Intelligent Robotics

Kinematics

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Differentiation of Rotation Matrix

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$$\mathbf{p} = \mathbf{R}\bar{\mathbf{p}}$$

$$\dot{\mathbf{p}} = \dot{\mathbf{R}}\bar{\mathbf{p}}$$

$$\dot{\mathbf{p}} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{p}$$

$$\downarrow$$

$$\dot{\mathbf{p}} = \omega \times \mathbf{p}$$

$$\omega \times \mathbf{p} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{p}$$

$$\downarrow$$

$$\omega \times \mathbf{p} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \omega_y p_z - \omega_z p_y \\ \omega_z p_x - \omega_x p_z \\ \omega_x p_y - \omega_y p_x \end{bmatrix}$$

Differentiation of Rotation Matrix

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$$\omega \times \mathbf{p} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \omega_y p_z - \omega_z p_y \\ \omega_z p_x - \omega_x p_z \\ \omega_x p_y - \omega_y p_x \end{bmatrix}$$

$$\omega \times \mathbf{p} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \equiv \mathbf{S}\mathbf{p}$$

$\mathbf{S}^T = -\mathbf{S}$ skew symmetric matrix

$$\omega \times \mathbf{p} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{p}$$

$(\dot{\mathbf{R}}\mathbf{R}^T)^T = -\dot{\mathbf{R}}\mathbf{R}^T$ skew symmetric matrix

Differentiation of Rotation Matrix

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$$\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

unskew: "wedge" ($\wedge$vee$) operation $\vee$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}^{\wedge} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

skew: "hat" ($\wedge$wedge$) operation $\wedge$

$$\omega \times \mathbf{p} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{p}$$

$$\hat{\omega} = \dot{\mathbf{R}}\mathbf{R}^T \qquad \omega = (\dot{\mathbf{R}}\mathbf{R}^T)^{\vee}$$

Matrix Exponential

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$$\hat{\omega} = \dot{\mathbf{R}}\mathbf{R}^T$$

$$\downarrow$$

$$\dot{\mathbf{R}} = \hat{\omega}\mathbf{R}$$ Basic Equation of Rotation

$$\downarrow$$

$$\mathbf{R}(t) = \mathbf{R}_0 e^{\hat{\omega}t}$$ Solution with **matrix exponential**

$$\boldsymbol{\omega} = \mathbf{a}\omega,\ \omega \equiv \|\boldsymbol{\omega}\|,\ \|\mathbf{a}\| = 1$$

$$\downarrow$$

$$e^{\hat{\omega}t} = \mathbf{I} + \hat{\mathbf{a}}\sin(\omega t) + \hat{\mathbf{a}}^2(1 - \cos(\omega t))$$ Rodrigues' formula

Rodrigues' Formula

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$$e^{\hat{\omega} t} = \mathbf{I} + \hat{\mathbf{a}} \sin(\omega t) + \hat{\mathbf{a}}^2 (1 - \cos(\omega t))$$

$$\boxed{e^{\hat{\mathbf{a}} \theta}} = \mathbf{I} + \hat{\mathbf{a}} \sin(\theta) + \hat{\mathbf{a}}^2 (1 - \cos(\theta))$$

Rotation matrix

Rodrigues' formula gives us the **rotation matrix** from an angular **velocity vector**.

Matrix Logarithm

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Angular velocity

$$\boxed{\boldsymbol{\omega}} = (\ln \boxed{\mathbf{R}})^{\vee}$$

Rotation matrix

$$(\ln \boldsymbol{R})^{\vee} = \begin{cases} [0\ 0\ 0]^T & (\text{if } \boldsymbol{R} = \boldsymbol{E}) \\ \frac{\pi}{2} \begin{bmatrix} r_{11}+1 \\ r_{22}+1 \\ r_{33}+1 \end{bmatrix} & (\text{else if } \boldsymbol{R} \text{ is diagonal}) \\ \theta \frac{\boldsymbol{l}}{\|\boldsymbol{l}\|} & (\text{otherwise}) \end{cases} \tag{2.39}$$

given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

$$\boldsymbol{l} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix},$$

$$\theta = \operatorname{atan2}(\|\boldsymbol{l}\|, r_{11} + r_{22} + r_{33} - 1)$$

Rotation Matrix <-> Angular Velocity

Log map (2.39)

$$\boldsymbol{\omega} = (\ln \mathbf{R})^{\vee}$$

Rotation Matrix

$$\mathbf{R}$$

Angular Velocity

$$\boldsymbol{\omega}$$

Rodrigues' formula (2.37)

$$q = \|\boldsymbol{\omega}\|, \ \mathbf{a} = \boldsymbol{\omega}/\|\boldsymbol{\omega}\|$$

$$\mathbf{R} = \mathbf{I} + \hat{\mathbf{a}}\sin(q) + \hat{\mathbf{a}}^2(1 - \cos(q))$$

Little Terminologies

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- **Group**
  - A group is a set equipped with an operation that combines any two elements of the set to produce a third element of the set, in such a way that the operation is associative, an identity element exists, and every element has an inverse.
  - Ex) $\mathbb{Z}$: 1) $\forall a, b, c \in \mathbb{Z}$, $(a + b) + c = a + (b + c)$, 2) $\forall a \in \mathbb{Z}$ $\exists 0 \in \mathbb{Z}$ s.t. $a + 0 = a$, 3) $a + b = b + a = 0$ where b is the inverse element of a.
- **Vector space**
  - A set of which is closed under addition and scaling.
  - Ex) The set of real numbers $\mathbb{R}$ is a vector space.
- **Field**
  - A field is a set of which addition and multiplication are defined.
  - Ex) The set of real numbers $\mathbb{R}$ is also a field.

Little Terminologies

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- **Field** vs. **Vector space**
  - The operations on a field $\mathbb{F}$ are
    - $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$
    - $\times : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$
  - The operations on a vector space $\mathbb{V}$ over a field $\mathbb{F}$ are
    - $+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$
    - $\times : \mathbb{F} \times \mathbb{V} \to \mathbb{V}$
  - For a field, any nonzero element $c \in \mathbb{F}$ has its inverse $c^{-1} \in \mathbb{F}$ s.t. $c \times c^{-1} = 1$, but there is no corresponding property among the vector space axioms.

Little Terminologies

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- **Algebra** (over a field)
  - An algebra is a vector space equipped with a bilinear product.

Algebra	vector space	bilinear operator	associativity	commutativity
complex numbers	$\mathbb{R}^2$	product of complex numbers $(a+ib)\cdot(c+id)$	Yes	Yes
cross product of 3D vectors	$\mathbb{R}^3$	cross product $\vec{a}\times\vec{b}$	No	No (anticommutative)
quaternions	$\mathbb{R}^4$	Hamilton product $(a+\vec{v})(b+\vec{w})$	Yes	No

- **Manifold**
  - A manifold is a topological space that locally resembles Euclidean space near each point. The mappings from a manifold the Euclidean space is called charts and the collection of charts is called an atlas.

Lie Group & Lie Algebra

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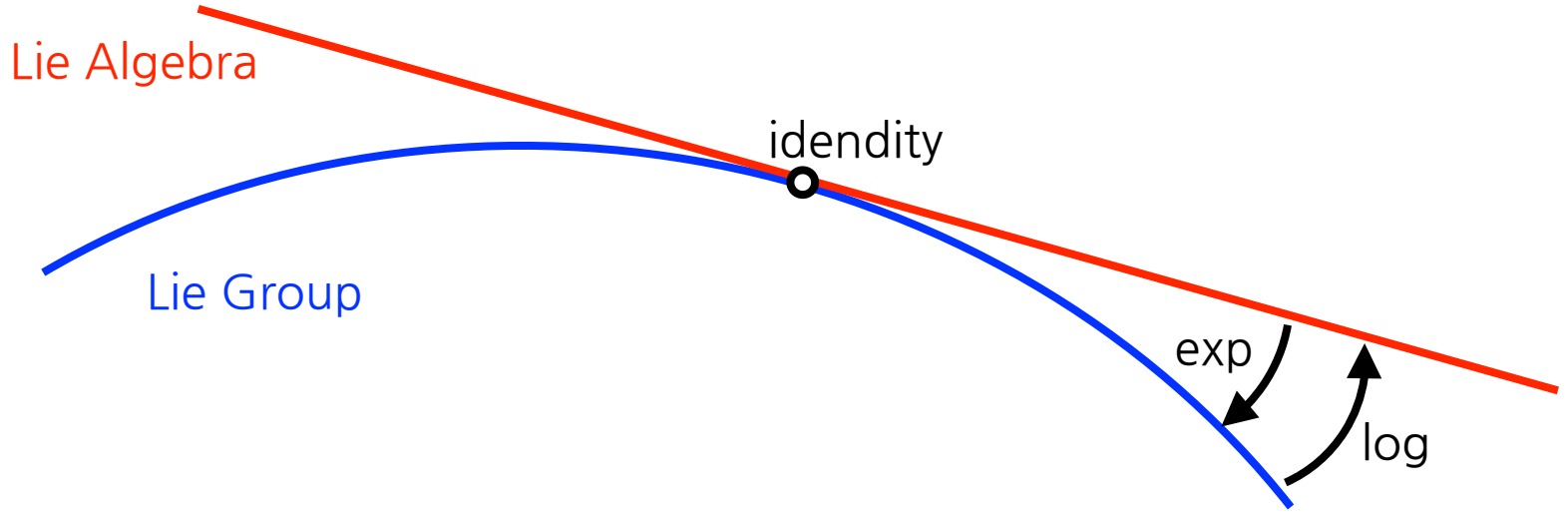

