Example) Show that $(W^{\perp})^{\perp} = W$

$\Rightarrow$ i) $W \subseteq (W^{\perp})^{\perp}$

· If $\vec{x} \in W$, then $\vec{x} \in (W^{\perp})^{\perp}$

ii) Let $r = \dim(W)$

$\dim(W^{\perp}) = n-r$, $\dim((W^{\perp})^{\perp}) = n-(n-r) = r$

$\because$ The only r-dimensional subspace of $(W^{\perp})^{\perp}$ is W.

Theorem (FTLA) $A \in \mathbb{R}^{m\times n}$ of rank r

- $\dim Col(A) = r$
- $\dim Col(A^T) = r$
- $\dim Nul(A) = n-r$
- $\dim Nul(A^T) = m-r$
- $Nul(A)$ is orthogonal to $Col(A^T)$ $\leftarrow$ $\underbrace{\dim Col(A^T)}_{=r} + \underbrace{\dim Nul(A)}_{=n-r} = n$
- $Nul(A^T)$ is orthogonal to $Col(A)$

• Example) Find all vectors orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

$\Rightarrow$ Find the orthogonal complement of $Col\left(\begin{bmatrix}1&0\\1&1\\1&1\end{bmatrix}\right)$

$\rightarrow Nul\left(\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}\right) \rightarrow \begin{bmatrix}1&1&1\\0&1&1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}\boxed{1}&0&0\\0&\boxed{1}&1\end{bmatrix}$ $\begin{cases}x_1 = 0\\ x_2 = -x_3\\ x_3 = x_3\end{cases} \rightarrow span\left(\begin{bmatrix}0\\-1\\1\end{bmatrix}\right)$

$\therefore$ Hence, $span\left\{\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$ is orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

Example) Let $V = \left\{\begin{bmatrix}a\\b\\c\end{bmatrix} : a+b = 2c\right\}$

Find a basis for the orthogonal complement of V.

$\Rightarrow V = Nul([1, 1, -2])$

$V^{\perp} = Col\left(\begin{bmatrix}1\\1\\-2\end{bmatrix}\right) = span\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}$

- A new perspective on $Ax=b$.

$Ax=b$ is solvable
$\Leftrightarrow$ $b \in Col(A)$ (direct)
$\Leftrightarrow$ b is orthogonal to $Nul(A^T)$ (indirect)

Example) Let $A=\begin{bmatrix}1&2\\3&1\\0&5\end{bmatrix}$. For which b does $Ax=b$ have a solution?

$\Rightarrow$ i) (old)

$$\left[\begin{array}{cc|c}1&2&b_1\\3&1&b_2\\0&5&b_3\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1&2&b_1\\0&-5&-3b_1+b_2\\0&5&b_3\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1&2&b_1\\0&-5&-3b_1+b_2\\0&0&-3b_1+b_2+b_3\end{array}\right]$$

$Ax=b$ is consistent $\Leftrightarrow$ $-3b_1+b_2+b_3=0$

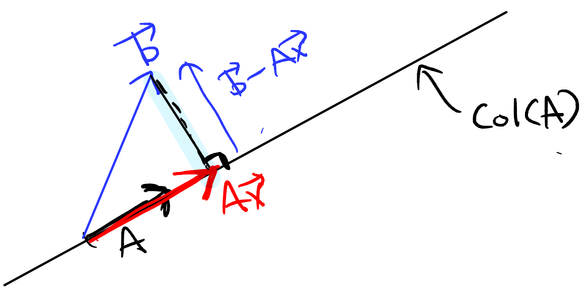
ii) (new)
$Ax=b$ is solvable $\Leftrightarrow$ b is orthogonal to $Nul(A^T)$

$$Nul(A^T): \begin{bmatrix}1&3&0\\2&1&5\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1&0&3\\0&1&-1\end{bmatrix} \Rightarrow \begin{cases}x_1=-3x_3\\x_2=x_2\\x_3=x_3\end{cases} \Rightarrow span\left\{\begin{bmatrix}-3\\1\\1\end{bmatrix}\right\}$$

$$[-3,1,1]\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=0 \rightarrow -3b_1+b_2+b_3=0$$

- Motivation

* Not all linear systems have solutions to $Ax=b$.



