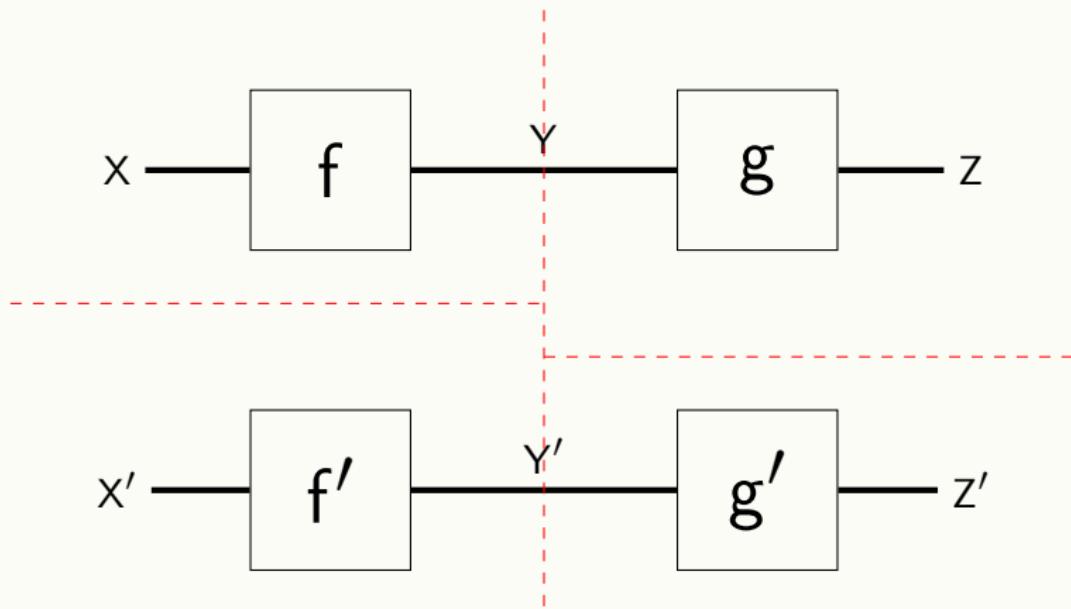
$$(f \otimes f') \cdot (g \otimes g') = (f \cdot g) \otimes (f' \cdot g')$$



String diagrams

The interchange law:

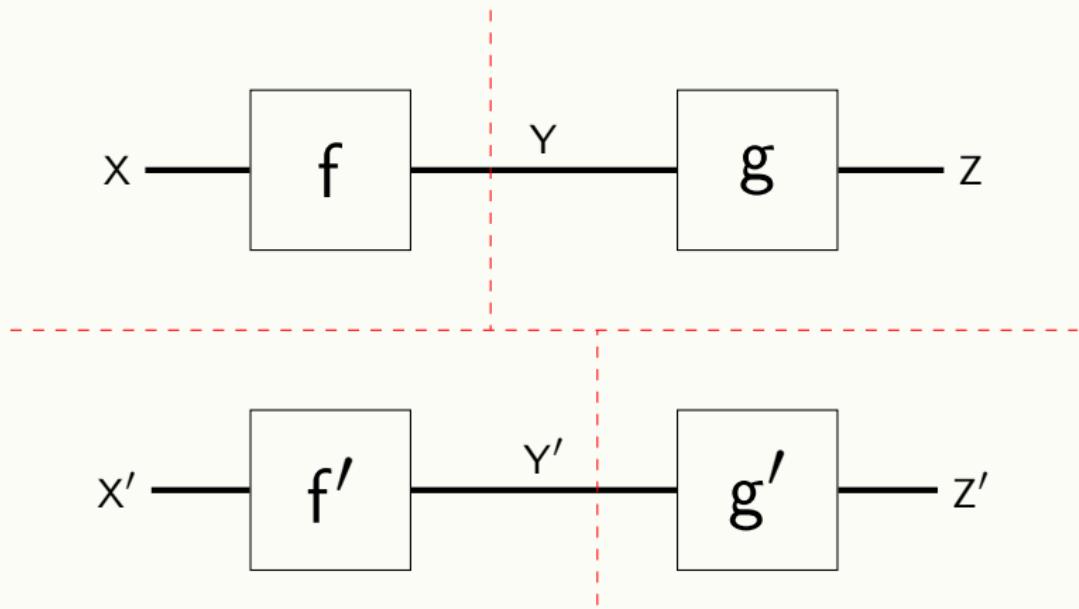
$$(f \otimes f') \cdot (g \otimes g') = (f \cdot g) \otimes (f' \cdot g')$$



String diagrams

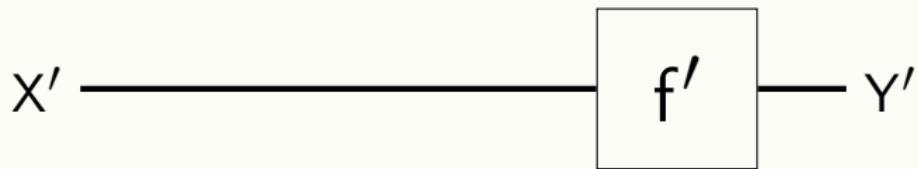
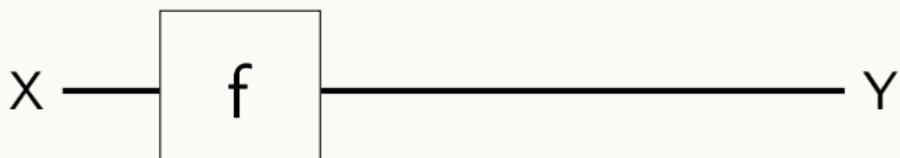
The interchange law:

$$(f \otimes f') \cdot (g \otimes g') = (f \cdot g) \otimes (f' \cdot g')$$



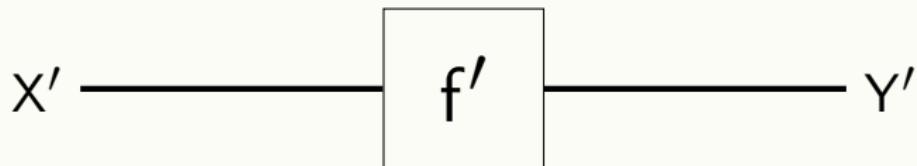
String diagrams

Logical time only!



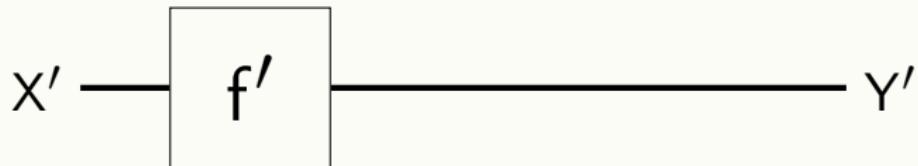
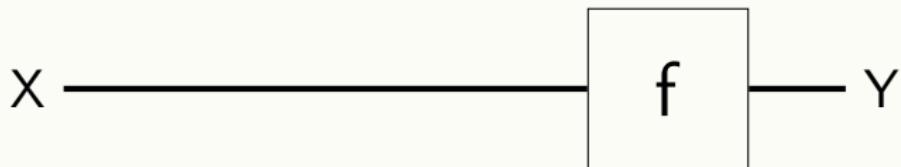
String diagrams

Logical time only!



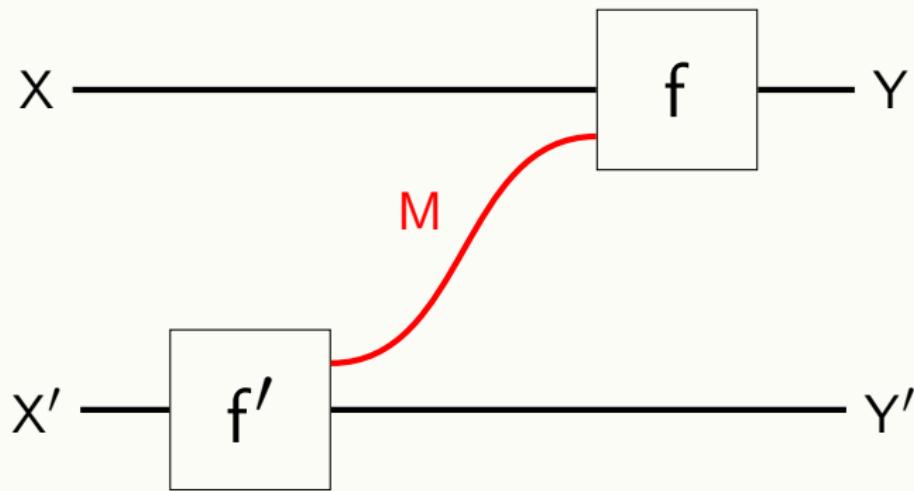
String diagrams

Logical time only!

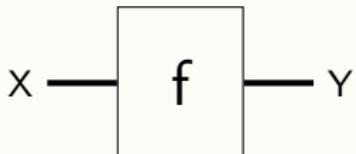
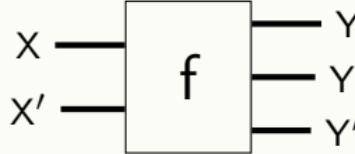
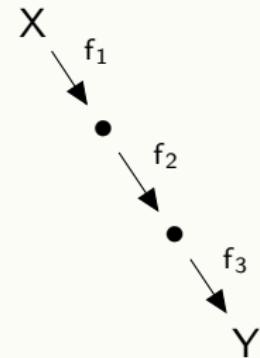
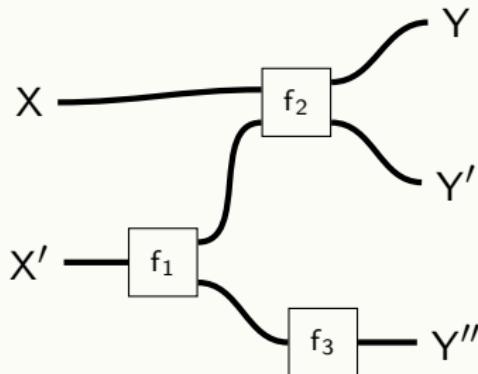


String diagrams

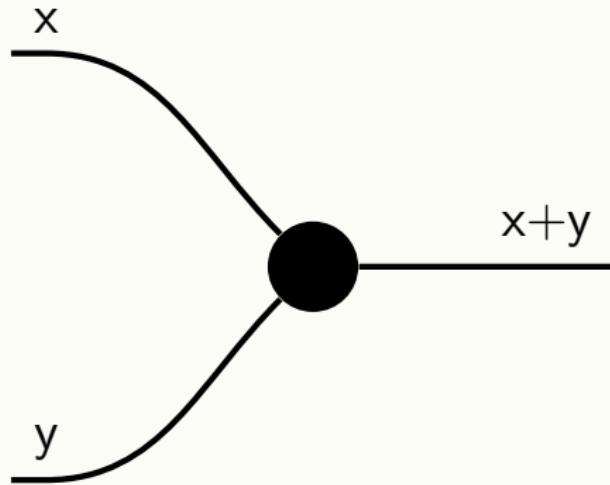
Logical time only! Sequence by messaging



Many in, many out

| Context | Vanilla categories | Monoidal categories |
|------------|--|--|
| Basic unit | $X \xrightarrow{f} Y$  | $X \otimes X' \xrightarrow{f} Y \otimes Y' \otimes Y''$  |
| Composite | $X \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \xrightarrow{f_3} Y$  | $X \xrightarrow{f_2} Y$ $X' \xrightarrow{f_1} Y'$ $f_3 \xrightarrow{} Y''$  |

Values on the wire



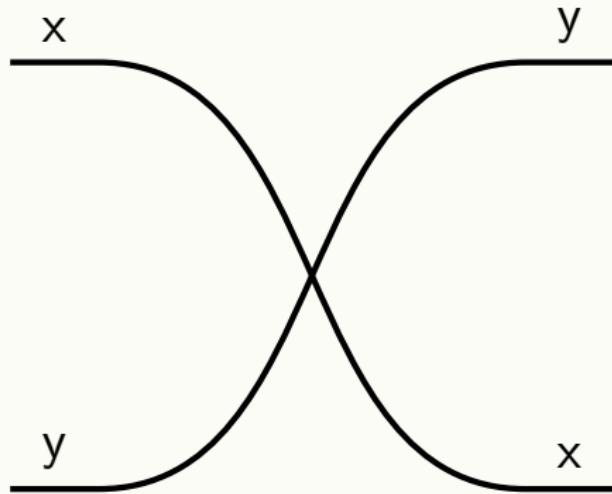
Merge

Values on the wire



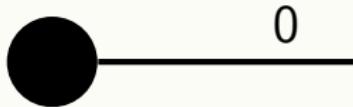
Identity

Values on the wire



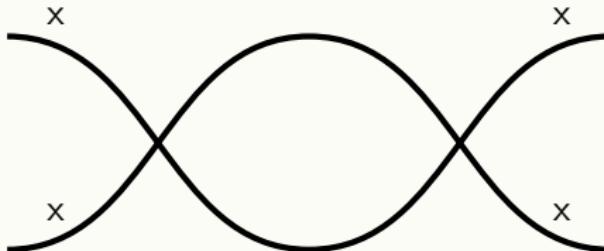
Swap

Values on the wire



Create

Following the rules

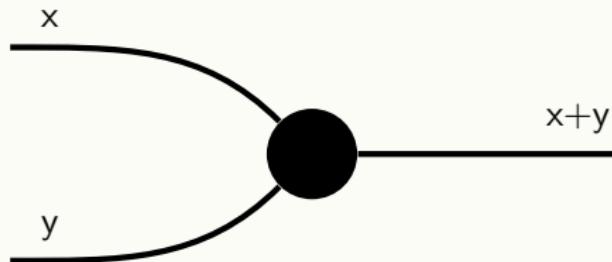


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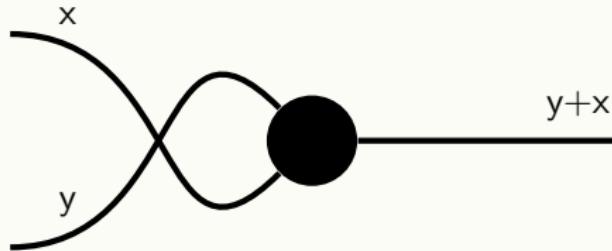


Double swap

Following the rules

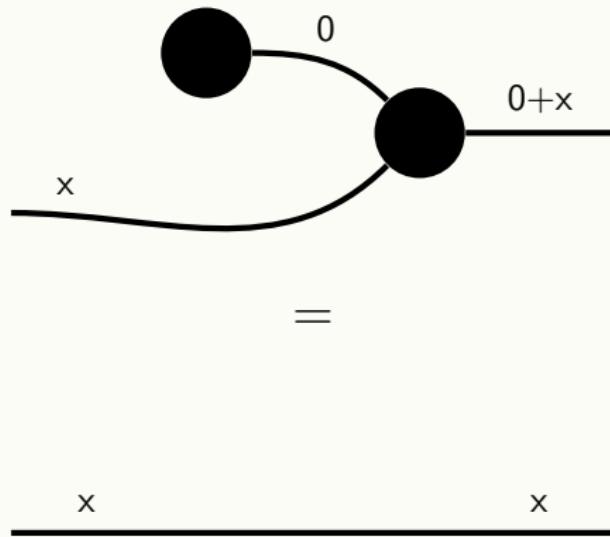


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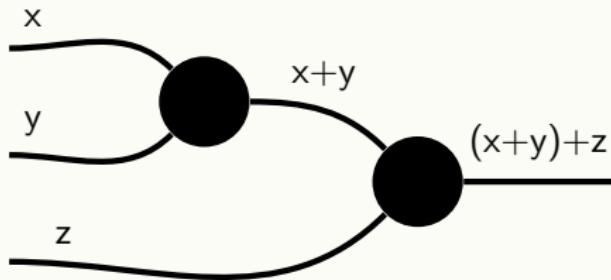
Commutativity

Following the rules

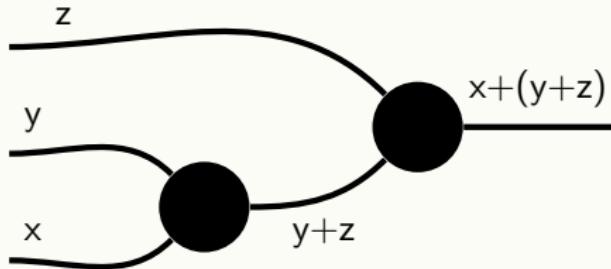


Unit

Following the rules

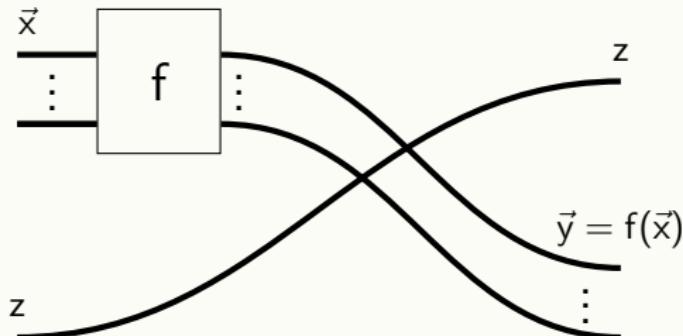


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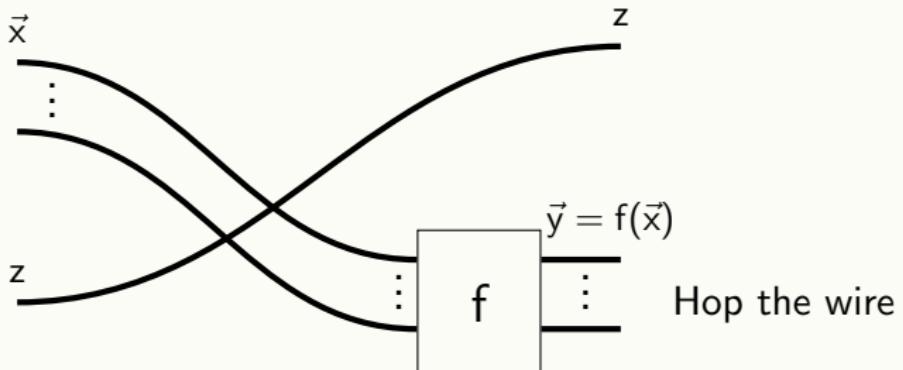


Associativity

Following the rules

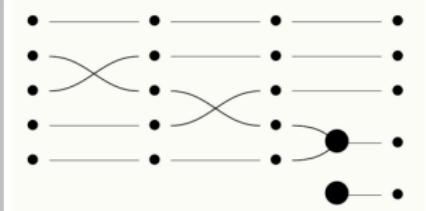


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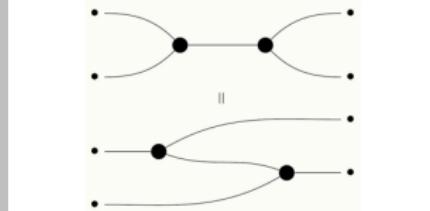


Questions?

