- Matrix times matrix

$$AB \triangleq \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} \triangleq \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{bmatrix}$$

matrix times vector

⇒ Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B.

Remark) Show that $(AB)\vec{x} = A(B\vec{x})$ (associative property)

→ $(AB)\vec{x} = x_1 \underbrace{[AB]_1}_{\text{first column of }(AB)} + x_2 [AB]_2 + \cdots$

$* \ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$

$= x_1 A b_1 + x_2 A b_2 + \cdots$

$= A(\underbrace{x_1 b_1 + x_2 b_2 + \cdots}_{= Bx})$

$= A(B\vec{x})$

Example) Suppose that $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$

(a) Under which condition does AB make sense?

→ $n = p$

(b) What are the dim. of AB?

$AB \in \mathbb{R}^{m\times q}$

Theorem
- $A(BC) = (AB)C$ (associative) HW.
- $A(B+C) = AB+AC$ (left distributive)
- $(A+B)C = AC+BC$ (right " — ")

- Transpose of a matrix

Definition The transpose A^T of a matrix A is the matrix whose columns are formed from the corresponding row of A.

Example) $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$

* Symmetric : $A = A^T$

Example) Show that $(AB)^T = B^T A^T$

$\Rightarrow$ $\underbrace{(i,j)\text{-th element of } AB}_{[AB]_{(i,j)}}$ is $\langle \underbrace{[A]_{(i)}}_{\text{ith row vector of } A}, \underbrace{[B]_j}_{\text{jth column vector}} \rangle$

$$[(AB)^T]_{(i,j)} = [(AB)]_{(j,i)} = \langle [A]_{(j)}, [B]_i \rangle \quad \text{— } (*)$$

$$[B^T A^T]_{(i,j)} = \langle [B^T]_{(i)}, [A^T]_j \rangle = \langle [B]_i, [A]_{(j)} \rangle \quad \text{— } (**)$$

$\because$ $(*) = (**)$ $\Rightarrow$ $(AB)^T = B^T A^T$

Theorem
- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Example) Deduce that $(ABC)^T = C^T B^T A^T$.

$\Rightarrow$ $(ABC)^T = ((AB)C)^T = C^T (AB)^T = C^T B^T A^T$.

- LU Decomposition

- Definition) An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

→ The result of an elementary row operation on A is EA. elementary matrix

* $\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{E} \Rightarrow \underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{E}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}c&d\\a&b\end{bmatrix}$

* $\begin{bmatrix}a&b\\c&d\end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix}c&d\\a&b\end{bmatrix}$ the same

* $\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = \begin{bmatrix}a&b&c\\2d&2e&2f\\g&h&i\end{bmatrix}$

* $\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix} \xrightarrow{R3 \leftarrow R3+3R1} \begin{bmatrix}1&&\\&1&\\3&&1\end{bmatrix} \Rightarrow \begin{bmatrix}1&&\\&1&\\3&&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = \begin{bmatrix}a&b&c\\d&e&f\\g+3a&h+3b&i+3c\end{bmatrix}$

- Elementary matrices are invertible because elementary row operations are reversible.

- $\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}^{-1} = \begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}$ $R3 \leftarrow R3+3R1$

- $\begin{bmatrix}1&&\\2&1&\\&&1\end{bmatrix}^{-1} = \begin{bmatrix}1&&\\-2&1&\\&&1\end{bmatrix}$ $R2 \leftarrow R2+2R1$

- $\begin{bmatrix}1&&\\&2&\\&&1\end{bmatrix}^{-1} = \begin{bmatrix}1&&\\&\frac{1}{2}&\\&&1\end{bmatrix}$ $R2 \leftarrow 2R2$

- $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}^{-1} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

Example) pre-multiplication vs. post-multiplication

- $\boxed{\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = \begin{bmatrix}a&b&c\\d&e&f\\g-d&h-e&i-f\end{bmatrix}$ $R3 \leftarrow R3-R2$

