Reccurance Relations & Generating Functions

An important use of an reccurance relation is the analysis of the complexity of algorithms. A reccurence relation that defines a sequence entry can be directly converted to an algorithm to compute the sequence.

Generating functions are important tools in discrete mathematics and their use is by to solve linear reccurence relation. The function can be used to solve many type of counting problems.

Def^n of reccurence relation:-

A reccurence relation is an equation that reccursively defined as sequence based and a rule that gives the next term in the sequence as a function of the previous terms when 1 or more initial terms are given.

(or)

A reccurence relation for the sequence a_n is an equation that relates a_n in terms of 1 or more of the previous terms of the sequence $a_0, a_1, a_2, \ldots, a_{n-1}$ for all integers $n \geq n_0$.

The specification of the values of a_n is called the initial conditions of reccurence relation.

Ex:- The sequence F_n is defined by using the reccurence relation

$F_n = F_{n-1} + F_{n-2}$ with initial conditions

$F_0 = 0$ and $F_1 = 1$.

In the same way we can find the next succeeding terms in the sequence such as

$$F_2 = F_{2-1} + F_{2-2} = F_1 + F_0 = 1+0 = 1$$

$$F_3 = F_{3-1} + F_{3-2} = F_2 + F_1 = 1+1 = 2$$

$$F_4 = F_3 + F_2 = 2+1 = 3$$

$$F_5 = F_4 + F_3 = 3+2 = 5$$

$\vdots$

$$F_n = F_{n-1} + F_{n-2} \text{ where, } F_0 = 0,\ F_1 = 1$$

We obtain the sequence of numbers by using the above reccurence relation $0, 1, 2, 3, 5, \ldots$

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① Find the sequence generated by the reccurence relations.

(i) $T_n = 2 \times T_{n-1}$ with $T_1 = 4$

Sol:- given $T_1 = 4$,

$$T_n = 2 \cdot T_{n-1} \longrightarrow ①$$

put $n=2$ in eq ① $\Rightarrow T_2 = 2\, T_{2-1} = 2 \times T_1 = 2 \times 4 = 8$

Ily $n=3$, $T_3 = 2T_2 = 2\times 8 = 16$

$n=4$, $T_4 = 2T_3 = 2\times 16 = 32$

⋮

The generated sequence is $4, 8, 16, 32, 64, \ldots$

$2^2, 2^3, 2^4, 2^5, 2^6, \ldots$

(or) 2^n where $n = 2, 3, \ldots$

(or) $n \geq 2$

(ii) . $T_n = 3T_{n-1} - 4$ with $T_1 = 3$.

Sol:- given $T_n = 3\cdot T_{n-1} - 4$

with $T_1 = 3$.

put $n=2$, $T_2 = 3T_1 - 4$

$T_2 = 3(3) - 4 = 9 - 4 = 5$

$n=3$, $T_3 = 3T_2 - 4 = 3\times 5 - 4 = 15 - 4 = 11$

$n=4$, $T_4 = 3T_3 - 4 = 3\times 11 - 4 = 33 - 4 = 29$

$n=5$, $T_5 = 3T_4 - 4 = 3\times 29 - 4 = 87 - 4 = 83$.

∴ The reccurence sequence generated by the reccurence relation is $3, 5, 11, 29, 83, \ldots$

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(2) . Find the reccurence relation for the following sequences.

(i) $2, 6, 18, 54, 162, \ldots$

Sol:- given $2, 6, 18, 54, 162, \ldots$

$T_1 = 2$,

$T_2 = 6 = 3T_1 = 3\times 2$

$T_3 = 18 = 3T_2 = 3\times 6$

⋮

$T_n = 3T_{n-1}$ ∴ $\boxed{T_n = 3T_{n-1} \text{ with } T_1 = 2}$

(ii) 20, 17, 14, 11, 8 ....

Sol: given, 20, 17, 14, 11, 8, ...

$T_1 = 20$, $T_2 = 20 - 3$

$T_2 = T_1 - 3$

$T_3 = T_2 - 3$

$T_4 = T_3 - 3$

⋮

$$\boxed{T_n = T_{n-1} - 3 \quad \text{with} \quad T_1 = 20}$$

(iii) 1, 3, 6, 10, 15, 21, ...

Sol: given, $T_1 = 1$

$T_2 = 3 = 1 + 2 \Rightarrow T_1 + 2$

$T_3 = 6 = 3 + 3 \Rightarrow T_2 + 3$

$T_4 = T_3 + 4$

$T_5 = T_4 + 5$

⋮

$$\boxed{T_n = T_{n-1} + n \quad \text{with} \quad T_1 = 1}$$

First order linear (or) Homogeneous Reccurrence Relation:

The linear reccurrence relation of first order with constant coefficients is $a_n = C\,a_{n-1} + f(n) \to (1)$ for $n \geq 1$. where 'c' is a constant and $f(n)$ is a given function.

The above function is called linear reccurrence relation of first order with constant coefficient.

In eq ①, if $f(x) = 0$ then the equation is called homogeneous recurrence relation, otherwise non-homogeneous recurrence relation.

Solution of first order linear homogeneous recurrence relation:-

An equation of the form of $a_n = c a_{n-1}$ for $n \geq 1$ is called first order homogeneous recurrence relation. The solution of first order homogeneous equation which is in the form of $a_n = c^n a_0$.

① Solve the recurrence relation $a_{n+1} = 4a_n$ for $n \geq 0$ given $a_0 = 3$.

Sol:- given recurrence relation is

$$a_{n+1} = 4a_n \rightarrow ① \text{ for } n \geq 0 \text{ and } a_0 = 3.$$

Eq ① is first order homogeneous recurrence relation.

The required solution is the form of $a_n = c^n a_0 \rightarrow ②$

Let $a_n = c^n \rightarrow ③$

If $n = n+1$ then

$$a_{n+1} = c^{n+1}$$

$$a_{n+1} = c^n \cdot c^1$$

from eq ①, $4a_n = c^n c^1$

from eq ③, $\cancel{4a_n}\ 4 \cancel{c^n} = \cancel{c^n} c^1$

$$\boxed{c = 4}$$

$\therefore$ from eq ②, $\boxed{a_n = 4^n \cdot (3)}$

(2) $a_{n+1} - 4a_n = 0$, $n \geq 0$, $a_1 = 5$.

Sol: given $a_{n+1} - 4a_n = 0$ for $n \geq 0$ and $a_1 = 5$

$$a_{n+1} = 4a_n \rightarrow (1)$$

Eq (1) is first order HRR (Homogeneous Recurrence Relation).

The required solution is in the form of

$$a_n = c^n a_0 \rightarrow (2).$$

Let $a_n = c^n \rightarrow (3)$

let $n = n+1$

$$a_{n+1} = c^{n+1}$$

from eq (1) $\Rightarrow$ $4a_n = c^n \cdot c^1$

from eq (3) $\Rightarrow$ $4\cancel{c^n} = \cancel{c^n} \cdot c^1$

$$\boxed{c = 4}$$

from eq (2) $\Rightarrow$ $a_n = 4^n \cdot a_0 \rightarrow (4)$

but given $a_1 = 5$.

put $n = 1$ in eq (4).

$a_1 = 4^1 \cdot a_0 \Rightarrow 5 = 4 \cdot a_0$

$$a_0 = \frac{5}{4}.$$

Now, from eq (4), $a_n = 4^n \cdot \frac{5}{4}$

$$\boxed{a_n = 5 \cdot 4^{n-1}}$$

(3). Solve the RR $a_n = 7a_{n-1}$ for $n \geq 1$ and given that $a_2 = 98$.

