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Beta and Gamma Functions

Beta Function:-

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called "Beta Function." and it is denoted by $B(m,n)$ and read as "Beta m,n". The above integral converges for $m>0$ and $n>0$.

Thus $B(m,n) = \int_0^{\infty} x^{m-1}(1-x)^{n-1}dx$ and also is called "Eulerian integral of the first kind".

Properties of Beta Function:-

(i) Symmetry of Beta Function i.e, $B(m,n) = B(n,m)$.

Proof:- By the definition of Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

put $1-x = t$ & diff wr to 't' on both sides.

$$-\frac{dx}{dt} = (1) \Rightarrow dx = -dt$$

$LL \Rightarrow$ if $x=0 \Rightarrow t = 1-0 = 1$

$UL \Rightarrow$ if $x=1 \Rightarrow t = 1-1 = 0$

$$\therefore B(m,n) = \int_1^0 (1-t)^{m-1}(t)^{n-1}(-dt)$$

$$B(m,n) = -\int_1^0 (t)^{n-1}(1-t)^{m-1}dt$$

$$B(m,n) = \int_0^1 (t)^{n-1}(1-t)^{m-1}dt = B(n,m)$$

$\therefore \boxed{B(m,n) = B(n,m)}$

We know that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii). Prove that, $B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2n-1}\, d\theta$

Proof :- By the definition $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, d$

given , $B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta.$

put $x = \sin^2\theta$

$\frac{dx}{d\theta} = 2\sin\theta\cos\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta.$

LL $\Rightarrow$ if $x = 0 \Rightarrow 0 = \sin^2\theta \Rightarrow \cancel{\sin}\theta = 0 = \cancel{\sin}\, 0^\circ$

$\boxed{\theta = 0^\circ}$

UL $\Rightarrow$ if $x = 1 \Rightarrow 1 = \sin^2\theta \Rightarrow \cancel{\sin}\theta = 1 = \cancel{\sin}\, 90^\circ$

$\boxed{\theta = \frac{\pi}{2}}$

$\therefore B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$

$B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-2} \sin\theta \cdot (\cos\theta)^{2n-2} \cdot \cos\theta\, d\theta$

$B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-2+1} (\cos\theta)^{2n-2+1} \cdot d\theta$

$B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2n-1} \cdot d\theta.$

$$\therefore B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2n-1}\, d\theta.$$

(iii). $B(m,n) = B(m+1,n) + B(m,n+1)$.

Sol:- Given that $B(m,n) = B(m+1,n) + B(m,n+1)$

$$RHS = B(m+1,n) + B(m,n+1)$$

$$RHS = \int_0^1 x^{m+\cancel{1}-\cancel{1}} (1-x)^{n-1}\, dx + \int_0^1 x^{m-1} (1-x)^{n+\cancel{1}-\cancel{1}} \cdot dx$$

$$RHS = \int_0^1 x^m (1-x)^{n-1}\, dx + \int_0^1 x^{m-1} (1-x)^n\, dx$$

$$RHS = \int_0^1 x^m (1-x)^n\, dx \left(\frac{1}{1-x} + \frac{1}{x}\right)$$

$$RHS = \int_0^1 x^m (1-x)^n\, dx \cdot \left(\frac{\cancel{x}+1-\cancel{x}}{x(1-x)}\right)$$

$$RHS = \int_0^1 x^m \cdot x^{-1} \cdot (1-x)^n \cdot (1-x)^{-1} \cdot dx$$

$$RHS = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx.$$

$RHS = B(m,n) = LHS.$

$\therefore LHS = RHS.$

$\therefore B(m,n) = B(m+1,n) + B(m,n+1)$ //

(iv). To show that $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dx.$$

Sol: By the definition $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

(or) $B(m,n) = \int_0^1 t^{m-1} (1-t)^{n-1} dt.$

put $t = \frac{1}{1+x} \Rightarrow \frac{1}{t} = 1+x \Rightarrow x = \frac{1}{t} - 1$

$\frac{-1}{t^2} \cdot \frac{dt}{dx} = 1 \Rightarrow -dt = t^2 dx \Rightarrow dt = \frac{-1}{(1+x)^2} dx.$

LL $\Rightarrow$ if $t = 0$, $x = \infty$

UL $\Rightarrow$ if $t = 1$, $x = 0$

$$\therefore B(m,n) = \int_\infty^0 \left(\frac{1}{1+x}\right)^{m-1} \left(1 - \frac{1}{1+x}\right)^{n-1} \cdot \frac{-1}{(1+x)^2} dx$$

$$B(m,n) = \int_\infty^0 \left(\frac{1}{1+x}\right)^{m-1} \cdot \left(\frac{\cancel{1}+x-\cancel{1}}{1+x}\right)^{n-1} \cdot \frac{-1}{(1+x)^2} dx$$

$$B(m,n) = \int_0^\infty \frac{1}{(1+x)^{m-1}} \cdot \frac{x^{n-1}}{(1+x)^{n-1}} \cdot \frac{1}{(1+x)^2} dx$$

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m\cancel{-1}+n\cancel{-1}+\cancel{2}}} dx. = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\therefore B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

① Express the following integrals in terms of Beta function.

(i) $\int_0^1 \frac{x}{\sqrt{1-x^2}} dx$

Sol: $= \int_0^1 x (1-x^2)^{-1/2} dx$

put $x^2 = t \Rightarrow x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}} \cdot dt$

LL $\Rightarrow$ if $x=0$, $t=0$

UL $\Rightarrow$ if $x=1$, $t=1$

$= \int_0^1 \cancel{\sqrt{t}} \cdot (1-t)^{-1/2} \cdot \frac{1}{2\cancel{\sqrt{t}}} \cdot dt$

$= \frac{1}{2} \int_0^1 t^0 \cdot (1-t)^{-1/2} dt \rightarrow ①$

Comparing the eq ① with $\int_0^1 x^{m-1} (1-x)^{n-1} dx$

we get, $m-1=0$, $n-1 = \frac{-1}{2}$

$\boxed{m=1}$, $n = 1-\frac{1}{2} \Rightarrow \boxed{n=\frac{1}{2}}$

$= \frac{1}{2} \int_0^1 t^{1-1} (1-t)^{\frac{1}{2}-1} dt = \frac{1}{2} B(1, \frac{1}{2})$.

$\therefore \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \frac{1}{2} B(1, \frac{1}{2})$ //

(ii) $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx.$

Sol: $\sqrt{9-x^2} = \sqrt{9\left(1-\left(\frac{x}{3}\right)^2\right)} = 3\sqrt{1-\left(\frac{x}{3}\right)^2}$

$$= \int_0^3 \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\cdot dx = \frac{1}{3}\int_0^3 \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx.$$

$$= \frac{1}{3}\int_0^3 \left[1-\left(\frac{x}{3}\right)^2\right]^{-1/2} dx$$

put $\left(\frac{x}{3}\right)^2 = t \Rightarrow x^2 = 3^2 t \Rightarrow x = 3\sqrt{t}.$

$dx = \frac{3}{2\sqrt{t}}\cdot dt.$

LL $\Rightarrow$ if $x=0$, $t=0$

UL $\Rightarrow$ if $x=3$, $t=1$

$$= \frac{1}{\cancel{3}}\int_0^1 (1-t)^{-1/2}\frac{\cancel{3}}{2\sqrt{t}}\,dt.$$

$$= \frac{1}{2}\int_0^1 t^{-1/2}\cdot(1-t)^{-1/2}\,dt \rightarrow (1)$$

Comparing eq (1) with $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$m-1 = \frac{-1}{2}$, $\quad n-1 = \frac{-1}{2}$

$\boxed{m=\frac{1}{2}}$ $\quad \boxed{n=\frac{1}{2}}$

$$= \frac{1}{2}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{1}{2}-1}\,dt = \frac{1}{2}B\left(\frac{1}{2},\frac{1}{2}\right).$$

$$\therefore \int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \frac{1}{2}B\left(\frac{1}{2},\frac{1}{2}\right)$$ //

② PT, $\int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx = \frac{1}{5}B\left(\frac{2}{5},\frac{1}{2}\right)$.

