Unit-3: Elementary Combinatorics

Combinatorics is an important part of discrete mathematics that solves counting problems without all possible cases.

Combinatorics deals with counting the number of ways "arranging" (permutations) or choosing (combinations) objects from a finite set according to certain specified rules.

Permutations:-

An ordered arrangement of r-elements of a set containing n-distinct elements is called an r-permutations of n-elements.

It is denoted by $P(n,r)$ or ${}^nP_r$.

It is defined by ${}^nP_r = \frac{n!}{(n-r)!}$

Combinations:-

An ordered choosing of r-elements of a set n-distinct elements is called r-combinations of n elements.

It is denoted by $C(n,r)$ or ${}^nC_r$.

It is defined by ${}^nC_r = \frac{n!}{(n-r)!\, r!}$

Basics of Counting:-

There are 2 rules of counting.

(i) Sum Rule

(ii) Product Rule.

Sum Rule:-

Suppose 2-tasks, t_1 and t_2 are to be performed in the same. If the task t_1 can be performed in m-different ways and the task t_2 can be performed in n-different ways. If the two tasks can not be performed simultaneously, then one of the two tasks can be performed in $m+n$ ways.

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(1) Suppose there are 16 boys and 18 girls in a class. How many ways to select one of the student either a boy (or) a girl as a class representative (CR).

Sol: Number of boys $= 16$

IIIy girls $= 18$

Number of ways for selecting a boy as a $CR = m = 16$ ways

IIIy a girl $= n = 18$ ways.

Number of ways for selecting a boy (or) girl as a $CR = m+n = 16+18 = 34$ ways.

③ Suppose a hostel library has 12 books on mathematics, 10 books on physics, 16 books on computer science and 11 books on electronics. How many ways a student to choose one of these books for study.

Sol:- Number of mathematics books = 12

Ily physics books = 10

CS books = 16

Electronics books = 11

Number of ways for selecting mathematics books for study = 12 ways

Ily, for physics, CS, electronics = 10, 16, 11 ways respectively

Number of ways for selecting either mathematics, physics, CS & electronics = 12 + 10 + 16 + 11

= 49 ways.

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③. Find how many number of ways for selecting a prime number ~~number~~ less than 10 and even number less than 10.

Sol:- Number of ways for selecting prime number < 10 = 4 ways (2, 3, 5, 7).

Ily for even numbers = 4 ways (2, 4, 6, 8).

Number of ways for selecting a prime and even number = 4 + 4 = 8 ways.

Product Rule:-

Suppose that 2 tasks, t_1 and t_2 are to be performed one after the other. If t_1 can be performed in 'm' different ways and for each of these ways, t_2 can be performed in 'n' different ways then both of those performed in $m \times n$ different ways.

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① Suppose a person has 5 shirts and 7 ties. How many ways a person can choose a shirt and tie.

Sol:- $S_1 \rightarrow$ he can choose one tie $= 7$ ways

$S_2 \rightarrow$ he can choose one tie $= 7$ ways

lly, $S_3, S_4, S_5, \ldots$

$\therefore$ A person can choose shirts and ties $= 5 \times 7 = 35$ ways

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② How many ways to construct sequence of 5 letters in which 3 letters are english letters. The rest is 2 letters are single digit numbers. If know letter are digit can be represent.

0 to 9

10 num

Sol:- case-i:- Repeatations are not allowed

The total number of ways $= 26 \times 25 \times 24 \times 10 \times 9$

$= 1404000$

case-ii:- Repeatations are allowed

The total number of ways $= 26 \times 26 \times 26 \times 10 \times 10$

$= 1757600$

(3) There are 30 married couple in a party. Find the no. of ways of choosing 1 man & 1 woman from the party such that the two are not married to each other.

Sol: Given that 30 married couple $= 30 \times 2 = 60$ persons

where men $= 30$ & women $= 30$

The total no. of ways $= 30 \times 29$

$= 870$

No. of Permutations of n-distinct objects (without repetation):-

The no. of distinct arrangements (permutations) with n different objects taken all at a time is

$$p(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

No. of permutations of n-objects (with repetation):-

It is required to find the no. of permutations to that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of second type n_3 are of 3rd type and so on n_n are of k^{th} type then the no. of permutations of

$$n \text{ objects} = \frac{n!}{n_1! \cdot n_2! \cdot n_2! \cdots \cdot n_k!}$$

No. of permutations of r-objects n distinct objects

Suppose given n objects and arrange 'r' of the objects denoted by $P(n,r) = \frac{n!}{(n-r)!} = {}^nP_r$

Circular Permutations:-

Permutations are in a circle are called circular permutation. The total no. of ways of arrange the n persons in a circle $= (n-1)!$.

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① How many ways are there to sit 10 boys and 10 girls around a circular table.

Sol:- GT, 10 boys and 10 girls sit around a circular table.

Total number of persons $= 10+10 = 20 = n$.

∴ The total number of ways of circular permutation $= (n-1)! = (20-1)! = 19!$.

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② How many ways are 3 persons sit around a round table.

Sol:- given that, $n=3$

∴ The total number of ways $= 2! = 2$ // $(3-1)! = 2!$,

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③ How many different strings are length 4 can be formed using the letters of the word "PROBLEM"

Sol:- given word is "PROBLEM".

It contains 7 letters, $n=7$.

given string length $= 4 = r$.