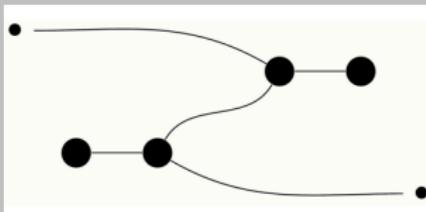
Generators



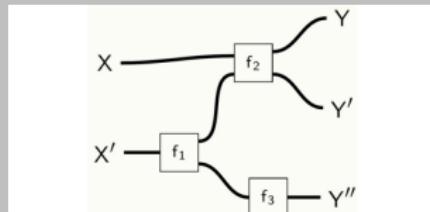
Rules



Proof



String diagrams



Linear Algebra

Monoidal Functors

$$(\mathbb{F}\mathbf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathbf{et}, \times, 1)$$

Monoidal Functors

$$(\mathbb{F}\mathbf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathbf{et}, \times, 1)$$

1

Monoidal Functors

$$(\mathbb{F}\mathbf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathbf{et}, \times, 1)$$

$$1 \mapsto X := \mathbb{M}(1)$$

Monoidal Functors

$$(\mathbb{F}\mathbf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathbf{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$2 \cong 1 + 1$$

Monoidal Functors

$$(\mathbb{F}\mathbf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathbf{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$2 \cong 1 + 1 \longleftarrow X \times X \cong X^2$$

Monoidal Functors

$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\text{et}, \times, 1)$$

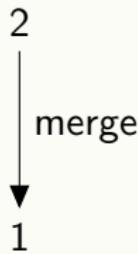
$$1 \longmapsto X := \mathbb{M}(1)$$

$$| \cong 1 + \dots + 1 \longleftarrow X \times \dots \times X \cong X^{|}$$

Monoidal Functors

$$(\mathbb{F}\mathbf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathbf{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$



Monoidal Functors

$$(\mathbb{F}\mathbf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathbf{et}, \times, 1)$$

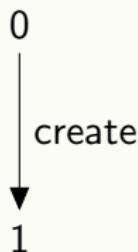
$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} 2 & & X^2 \\ \downarrow & \text{merge} \longmapsto & \downarrow \\ 1 & & X \end{array}$$

Monoidal Functors

$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\text{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$



Addition defines a *monoidal functor*

Monoidal Functors

$$(\mathbb{F}\mathbf{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\mathbf{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} 0 & & 1 \\ \downarrow & \text{create} \longleftarrow & \downarrow \\ 1 & & X \end{array}$$

Monoidal Functors

$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\text{et}, \times, 1)$$

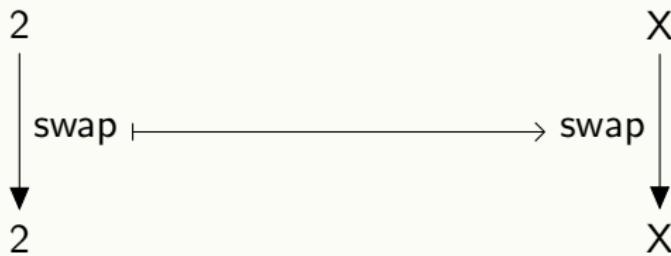
$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} 1 & \xrightarrow{\quad\quad\quad} & X \\ \downarrow id & \longleftarrow & \downarrow id \\ 1 & & X \end{array}$$

Monoidal Functors

$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\text{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$



Monoidal Functors

$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\text{et}, \times, 1)$$

$$1 \mapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} I & & X^I \\ \downarrow f & \longmapsto & \downarrow \sum_f \\ J & \longrightarrow & X^J \end{array}$$

Monoidal Functors

$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\text{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} I & & x^I \\ \downarrow f & \longrightarrow & \downarrow \sum_f \\ J & & \left\langle \sum_{f(i)=j} x_i \right\rangle \end{array}$$

Monoidal Functors

$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\mathbb{M} := (\text{Int}, +, 0)} (\mathbb{S}\text{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} I & & x^I \\ \downarrow f & \longmapsto & \downarrow \sum_f \\ J & & \left\langle \sum_{f(i)=j} x_i \right\rangle \end{array}$$

Monoidal Functors

$$(\mathbb{FSet}, +, \emptyset) \xrightarrow{\quad \mathbb{M} := (\text{Float}, +, 0) \quad} (\mathbb{Set}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} I & & x^I \\ \downarrow f & \longrightarrow & \downarrow \sum_f \\ J & & \left\langle \sum_{f(i)=j} x_i \right\rangle \end{array}$$

Monoidal Functors

$$(\mathbb{FSet}, +, \emptyset) \xrightarrow{\quad} (\mathbb{SSet}, \times, 1)$$

$$1 \longmapsto X := M(1)$$

$$\begin{array}{ccc} I & & x^I \\ \downarrow f & \longmapsto & \downarrow \sum_f \\ J & & \left\langle \sum_{f(i)=j} x_i \right\rangle \end{array}$$

Monoidal Functors

$$(\mathbb{FSet}, +, \emptyset) \xrightarrow{\quad \mathbb{M} := (\text{Float}^{n \times n}, +, 0) \quad} (\mathbb{Set}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} I & & x^I \\ \downarrow f & \longrightarrow & \downarrow \sum_f \\ J & & \left\langle \sum_{f(i)=j} x_i \right\rangle \end{array}$$

Monoidal Functors

$$(\mathbb{FSet}, +, \emptyset) \xrightarrow{\mathbb{M} := (\text{Float}, \times, 0)} (\mathbb{SSet}, \times, 1)$$

$$1 \mapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} I & & x^I \\ \downarrow f & \longrightarrow & \downarrow \\ J & \xrightarrow{\prod_f} & \left\langle \prod_{f(i)=j} \{x_i\} \right\rangle \end{array}$$

Monoidal Functors

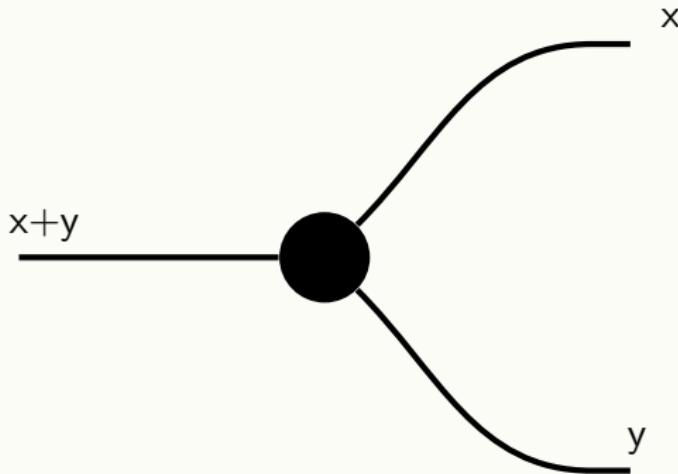
$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\quad \mathbb{M} := (\text{Float}^{n \times n}, \times, 0) \quad} (\mathbb{S}\text{et}, \times, 1)$$

$$1 \longmapsto X := \mathbb{M}(1)$$

$$\begin{array}{ccc} I & & x^I \\ \downarrow f & \longrightarrow & \downarrow \\ J & \longrightarrow & \prod_f \end{array}$$
$$x^J \quad \left\langle \prod_{f(i)=j} \{x_i\} \right\rangle$$

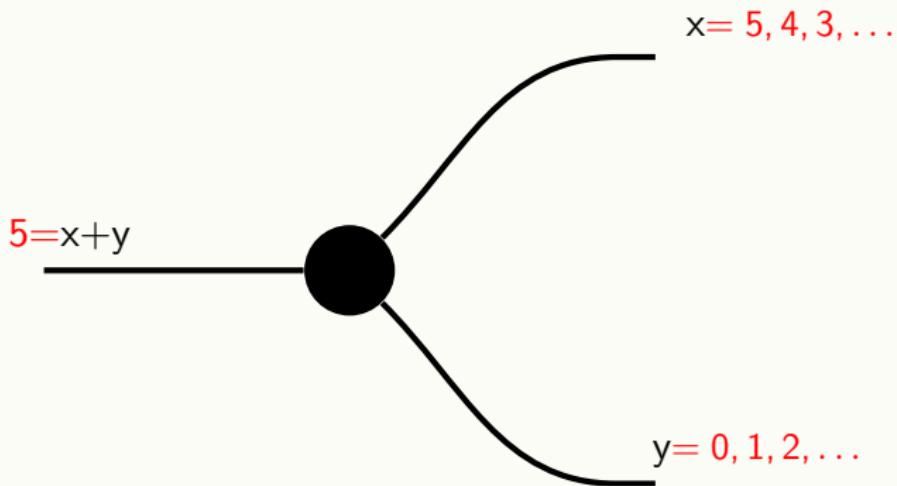
Put it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{???} (\text{Set}, \times, 1)$$



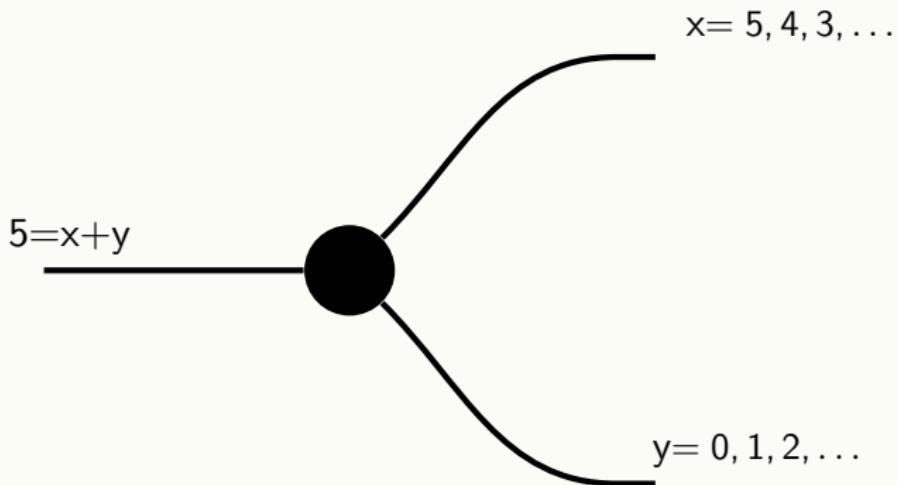
Put it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{???} (\cancel{\text{Set}}, \times, 1)$$



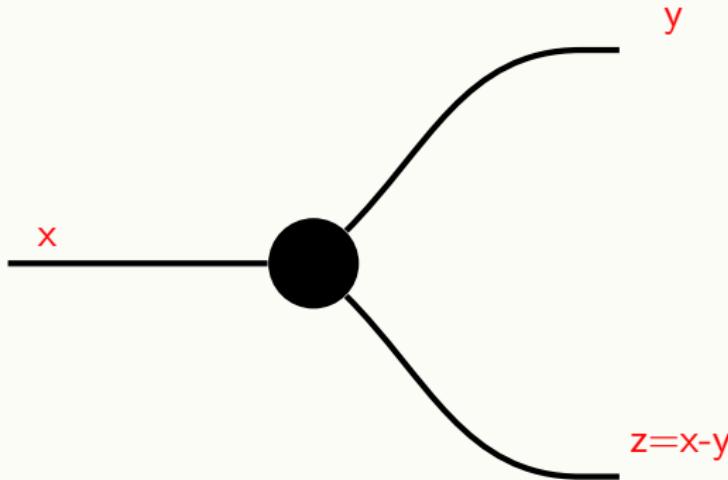
Put it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{\quad \checkmark \quad} (\mathbb{R}\text{el}, \times, 1)$$

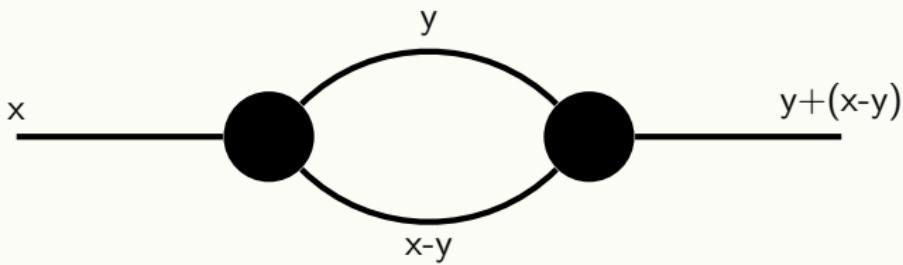


Put it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{\quad \checkmark \quad} (\mathbb{R}\text{el}, \times, 1)$$



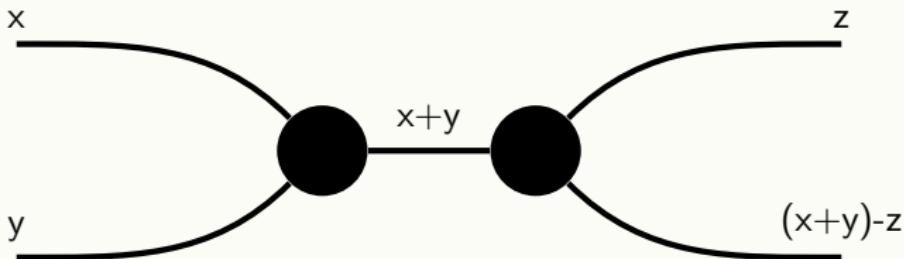
More rules



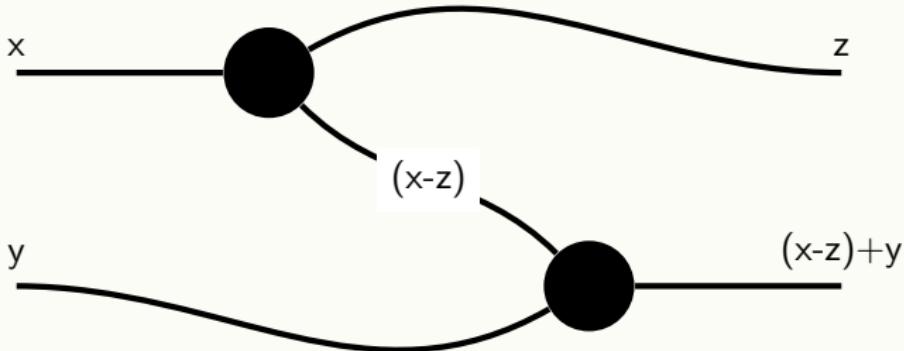
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More rules



=



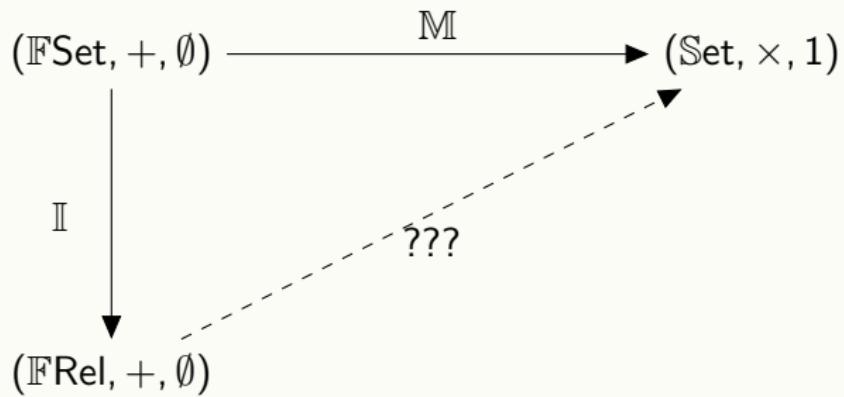
Semantic extension

$$(\mathbb{F}\text{Set}, +, \emptyset) \xrightarrow{\mathbb{M}} (\mathbb{S}\text{et}, \times, 1)$$

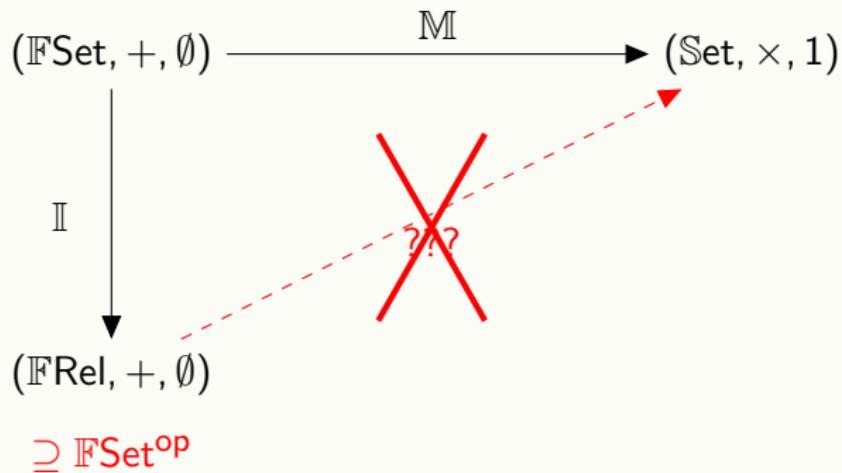
Semantic extension

$$\begin{array}{ccc} (\mathbb{F}\text{Set}, +, \emptyset) & \xrightarrow{\mathbb{M}} & (\mathbb{S}\text{et}, \times, 1) \\ \downarrow \mathbb{I} & & \\ (\mathbb{F}\text{Rel}, +, \emptyset) & & \end{array}$$

Semantic extension



Semantic extension



Semantic extension

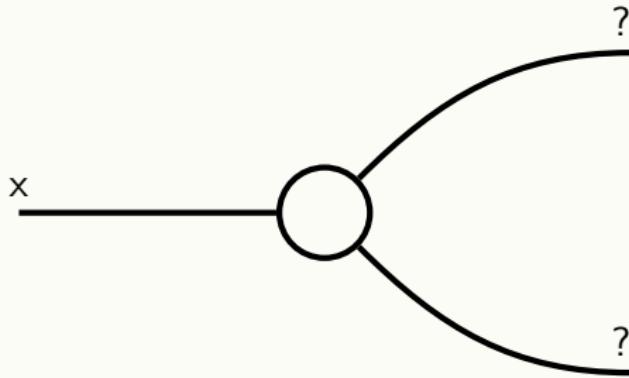
$$\begin{array}{ccc} (\mathbb{F}\text{Set}, +, \emptyset) & \xrightarrow{\mathbb{M}} & (\mathbb{S}\text{et}, \times, 1) \\ \downarrow \mathbb{I}^+ & & \downarrow \mathbb{I}^\times \\ (\mathbb{F}\text{Rel}, +, \emptyset) & & (\mathbb{R}\text{el}, \times, 1) \end{array}$$

Semantic extension

$$\begin{array}{ccc} (\mathbb{F}\text{Set}, +, \emptyset) & \xrightarrow{\mathbb{M}} & (\mathbb{S}\text{et}, \times, 1) \\ \downarrow \mathbb{I}^+ & & \downarrow \mathbb{I}^\times \\ (\mathbb{F}\text{Rel}, +, \emptyset) & \xrightarrow{\mathbb{M}^\dagger} & (\mathbb{R}\text{el}, \times, 1) \end{array}$$

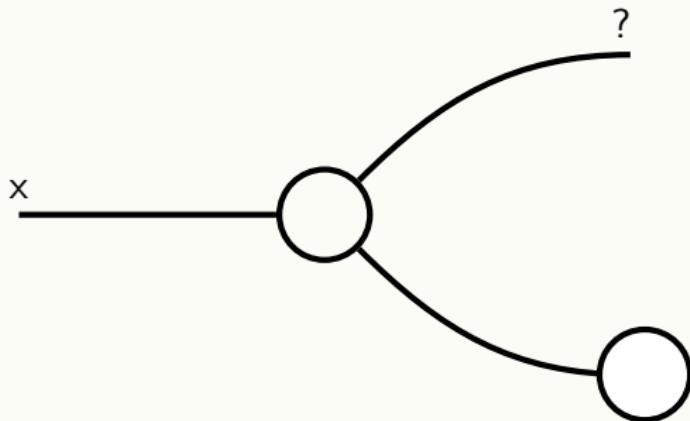
Force it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{???, \rightarrow} (\mathbb{S}\text{et}, \times, 1)$$



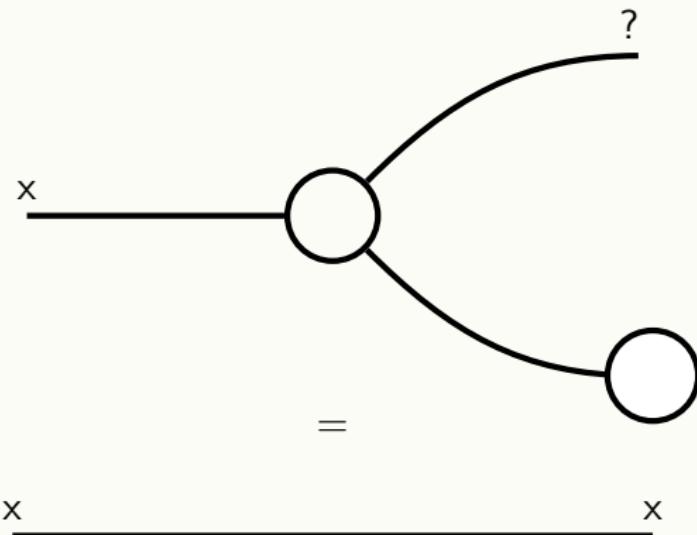
Force it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{???) (\mathbb{S}\text{et}, \times, 1)$$



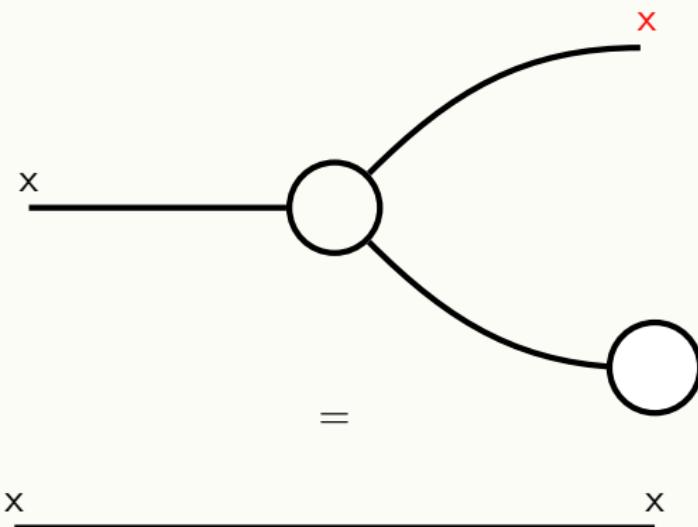
Force it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{???) (\mathbb{S}\text{et}, \times, 1)$$



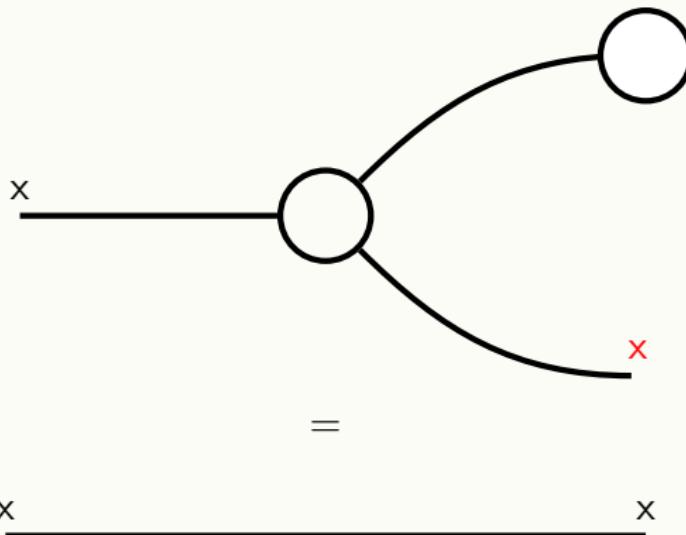
Force it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{???) (\mathbb{S}\text{et}, \times, 1)$$



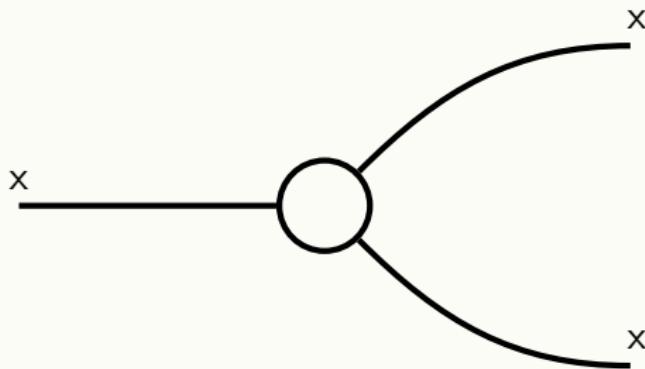
Force it in backwards?