Instead of giving up, we want $\vec{x}$ which makes $A\vec{x}$ and $\vec{b}$ as close as possible $\longrightarrow$ $(\vec{b}-A\vec{x})$ minimize

Such $\vec{x}$ is characterized by $A\vec{x}$ being orthogonal to the error $\vec{b}-A\vec{x}$!

• Orthogonal bases

Definition: A basis $\vec{v}_1, \dots, \vec{v}_n$ of a vector space V is an orthogonal basis if the vectors are pairwise orthogonal.

Example : The standard basis $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ is an orthogonal basis for $\mathbb{R}^3$.

Example) Are the vectors $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$?

$\Rightarrow$ Yes

Example) Suppose that $\vec{v}_1, \dots, \vec{v}_n$ is an orthogonal basis of V. Then, find $c_1, \dots, c_n$ for $\vec{w} \in V$ such that

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \quad \text{—} (*)$$

$\Rightarrow$ Take the dot product of $\vec{v}_i$ on $(*)$

$$\vec{v}_i \cdot \vec{w} = c_i \|\vec{v}_i\|^2 \rightarrow c_i = \frac{\vec{v}_i \cdot \vec{w}}{\vec{v}_i \cdot \vec{v}_i}$$

Definition) A basis $\vec{v}_1, \dots, \vec{v}_n$ of a vectorspace V is an orthonormal basis if the vectors are orthogonal and have length 1.

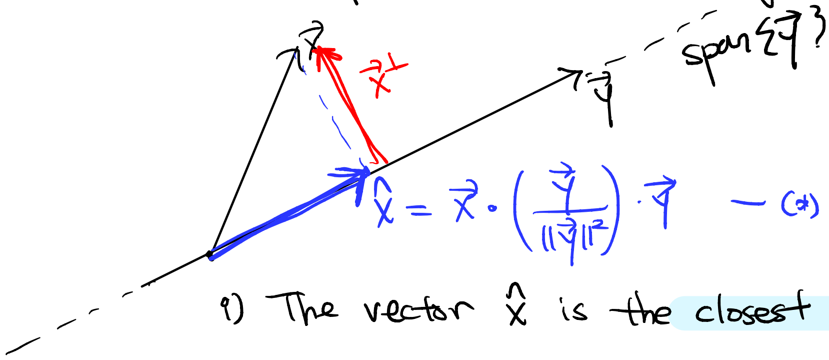
• Orthogonal projection

Definition) The orthogonal projection of a vector $\vec{x}$ onto vector $\vec{y}$ is

$$\hat{x} \triangleq \left(\frac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\right)\vec{y} \quad \text{—} (*)$$

What does this mean?

• Geometric interpretation of the orthogonal projection



i) The vector $\hat{x}$ is the closest vector to $\vec{x}$ which is in span$\{\vec{y}\}$.

ii) The error $\vec{x}^{\perp} = \vec{x} - \hat{x}$ is orthogonal to span$\{\vec{y}\}$

iii) (*) can be derived from i) and ii)

$\rightarrow$ 1. $\hat{x}$ can be written as $c\cdot\vec{y}$ ($\because$ i))

2. Then, the error $(\vec{x} - \hat{x})$ is in Nul$(\vec{y})$.

$\rightarrow (\vec{x} - \hat{x})\cdot\vec{y} = (\vec{x} - c\vec{y})\cdot\vec{y} = \vec{x}\cdot\vec{y} - c\,\vec{y}\cdot\vec{y} = 0$

$\therefore c = \frac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}} \rightarrow \hat{x} = c\vec{y} = \frac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\cdot\vec{y}$ — (*)

Example) What is the orthogonal projection of $\vec{x} = \begin{bmatrix}-8\\4\end{bmatrix}$ onto $\vec{y} = \begin{bmatrix}3\\1\end{bmatrix}$?

$$\Rightarrow \quad \hat{x} = \frac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\vec{y} = \frac{-24+4}{10}\vec{y} = -2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-6\\-2\end{bmatrix}$$

- Orthogonal projection on subspaces

Theorem) Let W be a subspace of $\mathbb{R}^n$. Then, each $\vec{x} \in \mathbb{R}^n$ can be uniquely written as

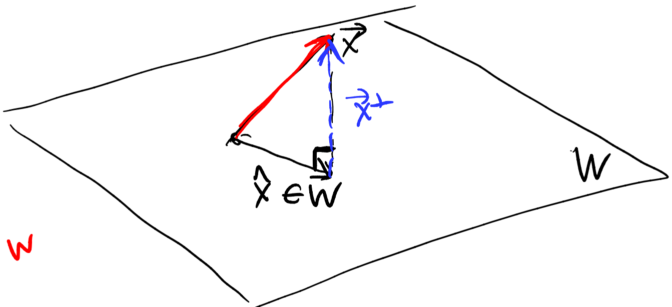
$$\vec{x} = \hat{x} + \vec{x}^{\perp}$$

where $\hat{x} \in W$ and $\vec{x}^{\perp} \in W^{\perp}$.