∴ The number of different strings of length 4 can be formed by using the letters of the word "PROBLEM" is $^nP_r = P(n,r) = {}^7P_4 = \cancel{7\times6\times5} = \frac{7!}{(7-4)!} = \frac{7!}{3!}$

$$^nP_r = \frac{7\times6\times5\times4\times3!}{3!} = 840.$$

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(4) How many ways of there 3 distinct letters can be formed the letters of the word "PASCAL".

Sol:- given word is "PASCAL" ~~A is repeated 2 times~~

$n=6$.

given $r=3$

Now, $^nP_r = {}^6P_3 = \frac{6!}{3!} = 6\times5\times4 = 120$

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(5) How many different arrangements of letters of the word "BOUGHT".

Sol:- The given word is "BOUGHT"

$n=6$. (all are distinct letters).

∴ The total number of arrangements of the letter in the word "BOUGHT" $= n! = 6! = 720$

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(6) Find the number of permutations of the letters of the word "ENGINEERING".

Sol:- The given word is "ENGINEERING".

$n=11$.

In this word, 3 E's, 3 N's, 2 I's, 2 G's, 1 R.

∴ The total number of permutations of the word "ENGINEERING" $= \dfrac{n!}{n_1! \cdot n_2! \, n_3! \, n_4! \, n_5!}$

$$= \frac{11!}{3! \times 3! \times 2! \times 2! \times 1!} = 277200$$

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⑦ Find the number of permutations of the letters of the word "SUCCESS".

Sol:- The given word is "SUCCESS" $n = 7!$

In this word, 3 s's, 2 c's, remaining u, e one time.

$\therefore$ The total number of permutations $= \dfrac{7!}{3! \times 2! \times 1! \times 1!}$

$$= 420$$

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⑧. Find the number of permutations of the letters of the word "MATHEMATICS".

Sol:- "MATHEMATICS" $= \dfrac{11!}{2! \times 2! \times 2! \times 1! \times 1! \times 1! \times 1! \times 1!}$

$$= 4989600$$

lly for "STRUCTURES" $= \dfrac{10!}{2! \times 2! \times 2! \times 2! \times 1! \times 1!}$

$$= 226800$$

lly for "DIFFICULT" $= \dfrac{9!}{2! \times 2! \times 1! \times 1! \times 1! \times 1! \times 1!}$

$$= 90720$$

lly for "GREAT" $= 5! = 120.$

(9) Find the number of permutations of the letters of the word MASSASAUGA. In how many of these all 4 A's together? How many of them begins with S?

Sol:- The no. of permutations of the given word is $\frac{n!}{n_1! n_2! \cdots n_k!}$

The given word is MASSASAUGA has contain 10 letters $n = 10$.

where, A is 4 is times, S is 3 times, M, U, G are 1 time repeated.

The required number of permutations

$$= \frac{10!}{4! \times 3!} = 25200$$

In a permutation all A's are to be together. Here we treated all A's are single letter then the no. of permutations of the required word is (AAAA) MSSSUG has contain 7 letters

$$\underbrace{(AAAA)}_{1} \quad n = 7.$$

The required no. of permutations = $\frac{7!}{3! \times 1!} = 840$.

In a permutations begining with S then occur 9 open positions to fill. where A is 4 times, S is 2 times and M, U, G are 1 time repeated.

∴ The required no. of permutations = $\frac{9!}{4! \times 2!} = 7560$

(10) It is required to seat 5 men and 4 women in a row so that the women occupying even places. How many such arrangements are possible?

Sol:- 5 men may be seated in odd places $= 5!$

4 women may be seated in even places $= 4!$

The total no. of such permutations are $5! \times 4! = 2880$

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(11) In how many ways can 6 men and 6 women are seated in a row.

(i) If any person may sit. ~~next to any other~~.

(ii) If men and women occupying alternates seates.

Sol:- The number of odd places $= 6$

even places $= 6$

Case-i:- The number of ways to sit 6 men ~~and~~ in 6 odd places $= 6!$

The no. of ways to sit 6 women in 6 even places $= 6!$

The total no. of ways $= 6! \times 6! = 518400$

Case-ii:-

The no. of ways to sit 6 men in 6 even places $= 6!$

lly 6 women in 6 odd places $= 6!$

The total no. of ways $= 6! \times 6! = 518400$

$\therefore$ No. of ways men and women occupy the alternative places $= 518400 + 518400 = 1036800$

(12) How many ways are there to sit 10 boys and 10 girls around a circular table? How many ways are there if boys and girls sit alternatively

Sol: The required no. of ways to sit 10 boys and 10 girls around a circular table $= (20-1)!$
$= 19!$

If the boys and girls are sit alternatively then the required no. of ways $= 10! \times 9!$

Combinations:

(1) A certain question paper contains 2 parts A and B each containing 4 questions. How many different ways a student can answer 5 Q by selecting at least 2 Q from each part.

Sol: In a question paper consist 2 parts part-A has 4 Q and part-B has 4 Q.

A student can answer 5 Q at least 2 Q from each part.

case-i: Student can select 3 Q from part-A and 2 Q from part-B.

3 Q from part-A to be selected $= C(4,3)$

$$= {}^4C_3 = \frac{4!}{1! \times 3!} = 4$$

2 Q from part-B $= C(4,2) = {}^4C_2 = \frac{4!}{2! \cdot 2!} = 6.$

$\therefore$ The total no. of ways $= 4 \times 6 = 24$

case ii:-

A student can select 2Q from part-A and 3Q from part-B.