Sol:- $= \int_0^1 x(1-x^5)^{-1/2}\,dx$.

put $x^5 = t \Rightarrow x = (t)^{1/5}$

$dx = \frac{1}{5}t^{-4/5}\,dt$

LL $\Rightarrow$ if $x=0$, $t=0$

UL $\Rightarrow$ if $x=1$, $t=1$

$$= \int_0^1 t^{1/5}(1-t)^{-1/2} \cdot \frac{1}{5} \cdot t^{-4/5}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{\frac{1}{5}-\frac{4}{5}}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{5}\int_0^1 t^{-3/5}(1-t)^{-1/2}\,dt \rightarrow ①$$

Comparing eq ① with $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$.

$m-1 = \frac{-3}{5}$, $n-1 = \frac{-1}{2}$

$m = 1-\frac{3}{5} \Rightarrow \boxed{m = \frac{2}{5}}$ $\boxed{n = \frac{1}{2}}$

$$= \frac{1}{5}\int_0^1 t^{\frac{2}{5}-1}(1-t)^{\frac{1}{2}-1}\,dt = \frac{1}{5}B\left(\frac{2}{5},\frac{1}{2}\right)$$

$$\therefore \int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx = \frac{1}{5}B\left(\frac{2}{5},\frac{1}{2}\right)$$

③. $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx = \frac{1}{5} B\left(\frac{3}{5}, \frac{1}{2}\right)$.

Sol:- $= \int_0^1 x^2 (1-x^5)^{-1/2}\,dx$

put $x^5 = t \Rightarrow x = (t)^{1/5}$

$dx = \frac{1}{5} t^{-4/5} \cdot dt \Rightarrow x^2 = t^{2/5}$.

LL $\Rightarrow$ if $x=0$, $t=0$

UL $\Rightarrow$ if $x=1$, $t=1$

$= \frac{1}{5}\int_0^1 (t)^{-2/5} (1-t)^{-1/2}\,dt \rightarrow ①$

Comparing eq ① with $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$.

$m-1 = \frac{-2}{5}$ $\qquad$ $n-1 = \frac{-1}{2}$

$m = 1 - \frac{2}{5} = \frac{3}{5}$ $\qquad$ $\boxed{n = \frac{1}{2}}$

$\boxed{m = \frac{3}{5}}$

$= \frac{1}{5}\int_0^1 t^{\frac{3}{5}-1}(1-t)^{\frac{1}{2}-1}\,dt = \frac{1}{5} B\left(\frac{3}{5}, \frac{1}{2}\right)$.

$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx = \frac{1}{5} B\left(\frac{3}{5}, \frac{1}{2}\right)$ //

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④. Show that $\int_a^b (x-a)^m (b-x)^n\,dx = (b-a)^{m+n+1} \cdot B(m+1, n+$

Sol:- given $\int_a^b (x-a)^m (b-x)^n\,dx = (b-a)^{m+n+1} \cdot B(m+1, n+1)$

put $x = a + (b-a)t \Rightarrow dx = (b-a)\,dt$.

$LL \rightarrow$ if $x = a$, $\quad \cancel{a} = \cancel{a} + (b-a)t$

$$(b-a)t = 0 \Rightarrow t = 0$$

$UL \Rightarrow$ if $x = b$, $\quad b = a + (b-a)t \Rightarrow t = \frac{\cancel{b-a}}{\cancel{b-a}} \Rightarrow t = 1.$

$$= \int_0^1 (\cancel{a} + (b-a)t - \cancel{a})^m \, (b - a - (b-a)t)^n \, (b-a) \, dt$$

$$= \int_0^1 (b-a)^m t^m ((b-a)^n (1-t)^n) \, (b-a) \, dt$$

$$= \int_0^1 (b-a)^{m+n+1} \, t^m \, (1-t)^n \, dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt \rightarrow (1)$$

Comparing eqn (1) with $\int_0^1 x^{m-1} (1-x)^{n-1} dx$

we get $\quad m = m-1 \quad | \quad n = n-1$

$\quad\quad\quad\quad m = m+1 \quad | \quad n = n+1$

$$= (b-a)^{m+n+1} \int_0^1 t^{m+1-1} \, (1-t)^{n+1-1} \, dt$$

$$= (b-a)^{m+n+1} \cdot B(m+1, n+1).$$

$$\therefore \int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} \cdot B(m+1, n+1)$$

Gamma Function :-

The definite integral $\int_0^\infty e^{-x} \cdot (x^{n-1}) \cdot dx$ is called as Gamma function and it is denoted by $\Gamma(n)$ for and read as "Gamma of n". Thus

$\Gamma(n) = \int_0^\infty e^{-x} (x^{n-1})\, dx$. the integral converges for $n>0$. Also is called as "Elerian Integral of the second kind".

Important Results:-

① Prove that, $\Gamma(1) = 1$.

Proof:- By the definition $\Gamma(n) = \int_0^\infty e^{-x} (x^{n-1})\, dx$.

put $n=1$.

$$\Gamma(1) = \int_0^\infty e^{-x} (x^{1-1})\, dx \qquad \int e^{ax}\, dx = \frac{e^{ax}}{a}$$

$$\Gamma(1) = \int_0^\infty e^{-x}\, x^0\, dx$$

$$\Gamma(1) = \left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$\Gamma(1) = -\left[e^{-\infty} - e^0\right] = -\left[\frac{1}{e^\infty} - e^0\right] = -[0-1] = 1$$

$\therefore \Gamma(1) = 1$ //

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② To prove that $\Gamma(n) = (n-1)\Gamma(n-1)$.

Sol:- By the definition $\int_0^\infty e^{-x} (x^{n-1})\, dx$.

By using ILATE

$$\begin{array}{lcccc} D = & x^2 & 2x & 2 & 0 \\ I = & e^x & e^x & e^x & e^x \end{array}$$

for $\int e^x x^2\, dx$

$= x^2 e^x - 2e^x x + 2e^x$

$\Rightarrow \int u\,dv = uv - \int v\,du$

$$\begin{array}{l|l} D = x^{n-1} & (n-1)x^{n-2} \\ I = e^{-x} & -e^{-x} \end{array}$$

$$\therefore \Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$$\Gamma(n) = \left[-x^{n-1} e^{-x}\right]_0^\infty + (n-1)\int_0^\infty e^{-x} x^{n-2}\, dx.$$

$$\Gamma(n) = -\left[\infty^{n-1} e^{-\infty} - 0^{n-1} e^{-0}\right] + (n-1)\int_0^\infty e^{-x} x^{n-2}\, dx$$

$$\Gamma(n) = -[0-0] + (n-1)\,\Gamma(n-1)$$

$$\therefore \Gamma(n) = (n-1)\,\Gamma(n-1) \;//$$

$\Rightarrow \Gamma(n) = (n-1)\,\Gamma(n-1)$

$\Gamma(n+1) = n\,\Gamma(n)$ (or) $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

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③ prove that $\Gamma(n+1) = n!$

Sol: LHS $= \Gamma(n+1)$ $\qquad \Gamma(n+1) = n\,\Gamma(n)$

$= n\,\Gamma(n)$ $\qquad \Gamma(0) = \infty.$

$= n(n-1)\,\Gamma(n-1)$

$= n(n-1)(n-2)\,\Gamma(n-2)$

$= n(n-1)(n-2)(n-3)\,\Gamma(n-3)$

$= n(n-1)(n-2)(n-3) \cdots\cdots \Gamma(n-r)$

$= n(n-1)(n-2)(n-3) \cdots \times 3 \times 2 \times 1$

$= n! =$ RHS.

$\therefore \Gamma(n+1) = n!$

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(4). An important relation between Beta and Gamma functions.

(or)

Prove that $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

Sol: By the defn $\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx$

put $x = yt$

$dx = y(1)\,dt \rightarrow dx = y\,dt$

$$\Gamma(m) = \int_0^\infty e^{-yt} \cdot (yt)^{m-1} \cdot y\,dt$$

$$\Gamma(m) = \int_0^\infty e^{-yt} \cdot t^{m-1} \cdot dt \cdot y^m$$

$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yt}\, t^{m-1} dt$$

Multiplying with $e^{-y} \cdot y^{m+n-1}$ and integrating w.r.to y and taking limits as 0 to ∞.

$$\Rightarrow \frac{\Gamma(m)}{y^m} \cdot \int_0^\infty e^{-y} \cdot y^{m+n-1} dy = \int_0^\infty\int_0^\infty e^{-yt} \cdot t^{m-1} \cdot dt \cdot e^{-y} \cdot y^{m+n-1} \cdot d$$

$$LHS = \Gamma(m) \int_0^\infty e^{-y} \cdot y^{n-1} \cdot dy = \Gamma(m) \cdot \Gamma(n).$$

$$RHS = \int_0^\infty e^{-yt} \cdot t^{m-1} \cdot dt \int_0^\infty e^{-y} \cdot y^{m+n-1} dy$$

$$RHS = \int_0^\infty t^{m-1} \cdot dt \int_0^\infty e^{-(1+t)y} \cdot y^{m+n-1} \cdot dy$$

put $(1+t)y = x \Rightarrow y = \frac{x}{1+t}$.

