Multiple Integrals:

Double Integration:-

Double integrals over a region R may be evaluated by two successive integrals as follows.

Suppose that R can be described by inequalities of the form $a \le x \le b$, $y_1(x) \le y_2(x)$, so that $y = y_1(x)$, $y = y_2(x)$ represent the boundary of R,

then $$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dx\,dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\right] dx$$

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① Evaluate $$\int_{y=0}^{2} \int_{x=0}^{3} xy\,dx\,dy$$

Sol:- $$= \int_0^2 \left[\int_0^3 xy\,dx\right] dy$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= \int_0^2 \left[y \int_0^3 x\,dx\right] dy = \int_0^2 \left[y \cdot \left[\frac{x^2}{2}\right]_0^3\right] dy$$

$$= \int_0^2 \left[y\left(\frac{9}{2} - \frac{0}{2}\right)\right] dy = \frac{9}{2}\int_0^2 y\,dy = \frac{9}{2}\left[\frac{y^2}{2}\right]_0^2$$

$$= \frac{9}{2}\left[\frac{4}{2} - \frac{0}{2}\right] = \frac{9}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{4}} = 9.$$

$$\therefore \int_{y=0}^{2} \int_{x=0}^{3} xy\,dx\,dy = 9$$

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② Evaluate $$\int_0^3 \int_1^2 xy(1+x+y)\,dx\,dy.$$

Sol:- $$\int_0^3 \int_1^2 (xy + x^2y + xy^2)\,dx\,dy$$

$$= \int_0^3 \left[\int_1^2 (xy + x^2y + xy^2)\, dx \right] dy$$

$$= \int_0^3 \left[y\int_1^2 x\, dx + y \int_1^2 x^2 dx + y^2 \int_1^2 x\, dx \right] dy$$

$$= \int_0^3 \left[y \cdot \left(\frac{x^2}{2}\right)_1^2 + y \cdot \left(\frac{x^3}{3}\right)_1^2 + y^2 \cdot \left(\frac{x^2}{2}\right)_1^2 \right] dy$$

$$= \int_0^3 \left[y\left(\frac{4}{2} - \frac{1}{2}\right) + y\left(\frac{8}{3} - \frac{1}{3}\right) + y^2\left(\frac{4}{2} - \frac{1}{2}\right) \right] dy$$

$$= \int_0^3 \left[\frac{3}{2}y + \frac{7}{3}y + \frac{3}{2}y^2 \right] dy$$

$$= \int_0^3 \left(\frac{9y + 14y + 9y^2}{6} \right) dy$$

$$= \frac{1}{6}\left[9\int_0^3 y\, dy + 14\int_0^3 y\, dy + 9\int_0^3 y^2 dy \right]$$

$$= \frac{1}{6}\left[9\left[\frac{y^2}{2}\right]_0^3 + 14\left[\frac{y^2}{2}\right]_0^3 + 9\left[\frac{y^3}{3}\right]_0^3 \right]$$

$$= \frac{1}{6}\left[9\left(\frac{9}{2} - 0\right) + 14\left(\frac{9}{2} - 0\right) + 9\left(\frac{27}{3} - 0\right) \right]$$

$$= \frac{1}{6}\left[\frac{81}{2} + \frac{63}{1} + \frac{81}{1} \right] = \frac{1}{6}\left[\frac{81 + 126 + 162}{3} \right] = \frac{369}{12} = \frac{123}{4}$$

$$\therefore \int_0^3 \int_1^2 xy(1 + x + y)\, dx\, dy = \frac{123}{4} \;//$$

(or)

$$\int_0^3 \int_1^2 (xy + x^2y + xy^2)\, dx\, dy$$

$$= \int_0^3 \left[\int_1^2 xy\,dy + \int_1^2 x^2 y\,dy + \int_1^2 xy^2\,dy \right] dx.$$

$$= \int_0^3 \left[x\left[\frac{y^2}{2}\right]_1^2 + x^2\left[\frac{y^2}{2}\right]_1^2 + x\left[\frac{y^3}{3}\right]_1^2 \right] dx$$

$$= \int_0^3 \left[x\left(\frac{4}{2}-\frac{1}{2}\right) + x^2\left(\frac{4}{2}-\frac{1}{2}\right) + x\left(\frac{8}{3}-\frac{1}{3}\right) \right] dx$$

$$= \int_0^3 \left(\frac{3x}{2} + \frac{3x^2}{2} + \frac{7x}{3} \right) dx$$

$$= \frac{3}{2}\int_0^3 x\,dx + \frac{3}{2}\int_0^3 x^2\,dx + \frac{7}{3}\int_0^3 x\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_0^3 + \frac{3}{2}\left[\frac{x^3}{3}\right]_0^3 + \frac{7}{3}\left[\frac{x^2}{2}\right]_0^3$$

$$= \frac{3}{2}\left(\frac{9}{2}-0\right) + \frac{3}{2}\left(\frac{27}{3}-0\right) + \frac{7}{3}\left(\frac{9}{2}-0\right)$$

$$= \frac{27}{4} + \frac{27}{2} + \frac{21}{2} = \frac{27+54+42}{4} = \frac{123}{4} \,//$$

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③ Evaluate $\int_0^1 \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\,dx\,dy$.

Sol: $\int_0^1 \frac{dx}{\sqrt{1-x^2}} \times \int_0^1 \frac{dy}{\sqrt{1-y^2}}$

$$= \left[\sin^{-1}(x)\right]_0^1 \left[\sin^{-1}(y)\right]_0^1$$

$$\{\sin^{-1}(1) - \sin^{-1}(0)\} \cdot [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-0\right) = \frac{\pi^2}{4} \,//.$$

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(4) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

Sol: $= \int_0^\infty \left[\int_0^\infty e^{-(x^2+y^2)}\, dx \right] dy$

$= \int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2} \cdot dx \cdot dy$

$= \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$

put $x^2 = t$	put $y^2 = s$
$x = \sqrt{t}$	$y = \sqrt{s}$
$dx = \frac{1}{2\sqrt{t}}\, dt$	$dy = \frac{1}{2\sqrt{s}}\, ds$

$= \int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{t}}\, dt \cdot \int_0^\infty e^{-s} \cdot \frac{1}{2\sqrt{s}}\, ds$

$= \frac{1}{4} \int_0^\infty e^{-t} (t)^{-1/2} dt \cdot \int_0^\infty e^{-s} \cdot (s)^{-1/2} ds$

we know, $\int_0^\infty e^{-x} (x)^{n-1} dx = \Gamma(n)$

$= \frac{1}{4} \int_0^\infty e^{-t} \cdot (t)^{\frac{1}{2}-1} dt \int_0^\infty e^{-s} \cdot (s)^{\frac{1}{2}-1} ds$

$= \frac{1}{4} \cdot \Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{2}) = \frac{1}{4} \cdot \sqrt{\pi} \cdot \sqrt{\pi} = \frac{\pi}{4}$

$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \frac{\pi}{4}$ //

5. Evaluate $\int_0^2 \int_0^x y\,dy\,dx$.

Sol: $= \int_0^2 \left[\int_0^x y\,dy \right] dx$

$$= \int_0^2 \left[\frac{y^2}{2}\right]_0^x dx = \int_0^2 \left(\frac{x^2}{2} - \frac{0}{2}\right) dx = \frac{1}{2} \int_0^2 x^2\,dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left[\frac{8}{3} - \frac{0}{3}\right] = \frac{4}{3}.$$

$$\therefore \int_0^2 \int_0^x y\,dy\,dx = \frac{4}{3} \,//$$

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6. Evaluate $\int_0^2 \int_0^x e^{x+y}\,.dx\,dy$

Sol: $\int_0^2 \int_0^x e^x e^y\,dx\,dy$

$$= \int_0^2 \left[\int_0^x e^x . e^y\,dy \right] dx$$

$$= \int_0^2 \left[e^x . \left[e^y\right]_0^x \right] dx = \int_0^2 \left[e^x (e^x - e^0)\right] dx$$

$$= \int_0^2 (e^{2x} - e^x)\,dx = \int_0^2 e^{2x}\,dx - \int_0^2 e^x\,dx$$

$$= \left[\frac{e^{2x}}{2}\right]_0^2 - \left[e^x\right]_0^2 = \left[\frac{e^4}{2} - \frac{e^{2(0)}}{2}\right] - \left[e^2 - e^0\right]$$

$$= \frac{e^4}{2} - \frac{1}{2} - e^2 + 1 = \frac{e^4}{2} - e^2 + \frac{1}{2} = \frac{e^4 - 2e^2 + 1}{2}$$

$$= \frac{(e^2-1)^2}{2} \,//$$

(7) Evaluate $\int_0^1 \int_0^x e^{x+y} \, dx\,dy$.