2Q from part A = $C(4,2) = {}^4C_2 = 6$

3Q from part B = $C(4,3) = {}^4C_3 = 4$

∴ The total no. of ways = $6 \times 4 = 24$

Total no. of ways = $24 + 24 = 48$

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(2) A question paper contains 3 parts A, B, C with 4Q in part-A, 5Q in part-B, 6Q in part-C. It is require to answer 7Q at least 2Q from each part? How many different ways can a student selecting is 7Q for answering?

Sol:- A question paper contains 3 parts A, B, C with 4Q, 5Q, 6Q respectively.

A student can select 7Q at least 2Q from each part then

case-i:- A → 3
B → 2
C → 2

3Q from part A $= C(4,3) = {}^4C_3 = 4$

Illy 2Q from part B $= C(5,2) = {}^5C_2 = \frac{5 \times 4}{1 \times 2} = 10$

3Q from part C $= C(6,2) = {}^6C_2 = \frac{6 \times 5}{1 \times 2} = 15$

∴ The total no. of ways $= 4 \times 10 \times 15 = 600$

case-iii:- A→2, B→3, C→2

2Q from part A = $C(4,2) = {}^4C_2 = \frac{4\times3}{1\times2} = 6$

3Q from part-B = $C(5,3) = {}^5C_3 = \frac{5\times4}{1\times2} = 10$

2Q from part-C = $C(6,2) = {}^6C_2 = \frac{6\times5}{1\times2} = 15$

∴ The total no. of ways = $6\times10\times15 = 900$

case-iv:- A→2, B→2, C→3

2Q from part A = $C(4,2) = {}^4C_2 = \frac{4\times3}{1\times2} = 6$

2Q from part-B = $C(5,2) = {}^5C_2 = \frac{5\times4}{1\times2} = 10$

3Q from part-C = $C(6,3) = {}^6C_3 = \frac{6\times5\times4}{1\times2\times3} = 20$

∴ The total no. of ways = $6\times10\times20 = 1200$

Total no. of ways = $600+900+1200 = 2700$

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③ From a group of 7 men and 6 women. 5 persons are to be selected from a committee. So that at least 3 members are men on the committee. In how many ways it can be done!

sol:- given a group contains of 7 men and 6 women.

5 persons are to be selected from a committee.

given, there at least 3 members are men on the committee.

Men = 7, Women = 6

case-i:- Men → 3
Women → 2

selecting 3 mens from 7 men = ${}^7C_3 = \frac{7\times6\times5}{1\times2\times3} = 35$

selecting 2 women from 6 women = $^6C_2 = \frac{\cancel{6}^3 \times 5}{1 \times \cancel{2}} = 15$

$\therefore$ Total no. of ways $= 35 \times 15 = 525$

Case-ii :- Men $\rightarrow 4 = {}^7C_4 = \frac{7 \times \cancel{6} \times 5}{1 \times \cancel{2} \times 3} = 35$

Women $\rightarrow 1 = {}^6C_1 = 6$

$\therefore$ Total no. of ways $= 35 \times 6 = 210$

Case-iii :- Men $\rightarrow 5 = {}^7C_5 = \frac{7 \times \cancel{6}^3}{1 \times \cancel{2}} = 21$ | $^nC_0 = 1, {}^nC_n = 1$

Women $\rightarrow 0 = {}^6C_0 = 1$

$\therefore$ Total no. of ways $= 21 \times 1 = 21$

$\therefore$ The total no. of ways $= 525 + 210 + 21$
$= 756$

$$\begin{array}{r} 525 \\ 231 \\ \hline 756 \end{array}$$

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(4) In how many ways can a committee of 5 persons be formed out of 6 men and 4 women which at least 1 women has to be necessary selected.?

Men = 6, Women = 4

Sol :- given that there are 6 men and 4 women. The Committee contain 5 persons. There at least 1 women has to be necessary.

Case-i :- Men $\rightarrow 4 = {}^6C_4 = \frac{\cancel{6}^3 \times 5}{1 \times \cancel{2}} = 15$

Women $\rightarrow 1 = {}^4C_1 = 4$

$\therefore$ Total no. of ways $= 15 \times 4 = 60$

Case-ii :- Men $\rightarrow 3 = {}^6C_3 = \frac{\cancel{6} \times 5 \times 4}{1 \times 2 \times \cancel{3}} = 20$

Women $\rightarrow 2 = {}^4C_2 = \frac{\cancel{4}^2 \times 3}{1 \times \cancel{2}} = 6$

$\therefore$ Total no. of ways $= 20 \times 6 = 120$

Case-iii: Men → 2 $= {}^6C_2 = \frac{\cancel{6}^3 \times 5}{1 \times \cancel{2}} = 15$

Women → 3 $= {}^4C_3 = 4$

∴ Total no. of ways $= 15 \times 4 = 60$

Case iv: Men → 1 $= {}^6C_1 = 6$

Women → 4 $= {}^4C_4 = 1$

∴ Total no. of ways $= 6 \times 1 = 6$.

~~Case v~~

∴ Total no. of ways $= 60 + 120 + 60 + 6$

$= 246$

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⑤ Find the no. of committees of 5 that can be selected from 7 men and 5 women. If the committee is consist of at least 1 men and at least 1 women?

