Mean Value Theorems

Rolle's Theorem:-

Let $f(x)$ be a function such that

(i) It is continuous in closed interval $[a,b]$.

(ii) It is differentiable in open interval (a,b) and

(iii) $f(a) = f(b)$, then there exist at least one point $c \in (a,b)$ such that $f'(c) = 0$.

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① Verify the rolles theorem for $f(x) = x^2 - 2x - 3$. in the interval $(-1,3)$.

Sol:- G.T, $f(x) = x^2 - 2x - 3$, in $(-1,3)$

(i) $f(x)$ is continuous in $[-1,3]$.

since, $f(x)$ is a polynomial.

(ii) $f(x)$ is differentiable in $(-1,3)$.

since, $f'(x)$ is exist in $(-1,3)$.

$$f(x) = x^2 - 2x - 3$$

$f(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0,$

$f(3) = (3)^2 - 2(3) - 3 = 9 - 6 - 3 = 0.$

$\therefore f(-1) = f(3)$.

(iii) $c \in (-1,3)$ such that $f'(c) = 0$

$$f'(x) = 2x - 2 = 0$$
$$f'(c) = 2c - 2 = 0$$
$$2c = 2$$
$$c = 1$$

$\therefore c = 1 \in (-1,3)$.

$\therefore f(x)$ is verified the rolles theorem in $(-1,3)$.

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② Verify Rolle's theorem for $g(x) = 8x^3 - 6x^2 - 2x + 1$ in the interval $(0,1)$.

Sol:- G.T, $g(x) = 8x^3 - 6x^2 - 2x + 1$

(i) $g(x)$ is continuous in $[0,1]$.

since, $g(x)$ is a polynomial.

(ii) $g(x)$ is differentiable in $(0,1)$.

since, $g'(x)$ is exist in $(0,1)$.

$$g(x) = 8x^3 - 6x^2 - 2x + 1$$

$$g(0) = 0 - 0 - 0 + 1 = 1$$

$$g(1) = 8 - 6 - 2 + 1 = 9 - 8 = 1$$

$$\therefore g(0) = g(1)$$

(iii) $\exists\, c \in (0,1)$ such that $g'(c) = 0$

$$g'(x) = 24x^2 - 12x - 2 = 0$$

$$g'(c) = 24c^2 - 12c - 2 = 0$$

$$12c^2 - 6c - 1 = 0$$

$$c = \frac{3+\sqrt{21}}{3}, \frac{3-\sqrt{21}}{3}$$

$\frac{3-\sqrt{21}}{3}$ is neglected.

$$\therefore c = \frac{3+\sqrt{21}}{3} \in (0,1).$$

$\therefore$ $g(x)$ is verified the Rolle's theorem in $(0,1)$.

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③ Verify whether Rolle's theorem can be applied to the following functions in the interval.

(i) $f(x) = \tan x$ in $[0, \pi]$.

Sol:- $f'(x) = \sec^2 x$.

$\therefore f'(x)$ does not exist at $x = \frac{\pi}{2}$.

$\therefore f(x) = \tan x$ is not differentiable in $(0, \pi)$

$\therefore$ f(x) is not verified the Rolle's theorem in $[0,\pi]$

(ii). $f(x) = \frac{1}{x^2}$ in $[-1,1]$.

Sol: $f'(x) = \frac{-2}{x^3}$

$\therefore f'(x) = \frac{-2}{x^3}$ does not exist at $x=0$.

$\therefore$ f(x) is not derivable in $(-1,1)$.

$\therefore$ Hence, f(x) is not verified the Rolle's theorem in $[-1,1]$.

(iii) $f(x) = x^3$ in $[1,3]$.

Sol: $f'(x) = 3x^2$

$\therefore f'(x) = 3x^2$ is exist in $(1,3)$

$\therefore$ f(x) is differentiable in $(1,3)$.

$\therefore$ f(x) is continuous in $[1,3]$.

$$f(x) = x^3$$

$$f(1) = (1)^3 = 1, \quad f(3) = (3)^3 = 27$$

$$\therefore f(1) \neq f(3).$$

$\therefore$ f(x) is not verified the Rolle's therein in $[1,3]$.

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4. Verify Rolle's theorem for $f(x) = \frac{x^2-x-6}{x-1}$ in the interval $(2,3)$.

Sol: $f(x) = \frac{x^2-x-6}{x-1} = \frac{(x+2)(x-3)}{x-1}$ | $x \in [-2,3] \setminus \{1\}$.

$$f'(x) = \frac{(x-1)(2x-1) - (x^2-x-6)(1)}{(x-1)^2}.$$

$\therefore f'(x)$ does not exist at $x=1$

$\therefore$ f(x) is not differentiable in $(-2,3)$.

Hence, f(x) is not verified the Rolle's theorem.

⑤ Verify the Rolle's theorem for $f(x) = (x-a)^m (x-b)^n$ where, m, n are +ve integers in [a,b].

Sol:- $f'(x) = (x-a)^m \cdot n \cdot (x-b)^{n-1} + (x-b)^n \cdot m (x-a)^{m-1}$

$\therefore$ f(x) is continuous in [a,b].

$$f(a) = (a-a)^m (a-b)^n = 0$$
$$f(b) = (b-a)^m (b-b)^n = 0$$

$\therefore$ f(x) is differentiable in (a,b).

$\therefore$ f(a) = f(b) then there exist at least one point $f'(c) = 0$

$$f'(c) = m(c-a)^{m-1}(c-b)^n + n(c-a)^m (c-b)^{n-1} = 0$$

$$\frac{m\cancel{(c-a)^m}\cancel{(c-b)^n}}{(c-a)} = \frac{-n\cancel{(c-a)^m}\cancel{(c-b)^n}}{(c-b)}$$

$$\frac{m}{c-a} = \frac{-n}{c-b} \Rightarrow mc - mb = -nc + na$$
$$mc + nc = mb + na$$
$$c(m+n) = mb + na$$

$$\therefore c = \frac{mb+na}{m+n}$$

$\therefore$ f(x) is verified the Rolle's theorem in [a,b]

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⑥. Verify Rolles theorem for the function $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in [a,b], a > 0, b > a

Sol:- $\frac{d}{dx}(\log(x)) = \frac{1}{x}$, $\frac{d}{d}\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

$$f'(x) = \frac{x(a+b)}{x^2+ab}\left[\frac{x(a+b)\cdot 2x - (x^2+ab)(b)}{(x(a+b))^2}\right]$$

$$f'(x) = \frac{\cancel{x(a+b)}\,[(ax+bx)2x - x^2 - ab]}{(x^2+ab)(x(a+b))^{\cancel{2}}}$$

$$f'(x) = \frac{2x^2(a+b) - (x^2+ab)}{(x^2+ab)(x(a+b))}$$

$$f(x) = \log(x^2+ab) - [\log x + \log(a+b)]$$

$$f(x) = \log(x^2+ab) - \log x - \log(a+b)$$

$$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} - 0. \Rightarrow f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}.$$

$\therefore$ $f'(x)$ is exist in $[a,b]$

(i) $f(x)$ is continuous in $[a,b]$.

(ii) $f(x)$ is differentiable in (a,b).

$$f(a) = \log\left(\frac{a^2+ab}{a(a+b)}\right) = 0$$

$$f(b) = \log\left(\frac{b^2+ab}{b(a+b)}\right) = 0.$$

$\therefore$ $f(a) = f(b)$.

