The inverse of a matrix

- Definition An $n \times n$ matrix A is invertible if there is a matrix B such that

$$AB = BA = I$$

In that case, B is the inverse of A and we write $A^{-1} = B$

Example)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note • The inverse of a matrix is unique.

→ Suppose that B and C are both inverses of A

$$\rightarrow C = \underbrace{CI = CAB}_{I=AB} = IB = B$$

• Why we do not write $\frac{A}{B}$?

$\Rightarrow AB^{-1} \neq B^{-1}A$

• If $AB = I$, then $BA = I$?

$\Rightarrow$ Yes.

pf) First, we will borrow some concepts such as basis (linearly independent vectors)

i) Let $\{\vec{x}_1, \cdots, \vec{x}_n\}$ be bases of the space. We first show that $\{B\vec{x}_1, B\vec{x}_2, \cdots, B\vec{x}_n\}$ is also a basis.

Suppose $\{B\vec{x}_1, \cdots, B\vec{x}_n\}$ is not a basis, Then, there exist $c_1, c_2, \cdots, c_n \in \mathbb{R}$ not all equal to zero such that

$$c_1 B\vec{x}_1 + c_2 B\vec{x}_2 + \cdots + c_n B\vec{x}_n = 0 \quad \text{— } (*)$$

Pre-multiplying $(*)$ with A, we get

$$c_1 \underbrace{AB}_{=I}\vec{x}_1 + c_2 AB\vec{x}_2 + \cdots + c_n AB\vec{x}_n = 0$$

$$\hookrightarrow c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = 0$$

This is contradiction with $\{\vec{x}_1, \cdots, \vec{x}_n\}$ being a basis.

* Hence, $\{B\vec{x}_1, \cdots, B\vec{x}_n\}$ is a basis, and every vector $\vec{y}$ can be represented as a linear combination of those vectors.

$\Rightarrow \forall \vec{y}, \exists \vec{x}$ s.t. $B\vec{x} = \vec{y}$ — $(**)$

ii) Now, to prove $BA = I$, it is identical to show that,

$\forall \vec{y}, \; BA\vec{y} = \vec{y}$ — $(***)$

Then, substituting $(**)$ to the LHS (left-hand-side) of $(***)$ we get

$$\forall \vec{y}, \; BA\vec{y} \underset{(**)}{=} BAB\vec{x} \underset{\text{(associativity)}}{=} B(AB)\vec{x} = B\vec{x} = \vec{y},$$ which completes the proof.

- Solving systems using matrix inverse.

Theorem) Let A be invertible, then the system $A\vec{x}=\vec{b}$ has the unique solution $\vec{x}=A^{-1}\vec{b}$.

$\rightarrow A\vec{x}=\vec{b} \rightarrow A^{-1}A\vec{x}=A^{-1}\vec{b} \rightarrow \vec{x}=A^{-1}\vec{b}$

Example) Solve $\begin{cases} -7x_1+3x_2=2 \\ 5x_1-2x_2=1 \end{cases}$ using matrix inversion.

$* \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\rightarrow \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\vec{x}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{x}=\begin{bmatrix} -7 & 3 \\ 5 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x}=\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 7 \\ 17 \end{bmatrix}$

- Recipe for computing the inverse

  - To solve $A\vec{x}=\vec{b}$, we do row reduction on $[A \mid \vec{b}]$
  - To solve $AX=I$, we do row reduction on $[A \mid I]$

- To compute A^{-1}
  - Form the augmented matrix $[A \mid I]$
  - Compute the reduced echelon form (aka Gauss-Jordan elimination)
  - If A is invertible, the result is of the form $[I \mid A^{-1}]$

$$[A \mid I] \xrightarrow{G.J.} [I \mid A^{-1}]$$

- Example) Find the inverse of $A=\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -3 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R3 \leftarrow R3+\frac{3}{2}R1} \left[\begin{array}{ccc|ccc} \boxed{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & \frac{3}{2} & 1 & 0 \end{array}\right]$