Sol: given committee contains 5 persons

There are totally 7 mens and 5 womens.

The committee must contains at least 1 men and 1 women.

Case-i: Men → 1 $= {}^7C_1 = 7$

Women → 4 $= {}^5C_4 = 5$

Total no. of ways $= 7 \times 5 = 35$

Case-iii: Men → 2 $= {}^7C_2 = \frac{7 \times \cancel{6}^3}{1 \times \cancel{2}} = 21$

Women → 3 $= {}^4C_3 = 4$

Total no. of ways $= 21 \times 4 = 84$

case-iii: Men $\rightarrow$ 3 $= {}^7C_3 = \frac{7\times\cancel{6}\times5}{1\times\cancel{2}\times\cancel{3}} = 35$

Women $\rightarrow$ 2 $= {}^4C_2 = \frac{\cancel{4}^2\times3}{1\times\cancel{2}} = 6$

Total no. of ways $= 35\times6 = 210$

case-iv: Men $\rightarrow$ 4 $= {}^7C_4 = \frac{7\times\cancel{6}\times5}{1\times\cancel{2}\times\cancel{3}} = 35$

Women $\rightarrow$ 1 $= {}^4C_1 = 4$

Total no. of ways $= 35\times4 = 140$

$\therefore$ Total no. of ways $= 35+84+210+140$

$= 469$

$$\begin{array}{r} 210 \\ 140 \\ 119 \\ \hline 469 \\ \hline \end{array}$$

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(6) If $p(n,r) = 2520$ and $c(n,r) = 21$ then find the value of $c(n+1, r+1)$?

Sol: given $P(n,r) = 2520$

${}^nP_r = 2520 \Rightarrow \frac{n!}{(n-r)!} = 2520 \rightarrow (1)$

and $c(n,r) = 21$

${}^nC_r = 21 \Rightarrow \frac{n!}{(n-r)!\,r!} = 21 \rightarrow (2)$

Now, $\frac{(1)}{(2)} \Rightarrow \frac{\cancel{n!}}{\cancel{(n-r)!}} \times \frac{\cancel{(n-r)!}\,r!}{\cancel{n!}} = \frac{2520}{21}$

$\therefore r! = 120$

$\boxed{r=5}$

from (1) $\Rightarrow \frac{n!}{(n-r)!} = 2520 \Rightarrow \cancel{\frac{n!}{n-5}}$

$$\frac{n!}{(n-5)!} = 2520$$

$$\frac{n(n-1)(n-2)(n-3)(n-4)\cancel{(n-5)!}}{\cancel{(n-5)!}} = 2520$$

$$n(n-1)(n-2)(n-3)(n-4) = 7 \times 6 \times 5 \times 4 \times 3$$

$$\therefore \boxed{n=7}$$

Now, $n=7$, $r=5$

$$C(n+1, r+1) = C(7+1, 5+1)$$
$$= C(8,6) = {}^{8}C_{6} = \frac{8 \times 7}{1 \times 2} = 28$$

$$\therefore C(n+1, r+1) = C(8,6) = 28$$

Properties of Combinations:-

Suppose we wish to select a combination of 'r' objects with repetation from a set of 'n' distinct objects. The number of such selections is given by $C(r+n-1, r) = C(r+n-1, n-1)$

The following are the other interpretations of this number.

(i) $C(r+n-1, r) = C(r+n-1, n-1)$ it represents The number of ways in which 'r' identical objects can be distributed among 'n' distinct containers

(ii) $C(r+n-1, r) = C(r+n-1, n-1)$ represents the number non-negative integer solutions of the equation.

Note: A non-negative integer solution of the equation $x_1+x_2+x_3+\cdots+x_n = r$

where, $x_1, x_2, x_3, \cdots, x_n$ are non-negative integers.

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① In how many ways can be distribute 10 identical marbles among 6 distinct containers.

Sol: given, $r=10$, $n=6$.

$$C(r+n-1, r) = C(10+6-1, 10)$$
$$= C(15, 10) = {}^{15}C_{10} = 3003$$
$$= \frac{15\times14\times13\times12\times11}{1\times2\times3\times4\times5} = 3003$$

∴ The required no. of ways can be distributed is 3003

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② Find the number of non-negative integers solutions of the equation $x_1+x_2+x_3+x_4+x_5 = 8$.

Sol: given $n=5$, $r=8$

$$C(r+n-1, r) = C(8+5-1, 8)$$
$$= C(12, 8) = {}^{12}C_8 = \frac{12\times11\times10\times9}{1\times2\times3\times4}$$
$$= 55\times9 = 495$$

∴ The number of non-negative integer solutions of the given equation is 495.

③ How many ways can be distribute 12 identical pencils to 5 childrens so that every child gets atleast one pencil.

Sol: Given that no. of pencils = 12

no. of childrens = 5

Every child get atleast one pencil, it means each child may greater than (or) equal to one pencil.

∴ The no. of ways to distribute 7 identical pencils to 5 childrens = $c(r+n-1, r)$

where, $r = 7, n = 5$.

$= {}^{7+5-1}C_7 = {}^{11}C_7 = 330$ ways

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④ In how many ways can be distribute 7 apples and 6 Oranges among 4 children so that each child get atleast one apple.

