- Solution of linear systems via row reduction

Example)
$$\begin{bmatrix} \boxed{1} & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{bmatrix} \rightarrow \begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 8x_4 = 5 \\ x_5 = 7 \end{cases}$$

- The pivots are located in columns 1, 3, 5. ⇒ x_1, x_3, x_5 are pivot variable.
- The remaining x_2, x_4 are free variables.
- Then, we can write down pivot variables in terms of free variable.

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 = \text{free} \\ x_3 = 8x_4 + 5 \\ x_4 = \text{free} \\ x_5 = 7 \end{cases}$$

← This is general solution in parametric form.

free variable

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 8 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix}$$

basis

Questions of existence and uniques

Theorem) A linear system is consistent iff and only if an echelon form of the augmented matrix has no row of the form
$$[0 \ 0 \ \cdots \ 0 \mid b]$$
where $b \neq 0$.

If a linear system is consistent, it has
- one unique solution (= no free variable) or
- infinitely many solutions (= at least one free variable)

Example) For what value of h will the following system be consistent?
$$\begin{cases} 3x_1 - 9x_2 = 4 \\ -2x_1 + 6x_2 = h \end{cases}$$

$$\Rightarrow \left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + \frac{2}{3}R_1} \left[\begin{array}{cc|c} 3 & -9 & 4 \\ 0 & 0 & h+\frac{8}{3} \end{array}\right]$$

$$\therefore h = -\frac{8}{3}$$

Summary) · Each linear system corresponds to an augmented matrix.
· From Gaussian elimination, we can
- read off, whether the system has no, one, or infinitely many solutions.

True or False) · There is no more than one pivot in any row
: True
· There cannot be more free variables than pivot variables
: False

∘ The geometry of linear equations

Example)
$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & -1 & 2 & 0 \\ 3 & 2 & -2 & 0 \end{bmatrix}$$
column vector

Example) A fundamental property of vectors is that vectors of the same kind can be added and scaled!

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}$$

∘ Definition) Given vectors $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$ and scalars $c_1, \ldots, c_m \in \mathbb{R}$, the vector
$$c_1v_1 + c_2v_2 + \cdots + c_mv_m$$
is a linear combination of $v_1, \ldots, v_m$. The scalars $c_1, \ldots, c_m$ are coefficients.

· Example) Express $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

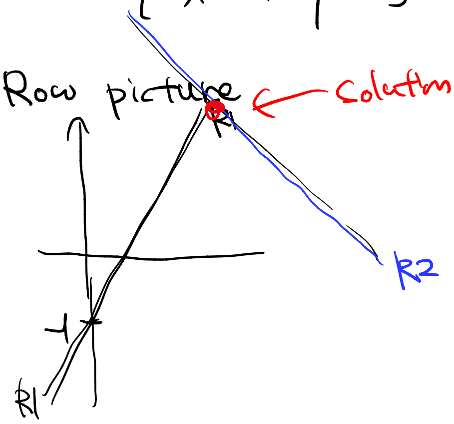
$$\rightarrow \quad c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} \boxed{2} & -1 & 1 \\ 1 & 1 & 5 \end{array}\right] \xrightarrow{R2 \leftarrow R2 - \frac{1}{2}R1} \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 1.5 & 4.5 \end{array}\right] \quad \therefore \begin{array}{l} c_2 = 3 \\ c_1 = 2 \end{array}$$

• The row and column picture

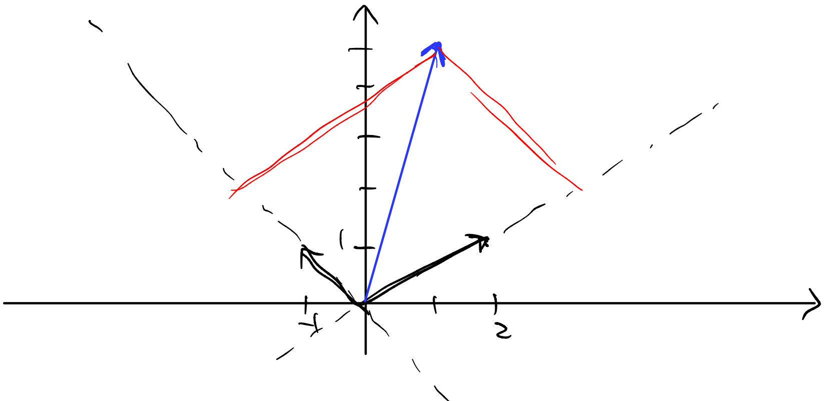
Example)
$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