$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{???) (\mathbb{S}\text{et}, \times, 1)$$

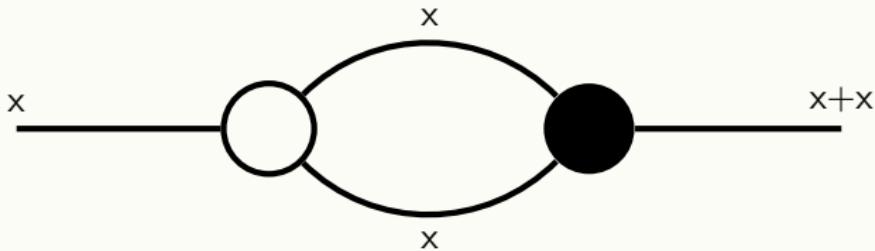


Force it in backwards?

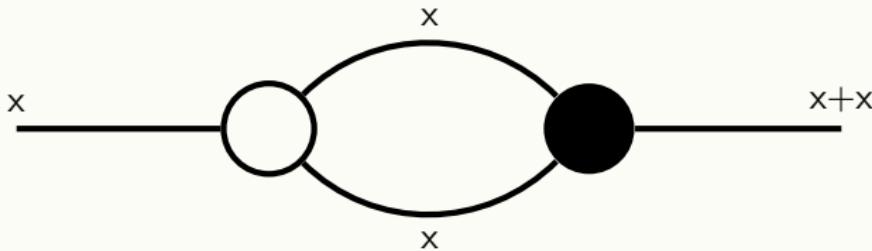
$$(\mathbb{F}\text{Set}^{\text{op}}, +, \emptyset) \xrightarrow{(\mathbf{X}, \text{copy})} (\mathbb{S}\text{et}, \times, 1)$$



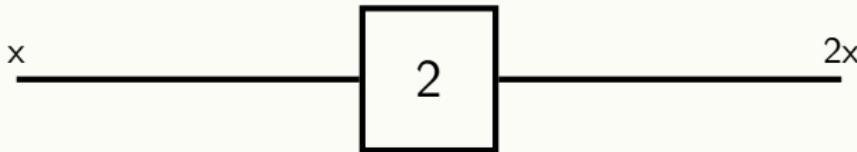
Add meets copy



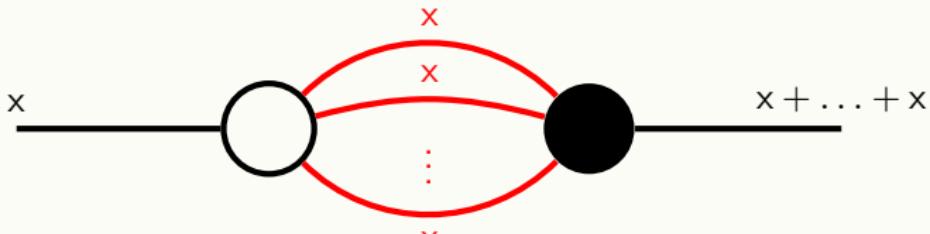
Add meets copy



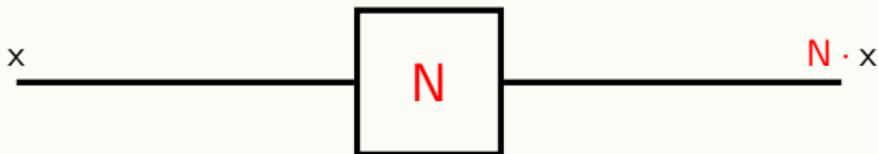
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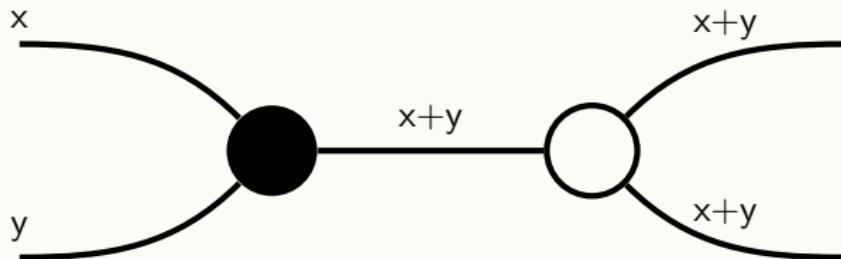
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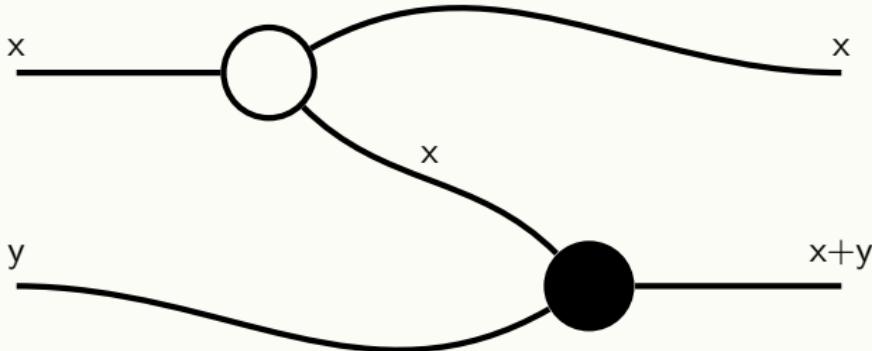
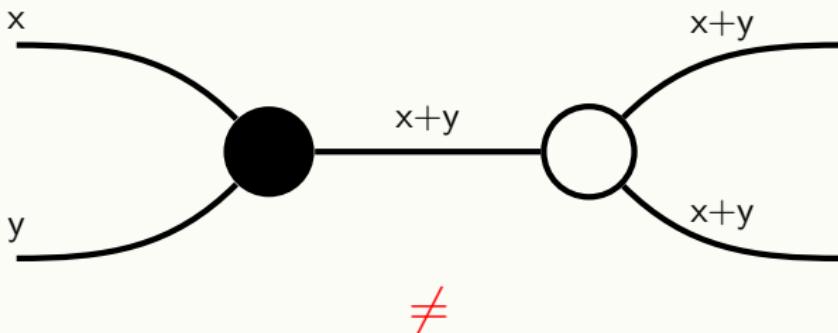
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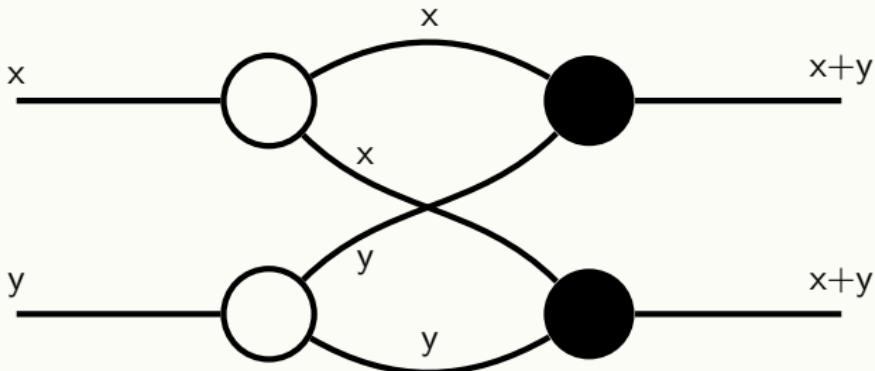
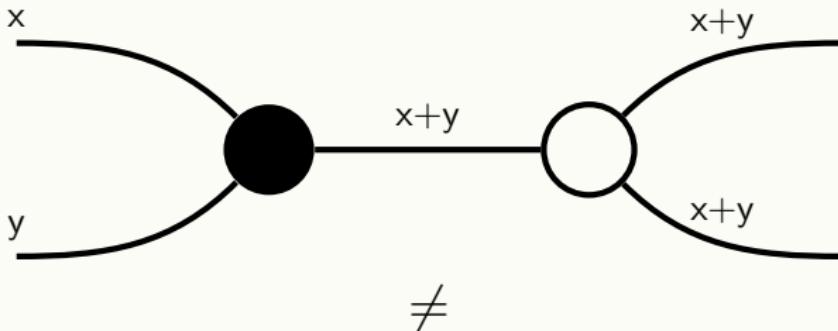
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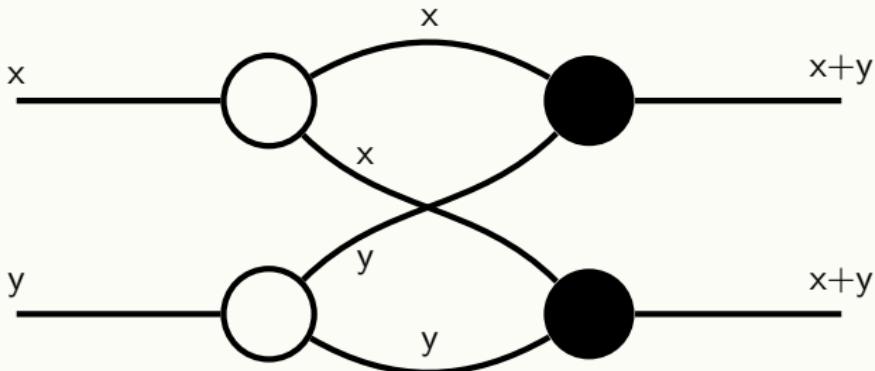
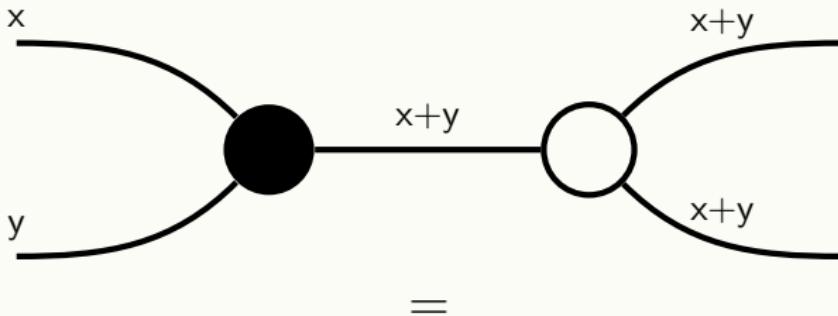
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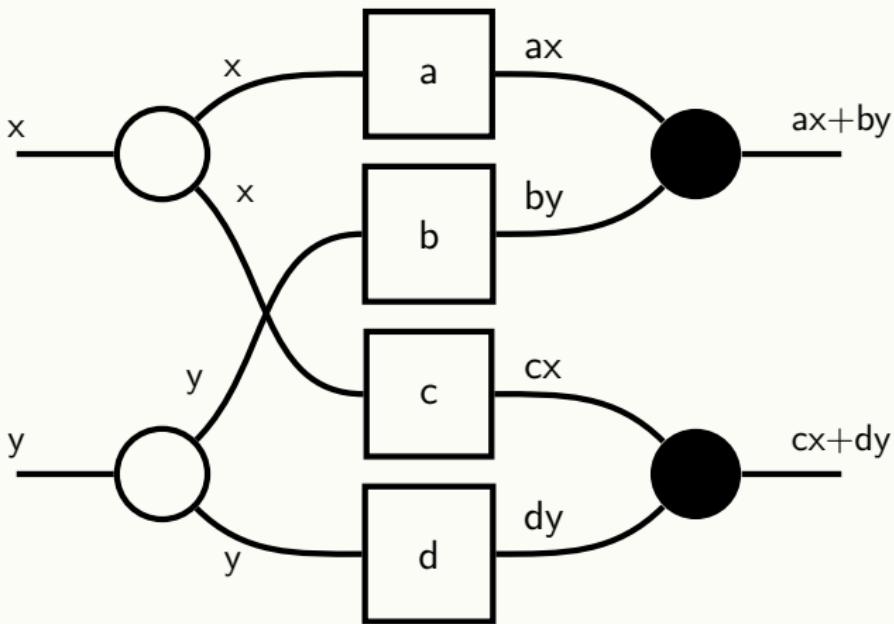
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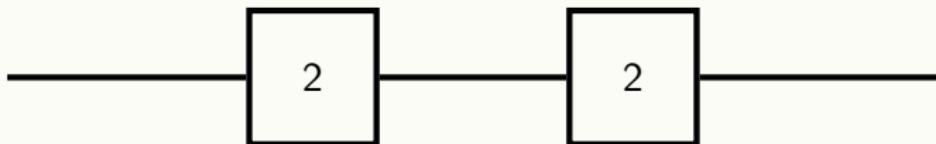
Add meets copy



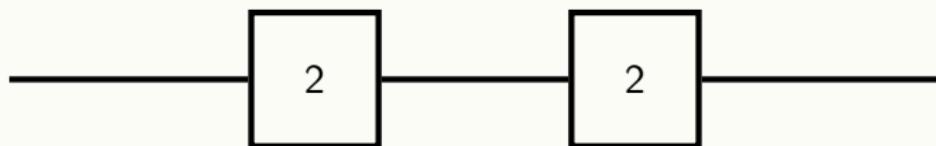
Matrices



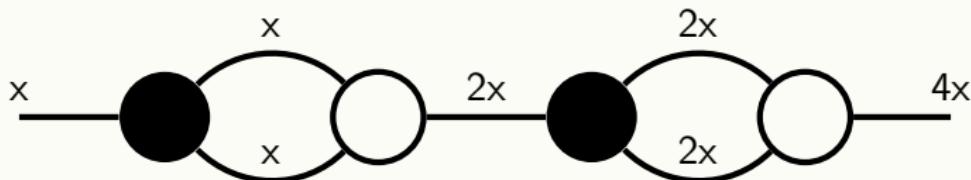
Diagrams multiply



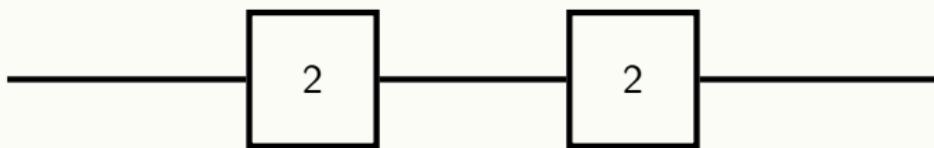
Diagrams multiply



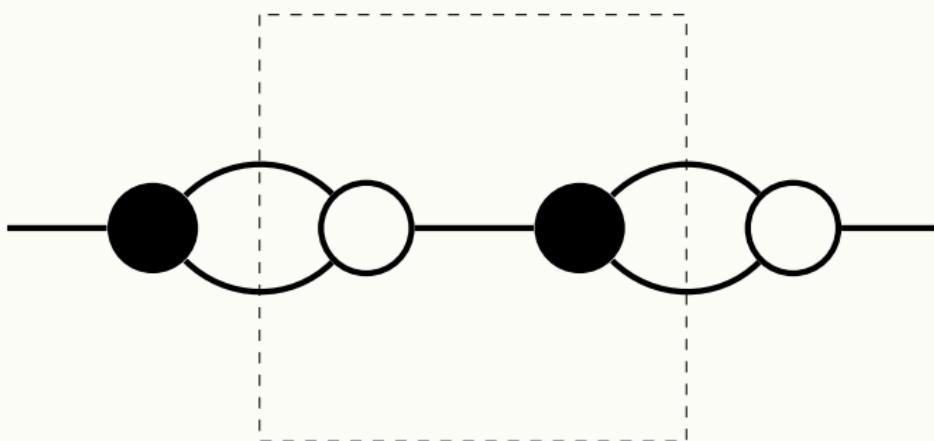
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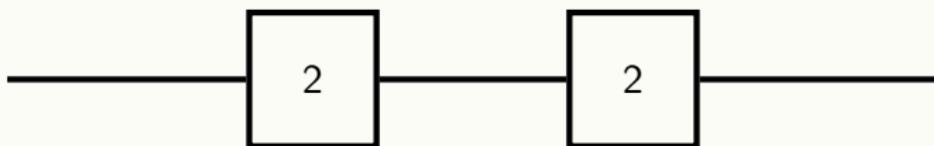
Diagrams multiply



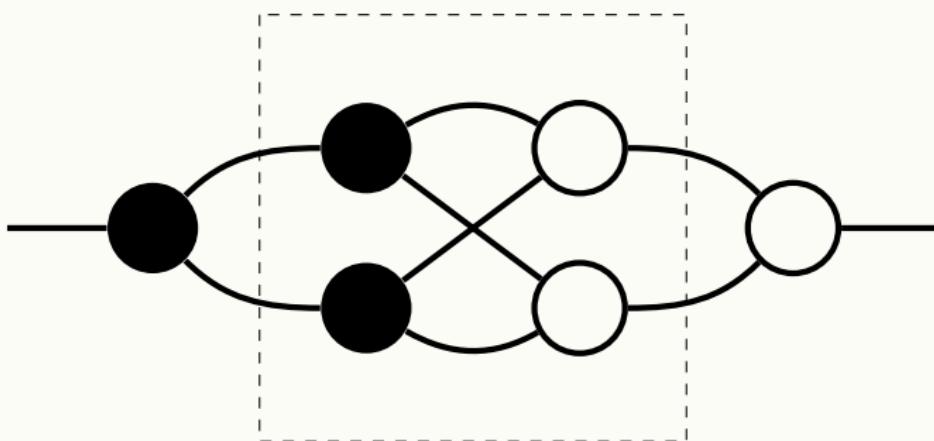
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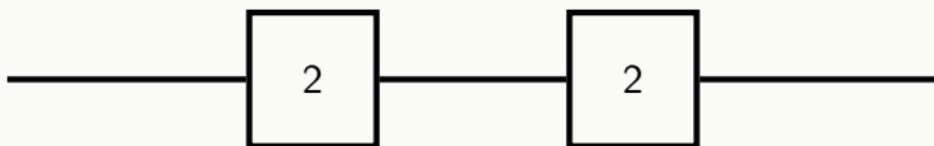
Diagrams multiply



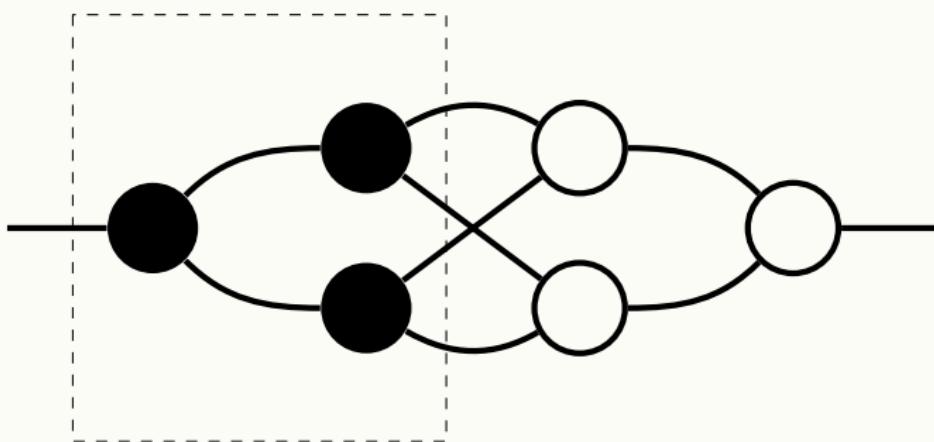
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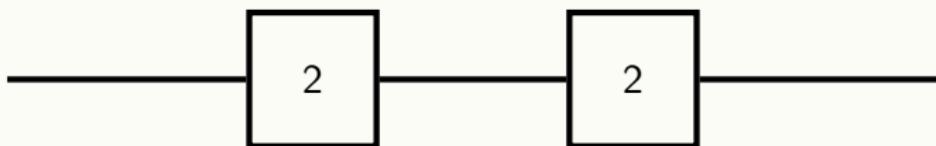
Diagrams multiply