(iii) $f(a) = f(b)$ then $\exists$ at least one point $c \in (a,b)$

$\Rightarrow f'(c) = 0.$

$$\Rightarrow f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} = 0$$

$$\frac{2c^2 - c^2 - ab}{c(c^2+ab)} = 0$$

$$c^2 - ab = 0$$

$$c^2 = ab \Rightarrow c = \pm\sqrt{ab}$$

$$c = \sqrt{ab} \in (a,b).$$

$\therefore$ $f(x)$ is verifies the rolle's theorem in $[a,b]$.

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(7) Verify the Rolle's theorem for $f(x) = \frac{\sin x}{e^x}$ (or) $e^{-x}\sin x$ in $[0,\pi]$.

Sol: $f(x) = e^{-x}\sin x$.

$f'(x) = e^{-x}\cos x + \sin x \cdot e^{-x}(-1)$

$f'(x) = e^{-x}\cos x - \sin x \cdot e^{-x}$

$f'(x) = e^{-x}(\cos x - \sin x) = \dfrac{\cos x - \sin x}{e^{x}}$

$\therefore f'(x)$ is exist in $[0,\pi]$.

(i) $f(x)$ is continuous in $[0,\pi]$.

(ii) $f(x)$ is differentiable in $(0,\pi)$.

$f(0) = e^{-0}\sin(0) = 0$

$f(\pi) = e^{-\pi}\sin(\pi) = 0.$

$\therefore f(0) = f(\pi).$

(iii) $f(0) = f(\pi)$, then $\exists$ at least one point in $c \in (0,\pi) \ni f'(c) = 0$

$$\frac{\cos c - \sin c}{e^{c}} = 0$$

$\cos c - \sin c = 0$

$\sin c = \cos c$

$\dfrac{\sin c}{\cos c} = 1$

$\tan c = 1$

$\tan c = \tan\frac{\pi}{4}$

$c = \frac{\pi}{4}$

$\therefore c \in (0,\pi).$

$\therefore f(x)$ is verified the rolles theorem in $[0,\pi]$

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8. Apply rolle's theorem for $\sin x\sqrt{\cos 2x}$ in $[0,\frac{\pi}{4}]$ Find x such that $0 < x < \frac{\pi}{4}$.

Sol:- $f(x) = \sin x\sqrt{\cos 2x}$ $\quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$f'(x) = \sin x \cdot \dfrac{-\sin 2x}{\cancel{2}\sqrt{\cos 2x}} \cdot \cancel{2} + \sqrt{\cos 2x}\cdot\cos x$

$f'(x) = -\dfrac{\sin x \cdot \sin 2x}{\sqrt{\cos 2x}} + \cos x\cdot\sqrt{\cos 2x}$

(i) $f(x)$ is continuous in $[0, \frac{\pi}{4}]$.

(ii) $f(x)$ is differentiable in $(0, \frac{\pi}{4})$.

$f'(x)$ is exist in $(0, \frac{\pi}{4})$.

$f(0) = \sin 0 . \sqrt{\cos 2(0)} = 0$

$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} \sqrt{\cos \cancel{2} . \frac{\pi}{\cancel{4}\,2}} = 0$

(iii) $f(0) = f\left(\frac{\pi}{4}\right) \Rightarrow$ at least one point $c \in (0, \frac{\pi}{4})$

then $f'(c) = 0$

$$-\frac{\sin c . \sin 2c}{\sqrt{\cos 2c}} + \cos c . \sqrt{\cos 2c} = 0$$

$$\cancel{-}\frac{\sin c . \sin 2c}{\sqrt{\cos 2c}} = \cancel{+}\cos c . \sqrt{\cos 2c}$$

$\sin c . \sin 2c = \cos c . \cos 2c$

$\cos c \cos 2c - \sin c \sin 2c = 0$

$\cos(c + 2c) = 0$

$\cos 3c = 0 = \cos\frac{\pi}{2}$

$3c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{6} \in (0, \frac{\pi}{4})$.

∴ Hence, the given function is verified the Rolle's theorem.

(i) Verify Rolle's theorem for $f(x) = e^x[\sin x - \cos x]$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Sol: $f'(x) = e^x[\cos x + \sin x] + [\sin x - \cos x]e^x$

$f'(x) = e^x[\cancel{\cos x} + \sin x + \sin x - \cancel{\cos x}] = 2e^x \sin x$.

$f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] = 0.$ | $f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = \frac{2e}{\sqrt{2}}$

~~f(x)~~ is continuous ~~on~~

$f(\frac{\pi}{4}) \neq f(\frac{5\pi}{4})$, it is not exist.

$\therefore$ Hence, Rolles theorem is not verified.

⑩. Verify Rolle's theorem for $f(x) = |x|$ in $[-1,1]$.

Sol: G.T, $f(x) = |x|$ in $[-1,1]$.

i.e, $f(x) = x$ for $x > 0$

$f(x) = -x$ for $x < 0$.

$$LH.L = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} |x| = \lim_{x \to 0} (-x) = 0$$

$$RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x| = \lim_{x \to 0} (x) = 0.$$

$\therefore$ LHL = RHL

$\therefore f(x)$ is continuous in $[-1,1]$.

$$LHD = f'(x) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{|x| - 0}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0} \left(\frac{-x}{x}\right) = -1$$

$$RHD = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = f'(x) = \lim_{x \to 0^+} \frac{|x| - 0}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x}$$

$$= \lim_{x \to 0} \left(\frac{x}{x}\right) = 1$$

$\therefore$ LHD $\neq$ RHD

$\therefore f(x)$ is not differentiable in $(-1,1)$.

$\therefore$ Hence, $f(x)$ is not verified the rolle's theore

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Laggange's Mean Value theorem:-

Let $f(x)$ be a function such that

(i) It is continuous in $[a,b]$.

(ii) It is differentiable in (a,b) and there exist a point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

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① Verify lagrange's mean value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$.

Sol:- $f'(x) = 3x^2 - 2x - 5$

$f(4) = (4)^3 - (4)^2 - 5(4) + 3 = 31$

$f(0) = 0 - 0 - 0 + 3 = 3$

(i) $f(x)$ is continuous in $[0,4]$.

(ii) $f(x)$ is differentiable in $(0,4)$, and $\exists$ a point $c \in (0,4)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$f'(c) = \frac{31-3}{4-0} = \frac{28}{4} = 7.$$

$$\Rightarrow 3c^2 - 2c - 5 = 7$$

$$3c^2 - 2c - 12 = 0. \Rightarrow c = \frac{1 \pm \sqrt{37}}{3}.$$

$$c = \frac{1 \pm 6.082}{3} \Rightarrow c = 2.36, -1.69$$

$c = 2.36 \in (0,4)$.

$\therefore$ Hence, $f(x)$ is verified the lagrange's mean value theorem.

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②. Verify Lagrange's mean value theorem for $f(x) = \log_e x$ in $[1,e]$.

Sol :- $f'(x) = \frac{1}{x}$.

$f(1) = \log_e 1 = 0$, $f(e) = \log_e e = 1$.

(i) $f(x)$ is continuous in $[1, e]$.

(ii) $f(x)$ is differentiable in $(1, e)$ and $\exists$ a point $c \in (1, e)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f'(c) = \frac{1-0}{e-1} = \frac{1}{e-1}$$

$$\frac{1}{c} = \frac{1}{e-1} \Rightarrow c = e-1 = 2.718 - 1$$

$$c = 1.718 \in (1, e).$$

$\therefore$ Hence, $f(x)$ is verified the lagrange's mean value theorem.