Sol: $\int_0^1 \int_0^x e^x \cdot e^y \cdot dx\,dy$

$$= \int_0^1 e^x [e^y]_0^x \, dx = \int_0^1 e^x (e^x - e^0) \, dx$$

$$= \int_0^1 (e^{2x} - e^x) \, dx = \left[\frac{e^{2x}}{2}\right]_0^1 - [e^x]_0^1$$

$$= \left(\frac{e^2}{2} - \frac{e^0}{2}\right) - (e^1 - e^0)$$

$$= \frac{e^2}{2} - \frac{1}{2} - e + 1 = \frac{e^2}{2} - e + \frac{1}{2} = \frac{e^2 - 2e + 1}{2}$$

$$= \frac{(e-1)^2}{2}$$

$$\therefore \int_0^1 \int_0^x e^{x+y} \cdot dx\,dy = \frac{(e-1)^2}{2}$$

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(8). Evaluate $\int_0^2 \int_0^x (x+y) \, dx\,dy$.

Sol: $\int_0^2 \left[\int_0^x x\,dy + \int_0^x y\,dy\right] dx$

$$= \int_0^2 \left[x [y]_0^x + \left[\frac{y^2}{2}\right]_0^x\right] dx$$

$$= \int_0^2 \left[x(x-0) + \left(\frac{x^2}{2} - \frac{0}{2}\right)\right] dx = \int_0^2 \left(x^2 + \frac{x^2}{2}\right) dx$$

$$= \int_0^2 \frac{3x^2}{2} dx = \frac{3}{2}\int_0^2 x^2 dx = \frac{3}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{\cancel{3}}{\cancel{2}}\left[\frac{\cancel{8}^4}{\cancel{3}} - \frac{0}{3}\right] = 4$$

$$\therefore \int_0^2\int_0^x (x+y)\, dx dy = 4$$

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(9) Evaluate $\int_0^5\int_0^{x^2} x(x^2+y^2)\, dx dy$.

Sol:-

$$= \int_0^5 \left[\int_0^{x^2} (x^3 + xy^2)\, dy\right] dx$$

$$= \int_0^5 \left[\int_0^{x^2} x^3 dy + \int_0^{x^2} xy^2 dy\right] dx$$

$$= \int_0^5 \left[x^3 \left[y\right]_0^{x^2} + x\left[\frac{y^3}{3}\right]_0^{x^2}\right] dx$$

$$= \int_0^5 \left[x^3(x^2-0) + x\left(\frac{x^6}{3} - \frac{0}{3}\right)\right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \int_0^5 x^5 dx + \frac{1}{3}\int_0^5 x^7 dx$$

$$= \left[\frac{x^6}{6}\right]_0^5 + \frac{1}{3}\left[\frac{x^8}{8}\right]_0^5$$

$$= \frac{5^6}{6} + \frac{1}{3}\cdot\frac{5^8}{8} = \frac{5^6}{6} + \frac{5^8}{24} = \frac{4\times 5^6 + 5^8}{24}$$

$$= \frac{5^6(4+5^2)}{24} = \frac{5^6 \times 29}{24}$$

(10) Evaluate $\int_0^4 \int_0^{x^2} e^{\frac{y}{x}} \, dy \, dx$.

Sol:- $\int_0^4 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}} \right]_0^{x^2} dx$ $\quad \Big\downarrow \int e^{ax} dx = \frac{e^{ax}}{a} + c$

$= \int_0^4 \left[x \cdot e^{\frac{y}{x}} \right]_0^{x^2} dx$

$= \int_0^4 \left[x \left(e^{\frac{x^2}{x}} - e^0 \right) \right] dx = \int_0^4 x e^x \, dx - \int_0^4 x \, dx$

ILATE $\Rightarrow$ $D = x \quad 1$
$I = e^x \quad e^x$ $\quad \Rightarrow e^x x - e^x$
$= e^x (x - 1)$

$\Big\downarrow \int u \, dv = uv - \int v \, du$

$= (x e^x - e^x)_0^4 - \left[\frac{x^2}{2} \right]_0^4$

$= \left[4 e^4 - e^4 - (0 \cdot e^0 - e^0) \right] - \left[\frac{\cancel{16}^{8}}{\cancel{2}} - \frac{0}{2} \right]$

$= 4e^4 - e^4 + 1 - 8 = 3e^4 - 7$

$\therefore \int_0^4 \int_0^{x^2} e^{\frac{y}{x}} \, dx \, dy = 3e^4 - 7$ //

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(11) Evaluate $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}} \, dy \, dx$

Sol:- Same as above 10th problem. but limits are change

$= \left[x e^x - e^x \right]_0^1 - \left[\frac{x^2}{2} \right]_0^1$

$$= \left[1e^1 - e^1 - (0 \cdot e^0 - e^0)\right] - \left[\frac{1}{2} - \frac{0}{2}\right]$$

$$= \cancel{e^1} - \cancel{e^1} + 1 - \frac{1}{2} = \frac{1}{2} \,//$$

$$\therefore \int_0^1 \int_0^{x^2} e^{\frac{y}{x}} \, dx\,dy = \frac{1}{2} \,//$$

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(12). $\displaystyle\int_0^1 \int_x^{\sqrt{x}} x^2 y^2 (x+y) \, dx\,dy$.

Soln:

$$\int_0^1 \left[\int_x^{\sqrt{x}} x^3 y^2 \, dy + \int_x^{\sqrt{x}} x^2 y^3 \, dy \right] dx$$

$$= \int_0^1 \left[x^3 \cdot \left[\frac{y^3}{3}\right]_x^{\sqrt{x}} + x^2 \left[\frac{y^4}{4}\right]_x^{\sqrt{x}} \right] dx$$

$$= \int_0^1 \left[x^3 \left(\frac{(\sqrt{x})^3}{3} - \frac{x^3}{3} \right) + x^2 \cdot \left(\frac{(\sqrt{x})^4}{4} - \frac{x^4}{4} \right) \right] dx$$

$$= \int_0^1 \left[\frac{x^3 \cdot x\sqrt{x}}{3} - \frac{x^6}{3} + \frac{x^4}{4} - \frac{x^6}{4} \right] dx$$

$$= \int_0^1 \left(\frac{x^4 \sqrt{x}}{3} + \frac{x^4}{4} - \frac{x^6}{3} - \frac{x^6}{4} \right) dx$$

$$= \int_0^1 \left(\frac{x^{9/2}}{3} + \frac{x^4}{4} - \frac{7x^6}{12} \right) dx$$

$$= \left[\frac{1}{3} \left(\frac{x^{\frac{11}{2}}}{\frac{11}{2}} \right) + \frac{x^5}{20} - \frac{x^7}{12} \right]_0^1$$

$$= \frac{2(1)^{\frac{11}{2}}}{33} + \frac{1}{20} - \frac{1}{12} = \frac{40+33-55}{660} = \frac{18}{660} = \frac{3}{110}.$$

$$\therefore \int_0^1 \int_x^{\sqrt{x}} x^2y^2(x+y)\,dx\,dy = \frac{3}{110} //$$

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(13). Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$.

Sol: $\int_0^1 \left[\int_x^{\sqrt{x}} x^2 dy + \int_x^{\sqrt{x}} y^2 dy \right] dx$

$$= \int_0^1 \left[x^2 [y]_x^{\sqrt{x}} + \left[\frac{y^3}{3}\right]_x^{\sqrt{x}} \right] dx$$

$$= \int_0^1 \left[x^2(\sqrt{x}-x) + \left(\frac{(\sqrt{x})^3}{3} - \frac{x^3}{3} \right) \right] dx$$

$$= \int_0^1 \left(x^2\sqrt{x} - x^3 - \frac{x^3}{3} + \frac{x\sqrt{x}}{3} \right) dx$$

$$= \int_0^1 \left(x^2\sqrt{x} + \frac{x\sqrt{x}}{3} - \frac{4x^3}{3} \right) dx$$

$$= \int_0^1 \left(x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - \frac{4}{3}x^3 \right) dx$$

$$= \left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^{\frac{5}{2}}}{3 \times \frac{5}{2}} - \frac{\cancel{4}}{3} \cdot \frac{x^4}{\cancel{4}} \right]_0^1$$

$$= \frac{2}{7} \times (1)^{\frac{7}{2}} + \frac{2}{15} \times (1)^{\frac{5}{2}} - \frac{1}{3} = \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{30+14-35}{105} = \frac{44-35}{105} = \frac{9}{105} = \frac{3}{35}$$

$$\therefore \int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy = \frac{3}{35}$$

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(14). Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx.$