Sol: given apples = 7

Oranges = 6

Childrens = 4

Each child get atleast one apple. It means every child may get ≥ one apple

case(i): ~~r=3~~, n=4

First we distribute one apple to each children the remaining apples can be distibute to 4 childrens.

where. $n=4, r=3$.

$\therefore$ No. of ways to distribute 3 apples to 4 childrens $= {}^{r+n-1}C_r = {}^{3+4-1}C_3 = {}^6C_3 = 20$ ways

case iii: The no. of ways 6 oranges distribute to 4 childrens, where $n=4, r=6$

$= {}^{r+n-1}C_r = {}^{6+4-1}C_6 = {}^9C_6 = 84$ ways.

$\therefore$ Total no. of ways $= 20 \times 84 = 1680$ ways

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(5) In how many ways can we distribute identically 12 balls among the 7 baskets

Sol: given no. of balls $= r = 12$

baskets $= n = 7$

$\therefore$ Total no. of ways we can distribute identically 12 balls among the 7 baskets

$= {}^{r+n-1}C_r = {}^{12+7-1}C_{12} = {}^{18}C_{12} = 18564$.

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(6) A bag contains 7 different denominations with atleast one dozen coins in each denomination. In how many ways can we select a dozen coins from a bag.

Sol: No. of denominations, $n=7$.

No. of coins to be selected with repetation $= 12 = r$.

∴ The no. of ways to be selected a dozen coins from a bag $= {}^{r+n-1}C_{r} = {}^{12+7-1}C_{12}$

$= {}^{18}C_{12} = 18564.$

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(7) Find the no. of positive integers solutions of the equation $x_1+x_2+x_3=17$ where $x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

Sol: Given $x_1+x_2+x_3=17 \longrightarrow (1)$

$x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

let $y_1 = x_1 - 1, \quad y_2 = x_2 - 1, \quad y_3 = x_3 - 1$

$x_1 = y_1 + 1, \quad x_2 = y_2 + 1, \quad x_3 = y_3 + 1$

from eq (1)

$y_1 + 1 + y_2 + 1 + y_3 + 1 = 17$

$y_1 + y_2 + y_3 = 14.$

where, $n=3, r=14$

The no. of +ve integer solutions of the equation is $= {}^{r+n-1}C_{r} = {}^{14+3-1}C_{14} = {}^{16}C_{14} = 120.$

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(8) Find the no. of integer solutions of the equation $x_1+x_2+x_3+x_4+x_5=30$, where $x_1 \geq 2$, $x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0.$

Sol:- Given $x_1+x_2+x_3+x_4+x_5 = 30 \rightarrow (1)$

where $x_1 \geq 2,\ x_2 \geq 3,\ x_3 \geq 4,\ x_4 \geq 2,\ x_5 \geq 0$.

let us consider 5 non-negative integers $y_1, y_2, y_3, y_4 \;\&\; y_5$

let $y_1 = x_1 - 2 \Rightarrow x_1 = y_1 + 2$

$y_2 = x_2 - 3 \Rightarrow x_2 = y_2 + 3$

$y_3 = x_3 - 4 \Rightarrow x_3 = y_3 + 4$

$y_4 = x_4 - 2 \Rightarrow x_4 = y_4 + 2$

$y_5 = x_5 - 0 \Rightarrow x_5 = y_5$

from eq (1),

$x_1 + x_2 + x_3 + x_4 + x_5 = 30$

$y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$

$y_1 + y_2 + y_3 + y_4 + y_5 = 30 - 11 = 19.$

where, $n = 5,\ r = 19$

$\therefore$ The no. of the integer solution of the equation

$= {}^{r+n-1}C_r = {}^{19+5-1}C_{19} = {}^{23}C_{19} = 8855$

Binomial Theorem:-

A binomial theorem describes the algebraic expansion of powers of a binomial with two variables

$$(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \cdots$$
$$\cdots + {}^nC_{n-1}\, x^1 \cdot y^{n-1} + {}^nC_n\, x^0 y^n$$

It can be written as

$$\therefore (x+y)^n = \sum_{r=0}^{\infty} {}^nC_r\, x^{n-r} \cdot y^r$$

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① Find the coefficient of $x^9 y^3$ in the expansion of $(x+2y)^{12}$

Sol: By the defn of BT,

$$(x+y)^n = \sum_{r=0}^{\infty} {}^nC_r\, x^{n-r}\, y^r \rightarrow ①$$

let $x=x$, $y=2y$ and $n=12$

$$(x+2y)^{12} = \sum_{r=0}^{\infty} {}^{12}C_r\, x^{12-r} (2y)^r$$

$$= \sum_{r=0}^{\infty} {}^{12}C_r\, x^{12-r} \cdot 2^r \cdot y^r \rightarrow ②$$

Comparing the coefficient of x^9 with x^{12-r}

$$x^9 = x^{12-r}$$

$$12 - r = 9$$

$$r = 12 - 9 \Rightarrow r = 3$$

from eqn ②,

$$(x+2y)^{12} = {}^{12}C_3\, x^{12-3} \cdot 2^3 \cdot y^3$$

$$= 1760 \times x^9 \cdot y^3$$

② Find the coefficient of $x^5 y^2$ in the expansion of $(2x-3y)^7$.