$x = (1+t)y$

diff wr to y.

$$\frac{dx}{dy} = (1+t) \Rightarrow dx = (1+t)dy$$

$$dy = \frac{dx}{1+t}.$$

$$RHS = \int_0^\infty t^{m-1}.dt \int_0^\infty e^{-x}.\left(\frac{x}{1+t}\right)^{m+n-1} . \frac{dx}{(1+t)}$$

$$RHS = \int_0^\infty t^{m-1}.dt . \frac{1}{(1+t)^{m+n}} . \int_0^\infty e^{-x}(x)^{m+n-1}.dx$$

$$RHS = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt \int_0^\infty e^{-x} . x^{m+n-1}.dx$$

$$RHS = B(m,n) . \Gamma(m+n)$$

$\Rightarrow$ LHS = RHS

$$\Gamma(m)\,\Gamma(n) = B(m,n) . \Gamma(m+n)$$

$$\therefore \quad B(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)} \quad //$$

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(5) Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Sol: We have $\quad B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

$\Gamma(1) = 1$

put $m = n = \frac{1}{2}$

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2}).\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{(\Gamma(\frac{1}{2}))^2}{\Gamma(1)} = (\Gamma(\tfrac{1}{2}))^2$$

$$\therefore B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = (\Gamma(\tfrac{1}{2}))^2 \rightarrow (1)$$

We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

put $m = n = \frac{1}{2}$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx$$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{-1/2}(1-x)^{-1/2}dx = \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}dx.$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

LL $\Rightarrow$ if $x = 0$, $\theta = 0$

UL $\Rightarrow$ if $x = 1$, $\theta = \frac{\pi}{2}$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^{\pi/2} \frac{1}{\cancel{\sin\theta}\cdot\cancel{\cos\theta}} \cdot 2\cancel{\sin\theta}\cancel{\cos\theta}\, d\theta = 2\int_0^{\pi/2} d\theta$$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \cancel{2}\left[\frac{\pi}{\cancel{2}} - 0\right] = \pi \rightarrow (2)$$

from eq^n (1) & (2), we get

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \Rightarrow \therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$ //

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(6) Prove that $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Sol: $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2.\Gamma\left(\frac{1}{2}\right)$$

$$\therefore \Gamma\left(-\frac{1}{2}\right) = -2.\sqrt{\pi}$$ //

7. Prove that $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$.

Sol:- we have $B(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$ and

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

from the above two equations,

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

put $m+n=1$

$m = 1-n$

$$\frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{1}}\,dx$$

$\downarrow\ \Gamma(1)=1$

$$\Gamma(1-n)\,\Gamma(n) = \int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx \rightarrow (1)$$

we know that $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n}\operatorname{cosec}\left(\frac{2m+1}{2n}\right)\pi.$

put $x^{2n} = t \Rightarrow x = t^{\frac{1}{2n}}$

$$dx = \frac{1}{2n}\cdot t^{\frac{1}{2n}-1}\,dt.$$

$$\int_0^\infty \frac{\left(t^{\frac{1}{2n}}\right)^{2m}}{(1+t)}\cdot\frac{1}{2n}\cdot t^{\frac{1}{2n}-1}\cdot dt = \frac{\pi}{2n}\operatorname{cosec}\left(\frac{2m+1}{2n}\right)\pi$$

$$\cancel{\frac{1}{2n}}\int_0^\infty \frac{t^{\frac{2m}{2n}+\frac{1}{2n}-1}}{(1+t)}\cdot dt = \cancel{\frac{1}{2n}}\cdot\pi\operatorname{cosec}\left(\frac{2m+1}{2n}\right)\pi$$

$$\int_0^\infty \frac{t^{\frac{2m+1}{2n}-1}}{(1+t)}\,dt = \frac{\pi}{\sin\left(\frac{2m+1}{n}\right)\pi}$$

$$\int_0^\infty \frac{t^{\frac{2m+1}{2n}-1}}{(1+t)}\,dt = \frac{\pi}{\sin\left(\frac{2m+1}{2n}\right)\pi}$$

$\downarrow \Gamma(\tfrac{1}{2})$.

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi} \rightarrow (2)$$

Comparing eq's (1) & (2) we get

$$\therefore \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

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(8) Find the values of

(i) $\Gamma\left(\frac{11}{2}\right)$

Sol:- we know that, $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\Gamma\left(\frac{11}{2}\right) = \left(\frac{11}{2}-1\right)\Gamma\left(\frac{11}{2}-1\right)$$

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\Gamma\left(\frac{9}{2}\right)$$

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)$$

$$\therefore \Gamma\left(\frac{11}{2}\right) = \frac{945\sqrt{\pi}}{32}$$

(ii) $\Gamma\left(\frac{5}{2}\right)$

Sol:- $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

(iii). $\Gamma(4.5)$

Sol:- $4.5 = \frac{45}{10} = \frac{9}{2}$

$$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{9}{2}\right) = \frac{105\sqrt{\pi}}{16}$$

8v) $\Gamma(10)$.

Sol:- $\Gamma(9+1) = 9!$ $\quad\downarrow\quad$ $\Gamma(n+1) = n!$

$$\Gamma(10) = (10-1)\Gamma(9-1)$$

$$\Gamma(10) = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times \Gamma(1) \times 1$$

$$\Gamma(10) = 9! \;//$$

---

9) Find the values of $B(2.5, 1.5)$

Sol:- $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ $\quad\downarrow\quad$ $\frac{25}{10} = \frac{5}{2}, \; \frac{15}{10} = \frac{3}{2}$

$$B(2.5, 1.5) = \frac{\Gamma(2.5)\,\Gamma(1.5)}{\Gamma(2.5+1.5)} = \frac{\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}$$

$$\therefore \; B(2.5, 1.5) = \frac{\frac{3}{4}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{3 \times 2 \times 1} = \frac{\pi}{16} \;//$$

---

10). To show that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Sol:- weknow that $\int_0^\infty e^{-x} x^{n-1}\,dx = \Gamma(n)$

put $x^2 = t \Rightarrow x = \sqrt{t}$

$$dx = \frac{1}{2\sqrt{t}}\,dt$$

$$= \int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^\infty e^{-t} \cdot (t)^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} \cdot (t)^{\frac{1}{2}-1}\,dt = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \;//$$

---

(11) Show that $\int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}$

Sol:- put $x^2 = t \Rightarrow x = \sqrt{t}$

$$dx = \frac{1}{2\sqrt{t}} dt$$

$$x^4 = (\sqrt{t})^4 = t^2$$

$$= \int_0^\infty t^2 \cdot e^{-t} \cdot \frac{1}{2\sqrt{t}} dt = \frac{1}{2}\int_0^\infty t^{2-\frac{1}{2}} \cdot e^{-t} dt$$

$$= \frac{1}{2}\int_0^\infty t^{\frac{3}{2}} e^{-t} dt = \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{\frac{5}{2}-1} dt$$ $n-1$ n

$$= \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{1}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{8}$$

$$\therefore \int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}$$

---

(12) Prove that $\int_0^\infty e^{-y^{1/m}} dy = m\Gamma(m)$.