i) Row picture

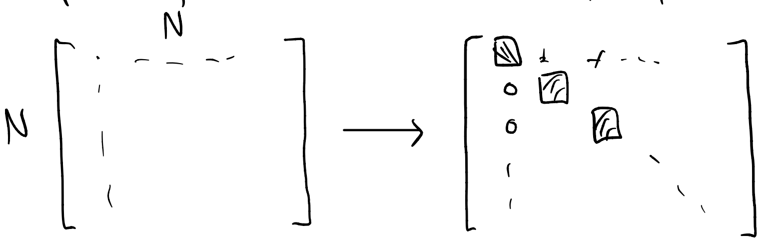


ii) Column picture

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$



• Question) How fast can we solve N linear equations in N unknowns?



i) To make a pivot: to create the zeros under the pivot

→ We have $(N-1)$ rows to process $\times$ per each (N) multiplications and (N) subtractions $\Rightarrow O(N^2)$ for a pivot

→ To make upto N pivots $\Rightarrow O(N^3)$

Summary) A vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m = \vec{b}$$

has the same solution set as the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m & \vec{b} \end{array}\right]$$

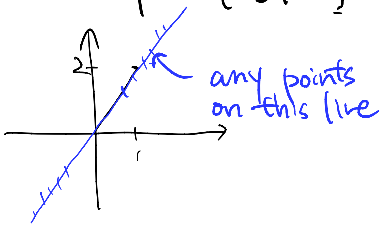
$\vec{b}$ is a linear combination of $\vec{a}_1, \ldots, \vec{a}_m$ $\Leftrightarrow$ The system is consistent.

The span of a set of vectors

• Definition) The span of vectors $v_1, \dots, v_m$ is the set of all their linear combination. We denote it by $\text{span}\{v_1, \dots, v_m\}$.

$$\text{span}\{v_1, \dots, v_m\} = \left\{ c_1 v_1 + \cdots + c_m v_m \;\middle|\; c_1, c_2, \dots, c_m \in \mathbb{R} \right\}$$

Example) (a) $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$



(b) Describe $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$.

$= \mathbb{R}^2$

More rigorously, let $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be any vector in $\mathbb{R}^2$.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} c_1 + \begin{bmatrix} 4 \\ 1 \end{bmatrix} c_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 1 & 1 & b_2 \end{array}\right]$$

$$\xrightarrow{R2 \leftarrow R2 - \frac{1}{2}R1} \left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 0 & -1 & b_2 - \frac{1}{2} b_1 \end{array}\right] \leftarrow \text{consistent!}$$

Example) Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Is b in the plane $\underbrace{\text{spanned by the columns}}_{\text{column space}}$ of A?

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{array}\right] \xrightarrow{R2 \leftarrow R2 - 3R1} \left[\begin{array}{cc|c} \boxed{1} & 2 & 8 \\ 0 & \boxed{-5} & -21 \\ 0 & 5 & 17 \end{array}\right] \xrightarrow{R3 \leftarrow R3 + R2} \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{array}\right]$$

the system is inconsistent. Hence b is not in the plane spanned by the columns of A.

- Matrix operation

For $(m\times n)$ matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \\ a_{m1} & & & a_{mn} \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

- Matrices, just like vectors, are added and scaled componentwise.

- Matrix times vector

→ Recall that $(x_1, \dots, x_n)$ solves the linear system with augmented matrix

$$\left[\begin{array}{c|c} A & b \end{array}\right] = \left[\begin{array}{cccc|c} | & | & & | & | \\ a_1 & a_2 & \cdots & a_n & b \\ | & | & & | & | \end{array}\right]$$

if and only if

$$x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{b}$$

It is therefore natural to define the product of matrix times vector as

$$A\vec{x} \triangleq x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n}$$

* $A\vec{x}$ is a linear combination of the columns of A with weights given by the entries of $\vec{x}$.

Example) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}x_1 + \begin{bmatrix} 2 \\ 4 \end{bmatrix}x_2 = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$

$\Downarrow$

$\begin{cases} x_1 + 2x_2 = b_1 \\ 3x_1 + 4x_2 = b_2 \end{cases}$: A linear system can be represented as $Ax = b$.