Sol: By the defn of BT,

$$(x+y)^n = \sum_{r=0}^{\infty} {}^nC_r \, x^{n-r} \, y^r \rightarrow ①$$

from ①, Here, $x = 2x$, $y = -3y$ & $n = 7$

$$(2x-3y)^7 = \sum_{r=0}^{\infty} {}^7C_r \, (2x)^{7-r} (-3y)^r$$

$$= \sum_{r=0}^{\infty} {}^7C_r \, (2)^{7-r} . (x)^{7-r} . (-3)^r . y^r \rightarrow ②$$

Comparing the x^5 coefficient.

$$x^{7-r} = x^5$$

$$7 - r = 5 \rightarrow r = 7 - 5$$

$$\boxed{r=2}$$

from eqn ②,

$$(2x-3y)^7 = {}^7C_2 \, (2)^{7-2} . (x)^{7-2} . (-3)^2 . y^2$$

$$= {}^7C_2 \, 2^5 . x^5 . 9 . y^2$$

$$= 6048 \, x^5 y^2.$$

---

③ Find the coefficient of ~~...~~ x^0 in the expansion of $\left(3x^2 - \frac{2}{x}\right)^{15}$

Sol: The BT is $(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} . y^r$.

Here, $x = 3x^2$, $y = \frac{-2}{x}$ & $n = 15$

$$\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=0}^{\infty} {}^{15}C_r \, (3x^2)^{15-r} . \left(\frac{-2}{x}\right)^r$$

$$= \sum_{r=0}^{\infty} {}^{15}C_r \, (3)^{15-r} . (x^2)^{15-r} . (-2)^r . x^{-r}$$

$$= \sum_{r=0}^{\infty} {}^{15}C_r \, (3)^{15-r} . (x)^{30-2r} \, (-2)^r . x^{-r} \rightarrow ①$$

To find the coefficient of x^0.

$$x^0 = x^{30-2r} \times x^{-r}.$$

$$30 - 3r = 0$$

$$3r = 30 \Rightarrow \boxed{r = 10}$$

Sub r in eq ①,

$$① \Rightarrow {}^{15}C_{10} \, (3)^{15-10} \, (-2)^{10} \, (x)^{30-3(10)}$$

$$= 3003 \times 243 \times 1024$$

$$= 747167424 \, x^0.$$

---

④ Find the coefficient of x^5 in the expansion of $(1-2x)^{-7}$.

Sol: The binomial theorem is given by $(1-x)^{-n}$

$$= \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r \rightarrow ①$$

Here, $x = 2x$, $n = 7$

from ①, $(1-2x)^{-7} = \sum_{r=0}^{\infty} {}^{7+r-1}C_r \, (2x)^r$

$(1-2x)^{-7} = \sum_{r=0}^{\infty} {}^{6+r}C_r \, (2)^r \, (x)^r$

To find coefficient of x^5.

from ②, $x^5 = x^r$

$\boxed{r=5}$

from eq ②, ${}^{6+5}C_5 \, (2)^5 \, x^5$

$= 462 \times 32 \times x^5$

$= 14784 \, x^5$.

---

⑤. Find the coefficient of x^{27} in the expansion of $(x^4 + x^5 + x^6 + x^7 + \ldots\ldots)^5$.

Sol: We know BT,

$(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r$

$= (x^4(1+x+x^2+\ldots))^5$

$= (x^4)^5 \cdot (1+x+x^2+\ldots)^5$

$= x^{20}((1-x)^{-1})^5$

$= x^{20} \cdot (1-x)^{-5}$.

Here, $x = x$, $n = 5$

$(1-x)^{-5} = \sum_{r=0}^{\infty} {}^{5+r-1}C_r \, (x)^r$

$x^{20} \cdot (1-x)^{-5} = \sum_{r=0}^{\infty} {}^{5+r-1}C_r \, x^r \cdot x^{20}$

$= \sum_{r=0}^{\infty} {}^{5+r-1}C_r \cdot x^{20+r}$.

Comparing the coefficient with x^{27}

$$x^{20+r} = x^{27}$$

$$20+r = 27$$

$$\boxed{r=7}$$

Now, $= {}^{4+7}C_7 \, x^{7+20}$

$= {}^{11}C_7 \, x^{27} = 330\, x^{27}$.

Binomial Expressions:-

$(1-x)^{-1} = 1+x+x^2+ \cdots$

$(1+x)^{-1} = 1-x+x^2-x^3+ \cdots$

$(1-x)^{-2} = 1+2x+3x^2+4x^3+ \cdots$

$(1+x)^{-2} = 1-2x+3x^2-4x^3+ \cdots$

Multinomial Theorem:-

Multinomial theorem is a generalization of the binomial theorem with more than two variables.

$$(x_1+x_2+x_3+\cdots+x_k)^n = \sum_{n_1,n_2 \cdots n_k} \frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}$$

where, $n_1+n_2+n_3+\cdots+n_k = n$

(1) Find the following values of

(i) $\binom{7}{2,3,2}$ (ii) $\binom{8}{4,2,2,0}$

Sol:- (i) $\binom{7}{2,3,2} = \frac{n!}{n_1! \times n_2! \times n_3!} = \frac{7!}{2! \times 3! \times 2!} = 210$

(ii) $\binom{8}{4,2,2,0} = \frac{8!}{4! \times 2! \times 2! \times 0!} = 420$

---

(2) What is the coefficient of $x^{101} y^{99}$ in the expansion of $(2x-3y)^{200}$.

Sol:- wkt, $(x+y)^n = \sum_{r=0}^{\infty} {}^nC_r \, x^{n-r} \, y^r$.