Sol: put $t = \sqrt{1+x^2}$

$$t^2 = 1+x^2$$

$$= \int_0^1 \int_0^t \frac{1}{t^2+y^2}\,dy\,dx \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$= \int_0^1 \left[\frac{1}{t} \cdot \left[\tan^{-1}\left(\frac{y}{t}\right)\right]_0^t \right] dx$$

$$= \int_0^1 \left[\frac{1}{t} \left(\tan^{-1}(1) - \tan^{-1}(0)\right)\right] dx$$

$$= \int_0^1 \frac{1}{t}\left(\frac{\pi}{4}\right) dx = \int_0^1 \frac{\pi}{4\sqrt{1+x^2}}\,dx \qquad t = \sqrt{1+x^2}$$

$$= \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx \qquad \int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}(x)$$

(or)

$$\log\left(x+\sqrt{1+x^2}\right).$$

$$= \left[\frac{\pi}{4}\sinh^{-1}(x)\right]_0^1 \quad (\text{or}).\left[\frac{\pi}{4}\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\sinh^{-1}(1) - \sinh^{-1}(0)\right] \quad (\text{or}) \quad \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log 1\right]$$

$$= \frac{\pi}{4}\sinh^{-1}(1) \quad (\text{or}) \quad \frac{\pi}{4}\log(1+\sqrt{2}) \; //$$

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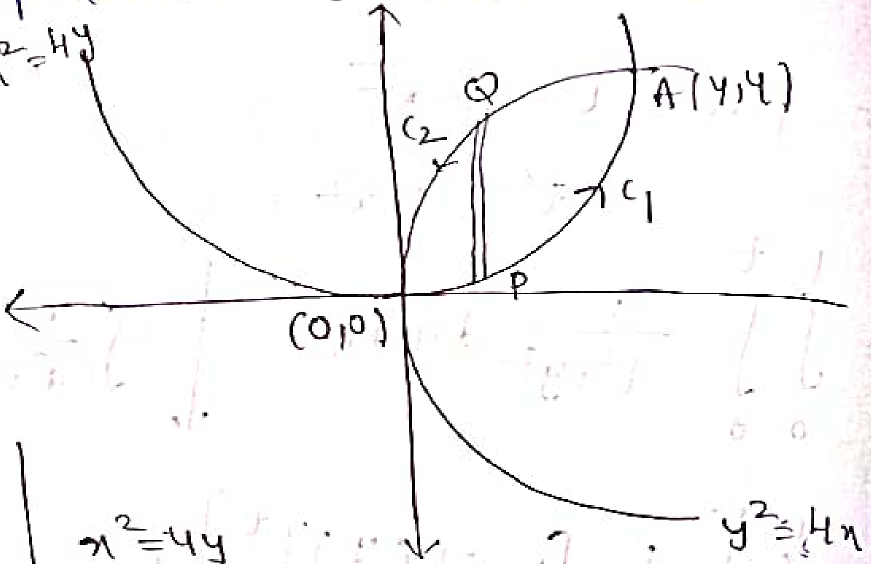
15. Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by the parabolas $y^2 = 4x$, $x^2 = 4y$.

Sol:-



$y^2 = 4x \rightarrow (1)$

$x^2 = 4y \rightarrow (2)$

Solving (1) & (2)

$y^2 = 4x$

S.O.B.S

$y^4 = 4^2 x^2$

$y^4 = 4^2 \cdot 4y = 16y \times 4$

$y^{\cancel{4}} = 16\cancel{y}\times 4 \Rightarrow y^3 = 4^3$

$\boxed{y=4}$

$x^2 = 4y$

$x^2 = 16$

$\boxed{x=4}$

Integrating with respect to y first, the strip is parallel to y-axis

limits :- $y \rightarrow \frac{x^2}{4}$ to $2\sqrt{x}$

$x \rightarrow 0$ to 4.

$$\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\,dx = \int_0^4 \left[\int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\right] dx$$

$$= \int_0^4 \left[\frac{(2\sqrt{x})^2}{2} - \frac{\left(\frac{x^2}{4}\right)^2}{2}\right] dx = \int_0^4 \left(\frac{\cancel{4}x}{\cancel{2}} - \frac{x^2}{32}\right) dx$$

$$= 2\int_0^4 x\,dx - \frac{1}{32}\int_0^4 x^4\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^4 - \frac{1}{32}\left[\frac{x^5}{5}\right]_0^4$$

$$= 2\left[\frac{16}{2} - \frac{0}{2}\right] - \frac{1}{\cancel{32}_{\,2}}\left[\frac{\cancel{16}\times\cancel{16}^{\,8}\times 4}{5} - \frac{0}{5}\right]$$

$$= 16 - \frac{32}{5} = \frac{80-32}{5} = \frac{48}{5}$$

$$\therefore \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\,dx = \frac{48}{5}$$

Note:- Integrating w.r.to x first, the finite number of strips are parallel to x-axis and w.r.to y-first the strip is parallel to y-axis.

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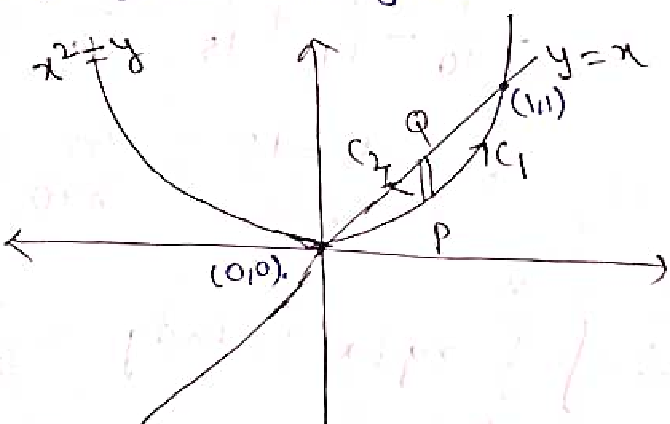
(16) Evaluate $\iint xy(x+y)\,dx\,dy$ over the region R bounded by the parabolas $y=x^2$ and $y=x$

Sol:-



$y = x^2 \rightarrow (1)$

$y = x \rightarrow (2)$

Solving (1) & (2)

$x = x^2$

$\boxed{x=1}$, $\boxed{y=1}$

Integrating w.r.to y-first, then the strip is parallel to y-axis.

limits:-

$y \to x^2$ to x

$x \to 0$ to 1.

$$\int_0^1 \int_{x^2}^{x} xy(x+y)\,dx\,dy$$

$$= \int_0^1 \left[\int_{x^2}^{x} (x^2y + xy^2)\,dy \right] dx$$

$$= \int_0^1 \left[x^2 \left[\frac{y^2}{2} \right]_{x^2}^{x} + x \left[\frac{y^3}{3} \right]_{x^2}^{x} \right] dx$$

$$= \int_0^1 \left[x^2 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) + x \left(\frac{x^3}{3} - \frac{x^6}{3} \right) \right] dx$$

$$= \int_0^1 \left[\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3} \right] dx$$

$$= \frac{1}{2} \left(\frac{x^5}{5} \right]_0^1 - \frac{1}{2} \left[\frac{x^7}{7} \right]_0^1 + \frac{1}{3} \left[\frac{x^5}{5} \right]_0^1 - \frac{1}{3} \left[\frac{x^8}{8} \right]_0^1$$

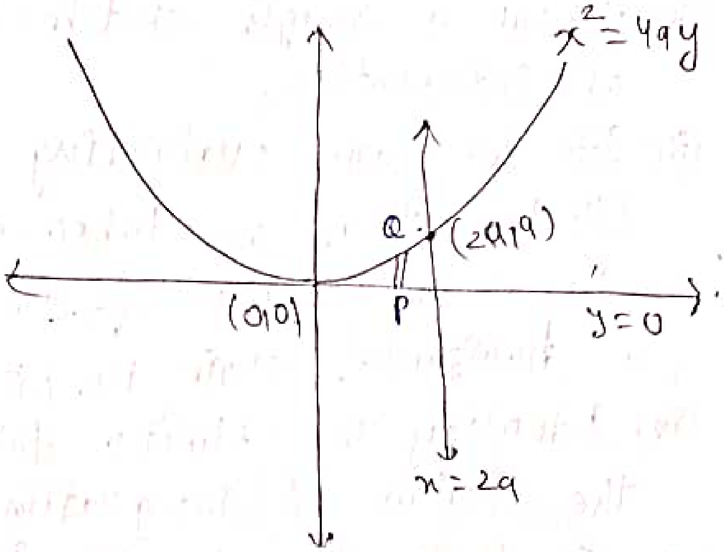
$$= \frac{1}{2}\left(\frac{1}{5} - 0\right) - \frac{1}{2}\left(\frac{1}{7} - 0\right) + \frac{1}{3}\left(\frac{1}{5} - 0\right) - \frac{1}{3}\left(\frac{1}{8} - 0\right)$$

$$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{84 - 60 + 54 - 35}{840}$$

$$= \frac{140 - 95}{840} = \frac{45}{840} = \frac{9}{168} \,// = \frac{3}{56}$$

$$\therefore \int_0^1 \int_{x^2}^{x} xy(x+y)\,dx\,dy = \frac{3}{56} \,//$$

(17). Evaluate $\iint_R xy\,dx\,dy$, where R is the region bounded by X-axis, ordinate $x = 2a$ and the $x^2 = 4ay$