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(3) Verify Lagrange's mean value theorem for $f(x) = x(x-2)(x-3)$ in $(0, 4)$.

Sol :- $f(x) = x[x^2 - 5x + 6]$

$f(x) = x^3 - 5x^2 + 6x$.

$f'(x) = 3x^2 - 10x + 6$

$f(0) = 0(0-2)(0-3) = 0$

$f(4) = 4(2)(1) = 8$.

(i) $f(x)$ is continuous in $[0, 4]$.

(ii) $f(x)$ is differentiable in $(0, 4)$ and $\exists$ a point $c \in (0, 4)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{8 - 0}{4 - 0}$

$$= \frac{8}{4} = 2$$

$$3c^2 - 10c + 6 = 2 \Rightarrow 3c^2 - 10c + 4 = 0$$

$$c = \frac{5 \pm \sqrt{13}}{3} , \; c = 2.86, 0.46 \in ($$

$\therefore$ Hence, $f(x)$ is verified the lagrange's mean val theorem.

(4). Explain why mean value theorem does not hold for $f(x) = x^{2/3}$ in $[-1,1]$.

Sol:- $f(x) = x^{2/3}$.

$$f'(x) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} \frac{1}{x^{1/3}}.$$

$\therefore$ $f'(x)$ does not exist at $x=0$.

So, $f(x)$ does not verified the Lagrange's mean value theorem for $[-1,1]$.

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(5) If $a<b$, prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ using Lagrange's mean value theorem. Deduce that

(i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

(ii) $\frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}$.

Sol:- $f(x) = \tan^{-1}(x)$ in $[a,b]$.

$$f'(x) = \frac{1}{1+x^2}$$

$f(a) = \tan^{-1}(a)$, $f(b) = \tan^{-1}(b)$.

(i) $f(x)$ is continuous in $[a,b]$.

(ii) $f(x)$ is differentiable in (a,b). then

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad c \in (a,b).$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a}$$

$$\tan^{-1}(b) - \tan^{-1}(a) = \frac{b-a}{1+c^2} \longrightarrow (1)$$

i.e, $c \in (a,b)$ then $a<c<b$

$a^2 < c^2 < b^2$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \frac{b-a}{1+c^2} > \frac{b-a}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \tan^{-1}b - \tan^{-1}a > \frac{b-a}{1+b^2} \quad \text{from eq}\ (1)$$

$$\therefore \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} \rightarrow (2)$$

(i) put $b = \frac{4}{3}$, $a = 1$, from (1), we get,

$$\frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+(1)^2}$$

$$\frac{1}{3}\times\frac{9}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{3}\times\frac{1}{2}$$

$$\therefore \frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4}+\frac{1}{6}.$$

(ii) put $b = 2$, $a = 1$, from (1), we get,

$$\frac{2-1}{1+2^2} < \tan^{-1}(2) - \tan^{-1}(1) < \frac{2-1}{1+1^2}$$

$$\frac{1}{5} < \tan^{-1}(2) - \frac{\pi}{4} < \frac{1}{2}$$

$$\frac{\pi}{4}+\frac{1}{5} < \tan^{-1}(2) < \frac{\pi}{4}+\frac{1}{2}$$

$$\therefore \frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}$$

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(6). Prove that $\frac{\pi}{6}+\frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6}+\frac{1}{8}$.

Sol:- Let $f(x) = \sin^{-1}(x)$ in $[a, b]$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$f(a) = \sin^{-1}(a)$, $f(b) = \sin^{-1}(b)$

then $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}(b) - \sin^{-1}(a)}{b-a}$$

$$\sin^{-1}(b) - \sin^{-1}(a) = \frac{b-a}{\sqrt{1-c^2}} \rightarrow (1)$$

$$\Rightarrow a < c < b$$

$$a^2 < c^2 < b^2$$

$$-a^2 > -c^2 > -b^2$$

$$1-a^2 > 1-c^2 > 1-b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} < \frac{b-a}{\sqrt{1-c^2}} < \frac{b-a}{\sqrt{1-b^2}}$$

from eq^n (1),

$$\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}(b) - \sin^{-1}(a) < \frac{b-a}{\sqrt{1-b^2}}$$

$b = \frac{3}{5}$, $a = \frac{1}{2}$ $\downarrow$ $\sin\frac{\pi}{6} = \frac{1}{2}$.

$$\frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} < \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{1}{2}\right) < \frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\left(\frac{3}{5}\right)^2}}$$

$$\frac{\frac{6-5}{10}}{\sqrt{\frac{4-1}{4}}} < \sin^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{6} < \frac{\frac{6-5}{10}}{\sqrt{\frac{25-9}{25}}}$$

$$\frac{1}{10} \times \frac{2}{\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{6} < \frac{1}{10} \times \frac{5}{4}$$

$$\therefore \frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8} \;//$$

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7. Prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$ using Lagrange's mean value theorem.

Sol:

$$f(x) = \cos^{-1}(x)$$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$f(a) = \cos^{-1}(a), \quad f(b) = \cos^{-1}(b)$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}(b) - \cos^{-1}(a)}{b-a}$$

$$\cos^{-1}(b) - \cos^{-1}(a) = \frac{-(b-a)}{\sqrt{1-c^2}}$$

$$\Rightarrow a < c < b$$

$$a^2 < c^2 < b^2$$

$$-a^2 > -c^2 > -b^2$$

$$1-a^2 > 1-c^2 > 1-b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} < \frac{b-a}{\sqrt{1-c^2}} < \frac{b-a}{\sqrt{1-b^2}}$$

$$-\frac{(b-a)}{\sqrt{1-a^2}} > \frac{-(b-a)}{\sqrt{1-c^2}} > \frac{-(b-a)}{\sqrt{1-b^2}}$$

$$-\frac{(b-a)}{\sqrt{1-a^2}} > \cos^{-1}(b) - \cos^{-1}(a) > \frac{-(b-a)}{\sqrt{1-b^2}}$$

$b=\frac{3}{5}, \quad a=\frac{1}{2}$.

$$-\frac{\left(\frac{3}{5}-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} > \cos^{-1}\left(\frac{3}{5}\right)-\frac{\pi}{3} > \frac{-\left(\frac{3}{5}-\frac{1}{2}\right)}{\frac{4}{5}}$$

$$-\frac{1}{10}\times\frac{2}{\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right)-\frac{\pi}{3} > \frac{-1}{10}\times\frac{5}{4}$$

$$\frac{-1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right)-\frac{\pi}{3} > \frac{-1}{8}$$

$$\therefore \frac{\pi}{3}-\frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3}-\frac{1}{8}$$

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⑧. Show that for any $x>0$, $1+x<e^x<1+x\cdot e^x$.

Sol. Let $f(x)=e^x$ in $[0,x]$

$$f'(x)=e^x$$

$$f(0)=e^0=1, \quad f(x)=e^x$$

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$e^c=\frac{e^x-1}{x-0}$$

$$\Rightarrow 0<c<x$$

$$e^0<e^c<e^x \Rightarrow 1<e^c<e^x$$

$$1<\frac{e^x-1}{x}<e^x$$

$$x<e^x-1<xe^x$$

$$\therefore 1+x<e^x<1+x\cdot e^x$$

---

⑨. Using mean value theorem for $0<a<b$, PT, $1-\frac{a}{b}<\log\left(\frac{b}{a}\right)<\frac{b}{a}-1$ and hence show that, $\frac{1}{6}<\log\left(\frac{6}{5}\right)<\frac{1}{5}$.

