- $A(BC) = (AB)C$

$$\Rightarrow A\begin{bmatrix} [BC]_1 & [BC]_2 & \cdots \end{bmatrix} = \begin{bmatrix} A[BC]_1 & A[BC]_2 & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} A(Bc_1), & A(Bc_2), & \cdots \end{bmatrix}$$

$\because (AB)x = A(Bx)$

$$= \begin{bmatrix} (AB)c_1, & (AB)c_2, & \cdots \end{bmatrix}$$

$$= (AB)C$$

$\therefore A(BC) = (AB)C$

- The inverse of a matrix

- Definition A $n \times n$ matrix A is invertible if there is a matrix B such that $AB = BA = I$.
  In this case, B is the inverse of A and we write $A^{-1} = B$.

Example)
$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Note : The inverse of a matrix is unique.
  → Assume B and C are both inverses of A, then ($\because AB = BA = AC = CA = I$)

$$C = CI \underset{I=AB}{=} CAB \underset{CA=I}{=} IB = B$$

- Why we do not write $\frac{A}{B}$? ⇒ $AB^{-1} \neq B^{-1}A$
  (A/B) $(B \backslash A)$

- If $AB = I$, then $BA = I$?

⇒ basis: independent vector,

i) Let $\{x_1, x_2, \ldots, x_n\}$ be bases of the space. We first show that $\{Bx_1, Bx_2, \ldots, Bx_n\}$ is also a basis.

→ Suppose that $\{Bx_1, \ldots, Bx_n\}$ is not a basis. Then, there exist $c_1, \ldots, c_n \in \mathbb{R}$ not all equal to zero, such that

$$c_1Bx_1 + c_2Bx_2 + \cdots + c_nBx_n = 0 \quad — (*)$$

Premultiply $(*)$ with A,

$$c_1ABx_1 + \cdots + c_nABx_n = 0 \quad AB=I$$

hence

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$$

But, this is a contradiction with $\{x_1, \ldots, x_n\}$ being a basis.

Hence, $\{Bx_1, \cdots Bx_n\}$ is a basis and every vector y can be represented as a linear combination of these vectors.

$\Rightarrow \forall y, \exists x$ s.t. $Bx = y$ — $(**)$

ii) Now to prove $BA = I$, it is identical to show that

$\forall y, \ BAy = y$ — $(***)$

Then, substituting $(**)$ to the LHS of $(***)$, we get

$$\forall y, \ BAy \underset{(**)}{=} BABx \underset{(\text{associativity})}{=} Bx = y$$, which completes the proof.

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

o Solving systems using matrix inverse.

Theorem. Let A be invertible, Then the system $Ax = b$ has the unique solution $x = A^{-1}b$.

$\Rightarrow Ax = b, \longrightarrow A^{-1}Ax = A^{-1}b \Rightarrow \quad x = A^{-1}b.$

Example) Solve $\begin{cases} -7x_1 + 3x_2 = 2 \\ 5x_1 - 2x_2 = 1 \end{cases}$ using matrix inversion.

$$\Rightarrow \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}^{-1} = -1\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

o Recipe for computing the inverse.

- To solve $Ax = b$, we do row reduction on $[A \mid b]$.
- To solve $AX = I$, we do row reduction on $[A \mid I]$

- To compute A^{-1}
  - Form the augmented matrix $[A \mid I]$
  - Compute the reduced echelon form (Gauss-Jordan elimination)
  - If A is invertible, then the result is of the form $[\, I \mid A^{-1}]$

solution A^{-1}.

• Example) Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -3 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R3 \leftarrow R3 + \frac{3}{2}R1} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R1 \leftarrow R1/2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

$\shortparallel I$ $\qquad\qquad$ $\rightarrow A^{-1}$

Hence, $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$.