Sol:- given RR is $a_n = 7a_{n-1}$ for $n \geq 1$. and $a_2 = 98$.

$$a_n = 7a_{n-1} \rightarrow (1)$$

eq (1) is first order HRR.

put $n = n+1$ in eq (1)

$$a_{n+1} = 7a_{n+\cancel{1}-\cancel{1}}$$

$$a_{n+1} = 7a_n \rightarrow (2)$$

The required solution is in the form of

$$a_n = c^n a_0 \rightarrow (3)$$

Let assume $a_n = c^n \rightarrow (4)$

let $n = n+1$

$$a_{n+1} = c^{n+1}$$

from eq (2) $\Rightarrow 7a_n = c^n \cdot c$

from eq (4) $\Rightarrow 7\cancel{c^n} = \cancel{c^n} c$

$$\boxed{c = 7}$$

from eq (3), $a_n = 7^n a_0 \rightarrow (5)$

put $n = 2$

$$a_2 = 7^2 a_0 \Rightarrow$$

given $a_2 = 98 \Rightarrow \cancel{98}^{2} = \cancel{49} a_0$

$$\boxed{a_0 = 2}$$

Now from (5),

$$a_n = 7^n \cdot 2$$

Second order linear Homogeneous RR:

A relation which is in the form of

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0 \text{ for } n \geq 2 .$$

where, c_n, c_{n-1}, c_{n-2} are called real constants is called second order linear homogeneous RR.

Solution of second order linear HRR:-

Let us consider second order HRR $\rightarrow$ ①

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0 \text{ for } n \geq 2$$

~~Multiply~~ Let sub $a_n = r^n$ in eq ①

$$c_n r^n + c_{n-1} (r)^{n-1} + c_{n-2} (r)^{n-2} = 0$$

$$r^{n-2} \left[c_n \cdot r^2 + c_{n-1} \cdot r + c_{n-2} \right] = 0$$

$$c_n r^2 + c_{n-1} r + c_{n-2} = 0$$

Here, 3 cases arise.

Case-i:- If 2 roots are real and distinct then the general solution is

$$a_n = b_1 r_1^n + b_2 r_2^n$$

where, b_1, b_2 are constants and

r_1, r_2 are the roots of the equation.

Case-ii:- If 2 roots are same and real then the general solution is.

$$a_n = (b_1 + b_2 n) r_1^n$$

where, b_1, b_2 are constants and
r is the roots of the equation.

Case-iii:- If the 2 roots are complex, it means
$\alpha \pm ib$. where $r_1 = a+ib$, $r_2 = a-ib$
then the general solution is
$$a_n = r^n [b_1 \cos n\theta + b_2 \sin n\theta]$$
where, $r = \sqrt{a^2+b^2}$
$$\theta = \tan^{-1}\left(\frac{a}{b}\right)$$

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① Solve $a_n = a_{n-1} + 2a_{n-2}$, where $n \geq 2$ and
The given initial conditions $a_0 = 0$, $a_1 = 1$

Sol:- Given RR is $a_n = a_{n-1} + 2a_{n-2}$

$$a_n - a_{n-1} - 2a_{n-2} = 0 \rightarrow ①$$

Eq ① is second order linear HRR.

Let $a_n = r^n$

from ①, $r^n - (r)^{n-1} - 2(r)^{n-2} = 0$
$$(r)^{n-2}[r^2 - r - 2] = 0$$
$$r^2 - r - 2 = 0$$
$$\boxed{r = 2, -1}$$

The 2 roots are real and distinct.
Here, $r_1 = 2$, $r_2 = -1$
Then the general solution is
$$a_n = b_1 r_1^n + b_2 r_2^n$$

Now, $a_n = b_1 (2)^n + b_2 (-1)^n \rightarrow (2)$

put $n=0$, in eq (2)

$$a_0 = b_1 (2)^0 + b_2 (-1)^0$$

$$0 = b_1 + b_2 \Rightarrow b_1 + b_2 = 0 \rightarrow (3)$$

$$b_2 = -b_1$$

put $n=1$ in eq (2)

$$a_1 = b_1 (2)^1 + b_2 (-1)^1$$

$$2b_1 - b_2 = 1 \rightarrow (4)$$

Solving (3) & (4) we get

$$2b_1 - (-b_1) = 1$$

$$3b_1 = 1 \Rightarrow b_1 = \frac{1}{3}$$

$$b_2 = -\frac{1}{3} \quad \text{from (3)}$$

Sub, b_1, b_2 in eq (2),

$$a_n = \frac{1}{3}(2)^n + \left(-\frac{1}{3}\right)(-1)^n$$

$$a_n = \frac{2^n}{3} - \frac{(-1)^n}{3} = \frac{2^n - (-1)^n}{3}$$

$$\therefore \; a_n = \frac{2^n}{3} + \frac{(-1)^n}{3}$$

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(2) Solve $f_n = f_{n-1} + f_{n-2}$, where $n \geq 2$ and
HW the given initial conditions $f_0 = 0$, $f_1 = 1$

(3) Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$ and
HW given $a_0 = 5$, $a_1 = 12$.

Sol:- given $a_n - 6a_{n-1} + 9a_{n-2} = 0$

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \rightarrow (1)$$

Eq (1) is second order linear HRR

Let put $a_n = r^n$ in eq (1),

from (1), $r^n - 6(r)^{n-1} + 9(r)^{n-2} = 0$

$$r^{n-2}[r^2 - 6r + 9] = 0.$$

$$\therefore r = 3, 3$$

The roots are same and real then the general solution is

$$a_n = (b_1 + b_2 n) r^n.$$

$$a_n = (b_1 + b_2 n)(3)^n. \rightarrow (2).$$

put $n = 0$ in eq (2),

$$a_0 = (b_1 + b_2(0))(3)^0$$

$$(b_1 + 0)1 = 5$$

$$\boxed{b_1 = 5}$$

put $n = 1$, in eq (2),

$$a_1 = (b_1 + b_2(1))(3)^1$$

$$\cancel{3}(b_1 + b_2) = \cancel{12}\,4$$

$$b_1 + b_2 = 4$$

$$5 + b_2 = 4$$

$$b_2 = 4 - 5 \Rightarrow \boxed{b_2 = -1}$$

sub, b_1, b_2 in eq (2),

$$a_n = [5 + (-1)n](3)^n \Rightarrow a_n = (5 - n)3^n.$$

3/1/5 Sol :- given, $f_n = f_{n-1} + f_{n-2}$

$$f_n - f_{n-1} - f_{n-2} = 0 \rightarrow (1)$$

Eq (1) is second order HRR

Let put $f_n = r^n$ in eq (1),

from (1), $r^n - (r)^{n-1} - (r)^{n-2} = 0$

$$r^{n-2}[r^2 - r - 1] = 0$$

$$r^2 - r - 1 = 0$$

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

The roots are real and distinct then the general solution is

$$f_n = b_1 r_1^n + b_2 r_2^n$$

Sub r_1, r_2 in the above equation

$$f_n = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_2\left(\frac{1-\sqrt{5}}{2}\right)^n \rightarrow (2)$$

put $n=0$, in eq (2),

$$f_0 = b_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + b_2\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$b_1 + b_2 = 0 \rightarrow (3)$$

put $n=1$, in eq (2),

$$f_1 = b_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + b_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$b_1\left(\frac{1+\sqrt5}{2}\right)+b_2\left(\frac{1-\sqrt5}{2}\right)=1$$

$$b_1(1+\sqrt5)+b_2(1-\sqrt5)=2 \rightarrow (4)$$

$$\cancel{b_1+\sqrt5 b_1+b_2-\sqrt5 b_2=2}$$

$$\cancel{b_1+b_2+\sqrt5(b_1-b_2)=2}$$

solving eq (3) & (4) we get

$$b_1=0.4472135$$

$$b_2=-0.4472135$$

sub b_1 & b_2 in eq (2)

$$\therefore f_n=(0.4472135)\left(\frac{1+\sqrt5}{2}\right)^n+(-0.4472)\left(\frac{1-\sqrt5}{2}\right)^n$$

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(4) Solve the RR $a_n=10a_{n-1}+29a_{n-2}$ for $n\geq3$.
given $a_1=10$, $a_2=100$.