- Definition: A Lie group is a smooth manifold that is also a group, such that the group operations mulitplication and inversion are smooth maps.
- Definition: The tangent space to a Lie group at the identity element is called the associated Lie algebra.
- The mapping from the Lie algebra to the Lie group is called the exponential map. Its inverse is called the logarithm map.

Lie Group & Lie Algebra

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- **Lie Algebra,** $so(3)$
  - $\hat{\omega} \in so(3) = \{\hat{\omega} \,|\, \omega \in \mathbb{R}^3\}$
  - An algebra over a field K is a vector space V over K with multiplication on the space V.
  - One can also define the Lie bracket: $[\hat{w}, \hat{v}] = \hat{w}\hat{v} - \hat{v}\hat{w}$
- **Lie Group,** $SO(3)$
  - $R \in SO(3) = \{R \in \mathbb{R}^{3\times3} \,|\, R^T R = RR^T = I,\ \det(R) = 1\}$
  - More precisely, a **Lie Group** is a smooth manifold that is also a group, such that the group operations multiplication and inversion are smooth maps.
- The **Lie algebra** $so(3)$ is the tangent space at the identity of the rotation group $SO(3)$
  - $R(t) = R_0 e^{\hat{\omega}t}$ and for small dt, $R(dt) = R(0) + \hat{\omega}(0)dt$ which shows that $\hat{\omega} \in so(3)$ gives the first order apporximation of a rotation.
  - It also shows that $\hat{\omega}$ is the tangent space at the identity of $SO(3)$ $(R(0) = I)$.

Rotation Matrix <-> Angular Velocity


```
R = rpy2r(360*D2R*rand(3,1)); % random rotation matrix
w = r2w(R); % log map: w = wedge(ln(R))
R2 = rodrigues(uv(w),norm(w)); % exp map
```

```
function w = r2w(R)
%
% Get angular velocity [in radian] vector from a rotation matrix
% - This operation corresponds to w = wedge(ln(R)), a.k.a. a log map.
% - p.36 in 'Introduction to Humanoid Robotics' by Kajita et al.
%
% The reverse can be computed by 'rodrigues'
% ex)
% w = r2w(R);
% R = rodrigues(uv(w),norm(w));
%
el = [R(3,2)-R(2,3); R(1,3)-R(3,1); R(2,1)-R(1,2)];
norm_el = norm(el);
if norm_el > eps
    w = atan2(norm_el, trace(R)-1)/norm_el * el;
elseif R(1,1)>0 && R(2,2)>0 && R(3,3)>0
    w = [0, 0, 0]';
else
    w = pi/2*[R(1,1)+1; R(2,2)+1; R(3,3)+1];
end
```

```
function R = rodrigues(a,q)
%
% Compute the rotation matrix from an angular velocity vector
%  The reverse (R -> w) can be achieve by the w = r2w(R) function where w = a*q
%

if abs(norm(a)) < 1e-6
    R = eye(3,3);
    return;
end
if abs(norm(a)-1) > 1e-6
    warning('[rodrigues] a is not a unit vector (%.4e).\n',norm(a));
end

% Resolve some numerical issues
norm_a = norm(a);
a = a / norm_a;
q = q * norm_a;

% Get R
a_hat = skew(a);
R = eye(3,3) + a_hat*sin(q) + a_hat*a_hat*(1-cos(q));
```

Interpolate Two Rotation Matrices

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1. Get the rotation matrix which links the two matrices. $\boldsymbol{R} = \boldsymbol{R}_1^T \boldsymbol{R}_2$
2. Get the equivalent angular velocity vector from this rotation matrix. $\boldsymbol{\omega} = (\ln \boldsymbol{R})^\vee$
3. The angular velocity vector in world coordinates is: $\boldsymbol{R}_1 \boldsymbol{\omega}$
4. The interpolation becomes $\boldsymbol{R}(t) = \boldsymbol{R}_1 e^{\hat{\omega} t}, \quad t \in [0, 1].$

We can scale $\hat{\omega} \in so(3)$ (i.e., $0.5\hat{\omega}$), but we cannot scale $R \in SO(3)$.

Interpolate Two Rotation Matrices

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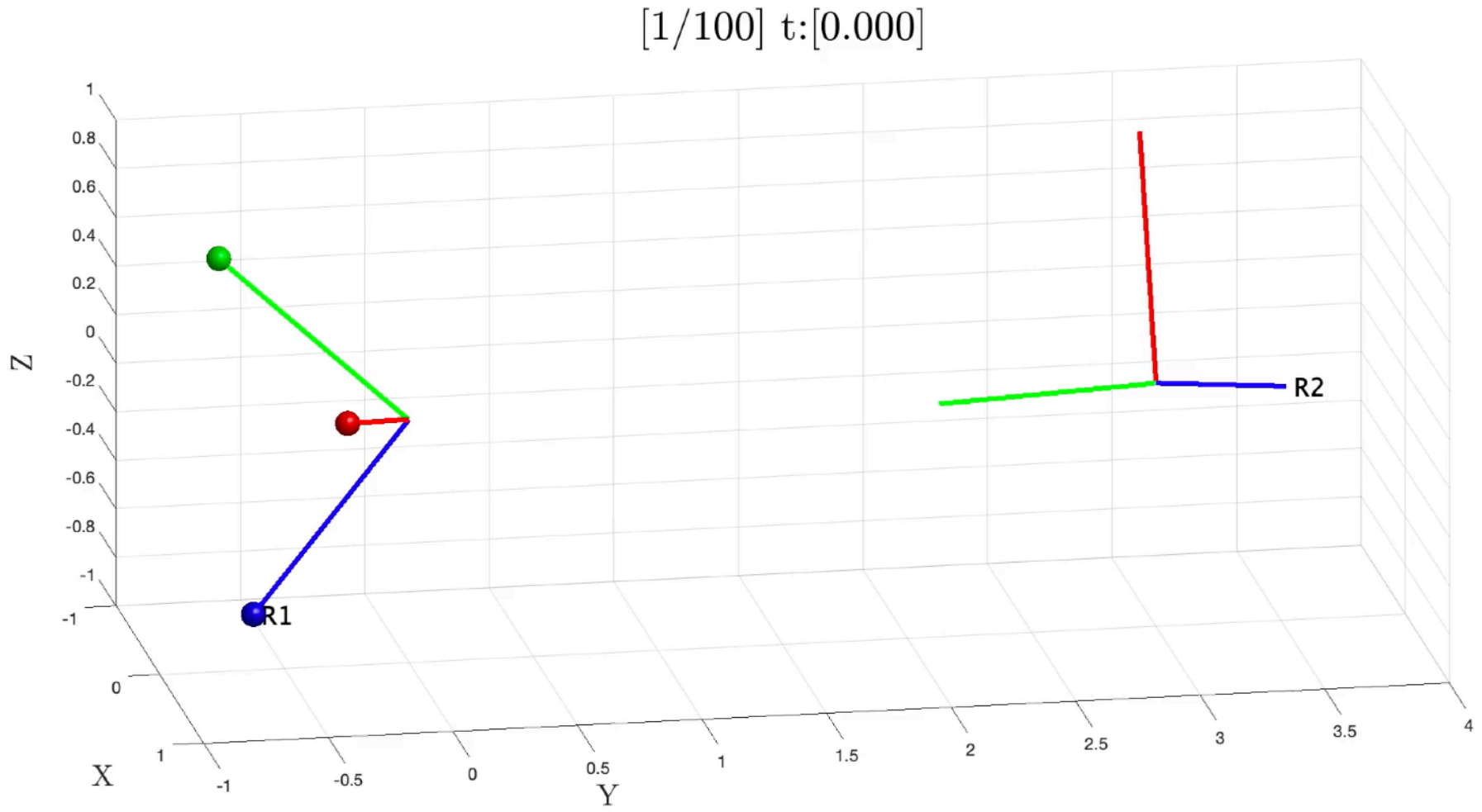