Sol: put $y^{\frac{1}{m}} = t \Rightarrow y = t^m$

$$dy = m t^{m-1} dt$$

$$= \int_0^\infty e^{-t} m \cdot t^{m-1} dt = m\int_0^\infty e^{-t} \cdot t^{m-1} dt = m\Gamma(m)$$

$$\therefore \int_0^\infty e^{-y^{1/m}} dy = m\Gamma(m)$$

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(13) Evaluate $\int_0^\infty x^2 e^{-x^2} dx$

Sol:- put $x^2 = t \Rightarrow x = \sqrt{t}$

$$dx = \frac{1}{2\sqrt{t}} dt$$

$$\int_0^\infty e^{-t} . (\sqrt{t})^2 . \frac{1}{2\sqrt{t}} dt = \frac{1}{2} \int_0^\infty e^{-t} . \sqrt{t}\, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} . t^{\frac{1}{2}} dt \rightarrow (1)$$

Comparing eq (1) with Gamma Function

$$n - 1 = \frac{1}{2} \Rightarrow n = 1 + \frac{1}{2} = \frac{3}{2}$$

$$= \frac{1}{2} \int_0^\infty e^{-t} . t^{\frac{3}{2}-1} dt = \frac{1}{2} \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}$$

---

(14). $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$,

Sol:- put $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow \sqrt{x} = \left(t^{\frac{1}{2}}\right)^{\frac{1}{2}} = t^{\frac{1}{4}}$

$$dx = \frac{1}{2\sqrt{t}} dt \qquad \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

$$\int_0^\infty t^{\frac{1}{4}} . e^{-t} . \frac{1}{2\sqrt{t}} dt = \frac{1}{2} \int_0^\infty t^{\frac{1}{4} - \frac{1}{2}} . e^{-t} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} . t^{\frac{3}{4}-1} dt = \frac{1}{2} \int_0^\infty e^{-t} . t^{\frac{3}{4}-1} dt = \frac{1}{2} \Gamma\left(\frac{3}{4}\right)$$

$$\Rightarrow \frac{1}{2} \Gamma\left(\frac{3}{4}\right) \Rightarrow \int_0^\infty \sqrt{x}\, e^{-x^2} dx = \frac{1}{2} \Gamma\left(\frac{3}{4}\right)$$

(15) (i) $\int_0^1 x^4 (1-x)^2 dx$.

Sol: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$m-1=4 \quad | \quad n-1=2$
$m=5 \quad | \quad n=3$

$\therefore \int_0^1 x^{5-1}(1-x)^{3-1} = B(5,3) = \dfrac{\Gamma(5)\,\Gamma(3)}{\Gamma(5+3)}$

$= \dfrac{4! \times 2!}{7!} = \dfrac{1}{105}$

$\therefore \int_0^1 x^4 (1-x)^2 dx = \dfrac{1}{105}$ //

(ii) $\int_0^1 x^{5/2}(1-x^2)^{3/2} dx$.

Sol: put $x^2 = t \Rightarrow x = \sqrt{t}$

$dx = \dfrac{1}{2\sqrt{t}} dt$

LL $\Rightarrow$ if $x=0$, $t=0$

UL $\Rightarrow$ if $x=1$, $t=1$

$\int_0^1 (t^{\frac{1}{2}})^{5/2} (1-t)^{3/2} \cdot \dfrac{1}{2\sqrt{t}} dt$

$= \dfrac{1}{2}\int_0^1 t^{\frac{5}{4}-\frac{1}{2}} \cdot (1-t)^{\frac{3}{2}} dt$

$= \dfrac{1}{2}\int_0^1 t^{\frac{3}{4}} (1-t)^{\frac{3}{2}} dt$

$m-1 = \dfrac{3}{4} \Rightarrow m = \dfrac{7}{4}$, $n-1 = \dfrac{3}{2} \Rightarrow n = \dfrac{5}{2}$.

$$= \frac{1}{2}\int_0^1 t^{\frac{7}{4}-1}(1-t)^{\frac{5}{2}-1}\,dt$$

$$= \frac{1}{2} B\left(\frac{7}{4}, \frac{5}{2}\right) = \frac{1}{2} \times \frac{\Gamma\left(\frac{7}{4}\right).\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{4}+\frac{5}{2}\right)}$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{7}{4}\right)\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{17}{4}\right)}$$

$$= \frac{1}{2}\cdot\frac{\frac{3}{4}\cdot\Gamma\left(\frac{3}{4}\right)\cdot\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)}{\frac{13}{4}\times\frac{9}{4}\times\frac{5}{4}\times\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{3^2}{25}\times\frac{\Gamma\left(\frac{3}{4}\right).\Gamma\left(\frac{1}{2}\right)\times 2^8}{13\times 9\times 5\times 1\times\Gamma\left(\frac{1}{4}\right)} = \frac{3^2\times \cancel{2^8}\,2^3}{\cancel{2^8}\times 3^2\times 65}\times\frac{\Gamma\left(\frac{3}{4}\right)\times\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{8}{65}\times\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)}$$

$$\therefore \int_0^1 x^{\frac{5}{2}}(1-x^2)^{3/2}\,dx = \frac{8}{65}\,\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)} \;//$$

(iii) $\int_0^2 x(8-x^3)^{\frac{1}{3}}\,dx$

Sol: $\int_0^2 x\cdot 8^{\frac{1}{3}}\left(1-\left(\frac{x}{2}\right)^3\right)^{\frac{1}{3}}dx$

put $\left(\frac{x}{2}\right)^3 = t \Rightarrow x^3 = 8t$

$x = 2t^{1/3}$

$dx = \frac{2}{3}\cdot t^{-\frac{2}{3}}\,dt$ LL $\Rightarrow$ if $x=0,\ t=0$

UL $\Rightarrow$ if $x=2,\ t=1$

$$= 8^{\frac{1}{3}} \int_0^1 2t^{\frac{1}{3}} (1-t)^{\frac{1}{3}} \cdot \frac{2}{3} t^{-\frac{2}{3}} dt$$

$$= 2 \times 2 \times \frac{2}{3} \int_0^1 t^{\frac{1}{3}-\frac{2}{3}} (1-t)^{\frac{1}{3}} dt .$$

$$= \frac{8}{3} \int_0^1 t^{-\frac{1}{3}} (1-t)^{\frac{1}{3}} dt$$

$m-1 = \frac{-1}{3}$ | $n-1 = \frac{1}{3}$

$m = \frac{2}{3}$ | $n = \frac{4}{3}$

$$= \frac{8}{3} \int_0^1 t^{\frac{2}{3}-1} (1-t)^{\frac{4}{3}-1} dt$$

$$= \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{8}{3} \times \frac{\Gamma\left(\frac{2}{3}\right) \cdot \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)}$$

$$= \frac{8}{3} \cdot \frac{\Gamma\left(\frac{2}{3}\right) \cdot \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\cancel{\frac{6}{3}}^{2}\right)} = \frac{8}{3} \times \frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}$$

$$= \frac{8}{3} \times \frac{\Gamma\left(\frac{2}{3}\right) \cdot \frac{1}{3} \cdot \Gamma\left(\frac{1}{3}\right)}{1}$$

$\Gamma(2) = 1! = 1$

$\Gamma(1) = 1$

$$= \frac{8}{9} \Gamma\left(\frac{2}{3}\right) \cdot \Gamma\left(\frac{1}{3}\right).$$

we know that $\Gamma(n) \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$= \frac{8}{9} \Gamma\left(\frac{2}{3}\right) \cdot \Gamma\left(1-\frac{2}{3}\right) = \frac{8}{9} \frac{\pi}{\sin n\pi}$$

$n = \frac{1}{3}$

$$= \frac{8}{9} \times \frac{\pi}{\sin \frac{\pi}{3}} = \frac{8}{9} \times \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{16\pi}{9\sqrt{3}}$$

(16). show that $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} \cdot dx \ , n>0.$

Sol:- put $\log\left(\frac{1}{x}\right) = t$

$$\log 1 - \log x = t$$

$$\log x = -t$$

$$x = e^{-t}.$$

$$dx = -e^{-t} dt$$

LL $\Rightarrow$ if $x = 0, t = \infty$ | $\log\left(\frac{1}{0}\right) = \infty$

UL $\Rightarrow$ if $x = 1, t = 0$ | $\log\left(\frac{1}{1}\right) = 0$

$$\int_\infty^0 (t)^{n-1} (-e^{-t}) \cdot dt$$

$$= -\int_\infty^0 t^{n-1} e^{-t} dt$$

$$= \int_0^\infty e^{-t} \cdot t^{n-1} dt = \Gamma(n)$$

$$\therefore \Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx = \int_0^\infty e^{-t} t^{n-1} dt \ //$$