Sol: given RR is $a_n=10a_{n-1}+29a_{n-2}$

$$a_n-10a_{n-1}-29a_{n-2}=0 \rightarrow (1)$$

The given Eq (1) is second order HRR

Let put $a_n=r^n$ in eq (1),

from (1), $r^n-10(r)^{n-1}-29(r)^{n-2}=0$

$$r^{n-2}[r^2-10r-29]=0$$

$$r^2-10r-29=0.$$

$$r=5+3\sqrt6,\ 5-3\sqrt6$$

$$r_1=5+3\sqrt6,\ r_2=5-3\sqrt6$$

The roots are real and distinct then the general solution is

$$a_n = b_1 r_1^n + b_2 r_2^n$$

sub r_1, r_2 in the above equation:

$$a_n = b_1 (5+3\sqrt{6})^n + b_2 (5-3\sqrt{6})^n \rightarrow (2)$$

put $n=1$, in eq (2),

$$a_1 = b_1 (5+3\sqrt{6})^1 + b_2 (5-3\sqrt{6})^1$$

$\cancel{b_1 + b_2 = 10 \rightarrow (3)} \downarrow$ $a_1 = 10$

~~put~~ $\cancel{n=2, \text{ in eq } (2)}$

$\cancel{a_2 = b_1 (5+3\sqrt{6})^2}$

$$10 = 5b_1 + 3\sqrt{6}b_1 + 5b_2 - 3\sqrt{6}b_2$$

$$b_1(5+3\sqrt{6}) + b_2(5-3\sqrt{6}) = 10 \rightarrow (3).$$

put $n=2$ in eq (2),

$$a_2 = b_1 (5+3\sqrt{6})^2 + b_2 (5-3\sqrt{6})^2$$

$$b_1 (79+30\sqrt{6}) + b_2(79-30\sqrt{6}) = 100 \rightarrow (4) \quad a_2 = 100$$

Solving (3) & (4).

$$b_1 = 0.68041, \quad b_2 = -0.68041$$

sub b_1, b_2 values in eq (2),

$$a_n = (0.68041)(5+3\sqrt{6})^n + (-0.68041)(5-3\sqrt{6})^n$$

(5) Solve the RR $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$.

Sol:- given RR is

$$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0 \rightarrow (1)$$

Eq(1) is third order IIRR.

let put $a_n = r^n$ in eq(1)

from (1), $r^n - 8(r)^{n-1} + 21(r)^{n-2} - 18(r)^{n-3} = 0$

$$r^{n-3}[r^3 - 8r^2 + 21r - 18] = 0$$

$$r^3 - 8r^2 + 21r - 18 = 0$$

$$r = 2, 3, 3$$

2	1	-8	21	-18
	0	2	-12	18
3	1	-6	9	0
	0	3	-9	
	1	-3	0	
	0	3		
	1	0		

Synthetic Division

$$r_1 = 2,\ r_2 = 3,\ r_3 = 3.$$

The roots are real and distinct, then the general solution is

$$a_n = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n$$

$$a_n = b_1(2)^n + b_2(3)^n + b_3(3)^n$$

$$\Rightarrow a_n = b_1(2)^n + (b_2 + b_3)(3)^n.$$

$$\Rightarrow a_n = (b_1 + b_2 n) r^n + b_3 r_1^n$$

$$a_n = (b_1 + b_2 n)(3)^n + b_3(2)^n$$

Note:- If 3 roots are same then the general solution is

$$a_n = (b_1 + b_2 n + b_3 n^2) r^n$$

(6) Solve the RR $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$ for $n \geq 0$ and given that $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

Sol:- given RR is

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$$

$$a_n - 2a_{n+1} - a_{n+2} + 2a_{n+3} = 0 \rightarrow (1)$$

eq (1) is third order HRR.

Let put $a_n = r^n$ in eq (1),

from (1), $r^n - 2(r)^{n+1} - a(r)^{n+2} + 2(r)^{n+3} = 0$

The given RR written as.

$$2a_{n+3} - a_{n+2} - 2a_{n+1} + a_n = 0$$

put $n = n-3$

$$2a_n - a_{n-1} - 2a_{n-2} + a_{n-3} = 0. \rightarrow (1)$$

eq (1) is third order HRR

put $a_n = r^n$ in eq (1),

from (1), $2r^n - (r)^{n-1} - 2(r)^{n-2} + (r)^{n-3} = 0$

$$r^{n-3}[2r^3 - r^2 - 2r + 1] = 0$$

$$2r^3 - r^2 - 2r + 1 = 0$$

$$r_1 = -1,\ r_2 = 1,\ r_3 = \frac{1}{2}$$

Here the all roots are real and distinct

Then the general solution is

$a_n = b_1 a_1^n + b_2 a_2^n + b_3 a_3^n$

$a_n = b_1(-1)^n + b_2(1)^n + b_3\left(\frac{1}{2}\right)^n \rightarrow$ (2)

Now sub $n=0$ in eq (2)

$a_0 = b_1(-1)^0 + b_2(1)^0 + b_3\left(\frac{1}{2}\right)^0$

$b_1 + b_2 + b_3 = 0 \rightarrow$ (3)

put $n=1$ in eq (2)

$a_1 = b_1(-1)^1 + b_2(1)^1 + b_3\left(\frac{1}{2}\right)^1$

$-b_1 + b_2 + b_3(0.5) = 1 \rightarrow$ (4)

put $n=2$ in eq (3).

$a_2 = b_1(-1)^2 + b_2(1)^2 + b_3\left(\frac{1}{2}\right)^2$

$b_1 + b_2 + (0.25)b_3 = 2 \rightarrow$ (5)

Solving (3), (4) and (5), we get.

$b_1 = \frac{1}{6}, \quad b_2 = \frac{5}{2}, \quad b_3 = \frac{-8}{3}$

Sub b_1, b_2 and b_3 in eq (2),

$\therefore a_n = \left(\frac{1}{6}\right)(-1)^n + \left(\frac{5}{2}\right)(1)^n + \left(\frac{-8}{3}\right)\left(\frac{1}{2}\right)^n$ //

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(7) Solve the RR (i) $a_n + 7a_{n-1} + 8a_{n-2} = 0$
where $a_0 = 1, a_1 = -2$.

(ii) $a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0, a_0 = 1, a_1 = 5, a_2 = 1$

(iii) $s_n = 10 s_{n-1} - 25 s_{n-2}$ where $s_0 = 3, s_1 = 17$.

(7) Sol :-

(i) $a_n + 7a_{n-1} + 8a_{n-2} = 0$, where, $a_0 = 1, a_1 = 2$

$a_n + 7a_{n-1} + 8a_{n-2} = 0 \rightarrow (1)$

eq (1) is second order HRR.

put $a_n = r^n$ in eq (1).

from (1), $r^n + 7(r)^{n-1} + 8(r)^{n-2} = 0$

$r^{n-2}[r^2 + 7r + 8] = 0$

$r^2 + 7r + 8 = 0$

$r_1 = \frac{-7+\sqrt{17}}{2}, \quad r_2 = \frac{-7-\sqrt{17}}{2}$

Here roots are real and distinct.