Interpolate Two Rotation Matrices

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```
% Get two random rotational matrices
p1 = cv([0,0,0]);
p2 = cv([0,3,0]);
R1 = rpy2r(360*rand(3,1)*D2R);
R2 = rpy2r(360*rand(3,1)*D2R);
R_link = R1'*R2; % rotation matrix which links R1 and R2
w_link = r2w(R_link); % equivalent velocity vector fron the rotation matrix

x_traj = []; y_traj = []; z_traj = [];
all = 1.0; % axis line length
axis_info = [-1,+1,-1,+4,-1,+1];
ts = linspace(0,1,100); % t:[0,1]
for tick = 1:length(ts)

    % Interpolate R1 and R2
    t = ts(tick);
    p_t = (1-t)*p1 + t*p2;
    R_t = R1*rodrigues(w_link/norm(w_link),norm(w_link)*t); % interpolate

    % Append the trajectory
    x_traj = cat(1,x_traj, rv(p_t)+all*rv(R_t(:,1)));
    y_traj = cat(1,y_traj, rv(p_t)+all*rv(R_t(:,2)));
    z_traj = cat(1,z_traj, rv(p_t)+all*rv(R_t(:,3)));

    % Animate
    if (tick==1) || (mod(tick,2)==0) || (tick==length(ts))
        view_info = [80,16];
        fig = set_fig(figure(1),'pos',[0.0,0.5,0.5,0.4],...
            'view_info',view_info,'axis_info',axis_info,...
            'AXIS_EQUAL',1,'GRID_ON',1,'REMOVE_MENUBAR',1,'USE_DRAGZOOM',1,...
            'SET_CAMLIGHT',1,'SET_MATERIAL','METAL','SET_AXISLABEL',1,'afs',18); % make figure
        plot_T(pr2t(p1,R1),'fig_idx',1,'subfig_idx',1,...
            'PLOT_AXIS',1,'all',all,'alw',3,'alc','','PLOT_SPHERE',0,'text_str','R1','TEXT_AT_ZTIP',1); % R1
        plot_T(pr2t(p2,R2),'fig_idx',1,'subfig_idx',2,...
            'PLOT_AXIS',1,'all',all,'alw',3,'alc','','PLOT_SPHERE',0,'text_str','R2','TEXT_AT_ZTIP',1); % R2
        plot_T(pr2t(p_t,R_t),'fig_idx',1,'subfig_idx',3,...
            'PLOT_AXIS',1,'all',all,'alw',2,'alc','','PLOT_SPHERE',0,'PLOT_AXIS_TIP',1); % R_t
        plot_traj(x_traj,'fig_idx',1,'subfig_idx',2,'tlc','r','tlw',1,'tls','--');
        plot_traj(y_traj,'fig_idx',1,'subfig_idx',3,'tlc','g','tlw',1,'tls','--');
        plot_traj(z_traj,'fig_idx',1,'subfig_idx',4,'tlc','b','tlw',1,'tls','--');
        title_str = sprintf('[%d/%d] t:[%.3f]',tick,length(ts),t);
        plot_title(title_str,'tfs',25,'interpreter','latex');
        drawnow; if ~ishandle(fig), break; end
    end

    if tick == 1, pause; end
end
```

Velocity in Three Dimensional Space

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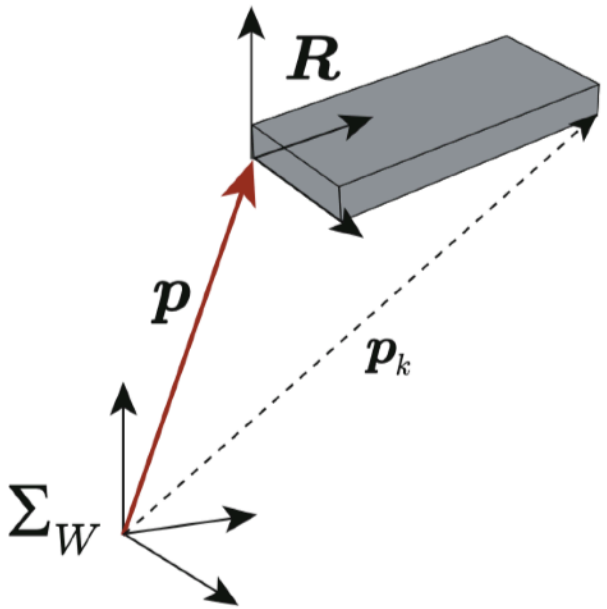

$$p_k = p + R\bar{p}_k. \tag{2.40}$$

Let us assume that this object is tumbling through space combining both translational and rotational motion. The velocity of the point p_k can be derived by differentiating the previous equation with respect to time

$$\begin{aligned} \dot{p}_k &= \dot{p} + \dot{R}\bar{p}_k \\ &= v + \hat{\omega}R\bar{p}_k \\ &= v + \omega \times (R\bar{p}_k), \end{aligned} \tag{2.41}$$

Local coordinate

where v, ω have been defined as follows

$$v \equiv \dot{p} \tag{2.42}$$

$$\omega \equiv (\dot{R}R^T)^\vee. \tag{2.43}$$

By making substitutions in (2.41) using (2.40) we obtain

$$\dot{p}_k = v + \omega \times (p_k - p). \tag{2.44}$$

Velocity in Three Dimensional Space

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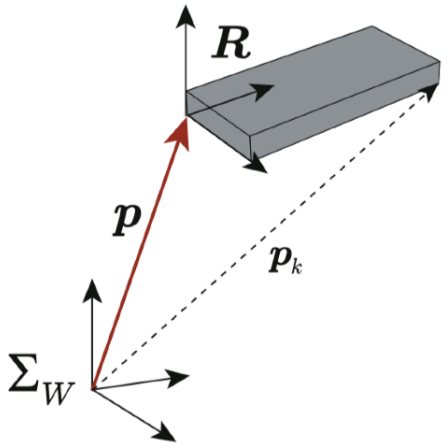

Let us assume that this object is tumbling through space combining both translational and rotational motion. The velocity of the point $\boldsymbol{p}_k$ can be derived by differentiating the previous equation with respect to time

$$\begin{aligned}\dot{\boldsymbol{p}}_k &= \dot{\boldsymbol{p}} + \dot{\boldsymbol{R}}\bar{\boldsymbol{p}}_k \\ &= \boldsymbol{v} + \hat{\boldsymbol{\omega}}\boldsymbol{R}\bar{\boldsymbol{p}}_k \\ &= \boldsymbol{v} + \boldsymbol{\omega} \times (\boldsymbol{R}\bar{\boldsymbol{p}}_k),\end{aligned} \tag{2.41}$$

where $\boldsymbol{v}, \boldsymbol{\omega}$ have been defined as follows

$$\boldsymbol{v} \equiv \dot{\boldsymbol{p}} \tag{2.42}$$

$$\boldsymbol{\omega} \equiv (\dot{\boldsymbol{R}}\boldsymbol{R}^T)^\vee. \tag{2.43}$$

By making substitutions in (2.41) using (2.40) we obtain

$$\dot{\boldsymbol{p}}_k = \boldsymbol{v} + \boldsymbol{\omega} \times (\boldsymbol{p}_k - \boldsymbol{p}). \tag{2.44}$$

```
% Get position and orientation
[p_A_in_W,R_A_in_W] = t2pr(T_A_in_W);
% Update position
p_A_in_W = p_A_in_W + v_A_in_W*dt;
% Update rotaton
a_in_W = R_A_in_W(:,3); % z-axis to be the axis of rotation
R_rot = rodrigues(a_in_W,omega_A_in_W*dt); % rotate w.r.t z-axis of {A}
R_A_in_W = R_rot*R_A_in_W;
% Update coordinate transform
T_A_in_W = pr2t(p_A_in_W,R_A_in_W);

% Point in the world coordinates {W}
p_X_in_W = t2p(T_A_in_W*pr2t(p_X_in_A,''));
% Compute the linear velocity of p_X in {W}
w_A_in_W = r2w(rodrigues(a_in_W,omega_A_in_W)); % angular velocity vector of {A} (2.43)
p_X_dot_in_W = v_A_in_W + cross(w_A_in_W,p_X_in_W-p_A_in_W); % (2.44)