---

(17). Evaluate $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx.$

Sol:- put $\log \frac{1}{x} = t \Rightarrow -\log x = t$

$$\log x = -t$$

$$x = e^{-t} \Rightarrow x^4 = e^{-4t}$$

$$dx = -e^{-t} dt.$$

LL $\Rightarrow$ if $x = 0, t = \infty$

UL $\Rightarrow$ if $x = 1, t = 0$

$$\int_{\infty}^{0} e^{-4t}\cdot (t)^3 (-e^{-t}dt)$$

$$= \int_0^{\infty} e^{-4t-t}\cdot t^3 dt = \int_0^{\infty} e^{-5t}\cdot (t)^3 dt.$$

$$\Rightarrow \int_0^{\infty} e^{-5t} (t)^3 dt$$

put $5t = y \Rightarrow t = \frac{y}{5}$

$$dt = \frac{dy}{5}$$

$$= \int_0^{\infty} e^{-y}\cdot \left(\frac{y}{5}\right)^3 \frac{dy}{5}.$$

$$= \frac{1}{5^4}\cdot \int_0^{\infty} e^{-y}\cdot y^{4-1} dy.$$

$$\therefore \frac{1}{5^4}\cdot \Gamma(4) = \frac{\Gamma(4)}{625} = \frac{3!}{625} = \frac{6}{625}$$

$$\therefore \int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 = \frac{6}{625}$$

---

(18). PT $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$

Sol :- put $\log_e x = -t$

$$x = e^{-t}$$

$$dx = -e^{-t} dt.$$

LL $\Rightarrow$ if $x=0$, $t=\infty$

UL $\Rightarrow$ if $x=1$, $t=0$

$$= \int_{\infty}^{0} (e^{-t})^m \cdot (-t)^n \, (-e^{-t} dt).$$

$$= \int_{0}^{\infty} (e^{-t})^m \cdot (-t)^n \cdot dt \, (e^{-t})$$

$$= \int_{0}^{\infty} e^{-t(1+m)} \cdot (-t)^n \cdot dt$$

put $y = t(1+m)$.

$$t = \frac{y}{1+m} \rightarrow dt = \frac{dy}{1+m}.$$

$$= \int_{0}^{\infty} e^{-t} (-1)^n \cdot (t)^n \, dt$$

$$= \int_{0}^{\infty} e^{-y} (-1)^n \left(\frac{y}{1+m}\right)^n \cdot \frac{dy}{1+m}$$

$$= (-1)^n \int_{0}^{\infty} e^{-y} \cdot \frac{y^n}{(1+m)^{n+1}} \cdot$$

$$= \frac{(-1)^n}{(1+m)^{n+1}} \int_{0}^{\infty} e^{-y} \cdot y^n \, dy$$

$$= \frac{(-1)^n}{(1+m)^{n+1}} \cdot \Gamma(n+1)$$

$$= \frac{(-1)^n \cdot n!}{(1+m)^{n+1}}$$

$$\therefore \int_{0}^{1} x^m (\log x)^n \, dx = \frac{(-1)^n \, n!}{(1+m)^{n+1}} \; //$$

(19). Express the integral $\int_0^\infty \frac{x^c}{c^x}\,dx$ $(c>1)$ in terms of Gamma function

Show that $\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$.

Sol:-

$$a^{\log_e x} = x^{\log_e a}$$

$$\log_n m = \frac{\log m}{\log n}$$

$$c = e^{\log_e c}$$

$$\int_0^\infty \frac{x^c}{c^x}\,dx$$

put $c = e^{\log_e c} = e^{\log c}$

$$c^x = (e^{\log c})^x = e^{x\log c}$$

$$c^x = e^{\log c^x}$$

$$= \int_0^\infty e^{-\log c^x}\cdot x^c\,dx = \int_0^\infty e^{-x\log c}\cdot x^c\,dx$$

put $t = x\log c$

$$x = \frac{t}{\log c}$$

$$dx = \frac{dt}{\log c}$$

$$= \int_0^\infty e^{-t}\cdot\left(\frac{t}{\log c}\right)^c\cdot\frac{dt}{\log c} = \frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-t}\cdot t^c\,d$$

$$= \frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-t}\cdot(t)^{(c+1)-1}\,dt$$

$$= \frac{\Gamma(c+1)}{(\log c)^{c+1}}$$

$$\therefore \int_0^\infty \frac{x^c}{c^x}\, dt = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$$

---

(20). Evaluate (i) $\int_0^\infty 3^{-4x^2} dx$. (ii) $\int_0^\infty a^{-bx^2} dx$

Sol: (ii) $\int_0^\infty a^{-bx^2} dx$

Sol:
$$a = e^{\log a}$$

$$a^{-bx^2} = e^{-bx^2 \log a}$$

$$\int_0^\infty e^{-bx^2 \log a}\, dx.$$

put $bx^2 \log a = t \Rightarrow x^2 = \frac{t}{b \log a} \Rightarrow x = \frac{\sqrt{t}}{\sqrt{b \log a}}$

$$dx = \frac{1}{\sqrt{b \log a}} \cdot \frac{1}{2\sqrt{t}}\, dt$$

LL $\Rightarrow$ if $x = 0,\ t = 0$

UL $\Rightarrow$ if $x = \infty,\ t = \infty$

$$\int_0^\infty e^{-t} \cdot \frac{1}{\sqrt{b \log a}} \cdot \frac{1}{2\sqrt{t}}\, dt.$$

$$= \frac{1}{2\sqrt{b \log a}} \int_0^\infty e^{-t} \cdot t^{-\frac{1}{2}}\, dt$$

$n - 1 = -\frac{1}{2} \Rightarrow n = \frac{1}{2}$

$$= \frac{1}{2\sqrt{b \log a}} \cdot \int_0^\infty e^{-t} \cdot t^{\frac{1}{2}-1}\, dt.$$

$$= \frac{1}{2\sqrt{b\log a}} \Gamma\left(\frac{1}{2}\right)$$

$$\therefore \int_0^\infty a^{-bx^2} dx = \frac{\sqrt{\pi}}{2\sqrt{b\log a}}$$

(9) $\int_0^\infty 3^{-4x^2} dx$.

Sol: $a = 3, \ b = 4$

$$\int_0^\infty 3^{-4x^2} dx = \frac{\sqrt{\pi}}{2\sqrt{4\log 3}} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}} \text{ (or) } \sqrt{\frac{\pi}{16\log 3}}$$

(21). Prove that $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx = 0$ using $\beta-\Gamma$ function

Sol: we KT, $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \cdot dx = B(m,n)$

$$\int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}} dx$$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}} dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}} dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{\cancel{15}+9}} dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{9+15}} dx$$

$$= B(15,9) - B(9,15)$$

$$= \cancel{B(15,9)} - \cancel{B(15,9)} = 0$$

$B(m,n) = B(n,m)$

$$\therefore \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx = 0$$

(22). Evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx$ using B-Γ functions?

Sol: $\int_0^\infty \frac{x^4}{(1+x)^{15}}\,dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}\,dx$

$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{10+5}} + \int_0^\infty \frac{x^{10-1}}{(1+x)^{5+10}}\,dx$

$= B(10,5) + B(5,10)$

$= B(10,5) + B(10,5)$

$= 2\,B(10,5)$

$= 2\cdot\frac{\Gamma(10)\cdot\Gamma(5)}{\Gamma(10+5)} = 2\cdot\frac{9!\times 4!}{14!}$

$= \frac{2\times \cancel{9!}\times 4\times 3\times 2\times 1}{14\times 13\times 12\times 11\times 10\times \cancel{9!}} = \frac{2}{10010} = \frac{1}{5005}.$

$\therefore \int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \frac{1}{5005}$ //

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(23) ST, $B(m,\frac{1}{2}) = 2^{2m-1}\cdot B(m,m)$..

Sol: we k.T, $B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}\cdot(\cos\theta)^{2n-1}\,d\theta$

Case-1:- put $n=\frac{1}{2}$.

$B(m,\frac{1}{2}) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{\cancel{2}\cdot\frac{1}{\cancel{2}}-1}\,d\theta$

$B(m,\frac{1}{2}) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}\,d\theta \longrightarrow (1).$

case-2: put $n = m$

$$B(m,m) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2m-1}\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta \cos\theta)^{2m-1}\, d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{2}{2}\sin\theta\cos\theta\right)^{2m-1}\, d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1}\, d\theta.$$

put $2\theta = \phi$

$\theta = \frac{\phi}{2}$ + $d\theta = \frac{d\phi}{2}$

LL $\Rightarrow$ if $\theta = 0$, $\phi = 0$

UL $\Rightarrow$ if $\theta = \pi/2$, $\phi = \pi$.

$$= \frac{2}{2^{2m-1}} \cdot \int_0^{\pi} (\sin\phi)^{2m-1} \cdot \frac{d\phi}{2}$$

$$= \frac{\cancel{2}}{2^{2m-1}} \cdot \frac{1}{\cancel{2}} \int_0^{\pi} (\sin\phi)^{2m-1}\, d\phi$$

$$B(m,m) = \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} (\sin\phi)^{2m-1}\, d\phi \rightarrow (2)$$

from eq^n (1) & (2)

$$B(m,m) = \frac{1}{2^{2m-1}} \cdot B\left(m, \frac{1}{2}\right)$$

$$\therefore B\left(m, \frac{1}{2}\right) = 2^{2m-1} \cdot B(m,m).$$ //

(24) · Show that $B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2n-1} \, d\theta$ and hence deduce that $\int_0^{\pi/2} (\sin\theta)^n d\theta = \int_0^{\pi/2} (\cos\theta)^n d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$

Sol:- By the definition $B(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx$.

put $x = \sin^2\theta$.