~~sub r_1, r_2 in eq~~ Then the general solution is

$a_n = b_1(r_1)^n + b_2 r_2^n \rightarrow (2)$

sub r_1, r_2 in eq (2),

$a_n = b_1\left(\frac{-7+\sqrt{17}}{2}\right)^n + b_2\left(\frac{-7-\sqrt{17}}{2}\right)^n \rightarrow (3)$

sub $n=0$ in eq (3)

$a_0 = b_1\left(\frac{-7+\sqrt{17}}{2}\right)^0 + b_2\left(\frac{-7-\sqrt{17}}{2}\right)^0$

$a_0 = b_1 + b_2$

$b_1 + b_2 = 1 \rightarrow (4)$

sub $n=1$ in eq (3),

$$a_1 = b_1\left(\frac{-7+\sqrt{17}}{2}\right)^1 + b_2\left(\frac{-7-\sqrt{17}}{2}\right)^1$$

$$b_1\left(\frac{-7+\sqrt{17}}{2}\right) + b_2\left(\frac{-7-\sqrt{17}}{2}\right) = -2 \rightarrow (4)$$

solving (4) & (5) we get $\quad -1.4384(b_1) - (5.5615)b_2 = -2 \rightarrow (5)$

$\cancel{b_1 = 1.7126}$, $\cancel{b_2 = -0.7126}$

$$b_1 = 0.86379, \; b_2 = 0.136208$$

sub b_1, b_2 in eq (3),

$$a_n = (0.86379)\left(\frac{-7+\sqrt{17}}{2}\right)^n + (0.136208)\left(\frac{-7-\sqrt{17}}{2}\right)^n$$ //.

(ii) $a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0$,

where, $a_0 = 1, a_1 = 5, a_2 = 1$.

Sol: $a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0 \rightarrow (1)$

Eq (1) is third order HRR.

put $a_n = r^n$ in eq (1)

from (1), $r^n + (r)^{n-1} - 8(r)^{n-2} - 12(r)^{n-3} = 0$

$$r^{n-3}\left[r^3 + r^2 - 8r - 12\right] = 0$$

$$r^3 + r^2 - 8r - 12 = 0$$

$$r_1 = 3, \; r_2 = -2, \; r_3 = -2$$

3	1	1	-8	-12
	0	3	12	12
-2	1	4	4	0
	0	-2	-4	
-2	1	2	0	
	0	-2		
	1	0		

Here all roots are real and distinct, then the general solution is $\cancel{a_n = (b_1 + b_2 n)3^n + b_3(2)^n}$

$$a_n = b_1 r_1^n + (b_2 + b_3 n)\, r_2^n$$

$$a_n = b_1 (3)^n + (b_2 + b_3 n)(-2)^n \;// \longrightarrow (2)$$

put $n=0$, in eq$_n$ (2)

$$a_0 = b_1 (3)^0 + (b_2 + b_3 (0))(-2)^0$$

$$a_0 = b_1 + (b_2 + 0) - 2$$

$$a_0 = b_1 + (b_2) - 2$$

$$b_1 - 2b_2 = 1 \longrightarrow (3)$$

put $n=1$ in eq$_n$ (2),

$$a_1 = b_1 (3)^1 + (b_2 + b_3 (1))(-2)^1$$

$$3b_1 + (b_2 + b_3)(-2) = 5$$

$$3b_1 - 2b_2 - 2b_3 = 5 \rightarrow (4)$$

put $n=2$ in eq$_n$ (2),

$$a_2 = b_1 (3)^2 + (b_2 + b_3 (2))(-2)^2$$

$$9b_1 + (b_2 + 2b_3)\,4 = 1$$

$$9b_1 + 4b_2 + 8b_3 = 1 \rightarrow (5)$$

Solving (3), (4), & (5)

$$b_1 = 1,\; b_2 = 0,\; b_3 = -1$$

sub b_1, b_2, b_3 in eq$_n$ (2),

$$a_n = (1)(3)^n + (0 + (-1)n)(-2)^n$$

$$\therefore a_n = (3)^n - n(-2)^n \;//$$

(ii) $a_n = 10 a_{n-1} - 25 a_{n-2}$ where, $a_0 = 3$, $a_1 = 17$.

Sol: $a_n - 10 a_{n-1} + 25 a_{n-2} = 0 \rightarrow ①$

eq ① is second order HRR

put $a_n = r^n$ in eq ①

$r^n - 10 (r)^{n-1} + 25 (r)^{n-2} = 0$

$r^{n-2} [r^2 - 10r + 25] = 0$

$r^2 - 10r + 25 = 0$

$$\begin{array}{c|ccc} 5 & 1 & -10 & 25 \\ & 0 & 5 & -25 \\ \hline 5 & 1 & -5 & (0) \\ & 0 & 5 & \\ \hline & 1 & (0) & \end{array}$$

$\therefore r = 5, 5$

$r_1 = 5$, $r_2 = 5$

$\therefore$ Hence, roots are real and same.

Then the general solution is

$a_n = (b_1 + b_2 n) r^n \rightarrow ②$

sub 'r' in eq ②

$a_n = (b_1 + b_2 n)(5)^n \rightarrow ③$

put $n = 0$, in eq ③,

$a_0 = (b_1 + b_2(0))(5)^0$

$3 = (b_1 + 0) 1 \Rightarrow b_1 = 3$

put $n = 1$ in eq ③,

$a_1 = (b_1 + b_2(1))(5)^1$

$17 = (b_1 + b_2) 5$

$5b_1 + 5b_2 = 17 \rightarrow ④$

sub $b_1 = 3$ in ④ $\Rightarrow$ $15 + 5b_2 = 17 \Rightarrow 5b_2 = 2$

$\boxed{b_2 = 2/5}$

Here, $b_1 = 3$, $b_2 = 2/5$

sub b_1, b_2 in eq_n ⑤,

$\therefore f_n = \left(3 + \frac{2}{5} n\right) (5)^n$. //

---

⑧ Solve the reccurrence relation

$$a_{n+3} = 3a_{n+2} + 4a_{n+1} - 12a_n \text{ for } n \geq 0$$

Given that $a_0 = 0$, $a_1 = -11$ and $a_2 = -15$.

Sol:- given RR is

$$a_{n+3} = 3a_{n+2} + 4a_{n+1} - 12a_n$$

$$a_{n+3} - 3a_{n+2} - 4a_{n+1} + 12a_n = 0$$

put $n = n-3$

$$a_n - 3a_{n-1} - 4a_{n-2} + 12a_{n-3} = 0 \rightarrow ①$$

Eqn ① is third order HRR.

put $a_n = r^n$ in eqn ①,

$$(r)^n - 3(r)^{n-1} - 4(r)^{n-2} + 12(r)^{n-3} = 0$$

$$r^{n-3}\left[r^3 - 3r^2 - 4r + 12\right] = 0$$

$$r^3 - 3r^2 - 4r + 12 = 0$$

$$r = -2, 3, 2$$

$$r_1 = -2, \quad r_2 = 3, \quad r_3 = 2$$

Hence the all roots are real and distinct.

Then the general solution is

$$a_n = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n$$

$$a_n = b_1(-2)^n + b_2(3)^n + b_3(2)^n \rightarrow (2)$$

Now sub $n=0$ in eq (2) $\downarrow a_0 = 0$

$$a_0 = b_1(-2)^0 + b_2(3)^0 + b_3(2)^0$$

$$b_1 + b_2 + b_3 = 0 \rightarrow (3)$$

Now, sub ~~a~~ $n=1$ in eq (2) $\downarrow a_1 = -11$

$$a_1 = b_1(-2)^1 + b_2(3)^1 + b_3(2)^1$$

$$-2b_1 + 3b_2 + 2b_3 = -11$$

$$2b_1 - 3b_2 - 2b_3 = 11 \rightarrow (4)$$

sub $n=2$ in eq (2), $\downarrow a_2 = -15$

$$a_2 = b_1(-2)^2 + b_2(3)^2 + b_3(2)^2$$

$$4b_1 + 9b_2 + 4b_3 = -15 \rightarrow (5)$$

solving (3), (4), and (5) we get

$$b_1 = 2,\ b_2 = -3,\ b_3 = 1$$

sub, b_1, b_2 and b_3 in eq (2)

from (2),

$$a_n = (2)(-2)^n + (-3)(3)^n + (1)(2)^n$$

$$\therefore a_n = (2)(-2)^n - 3(3)^n + (2)^n$$

Non Homogeneous second order linear RR:-

A relation which is in the form of

$a_n + a_{n-1} + a_{n-2} = f(n)$ then is called general form of second order non-HRR.. [when $f(n) \neq 0$].