% Concat 'p_X_in_W'
p_X_in_W_traj = cat(1,p_X_in_W_traj,rv(p_X_in_W));
```

Velocity in Three Dimensional Space

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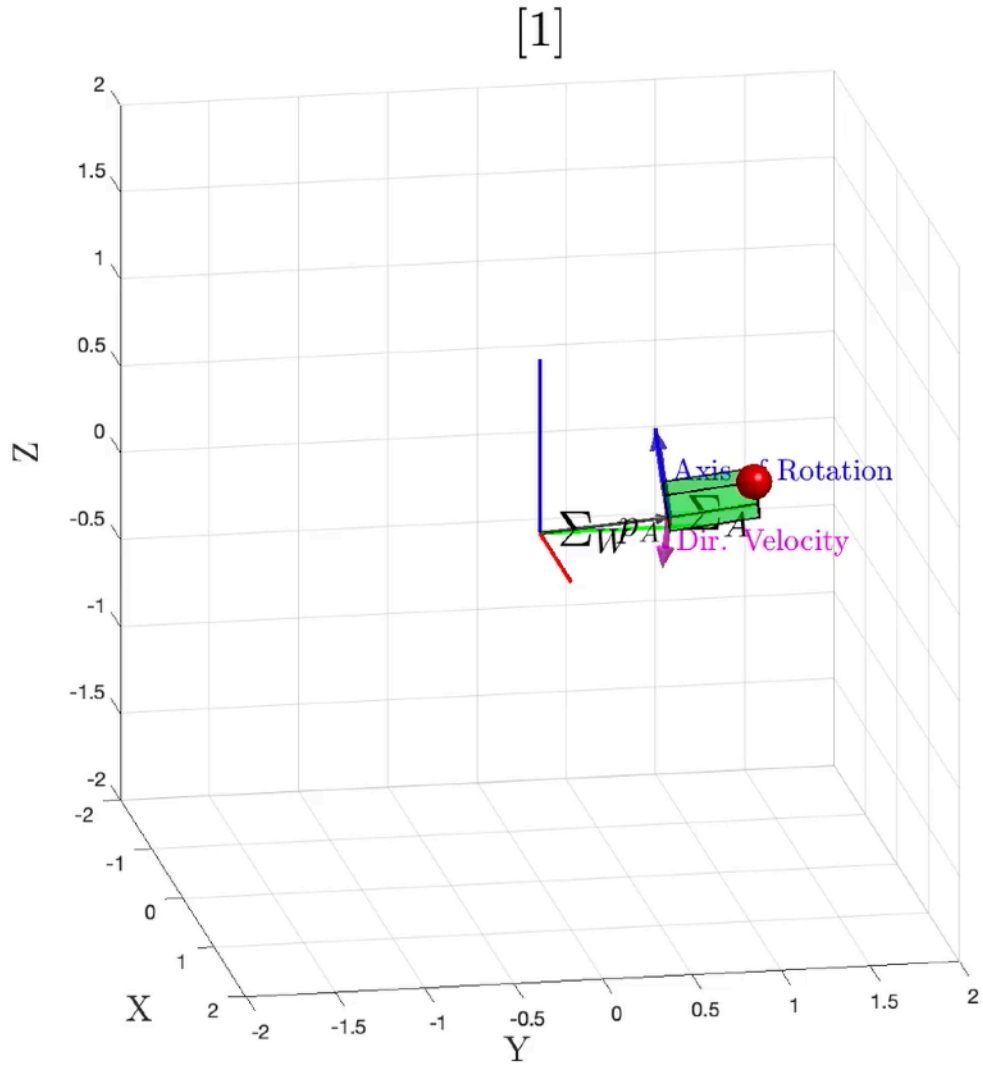

Two Object Case

Next we will think about two objects moving in 3D space. Let us assume that we have been given the local coordinates of these objects as follows

$$\boldsymbol{T}_1 = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{p}_1 \\ 0\,0\,0 & 1 \end{bmatrix} \tag{2.45}$$

$$^1\boldsymbol{T}_2 = \begin{bmatrix} \boldsymbol{R}_d & \boldsymbol{p}_d \\ 0\,0\,0 & 1 \end{bmatrix}. \tag{2.46}$$

The position and attitude of the second object is given to you relative to the first object as $^1\boldsymbol{T}_2$. So,

$$\begin{aligned} \boldsymbol{T}_2 &= \boldsymbol{T}_1\, {}^1\boldsymbol{T}_2 \\ &= \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{p}_1 \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_d & \boldsymbol{p}_d \\ 0\,0\,0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (\boldsymbol{R}_1\boldsymbol{R}_d) & (\boldsymbol{p}_1 + \boldsymbol{R}_1\boldsymbol{p}_d) \\ 0\,0\,0 & 1 \end{bmatrix}. \end{aligned} \tag{2.47}$$

Therefore the position and attitude of the second object in world coordinates is

$$\boldsymbol{p}_2 = \boldsymbol{p}_1 + \boldsymbol{R}_1\boldsymbol{p}_d \tag{2.48}$$

$$\boldsymbol{R}_2 = \boldsymbol{R}_1\boldsymbol{R}_d. \tag{2.49}$$

The linear velocity of the second object can be obtained by taking the time derivative of (2.48).

$$\begin{aligned} \boldsymbol{v}_2 &= \frac{d}{dt}(\boldsymbol{p}_1 + \boldsymbol{R}_1\boldsymbol{p}_d) \\ &= \dot{\boldsymbol{p}}_1 + \dot{\boldsymbol{R}}_1\boldsymbol{p}_d + \boldsymbol{R}_1\dot{\boldsymbol{p}}_d \\ &= \boldsymbol{v}_1 + \widehat{\boldsymbol{\omega}}_1\boldsymbol{R}_1\boldsymbol{p}_d + \boldsymbol{R}_1\boldsymbol{v}_d \\ &= \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times (\boldsymbol{R}_1\boldsymbol{p}_d) + \boldsymbol{R}_1\boldsymbol{v}_d \end{aligned}$$

where, $\boldsymbol{v}_1 \equiv \dot{\boldsymbol{p}}_1$, $\boldsymbol{v}_d \equiv \dot{\boldsymbol{p}}_d$.

Here, using (2.48) and rearranging its elements we obtain

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{R}_1\boldsymbol{v}_d + \boldsymbol{\omega}_1 \times (\boldsymbol{p}_2 - \boldsymbol{p}_1). \tag{2.50}$$

The angular velocity of the second object can be calculated from (2.49) as follows

$$\begin{aligned} \widehat{\boldsymbol{\omega}}_2 &= \dot{\boldsymbol{R}}_2\boldsymbol{R}_2^T \\ &= \frac{d}{dt}(\boldsymbol{R}_1\boldsymbol{R}_d)\boldsymbol{R}_2^T \\ &= (\dot{\boldsymbol{R}}_1\boldsymbol{R}_d + \boldsymbol{R}_1\dot{\boldsymbol{R}}_d)\boldsymbol{R}_2^T \\ &= (\widehat{\boldsymbol{\omega}}_1\boldsymbol{R}_1\boldsymbol{R}_d + \boldsymbol{R}_1\widehat{\boldsymbol{\omega}}_d\boldsymbol{R}_d)\boldsymbol{R}_2^T \\ &= \widehat{\boldsymbol{\omega}}_1 + \boldsymbol{R}_1\widehat{\boldsymbol{\omega}}_d\boldsymbol{R}_d\boldsymbol{R}_d^T\boldsymbol{R}_1^T \\ &= \widehat{\boldsymbol{\omega}}_1 + \boldsymbol{R}_1\widehat{\boldsymbol{\omega}}_d\boldsymbol{R}_1^T \end{aligned}$$