$dx = 2\sin\theta \cos\theta \, d\theta$.

LL ⇒ if $x = 0$, $\theta = 0$.

UL ⇒ if $x = 1$, $\theta = \frac{\pi}{2}$.

$$B(m,n) = \int_0^{\pi/2} (\sin\theta)^{2(m-1)} \cdot (1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta \, d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-2+1} \cdot (\cos\theta)^{2n-2+1} d\theta$$

$$\therefore B(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2n-1} \cdot d\theta$$

Case-I:- $\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2n-1} d\theta = \frac{1}{2} B(m,n)$

Here, $2m-1 = n$, $2n-1 = 0$

$m = \frac{n+1}{2}$, $n = \frac{1}{2}$

$$\int_0^{\pi/2} (\sin\theta)^n \cdot (\cos\theta)^0 d\theta = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

$$\int_0^{\pi/2} (\sin\theta)^n d\theta = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2} + \frac{1}{2}\right)} = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.$$

$$\therefore \int_0^{\pi/2} (\sin\theta)^n d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \quad //$$

Case-2: $\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2n-1} d\theta = \frac{1}{2} B(m,n)$

Here, $2m-1=0$, $2n-1=n$

$m=\frac{1}{2}$ $\quad$ $n=\frac{n+1}{2}$.

$$\int_0^{\pi/2} (\sin\theta)^0 \cdot (\cos\theta)^n d\theta = \frac{1}{2} \cdot B\left(\frac{1}{2}, \frac{n+1}{2}\right).$$

$$\int_0^{\pi/2} (\cos\theta)^n d\theta = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n+1}{2}\right)}$$

$$\therefore \int_0^{\pi/2} (\cos\theta)^n d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \text{ //}$$

---

25. Prove that, $\int_0^{\pi/2} (\sin\theta)^2 (\cos\theta)^4 d\theta = \frac{\pi}{32}$.

Sol: WKT, $\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta = \frac{1}{2} B(m,n)$.

Here, $2m-1=2$ $\quad$ | $\quad$ $2n-1=4$

$2m=3$ $\quad$ | $\quad$ $n=\frac{5}{2}$

$m=\frac{3}{2}$

$$= \frac{1}{2} B\left(\frac{3}{2}, \frac{5}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{5}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{3!}$$

$$= \frac{3\pi}{16 \times 3 \times 2 \times 1} = \frac{\pi}{32}.$$

$$\therefore \int_0^{\pi/2} (\sin\theta)^2 \cdot (\cos\theta)^4 d\theta = \frac{\pi}{32} \text{ //}$$

(26). Evaluate $\int_0^{\pi/2} (\sin\theta)^6 (\cos\theta)^7 \, d\theta$.

Sol: WKT, $\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \, d\theta = \frac{1}{2} B(m,n)$.

Here, $2m-1=6$, $m=\frac{7}{2}$ | $2n-1=7$, $n=4$.

$$= \frac{1}{2} B\left(\frac{7}{2}, 4\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{7}{2}\right) \cdot \Gamma(4)}{\Gamma\left(\frac{7}{2}+4\right)} = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{7}{2}\right) \cdot \Gamma(4)}{\Gamma\left(\frac{15}{2}\right)}.$$

$$= \frac{\cancel{\frac{1}{2}} \cdot \cancel{\frac{5}{2}} \cdot \cancel{\frac{3}{2}} \cdot \cancel{\frac{1}{2}} \cdot \cancel{\sqrt{\pi}} \cdot 3 \times \cancel{2} \times 1}{\frac{13}{2} \times \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \cancel{\frac{5}{2}} \cdot \cancel{\frac{3}{2}} \cdot \cancel{\frac{1}{2}} \cdot \cancel{\sqrt{\pi}}} = \frac{3}{\frac{9009}{16}} = \frac{16}{3003}$$

$$\therefore \int_0^{\pi/2} (\sin\theta)^6 (\cos\theta)^7 \, d\theta = \frac{16}{3003} \quad //$$

---

(27). Evaluate $\int_0^{\pi/2} (\sin\theta)^{7/2} (\cos\theta)^{3/2} \, d\theta$.

Sol: WKT, $\int_0^{\pi/2} (\sin\theta)^{2m-1} \cdot (\cos\theta)^{2n-1} \cdot d\theta = \frac{1}{2} B(m,n)$.

Here, $2m-1=\frac{7}{2}$ | $2n-1=\frac{3}{2}$

$2m = \frac{7}{2}+1 = \frac{9}{2}$ | $2n = \frac{3}{2}+1 = \frac{5}{2}$

$m = \frac{9}{4}$ | $n = \frac{5}{4}$

$$= \frac{1}{2} B\left(\frac{9}{4}, \frac{5}{4}\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{9}{4}\right) \cdot \Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{9}{4}+\frac{5}{4}\right)} = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{9}{4}\right) \cdot \Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{4} \cdot \frac{1}{4} \cdot \Gamma\left(\frac{1}{4}\right) \cdot \frac{1}{4} \cdot \Gamma\left(\frac{1}{4}\right)}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)} = \frac{1}{48} \cdot \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\sqrt{\pi}}$$

$$\therefore \int_0^{\pi/2} (\sin\theta)^{\frac{7}{2}} (\cos\theta)^{\frac{3}{2}} \, d\theta = \frac{1}{48} \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\sqrt{\pi}} \quad //$$

(28) . Find $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$.

Sol: $\int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta = \int_0^{\pi/2} (\sin\theta)^{\frac{1}{2}} (\cos\theta)^{\frac{-1}{2}} d\theta$

$2m-1=\frac{1}{2}$	$2n-1=\frac{-1}{2}$
$2m=\frac{3}{2}$	$2n=\frac{1}{4}$
$m=\frac{3}{4}$	$n=\frac{1}{4}$.

$$\therefore \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{1}{2}\cdot B\left(\frac{3}{4},\frac{1}{4}\right)$$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{2}\cdot\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right) \quad \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{2}\cdot\frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}\cancel{\sqrt{2}}}\pi\times\cancel{\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}} \;//$$

---

(29) . Find $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

Sol: $\int_0^{\pi/2} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\, d\theta = \int_0^{\pi/2} (\sin\theta)^{-\frac{1}{2}} (\cos\theta)^{\frac{1}{2}} d\theta$

$2m-1=\frac{-1}{2}$	$2n-1=\frac{1}{2}$

$$2m = \frac{1}{2} \quad\Big|\quad 2n = \frac{3}{2}$$

$$m = \frac{1}{4} \quad\Big|\quad n = \frac{3}{4}.$$

$$= \frac{1}{2} B\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)} = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}} \;//$$

---

(30). prove that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \pi$

Sol: Now, $$\int_0^{\pi/2} (\sin\theta)^{\frac{1}{2}} d\theta = \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \cdot \frac{\sqrt{\pi}}{2} \to (1)$$

Now, $$\int_0^{\pi/2} (\sin\theta)^{-\frac{1}{2}} d\theta = \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \cdot \frac{\sqrt{\pi}}{2} \to (2)$$

multiplying (1) & (2)

$$= \frac{\cancel{\Gamma\left(\frac{3}{4}\right)}}{\frac{1}{4} \cdot \cancel{\Gamma\left(\frac{1}{4}\right)}} \cdot \frac{\sqrt{\pi}}{2} \times \frac{\cancel{\Gamma\left(\frac{1}{4}\right)}}{\cancel{\Gamma\left(\frac{3}{4}\right)}} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi}{\cancel{4}} \times \frac{\cancel{4}}{1} = \pi$$

$$\therefore \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \pi \;//$$

---

(31) Find $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}}\, d\theta.$

Sol: Now, $$\int_0^{\pi/2} (\cos\theta)^{\frac{1}{2}} d\theta = \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{\Pi\left(\frac{3}{4}\right)}{\Pi\left(\frac{5}{4}\right)} \cdot \frac{\sqrt{\pi}}{2} \rightarrow ①$$