General Solution of second order non-HRR:-

$$a_n = (a_n)^c + (a_n)^p \longrightarrow ①$$

where $(a_n)^c$ is called solution of RR by keeping $f(n) = 0$.

$(a_n)^p$ is a solution of RR for some special cases when $f(n) =$ Constant.

$f(n) =$ Polynomial

$f(n) = \alpha^n$, where 'α' is a constant

Case-i:- If $f(n)$ is a constant.

Let consider second order non-HRR.

$$a_n + a_{n-1} + a_{n-2} = f(n)$$

If $f(n)$ is a constant. That is $f(n) = k$

The particular solution is substitute $a_n = a_{n-1} = a_{n-2} = A_0$

in the given RR.

Simplify above relation to get A_0.

where A_0 is called particular solution.

(1) Solve $a_{n+2} - 5a_{n+1} + 6a_n = 2$ in the initial conditions $a_0 = 1$, $a_1 = -1$.

Sol:- given non-HRR is

$$a_{n+2} - 5a_{n+1} + 6a_n = 2 \rightarrow (1)$$

The given RR is converting into general form.

Substituting $n = n-2$

$$a_n - 5a_{n-1} + 6a_{n-2} = 2 \rightarrow (2)$$

It is in the form of $a_n + a_{n-1} + a_{n-2} = f(n)$

where $f(n) = 2 =$ constant.

The general solution is $a_n = (a_n)^c + (a_n)^p$

complimentary, particular

To find $(a_n)^c$:-

Let us consider $f(n) = 0$

then $a_n - 5a_{n-1} + 6a_{n-2} = 0$

put $a_n = r^n$

$$(r)^n - 5(r)^{n-1} + 6(r)^{n-2} = 0$$

$$(r)^{n-2}\left[r^2 - 5r + 6\right] = 0$$

$$r^2 - 5r + 6 = 0$$

$$\therefore r_1 = 3, \ r_2 = 2$$

Here all the roots are real and distinct

Then the general solution is

$$(a_n)^c = b_1 r_1^n + b_2 r_2^n$$

$$(a_n)^c = b_1 (3)^n + b_2 (2)^n \rightarrow (3)$$

To find $(a_n)^p$:

Substitute $a_n = a_{n-1} = a_{n-2} = A_0$ in eq ②

$$a_n - 5a_{n-1} + 6a_{n-2} = 2$$

$$A_0 - 5A_0 + 6A_0 = 2$$

$$2A_0 = 2$$

$$\boxed{A_0 = 1} = (a_n)^p$$

Then the general solution is

$$a_n = (a_n)^c + (a_n)^p$$

$$a_n = b_1 (3)^n + b_2 (2)^n + 1 \rightarrow ④$$

Now, sub $n = 0$ in eq ④, $a_0 = 1$

$$a_0 = b_1 (3)^0 + b_2 (2)^0 + 1$$

$$b_1 + b_2 + 1 = 1$$

$$b_1 + b_2 = 0 \rightarrow ⑤$$

sub $n = 1$ in eq ④ & $a_1 = -1$

$$a_1 = b_1 (3)^1 + b_2 (2)^1 + 1$$

$$3b_1 + 2b_2 + 1 = -1$$

$$3b_1 + 2b_2 = -2 \rightarrow ⑥$$

Solving ⑤ and ⑥

$b_1 = -2$, $b_2 = 2$

sub b_1, b_2 in eq^n (4) we get

$$\therefore a_n = (-2)(3)^n + (2)(2)^n + 1$$

---

(2) $u_n + 4u_{n-1} + 4u_{n-2} = 1$ for $n \geq 2$.

Sol: given $u_n + 4u_{n-1} + 4u_{n-2} = 1 \rightarrow$ (1)

Eq^n(1) is second order non HRR.

Then the general solution is $a_n = (a_n)^c + (a_n)^p$

$u_n = (u_n)^c + (u_n)^p$

To find $(u_n)^c$:

Let us consider $f(n) = 0$

then $u_n + 4u_{n-1} + 4u_{n-2} = 0$

put $u_n = r^n$

$$(r)^n + 4(r)^{n-1} + 4(r)^{n-2} = 0$$

$$(r)^{n-2}\left[r^2 + 4r + 4\right] = 0$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = -2, -2$$

The two roots are real and same Then

$$(a_n)^c = (b_1 + b_2 n) r^n$$

$$(u_n)^c = (b_1 + b_2 n)(-2)^n \rightarrow (2)$$

To find $(u_n)^p$:

substitute $u_n = u_{n-1} = u_{n-2} = A_0$ in eq^n (1)

then $A_0 + 4A_0 + 4A_0 = 1 \Rightarrow 9A_0 = 1$

$$\therefore A_0 = 1/9$$

$\therefore A_0 = \frac{1}{9} = (u_n)^p$

The general solution is

$$u_n = (u_n)^c + (u_n)^p$$

$$\therefore u_n = (b_1 + b_2 n)(-2)^n + \frac{1}{9}$$

Case-ii:- If $f(n) = b^n$

where, b is a constant and b is not a root of RR.

Consider a RR

$$a_n + a_{n-1} + a_{n-2} = b^n \longrightarrow ①$$

In this case substitute $a_n = A_0 b^n$

$a_{n-1} = A_0 b^{n-1}$

$a_{n-2} = A_0 b^{n-2}$

The particular solution is in the form of $(a_n)^p = A_0 b^n$.

In this consider sub a_n, a_{n-1}, a_{n-2}

and solve the above equation to find A_0 value.

i.e. the value of A_0 is called particular solution

---

① Solve $a_n - 2a_{n-1} - 3a_{n-2} = 5^n$, when $n \geq 2$ and given $a_0 = -2$, $a_1 = 1$.

Sol:- given $a_n - 2a_{n-1} - 3a_{n-2} = 5^n \longrightarrow ①$

Eq ① is second order non HRR.

Then the general solution is $a_n = (a_n)^c + (a_n)^p \rightarrow ②$

To find $(a_n)^c$:-

Let us consider $f(n)=0$

$$a_n - 2a_{n-1} - 3a_{n-2} = 0$$

substitute $a_n = r^n$

$$(r)^n - 2(r)^{n-1} - 3(r)^{n-2} = 0$$

$$(r)^{n-2}[r^2 - 2r - 3] = 0$$

$$r^2 - 2r - 3 = 0$$

$$r = 3, -1.$$

$$\therefore r_1 = 3, r_2 = -1$$

The roots are real and distinct

Then the general solution is

$$(a_n)^c = b_1 r_1^n + b_2 r_2^n$$

$$(a_n)^c = b_1(3)^n + b_2(-1)^n \rightarrow ③$$

To find $(a_n)^P$:- Here, $b=5$ is not a root of RR

substitute $a_n = A_0 b^n$ | $a_{n-2} = A_0 b^{n-2}$

$a_{n-1} = A_0 b^{n-1}$

in eq^n ① we get

$$a_n - 2a_{n-1} - 3a_{n-2} = 5^n$$

$$\cancel{A_0 b^n - 2A_0 b^{n-1} - 3A_0 b^{n-2} = 5^n}$$

$$\cancel{A_0 b^{n-2}}[b^2 - 2b - 3] = 5^n$$

$$A_0 5^n - 2A_0 5^{n-1} - 3A_0 5^{n-2} = 5^n$$

$$5^{n-2}\left[A_0 5^2 - 2A_0 \cdot 5 - 3A_0\right] = 5^n$$

$$\cancel{5^{n-2}}\left[25A_0 - 10A_0 - 3A_0\right] = \cancel{5^{n-2}} \cdot 5^2$$