* $\hat{x}$ is the orthogonal projection of $\vec{x}$ onto W.

$\hat{x}$ is the point in W closest to X

$\rightarrow$ For any other $\vec{y} \in W$, $\text{dist}(\vec{x}, \hat{x}) < \text{dist}(\vec{x}, \vec{y})$



orthogonal projection onto W

If $\vec{v}_1, \dots, \vec{v}_m$ is an orthogonal basis of W, then,

$$\hat{x} = \left(\frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \cdots + \left(\frac{\vec{x}\cdot\vec{v}_m}{\vec{v}_m\cdot\vec{v}_m}\right)\vec{v}_m$$

- First, we will show that:

Let W be a subspace of a vector space V ($W \subseteq V$) and $\vec{v} \in V$.
Then, $\vec{w} \in W$ is closest to V iff $(\vec{v} - \vec{w}) \in W^{\perp}$ (or $\vec{v} - \vec{w} \perp W$)

a) If $(\vec{v} - \vec{w}) \in W^{\perp}$ then $\|\vec{v} - \vec{w}\| \le \|\vec{v} - \vec{u}\|$ for all $\vec{u} \in W$ and $\|\vec{v} - \vec{w}\| = \|\vec{v} - \vec{u}\|$ iff $\vec{w} = \vec{u}$, thus, $\vec{w}$ is the closest point in W to $\vec{v}$.

$\Rightarrow$ First, remark that

$$\|\vec{v} - \vec{w}\| \le \|\vec{v} - \vec{u}\| \iff \|\vec{v} - \vec{w}\|^2 \le \|\vec{v} - \vec{u}\|^2$$

Hence

$$\begin{aligned}
\|\vec{v} - \vec{u}\|^2 &= \|(v-w) + (w-u)\|^2 \\
&= \|v-w\|^2 + \|w-u\|^2 + \underbrace{2\cdot\langle v-w, w-u\rangle}_{=0} \\
&\ge \|v-w\|^2
\end{aligned}$$

($\vec{v} \in V$, $\vec{w} \in W$, $\vec{u} \in W$, $\vec{v} - \vec{w} \in W^{\perp}$, $\vec{w} - \vec{u} \in W$)

where the equality holds when $\|w-u\|^2 = 0 \iff w = u$.

(b) If $\vec{w} \in W$ is the closest to $\vec{v} \in V$, then $\vec{v}-\vec{w} \in W^{\perp}$.

$\Rightarrow$ We know that $\|\vec{v}-\vec{w}\|^2 \le \|\vec{v}-\vec{u}\|^2$ for all $\vec{u} \in W$

Therefore the function $f: \mathbb{R} \to \mathbb{R}$

$$F(t) = \|\vec{v}-\vec{w} + t\vec{x}\|^2, \quad \vec{x} \in W$$

has a minimum at $t=0$ for all $\vec{x} \in W$.

$$\Rightarrow F(t) = \langle \vec{v}-\vec{w}+t\vec{x}, \vec{v}-\vec{w}+t\vec{x} \rangle$$
$$= \|\vec{v}-\vec{w}\|^2 + 2t\langle \vec{v}-\vec{w}, \vec{x} \rangle + t^2\|\vec{x}\|^2$$

and

$$F'(t) = 2\langle \vec{v}-\vec{w}, \vec{x} \rangle + 2t\|\vec{x}\|^2 = 0 \Big|_{t=0}$$

$$\to \langle \vec{v}-\vec{w}, \vec{x} \rangle = 0 \quad \text{for all } \vec{x} \in W.$$

which shows that $\vec{v}-\vec{w} \in W^{\perp}$

$\therefore$ $\vec{w} \in W$ is closest to $\vec{v} \in V$ iff $\vec{v}-\vec{w} \in W^{\perp}$.