Sol: Let $f(x) = \log x$ in $[a,b]$.

$$f'(x) = \frac{1}{x}$$

$$f(a) = \log a, \quad f(b) = \log b$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log b - \log a}{b-a}$$

$$\log b - \log a = \frac{b-a}{c}$$

$$\Rightarrow \quad a < c < b \quad \Rightarrow \quad \frac{1}{a} > \frac{1}{c} > \frac{1}{b}$$

$$\frac{b-a}{a} > \frac{b-a}{c} > \frac{b-a}{b}$$

$$\frac{b-a}{a} > \log\left(\frac{b}{a}\right) > \frac{b-a}{b}$$

$$\frac{b}{a} - 1 > \log\left(\frac{b}{a}\right) > 1 - \frac{a}{b}$$

$$\Rightarrow \quad 1 - \frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a} - 1$$

put $b = 6, \ a = 5$.

$$1 - \frac{5}{6} < \log\left(\frac{6}{5}\right) < \frac{6}{5} - 1$$

$$\therefore \quad \frac{1}{6} < \log\left(\frac{6}{5}\right) < \frac{1}{5} \ //$$

---

(10). Using mean value theorem, prove that, $\tan x > x$ in $0 < x < \frac{\pi}{2}$.

Sol: Let $f(x) = \tan x$, in $0 < x < \frac{\pi}{2}$ $\exists$ a point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$f'(x) = \sec^2 x$, Let, $a=0$, $b=x$

$f(a) = \tan a$, $f(b) = \tan b$

$f(0) = \tan 0 = 0$, $f(x) = \tan x$.

$$\therefore f'(c) = \frac{f(x) - f(0)}{x-0}$$

$$\therefore \sec^2 c = \frac{\tan x - 0}{x - 0}$$

$$\sec^2 c = \frac{\tan x}{x}$$

we have $\sec^2 x > 1$

$$\frac{\tan x}{x} > 1$$

$$\therefore \tan x > x$$

---

(11). Using mean value theorem $|\sin u - \sin v| \leq |u - v|$

Sol:- Let $f(x) = \sin x$ in $[u, v]$.

$$\exists \; c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$f'(c) = \frac{f(v) - f(u)}{v-u}$$

$f'(x) = \cos x$

$f(u) = \sin u$, $f(v) = \sin v$

$$\cos c = \frac{\sin v - \sin u}{v-u}$$

Taking mode on both side,

$|\cos x| \leq 1 \Rightarrow |\cos c| \leq 1$

$$\left|\frac{\sin v - \sin u}{v-u}\right| \leq 1 \Rightarrow \left|\frac{\sin v - \sin u}{v-u}\right| \leq 1$$

$$\frac{|\sin v - \sin u|}{|v-u|} \leq 1$$

$$|\sin v - \sin u| \leq |v-u|$$

$$\therefore \quad |\sin u - \sin v| \leq |u-v| \quad //$$

---

(12) PT, If $x>0$, $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$

Sol:-

(13). Calculate approximately $\sqrt[5]{245}$ by using Lagranges mean value theorem.

Sol: Let $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$

put $a = 243$, $b = 245$ $\quad \sqrt[5]{243} = 3.$

$f'(x) = \frac{1}{5} x^{\frac{1}{5}-1} = \frac{1}{5} x^{-4/5}$

$f'(x) = \frac{1}{5x^{4/5}}$

Now, $f'(c) = \frac{f(b)-f(a)}{b-a}$

$f'(c) = \frac{f(245)-f(243)}{245-243} = \frac{\sqrt[5]{245}-\sqrt[5]{243}}{2}$

$f'(c) = \frac{1}{5c^{4/5}} = \frac{\sqrt[5]{245}-3}{2}$ $\quad a = c = 243$

$\therefore \sqrt[5]{245} = \frac{2}{5c^{4/5}} + 3 = \frac{2}{5 \times (243)^{4/5}} + 3$

$\sqrt[5]{245} = \frac{2}{5 \times (3)^4} + 3 = \frac{2}{405} + 3 = 4.938 \times 10^{-3} + 3$

$\therefore \sqrt[5]{245} = 0.004938 + 3 = 3.004938$

$\therefore \sqrt[5]{245} = 3.004938$

This value is also calculated by using the formula $f(x+\Delta x) = f(x) + f'(x)\,\Delta x$.

---

(14). Calculate approximate value of $\sqrt[6]{45}$ by using lagrange's mean value theorem.

Sol: Let $f(x) = \sqrt[6]{x} = x^{1/6}$.

$$f'(x) = \frac{1}{6} x^{\frac{1}{6}-1} = \frac{1}{6 x^{5/6}}$$

put $a = 64,\ b = 65$ $\quad \sqrt[6]{64} = 2$

Now, $$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{6(c)^{5/6}} = \frac{f(65) - f(64)}{65-64} = \frac{\sqrt[6]{65} - \sqrt[6]{64}}{1}$$

$$\therefore \sqrt[6]{65} = \frac{1}{6 \cdot ((64)^{1/6})^5} + \sqrt[6]{64}$$

$$\therefore \sqrt[6]{65} = \frac{1}{6 \times (2^5)} + 2 = \frac{1}{192} + 2$$

$$\therefore \sqrt[6]{65} = 0.005208 + 2 = 2.005208$$

$$\therefore \sqrt[6]{65} = 2.005208$$

---

(15) Calculate approximate value of $\sqrt{85}$ by using Lagrange's mean value theorem.

Sol^n $\quad f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

put $a = 81,\ b = 85$ $\quad \sqrt{81} = 9 = \sqrt{a} = \sqrt{c} = 9$

Now, $$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{2\sqrt{c}} = \frac{f(85) - f(81)}{85-81} = \frac{f(85) - 9}{4}$$

$$f(85) = \frac{4^2}{2\sqrt{c}} + 9 = \frac{2}{\sqrt{c}} + 9 = \frac{2}{\sqrt{81}} + 9$$

$\sqrt{85} = \frac{2}{9} + 9 = 0.22222 + 9 = 9.22222$

$\therefore \sqrt{85} \simeq 9.222222$

Cauchye's Mean Value Theorem:-

If $f : [a,b] \to R$, $g : [a,b] \to R$ such that

(i) $f(x)$, $g(x)$ are continuous in $[a,b]$

(ii) $f(x)$, $g(x)$ are differentiable in (a,b) and

(iii) $g'(x) \neq 0$ $\forall$ $x \in (a,b)$ then there exist atleast one point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Q. Verify Cauchy's mean value theorem for $f(x) = x^2$, $g(x) = x^3$ in $[1,2]$.

Sol: given that $f(x) = x^2$, $f'(x) = 2x$

$g(x) = x^3$, $g'(x) = 3x^2$.

$f(1) = (1)^2 = 1$, $f(2) = 2^2 = 4$

$g(1) = (1)^3 = 1$, $g(2) = (2)^3 = 8$

(i) $f(x)$, $g(x)$ are continuous in $[1,2]$.

(ii) $f(x)$, $g(x)$ are differentiable in $(1,2)$.

