Theorem) If V has a basis with n elements, then every set of m vectors which has more than n elements is linearly dependent $(m > n)$

$\Rightarrow$ Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of V and let S be the subset of V with $m > n$ elements. We need to prove that S is linearly dependent.

Then, every element $\{\vec{s}_1, \ldots, \vec{s}_m\}$ of S is a linear combination of $\{\vec{v}_1, \ldots, \vec{v}_n\}$

$$\begin{cases} \vec{s}_1 = a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n \\ \vec{s}_2 = \cdots \\ \vdots \\ \vec{s}_m = a_{m1}\vec{v}_1 + \cdots + a_{mn}\vec{v}_n \end{cases}$$

coefidents

Consider the following n-vectors:

$$\begin{cases} \vec{t}_1 = (a_{11}, a_{12}, \ldots, a_{1n}) \in \mathbb{R}^n \\ \vdots \\ \vec{t}_m = (a_{m1}, a_{m2}, \ldots, a_{mn}) \in \mathbb{R}^n \end{cases}$$

We will show that $\{\vec{t}_1, \ldots, \vec{t}_m\}$ are linearly dependent.

We can show this by constructing

$$T = \begin{array}{c c} & m \\ n & \begin{bmatrix} | & | & & | \\ \vec{t}_1 & \vec{t}_2 & \cdots & \vec{t}_m \\ | & | & & | \end{bmatrix} \end{array} \in \mathbb{R}^{n \times m} \quad \text{where } m > n$$

Since T can have at most n pivots, $\{\vec{t}_i\}_{i=1}^m$ are linearly dependent.

It means that there exist non-trivial $\{x_i\}_{i=1}^m$ such that

$$x_1\vec{t}_1 + x_2\vec{t}_2 + \cdots + x_m\vec{t}_m = \vec{0}$$

$$\downarrow$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Recall that

$$\begin{cases} \vec{s}_1 = a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n \\ \vdots \\ \vec{s}_m = a_{m1}\vec{v}_1 + \cdots + a_{mn}\vec{v}_n \end{cases}$$

multiply i th entry with x_i

$$\begin{cases} x_1\vec{s}_1 = x_1a_{11}\vec{v}_1 + \cdots + x_1a_{1n}\vec{v}_n \\ \vdots \\ x_m\vec{s}_m = x_ma_{m1}\vec{v}_1 + \cdots + x_ma_{mn}\vec{v}_n \end{cases}$$

Then, add all entries

$$x_1\vec{s}_1 + x_2\vec{s}_2 + \cdots + x_m\vec{s}_m = \overset{\parallel 0}{(x_1a_{11} + x_2a_{21} + \cdots + x_ma_{m1})}\vec{v}_1 + \vdots \overset{\parallel 0}{(x_1a_{1n} + x_2a_{2n} + \cdots + a_ma_{mn})}\vec{v}_n$$

$$= \vec{0}$$

where $\vec{x} \neq \vec{0}$ $\quad\therefore$ We have shown that $\{\vec{s}_1, \ldots, \vec{s}_m\}$ are linearly dependent!

Theorem) If V has a basis with n elements, then every set of vectors with fewer than n elements, say $m < n$, does not span V.

$\Rightarrow$ Suppose that S is a basis, $S = \{\vec{s}_1, \dots, \vec{s}_n\}$ of V and T is a subset of V with m elements where $m < n$.
By contradiction, assume that $\text{span}(T) = V$. Then, S has n elements, $(n > m)$ and from the previous theorem, the vectors of S are linearly dependent.
$\hookrightarrow$ But this contradicts the fact that S is linearly independent ($\because$ basis).
This completes the proof.

Theorem) If V has a basis with n elements. Then, all bases of V have the same number of elements $(= n)$.

Definition: dimension $\triangleq$ the number of elements in a basis

Theorem) Suppose that V has dimension d.
a) A set of d vectors in V are a basis if they span V.
b) A set of d vectors in V are a basis if they are linearly independent!

Example) Are the following sets a basis for $\mathbb{R}^3$? (dim = 3)

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow$ No! $(\neq 3)$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \right\}$

$\Rightarrow$ No! $(\neq 3)$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$

$\Rightarrow$ We need to check whether they are linearly indep.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$\rightarrow$ It has 3 pivots $\rightarrow$ L.I $\rightarrow$ hence, it is a basis of $\mathbb{R}^3$.

Example) $\mathbb{R}^3$'s subspaces.

- The only 0-dimensional subspace (of $\mathbb{R}^3$) is $\{\vec{0}\}$.
- The 1-dimensional subspace is of the form span$\{\vec{v}\}$ where $\vec{v} \neq \vec{0}$.
- The 2-dimensional subspace is span$\{\vec{v}, \vec{w}\}$.
  where $\vec{v} \neq \vec{0}$, $\vec{w} \neq \vec{0}$ & $\vec{v}$ and $\vec{w}$ are not multiples of each other.
- The only 3-dim. subspace (of $\mathbb{R}^3$) is $\mathbb{R}^3$.

True & False

1) Suppose that $\dim(V) = n$. Then any set with more than n elements are linearly dependent.
   → true.
2) $\mathbb{P}_n$ of polynomials of degree at most n has dimension $n+1$.
   → true $\{1, t, \ldots, t^n\}$.
3) Consider $V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$. If one of the vectors, say $\vec{v}_k$, is a linear combination of the remaining $(p-1)$ vectors, then the remaining ones still span V.
   → True.