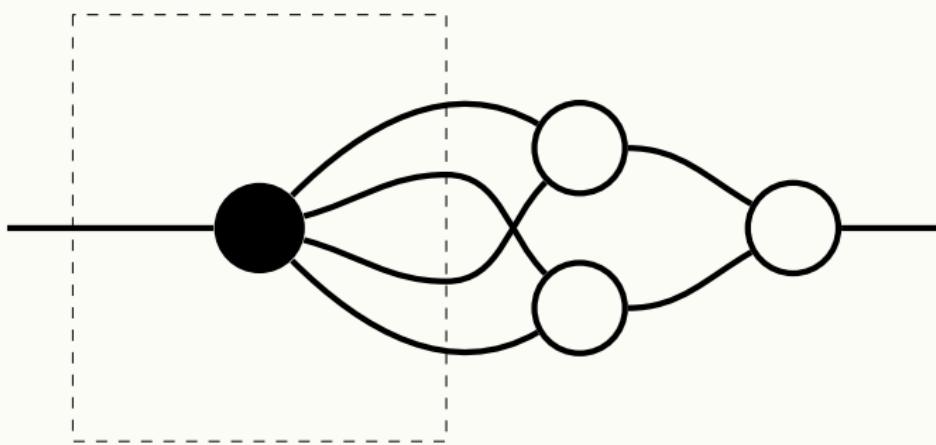
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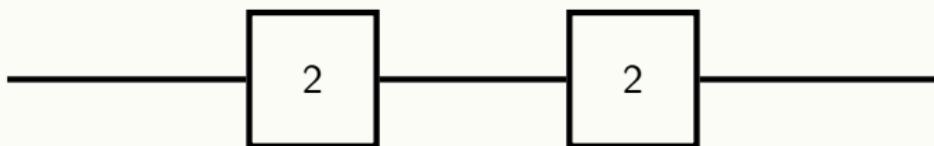
Diagrams multiply



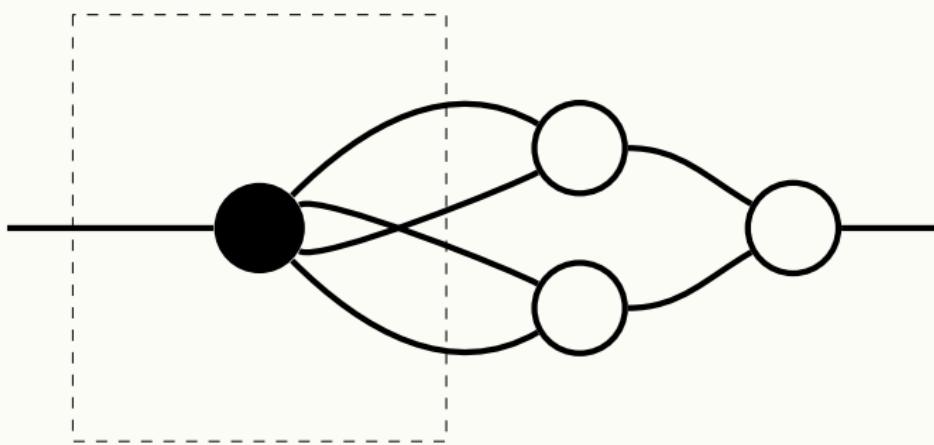
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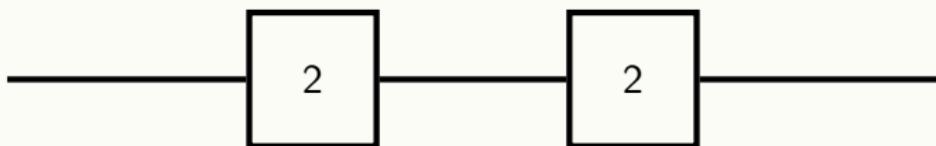
Diagrams multiply



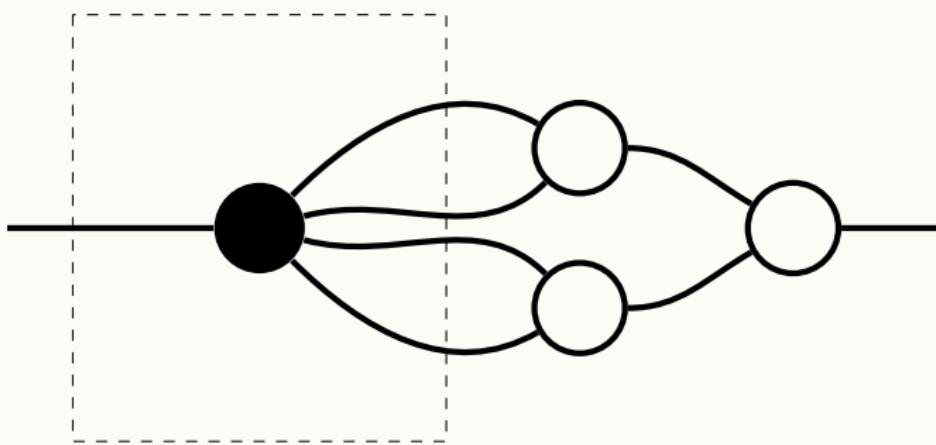
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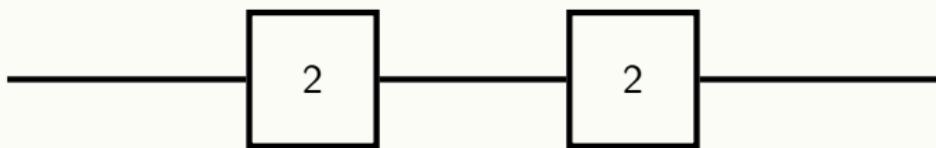
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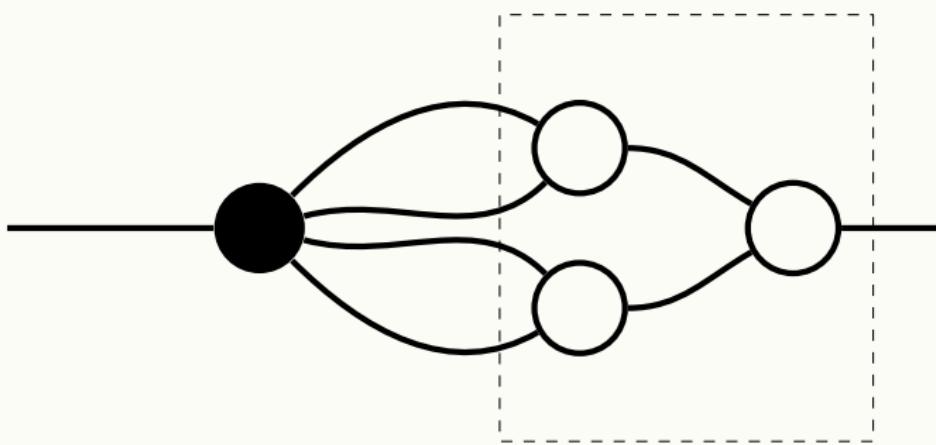
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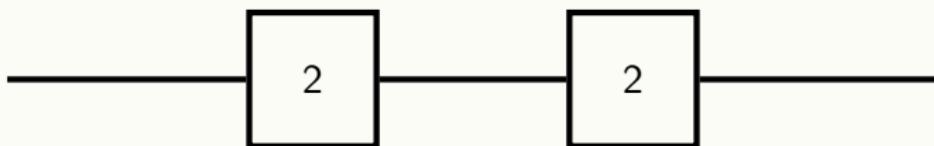
Diagrams multiply



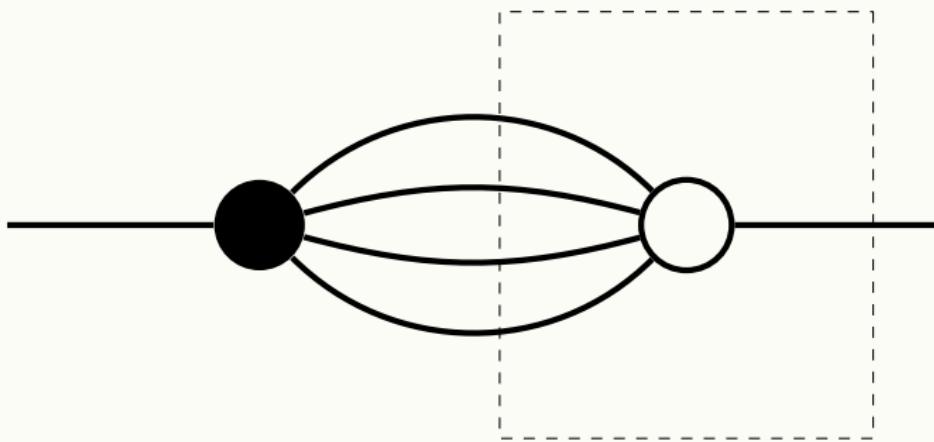
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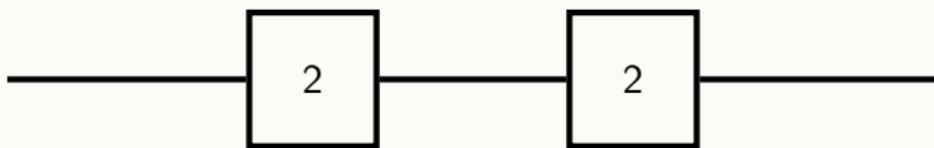
Diagrams multiply



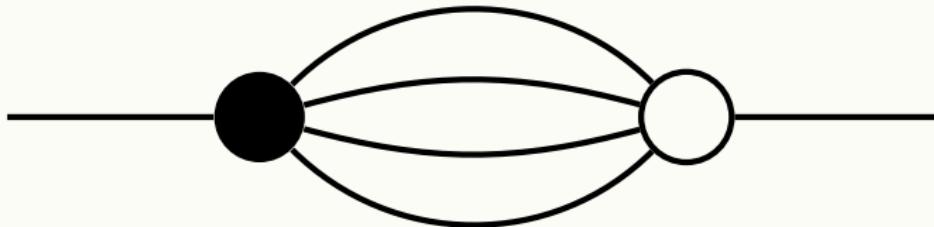
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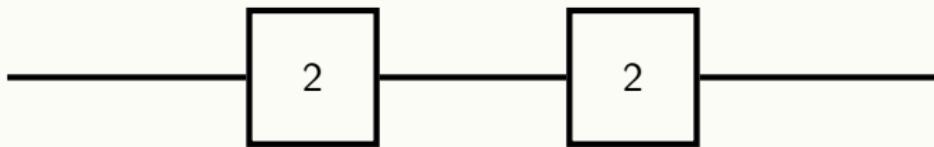
Diagrams multiply



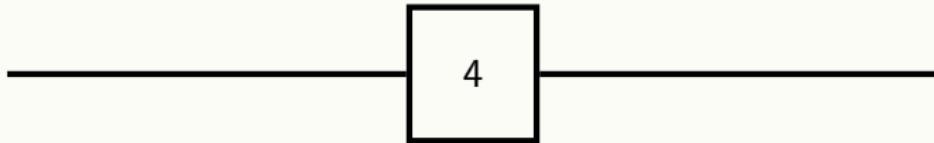
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Diagrams multiply

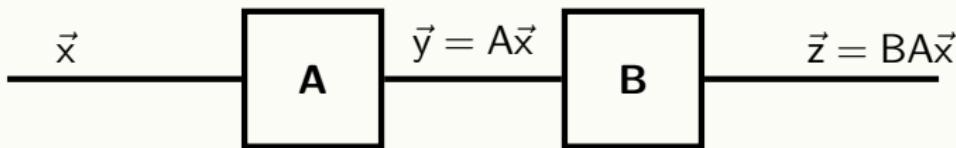


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Homework

For matrices A ($k \times n$) and B ($m \times k$), show

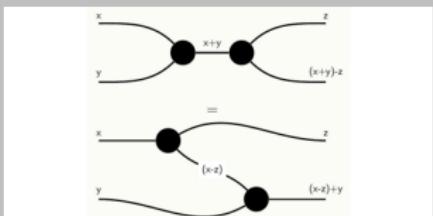


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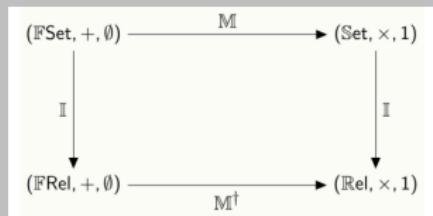


Questions?