Sol:- $x^2 = 4ay \rightarrow ①$

$x = 2a \rightarrow ②$

solving ① & ②

$\cancel{4}a^2 = \cancel{4}ay$

$\boxed{y = a}$

$x^2 = 4aa$

$x^2 = 4a^2 \Rightarrow \boxed{x = 2a}$

Limits:-

$y \rightarrow 0$ to $\frac{x^2}{4a}$

$x \rightarrow 0$ to $2a$

$$\int_0^{2a}\int_0^{\frac{x^2}{4a}} xy\,dx\,dy = \int_0^{2a}\int_0^{x^2/4a} xy\,dx\,dy$$

$$= \int_0^{2a}\left[\int_0^{\frac{x^2}{4a}} xy\,.dy\right]dx$$

$$= \int_0^{2a}\left[x\left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}}\right]dx = \int_0^{2a}\left[x.\left(\frac{\left(\frac{x^2}{4a}\right)^2}{2} - 0\right)\right]dx$$

$$= \int_0^{2a}\left[x\left(\frac{x^4}{32a^2}\right)dx\right] = \frac{1}{32a^2}\left[\frac{x^5}{5}\right]_0^{2a}$$

$$= \frac{1}{\cancel{32}\times a^2}\times\left(\frac{(\cancel{2})^5(a)^5}{5} - 0\right) = \frac{a^3}{5}$$

Change of order of Integration:-

Working procedure:-

(i) First identify the variables for the limits.

(ii) Draw a rough sketch of the given region of integration.

(iii) If we are evaluating the integral w.r.to 'y' first, then we takes a vertical strip, i.e, strip is parallel to y-axis otherwise take a horizontal strip i.e, strip is parallel to x-axis.

(iv) Identify the limits for other variables for the region of integration.

(v) Evaluate the double integral with new order of integration.

---

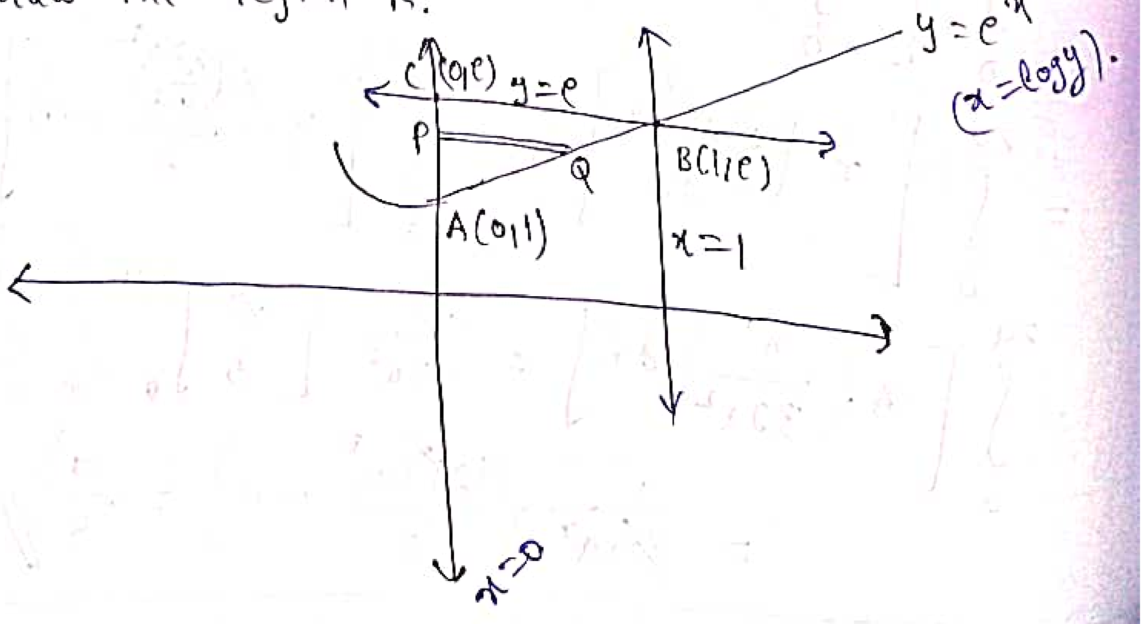
(18). By changing the order of integration evaluate

$$\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}.$$

Sol:- Given that $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$.

Let R be the region bounded by the lines $x=0$, $x=1$, $y=e^x$ and $y=e$

Draw the region R.



Now, integrating w.r.to x-first then the strip is parallel to X-axis.

$x \to 0$ to $\log y$

$y \to 1$ to e.

$$\int_0^1 \int_{e^x}^{e} \frac{dx\,dy}{\log y} = \int_1^e \int_0^{\log y} \frac{dx\,dy}{\log y}$$

$$= \int_1^e \left[\int_0^{\log y} \frac{1}{\log y}\, dx \right] dy = \int_1^e \frac{1}{\log y} [x]_0^{\log y}\, dy.$$

$$= \int_1^e \frac{1}{\log y} \cdot (\log y - 0) \cdot dy = [y]_1^e = e-1.$$

$$\therefore \int_0^1 \int_{e^x}^{e} \frac{dx\,dy}{\log y} = e-1 \,//.$$

---

(19). By changing the order of integration evaluate
$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dx\,dy.$$

Sol:- Given that $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dx\,dy$

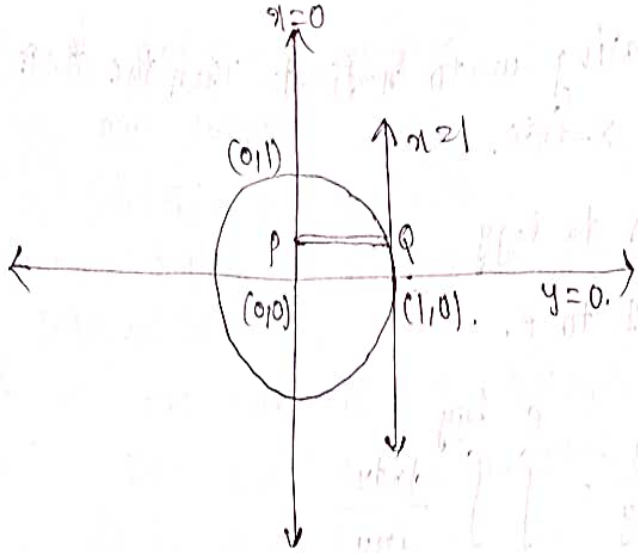
Let R be the region bounded the lines

$x=0,\ x=1,\ y=0$ & $y=\sqrt{1-x^2}$

S.O.B.S.

$y^2 = 1-x^2$

$x^2+y^2=1.$



Now, integrating w.r.to x - first then the strip is parallel to x-axis.

$x \to 0$ to $\sqrt{1-y^2}$

$y \to 0$ to 1.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx = \int_0^1 \left[\int_0^{\sqrt{1-y^2}} y^2 \, dx \right] dy$$

$$= \int_0^1 \left[y^2 \left[x \right]_0^{\sqrt{1-y^2}} \right] dy = \int_0^1 y^2 \left(\sqrt{1-y^2} \right) dy$$

put $y = \sin\theta$

$dy = \cos\theta \, d\theta$

LL $\Rightarrow$ if $y=0$, $\theta = 0$

UL $\Rightarrow$ if $y=1$, $\theta = \frac{\pi}{2}$.

$$= \int_0^{\pi/2} \sin^2\theta \cdot \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta \, d\theta$$

$2m-1=2$, $2n-1=2$

$2m=3$

$m=\frac{3}{2}$ $\quad$ $n=\frac{3}{2}$.

$$= \frac{1}{2} B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)}.$$

$$= \frac{1}{2} \cdot \frac{\frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} = \frac{1}{8} \cdot \frac{\sqrt{\pi} \cdot \sqrt{\pi}}{3 \times 2 \times 1} = \frac{\pi}{48}.$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx = \frac{\pi}{48}$$

---

(20). By changing the order of integration and evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$.