(iii) $g'(c) \neq 0$, then $\exists$ $c \in (1,2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\Rightarrow \frac{2c}{3c^2} = \frac{4-1}{8-1} \Rightarrow \frac{2}{3c} = \frac{3}{7}$$

$9c = 14 \Rightarrow \therefore c = 1.555 \in (1,2)$

$\therefore f(x), g(x)$ are verified by the cauchye's mean value theorem. for $[1,2]$.

②. Verify cauchy's mean value theorem for the functions, e^x and e^{-x} in the interval (0,2)

Sol:- $f(x) = e^x, \quad g(x) = e^{-x}$

$f'(x) = e^x, \quad g'(x) = -e^{-x}.$

$f(0) = e^0 = 1 \quad | \quad g(0) = e^{-0} = 1$

$f(2) = e^2 \quad | \quad g(2) = e^{-2} = \frac{1}{e^2}$

(i) $f(x), g(x)$ are continuous in $[0,2]$.

(ii) $f(x), g(x)$ are differentiable in $(0,2)$.

(iii) $g'(c) \neq 0$, then $\exists$ there exist $c \in (0,2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

$$\frac{e^c}{-e^{-c}} = \frac{e^2 - 1}{e^{-2} - 1} \Rightarrow -e^{2c} = \frac{e^2 - 1}{\frac{1 - e^2}{e^2}}$$

$$\cancel{-} e^{2c} = \cancel{-}\left(\frac{\cancel{e^2 - 1}}{\cancel{e^2 - 1}}\right) e^2 \Rightarrow e^{2c} = e^2$$

$$2c = 2$$

$$\boxed{c = 1} \in [0,2]$$

$\therefore f(x), g(x)$ are verified the cauchy's mean value theorem.

---

③. Verify cauchy's mean value theorem for the function e^x and e^{-x} in the interval $[3,7]$

Sol:- $f(x) = e^x, \quad g(x) = e^{-x}$

$f'(x) = e^x, \quad g'(x) = -e^{-x}.$

$f'(c) = e^c, \quad g'(c) = -e^{-c}.$

$f(3) = e^3 \quad | \quad g(3) = e^{-3}$

$f(7)\ e^7 \quad | \quad g(7) = e^{-7}$

(i) $f(x), g(x)$ are continuous in $[3,7]$

(ii) $f(x), g(x)$ are differentiable in $(3,7)$.

(iii) $g'(x) \neq 0$, then $\exists\ c \in (3,7)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^7-e^3}{e^{-7}-e^{-3}}$$

$$-e^{2c} = \frac{e^4}{\frac{1}{e^7}-\frac{1}{e^3}} \Rightarrow -e^{2c} = \frac{e^4}{\frac{e^3-e^7}{e^7 e^3}}$$

$$\Rightarrow \cancel{-}e^{2c} = e^4 \times \cancel{-}(e^7-e^3)^{-1} \times e^{10}$$

$$e^{2c} = \cancel{e^4} \times \cancel{e^{-4}} \times e^{10}$$

$$e^{2c} = e^{10}$$

$$2c = 10$$

$$\therefore\ c = 5 \in (3,7)$$

$\therefore f(x), g(x)$ are verified the cauchy's mean value theorem.

---

4. Find c by cauchy's mean value theorem for $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$, where $0<a<b$.

Sol: Given that, $f(x) = \sqrt{x}$.

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2x^{1/2}}.$$

$$g(x) = x^{-1/2} \Rightarrow g'(x) = \frac{-1}{2} x^{-\frac{1}{2}-1} = \frac{-1}{2x^{3/2}}$$

(i) $f(x), g(x)$ are continuous in $[a,b]$

(ii) $f(x), g(x)$ are differentiable in (a,b).

(iii) $g'(x) \neq 0$, then $\exists$ at least one point $c \in (a,b)$

then $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$f(x) = \sqrt{x}$, $f(a) = \sqrt{a}$, $f(b) = \sqrt{b}$

$g(x) = \frac{1}{\sqrt{x}}$, $g(a) = \frac{1}{\sqrt{a}}$, $g(b) = \frac{1}{\sqrt{b}}$

Now,
$$\frac{\frac{1}{\cancel{2}\sqrt{c}}}{\frac{-1}{\cancel{2}\, c^{3/2}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$-\frac{1}{c^{1/2}} \times \frac{c^{3/2}}{1} = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$$

$$\cancel{-}c^{3/2-\frac{1}{2}} = \frac{\cancel{\sqrt{b}-\sqrt{a}}}{1} \times \frac{\sqrt{a}\sqrt{b}}{\cancel{-(\sqrt{b}-\sqrt{a})}}$$

$c = \sqrt{a}\sqrt{b} \Rightarrow \therefore c = \sqrt{ab} \in (a,b)$

$\therefore f(x), g(x)$ functions are verified by cauchy's mean value theorem.

---

⑤ Verify cauchy's mean value theorem for the function $f(x)$ and $f'(x)$ in $[1,e]$ given $f(x) = \log x$.

Sol: $f(x) = \log x$, $f'(x) = \frac{1}{x}$

$g(x) = f'(x) = \frac{1}{x} \Rightarrow g'(x) = \frac{-1}{x^2}$

(i) $f(x), g(x)$ are continuous in $[1,e]$

(ii) $f(x), g(x)$ are differentiable in $(1,e)$

(iii) $g'(c) \neq 0$ then $\exists$ a point $c \in (1,e)$ then

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$f(1) = \log 1$, $f(e) = \log e$

$g(1) = \frac{1}{(1)} = 1$, $g(e) = \frac{1}{e}$.

Now, $$\frac{\frac{1}{c}}{\frac{-1}{c^2}} = \frac{\log e - \log 1}{\frac{1}{e}-1} = \frac{\log_e e - 0}{\frac{1-e}{e}} = \frac{1}{\frac{1-e}{e}}$$

$$\cancel{-}c = \frac{\cancel{-}e}{(e-1)} \rightarrow c = \frac{e}{e-1}$$

$\therefore c = 1.5819 \in (1, e)$.

$\therefore f(x), g(x)$ are verified by cauchy's mean value theorem.

---

⑥. verify generalize mean value theorem for $f(x) = e^x$, $g(x) = e^{-x}$ in $[2,6]$.

Sol: $f(x) = e^x \Rightarrow f'(x) = e^x$

$g(x) = e^{-x} \rightarrow g'(x) = -e^{-x}$.

$f(2) = e^2$, $f(6) = e^6$

$g(2) = -e^{-2}$, $g(6) = e^{-6}$.

(i) $f(x), g(x)$ are continuous in $[2,6]$

(ii) $f(x), g(x)$ are differentiable in $(2,6)$.

(iii) $g'(c) \neq 0$ then $\exists$ a point $c \in (2,6)$ then

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^6 - e^2}{\frac{1}{e^6} - \frac{1}{e^2}} \Rightarrow -e^{2c} = e^4 \times \frac{e^6 \times e^2}{e^2 - e^6}$$

$$\cancel{x} e^{2c} = e^6 \cancel{- e^2} \times \frac{e^6 \times e^2}{\cancel{x}(\cancel{e^6 - e^2})}$$

$$e^{2c} = e^8$$

$\therefore \quad 2c = 8 \Rightarrow \boxed{c = 4} \in (2,6)$

$\therefore$ $f(x)$, $g(x)$ functions are verified by cauchy's mean value theorem.