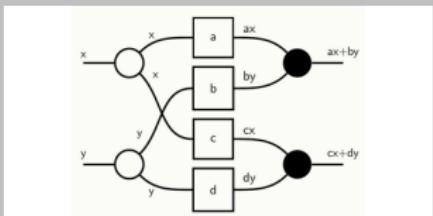
Addition^{op}



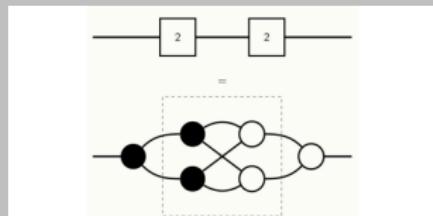
Semantic extension



Matrices



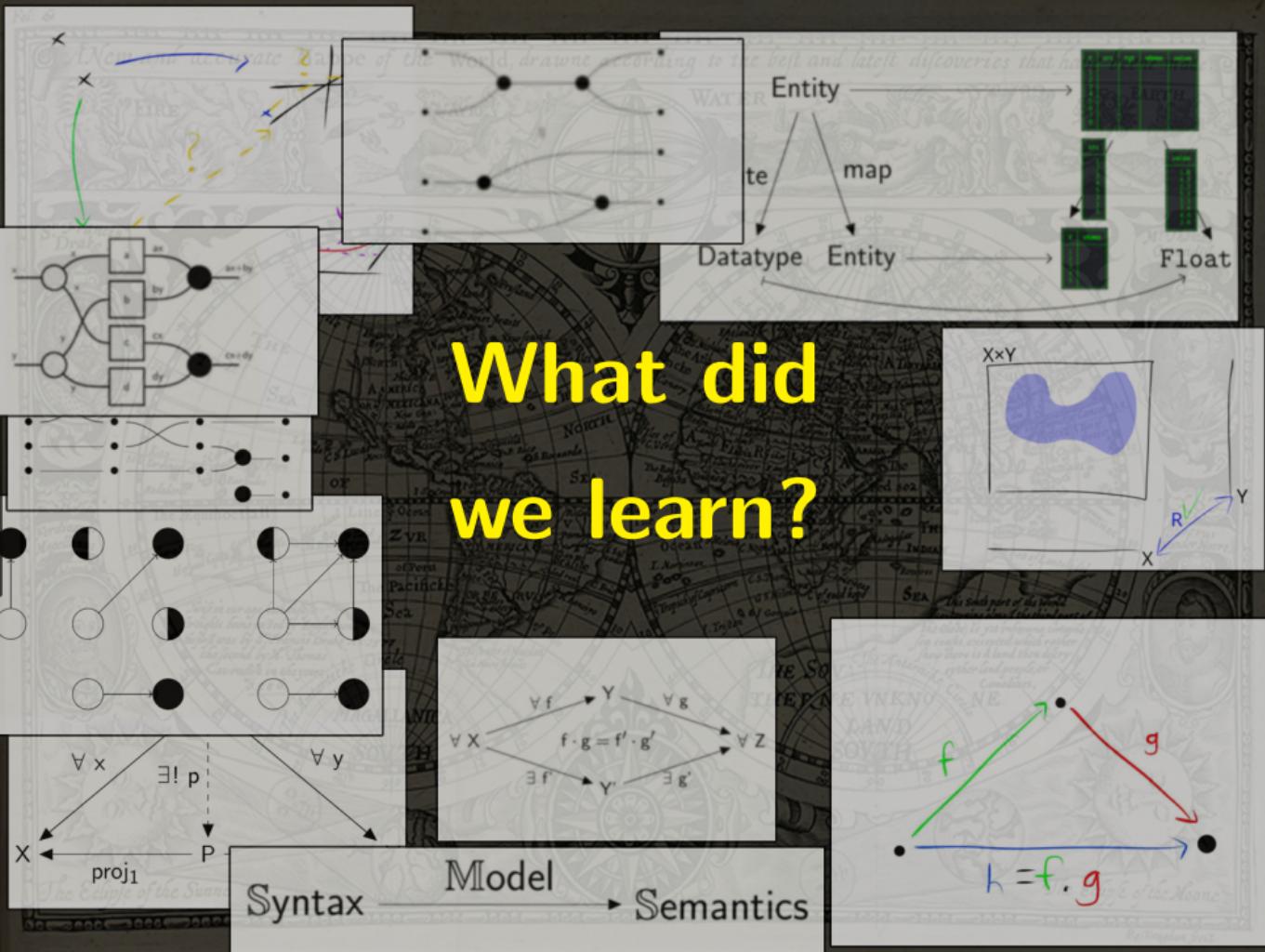
Multiplication

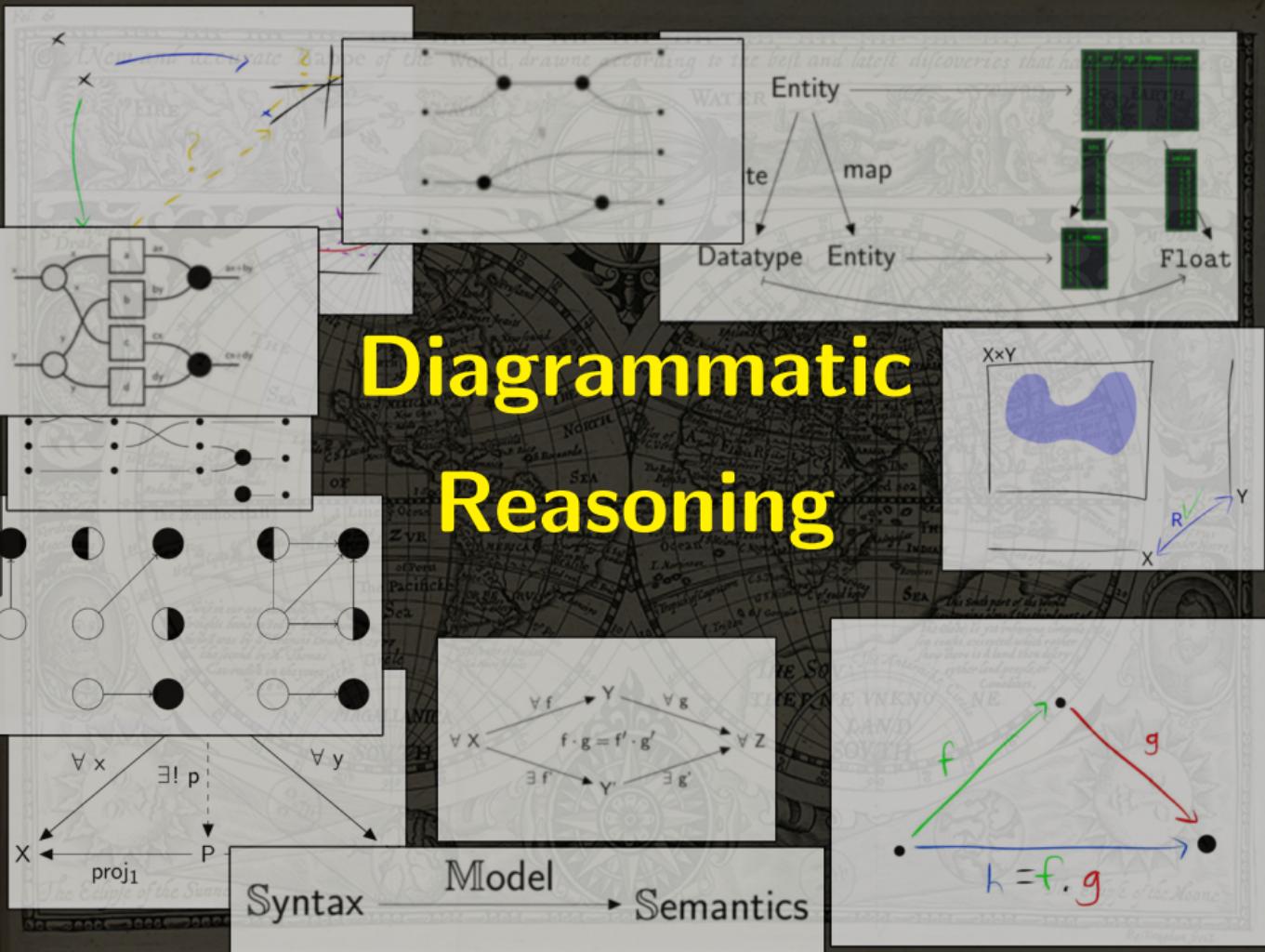


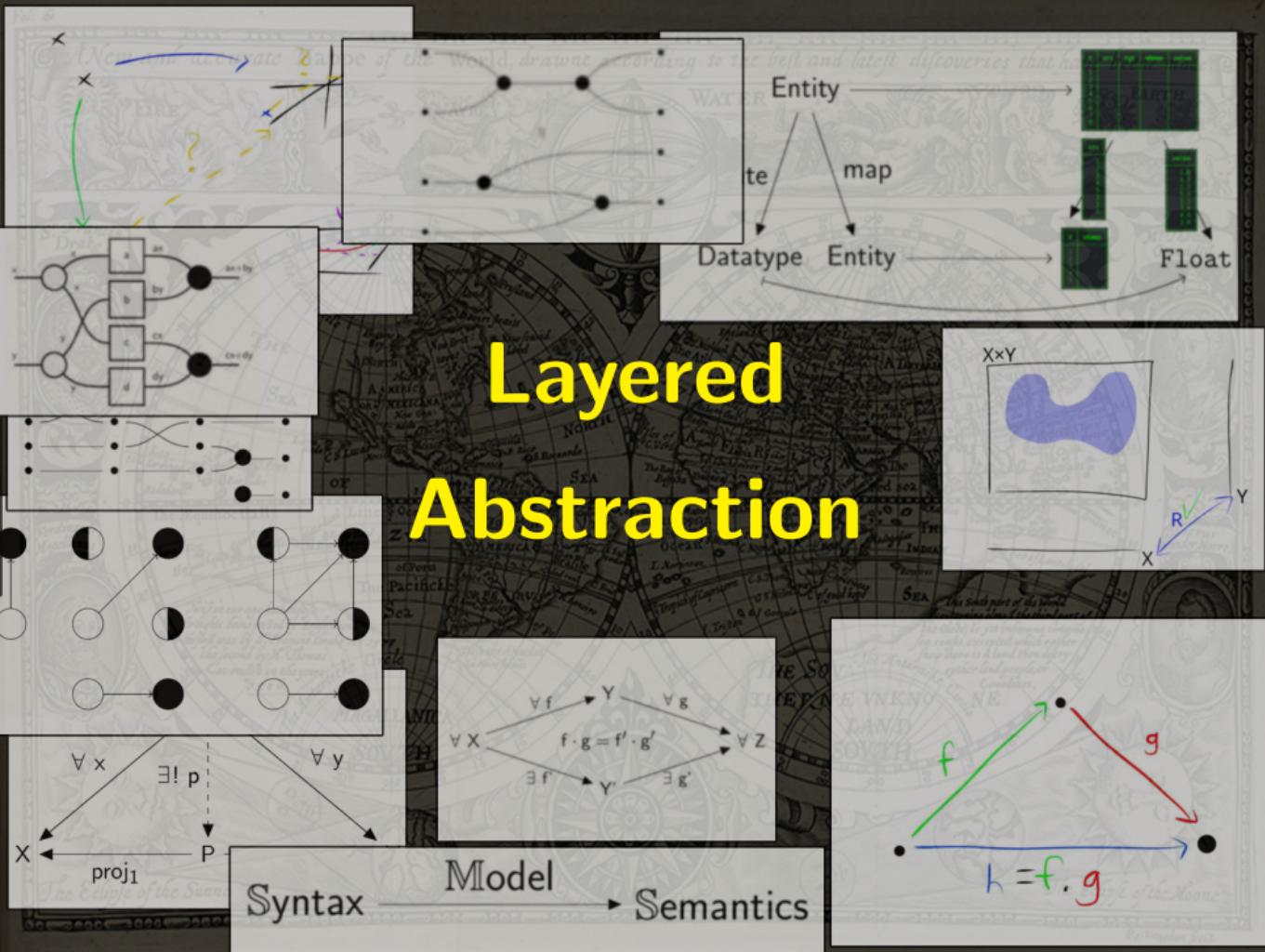
Wrapping up

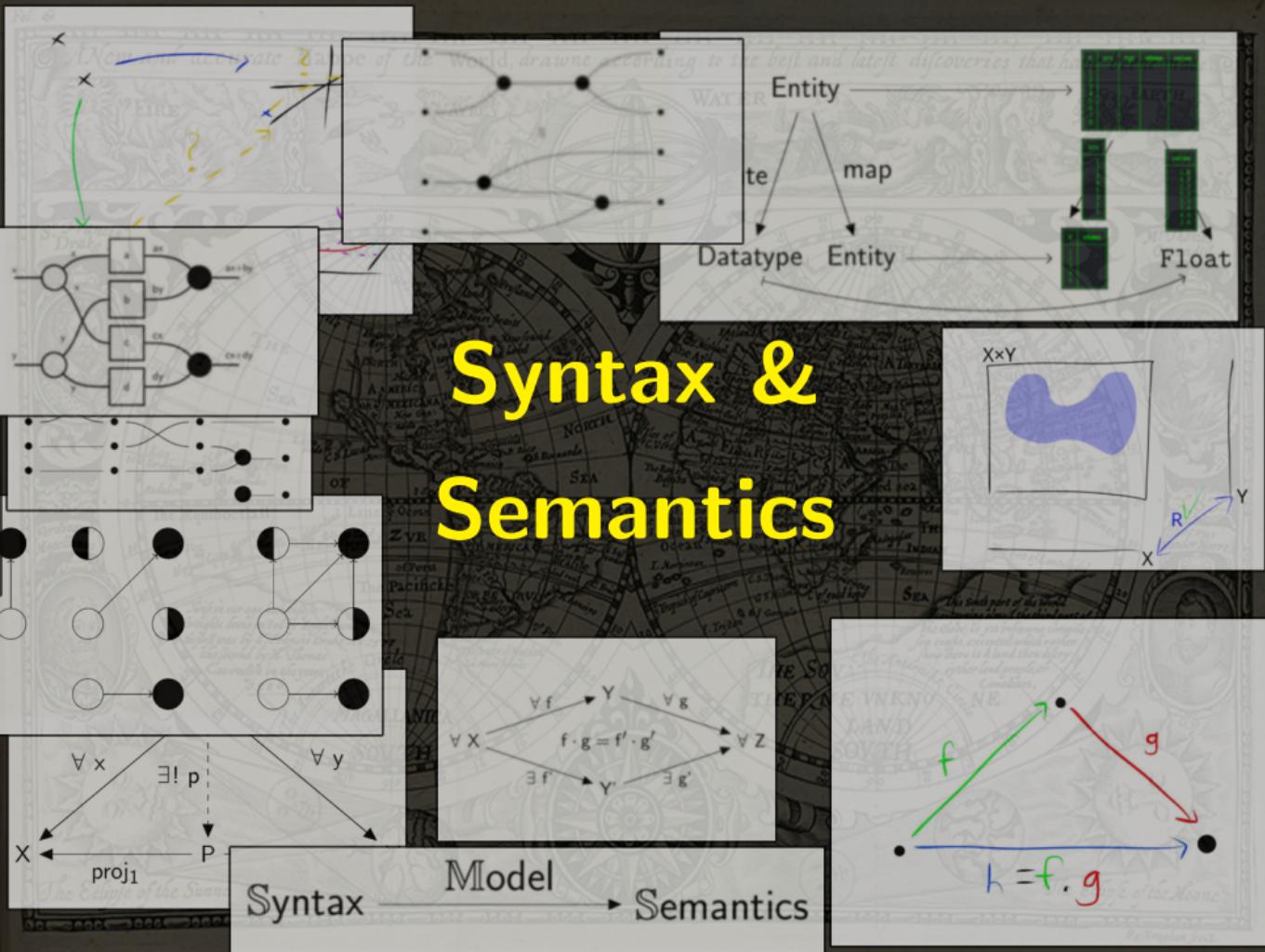
What we did

| | | |
|---------------|-------------|--|
| Part 1 | Composition | $\mathbb{S}et, \mathbb{R}el$ |
| | Categories | Diagrams |
| | Structure | $X \cong Y, X \times Y$ |
| | | |
| Part 2 | Models | $\mathbb{M} : \mathbb{S}yn \rightarrow \mathbb{S}em$ |
| | Data | Catlab |
| | Processes | Add \heartsuit Copy |









What we didn't do

Real applications

| | |
|------------------------|-------------|
| Cyber physical systems | Robotics |
| Graph transformation | Design |
| Data Analytics | Databases |
| Programming | Networks |
| Epidemiology | Ecology |
| Neural Networks | Language |
| Molecular Biology | Probability |

What we didn't do

Real proofs

- Books
 - Spivak & Fong - Seven Sketches
 - Lawvere & Schanuel - Conceptual Mathematics
 - Spivak - CT for the sciences

Papers

- Bradley - What is applied category theory?
 - Baez & Stay - A Rosetta Stone
 - Coecke, et al. - Mathematical theory of resources

More resources

Blogs

[Graphical Linear Algebra](#) [AlgebraicJulia](#)

[The n-Category Café](#) [Math3ma](#)

Videos

[Cheng - Category Theory in Life](#)

[Compositionality \(Simons Inst.\)](#)

[Compositional Robotics \(ICRA workshop\)](#)

What's still missing

User Interface

Use cases

Methodology

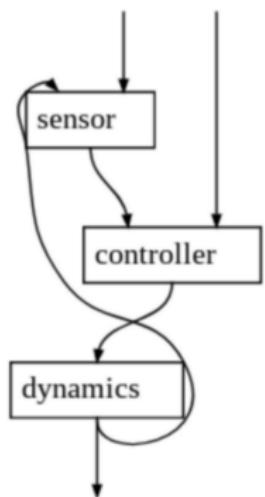
Community of Practice

Thank You!

Please join us for an open discussion on category theory and systems engineering after the break.

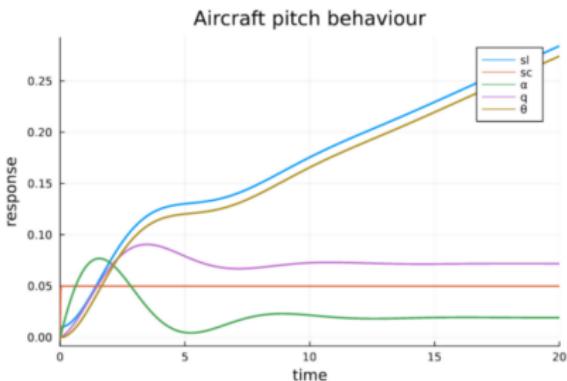
Endnotes

Cyber Physical Systems

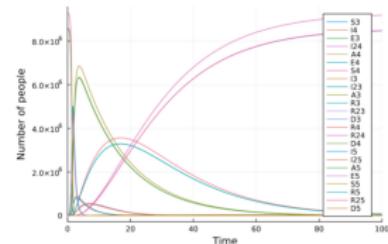
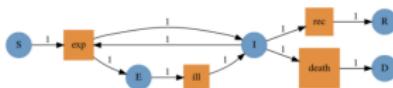
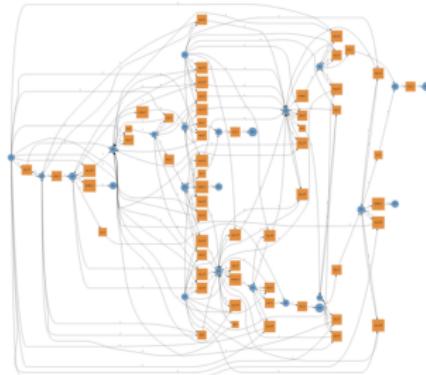
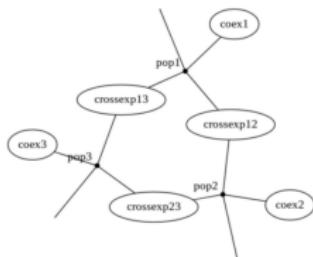


```
function L(w)
    L(u, x, p, t) = LVector( sc = -p.Rl * (u[1] - x[1] - x[2]) );
    C(u, x, p, t) = LVector( sl = -p.Rc * (u[1] + p.Bc*x[1] - x[2]) );
    D(u, x, p, t) = LVector( alpha = -0.313*u[1] + 56.7*u[2] + 0.232*x[1],
                                q = -0.013*u[1] - 0.426*u[2] + 0.0203*x[1],
                                theta = 56.7*u[2] );

```



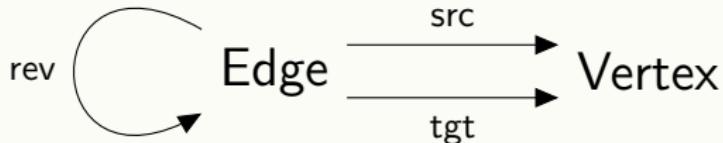
Epidemiology



Libkind, et al (code)

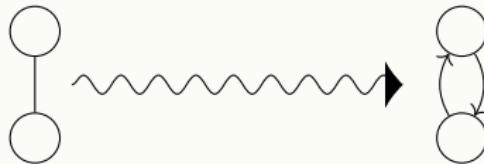
Symmetric Graphs

Schema:



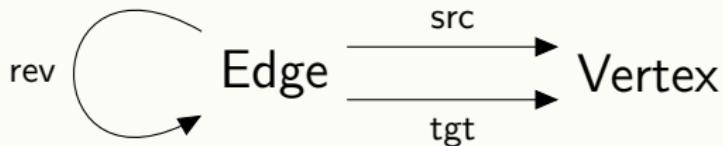
$$\text{rev} \cdot \text{rev} = \text{id}$$

$$\text{rev} \cdot \text{src} = \text{tgt}$$



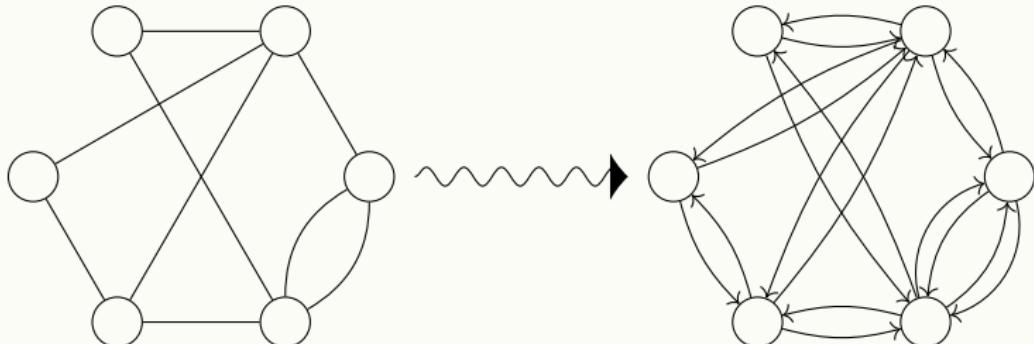
Symmetric Graphs

Schema:



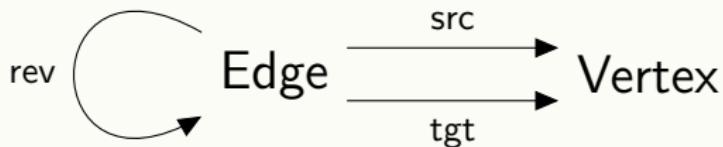
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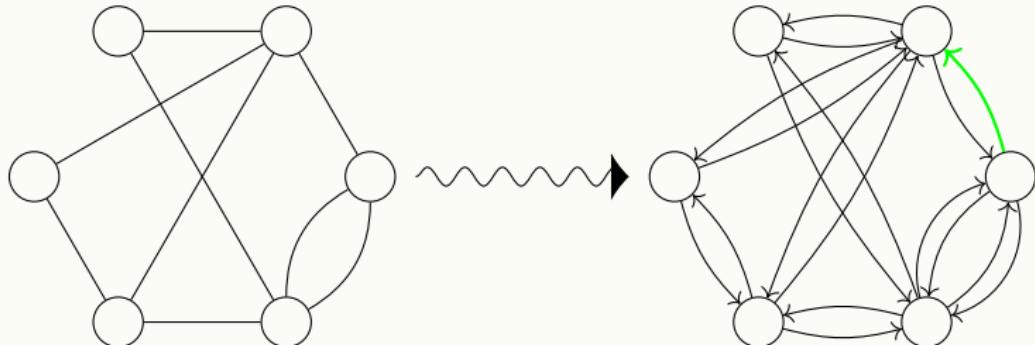
Symmetric Graphs

Schema:



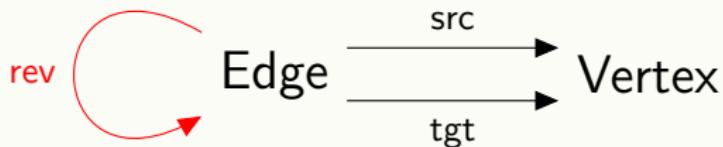
$$\text{rev} \cdot \text{rev} = \text{id}$$

$$\text{rev} \cdot \text{src} = \text{tgt}$$



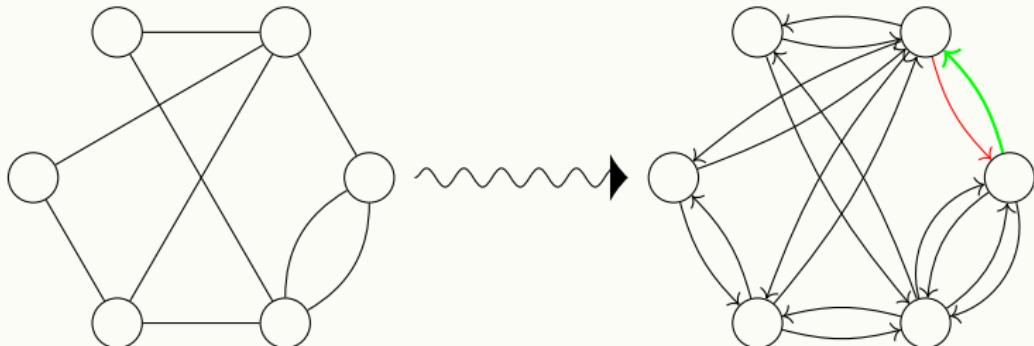
Symmetric Graphs

Schema:



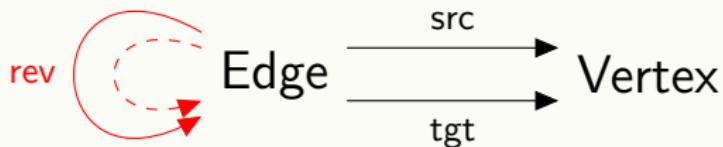
$$\text{rev} \cdot \text{rev} = \text{id}$$

$$\text{rev} \cdot \text{src} = \text{tgt}$$



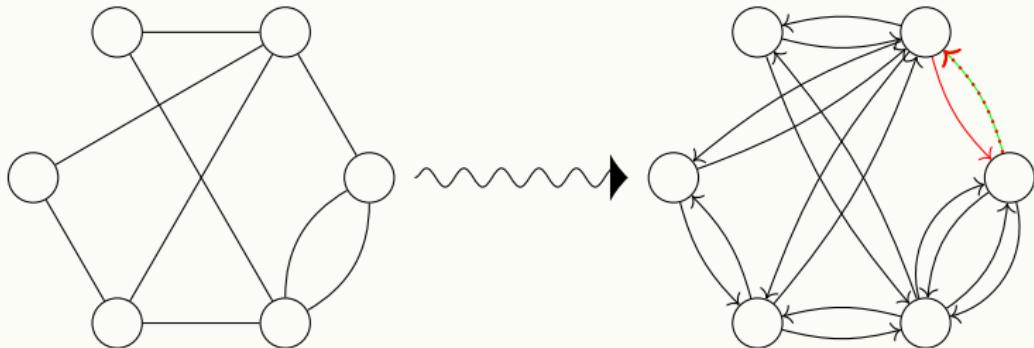
Symmetric Graphs

Schema:



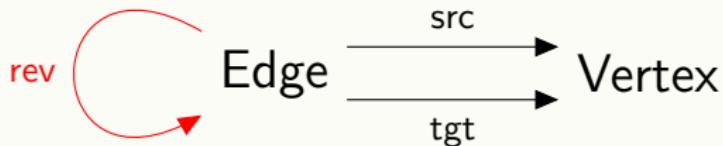
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$$\text{rev} \cdot \text{src} = \text{tgt}$$



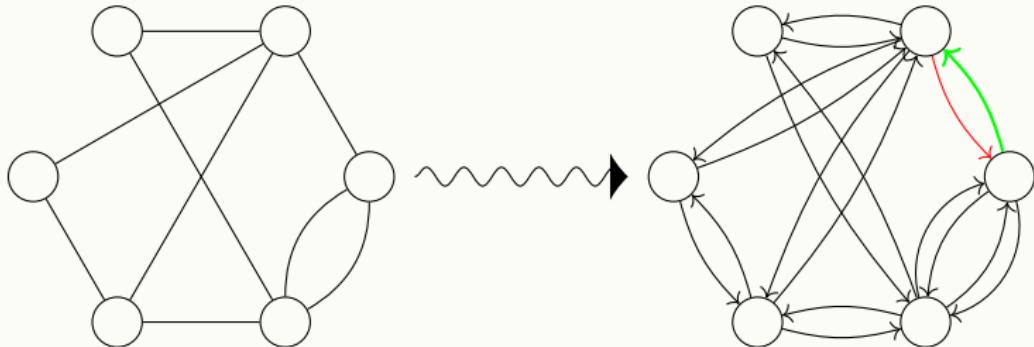
Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

$$\text{rev} \cdot \text{src} = \text{tgt}$$



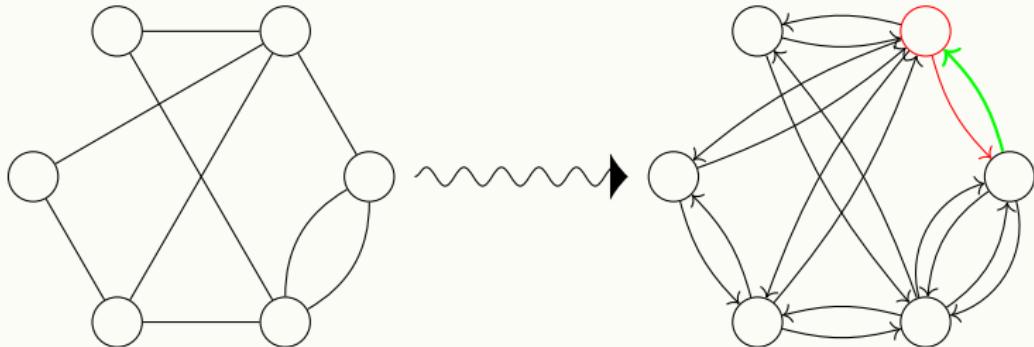
Symmetric Graphs

Schema:



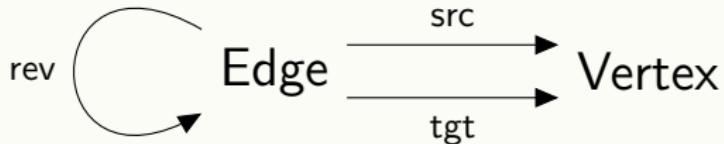
$$\text{rev} \cdot \text{rev} = \text{id}$$

$$\text{rev} \cdot \text{src} = \text{tgt}$$



Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

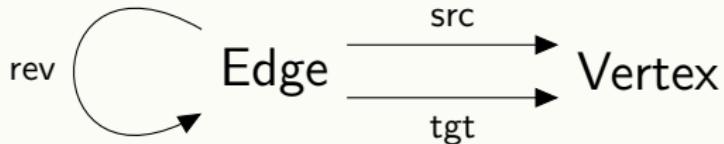
$$\text{rev} \cdot \text{src} = \text{tgt}$$

Equational reasoning:

$$\text{rev} \cdot \text{tgt}$$

Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

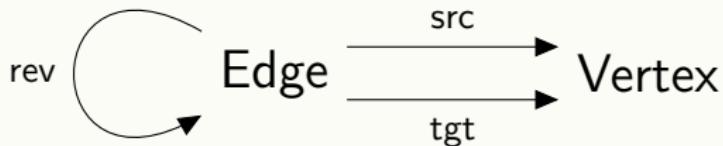
$$\text{rev} \cdot \text{src} = \text{tgt}$$

Equational reasoning:

$$\text{rev} \cdot \text{tgt} = \text{rev} \cdot (\text{rev} \cdot \text{src})$$

Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

$$\text{rev} \cdot \text{src} = \text{tgt}$$

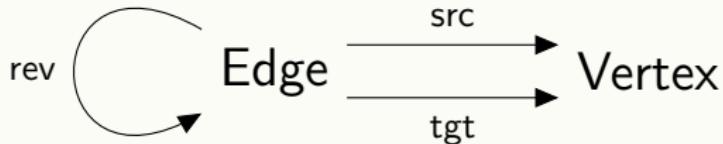
Equational reasoning:

$$\text{rev} \cdot \text{tgt} = \text{rev} \cdot (\text{rev} \cdot \text{src})$$

$$= (\text{rev} \cdot \text{rev}) \cdot \text{src}$$

Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

$$\text{rev} \cdot \text{src} = \text{tgt}$$

Equational reasoning:

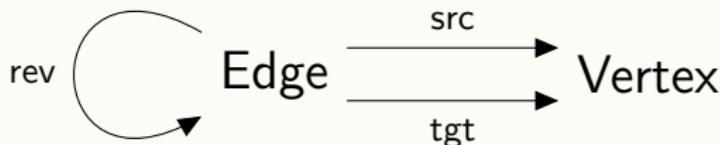
$$\text{rev} \cdot \text{tgt} = \text{rev} \cdot (\text{rev} \cdot \text{src})$$

$$= (\text{rev} \cdot \text{rev}) \cdot \text{src}$$

$$= \text{id} \cdot \text{src}$$

Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

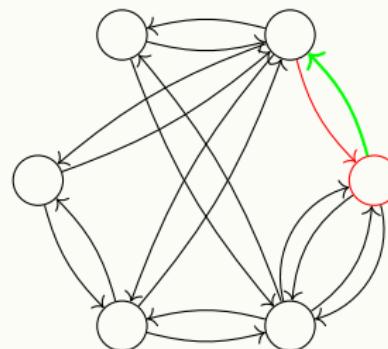
$$\text{rev} \cdot \text{src} = \text{tgt}$$

Equational reasoning:

$$\text{rev} \cdot \text{tgt} = \text{rev} \cdot (\text{rev} \cdot \text{src})$$

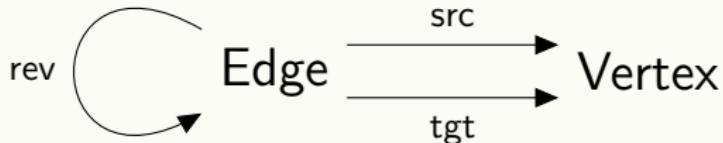
$$= (\text{rev} \cdot \text{rev}) \cdot \text{src}$$

$$= \text{id} \cdot \text{src} = \text{src}$$



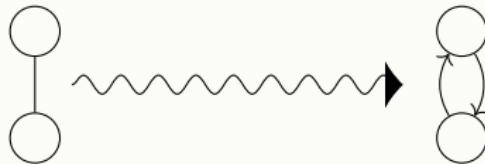
Symmetric Graphs

Schema:



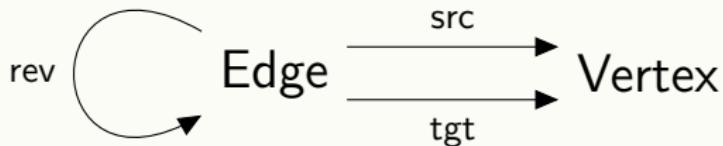
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$$\text{rev} \cdot \text{src} = \text{tgt}$$



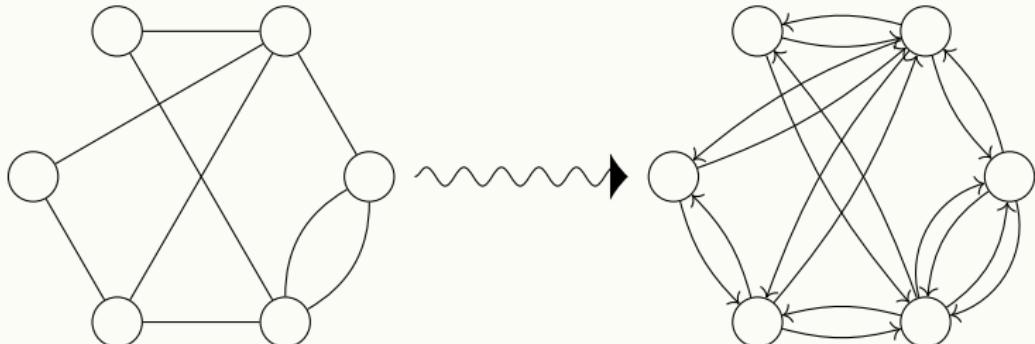
Symmetric Graphs

Schema:



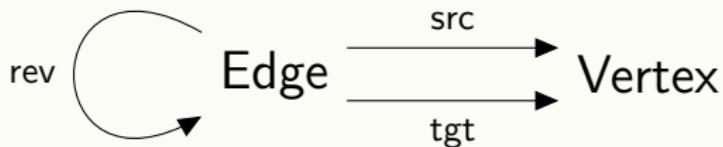
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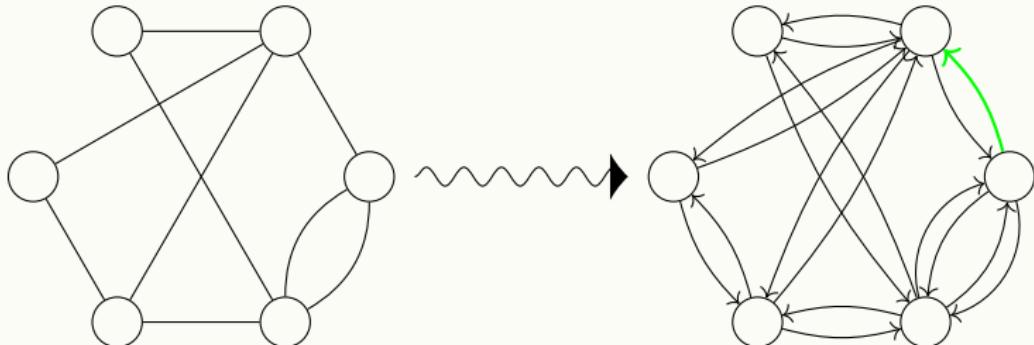
Symmetric Graphs

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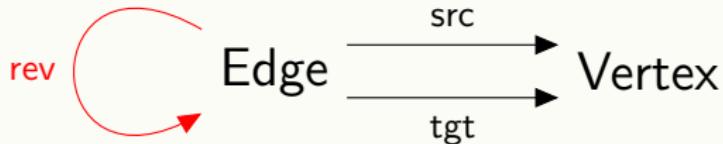
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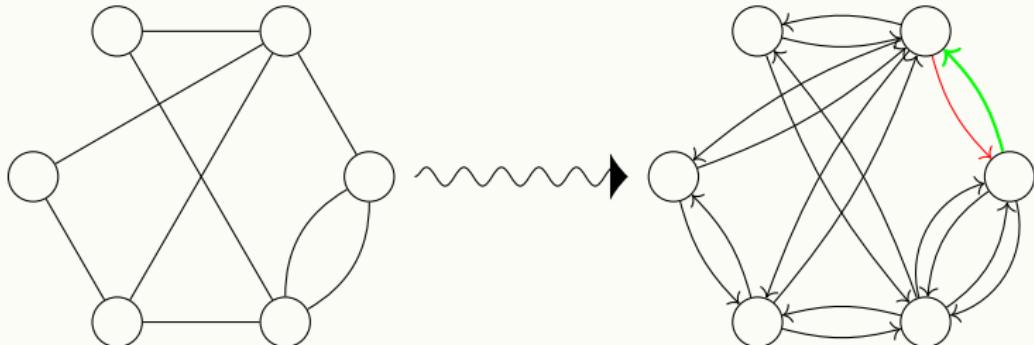
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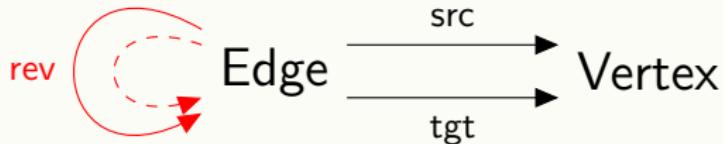
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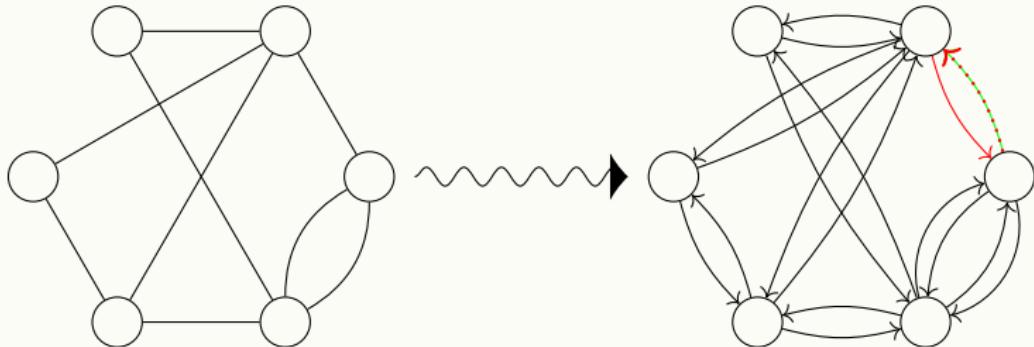
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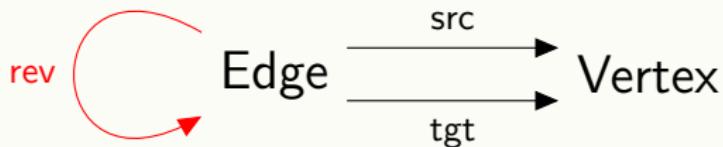
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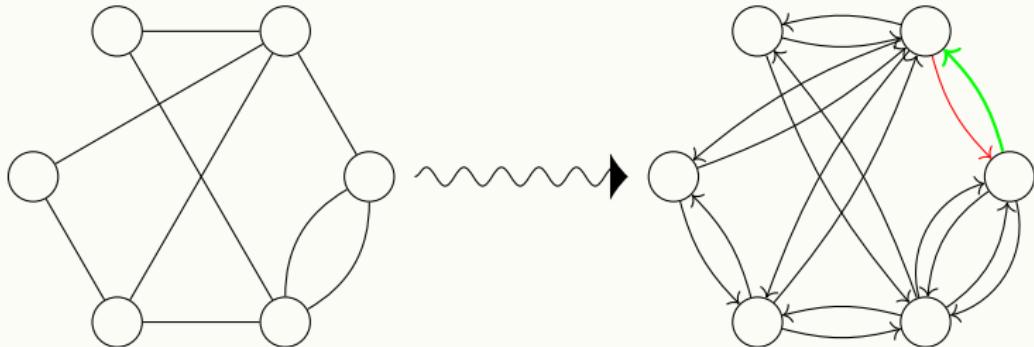
Symmetric Graphs

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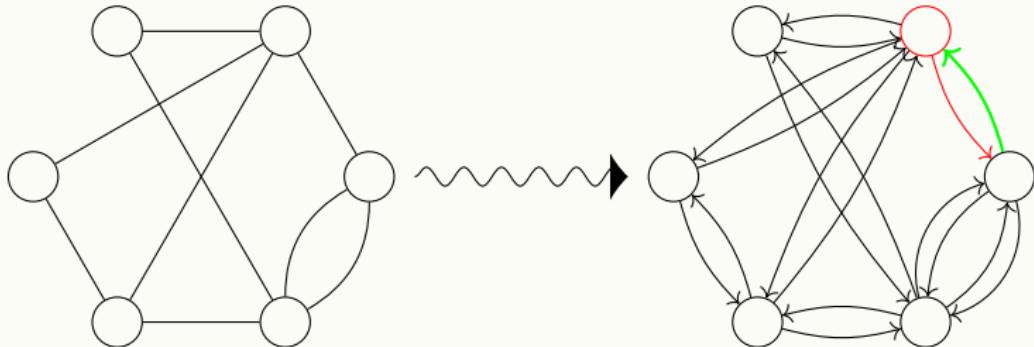
Symmetric Graphs

Schema:



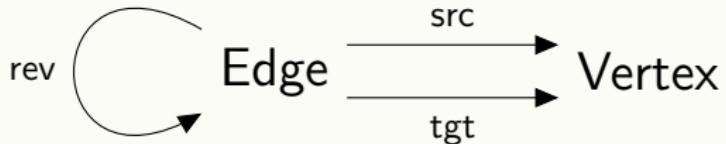
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Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

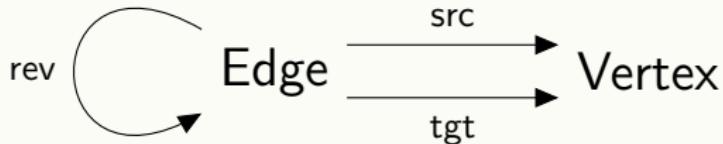
$$\text{rev} \cdot \text{src} = \text{tgt}$$

Equational reasoning:

$$\text{rev} \cdot \text{tgt}$$

Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

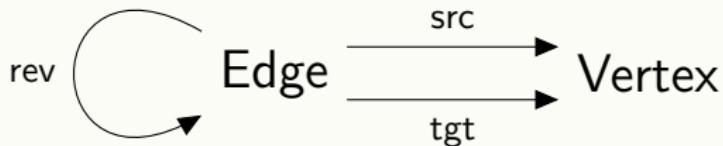
$$\text{rev} \cdot \text{src} = \text{tgt}$$

Equational reasoning:

$$\text{rev} \cdot \text{tgt} = \text{rev} \cdot (\text{rev} \cdot \text{src})$$

Symmetric Graphs

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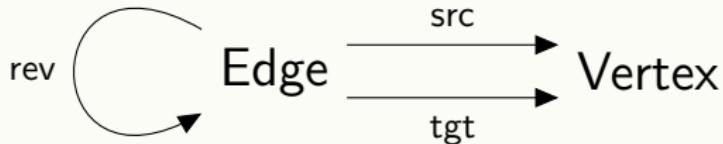
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$$= (\text{rev} \cdot \text{rev}) \cdot \text{src}$$

Symmetric Graphs

Schema:



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Equational reasoning:

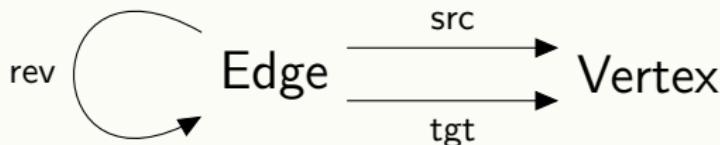
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Symmetric Graphs

Schema:



$$\text{rev} \cdot \text{rev} = \text{id}$$

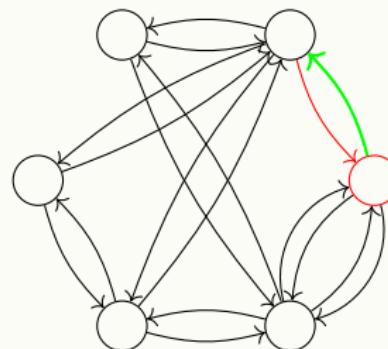
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Equational reasoning:

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$$= \text{id} \cdot \text{src} = \text{src}$$



Natural Transformations

Set of
Elements



Category
of Sets



Category of
Categories



Category of
Functors



Natural Transformations

Definition: Given two functors $\mathbb{F}, \mathbb{G} : \mathbb{X} \Rightarrow \mathbb{Y}$, a *natural transformation* $t : \mathbb{F} \Rightarrow \mathbb{G}$ is

- a function from objects $X \in \mathbb{X}$ to arrows $\mathbb{F}(X) \rightarrow \mathbb{G}(X) \in \mathbb{Y}$
- satisfying naturality $t(X) \cdot \mathbb{G}(h) = \mathbb{F}(h) \cdot t(X')$

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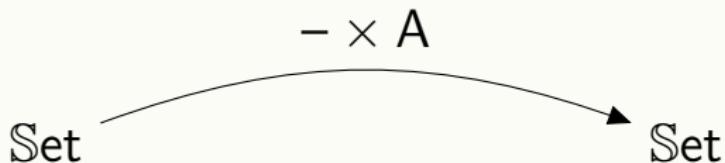
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$$\begin{array}{ccc} \mathbb{F}(X) & \xrightarrow{t(X)} & \mathbb{G}(X) \\ \mathbb{F}(h) \downarrow & & \downarrow \mathbb{G}(h) \\ \mathbb{F}(X') & \xrightarrow{t(X')} & \mathbb{G}(X') \end{array}$$

Projections are natural

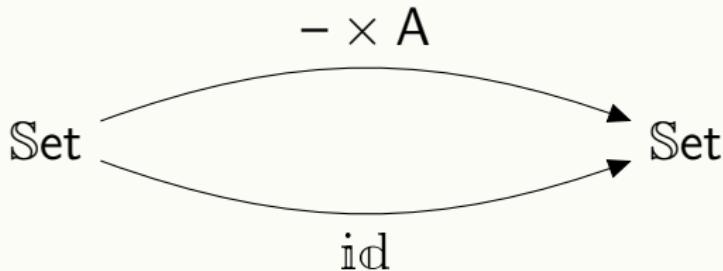
$\text{proj}_1 : - \times A \Rightarrow \text{id}$

Projections are natural



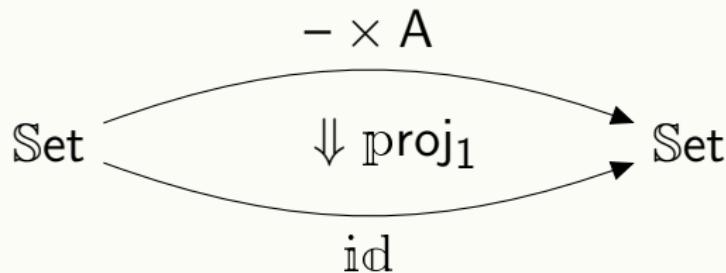
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Projections are natural



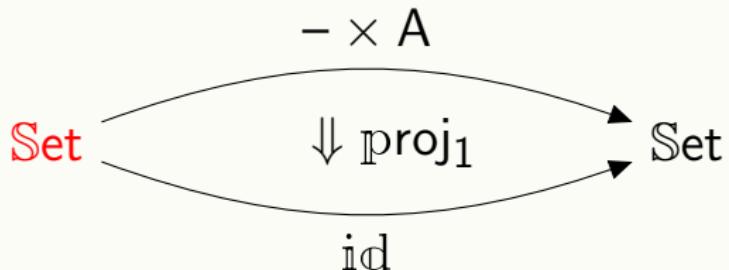
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Projections are natural



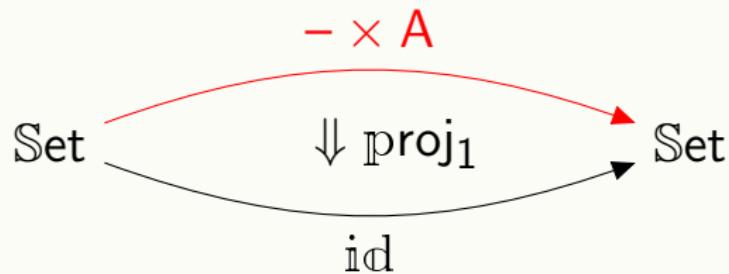
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Projections are natural