- $\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\boxed{\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}} = \begin{bmatrix}a&b-c&c\\d&e-f&f\\g&h-i&i\end{bmatrix}$ $c2 \leftarrow c2-c3$

* $\alpha(i,j)$ pre-multiplication : $R_i \leftarrow R_i + \alpha R_j$
  $\alpha(i,j)$ post- " — " : $C_j \leftarrow C_j + \alpha C_i$

Example) Elementary matrices in action:

- $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$ $\quad R3 \leftrightarrow R1$ (interchange)

- $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 7 \end{bmatrix}$: $R3 \leftarrow 7 \cdot R3$ (scaling)

- $\begin{bmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{bmatrix}$: $R3 \leftarrow R3 + 3R1$ (replacement)

- Gaussian elimination revisited.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} \xrightarrow[\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}]{R2 \leftarrow R2 - 2R1} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix} \Leftarrow \text{echelon form}$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

(elementary mtx)

$$\underset{}{E} \cdot A = \text{echelon form} = B$$

$$\Rightarrow A = E^{-1} B$$

$$\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{\substack{L \\ = E^{-1}}} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}}_{\substack{U \\ = \text{echelon form}}} = LU$$

- $A = LU$ is known as the LU decomposition of A.
  (L: lower triangular, U: upper triangular)

• Example) Factor $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ as $A = LU$.

$$\Rightarrow \quad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \xrightarrow[E_1]{R2 \leftarrow R2 - 2R1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \xrightarrow[E_2]{R3 \leftarrow R3 + R1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

$$\xrightarrow[E_3]{R3 \leftarrow R3 + R2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$E_3 E_2 E_1 A = U \quad \Rightarrow \quad A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$$

where $E_1 = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} \rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \rightarrow E_2^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix}$

$E_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \rightarrow E_3^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}$

$\therefore A = LU$, $L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

• Remark) Once we have $A = LU$, it is simple to solve $Ax = b$

$$A x = b$$
$$\Leftrightarrow \quad L(\underbrace{Ux}_{=c}) = b$$
$$\Leftrightarrow \quad Lc = b, \quad Ux = c$$

* Both of $(Lc = b)$ and $(Ux = c)$ are triangular and hence easily solved.
  - $Lc = b$ by forward substitution
  - $Ux = c$ by backward substitution to find x.

* This can be quickly repeated for many different b!

$$\begin{bmatrix} L \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

$$\begin{bmatrix} U \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} c \end{bmatrix}$$

* $O(N)$

Example) Solve $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} x = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}$ from $\underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix}}$

$\Rightarrow$ $Ax = b$, $A = LU$

$L(Ux) = b$, $c = Ux$ $\rightarrow$ 1) $Lc = b$

2) $Ux = c$

1) $\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix} \longrightarrow \begin{cases} c_1 = 4 \\ c_2 = 2 \\ c_3 = 3 \end{cases}$ (forward substitution)

2) $\begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \longrightarrow \begin{cases} x_3 = 3 \\ x_2 = -1 \\ x_1 = 1 \end{cases}$ (backward substitution)

Check $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}$ ✓

- Triangular factors for any matrix

"Can we always factor $A = LU$?" $\longrightarrow$ No.

1) Replacement $\rightarrow$ $E = \begin{bmatrix} 1 & & \\ & 1 & \\ & 3 & 1 \end{bmatrix}$

2) Interchange $\rightarrow$ $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\leftarrow$ problematic.

3) scaling $\rightarrow$ $E = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}$

Definition) A permutation matrix is one that is obtained by performing row exchanges on an identity matrix.

Theorem) For any matrix A, there is a permutation matrix P such that

$$PA = LU.$$

Example) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$, do LU decomposition.

$\Rightarrow$ $R1 \leftrightarrow R3$ $\Rightarrow$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (permutation matrix, P)

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$PA = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 - \frac{1}{2}R1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$

$EPA = U$, $PA = E^{-1}U = LU$

where $P = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$, $L = \begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ & & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 1 & 0 \\ & \frac{1}{2} & 0 \\ & & 1 \end{bmatrix}$