---

(7) If $f(x) = \log x$ and $g(x) = x^2$ in $[a,b]$ with $b > a > 1$ using cauchy's mean value theorem,

PT, $$\frac{\log b - \log a}{b-a} = \frac{a+b}{2c^2}$$

Sol: $f(x) = \log x$, $f'(x) = \frac{1}{x}$

$g(x) = x^2$, $g'(x) = 2x$

$f(a) = \log a$, $f(b) = \log b$

$g(a) = a^2$, $g(b) = b^2$.

(i) $f(x), g(x)$ are ~~differentiable~~ continuous in $[a,b]$

(ii) $f(x), g(x)$ are differentiable in (a,b)

(iii) $g'(x) \neq 0$ then $\exists$ a point $c \in (a,b)$ such that

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{c}}{2c} = \frac{\log b - \log a}{b^2 - a^2}$$

$$\frac{1}{2c^2} = \frac{\log b - \log a}{(b-a)(b+a)}$$

$$\therefore \frac{\log b - \log a}{b-a} = \frac{a+b}{2c^2} \quad //$$

Taylor's Theorem:-

If $f: [a,b] \to R$ is such that

(i) $f^{n-1}(x)$ is continuous in $[a,b]$.

(ii) $f^{n-1}(x)$ is differentiable in (a,b). (or)

$f^{n}(x)$ is exist in (a,b) and $p \in z^{+}$,

then there exist a point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots\cdots + R_n.$$

where, $R_n = \dfrac{(b-a)^n (b-c)^{n-p} f^n(c)}{p(n-1)!}$ is called as

Roche's form of remainder.

Note:-

→ If $p = n$ then

$$R_n = \frac{(b-a)^n f^n(c)}{n(n-1)!}$$ is called as Lagrange's

form of remainder.

→ If $p=1$ then

$$R_n = \frac{(b-a)^n (b-c)^{n-1} f^n(c)}{(n-1)!}$$ is called as

cauchy's form of remainder.

---

① Find the taylor's series expansion of e^x about $x=1$.

Sol:- Let $f(x) = e^x$ in $[1,x]$.

we know that,

$$f(x) = f(1) + \frac{(x-1)}{1!} f'(x) + \frac{(x-1)^2}{2!} f''(x) + \cdots\cdots \to (1)$$

$f(x) = e^x \rightarrow f(1) = e$
$f'(x) = e^x \rightarrow f'(1) = e$
$f''(x) = e^x \rightarrow f''(1) = e$
$f'''(x) = e^x \rightarrow f'''(1) = e$

from (1), we get

$$e^x = e + \frac{(x-1)}{1!}(e) + \frac{(x-1)^2}{2!}(e) + \frac{(x-1)^3}{3!}(e) + \cdots$$

$$\therefore e^x = e\left[1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$$

---

(2) Find the taylor's series expansion of e^x about $x = -1$.

Sol: Let $f(x) = e^x$ in $[-1, x]$.

Now,

$f(x) = e^x \rightarrow f'(x) = e^x$

$f''(x) = e^x \rightarrow f'''(x) = e^x$

$f(-1) = e^{-1} \rightarrow \frac{1}{e}$ | $f''(-1) = e^{-1} = \frac{1}{e}$

$f'(-1) = e^{-1} \rightarrow \frac{1}{e}$ | $f'''(-1) = e^{-1} = \frac{1}{e}$

Now, $$f(x) = f(-1) + \frac{(x+1)}{1!}f'(-1) + \frac{(x+1)^2}{2!}f''(-1) + \frac{(x+1)^3}{3!}f'''(-1) + \cdots$$

$$f(x) = \frac{1}{e} + \frac{x+1}{1!}\left(\frac{1}{e}\right) + \frac{(x+1)^2}{2!}\left(\frac{1}{e}\right) + \frac{(x+1)^3}{3!}\left(\frac{1}{e}\right) + \cdots$$

$$\therefore e^x = e^{-1}\left[1 + \frac{(x+1)}{1!} + \frac{(x+1)^2}{2!} + \cdots\right]$$

③. Find Taylor's series of $f(x) = \sin 2x$ in powers of $x - \frac{\pi}{4}$.

Sol: Let $f(x) = \sin 2x$ in $\left[\frac{\pi}{4}, x\right]$

$f'(x) = 2\cos 2x$

$f''(x) = -4\sin 2x$

$f'''(x) = -8\cos 2x$

$f^{IV}(x) = 16\sin 2x$

~~$f(0) = \sin 2(0) = 0$~~

~~$f'(0) = 2(1) = 2$~~

~~$f''(0) = -4\sin 2(0) = 0$~~

~~$f'''(0) = -8(1) = -8$~~

~~$f^{IV}(0) = 16(0) = 0$~~

$f\left(\frac{\pi}{4}\right) = \sin 2\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1$

$f'\left(\frac{\pi}{4}\right) = 2\cos 2\left(\frac{\pi}{4}\right) = 2(0) = 0$

$f''\left(\frac{\pi}{4}\right) = -4 \cdot \sin 2\left(\frac{\pi}{4}\right) = -4$

$f'''\left(\frac{\pi}{4}\right) = -8\cos 2\left(\frac{\pi}{4}\right) = 0$.

$f^{IV}\left(\frac{\pi}{4}\right) = 16\sin 2\left(\frac{\pi}{4}\right) = 16$

Now, $f(x) = f\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)}{1!} f'\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots$

$\therefore \sin 2x = 1 + \frac{\left(x - \frac{\pi}{4}\right)}{1!}(0) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!}(-4) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!}(0) + \frac{\left(x - \frac{\pi}{4}\right)^4}{4!}(16) + \cdots$

$\therefore \sin 2x = 1 - \frac{2^2}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{2^4}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots$

---

④. Expand $f(x) = \log \sin x$ about $x = 3$ using taylor's series expansion.

Sol: Let $f(x) = \log(\sin x) \Rightarrow f(3) = \log(\sin(3))$.

$f'(x) = \frac{\cos x}{\sin x} = \cot x \Rightarrow f'(3) = \cot 3. \Rightarrow f''(x) = -\operatorname{cosec}^2 x$

$f''(3) = -\operatorname{cosec}^2 3.$

$f'''(x) = \cancel{2\operatorname{cosec}^2 x \cot x} \quad -2\operatorname{cosec} x\,(-\operatorname{cosec} x \cdot \cot x).$

$f'''(x) = 2\operatorname{cosec}^2 x \cdot \cot x$

$f'''(3) = 2\operatorname{cosec}^2 3 \cot 3$

~~from~~ Now,

$$f(x) = f(3) + \frac{(x-3)}{1!} f'(3) + \frac{(x-3)^2}{2!} f''(3) + \frac{(x-3)^3}{3!} f'''(3) + \cdots$$

$$\therefore \log(\sin x) = \log(\sin 3) + \frac{(x-3)}{1!}\cot 3 - \frac{(x-3)^2}{2!}\operatorname{cosec}^2 3 + \frac{2(x-3)^3}{3!}\operatorname{cosec}^2 3 \cot 3 + \cdots\cdots$$

Maclaurin's Theorem:-

If $f : [0,x] \to R$ such that

(i) f^{n-1} is continuous in $[0,x]$.

(ii) f^{n-1} is differentiable in $(0,x)$ (or) f^n is differentiable in $(0,x)$ and $p \in \mathbb{Z}^+$,

then there exist at least a point $c \in (0,x)$ such that

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots + R_n$$

where,

$$R_n = \frac{(x)^p (x-c)^{n-p} f^n(c)}{p\,(n-1)!}$$ is called as Roche's form of remainder.