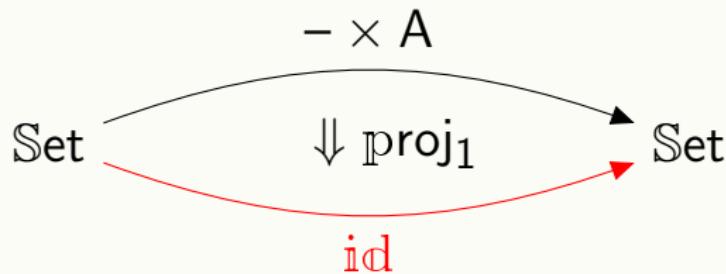
X

Projections are natural

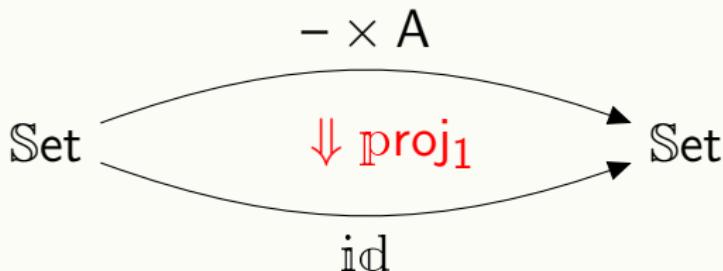


$$X \xrightarrow{\quad} X \times A$$

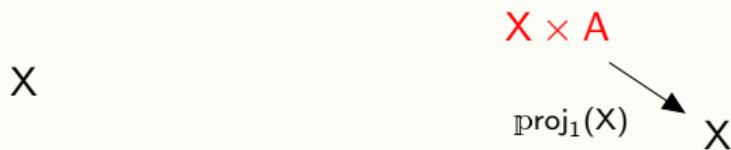
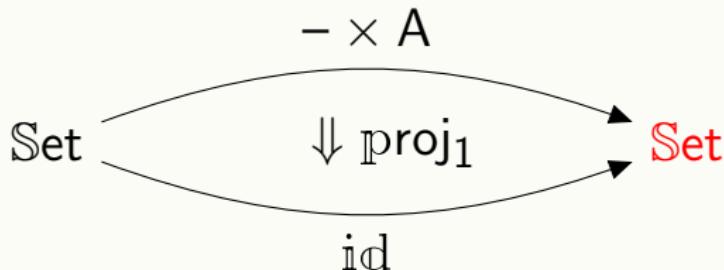
Projections are natural



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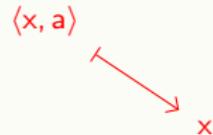
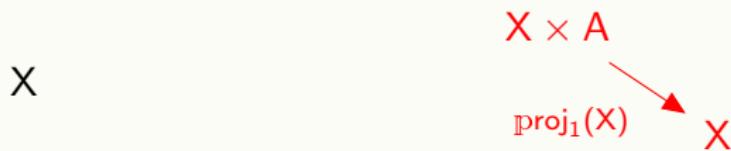
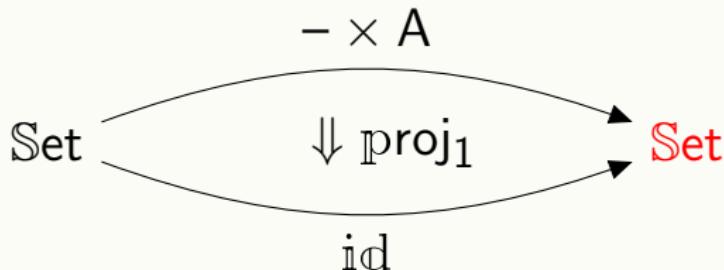


Projections are natural

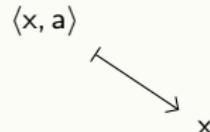
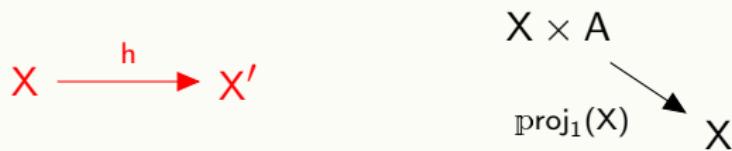
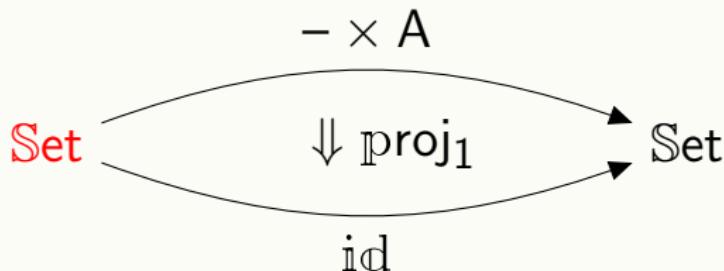


$\langle x, a \rangle$

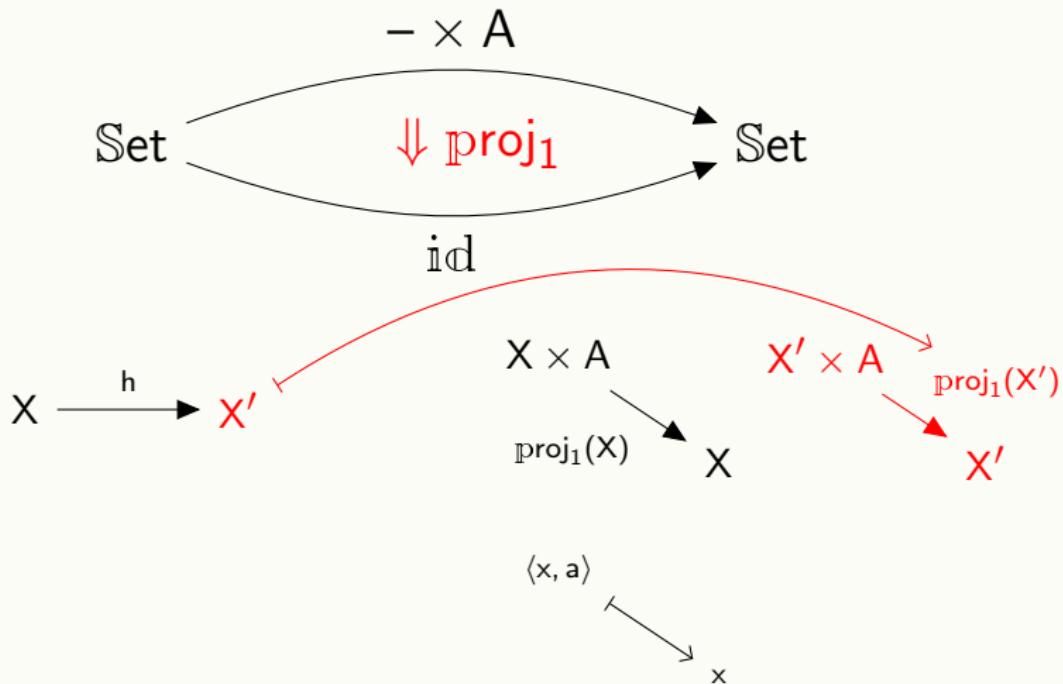
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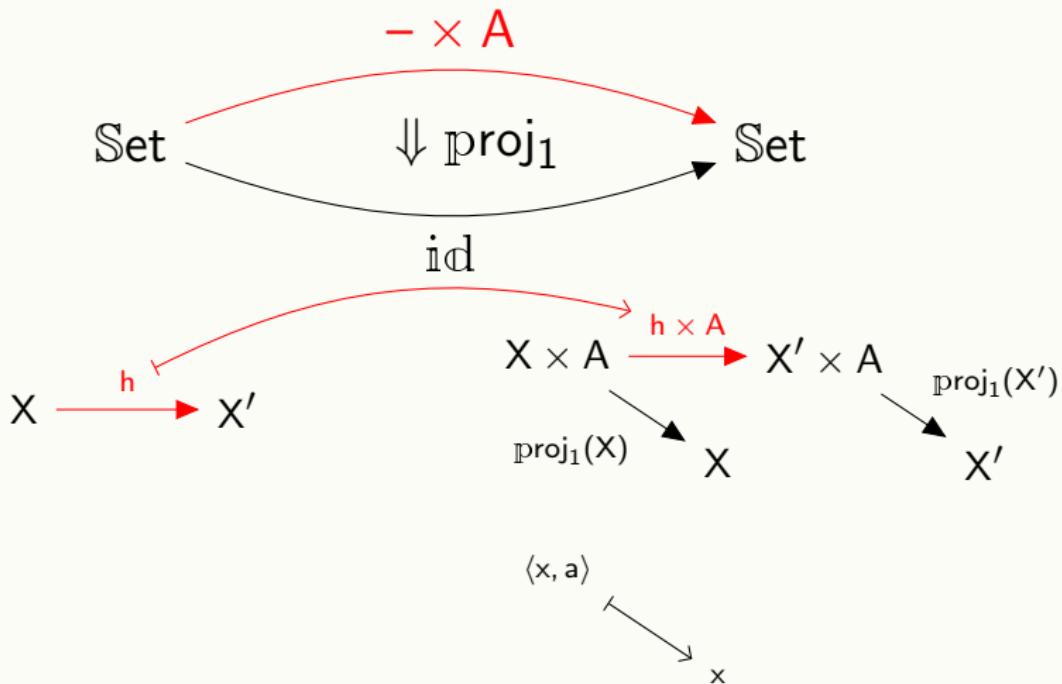
Projections are natural



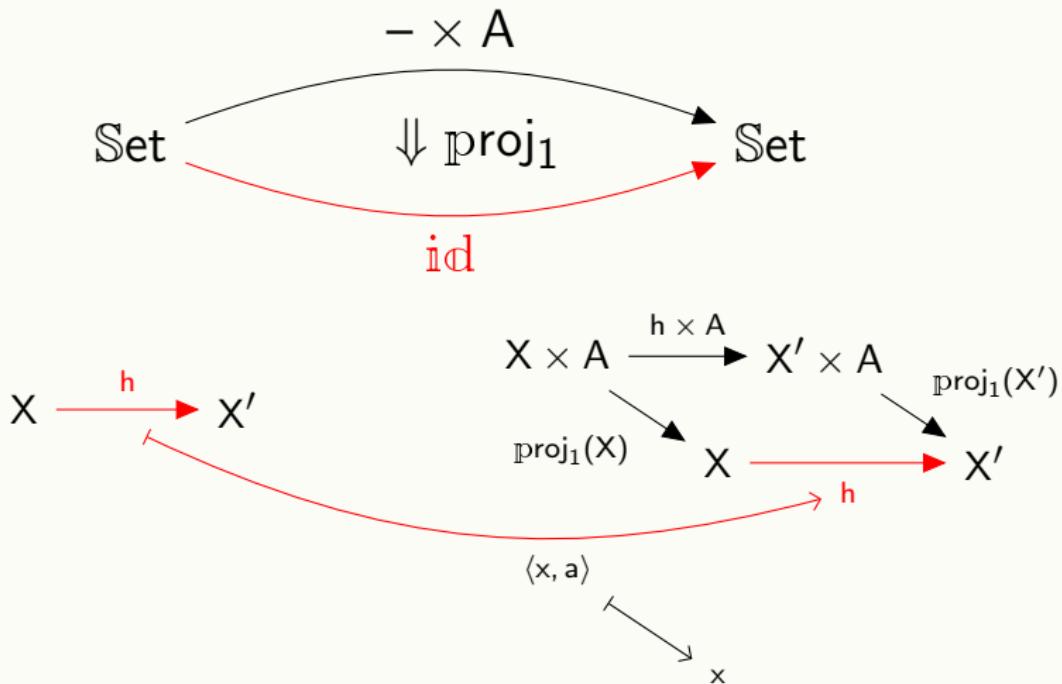
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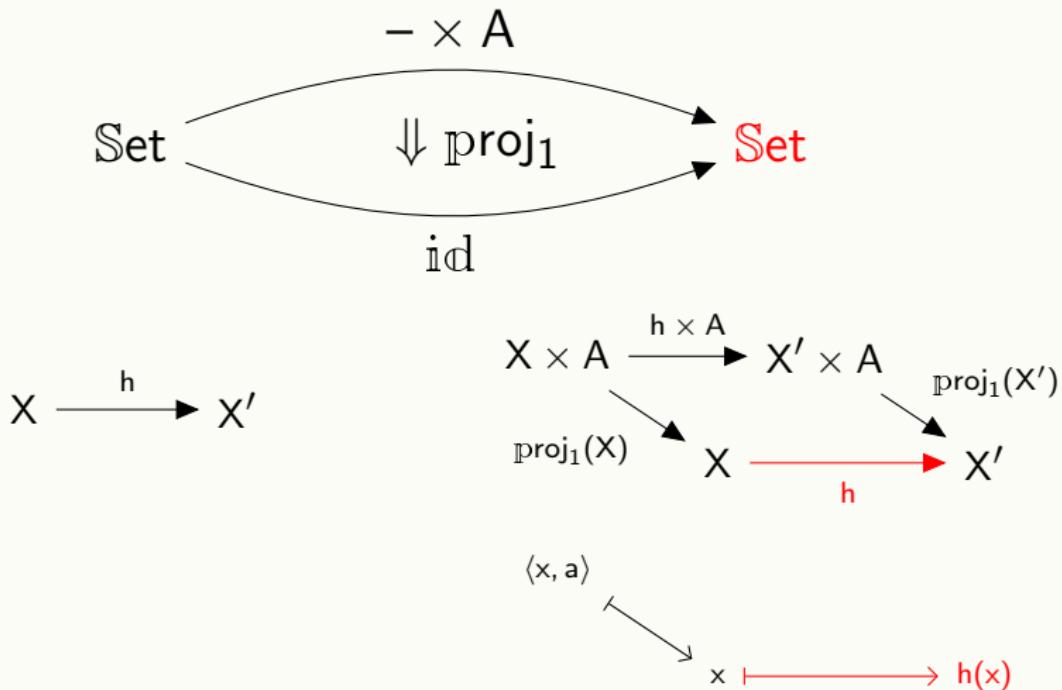
Projections are natural



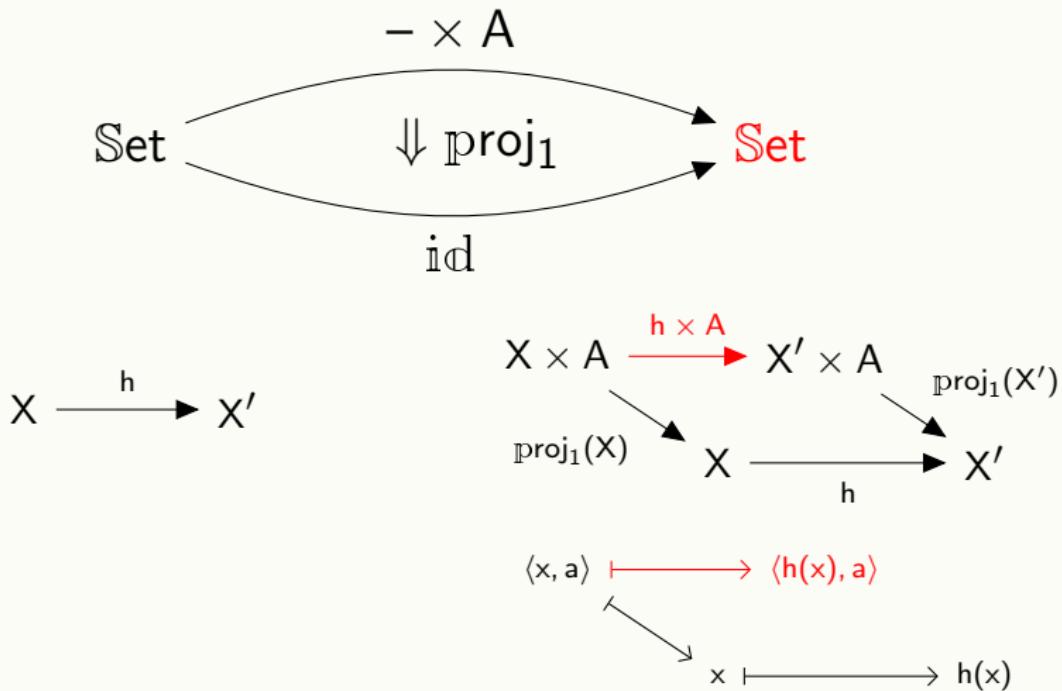
Projections are natural



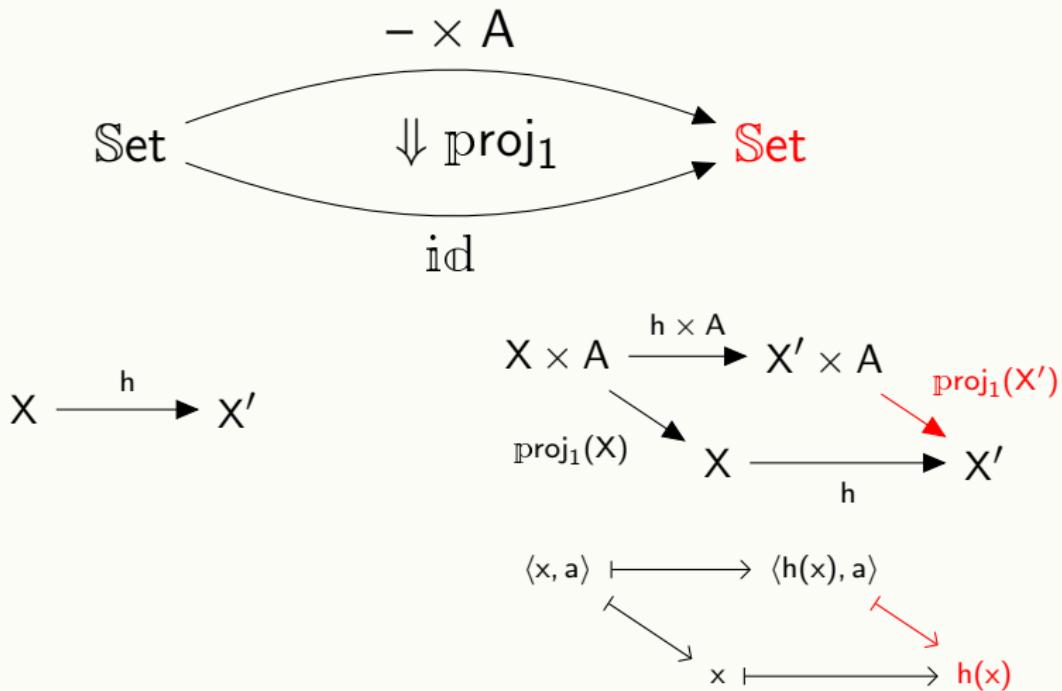
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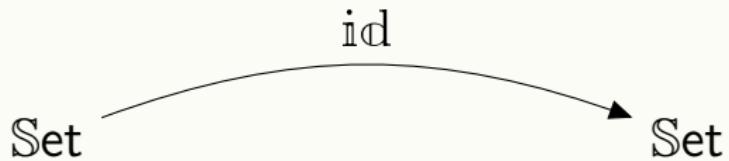
Projections are natural



Singletons are natural

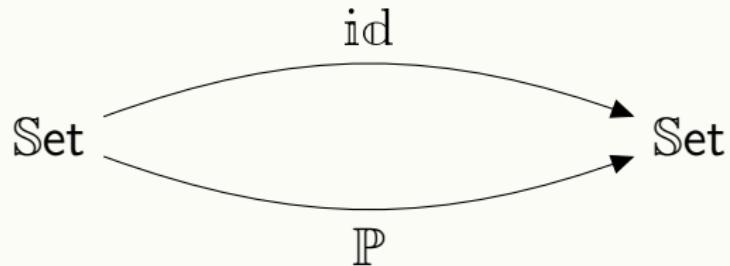
$$\{-\} : \text{id} \Rightarrow \mathbb{P}$$

Singl~~e~~tons are natural



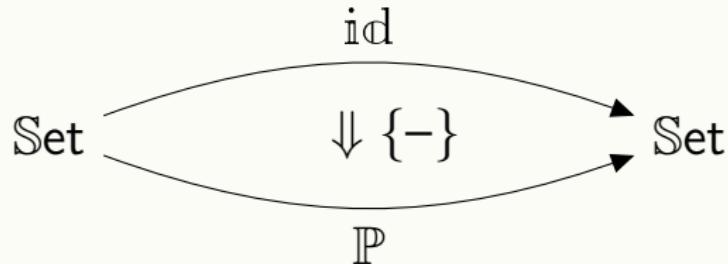
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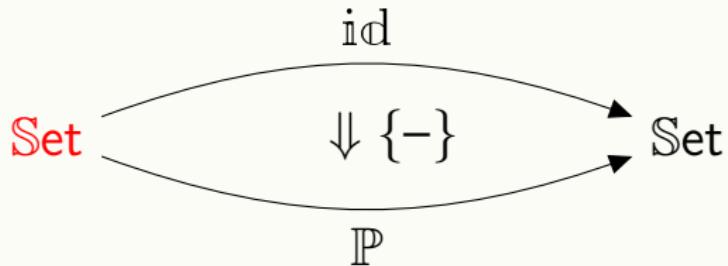
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Singlenton are natural



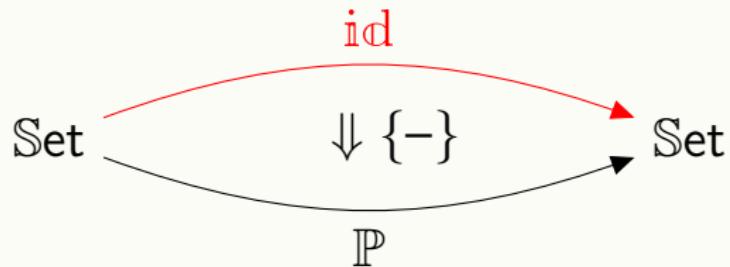
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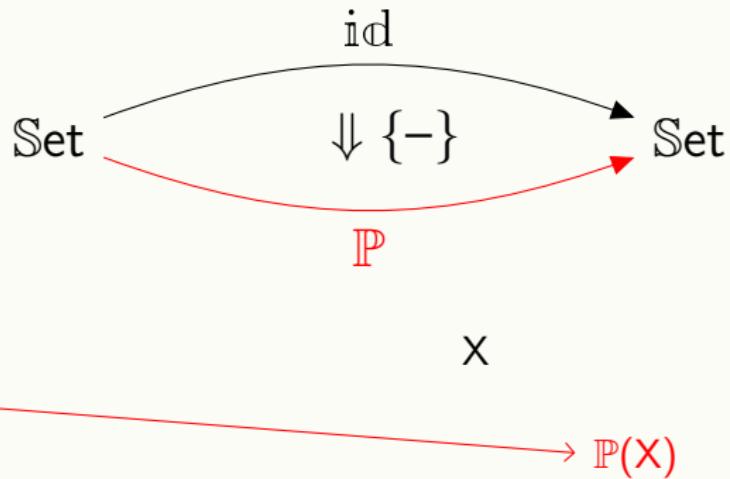