Here, $x = 2x$, $y = -3y$ and $n = 200$

Now, $(2x-3y)^{200} = \sum_{r=0}^{\infty} {}^{200}C_r \, (2x)^{200-r} \cdot (-3y)^r$

$= \sum_{r=0}^{\infty} {}^{200}C_r \, (2)^{200-r} \cdot (x)^{200-r} \cdot (-3)^r \cdot (y)^r \rightarrow (1)$

Comparing coefficients then.

$x^{200-r} \cdot y^r = x^{101} \, y^{99}$

$\boxed{r = 99}$ (or) $200 - r = 101$ $\boxed{r = 99}$

sub $r = 99$ in eq (1),

$(2x-3y)^{200} = {}^{200}C_{99} \, (2)^{200-99} \, (x)^{200-99} \cdot (-3)^{99} \, (y)^{99}$

$= {}^{200}C_{99} \, (2)^{101} \, (x)^{101} \cdot (-3)^{99} \cdot (y)^{99}$

$= {}^{200}C_{99} \, (2)^{101} \cdot (-3)^{99} \cdot x^{101} \, y^{99}$

---

③ Find the coefficient of x^5y^2 in the expansion of $(2x-3y)^7$.

Sol:- wkt, BT is given by

$$(x+y)^n = \sum_{r=0}^{\infty} {}^nC_r \, x^{n-r} \cdot y^r.$$

Here, $x = 2x$, $y = -3y$ and $n = 7$

Now, $(2x-3y)^7 = \sum_{r=0}^{\infty} {}^7C_r \, (2x)^{\cancel{7}-r} \cdot (-3y)^r$

$$= \sum_{r=0}^{\infty} {}^7C_r \, (2)^{n-r} (-3)^r \cdot x^{7-r} \cdot y^r \rightarrow ①$$

Comparing coefficients we get

$$x^{7-r} \cdot y^r = x^5 y^2$$

$$\boxed{r=2}.$$

Sub $r=2$ in eq ①.

$$(2x-3y)^7 = {}^7C_2 \, (2)^{7-2} (-3)^2 \, x^{7-2} \cdot y^2$$

$$= {}^7C_2 \, 2^5 \, 3^2 \cdot x^5 y^2$$

$$= 6048 \, x^5 y^2 \, //.$$

Multinomial Theorem:-

Multinomial theorem is a generalization of binomial theorem with more than 2 variables and it is defined by

$$(x_1+x_2+x_3+\cdots+x_k)^n = \sum_{n_1+n_2+\cdots n_k} \binom{n}{n_1\ n_2\ n_3 \cdots n_k}$$

where, $n = n_1 + n_2 + n_3 + \cdots + n_k$.

(ii)

$$(x_1+x_2+x_3+\cdots+x_k)^n = \sum_{n_1,n_2,\cdots n_k} \frac{n!}{n_1! \times n_2! \times n_3! \times \cdots \times n_k!}$$

(iii)

$$(x_1+x_2+x_3+\cdots+x_k)^n = \frac{n!}{n_1! \times n_2! \times \cdots \times n_k!} \times (x_1)^{n_1} (x_2)^{n_2} \cdots (x_k)^{n_k}$$

Here, $n = n_1 + n_2 + n_3 + \cdots + n_k$

---

① Find the coefficient of xyz^2 in the expansion of $(2x-y-z)^4$

Sol:- By the def^n of multinomial theorem.

$$(x_1+x_2+\cdots+x_k)^n = \frac{n!}{n_1! \times n_2! \times \cdots n_k!} (x_1)^{n_1} (x_2)^{n_2} \cdots (x_k)^{n_k}.$$

Here, $x_1 = 2x$, $x_2 = -y$, $x_3 = -z$ and $n = 4$.

Now, $$(2x-y-z)^4 = \frac{4!}{n_1! \times n_2! \times n_3!} (2x)^{n_1} (-y)^{n_2} (-z)^{n_3}$$

where, $n = n_1 + n_2 + n_3$.

$$(2x-y-z)^4 = \frac{4!}{n_1! \times n_2! \times n_3!} \times (2)^{n_1} (-1)^{n_2} (-1)^{n_3} (x)^{n_1} (y)^{n_2} (z)^{n_3} \quad \text{①}$$

Comparing coefficients then

$$x^{n_1} y^{n_2} z^{n_3} = xyz^2$$

$n_1 = 1,\ n_2 = 1,\ n_3 = 2$

from eq ①,

$$(2x - y - z)^4 = \frac{4!}{1! \times 1! \times 2!} \times (2)^1 (-1)^1 (-1)^2\ x y z^2$$

$$= -24 x y z^2$$

---

② Find the coefficient of $a^2 b^3 c^2 d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$.