Note:-

i) put $p=n$ then $R_n = \frac{x^n f^n(c)}{n(n-1)!}$ is called as lagrange's ~~mean~~ form of remainder.

ii) put $p=1$ then $R_n = \frac{x^n (x-c)^{n-1} f^n(c)}{(n-1)!}$ is called as cauchy's form of remainder.

---

1) Obtain the maclaurin's series of the following functions.

(i) e^x.

Sol: Let $f(x) = e^x$ in $[0,x]$

we have,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots \rightarrow (1).$$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1.$

from (1),

$$e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \cdots\cdots$$

$$\therefore e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots //$$

(ii) $\cos x$

Sol: Let $f(x) = \cos x$ in $[0,x]$

we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots \rightarrow (1)$$

$f(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -\cos(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = \sin 0 = 0$

from eqn ①, we get

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots\cdots,$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots //$$

(iii) $\sin x$.

Sol: Let $f(x) = \sin x$ in $[0, x]$

$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

Now,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots\cdots,$$

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \cdots\cdots$$

$$\therefore \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots //$$

(iv). $\cosh x$.

Sol: Let $f(x) = \cosh x$ in $[0, x]$.

$f(0) = \cosh(0) = 1$

$f'(x) = \sinh x \Rightarrow f'(0) = \sinh(0) = 0$

$f''(x) = \cosh x \Rightarrow f''(0) = \cosh(0) = 1$

$f'''(x) = \sinh x \Rightarrow f'''(0) = \sinh(0) = 0$

$f^{IV}(x) = \cosh x \Rightarrow f^{IV}(0) = \cosh(0) = 1.$

Now, $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$\cosh x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$

$\therefore \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ //

(v) $\sinh x$.

Sol: Let ~~sinhx~~ $f(x) = \sinh x$ in $[0, x]$

$f(x) = \sinh x \Rightarrow f(0) = \sinh(0) = 0$

$f'(x) = \cosh x \Rightarrow f'(0) = \cosh(0) = 1$

$f''(x) = \sinh x \Rightarrow f''(0) = \sinh(0) = 0$

$f'''(x) = \cosh x \Rightarrow f'''(0) = \cosh(0) = 1$

$f^{iv}(x) = \sinh x \Rightarrow f^{iv}(0) = \sinh(0) = 0$

Now,

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$\sinh x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) + \cdots$

$\therefore \sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ //

---

(2) Obtain the maclaurin's series expansion for $f(x) = (1+x)^n$.

Sol: Let $f(x) = (1+x)^n$ in $[0, x]$.

we have

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots \rightarrow (1)$

$f(x) = (1+x)^n \Rightarrow f(0) = (1+0)^n = 1^n = 1$

$f'(x) = n.(1+x)^{n-1} \Rightarrow f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2} \Rightarrow f''(0) = n(n-1)$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \Rightarrow f'''(0) = n(n-1)(n-2)$,

from eq ①, we get

$(1+x)^n = 1 + \frac{x}{1!}(n) + \frac{x^2}{2!}(n(n-1)) + \frac{x^3}{3!}(n(n-1)(n-2)) + \cdots$

$\therefore (1+x)^n = 1 + {}^nC_1(x) + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots\cdots$ $\quad {}^nC_0 = 1$

---

③. Show that $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{4x^3}{3!} + \cdots\cdots$

Sol: Let $f(x) = \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$ in $[0,x]$

we have $f(0) = 0$

$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots\cdots \rightarrow$ ①

$f(x) = \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} \Rightarrow f(x)(\sqrt{1-x^2}) = \sin^{-1}(x)$

$\Rightarrow$ Diff on BS

$f'(x)(\sqrt{1-x^2}) + f(x).(-\cancel{2}x)\frac{1}{\cancel{2}\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$

$f'(x)(1-x^2) + (-\cancel{x}x)f(x) = 1$

$\therefore f'(x)(1-x^2) - x f(x) = 1 \rightarrow$ ②

$f'(0)(1-0^2) - (0)f(0) = 1$

$f'(0) = 1$

Diff on B.S in eq ②,

$f''(x)(1-x^2) - 2x f'(x) - x f'(x) - f(x) = 0$

$f''(x)(1-x^2) - 3x f'(x) - f(x) = 0 \rightarrow$ ③

$f''(0)(1-0^2) - 3(0)f'(0) - f(0) = 0$

$\therefore f''(0) = 0$

Diff on B.S eq (5)

$f''(x)(-2x) + (1-x^2)f'''(x) - 3[xf''(x) + f'(x)] - f'(x) = 0$

$\therefore f'''(x)(1-x^2) - 5xf''(x) - 4f'(x) = 0$

$f'''(0)(1-0^2) - 5(0)f''(0) - 4f'(0) = 0$

$f'''(0) = 0 + 4(1) = 4$

from eq (1), we get

$$\therefore \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(4) + \cdots$$

$$\therefore \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} = \frac{x}{1!} + \frac{4x^3}{3!} + \cdots$$

$$\therefore \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} = x + \frac{4x^3}{3!} + \cdots$$

---

(4). Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$ and, Hence deduce that $\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$

Sol: Let $f(x) = \log_e(1+e^x)$ in $[0,x]$

$f(0) = \log(1+e^0) = \log(1+1) = \log 2$

Diff on both sides

$f'(x) = \frac{e^x}{1+e^x} \Rightarrow f'(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$

$\Rightarrow f'(x)[1+e^x] = e^x$

Diff on BS.

$f'(x)(e^x) + (1+e^x)f''(x) = e^x$

$f''(x)(1+e^x) + f'(x)e^x = e^x$ → (2)

Diff on B.S

$f''(x)e^x + (1+e^x)f'''(x) + f'(x)e^x + e^x f''(x) = e^x$

$f'''(x)(1+e^x) + 2e^x f''(x) + f'(x)e^x = e^x \rightarrow$ (a)

Now,

$\Rightarrow f''(0)(1+e^0) + f'(0)e^0 = e^0$

$2f''(0) = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow f''(0) = \frac{1}{4}$

$\Rightarrow f'''(0)(1+e^0) + 2e^0 f''(0) + f'(0)e^0 = e^0$

$f'''(0)(2) = \frac{-1}{2} - \frac{1}{2} + 1 = \cancel{\frac{-2}{2}} + \cancel{1} = 0$

$\therefore f'''(0) = 0$

Now,

$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \rightarrow$ (b)

Diff on BS eqn (a)

$f'''(x)e^x + (1+e^x)f^{iv}(x) + 2[e^x f'''(x) + f''(x)e^x]$
$+ e^x f''(x) + f'(x)e^x = e^x$

$f^{iv}(x)(1+e^x) + 3e^x f'''(x) + 3e^x f''(x) + f'(x)e^x = e^x$

Now, $f^{iv}(0)(1+e^0) + 3e^0 f'''(0) + 3e^0 f''(0) + f'(0)e^0 = e^0$

$f^{iv}(0)(2) + 3(1)(0) + 3(1)\left(\frac{1}{4}\right) + \frac{1}{2}(1) = 1$

$f^{iv}(0)(2) = \frac{-3}{4} - \frac{1}{2} + 1 = \frac{-5+4}{4} = \frac{-1}{4}$

$f^{iv}(0) = \frac{-1}{8}$.