Theorem) Suppose A and B are invertible,

- (A^{-1}) is invertible, $(A^{-1})^{-1} = A$
- A^T is invertible, $(A^T)^{-1} = (A^{-1})^T$

  $\Rightarrow (A^{-1})^T A^T = (A \cdot A^{-1})^T = I^T = I$

  inverse

  $\therefore (A^T)^{-1} = (A^{-1})^T$
- (AB) is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

  $\Rightarrow (B^{-1}A^{-1})(AB) = B^{-1}B = I$

Theorem) Let A be an $n \times n$ matrix. Then, the following statements are equivalent.

(a) A is invertible.
(b) A is row equivalent to I.
(c) A has n pivots.
(d) $Ax = b$ has a unique solution.
(e) $\exists B$ s.t. $AB = I$
(f) $\exists C$ s.t. $CA = I$

• Matrices that are not invertible are often called singular.

○ Vector spaces and subspaces

• Definition) A vector space is a non empty set V of vectors, which may be added and scaled.

∘ Axioms ($u, v, w \in V$, $c, d \in \mathbb{R}$)

(a) $u+v \in V$

(b) $u+v = v+u$ (commutativity)

(c) $(u+v)+w = u+(v+w)$ (associativity)

(d) $\exists 0 \in V$, s.t. $u+0 = u$, $\forall u \in V$

(e) $\exists -u$ s.t. $u+(-u)=0$

(f) $cu \in V$

(g) $c(u+v) = cu+cv$ (distributivity)

(h) $(c+d)\cdot u = cu+du$

(i) $(cd)u = c\cdot(du)$

(j) $1\cdot u = u$

tl;dr: A vector space is a collection of vectors which can be added and scaled without leaving the space.

Example) $M_{2\times2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$ is a vector space?

$\Rightarrow$ i) The zero vector becomes $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{2\times2}$

ii) Addition: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \triangleq \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \in M_{2\times2}$

iii) scaling: $\gamma \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \gamma a & \gamma b \\ \gamma c & \gamma d \end{bmatrix} \in M_{2\times2}$

$\therefore$ $M_{2\times2}$ is a vector space!

Example) Let $\mathbb{P}_n$ be the set of all polynomials of degree at most $n \geq 0$.
Is $\mathbb{P}_n$ a vector space?

$\Rightarrow$ Members of $\mathbb{P}_n$ are of the form

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

where $a_0, \ldots, a_n$ are in $\mathbb{R}$ and t is a variable.

Then, we can define the addition and scaling as follows.

i) addition: $[a_0 + a_1 t + \cdots + a_n t^n] + [b_0 + b_1 t + \cdots + b_n t^n]$
$= (a_0+b_0) + (a_1+b_1)t + \cdots + (a_n+b_n)t^n$

ii) scaling: $\gamma[a_0 + a_1 t + \cdots + a_n t^n]$
$= (\gamma a_0) + (\gamma a_1)t + \cdots + (\gamma a_n)t^n$

Note that

$$\mathbb{P}_n = \{a_0 + a_1 t + \cdots + a_n t^n\} \xleftrightarrow{\text{isomorphic}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{n+1}$$

Example) Let $\mathbb{V}$ be the set of all polynomials of degree 3.
Is $\mathbb{V}$ a vector space?

$\Rightarrow$ i) the zero polynomial $\vec{0} = \underbrace{0+0t+0t^2+0t^3}_{\text{degree } 0.} \notin \mathbb{V}$

2) It is not closed under addition:

$$\underbrace{(t^3+3)}_{\in \mathbb{V}} + \underbrace{(-t^3+2t)}_{\in \mathbb{V}} = 2t+3 \notin \mathbb{V}$$

No.

Example) Let $\mathbb{V}$ be the set of all functions $f: \mathbb{R} \to \mathbb{R}$,
Is $\mathbb{V}$ a vector space?

$\Rightarrow$ i) addition $\quad (f+g)(x) \triangleq f(x) + g(x), \quad f, g \in \mathbb{V}$

ii) scaling $\quad (\gamma f)(x) \triangleq \gamma \cdot f(x) \quad f \in \mathbb{V}.$