Sol: Here, $x_1 = a,\ x_2 = 2b,\ x_3 = -3c,\ x_4 = 2d,\ x_5 = 5$ and $n = 16$

Now, $(a + 2b - 3c + 2d + 5)^{16}$

$$= \frac{16!}{n_1! \times n_2! \times \cdots \times n_5!} \times (a)^{n_1} (2b)^{n_2} (-3c)^{n_3} (2d)^{n_4} (5)^{n_5}$$

$$= \frac{16!}{n_1! \times n_2! \times \cdots \times n_5!} \times (2)^{n_2} (-3)^{n_3} (5)^{n_5} (a)^{n_1} (b)^{n_2} (c)^{n_3} (d)^{n_4} \quad ①$$

Comparing then

$$a^2 b^3 c^2 d^5 = (a)^{n_1} b^{n_2} c^{n_3} d^{n_4}$$

$n_1 = 2,\ n_2 = 3,\ n_3 = 2,\ n_4 = 5$

wkt, $n_1 + n_2 + n_3 + n_4 + n_5 = 16$

$2 + 3 + 2 + 5 + n_5 = 16$

$n_5 = 16 - 12$

$\boxed{n_5 = 4}$

from (1),

$$= \frac{16!}{2!\times 3!\times 2!\times 5!\times 4!} \times (2)^3 (-3)^2 (2)^5 (5)^4 \; a^2 b^3 c^2 d^5$$

$$= 4.35891456 \times 10^{14} \; a^2 b^3 c^2 d^5$$

(3). Find the coefficient of $x^{11} y^4 z^2$ in the expansion of $(2x^3 - 3xy^2 + z^2)^6$

Sol^n: By the defn of multinominal theorem,

$x_1 = 2x^3$, $\cancel{x_2} = -3xy^2$, $x_3 = z^2$ and $n = 6$

Now, $(2x^3 - 3xy^2 + z^2)^6$

$$= \frac{6!}{n1!\times n_2!\times n_3!} \times (2x^3)^{n1} (-3xy^2)^{n_2} (z^2)^{n_3}$$

$$= \frac{6!}{n_1!\times n_2!\times n_3!} \times (2)^{n1} (x)^{3n1} (-3)^{n2} (x)^{n2} (y)^{2n2} (z)^{2n3}$$

$$= \frac{6!}{n_1!\times n_2!\times n_3!} \times (2)^{n1} (-3)^{n2} (x)^{3n1+n2} (y)^{2n2} (z)^{2n3} \quad \text{(1)}$$

Comparing coefficients then,

$$(x)^{3n1+n2} (y)^{2n2} (z)^{2n_3} = x^{11} y^4 z^2$$

$3n1 + n2 = 11$	$2n_2 = 4$	$2n_3 = 2$
$3n_1 + 2 = 11$	$n_2 = 2$	$n_3 = 1$
$3n_1 = 9$		
$n_1 = 3.$		

Now, from eqn ①.

$$= \frac{6!}{3! \times 2! \times 1!} \times (2)^3 (-3)^2 (1)^1 (x)^{3(3)+2} (y)^{2(2)} (z)^{2(1)}$$

$$= 4320\ x^{11} y^4 z^2.$$

---

④ Find the coefficient of $x^3y^3z^2$ in the expansion of $(2x-3y+5z)^8$.

Sol:- By the defn of multinominal theorem.

$x_1 = 2x,\ x_2 = -3y,\ x_3 = 5z$ and $n=8$

Now, $(2x-3y+5z)^8$

$$= \frac{8!}{n_1! \times n_2! \times n_3!} \times (2x)^{n1} (-3y)^{n2} (5z)^{n3}$$

$$= \frac{8!}{n_1! \times n_2! \times n_3!} \times (2)^{n1} (-3)^{n2} (5)^{n3} (x)^{n1} (y)^{n2} (z)^{n3} \quad \text{①}$$

Comparing with $x^3y^3z^2 = (x)^{n1} (y)^{n2} (z)^{n3}$

$n1 = 3,\ n2 = 3,\ n3 = 2.$

from eqn ①,

$$(2x-3y+5z)^8 = \frac{8!}{3! \times 3! \times 2!} \times (2)^3 (-3)^3 (5)^2\ x^3y^3z^2$$

$$= -3024000\ x^3y^3z^2$$

⑤ Find the coefficient of $x^3y^2z^2$ in the expansion of $(x+y+z)^9$.

Sol: By the def^n of multinomial theorem.

$$x_1 = x, \quad x_2 = y, \quad x_3 = z \quad \text{and} \quad n=7.$$

Now, $(x+y+z)^9 = \dfrac{7!}{n_1! \times n_2! \times n_3!} \times (x)^{n_1} (y)^{n_2} (z)^{n_3} \rightarrow ①$

Comparing with $x^3y^2z^2 = (x)^{n_1}(y)^{n_2}(z)^{n_3}$

$$n_1 = 3, \quad n_2 = 2, \quad n_3 = 2$$

from eq ①,

$$(x+y+z)^9 = \frac{7!}{3! \times 2! \times 2!} x^3y^2z^2$$

$$= 210\, x^3y^2z^2$$

⑥. Find the coefficient of xyz^5 and x^3z^4 in the expansion of $(x+y+z)^7$.

Sol: By the def^n of multinomial theorem

$$x_1 = x, \quad x_2 = y, \quad x_3 = z \quad \text{and} \quad n=7.$$

Now, $(x+y+z)^7 = \dfrac{7!}{n_1! \times n_2! \times n_3!} \times (x)^{n_1} (y)^{n_2} (z)^{n_3} \rightarrow ①$

Comparing with $xyz^5 = (x)^{n_1} \cdot (y)^{n_2} \cdot (z)^{n_3}$

we get $n_1 = 1, \quad n_2 = 1, \quad n_3 = 5$

from eq ①, $(x+y+z)^7 = \dfrac{7!}{1! \times 1! \times 5!} \times xyz^5 = 42xyz^5$