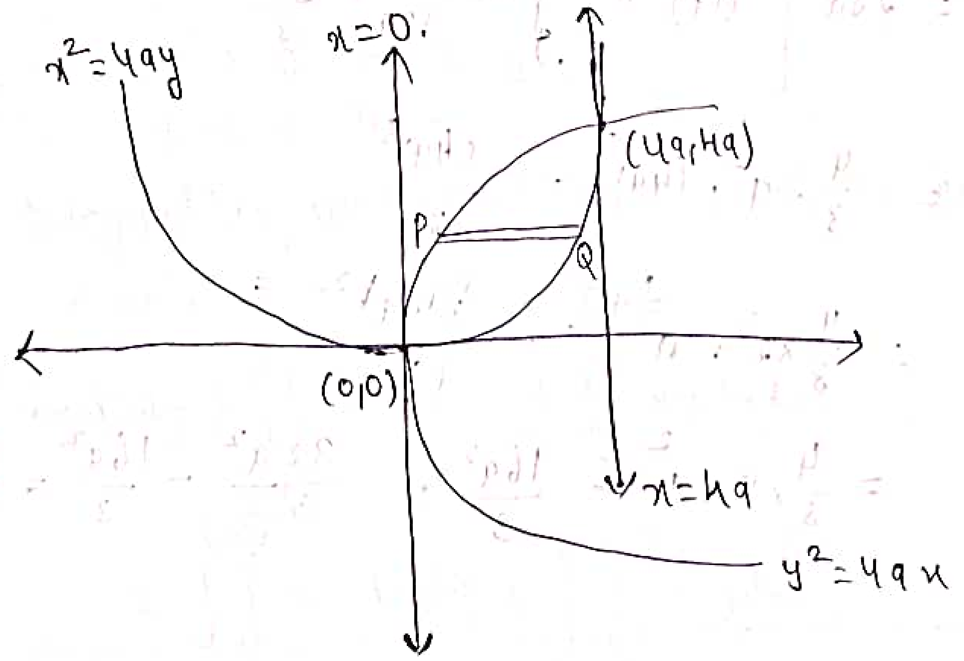
Sol: given that $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Let R be the region bounded by the lines

$x = 0, \; x = 4a; \; y = \frac{x^2}{4a}$ & $y = 2\sqrt{ax}$.

S.O.B.S.

$x^2 = 4ay \; ; \; y^2 = 4ax.$



Now, integrating w.r.to. x - first then the strip is parallel to X-axis

$$x \to \frac{y^2}{4a} \text{ to } 2\sqrt{ay}$$

$$y \to 0 \text{ to } 4a.$$

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right] dy$$

$$= \int_0^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy = \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= 2\sqrt{a}\int_0^{4a} \sqrt{y}\,dy - \frac{1}{4a}\int_0^{4a} y^2 dy$$

$$= 2\sqrt{a}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left[\frac{(4a)^3}{3}\right]$$

$$= 2\sqrt{a}\left[(4a)^{\frac{3}{2}} \times \frac{2}{3}\right] - \frac{1}{4a}\cdot\frac{(4a)^3}{3}$$

$$= \frac{4}{3}\cdot a^{\frac{1}{2}}\cdot(4a)^{\frac{3}{2}} - \frac{(4a)^3}{12a}$$

$$= \frac{4}{3}\times 2^3 \times a^{\frac{1}{2}+\frac{3}{2}} - \frac{\cancel{64}^{16} a^{\cancel{3}2}}{\cancel{12}_3 \cancel{a}}$$

$$= \frac{4}{3}\times 8 \times a^2 - \frac{16a^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

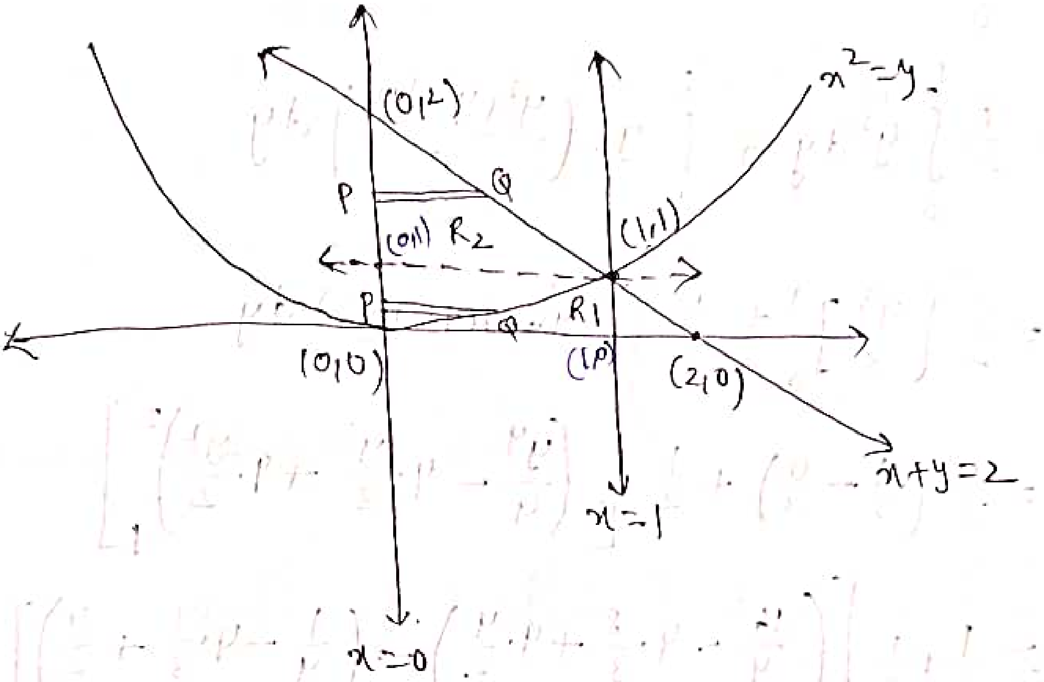
(21). By changing the order of integration in $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate double integral?

Sol: Given that $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$.

Let R be the region bounded by the lines

$y=x^2,\ y=2-x,\ x=0,\ x=1.$

$x+y=2 \Rightarrow \frac{x}{2}+\frac{y}{2}=1.$



$R_1 \Rightarrow\ x \to 0$ to $\sqrt{y}$

$y \to 0$ to 1.

$R_2 \Rightarrow\ x \to 0$ to $2-y$

$y \to 1$ to 2.

Now, integrating w.r to x first, strip is parallel to x-axis.

$$\iint_R xy\,dx\,dy = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy$$

$$= \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 \left[\int_0^{\sqrt{y}} xy\,dx \right] dy + \int_1^2 \left[\int_0^{2-y} xy\,dx \right] dy$$

$$= \int_0^1 \left[y \cdot \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} \right] dy + \int_1^2 \left[y \left(\frac{x^2}{2}\right)_0^{2-y} \right] dy$$

$$= \int_0^1 y \cdot \left(\frac{y}{2} - \frac{0}{2} \right) dy + \int_1^2 y \cdot \left(\frac{(2-y)^2}{2} - 0 \right) dy$$

$$= \frac{1}{2} \int_0^1 y^2 dy + \int_1^2 y \cdot \left(\frac{y^2+4-4y}{2} \right) dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1 + \frac{1}{2} \int_1^2 (y^3 + 4y - 4y^2)\, dy$$

$$= \frac{1}{2} \left(\frac{1}{3} - \frac{0}{3} \right) + \frac{1}{2} \left[\left(\frac{y^4}{4} - 4 \cdot \frac{y^3}{3} + 4 \cdot \frac{y^2}{2} \right)_1^2 \right]$$

$$= \frac{1}{6} + \frac{1}{2} \left[\left(\frac{16}{4} - 4 \cdot \frac{8}{3} + 4 \cdot \frac{4}{2} \right) - \left(\frac{1}{4} - 4 \cdot \frac{1}{3} + \frac{4}{2} \right) \right]$$

$$= \frac{1}{6} + \frac{1}{2} \left[\left(4 + 8 - \frac{32}{3} \right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{2}{1} \right) \right]$$

$$= \frac{1}{6} + \frac{1}{2} \left(12 - \frac{32}{3} - \frac{1}{4} + \frac{4}{3} - 2 \right)$$

$$= \frac{1}{6} + \frac{1}{2} \left[\frac{120 - 128 - 3 + 16}{12} \right]$$

$$= \frac{1}{6} + \frac{1}{2} \left[\frac{-131 + 136}{12} \right] = \frac{1}{6} + \frac{1}{2} \left(\frac{5}{12} \right) = \frac{1}{6} + \frac{5}{24} = \frac{4+5}{24}$$

$$= \frac{9}{24} = \frac{3}{8}.$$

$$\therefore \int \int_{x^2}^{2-x^2} xy\,dx\,dy = \frac{3}{8} \;//.$$

(22). Evaluate by changing the order of integration.