$$\left[25A_0 - 13A_0\right] = 25$$

$$12A_0 = 25$$

$$\therefore A_0 = 25/12$$

The particular solution is

$\Rightarrow (a_n)^P = 5^n A_0$

$\Rightarrow (a_n)^P = \frac{25}{12} \cdot 5^n$

Now, from eqn ② & ③,

$$a_n = b_1(3)^n + b_2(-1)^n + \frac{25}{12}5^n \rightarrow ④$$

Substitute, $n=0$ in eqn ④ & $a_0 = -2$

$$a_0 = b_1(3)^0 + b_2(-1)^0 + \frac{25}{12}(5)^0$$

$$b_1 + b_2 + \frac{25}{12} = -2$$

$$b_1 + b_2 = -\frac{49}{12} \rightarrow ⑤$$

Hly, $n=1$ in eqn ④ & $a_1 = 1$

$$a_1 = b_1(3)^1 + b_2(-1)^1 + \frac{25}{12}(5)^1$$

$$3b_1 - b_2 + \frac{125}{12} = 1$$

$$3b_1 - b_2 = \frac{-113}{12} \rightarrow ⑥$$

solving ⑤ & ⑥ we get

$$b_1 = \frac{-27}{8}, \quad b_2 = \frac{-17}{24}$$

sub, b_1, b_2 in eqn ④,

$$\therefore a_n = \left(\frac{-27}{8}\right)(3)^n + \left(\frac{-17}{24}\right)(-1)^n + \frac{25}{12}(5)^n$$

$$a_n - 4a_{n-1} + 4a_{n-2} = 2^{n-2}$$

$$A_0 b^n - 4A_0(b)^{n-1} + 4A_0(b)^{n-2} = 2^{n-2}$$

$$A_0(2^{n-2}) - 4A_0(2)^{n-2-1} + 4A_0(2)^{n-2-2} = 2^{n-2}$$

$$A_0 2^{n-2} - 4A_0(2)^{n-3} + 4A_0(2)^{n-4} = 2^{n-2}$$

$$2^{n-4}\left[A_0(2)^2 - 4(A_0)2 + 4A_0\right] = 2^{n-2}$$

$$\frac{2^n}{2^4}\left[4A_0 - 8A_0 + 4A\right] = 2^n$$

Case-ii:

If $f(n) = b^n$ and b is not a root of RR. then $(a_n)^p = A_0 b^n$

If $f(n) = b^n$ and b is a root of RR then $(a_n)^p = A_0 b^n n^2$.

---

① Solve $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$ for $n \geq 2$.

Sol: given $a_n - 7a_{n-1} + 10a_{n-2} = 4^n \rightarrow ①$

It is in the form of

$$a_n + a_{n-1} + a_{n-2} = f(n)$$

where, $f(n) = 4^n$

It is called second order non-HRR.

The general solution is $a_n = (a_n)^c + (a_n)^p \rightarrow ②$

To find $(a_n)^c$:

Let us consider $f(n)=0$

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

sub. $a_n = r^n$

$$(r)^n - 7(r)^{n-1} + 10(r)^{n-2} = 0$$

$$r^{n-2}[r^2 - 7r + 10] = 0$$

$$r^2 - 7r + 10 = 0$$

$$r = 5, 2$$

The roots are real and distinct.

Then the solution is $(a_n)^c = b_1(r_1)^n + b_2(r_2)^n$

$$(a_n)^c = b_1(5)^n + b_2(2)^n \rightarrow (3)$$

To find $(a_n)^p$:

given, $f(n) = 4^n = b^n$

$$\boxed{b=4}$$

$b=4$ is not a root of RR

Then $(a_n)^p = A_0 b^n = A_0 4^n$

$$a_{n+2} - 4a_{n+1} + 4a_n = 2^n$$

sub $a_n = A_0 4^n . n^2$

$$A_0(2)^{n+2}(n+2)^2 - 4A_0(2)^{n+1}(n+1)^2 + 4A_0(2)^n(n)^2 = 2^n$$

$$2^n[A_0 2^2(n^2+4n+4) - 4A_0 2(n^2+2n+1) + 4A_0 n^2] = 2^n$$

~~Case-III~~

~~If $f(n) = b^n$ and b is not a root of recurrence relation~~ then

~~$4A_0 n^2 + 16A_0 n + 16A_0 - 8A_0 n^2 - 16A_0 n - 8A_0 + 4A_0 n^2$~~

~~$A_0 [4n^2 + 16n + 16 - 8n^2 - 16n - 8 + 4n^2] = 1$~~

~~$8A_0 = 1$~~

~~$\boxed{A_0 = 1/8}$~~

~~Now, $(a_n)^p = \left(\frac{1}{8}\right)(2)^n (n)^2.$~~

~~The general solution is $a_n = (a_n)^c + (a_n)^p$~~

~~$a_n = b_1 + b_2$~~

Now, $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$

$A_0 (4)^n - 7A_0 (4)^{n-1} + 10A_0 . (4)^{n-2} = 4^n.$

$\cancel{4^{n-2}} [A_0 . 4^2 - 7A_0 . 4^1 + 10A_0] = \cancel{4^{n-2}} . 4^2$

$16A_0 - 28A_0 + 10A_0 = 16$

$-2A_0 = 16$

$\boxed{A_0 = -8}$

$\therefore (a_n)^p = A_0 4^n = (-8)(4)^n.$

Then the general solution from eq (2)

$a_n = (a_n)^c + (a_n)^p$

$\therefore\ a_n = b_1(5)^n + b_2(2)^n + (-8)\,4^n$ //.

---

(2). $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$

Sol: given $a_{n+2} - 4a_{n+1} + 4a_n = 2^n \rightarrow$ (1)

where, $f(n) = 2^n = b^n$

The general solution is $a_n = (a_n)^c + (a_n)^p \rightarrow$ (2)

To find $(a_n)^c$: let us consider $f(n) = 0$

put $n = n-2$

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

put $a_n = r^n$

$$(r)^n - 4(r)^{n-1} + 4(r)^{n-2} = 0$$

$$r^{n-2}\left[r^2 - 4r + 4\right] = 0$$

$$r^2 - 4r + 4 = 0$$

$$r = 2, 2.$$

The roots are real and same then the general solution is

$$(a_n)^c = (b_1 + b_2 n)\, r^n$$

$$(a_n)^c = (b_1 + b_2 n)\,(2)^n \rightarrow (3)$$

To find $(a_n)^p$:

Here, $f(n) = b^n = 2^n$

$b = 2$.

Here, $b = 2$ is a root of RR

Then $(a_n)^P = A_0 b^n n^2 = A_0 2^n . n^2$.

$$a_{n+2} - 4a_{n+1} + 4a_n = 2^n$$

sub $a_n = A_0 b^n . n^2$

$$A_0 . (2)^{n+2} . (n+2)^2 - 4A_0 (2)^{n+1} (n+1)^2 + 4A_0 2^n n^2 = 2^n$$

$$\cancel{2^n}[A_0 . 2^2 (n^2+4n+4) - 4A_0 2 (n^2+2n+1) + 4A_0 . n^2] = \cancel{2^n}$$

$$4A_0 . n^2 + 16A_0 . n + 16A_0 - 8A_0 . n^2 - 16A_0 n - 8A_0 + 4A_0 n^2 =$$

$$A_0[\cancel{4n^2} + \cancel{16n} + 16 - \cancel{8n^2} - \cancel{16n} - 8 + \cancel{4n^2}] = 1$$

$$8A_0 = 1$$

$$\boxed{A_0 = 1/8}$$

Now, $(a_n)^P = \frac{1}{8} (2)^n (n)^2$

The general solution is.

$$a_n = (a_n)^C + (a_n)^P$$

$$\therefore a_n = (b_1 + b_2 n)(2)^n + \frac{1}{8} (2)^n . (n)^2$$

Case-iii:-

If $f(n)$ = polynomial

(i) If $f(n) = n$ then the particular solution of a_n is defined as

$$(a_n)^P = A_0 + A_1 n.$$

(ii) If $f(n) = n^2$ then the particular solution of a_n is defined as $(a_n)^P = A_0 + A_1 n + A_2 . n^2$.

