- Subspaces

- Definition: A subset W of a vector space V is a subspace if W is itself a vectorspace,

Since the rules like associativity, commutativity, and distributivity still hold, we only need to check the followings:

$W \subseteq V$ is a subspace of V if

1) W contains the zero vector
2) W is closed under addition.
3) W is closed under scaling

Example) Is $W = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ a subspace of $\mathbb{R}^2$?

$\Rightarrow$ Yes,

1) $\begin{bmatrix}0\\0\end{bmatrix} \in W$
2) $\underbrace{\begin{bmatrix}a\\a\end{bmatrix}}_{\in W} + \underbrace{\begin{bmatrix}b\\b\end{bmatrix}}_{\in W} = \begin{bmatrix}a+b\\a+b\end{bmatrix} \in W$
3) $r\underbrace{\begin{bmatrix}a\\a\end{bmatrix}}_{\in W} = \begin{bmatrix}ra\\ra\end{bmatrix} \in W$

Example) Is $W = \left\{\begin{bmatrix}a\\0\\b\end{bmatrix} ; a,b \in \mathbb{R}\right\}$ a subspace of $\mathbb{R}^3$?

$\Rightarrow$ Yes,

Example) Is $W = \left\{\begin{bmatrix}a\\1\\b\end{bmatrix} ; a,b \in \mathbb{R}\right\}$ a subspace of $\mathbb{R}^3$?

$\Rightarrow$ No,

1) $\begin{bmatrix}0\\0\\0\end{bmatrix} \notin W$
2) $\begin{bmatrix}a\\1\\b\end{bmatrix} + \begin{bmatrix}c\\1\\d\end{bmatrix} = \begin{bmatrix}a+c\\2\\b+c\end{bmatrix} \notin W$

Example) Is $W = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ a subspace of $\mathbb{R}^2$?

$\Rightarrow$

1) $\begin{bmatrix}0\\0\end{bmatrix} \in W$
2) $\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \in W$
3) $r\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \in W$

Yes.

Example) Is $W = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

$\Rightarrow$ No.

◎ Span of vectors are subspaces

Review : The span of vectors $v_1, \dots, v_m$ is the set of all their linear combinations
We denote it by $\text{span}\{v_1, \dots, v_m\} = \{c_1v_1 + c_2v_2 + \dots + c_mv_m : c_1 \dots c_m \in \mathbb{R}\}$.

Theorem) If $v_1, \dots, v_m$ are in a vector space V, then $\text{span}\{v_1, \dots, v_m\}$ is a subspace of V.

$\Rightarrow$ 1) $\phi \in \text{span}\{v_1, \dots, v_m\}$

2) $\underbrace{[c_1v_1 + \dots + c_mv_m]}_{\in U} + \underbrace{[d_1v_1 + \dots + d_mv_m]}_{\in U}$

$= [(c_1+d_1)v_1 + \dots + (c_m+d_m)v_m] \in U$

3) $\gamma[c_1v_1 + \dots + c_mv_m] = (\gamma c_1)v_1 + \dots + (\gamma c_m)v_m \in U$

Example) Is $W = \left\{ \begin{bmatrix} a+3b \\ 2a-b \end{bmatrix} : a,b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$?

$\Rightarrow$ $W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} a + \begin{bmatrix} 3 \\ -1 \end{bmatrix} b : a,b \in \mathbb{R} \right\}$

$= \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right)$

Yes.

Example) Is $W = \left\{ \begin{bmatrix} -a & 2b \\ a+b & 3a \end{bmatrix} : a,b \in \mathbb{R} \right\}$ a subspace of $M_{2\times2}$?

$\Rightarrow$ $W = \text{span}\left\{ \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right\}$

Example) Are the following sets vector space?

(a) $W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+3b=0,\ 2a-c=1 \right\}$

$\Rightarrow$ i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$

No.

(b) $W_2 = \left\{ \begin{bmatrix} a+c & -2b \\ b+3c & c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

$\Rightarrow$ Yes, $W_2 = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}$

(c) $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : ab \geq 0 \right\}$

$\Rightarrow$ No, i) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W_3$ ii) $\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\in W_3} + \underbrace{\begin{bmatrix} -2 \\ -4 \end{bmatrix}}_{\in W_3} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \notin W_3$

(d) W_4 is the set of all polynomials $p(t)$ such that $p'(2)=1$.

$\Rightarrow$ i) $p_0(t)=0 = 0+0\cdot t+0\cdot t^2+\cdots \notin W_4$.

ii) $p_a(t), p_b(t) \in W_4 \rightarrow p_a'(2)=p_b'(2)=1$.

$p_c(t) = p_a(t)+p_b(t) \rightarrow p_c'(t) = p_a'(t)+p_b'(t)\Big|_{t=2} \rightarrow p_c'(2)=1+1=2$

hence $p_c(t) \notin W_4$.

$\therefore$ No

(e) W_5 is the set of all polynomials $p(t)$ s.t. $p'(2)=0$.