where $\boldsymbol{\omega}_1 \equiv (\dot{\boldsymbol{R}}_1\boldsymbol{R}_1^T)^\vee$, $\boldsymbol{\omega}_d \equiv (\dot{\boldsymbol{R}}_d\boldsymbol{R}_d^T)^\vee$.

From Fig. 2.12 we can see that $(\boldsymbol{R}\boldsymbol{\omega})^\wedge = \boldsymbol{R}\widehat{\boldsymbol{\omega}}\boldsymbol{R}^T$, so we get

$$\widehat{\boldsymbol{\omega}}_2 = \widehat{\boldsymbol{\omega}}_1 + (\boldsymbol{R}_1\boldsymbol{\omega}_d)^\wedge.$$

By applying $\vee$ to both sides of this equation we get,

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \boldsymbol{R}_1\boldsymbol{\omega}_d. \tag{2.51}$$

Two Object Case

Next we will think about two objects moving in 3D space. Let us assume that we have been given the local coordinates of these objects as follows

$$\boldsymbol{T}_1 = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{p}_1 \\ 0\,0\,0 & 1 \end{bmatrix} \tag{2.45}$$

Let's take a moment to think about (2.22) in Section 2.2.4,

$$\boldsymbol{R}(\boldsymbol{\omega} \times \boldsymbol{p}) = (\boldsymbol{R}\boldsymbol{\omega}) \times (\boldsymbol{R}\boldsymbol{p}) \qquad \ldots (a)$$

The left side of this equation can be rewritten as

$$\begin{aligned} \boldsymbol{R}(\boldsymbol{\omega} \times \boldsymbol{p}) &= \boldsymbol{R}\widehat{\boldsymbol{\omega}}\boldsymbol{p} \\ &= \boldsymbol{R}\widehat{\boldsymbol{\omega}}\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{p} \\ &= (\boldsymbol{R}\widehat{\boldsymbol{\omega}}\boldsymbol{R}^T)(\boldsymbol{R}\boldsymbol{p}). \qquad \ldots (b) \end{aligned}$$

Comparing equation (a) and equation (b) we can see that

$$(\boldsymbol{R}\boldsymbol{\omega})^\wedge = \boldsymbol{R}\widehat{\boldsymbol{\omega}}\boldsymbol{R}^T.$$

Fig. 2.12 Coordinate Transformation and the Angular Velocity Vector

The linear velocity of the second object can be obtained by taking the time derivative of (2.48).

$$\begin{aligned} \boldsymbol{v}_2 &= \frac{d}{dt}(\boldsymbol{p}_1 + \boldsymbol{R}_1\boldsymbol{p}_d) \\ &= \dot{\boldsymbol{p}}_1 + \dot{\boldsymbol{R}}_1\boldsymbol{p}_d + \boldsymbol{R}_1\dot{\boldsymbol{p}}_d \\ &= \boldsymbol{v}_1 + \widehat{\boldsymbol{\omega}}_1\boldsymbol{R}_1\boldsymbol{p}_d + \boldsymbol{R}_1\boldsymbol{v}_d \\ &= \boldsymbol{v}_1 + \boldsymbol{\omega}_1 \times (\boldsymbol{R}_1\boldsymbol{p}_d) + \boldsymbol{R}_1\boldsymbol{v}_d \end{aligned}$$

where, $\boldsymbol{v}_1 \equiv \dot{\boldsymbol{p}}_1$, $\boldsymbol{v}_d \equiv \dot{\boldsymbol{p}}_d$.

Here, using (2.48) and rearranging its elements we obtain

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{R}_1\boldsymbol{v}_d + \boldsymbol{\omega}_1 \times (\boldsymbol{p}_2 - \boldsymbol{p}_1). \tag{2.50}$$

The angular velocity of the second object can be calculated from (2.49) as follows