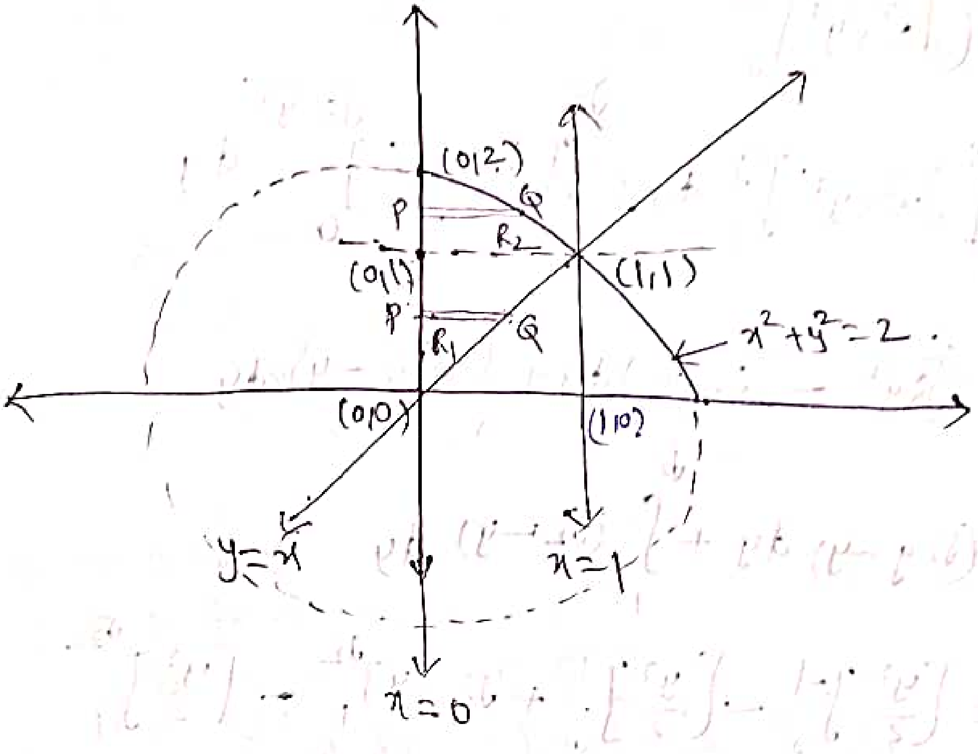
$$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

Sol: Given that $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$

Let R be the region bounded by the lines

$y=x$, $y=\sqrt{2-x^2}$, $x=0$, $x=1$.

$x^2+y^2=2$.



$R_1 \Rightarrow$ $x \to 0$ to y.

$y \to 0$ to 1.

$R_2 \Rightarrow$ $x \to 0$ to $\sqrt{2-y^2}$

$y \to 1$ to $\sqrt{2}$

$$\iint_R \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \iint_{R_1} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \iint_{R_2} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

$$= \int_0^1\int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}}\int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

Now, $\int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$

put $x^2+y^2=t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$= \frac{1}{2}\int_0^y \frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^y = \cancel{\frac{1}{2}}\left[\frac{\sqrt{x^2+y^2}}{\cancel{\frac{1}{2}}}\right]_0^y$$

$$= \left[\sqrt{x^2+y^2}\right]_0^y$$

$$= \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^y dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 \left(\sqrt{2y^2} - \sqrt{0^2+y^2}\right) dy + \int_1^{\sqrt{2}} (\sqrt{2}-y)\, dy$$

$$= \int_0^1 (\sqrt{2}y - y)\, dy + \int_1^{\sqrt{2}} (\sqrt{2}-y)\, dy$$

$$= \sqrt{2}\left[\frac{y^2}{2}\right]_0^1 - \left[\frac{y^2}{2}\right]_0^1 + \sqrt{2}\,[y]_1^{\sqrt{2}} - \left[\frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \sqrt{2}\left[\frac{1}{2}-0\right] - \frac{1}{2} + \sqrt{2}(\sqrt{2}-1) - \left(\frac{\cancel{2}}{\cancel{2}} - \frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2} + 2 - \sqrt{2} - \frac{1}{2}$$

$$= -\frac{2}{2} + 2 - \sqrt{2} + \frac{1}{\sqrt{2}} = -1 + 2 - \sqrt{2} + \frac{1}{\sqrt{2}} = 1 - \sqrt{2} + \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2}-2+1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$

$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} = 1 - \frac{1}{\sqrt{2}} \,//$$

~~23~~ $\cancel{\int_0 \int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}}$

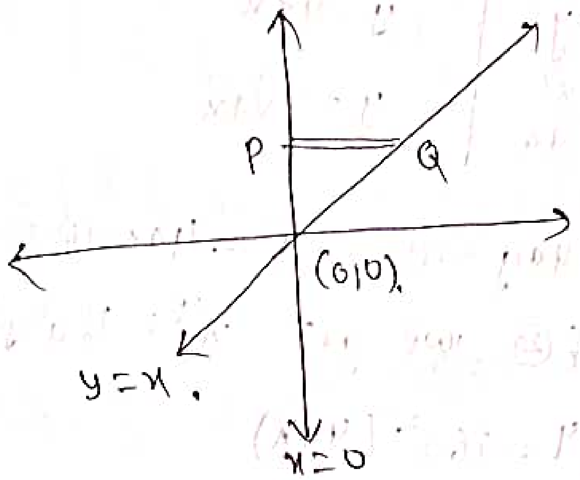
(23) Evaluate the integral by changing the order of integration.

$$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx.$$

Sol: Given that $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx.$

let R be the region bounded by the lines

$y=x,\ y=\infty,\ x=0,\ x=\infty.$



$x \to 0$ to y

$y \to 0$ to ∞

Now, $\int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\, dy\, dx = \int_0^{\infty}\left[\int_0^{y} \frac{e^{-y}}{y}\, dx\right] dy$

$$= \int_0^{\infty} \frac{e^{-y}}{y}\, [x]_0^y\, dy = \int_0^{\infty} \frac{e^{-y}}{\cancel{y}} \cdot (\cancel{y}-0)\, dy$$

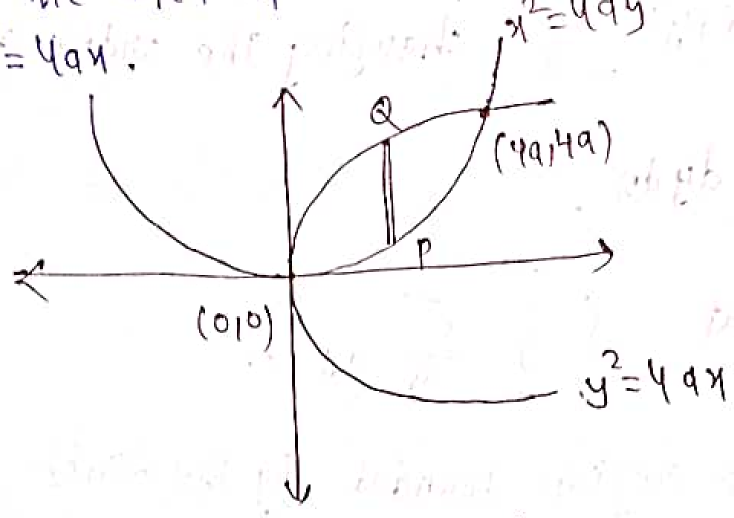
$$= \int_0^{\infty} e^{-y}\, dy = [-e^{-y}]_0^{\infty} = -[e^{-y}]_0^{\infty}$$

$$= -[e^{-\infty} - e^{-0}] = 1 - e^{-\infty} = 1-\frac{1}{e^{\infty}} = 1-\frac{1}{\infty}$$

$$= 1-0 = 1. \qquad \therefore \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx = 1$$

(24). Find the area b/w the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Sol:



The strip is parallel to y axis,

$$x^2 = 4ay \qquad\qquad y^2 = 4ax$$
$$y = \frac{x^2}{4a} \qquad\qquad y = 2\sqrt{ax}$$

Given that $x^2 = 4ay \rightarrow (1)$ $\quad y^2 = 4ax \rightarrow (2)$.

from eq's (1) & (2) we get $x^4 = 16a^2y^2$

$$x^4 = 16a^2(4ax)$$
$$x^3 = 64a^3 \cancel{x} \Rightarrow x^3 = 64a^3$$
$$\boxed{x = 4a}$$

put $x=4a$ in eq (1), we get $y = 4a$.

we know that $\iint_R dx\,dy$,

$y \rightarrow \frac{x^2}{4a}$ to $2\sqrt{ax}$

$x \rightarrow 0$ to $4a$.