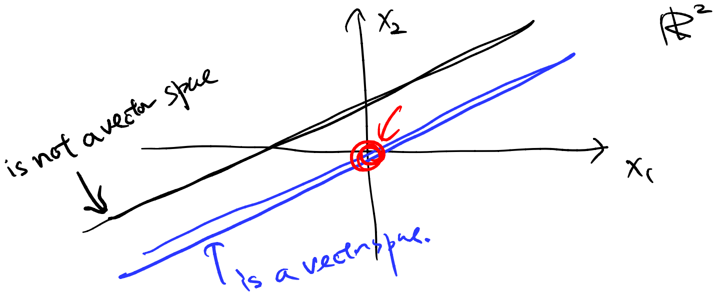
$\Rightarrow$ i) $p_0(t)=0$, $p_0'(t)=0 \rightarrow p_0(t) \in W_5$

ii) $p_a(t), p_b(t) \in W_5$, $p_c(t) = p_a(t)+p_b(t) \in W_5$ $\because p_c'(t)\Big|_{t=2} = p_a'(t)+p_b'(t)\Big|_{t=2}=0$

iii) $p_a(t) \in W_5$, $p_c(t) = \gamma\cdot p_a(t) \in W_5$, $\because p_c'(t)\Big|_{t=2} = \gamma\cdot p_a'(t)\Big|_{t=2}=0$

(f) $W_6 = \left\{ \begin{bmatrix} a+c & -2b \\ b+3c & c+7 \end{bmatrix}, a,b,c \in \mathbb{R} \right\}$

$\Rightarrow W_6 = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} + \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}$



⊙ Solving $Ax=0$ and $Ax=b$

- Column space

Definition) The column space $C(A)$ of a matrix A is the span of the columns of A.

If $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$, $\mathrm{Col}(A) = \mathrm{span}\{a_1, \dots, a_n\}$

- In other words, $Ax=b$ has a solution iff $b \in \mathrm{Col}(A)$.

↳ why? Because, $Ax = \vec{a}_1 x_1 + \vec{a}_2 x_2 + \cdots + \vec{a}_n x_n$ is a linear combination of columns of A with the coefficients given by x.

- If $A \in \mathbb{R}^{m\times n}$, then $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Example) Find a matrix A such that $W = \mathrm{Col}(A)$ where

$$W = \left\{ \begin{bmatrix} 2x-y \\ 3y \\ 7x+y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$\Rightarrow \left\{ \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix} x + \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} y : x, y \in \mathbb{R} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\} = \mathrm{Col}\left(\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 7 & 1 \end{bmatrix} \right)$$

$$\therefore A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 7 & 1 \end{bmatrix}$$

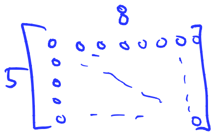
Is it unique? No, $\tilde{A} = \begin{bmatrix} 2 & -2 \\ 0 & 6 \\ 7 & 2 \end{bmatrix} \neq A$ but $\mathrm{Col}(\tilde{A}) = W$.

⊙ Null spaces

Definition) The nullspace of a matrix A is

$$\mathrm{Nul}(A) = \{ x : Ax = 0 \}$$

Theorem) If $A \in \mathbb{R}^{m\times n}$, then $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

* $A \in \mathbb{R}^{5\times 8}$ →  $\mathrm{Col}(A) \subseteq \mathbb{R}^5$ $\quad$ $\mathrm{Nul}(A) \subseteq \mathbb{R}^8$

⇒ i) $\phi \in \mathrm{Nul}(A)$ since $A\cdot\phi = \phi$

ii) $x, y \in \mathrm{Nul}(A) \rightarrow Ax = Ay = 0$

$z = x+y \rightarrow Az = Ax + Ay = 0$, $z \in \mathrm{Nul}(A)$ → closed under addition.

iii) $x \in \mathrm{Nul}(A) \rightarrow Ax = 0$

$z = \gamma x \rightarrow A\cdot\gamma x = \gamma Ax = 0 \rightarrow z \in \mathrm{Nul}(A)$ → closed under scaling

Example) Find an explicit description of Nul(A) where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

$\Rightarrow$ $\text{Nul}(A) = \{ x \mid Ax = 0 \}$

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 - 2R1} \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \xrightarrow{R1 \leftarrow \frac{1}{3}R1} \begin{bmatrix} \boxed{1} & 2 & 2 & 1 & 3 \\ 0 & 0 & \boxed{1} & -6 & -15 \end{bmatrix}$$

$$\xrightarrow{R1 \leftarrow R1 - 2R2} \begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = -2x_2 - 13x_4 - 33x_5 \\ x_2 = \text{free} \\ x_3 = 6x_4 + 15x_5 \\ x_4 = \text{free} \\ x_5 = \text{free} \end{cases}$$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} x_5 = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Another look at solutions to $Ax = b$.

Theorem) Let x_p be a solution of $Ax = b$. Then, every solution to $Ax = b$ is of the form $x = x_p + x_h$ where x_h is a solution to the homogeneous equation $Ax = 0$.

In other words, $\{x : Ax = b\} = \underbrace{x_p}_{\text{particular solution.}} + \text{Nul}(A)$

$\Rightarrow$ Let $\bar{x}$ be another solution to $A\bar{x} = b$,

Then, $(\bar{x} - x_p)$ satisfies $(\bar{x} - x_p) \in \text{Nul}(A)$ since

$$A(\bar{x} - x_p) = A\bar{x} - Ax_p = b - b = 0$$

Example) Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$. Find $Ax = b$.

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 7 & 5 \\ -1 & -3 & 3 & 4 & 5 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{cccc|c} \boxed{1} & 3 & 0 & -1 & -2 \\ 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = -2 - 3x_2 + x_4 \\ x_2 = \text{free} \\ x_3 = 1 - x_4 \\ x_4 = \text{free} \end{cases} \Rightarrow x = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{= x_p} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} x_4$$

$$\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

- True or False

a) The solutions to the equation $Ax=b$ form a vector space?

$\Rightarrow$ False

* $V=\{x \mid \underbrace{Ax=0\}}_{Nul(A)}$ is a vector space!

b) The solutions to $Ax=0$ is a vector space

$\Rightarrow$ True.