① Solve $a_{n+2} - a_{n+1} - 2a_n = n^2$.

Sol: given RR is

$$a_{n+2} - a_{n+1} - 2a_n = n^2$$

The general solution is $a_n = (a_n)^c + (a_n)^p \rightarrow ①$

To find $(a_n)^c$:

$$a_{n+2} - a_{n+1} - 2a_n = 0$$

put $n = n-2$

$$a_n - a_{n-1} - 2a_{n-2} = 0$$

put $a_n = r^n$

$$(r)^n - (r)^{n-1} - 2(r)^{n-2} = 0$$

$$r^{n-2}[r^2 - r - 2] = 0$$

$$r^2 - r - 2 = 0$$

$$r = 2, -1.$$

$\therefore$ Then $(a_n)^c = b_1(r_1)^n + b_2(r_2)^n$

$$(a_n)^c = b_1(2)^n + b_2(-1)^n.$$

To find $(a_n)^p$:

given, $f(n) = n^2$ then

$$(a_n)^p = A_0 + A_1 n + A_2 n^2$$

~~given~~ ~~RR~~ sub $a_n = A_0 + A_1 n + A_2 \cdot n^2$

$$a_{n+2} - a_{n+1} - 2a_n = n^2$$

$$\Rightarrow \left[A_0 + A_1(n+2) + A_2(n+2)^2\right] - \left[A_0 + A_1(n+1) + A_2(n+1)^2\right]$$
$$- 2\left[A_0 + A_1 n + A_2 n^2\right] = n^2$$

$$\Rightarrow A_0 - A_0 - 2A_0 + A_1(n+2) - A_1(n+1) - 2A_1 n$$
$$+ A_2(n+2)^2 - A_2(n+1)^2 - 2A_2 n^2 = n^2$$

$$\Rightarrow -2A_0 + \cancel{A_1 n} + 2A_1 - \cancel{A_1 n} - A_1 - 2A_1 n +$$
$$+ A_2(n^2+4+4n) - A_2(n^2+1+2n) - 2A_2 n^2 = n^2$$

$$\Rightarrow -2A_0 - 2A_1 n + A_1 + \cancel{A_2 n^2} + 4A_2 + 4nA_2 - \cancel{A_2 n^2}$$
$$- A_2 - 2A_2 n - 2A_2 n^2 = n^2$$

$$\Rightarrow -2A_0 - 2A_1 n + A_1 + 2.n.A_2 + 3A_2 - 2A_2 n^2 = n^2$$

$$\Rightarrow -2A_0 + A_1 - 2A_1 n + 3A_2 + 2nA_2 - 2A_2 n^2 = n^2$$

$\cancel{-2A_0 + A_1(1-2n) +}$

$$-2A_2 n^2 + 2nA_2 - 2A_1 n + 3A_2 + A_1 - 2A_0 = n^2$$

Comparing coefficients on both sides

$$-2A_2 = 1 \qquad \boxed{A_2 = \frac{-1}{2}}$$

$$2A_2 - 2A_1 = 0$$
$$2\left(\frac{-1}{2}\right) = 2A_1$$
$$\boxed{A_1 = \frac{-1}{2}}$$

$$3A_2 + A_1 - 2A_0 = 0$$
$$3\left(\frac{-1}{2}\right) - \frac{1}{2} = 2A_0$$
$$2A_0 = \frac{-3}{2} - \frac{1}{2} = \frac{-4}{2}$$
$$2A_0 = -2$$
$$\boxed{A_0 = -1}$$

Now, $(a_n)^P = A_0 + A_1 n + A_2 n^2$

$$(a_n)^P = -1 + \left(\frac{-1}{2}\right)n + \left(\frac{-1}{2}\right)n^2$$

The general solution is $a_n = (a_n)^c + (a_n)^p$

$$a_n = b_1(2)^n + b_2(-1)^n - 1 - \frac{1}{2}n - \frac{1}{2}n^2.$$

Standard Results :-

(i) If $f(n) = n + b^n$ then the particular solution of given RR is

$$(a_n)^p = (A_0 + A_1 n) + A_2\, b^n$$

(ii) If $f(n) = n \cdot b^n$ then the particular solution of given RR is

$$(a_n)^p = (A_0 + A_1 n) \cdot n \cdot b^n.$$

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① Find the RR satisfying the conditions

(i) $y_n = A\,(2)^n + B\,(-3)^n$

Sol :- given $y_n = A\,(2)^n + B\,(-3)^n$

It is in the form of $a_n = b_1(r_1)^n + b_2(r_2)^n$

Here, $r_1 = 2,\ r_2 = -3.$

Then, $(r-2)(r+3) = 0$

$$r^2 + 3r - 2r - 6 = 0$$

$$r^2 + r - 6 = 0.$$

$$r^{n-2}\left[r^2 + r - 6\right] = 0$$

$$r^n + r^{n-1} - 6\, r^{n-2} = 0$$

put $r^n = y_n.$

$$y_n + y_{n-1} - 6\, y_{n-2} = 0$$

(ii) $u_n = (A + B.n)(3)^n$.

Sol: given $u_n = (A + Bn)(3)^n$

It is in the form of $a_n = (b_1 + b_2 n)(r)^n$

Here, $\boxed{r = 3}$, $r_1 = 3$, $r_2 = 3$

Then, $(r-3)(r-3) = 0$

$$r^2 - 6r + 9 = 0$$

$$r^{n-2}[r^2 - 6r + 9] = 0$$

$$(r)^n - 6(r)^{n-1} + 9(r)^{n-2} = 0$$

put $r^n = u_n$

$$\therefore u_n - 6u_{n-1} + 9u_{n-2} = 0$$

(iii) $a_n = A(2)^n + B(1)^n$

Sol: given $a_n = A(2)^n + B(1)^n$.

It is in the form of $a_n = b_1(r_1)^n + b_2(r_2)^n$.

Here, $r_1 = 2$, $r_2 = 1$

Then, $(r-2)(r-1) = 0$

$$r^2 - r - 2r + 2 = 0$$

$$r^2 - 3r + 2 = 0$$

$$r^{n-2}[r^2 - 3r + 2] = 0$$

$$(r)^n - 3(r)^{n-1} + 2(r)^{n-2} = 0$$

put $r^n = a_n$

$$\therefore a_n - 3a_{n-1} + 2a_{n-2} = 0$$

Solution of Non-linear RR :-

The non-linear RR's can be solved by converting into linear RR's by substituting suitable terms.

① Solve $a^2_{n+2} - 5a^2_{n+1} + 4a^2_n = 0$ for $n \geq 0$ given $a_0 = 0$ and $a_1 = 13$.