Now, $\int_0^{\pi/2} (\cos\theta)^{-\frac{1}{2}} d\theta = \frac{\Pi\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Pi\left(\frac{-\frac{1}{2}+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} = \frac{\Pi\left(\frac{1}{4}\right)}{\Pi\left(\frac{3}{4}\right)} \cdot \frac{\sqrt{\pi}}{2}$

multiplying ① & ②

$$= \frac{\Pi\left(\frac{3}{4}\right)}{\frac{1}{4}\cdot\Pi\left(\frac{1}{4}\right)} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Pi\left(\frac{1}{4}\right)}{\Pi\left(\frac{3}{4}\right)} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi}{4} \times \frac{4}{1} = \pi$$

$$\therefore \int_0^{\pi/2} \sqrt{(\cos\theta)}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}}\, d\theta = \pi$$

---

(32). Find $\int_0^{\pi/2} \sqrt{\sec\theta}\, d\theta$

Sol: $\int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}} d\theta = \int_0^{\pi/2} (\cos\theta)^{-\frac{1}{2}} d\theta = \frac{\Pi\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Pi\left(\frac{-\frac{1}{2}+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$

$$= \frac{\Pi\left(\frac{1}{4}\right)}{\Pi\left(\frac{3}{4}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

---

(33). Find $\int_0^{\pi/2} \sqrt{\operatorname{cosec}\theta}\, d\theta$

Sol: $\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta = \int_0^{\pi/2} (\sin\theta)^{-1/2} d\theta = \frac{\Pi\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Pi\left(\frac{-\frac{1}{2}+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$

$$= \frac{\Pi\left(\frac{1}{4}\right)}{\Pi\left(\frac{3}{4}\right)} \cdot \frac{\sqrt{\pi}}{2} \quad (\text{or}) \quad \frac{1}{2\sqrt{2\pi}} \left(\Pi\left(\frac{1}{4}\right)\right)^2$$

$\times \frac{\Pi\left(\frac{1}{4}\right)}{\Pi\left(\frac{1}{4}\right)}$

34. Evaluate $\int_0^{\pi/2} (\sin\theta)^{9/2} (\cos\theta)^5 \, d\theta$

Sol :-

$$2m-1 = \frac{9}{2} \qquad 2n-1 = 5$$
$$2m = \frac{11}{2} \qquad 2n = 6$$
$$m = \frac{11}{4} \qquad n = 3$$

$$= \frac{1}{2} B\left(\frac{11}{4}, 3\right) = \frac{1}{2} \cdot \frac{\Pi\left(\frac{11}{4}\right) \cdot \Pi(3)}{\Pi\left(\frac{11}{4}+3\right)}$$

$$= \cancel{\frac{1}{2}} \cdot \frac{\frac{7}{4} \times \frac{3}{4} \cdot \Pi\left(\frac{3}{4}\right) \cdot \cancel{2}}{\Pi\left(\frac{23}{4}\right)} = \frac{\cancel{\frac{7}{4}} \times \cancel{\frac{3}{4}} \times \cancel{\Pi\left(\frac{3}{4}\right)}}{\frac{19}{4} \times \frac{15}{4} \times \frac{11}{4} \times \cancel{\frac{7}{4}} \cdot \cancel{\frac{3}{4}} \cdot \cancel{\Pi\left(\frac{3}{4}\right)}}$$

$$= \frac{4 \times 4 \times 4}{19 \times 15 \times 11} = \frac{64}{3135}$$

$$\therefore \int_0^{\pi/2} (\sin\theta)^{9/2} (\cos\theta)^5 \, d\theta = \frac{64}{3135}$$

---

35. Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}} dx = \frac{\pi}{4\sqrt{2}}$.

Sol :-

$$LHS = \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

Now, $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}}$

put $x^2 = \sin\theta \Rightarrow x = \sqrt{\sin\theta}$

$$dx = \frac{\cos\theta}{2\sqrt{\sin\theta}} d\theta$$

LL $\Rightarrow$ if $x=0$, $\theta = 0$

UL $\Rightarrow$ if $x=1$, $\theta = \frac{\pi}{2}$

$$\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \frac{\cos\theta}{2\sqrt{\sin\theta}}\, d\theta = \int_0^{\pi/2} \frac{\sin\theta}{2\sqrt{\sin\theta}} \cdot \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \frac{1}{2}\int_0^{\pi/2} (\sin\theta)^{\frac{1}{2}} d\theta$$

$$= \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \rightarrow ①$$

Now, $\int_0^1 \frac{1}{\sqrt{1+x^4}}\, d\theta$

put $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$

$$dx = \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta \cdot d\theta$$

LL $\Rightarrow$ if $x=0$, $\theta=0$

UL $\Rightarrow$ if $x=1$, $\theta = \pi/4$.

$$\int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\cancel{\sec\theta}} \cdot \frac{\sec^{\cancel{2}}\theta}{\sqrt{\tan\theta}}\, d\theta$$

$$= \frac{1}{2} \cdot \frac{2}{2}\int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}}\, d\theta = \frac{1}{4}\int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}\sqrt{\sin\theta}}\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} (\sin\theta)^{-1/2} (\cos\theta)^{-1/2}\, d\theta$$

$2m-1 = \frac{-1}{2}$	$2n-1 = \frac{-1}{2}$
$2m = \frac{1}{2}$	
$m = \frac{1}{4}$	$n = \frac{1}{4}$.

$$= \frac{1}{4} \cdot \frac{1}{2} \cdot B(m,n) = \frac{1}{8} \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{4}\right)} = \frac{1}{8} \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{1}{8} \cdot \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\sqrt{\pi}} \rightarrow (2)$$

multiplying (1) & (2)

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \times \frac{1}{8} \cdot \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\sqrt{\pi}}$$

$$= \frac{\cancel{\sqrt{\pi}}}{\cancel{4}} \cdot \frac{\Gamma\left(\frac{3}{4}\right)}{\cancel{\frac{1}{4}} \cdot \cancel{\Gamma\left(\frac{1}{4}\right)}} \cdot \frac{1}{8} \cdot \frac{\cancel{\Gamma\left(\frac{1}{4}\right)} \cdot \Gamma\left(\frac{1}{4}\right)}{\cancel{\sqrt{\pi}}}$$

$$= \frac{1}{8} \cdot \Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right) = \frac{1}{8} \cdot \Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(1-\frac{1}{4}\right) = \frac{1}{8} \cdot \frac{\pi}{\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{\pi}{8 \cdot \frac{1}{\sqrt{2}}} = \frac{\cancel{\sqrt{2}}\,\pi}{4 \times \sqrt{2} \cancel{\sqrt{2}}} = \frac{\pi}{4\sqrt{2}} = RHS.$$

$\therefore$ LHS = RHS

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx = \frac{\pi}{4\sqrt{2}}$$

---

(36) Evaluate $4\int_0^\infty \frac{x^2}{1+x^4}\,dx$ using Beta-Gamma functions?

Sol: put $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$

$$dx = \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta\, d\theta$$

$\tan(\infty) = \frac{\pi}{2}$

LL $\Rightarrow$ if $x=0$, $\theta = 0$, UL $\Rightarrow$ if $x = \infty$, $\theta = \frac{\pi}{2}$

$$= 4\int_0^{\pi/2} \frac{\tan\theta}{\cancel{\sec^2\theta}} \cdot \frac{1}{2\sqrt{\tan\theta}} \cdot \cancel{\sec^2\theta} \, d\theta$$

$$= \frac{4}{2}\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = 2\int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta$$

$$= \frac{\cancel{4}^{2}}{\cancel{4}} \cdot \frac{\sqrt{\pi}}{2} = \sqrt{2}\pi$$

$\downarrow \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

---

(37). Evaluate $\int_0^{\infty} \frac{x\,dx}{1+x^6}$ using β-Γ functions.