$$\begin{aligned} \widehat{\boldsymbol{\omega}}_2 &= \dot{\boldsymbol{R}}_2\boldsymbol{R}_2^T \\ &= \frac{d}{dt}(\boldsymbol{R}_1\boldsymbol{R}_d)\boldsymbol{R}_2^T \\ &= (\dot{\boldsymbol{R}}_1\boldsymbol{R}_d + \boldsymbol{R}_1\dot{\boldsymbol{R}}_d)\boldsymbol{R}_2^T \\ &= (\widehat{\boldsymbol{\omega}}_1\boldsymbol{R}_1\boldsymbol{R}_d + \boldsymbol{R}_1\widehat{\boldsymbol{\omega}}_d\boldsymbol{R}_d)\boldsymbol{R}_2^T \\ &= \widehat{\boldsymbol{\omega}}_1 + \boldsymbol{R}_1\widehat{\boldsymbol{\omega}}_d\boldsymbol{R}_d\boldsymbol{R}_d^T\boldsymbol{R}_1^T \\ &= \widehat{\boldsymbol{\omega}}_1 + \boldsymbol{R}_1\widehat{\boldsymbol{\omega}}_d\boldsymbol{R}_1^T \end{aligned}$$

where $\boldsymbol{\omega}_1 \equiv (\dot{\boldsymbol{R}}_1\boldsymbol{R}_1^T)^\vee$, $\boldsymbol{\omega}_d \equiv (\dot{\boldsymbol{R}}_d\boldsymbol{R}_d^T)^\vee$.

From Fig. 2.12 we can see that $(\boldsymbol{R}\boldsymbol{\omega})^\wedge = \boldsymbol{R}\widehat{\boldsymbol{\omega}}\boldsymbol{R}^T$, so we get

$$\widehat{\boldsymbol{\omega}}_2 = \widehat{\boldsymbol{\omega}}_1 + (\boldsymbol{R}_1\boldsymbol{\omega}_d)^\wedge.$$

By applying $\vee$ to both sides of this equation we get,

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \boldsymbol{R}_1\boldsymbol{\omega}_d. \tag{2.51}$$

Two Object Case

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In summary, we first defined the position and attitude of object 1 and object 2 as $(\boldsymbol{p}_1, \boldsymbol{R}_1)$, $(\boldsymbol{p}_2, \boldsymbol{R}_2)$. If the velocity of object 1 is $(\boldsymbol{v}_1, \boldsymbol{\omega}_1)$ and the velocity of object 2 relative to object 1 is $(\boldsymbol{v}_d, \boldsymbol{\omega}_d)$, the velocity of object 2 will be given as follows[8]

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{R}_1\boldsymbol{v}_d + \boldsymbol{\omega}_1 \times (\boldsymbol{p}_2 - \boldsymbol{p}_1) \tag{2.52}$$

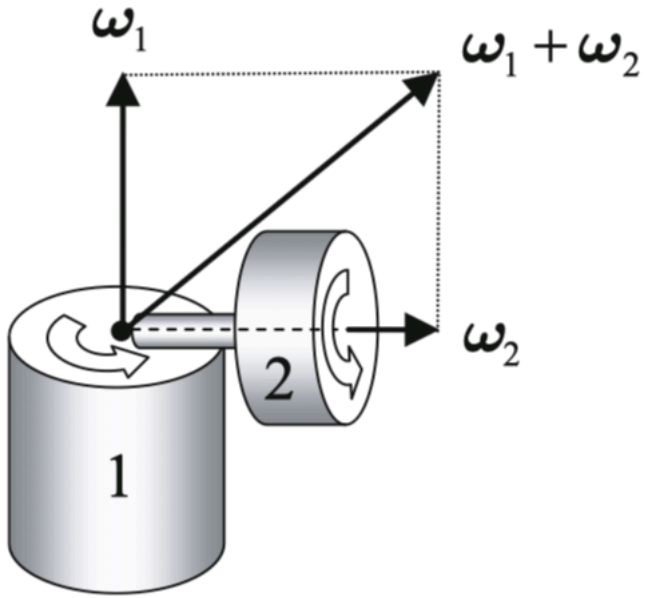
$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \boldsymbol{R}_1\boldsymbol{\omega}_d. \tag{2.53}$$

Here $\boldsymbol{R}_1\boldsymbol{v}_d$ and $\boldsymbol{R}_1\boldsymbol{\omega}_d$ describe the relative velocity between the objects in world coordinates. If we replace these using ${}^W\boldsymbol{v}_d, {}^W\boldsymbol{\omega}_d$ we get,

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 + {}^W\boldsymbol{v}_d + \boldsymbol{\omega}_1 \times (\boldsymbol{p}_2 - \boldsymbol{p}_1) \tag{2.54}$$

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + {}^W\boldsymbol{\omega}_d. \tag{2.55}$$

Therefore barring the fact that we get an influence of the rotational velocity the third element of (2.54), the velocity of the object in world coordinates is simply a sum.



Kinematics of a Humanoid Robot

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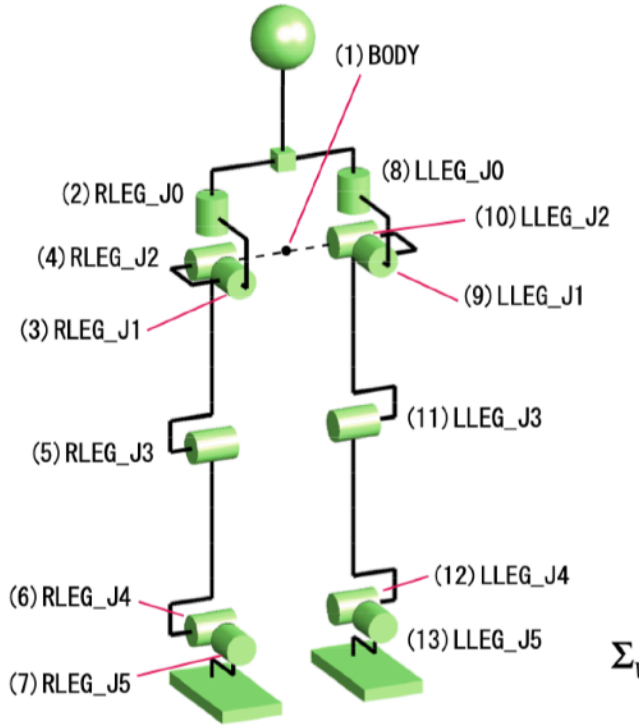
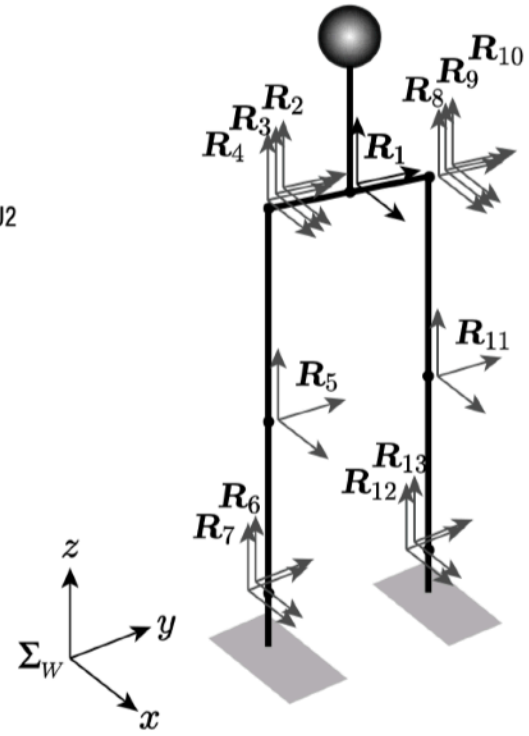
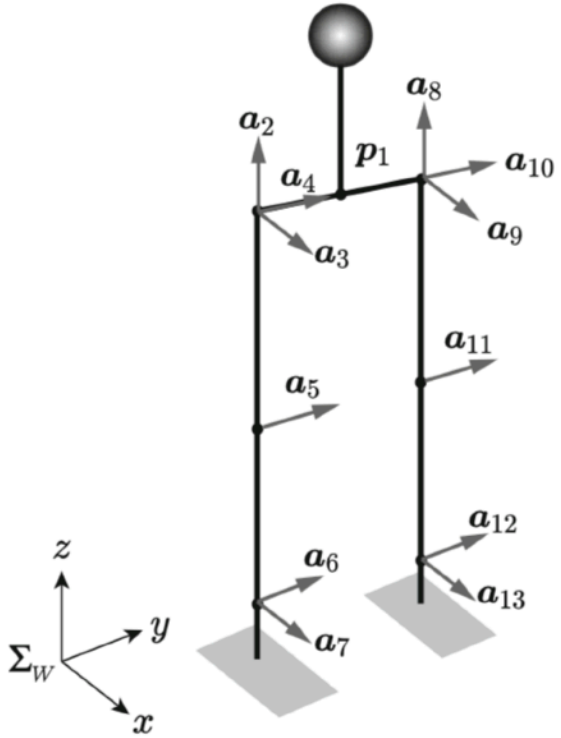


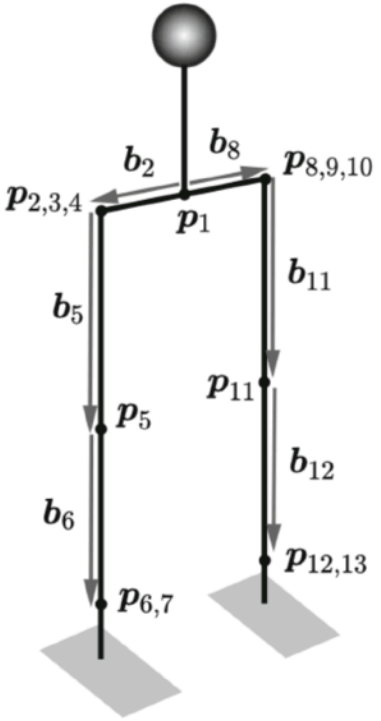
Structure of a robot



Rotation matrices



Joint axis vectors



Relative position and local coordinates.

Data Structures

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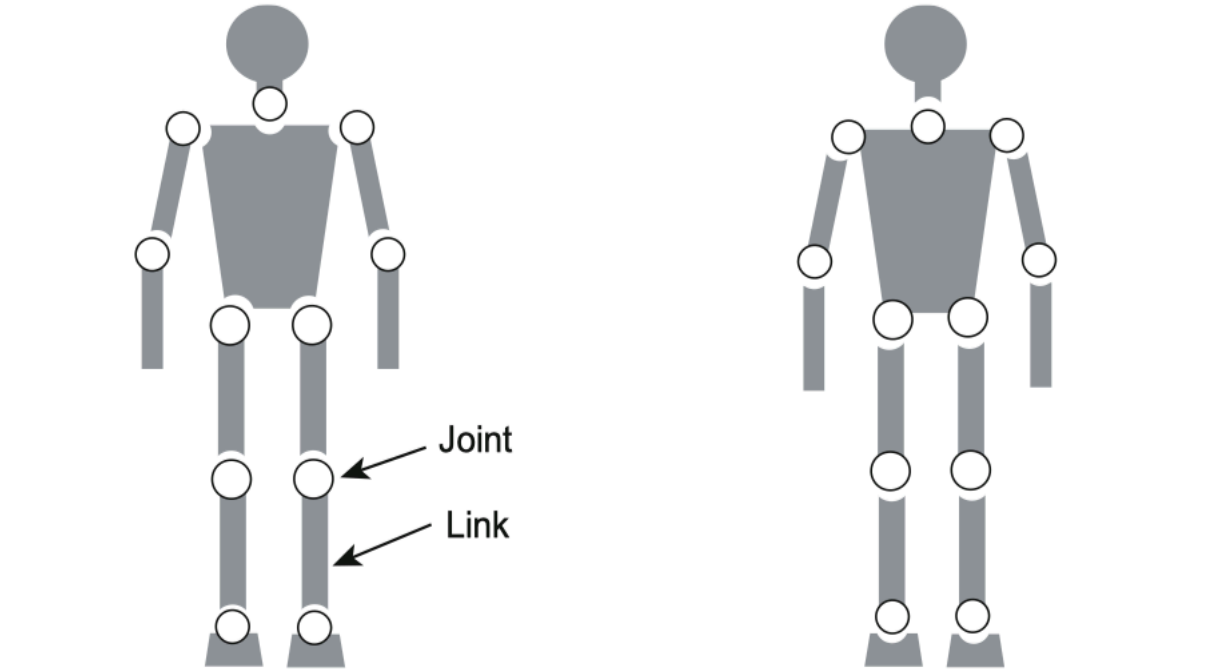

(a) Separate into joint+link units (b) Separate into link+joint units

What is a **proper data structure** for humanoid robots?

Data Structures

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Joints vs. Joints + Links?