X

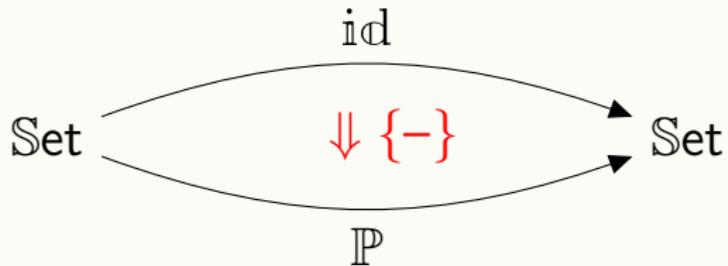
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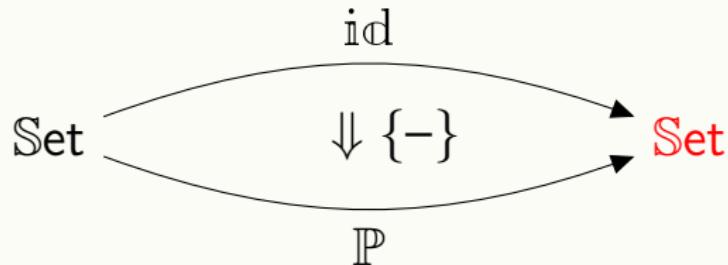
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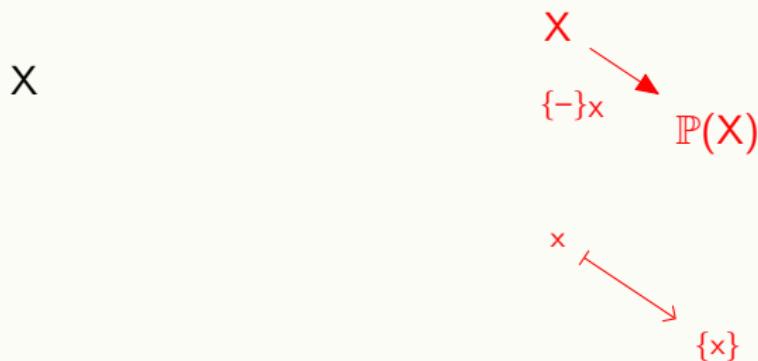
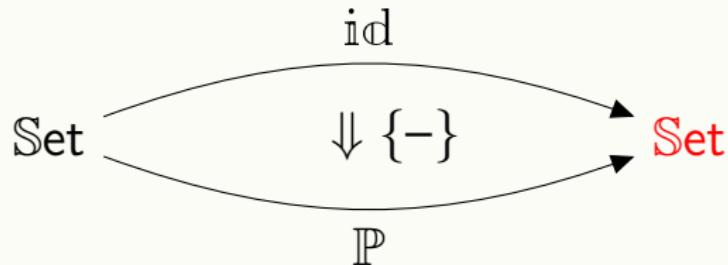


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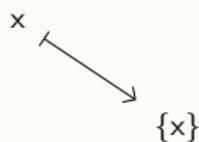
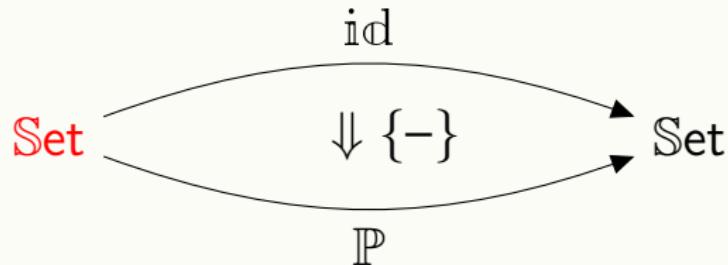


x

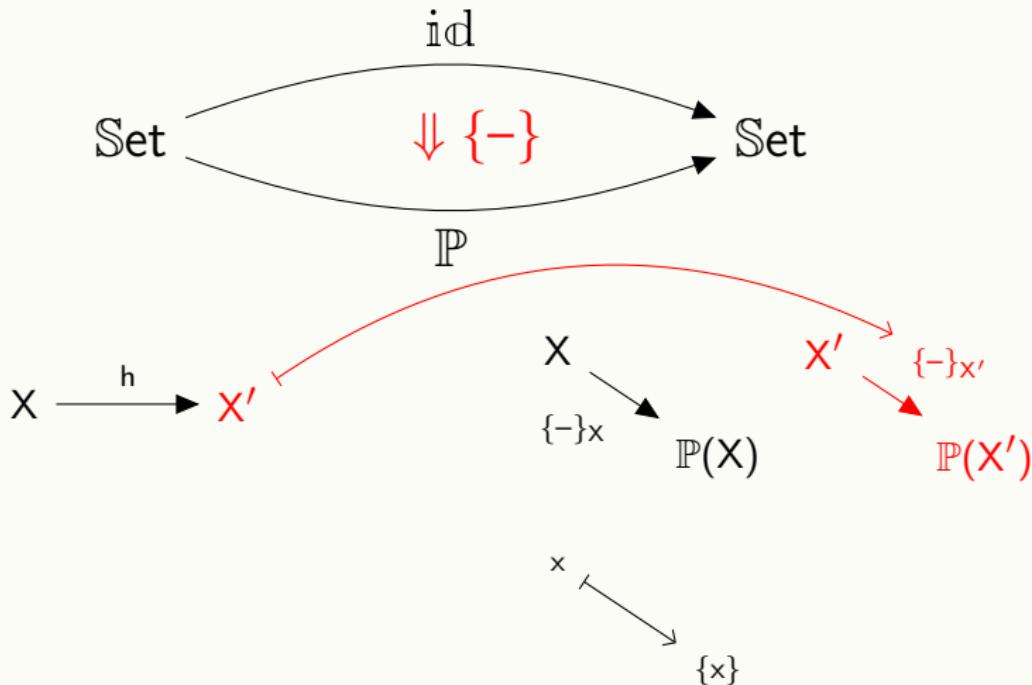
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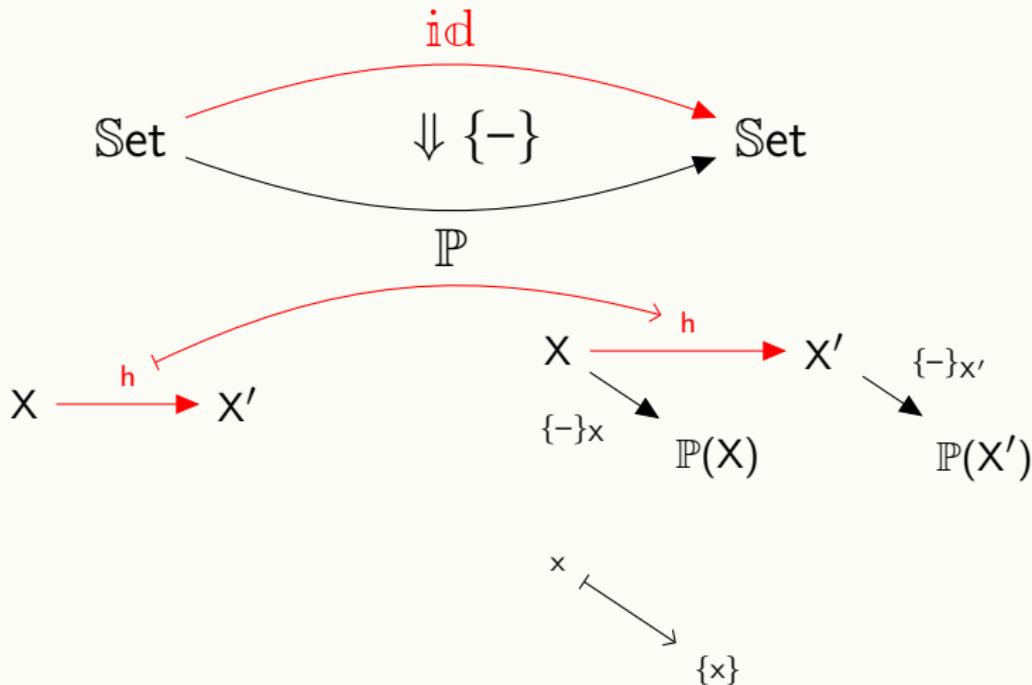
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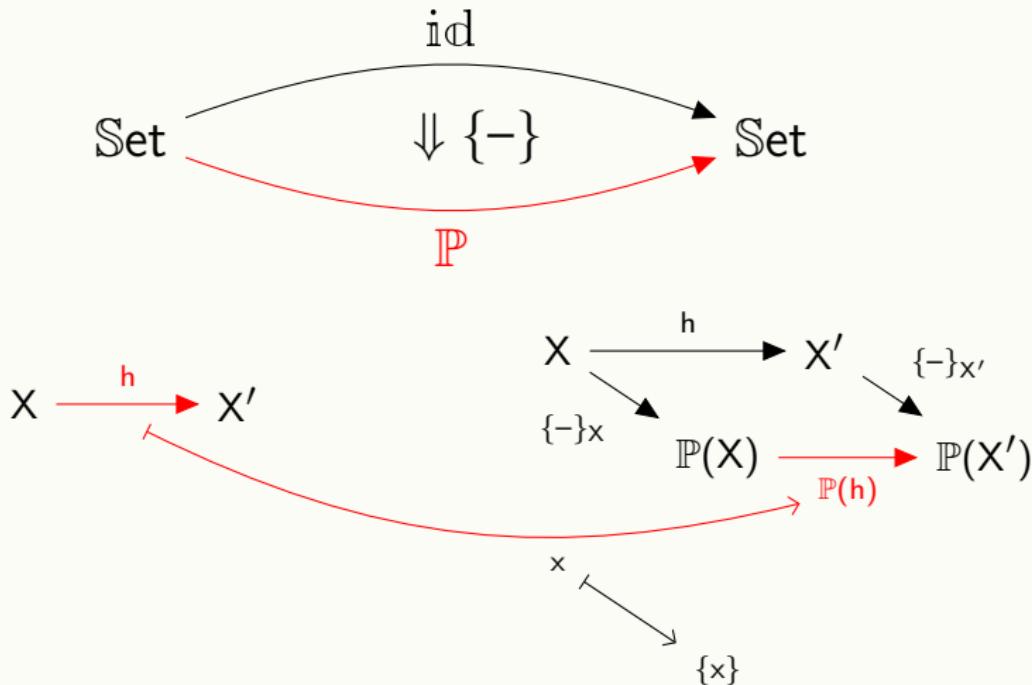
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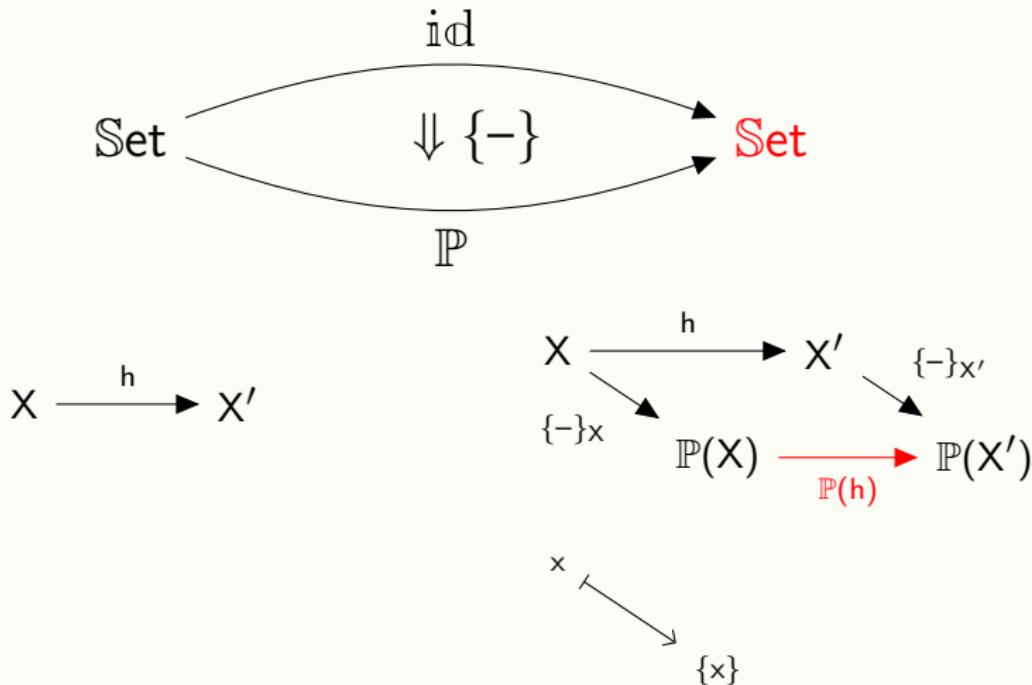
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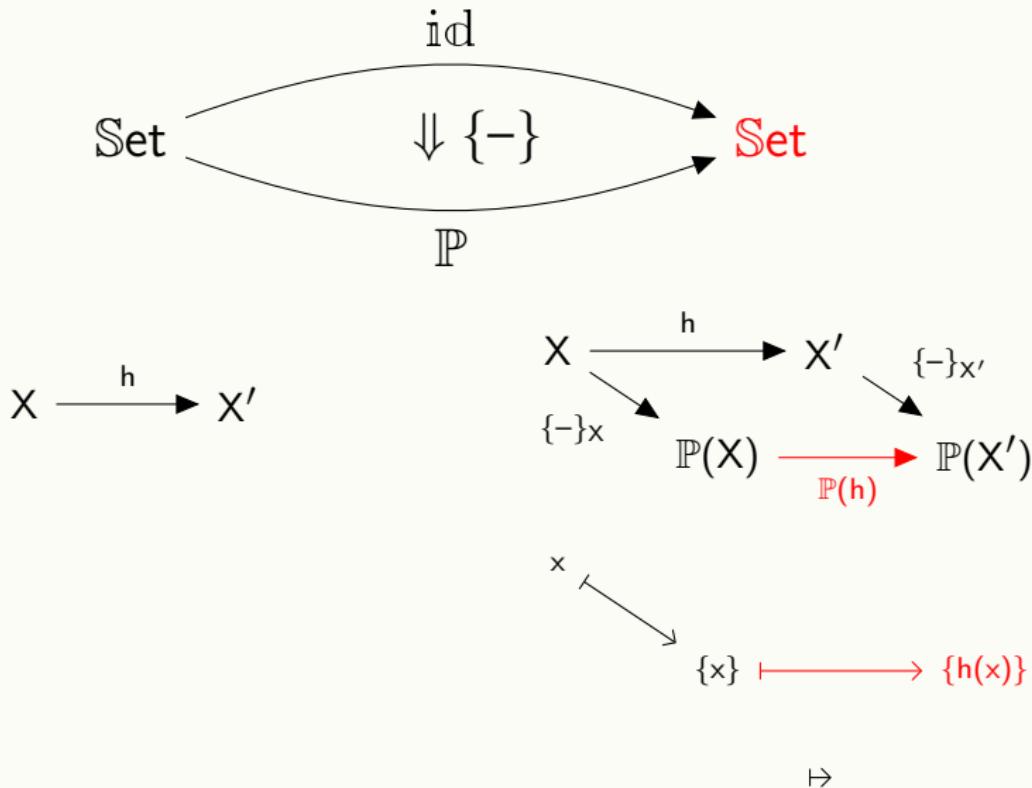


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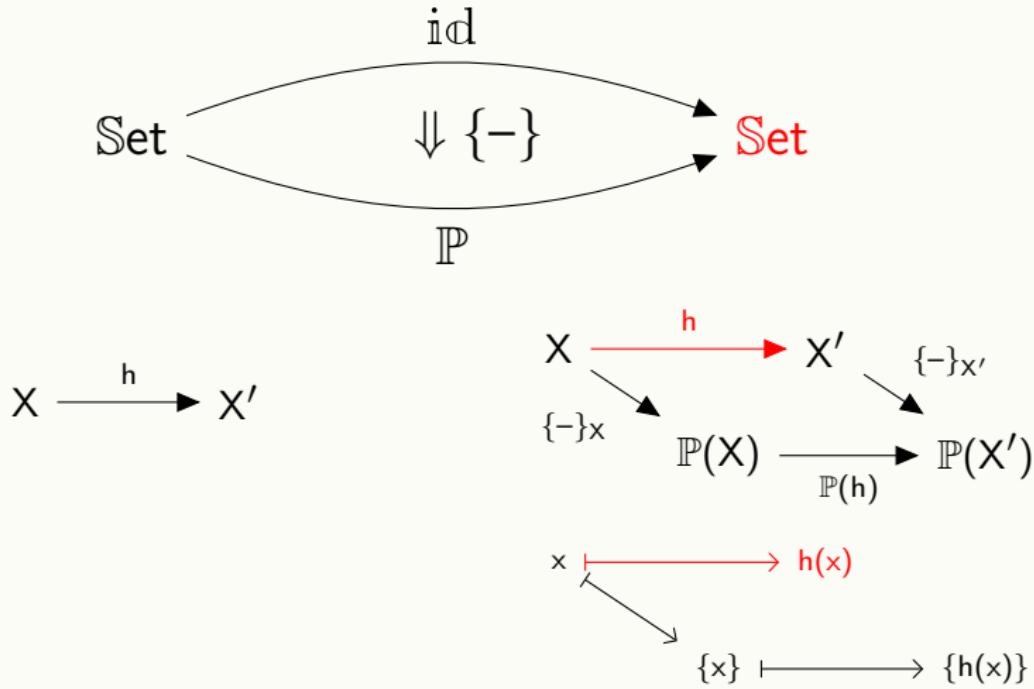


$$\{x_1, x_2, x_3\} \mapsto \{h(x_1), h(x_2), h(x_3)\}$$

Singletons are natural

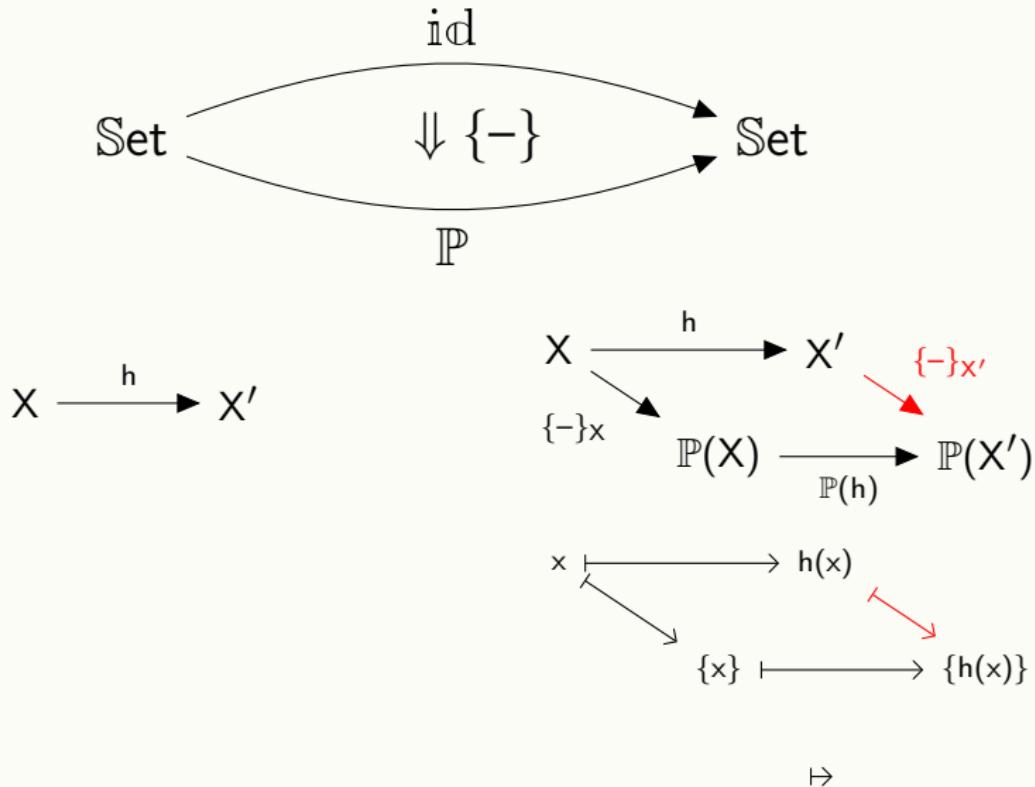


Singletons are natural



\mapsto

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Natural Isomorphism

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

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$$\begin{aligned}(X \times Y) \times Z &\neq X \times (Y \times Z) \\ \langle\langle x, y \rangle, z \rangle &\neq \langle x, \langle y, z \rangle \rangle\end{aligned}$$

Natural Isomorphism

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