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\left[\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\right]dx.$$

$$= \int_0^{4a} \left[[y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} \right] dx = \int_0^{4a} \left[\left(2\sqrt{ax} - \frac{x^2}{4a} \right) \right] dx$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a} \right) dx$$

$$= 2\sqrt{a} \int_0^{4a} \sqrt{x}\, dx - \frac{1}{4a} \int_0^{4a} x^2 dx \,..$$

$$= 2\sqrt{a} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{4a} - \frac{1}{4a} \left[\frac{x^3}{3} \right]_0^{4a}$$

$$= 2\sqrt{a} \left[\frac{2}{3} \times (4a)^{\frac{3}{2}} \right] - \frac{1}{\cancel{(4a)}} \cdot \frac{(4a)^{\cancel{3}\,2}}{3}$$

$$= 2\sqrt{a} \cdot \frac{2}{3} \cdot (2\sqrt{a})^3 - \frac{(4a)^2}{3}$$

$$= \frac{2 \cdot 2 \cdot \sqrt{a} \cdot 2\sqrt{a} \cdot (2\sqrt{a})^2}{3} - \frac{(4a)^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} //$$

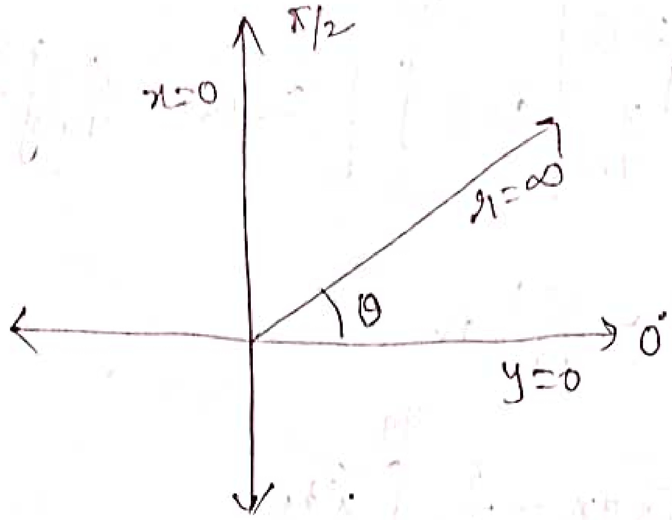
$\therefore$ The area bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $\frac{16a^2}{3}$ //

---

(25). Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar coordinates..

Sol:- given that $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$.

$x = 0,\ x = \infty,\ y = 0,\ y = \infty.$



Let $x^2+y^2=r^2$, $dx\,dy = r\,dr\,d\theta$

$r \to 0$ to ∞

$\theta \to 0$ to $\pi/2$.

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}.r.dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^{\infty} e^{-r^2}.r\,dr\right]d\theta$$

put $r^2 = t \Rightarrow 2r.dr = dt$

$r\,dr = \frac{dt}{2}$.

$$= \int_0^{\pi/2}\left[\int_0^{\infty} e^{-t}.\frac{dt}{2}\right]d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}.\left[-e^{-t}\right]_0^{\infty}.d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(-\left(e^{-\infty}-e^{-0}\right)\right).d\theta \qquad e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} =$$

$$= \frac{1}{2}\int_0^{\pi/2} (-(0-1))\,d\theta = \frac{1}{2}\left[\theta\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}.$$

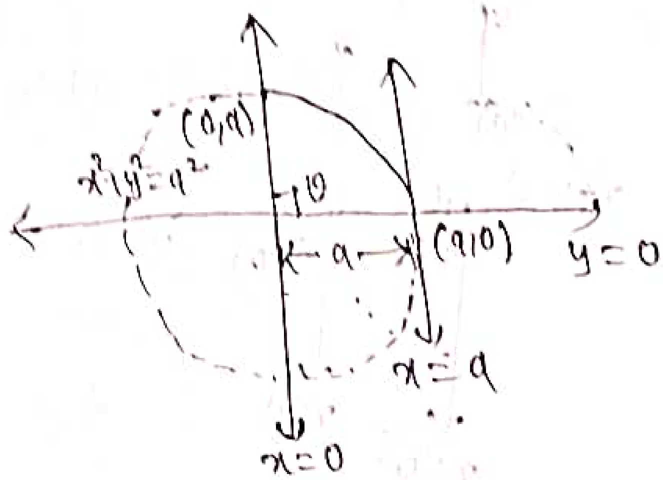
$$\therefore \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4} \,//$$

(29). Transform the integral into polar form coordinates and hence evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy$.

Sol:- given that $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy$

$y=0,\ y=\sqrt{a^2-x^2},\ x=0,\ x=a.$

$x^2+y^2=a^2$



$r \to 0$ to a

$\theta \to 0$ to $\frac{\pi}{2}$

$x^2+y^2=a^2$

$dx\,dy = r\,dr\,d\theta.$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy = \int_0^{\pi/2} \int_0^a r.r.dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\int_0^a r^2 d\theta\right] d\theta = \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (a^3-0)\, d\theta = \frac{a^3}{3}\left[\theta\right]_0^{\pi/2} = \frac{a^3}{3}\cdot\frac{\pi}{2} = \frac{\pi a^3}{6}.$$

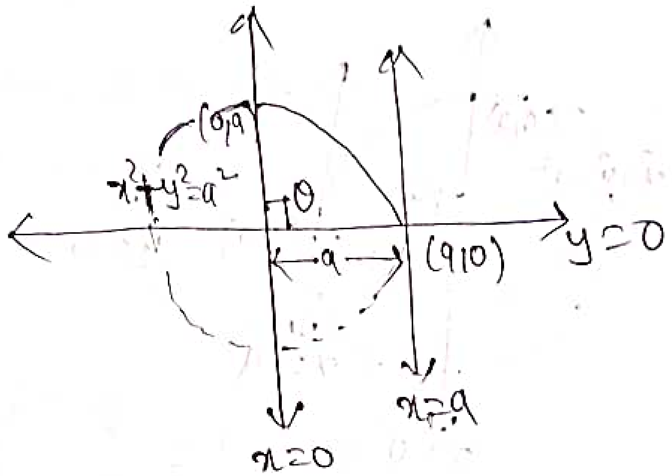
$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy = \frac{\pi a^3}{6}$$

(27). Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy$ by change into polar coordinates.

Sol: given that $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy$.

$y=0,\ y=\sqrt{a^2-x^2},\quad x=0,\ x=a.$

$x^2+y^2=a^2$



$r \to 0$ to a

$\theta \to 0$ to $\frac{\pi}{2}$

$x^2+y^2=r^2$

$dx\,dy = r.dr.d\theta$.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy = \int_0^{\pi/2} \int_0^a r^2.r.dr.d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta = \int_0^{\pi/2} \left(\frac{a^4}{4}-0\right) d\theta$$

$$= \frac{a^4}{4}\left[\theta\right]_0^{\pi/2} = \frac{\pi}{2}\cdot\frac{a^4}{4} = \frac{\pi a^4}{8}.$$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy = \frac{\pi a^4}{8}$$ //.

Triple Integration (Volume Integration) :-

Given integrals $\int_{x(x_1)}^{x(x_2)} \int_{y(y_1)}^{y(y_2)} \int_{z(z_1)}^{z(z_2)} dx\,dy\,dz$.

$$= \int_{a}^{b} \int_{y(y_1)}^{y(y_2)} \int_{z(y_1 z_1)}^{z(y_2 z_2)} dx\,dy\,dz$$

$$= \int_{a}^{b} \left[\int_{y(y_1)}^{y(y_2)} \left(\int_{z(y_1 z_1)}^{z(y_2 z_2)} dx \right) dy \right] dz.$$

---

(28). Evaluate $\int_0^1 \int_0^1 \int_0^1 dx\,dy\,dz$.

Sol:- given that $\int_0^1 \int_0^1 \int_0^1 dx\,dy\,dz$

$$= \int_0^1 \int_0^1 [x]_0^1 \, dy\,dz = \int_0^1 \int_0^1 (1-0)\,dy\,dz$$

$$= \int_0^1 [y]_0^1 \, dz = [z]_0^1 = (1-0) = 1.$$

$$\therefore \int_0^1 \int_0^1 \int_0^1 dx\,dy\,dz = 1$$

---

(29). $\int_0^1 \int_y^1 \int_0^{1-x} x\,dz\,dy\,dx$

Sol:- given that $\int_0^1 \int_y^1 \int_0^{1-x} (x\,dz).\,dy\,dx$

$$= \int_0^1 \int_y^1 x(1-x)\,dy\,dx = \int_0^1 \int_y^1 (x-x^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_y^1 dy = \int_0^1 \left[\frac{1}{2} - \frac{1}{3} - \frac{y^2}{2} + \frac{y^3}{3}\right] dy$$

$$= \frac{1}{2}[y]_0^1 - \frac{1}{3}[y]_0^1 - \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{3}\left[\frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{2} - \frac{1}{3} - \frac{1}{6} + \frac{1}{12}$$

$$= \frac{6-4-2+1}{12} = \frac{7-6}{12} = \frac{1}{12}$$

(30) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz.$

Sol: given that $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz$

$$= \int_0^1 \int_0^{1-x} [z]_0^{1-x-y}\, dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\, dx\,dy$$

$$= \int_0^1 \left[\int_0^{1-x} dy - \int_0^{1-x} x\,dy - \int_0^{1-x} y\,dy\right] dx$$

$$= \int_0^1 \left[[y]_0^{1-x} - x[y]_0^{1-x} - \left[\frac{y^2}{2}\right]_0^{1-x}\right] dx$$

$$= \int_0^1 \left[1-x - x(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \left[1-x - x + x^2 - \frac{1+x^2-2x}{2} \right] dx$$

$$= \int_0^1 \left(\frac{2 - \cancel{2x} - 2x + 2x^2 - 1 - x^2 + \cancel{2x}}{2} \right) dx$$

$$= \int_0^1 \left(\frac{x^2 - 2x + 1}{2} \right) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} - 2 \cdot \frac{x^2}{2} + x \right]_0^1$$

$$= \frac{1}{2} \left(\frac{1}{3} - \cancel{2} \cdot \frac{1}{\cancel{2}} + \cancel{1} \right] = \frac{1}{2} \left(\frac{1}{3} \right) = \frac{1}{6}$$

$$\therefore \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz = \frac{1}{6} \,//$$