Kinematic Chain

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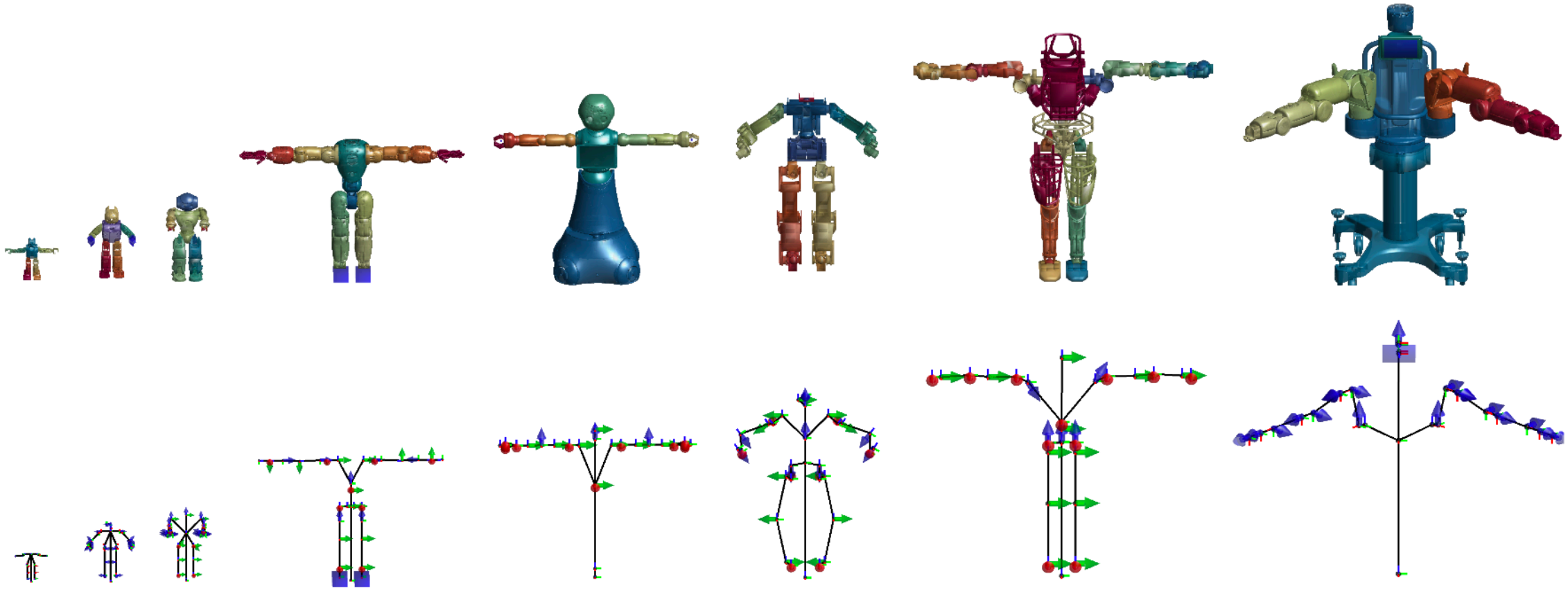

Kinematic Chain

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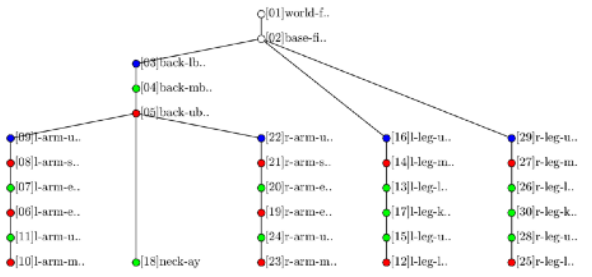
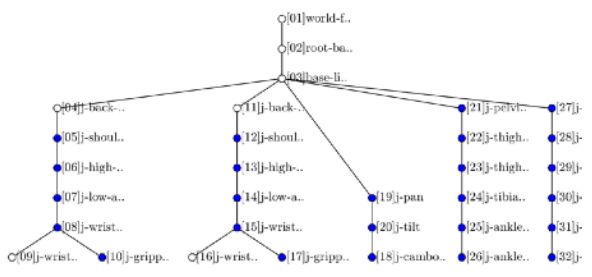
Atlas

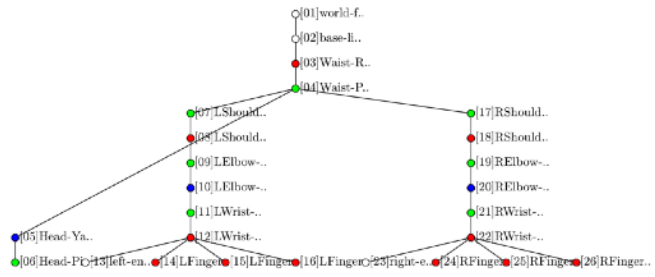


Coman

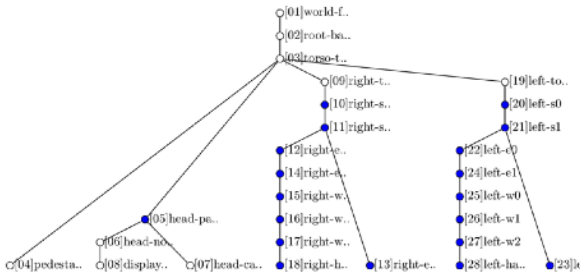
Darwin



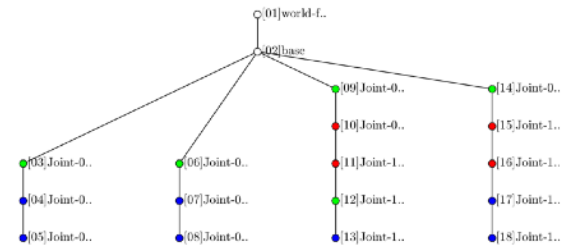
Social-robot



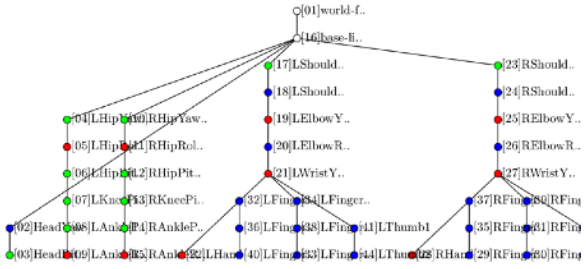
Baxter



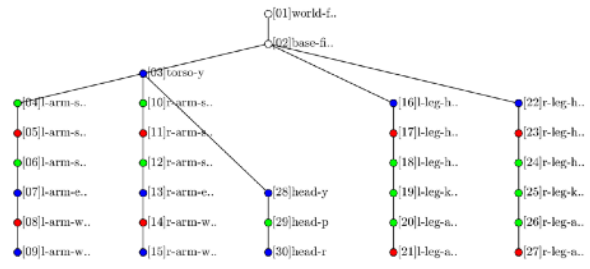
Mini



Nao



Thormang



Kinematics of a Humanoid Robot

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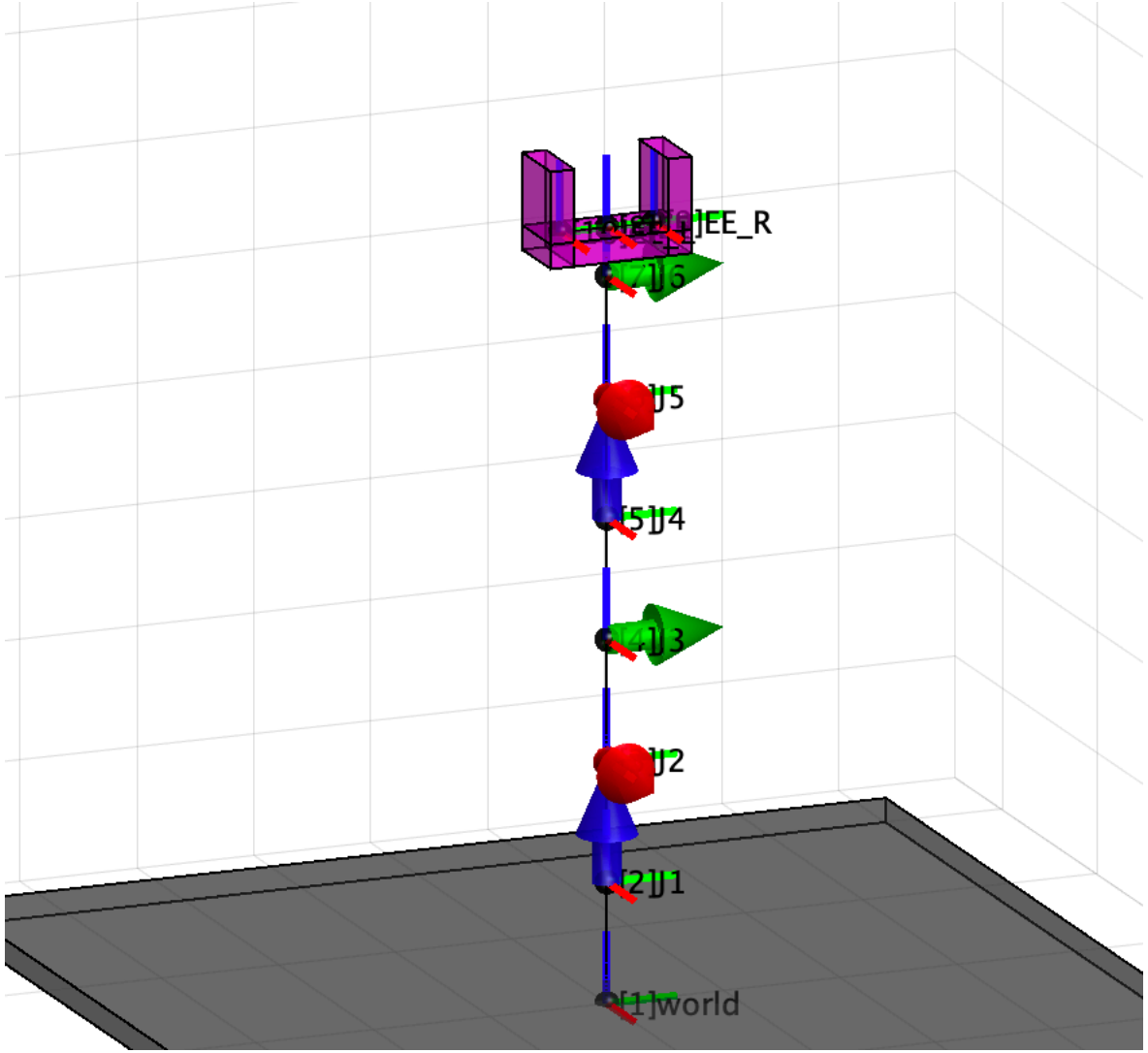

Kinematic Chain

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```
% Initialize a kinematic chain
chain = init_chain('name','kinematic_chain');

% Add joint to the chain
chain = add_joint_to_chain(chain,'name','world');
chain = add_joint_to_chain(chain,'name','J1','parent_name','world',...
    'p_offset',cv([0,0,0.5]),'a',cv([0,0,1]));
chain = add_joint_to_chain(chain,'name','J2','parent_name','J1',...
    'p_offset',cv([0,0,0.5]),'a',cv([1,0,0]));
chain = add_joint_to_chain(chain,'name','J3','parent_name','J2',...
    'p_offset',cv([0,0,0.5]),'a',cv([0,1,0]));
chain = add_joint_to_chain(chain,'name','J4','parent_name','J3',...
    'p_offset',cv([0,0,0.5]),'a',cv([0,0,1]));
chain = add_joint_to_chain(chain,'name','J5','parent_name','J4',...
    'p_offset',cv([0,0,0.5]),'a',cv([1,0,0]));
chain = add_joint_to_chain(chain,'name','J6','parent_name','J5',...
    'p_offset',cv([0,0,0.5]),'a',cv([0,1,0]));
chain = add_joint_to_chain(chain,'name','EE','parent_name','J6',...
    'p_offset',cv([0,0,0.2]),'a',cv([0,0,0]));
chain = add_joint_to_chain(chain,'name','EE_R','parent_name','EE',...
    'p_offset',cv([0,0.2,0]),'a',cv([0,0,0]));
chain = add_joint_to_chain(chain,'name','EE_L','parent_name','EE',...
    'p_offset',cv([0,-0.2,0]),'a',cv([0,0,0]));

% Add link to the chain
box_added = struct('xyz_min',[-2,-2,0],'xyz_len',[4,4,0.1],...
    'p_offset',cv([0,0,0]),'R_offset',rpy2r([0,0,0]*D2R),...
    'color',0.3*[1,1,1],'alpha',0.5,'ec','k');
chain = add_link_to_chain(chain,'name','base_link','joint_name','world','box_added',box_added);
box_added = struct('xyz_min',[-0.15,-0.3,-0.1],'xyz_len',[0.3,0.6,0.1],...
    'p_offset',cv([0,0,0]),'R_offset',rpy2r([0,0,0]*D2R),...
    'color','m','alpha',0.5,'ec','k');
chain = add_link_to_chain(chain,'name','EE_link','joint_name','EE','box_added',box_added);
box_added = struct('xyz_min',[-0.15,0,-0.1],'xyz_len',[0.3,0.1,0.4],...
    'p_offset',cv([0,0,0]),'R_offset',rpy2r([0,0,0]*D2R),...
    'color','m','alpha',0.5,'ec','k');
chain = add_link_to_chain(chain,'name','EE_R_link','joint_name','EE_R','box_added',box_added);
box_added = struct('xyz_min',[-0.15,-0.1,-0.1],'xyz_len',[0.3,0.1,0.4],...
    'p_offset',cv([0,0,0]),'R_offset',rpy2r([0,0,0]*D2R),...
    'color','m','alpha',0.5,'ec','k');
chain = add_link_to_chain(chain,'name','EE_L_link','joint_name','EE_L','box_added',box_added);
```