$\xrightarrow{R1 \leftarrow \frac{1}{2}R1} \left[\begin{array}{ccc|ccc} 1 & & & \frac{1}{2} & 0 & 0 \\ & 1 & & 0 & 0 & 1 \\ & & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$ $\quad$ Hence $A^{-1}=\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Theorem) Suppose A and B are invertible,
- A^{-1} is invertible, $(A^{-1})^{-1}=A$
- A^{T} is invertible, $(A^{T})^{-1}=(A^{-1})^{T}$

$\Rightarrow (A^{-1})^{T}A^{T}=(A\cdot A^{-1})^{T}=I^{T}=I$

- (AB) is invertible, $(AB)^{-1}=B^{-1}A^{-1}$

$\Rightarrow (B^{-1}A^{-1})(AB)=B^{-1}\underbrace{(A^{-1}A)}B=B^{-1}B=I$

Theorem) Let A be an $n \times n$ matrix, then the following statements are equivalent.

(a) A is invertible
(b) A is row equivalent to I
(c) A han n pivots.
(d) $A\vec{x} = \vec{b}$ has a unique solution
(e) $\exists B$ s.t. $AB = I$
(f) $\exists C$ s.t. $CA = I$

- Matrices that are not invertible are often called singular.

• Vector spaces and subspaces

• Definition) A vector space is a nonempty set V of vectors, which may be added and scaled.

• Axioms ($u, v, w \in V$, $c, d \in \mathbb{R}$)

(a) $u+v \in V$

(b) $v+u = u+v$ (commutativity)

(c) $(u+v)+w = u+(v+w)$ (associativity)

(d) $\exists\, 0 \in V$, s.t. $u+0 = u$

(e) $\exists -u \in V$, s.t. $u+(-u) = 0$

(f) $cu \in V$

(g) $c(u+v) = cu+cv$ (distributivity)

(h) $(c+d)u = cu+du$

(i) $(cd)u = c\cdot(du)$

(j) $1u = u$

Example) $M_{2\times2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$ is vector space?

→ i) zero vector : $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

ii) addition $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \triangleq \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \in M_{2\times2}$

iii) scaling $r\begin{bmatrix} a & b \\ c & d \end{bmatrix} \triangleq \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} \in M_{2\times2}$

Example) Let $\mathbb{P}_n$ be the set of all polynomials of degree at most $n \geq 0$.
Is $\mathbb{P}_n$ a vector space?

→ Members of $\mathbb{P}_n$ are of the form

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

where $a_1, \ldots, a_n \in \mathbb{R}$ and t is a variable.

Then, we can define the addition and scaling as follows:

i) addition : $[a_0 + a_1 t + \cdots + a_n t^n] + [b_0 + b_1 t + \cdots + b_n t^n]$
$\triangleq (a_0+b_0) + (a_1+b_1)t + \cdots + (a_n+b_n)t^n \in \mathbb{P}_n$

ii) scaling $r(a_0 + a_1 t + \cdots + a_n t^n) \triangleq (ra_0) + (ra_1)t + \cdots + (ra_n)t^n \in \mathbb{P}_n$

Note that

$$(a_0 + a_1 t + \cdots + a_n t^n) \xleftrightarrow{\text{isomorphic}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{n+1}$$

Example) Let $\mathbb{V}$ be the set of all polynomials of degree exactly 3.
Is $\mathbb{V}$ a vector space?

$\Rightarrow$ No. $\because$ reason 1: $\mathbb{V}$ does not contain the zero polynomial,
i.e., $0+0t+\cdots+0t^3 \notin \mathbb{V}$

reason 2: $\mathbb{V}$ is not closed under addition.

$(3t^3+t) \in \mathbb{V}$

$(-3t^3+3t+5) \in \mathbb{V}$

$\hookrightarrow (3t^3+t)+(-3t^3+3t+5) = 4t+5 \notin \mathbb{V}$