Sol:- given $a^2_{n+2} - 5a^2_{n+1} + 4a^2_n = 0 \longrightarrow (1)$

The given non-linear RR is converting into linear RR by sub suitable terms

put $a_n^2 = b_n$

$$b_{n+2} - 5b_{n+1} + 4b_n = 0$$

put $n = n-2$

$$b_n - 5b_{n-1} + 4b_{n-2} = 0$$

To find (gns):-

put $b_n = r^n$

$$(r)^n - 5(r)^{n-1} + 4(r)^{n-2} = 0$$

$$r^{n-2}[r^2 - 5r + 4] = 0$$

$$r^2 - 5r + 4 = 0$$

$$r = 4, 1.$$

$$r_1 = 4, \ r_2 = 1$$

The general solution is $b_n = b_1(r_1)^n + b_2(r_2)^n$

$$b_n = b_1(4)^n + b_2(1)^n \longrightarrow (2)$$

given initial conditions $a_0 = 0, a_1 = 13$.

wkt, $b_n = a_n^2$

put $n=0 \Rightarrow a_0^2 = b_0$ ↓, $a_0 = 0$

$b_0 = (0)^2 = 0 \Rightarrow b_0 = 0$ //.

put $n=1 \Rightarrow a_1^2 = b_1$ ↓, $a_1 = 13$

$b_1 = (13)^2 = 169 \Rightarrow b_1 = 169$ /

from eq (2),

$b_n = b_1 (4)^n + b_2 (1)^n$

put $n=0 \Rightarrow b_0 = b_1 (4)^0 + b_2 (1)^0$ ↓ $b_0 = 0$

$b_1 + b_2 = 0 \rightarrow (3)$

put $n=1 \Rightarrow b_1 = b_1 (4)^1 + b_2 (1)^1$

$4b_1 + b_2 = 169 \rightarrow (4)$ ↓ $b_1 = 169$

solving (3) and (4) we get,

$b_1 = \frac{169}{3}$, $b_2 = \frac{-169}{3}$.

from eq (2),

sub b_1, b_2 in eq (2),

$$b_n = \left(\frac{169}{3}\right)(4)^n + \left(\frac{-169}{3}\right)(1)^n$$

wkt, $a_n^2 = b_n$

$$a_n^2 = \frac{169}{3}\left[(4)^n - (1)^n\right]$$

$$a_n = \sqrt{\frac{169}{3}\left[(4)^n - (1)^n\right]}$$

$\therefore \quad a_n = 13 \left[\frac{4^n - 1}{3} \right]$ //.

Generating Functions

The generating function for the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$ of real numbers is infinite series which is given by

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_k x^k + \cdots$$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

where, $G(x)$ is called generating function in the sequence of $a_0, a_1, a_2, \ldots$

Ex:- The generating function of the sequence $1, 2, 3, 4, \ldots$ is

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

where, $\frac{1}{(1-x)^2}$ is called GF in the sequence of $1, 2, 3, 4, \ldots$

Some Binomial Expansions:-

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

(ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

(iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

(iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

① What is the G.F of the sequence

(i) $0, 1, -2, 3, -4, \ldots$

Sol: Let $a_0 = 0,\ a_1 = 1,\ a_2 = -2,\ a_3 = 3,\ a_4 = -4, \ldots$

w.k.t, $G(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

$G(x) = 0 + 1(x) + (-2)x^2 + (3)(x^3) + (-4)x^4 + \ldots$

$G(x) = x - 2x^2 + 3x^3 - 4x^4 + \ldots$

$G(x) = x\{1 - 2x + 3x^2 - 4x^3 + \ldots\}$

$G(x) = x\ (1+x)^{-2}$

$\therefore\ G(x) = \dfrac{x}{(1+x)^2}$ //

(ii) $0, 1, 0, -1, 0, 1, 0, -1, 0, \ldots$

Sol: Let $a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = -1,\ a_4 = 0, \ldots$
$a_5 = 1,\ a_6 = 0,\ a_7 = -1,\ a_8 = 0 \ldots$

w.k.t, $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$

$G(x) = 0 + (1)(x) + 0(x^2) + (-1)x^3 + (0)(x^4) + 1(x^5) + 0(x^6)$
$+ (-1)x^7 + 0(x^8) + \ldots$

$G(x) = x^1 - x^3 + x^5 - x^7 + x^9 - x^{11} + \ldots$

$G(x) = x[1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots]$

$G(x) = x\ [1 - (x^2)^1 + (x^2)^2 - (x^2)^3 + (x^2)^4 - (x^2)^5 + \ldots]$

$G(x) = x \cdot (1+x^2)^{-1}$

$\therefore\ G(x) = \dfrac{x}{1+x^2}$ //.

Counting Problems and Generating Functions:

To find the number of ways of selecting 'r' objects among 'n' objects with unlimited repetitions:

Suppose to find the coefficient of x^r in the given generating function is defined as

$$G(x) = (1+x+x^2+x^3+\cdots) = (1-x)^{-n}.$$

The coefficient of $(x)^r$ in $(1-x)^n$ is defined by $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$

where, $\binom{n+r-1}{r} = {}^{n+r-1}C_r$ is the coefficient of x^r.

① Find the coefficient of x^{27} in the following function. $(x^4+2x^5+3x^6+\cdots)^5$.

Sol: given, $(x^4+2x^5+3x^6+\cdots)^5$

$$= \left[(x^4)\left[1+2x+3x^2+4x^3+\cdots\right]\right]^5$$

$$= (x^4)^5\left[1+2x+3x^2+\cdots\right]^5$$

$$= x^{20}\left[(1-x)^{-2}\right]^5$$

$$= x^{20}(1-x)^{-10}$$

wkt, $(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \cdot x^r.$

$$= x^{20} \cdot \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r$$

$$= x^{20} \sum_{r=0}^{\infty} {}^{10+r-1}C_r \, x^r$$

$$= x^{20} \sum_{r=0}^{\infty} {}^{9+r}C_r \, x^r .$$

put $r=7$

$$= x^{20} \left(\sum_{r=0}^{\infty} {}^{9+7}C_7 \right) x^7$$

$$= x^{27} \cdot {}^{16}C_7 = 11440 \, x^{27}$$

GSKD Sri krishnadevarayulu.

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$\cancel{G(x)}$

$\cancel{(1-x)^{-1}}$

$\cancel{(1+x)^{-1}}$

$\cancel{(1-x)^{-2}}$

Graph Theory

Degree of a Vertex
Hand shaking property
Isomorphism
Planar Graph
Euler Circuits and ~~[illegible]~~ Euler Paths
Hamilton Graph.
Chromic Numbers. (Chromatic Numbers)
Spanning Tree

(1). Algorithms to find minimal spanning tree.

DFS - Deapth First Search :-

BFS - Breadth First Search :-

Kruskkal's algorithm :-

Generating Function $G(x)$. a_n.

1) $\frac{1}{1-x} \Rightarrow (1)^n$

2) $\frac{1}{1+x} \Rightarrow (-1)^n$

3) $\frac{1}{(1-x)^2} \Rightarrow n+1$

4) $\frac{x}{(1-x)^2} \Rightarrow n$

5) $\frac{2x}{(1-x)^3} \Rightarrow n(n+1)$

6) $\frac{2}{(1-x)^3} \Rightarrow (n+1)(n+2)$

7) $\frac{1}{1-ax} \Rightarrow a^n$

8) $\frac{1}{1+ax} \Rightarrow (-a)^n$

9) $e^x \longrightarrow \frac{1}{n!}$

10) $\frac{x(1+x)}{(1-x)^3} \longrightarrow n^2$

11) $\frac{x(x^2+4x+1)}{(1-x)^4} \Longrightarrow n^3$

12) $\frac{ax}{(1-ax)^2} \Rightarrow na^n.$

13) $\frac{x}{1+x^2} \Rightarrow \sin\frac{n\pi}{2}$

14) $\frac{1}{1+x^2} \Rightarrow \cos\frac{n\pi}{2}$

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15) $\frac{1-x^2}{(1+x^2)^2} \Rightarrow n.\sin\frac{n\pi}{2}$

16) $\frac{-2x^2}{(1+x^2)^2} \Rightarrow n.\cos\frac{n\pi}{2}$