Kinematic Chain

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```
chain =

  struct with fields:

              name: 'kinematic_chain'
                dt: 0.0100
             joint: [1×10 struct]
       joint_names: {1×10 cell}
           n_joint: 10
     rev_joint_names: {'J1'  'J2'  'J3'  'J4'  'J5'  'J6'}
       n_rev_joint: 6
              link: [1×4 struct]
        link_names: {1×4 cell}
            n_link: 4
```

Kinematic Chain

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```
>> chain.joint

ans =

  1×10 struct array with fields:

    name
    p
    R
    a
    type
    p_offset
    R_offset
    q
    dq
    ddq
    q_diff
    q_prev
    v
    vo
    w
    dvo
    dw
    u
    ext_f
    parent
    childs
    link_idx
    limit
```

```
>> chain.link

ans =

  1×4 struct array with fields:

    name
    joint_idx
    p_offset
    R_offset
    mesh_path
    scale
    fv
    box
    capsule
    box_added
    m
    I_bar
    com_bar
    v
    vo
    w
```

```
% Initialize a kinematic chain
chain = init_chain('name','kinematic_chain');

% Add joint to the chain
chain = add_joint_to_chain(chain,'name','world');
chain = add_joint_to_chain(chain,'name','J1','parent_name','world',...
    'p_offset',cv([0,0,0.5]),'a',cv([0,0,1]));
chain = add_joint_to_chain(chain,'name','J2','parent_name','J1',...
    'p_offset',cv([0,0,0.5]),'a',cv([1,0,0]));
chain = add_joint_to_chain(chain,'name','J3','parent_name','J2',...
    'p_offset',cv([0,0,0.5]),'a',cv([0,1,0]));
chain = add_joint_to_chain(chain,'name','J4','parent_name','J3',...
    'p_offset',cv([0,0,0.5]),'a',cv([0,0,1]));
chain = add_joint_to_chain(chain,'name','J5','parent_name','J4',...
    'p_offset',cv([0,0,0.5]),'a',cv([1,0,0]));
chain = add_joint_to_chain(chain,'name','J6','parent_name','J5',...
    'p_offset',cv([0,0,0.5]),'a',cv([0,1,0]));
chain = add_joint_to_chain(chain,'name','EE','parent_name','J6',...
    'p_offset',cv([0,0,0.2]),'a',cv([0,0,0]));
chain = add_joint_to_chain(chain,'name','EE_R','parent_name','EE',...
    'p_offset',cv([0,0.2,0]),'a',cv([0,0,0]));
chain = add_joint_to_chain(chain,'name','EE_L','parent_name','EE',...
    'p_offset',cv([0,-0.2,0]),'a',cv([0,0,0]));

% Add link to the chain
box_added = struct('xyz_min',[-2,-2,0],'xyz_len',[4,4,0.1],...
    'p_offset',cv([0,0,0]),'R_offset',rpy2r([0,0,0]*D2R),...
    'color',0.3*[1,1,1],'alpha',0.5,'ec','k');
chain = add_link_to_chain(chain,'name','base_link','joint_name','world','box_added',box_added);
box_added = struct('xyz_min',[-0.15,-0.3,-0.1],'xyz_len',[0.3,0.6,0.1],...
    'p_offset',cv([0,0,0]),'R_offset',rpy2r([0,0,0]*D2R),...
    'color','m','alpha',0.5,'ec','k');
chain = add_link_to_chain(chain,'name','EE_link','joint_name','EE','box_added',box_added);
box_added = struct('xyz_min',[-0.15,0,-0.1],'xyz_len',[0.3,0.1,0.4],...
    'p_offset',cv([0,0,0]),'R_offset',rpy2r([0,0,0]*D2R),...
    'color','m','alpha',0.5,'ec','k');
chain = add_link_to_chain(chain,'name','EE_R_link','joint_name','EE_R','box_added',box_added);
box_added = struct('xyz_min',[-0.15,-0.1,-0.1],'xyz_len',[0.3,0.1,0.4],...
    'p_offset',cv([0,0,0]),'R_offset',rpy2r([0,0,0]*D2R),...
    'color','m','alpha',0.5,'ec','k');
chain = add_link_to_chain(chain,'name','EE_L_link','joint_name','EE_L','box_added',box_added);

% Update chain mass, inertia, and com
chain = update_chain_mass_inertia_com(chain,'density',500);
```

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Thank You