Sol:- put $x^6 = t$

$x = (t)^{1/6} \Rightarrow dx = \frac{1}{6}(t)^{-5/6}\, dt.$

LL $\rightarrow$ if $x = 0$, $t = 0$

UL $\Rightarrow$ if $x = \infty$, $t = \infty$

$$\int_0^{\infty} \frac{(t)^{1/6}}{1+t} \cdot \frac{1}{6} \cdot (t)^{-5/6}\, dt$$

$$= \frac{1}{6}\int_0^{\infty} t^{\frac{1}{6}-\frac{5}{6}} \cdot (1+t)^{-1}\, dt$$

$$= \frac{1}{6}\int_0^{\infty} (t)^{-2/3} \cdot (1+t)^{-1}\, dt$$

$\downarrow \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} = \beta(m,n)$

$$= \frac{1}{6}\int_0^{\infty} \frac{(t)^{-2/3}}{(1+t)}\, dt$$

$n - 1 = \frac{-2}{3} \Rightarrow n = \frac{1}{3}$

$m + n = 1$

$m = 1 - n = 1 - \frac{1}{3} = \frac{2}{3}$

$$= \frac{1}{6}\int_0^{\infty} \frac{t^{\frac{1}{3}-1}}{(1+t)^{\frac{2}{3}+\frac{1}{3}}}\, dt = \frac{1}{6} B\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{1}{6} \frac{\Gamma\left(\frac{2}{3}\right) \cdot \Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3} + \frac{1}{3}\right)}$$

$$= \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)} = \frac{1}{6}\cdot\frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\cancel{6}\,3}\cdot\frac{\pi}{\frac{\sqrt{3}}{\cancel{2}}} = \frac{\pi}{3\sqrt{3}}$$

(or)

$$\int_0^\infty \frac{x\,dx}{1+x^6}\,dx$$

put $x^3 = \tan\theta$

$x = (\tan\theta)^{1/3} \Rightarrow dx = \frac{1}{3}(\tan\theta)^{-2/3}\,d\theta$

LL $\Rightarrow$ if $x=0,\ \theta=0$

UL $\Rightarrow$ if $x=\infty,\ \theta=\frac{\pi}{2}$.

$$\int_0^{\pi/2} \frac{(\tan\theta)^{1/3}}{1+\tan^2\theta}\cdot\frac{1}{3}\cdot(\tan\theta)^{-2/3}\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \frac{(\tan\theta)^{-2/3+\frac{1}{3}}}{\sec^2\theta}\cdot d\theta = \frac{1}{3}\int_0^{\pi/2} \cos^2\theta\cdot\left(\frac{\sin\theta}{\cos\theta}\right)^{-1/3} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^2\theta\cdot(\cos\theta)^{1/3}\cdot(\sin\theta)^{-1/3}\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (\sin\theta)^{-1/3}(\cos\theta)^{7/3}\,d\theta$$

$2m-1 = \frac{-1}{3} \Rightarrow 2m = 1-\frac{1}{3} = \frac{3-1}{3} = \frac{2}{3} \Rightarrow \cancel{2}m = \frac{\cancel{2}}{3} \Rightarrow m = \frac{1}{3}$

$2n-1 = \frac{7}{3} \Rightarrow 2n = 1+\frac{7}{3} = \frac{3+7}{3} = \frac{10}{3} \Rightarrow \cancel{2}n = \frac{\cancel{10}\,5}{3} \Rightarrow n = \frac{5}{3}$.

$$= \frac{1}{3}\,B\left(\frac{1}{3},\frac{5}{3}\right) = \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{5}{3}\right)} = \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\cdot\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{\Gamma(2)}$$

$$= \frac{1}{3}\cdot\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)\cdot\frac{\cancel{2}}{3}\times\frac{1}{\cancel{2}} = \frac{1}{6}\,\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)$$

$$= \frac{1}{6}\,\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(1-\frac{1}{3}\right) = \frac{1}{6}\cdot\frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\cancel{6}\,3}\cdot\frac{\pi}{\frac{\sqrt{3}}{\cancel{2}}} = \frac{\pi}{3\sqrt{3}}$$ //.

(38). Show that $\int_0^1 \frac{dx}{(1-x^n)^{1/2}} = \frac{\sqrt{\pi}\cdot\Gamma\left(\frac{1}{n}\right)}{n\cdot\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$.

Sol:- put $x^n = t$

$x = (t)^{1/n}$

$dx = \frac{1}{n}\cdot(t)^{\frac{1}{n}-1} dt$

$$\int_0^1 \frac{1}{(1-t)^{1/2}} \frac{1}{n}\cdot(t)^{\frac{1}{n}-1}\cdot dt$$

$$= \frac{1}{n}\int_0^1 \frac{(t)^{\frac{1}{n}-1}}{(1-t)^{\frac{1}{2}}} dt \qquad \int_0^1 x^{m-1}\cdot(1-x)^{n-1} dx = B(m,n)$$

$$= \frac{1}{n}\int_0^1 (t)^{\frac{1}{n}-1}\cdot(1-t)^{-1/2} dt.$$

$m-1 = \frac{1}{n}-1$ | $n-1 = \frac{-1}{2}$

$m = \frac{1}{n}$ | $n = 1-\frac{1}{2} = \frac{1}{2}$.

$$= \frac{1}{n}\int_0^1 (t)^{\frac{1}{n}-1}(1-t)^{\frac{1}{2}-1} dt = \frac{1}{n} B\left(\frac{1}{n},\frac{1}{2}\right).$$

$$= \frac{1}{n}\,\frac{\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{1}{n}\cdot\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$ //

(39). Prove that $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} dx \cdot \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4}$.

Sol:- Now, $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx$

$x^4 = t$

$x = (t)^{1/4} \Rightarrow dx = \frac{1}{4}(t)^{-3/4} dt.$

$$\int_0^1 (t)^{1/2} \frac{1}{(1-t)^{1/2}} \cdot \frac{1}{4} (t)^{-3/4} dt$$

$$= \frac{1}{4} \int_0^1 \frac{(t)^{\frac{1}{2}-\frac{3}{4}}}{(1-t)^{1/2}} dt = \frac{1}{4} \int_0^1 \frac{(t)^{-1/4}}{(1-t)^{1/2}} dt$$

$$= \frac{1}{4} \int_0^1 (t)^{-1/4} (1-t)^{-1/2} dt \quad \Big| \int_0^1 x^{m-1} (1-x)^{n-1} dx = B(m,n)$$

$$m - 1 = \frac{-1}{4} \quad \Big| \quad n-1 = \frac{-1}{2}$$

$$m = \frac{3}{4} \quad \Big| \quad n = \frac{1}{2}$$

$$= \frac{1}{4} \int_0^1 (t)^{\frac{3}{4}-1} \cdot (1-t)^{\frac{1}{2}-1} dt = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \cdot \frac{\Pi\left(\frac{3}{4}\right) \cdot \Pi\left(\frac{1}{2}\right)}{\Pi\left(\frac{3}{4}+\frac{1}{2}\right)} = \frac{1}{4} \cdot \frac{\Pi\left(\frac{3}{4}\right) \cdot \sqrt{\pi}}{\Pi\left(\frac{5}{4}\right)} \rightarrow (1)$$

Now,

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1-x^4}} dx$$

$$x^4 = t \Rightarrow dx = \frac{1}{4} (t)^{-3/4} dt$$

$$= \int_0^1 \frac{1}{(1-t)^{1/2}} \cdot \frac{1}{4} (t)^{-3/4} dt.$$

$$= \frac{1}{4} \int_0^1 (t)^{-3/4} (1-t)^{-1/2} dt$$

$$m-1 = \frac{-3}{4} \quad \Big| \quad n-1 = \frac{-1}{2} \quad \Big| \int_0^1 x^{m-1} (1-x)^{n-1} dx = B(m,n)$$

$$m = \frac{1}{4} \quad \Big| \quad n = \frac{1}{2}$$

$$= \frac{1}{4}\int_0^1 (t)^{\frac{1}{4}-1} (1-t)^{\frac{1}{2}-1}\, dt$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{1}{4} \cdot \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \rightarrow (2)$$

multiplying eq's (1) & (2)

$$= \frac{1}{4} \times \frac{\sqrt{\pi} \times \Gamma\left(\frac{1}{4}\right)}{\cancel{\Gamma\left(\frac{3}{4}\right)}} \times \frac{1}{4} \times \frac{\cancel{\Gamma\left(\frac{3}{4}\right)} \cdot \sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)}.$$

$$= \frac{\pi}{16} \times \cancel{\Gamma\left(\frac{1}{4}\right)} \times \frac{1}{\frac{1}{4} \cdot \cancel{\Gamma\left(\frac{1}{4}\right)}} = \frac{\pi}{4} \,//$$