Now, from eqn (b),

$\log(1+e^x) = \log 2 + \frac{x}{1!}\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(\frac{-1}{8}\right) + \cdots$

$\therefore \log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$

Diff on B.S.

$$\frac{e^x}{1+e^x} = 0 + \frac{1}{2} + \frac{2x}{8} - \frac{4x^3}{192} + \cdots$$

$$\therefore \frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

---

(5). Expand $e^{x\sin x}$ in powers of x.

Sol: Let $f(x) = e^{x\sin x}$ in $[0,x]$

we have $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots \longrightarrow (1)$

$f(x) = e^{x\sin x} \Rightarrow f(0) = e^{0\sin 0} = e^0 = 1$

$f'(x) = e^{x\sin x}[x\cos x + \sin x]$

$\therefore f'(0) = e^{0\sin 0}[0\cos 0 + \sin 0] = 0.$

$f''(x) = f(x)[-x\sin x + \cos x + \cos x] + [x\cos x + \sin x]f'(x)$

$f''(x) = f(x)[2\cos x - x\sin x] + f'(x)[x\cos x + \sin x]$

$f''(0) = f(0)[2\cos 0 - 0\sin 0] + f'(0)[0\cos 0 + \sin 0]$

$f''(0) = 1[2(1) - 0] + 0(0) = 2$

$\therefore f''(0) = 2.$

$f'''(x) = f(x)[-2\sin x - (x\cos x + \sin x)]$
$\quad + [2\cos x - x\sin x]f'(x) + f'(x)[-x\sin x + \cos x + \cos x]$
$\quad + (x\cos x + \sin x)f''(x).$

$f'''(x) = f(x)[-x\cos x - 3\sin x] + f'(x)[2\cos x - x\sin x]$
$\quad + f'(x)[2\cos x - x\sin x] + f''(x)[x\cos x + \sin x]$

$$f'''(x) = f(x)\,[-x\cos x - 3\sin x] + 2f'(x)\,[2\cos x - x\sin x] + f''(x)\,[x\cos x + \sin x]$$

$$f'''(0) = f(0)\,[-0\cos 0 - 3\sin 0] + 2f'(0)\,[3\cos 0 - 0\sin 0] + f''(0)\,[0\cos 0 + \sin 0]$$

$$f'''(0) = 1(0-0) + 2(0)\,[2-0) + 2\,[0+0].$$

$$f'''(0) = 0+0+0 = 0. \Rightarrow f'''(0) = 0$$

from eq ①,

$$e^{x\sin x} = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(0) + \cdots\cdots$$

$$e^{x\sin x} = 1 + \frac{2x^2}{2!} + \cdots\cdots$$

$$\therefore\ e^{x\sin x} = 1 + x^2 + \cdots\cdots$$

---

⑥. Verify taylor's theorem for $f(x) = (1-x)^{5/2}$ with lagrange's form of remainder up to 2 terms in the interval $[0,1]$.

Sol:- given that, $f(x) = (1-x)^{5/2}$ in $[0,x]$.

$\therefore$ $f'(x)$ is exist in $c \in (0,1)$.

$\therefore$ It is continuous in $[0,1]$.

It is differentiable in $(0,1)$. Then $\exists$ a point $c \in (0,1)$ such that

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(cx) \longrightarrow ①$$

$$f(x) = (1-x)^{5/2} \Rightarrow f(0) = (1-0)^{5/2} = 1^{5/2} = 1$$

$$f'(x) = \frac{5}{2}(1-x)^{5/2-1}(-1) = \frac{-5}{2}(1-x)^{3/2}$$

$$f'(0) = \frac{-5}{2}(1-0)^{3/2} = \frac{-5}{2}.$$

$f'''(x) = \frac{-5}{2} \times \frac{3}{2} \times (1-x)^{3/2-1} (-1)$

$f'''(x) = \frac{15}{4} (1-x)^{1/2}.$

$f''(0) = \frac{15}{4}((1-0))^{1/2} = \frac{15}{4}.$

$f''(cx) = \frac{15}{4}(1-cx)^{1/2} = \frac{15}{4}\sqrt{1-cx}$

from eq ①,

$$(1-x)^{5/2} = 1 + \frac{x}{1!}\left(\frac{-5}{2}\right) + \frac{x^2}{2!}\left(\frac{15}{4}\sqrt{1-cx}\right)$$

put $x=1$,

$$(1-1)^{5/2} = 1 + \frac{(1)}{1!}\left(\frac{-5}{2}\right) + \frac{(1)^2}{2!}\left(\frac{15}{4}\sqrt{1-c(1)}\right)$$

$$0 = 1 - \frac{5}{2} + \frac{1}{2} \times \frac{15}{4}\sqrt{1-c}$$

$$\frac{\cancel{15}^{5}}{\cancel{8}_{4}}\sqrt{1-c} = \frac{3}{2} \Rightarrow 5\sqrt{1-c} = 4$$

S.O.B.S.

$25(1-c) = 16$

$25 - 25c = 16$

$25c = 25 - 16$

$25c = 9$

$c = \frac{9}{25} \Rightarrow c = 0.36$ in $(0,1)$.

∴ Hence, $f(x)$ is verified the taylor's theorem with lagranges form of remainder up to 2 terms in the interval $[0,1]$.

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⑦. Write taylor's theorem (series) for $f(x) = (1-x)^{5/2}$ with lagrange's form of remainder upto 3 terms in $[0,1]$.

Sol: $f(x) = (1-x)^{5/2}$ in $[0,x]$.

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(cx) \rightarrow ①$$

$$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$$

$$f'(x) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}$$

$$f''(x) = \frac{15}{4}\sqrt{1-x} \Rightarrow f''(0) = \frac{15}{4}$$

$$f'''(x) = \frac{-15}{8} \frac{1}{\sqrt{1-x}}$$

$$f'''(cx) = \frac{-15}{8} \frac{1}{\sqrt{1-cx}}$$

from eq ①,

$$(1-x)^{5/2} = 1 + \frac{x}{1!}\left(\frac{-5}{2}\right) + \frac{x^2}{2!}\left(\frac{15}{4}\right) + \frac{x^3}{3!}\left(\frac{-15}{8} \times \frac{1}{\sqrt{1-cx}}\right)$$

put $x=1$,

$$(1-1)^{5/2} = 1 + \frac{1}{1!}\left(\frac{-5}{2}\right) + \frac{(1)^2}{2!}\left(\frac{15}{4}\right) + \frac{(1)^3}{3!}\left(\frac{-15}{8} \times \frac{1}{\sqrt{1-c(1)}}\right)$$

$$0 = 1 - \frac{5}{2} + \frac{15}{8} - \frac{15}{48} \times \frac{1}{\sqrt{1-c}}$$

$$\frac{15}{48} \times \frac{1}{\sqrt{1-c}} = \frac{15}{8} + 1 - \frac{5}{2} = \frac{15+8-20}{8}$$

$$\frac{\cancel{15}^{5}}{\cancel{48}_{6}} \times \frac{1}{\sqrt{1-c}} = \frac{\cancel{3}}{\cancel{8}} \Rightarrow \frac{5}{6\sqrt{1-c}} = 1$$

$$5 = 6\sqrt{1-c}$$

S.O.B.S

$$25 = 36(1-c)$$

$$36c = 36 - 25$$

$$c = 11/36$$

$$\therefore c = 0.305 \in (0,1)$$