---

(31). Evaluate $\iiint_V (xy + yz + zx)\, dx\,dy\,dz$, where V is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$.

Sol:-

$$\int_0^1 \int_0^2 \int_0^3 (xy + yz + zx)\, dz\,dy\,dx$$

$$= \int_0^1 \int_0^2 \left[xy \cdot z + y \cdot \frac{z^2}{2} + x \cdot \frac{z^2}{2} \right]_0^3 dy\,dx$$

$$= \int_0^1 \int_0^2 \left(3xy + \frac{9y}{2} + \frac{9x}{2} \right) dy\,dx$$

$$= \int_0^1 \left[3x \cdot \frac{y^3}{2} + \frac{9}{2} \cdot \frac{y^2}{2} + \frac{9}{2} \cdot x \cdot y \right]_0^2 dx$$

$$= \int_0^1 \left[\frac{3x \cdot \cancel{8}^{2}}{\cancel{2}} + \frac{9}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{2}} + \frac{9}{\cancel{2}} \cdot \cancel{2}x \right] dx$$

$$= \int_0^1 \left[6x + 9 + 9x \right] dx$$

$$= \left[\cancel{6}^{3} \cdot \frac{x^2}{\cancel{2}} + 9 \cdot x + 9 \cdot \frac{x^2}{2} \right]_0^1 = \left[3x^2 + 9x + \frac{9x^2}{2} \right]_0^1$$

$$= 3 + 9 + \frac{9}{2} = \frac{6+18+9}{2} = \frac{33}{2}$$

$$\therefore \iiint_V (xy + yz + zx)\, dx\, dy\, dz = \frac{33}{2}$$

---

(32) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^x \, dx\, dy\, dz$.

Sol: given that $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^x \, dx\, dy\, dz$

$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} e^x \, dz \right] dy\, dx = \int_0^1 \int_0^{1-x} e^x (1-x-y) \cdot dy\, dx$$

$$= \int_0^1 \left[\int_0^{1-x} e^x \, dy - \int_0^{1-x} x e^x \, dy - \int_0^{1-x} y e^x \, dy \right] dx$$

$$= \int_0^1 \left[e^x (1-x) - x e^x (1-x) - e^x \cdot \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \left(e^x - xe^x - xe^x + x^2 e^x - \frac{e^x (1 + x^2 - 2x)}{2} \right) dx$$

$$= \frac{1}{2} \int_0^1 \left(2e^x - 4xe^x + 2x^2 e^x - e^x - x^2 e^x + 2x e^x \right) dx$$

$$= \frac{1}{2} \int_0^1 \left(e^x - 2xe^x + x^2 e^x \right) dx$$

$$= \frac{1}{2} \left[\int_0^1 e^x dx - 2\int_0^1 xe^x dx + \int_0^1 x^2 e^x dx \right]$$

$$\begin{array}{lllll} D = x^2 & 2x & 2 & 0 \\ I = e^x & e^x & e^x & e^x \end{array} \qquad \begin{array}{lll} D = x & 1 & 0 \\ I = e^x & e^x & e^x \end{array}$$

$$= \frac{1}{2} \Big[(e^1 - e^0) - 2\left[e^1 (1-1) - e^0 (0-1) \right] + \left((1)^2 e^1 - 2(1) e^1 + 2e^1 - 0 - 0 - 2e^0 \right) \Big]$$

$$= \frac{1}{2} \left[e - 1 - 2(-(-1)) + e - \cancel{2e} + \cancel{2e} - 2 \right]$$

$$= \frac{1}{2} \left[e - 1 - 2 + e - 2 \right] = \frac{1}{2} (2e - 5)$$

$$= \frac{2e-5}{2}$$

$$\therefore \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^x \, dx\,dy\,dz = \frac{2e-5}{2} \; //$$

$$\int e^x x^n dx = e^x \left[x^n - n x^{n-1} + n(n-1) x^{n-2} + \dots \right].$$

(33) - Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz$.

Sol: given $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,.dx\, dy\, dz$,

$$= \int_0^{\log 2}\int_0^{x}\left[\int_0^{x+\log y} e^{z}\, dz\right] e^{x} e^{y}\, dy\, dx.$$

$$= \int_0^{\log 2}\int_0^{x} [e^{z}]_0^{x+\log y}\, .\, e^{x}.e^{y}\, dy\, dx$$

$$= \int_0^{\log 2}\int_0^{x} (e^{x+\log y} - e^{0})\, e^{x} e^{y}\, dx\, dy$$

$$= \int_0^{\log 2}\int_0^{x} (e^{x}.y - 1)\, e^{x} e^{y}\,.dy.dx \quad \Big\downarrow \; e^{\log_e y} = y \log_e e$$

$$= \int_0^{\log 2}\int_0^{x} (e^{x}.y.e^{y} - e^{y})\, dy\,.\, e^{x} dx.$$

$$= \int_0^{\log 2}\left[e^{x} (e^{y}(y-1)]_0^{x} - [e^{y}]_0^{x}\right] e^{x}\, dx$$

$$= \int_0^{\log 2}\left[e^{x}\left[e^{x}(x-1) - e^{0}(0-1)\right] - (e^{x} - e^{0})\right] e^{x} dx$$

$$= \int_0^{\log 2}\left[e^{x}(xe^{x} - e^{x} + 1) - e^{x} + 1\right].\, e^{x}\, dy$$

$$= \int_0^{\log 2} (e^{x}(xe^{x} - e^{x} + 1) - e^{x} + 1)\, e^{x} dy.$$

$$= \int_0^{\log 2} (xe^{2x} - e^{2x} + \cancel{e^x} - \cancel{e^x} + 1)\, e^x \, dx$$

$$= \int_0^{\log 2} (xe^{3x} - e^{3x} + e^x)\, dx$$

$$\begin{array}{llll} D = x & 1 & 0 \\ I = e^{3x} & \frac{e^{3x}}{3} & \frac{e^{3x}}{9} \end{array} = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$$

$$= \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^x \right]_0^{\log 2}$$

$$= \frac{\log 2 \cdot e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{\log 2}$$

$$- \left[0 - \frac{e^{3(0)}}{9} - \frac{e^{3(0)}}{3} + e^0 \right]$$

$$= \frac{\log 2}{3} \cdot e^{\log 8} - \frac{e^{\log 8}}{9} - \frac{e^{\log 8}}{3} + e^{\log 2} + \frac{e^0}{9} + \frac{e^0}{3} - e^0$$

$$= \frac{8\log 2}{3} - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \log 2 \cdot \frac{8}{3} - \frac{7}{9} - \frac{7}{3} + 1$$

$$= \frac{24\log 2 - 7 - 21 + 9}{9} = \frac{8}{3}\log 2 - \frac{19}{9}$$

$$\therefore \int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz = \frac{8}{3}\log 2 - \frac{19}{9}$$

(34). Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\,dy\,dz$

Sol:

$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right)+c$

put $p = \sqrt{1-x^2-y^2}$

$p^2 = 1-x^2-y^2$.

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\int_0^p \frac{1}{\sqrt{p^2-z^2}}\, dz\right] dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{p}\right)\right]_0^p dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0\right) dx\,dy$$

$$= \frac{\pi}{2}\int_0^1 \left[\int_0^{\sqrt{1-x^2}} dy\right] dx = \frac{\pi}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$$

$$= \frac{\pi}{2}\left[0 + \frac{1}{2}\sin^{-1}(1) - 0 - \frac{1}{2}\sin^{-1}(0)\right]$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{\pi^2}{8}$$