Definition (From linear maps to matrices)

Let $\{\vec{x}_i\}_{i=1}^{n}$ be a basis for V and $\{\vec{y}_i\}_{i=1}^{m}$ be a basis for W.

The matrix representing $T: V \to W$ with respect to these bases

- has n columns
- the j-th column has m entries $a_{1j}, a_{2j}, \ldots, a_{mj}$ determined by

$$T(\vec{x}_j) = a_{1j}\vec{y}_1 + a_{2j}\vec{y}_2 + \cdots + a_{mj}\vec{y}_m$$

$$T \in \mathbb{R}^{m \times n} \qquad \begin{array}{c c} & \begin{array}{ccccc} 1\ 2\ 3 & \cdots & j & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ \\ m \end{array} & \begin{bmatrix} & & a_{1j} & & \\ & & a_{2j} & & \\ & & a_{3j} & & \\ & & \vdots & & \\ & & & & \\ & & a_{mj} & & \end{bmatrix} \end{array}$$

Example Let $V = \mathbb{R}^2$, $W = \mathbb{R}^3$, Let $T: V \to W$ be the linear map such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}$$

What is the matrix B representing T, w.r.t

$$\vec{x}_1 = \begin{bmatrix}1\\1\end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix}-1\\2\end{bmatrix} \text{ for } V, \qquad \vec{y}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{y}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{y}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ for } W.$$

$$\Rightarrow T(\vec{x}_1) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\0\\7\end{bmatrix} = \begin{bmatrix}5\\2\\10\end{bmatrix}$$

$$\| \quad c_1 \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{=\vec{y}_1} + c_2 \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{=\vec{y}_2} + c_3 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}5\\2\\10\end{bmatrix} \Rightarrow c_1 = 5,\ c_2 = -3,\ c_3 = 5$$

$$B = \begin{bmatrix} 5 & \square \\ -3 & \square \\ 5 & \square \end{bmatrix}$$

$$T(\vec{x}_2) = -1 \cdot T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 2 \cdot T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}7\\-2\\11\end{bmatrix}$$

$$\| \quad c_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}0\\0\\1\end{bmatrix} \longrightarrow c_1 = 7,\ c_2 = -9,\ c_3 = 4$$

$$B = \begin{bmatrix} 5 & 7 \\ -3 & -9 \\ 5 & 4 \end{bmatrix}$$

Example) Suppose that $A \in \mathbb{R}^{5\times 5}$, $\vec{v} \in \mathbb{R}^5$ s.t. $\vec{v} \in \mathrm{Col}(A)$

Then, what can we say about $\mathrm{Nul}(A)$?

→ First, Rank of A cannot be 5.

Then, $\dim \mathrm{Nul}(A) > 0$ (FTLA)

Example) We consider linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\vec{x} \mapsto A\vec{x}$

1) $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$ elongation

2) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$

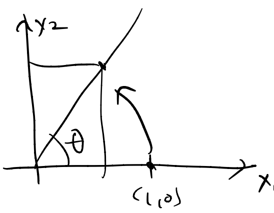
3) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ projection

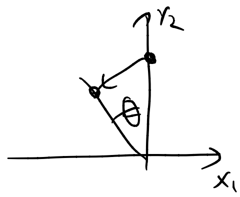
4) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$

Example) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the map which rotates a vector counter-clockwise by θ. Find A that represents T with the standard basis.

⇒ $X = \mathbb{R}^2$ has $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ as a basis.

$T_\theta\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $T_\theta\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$





∴ Hence $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Example) Let $T : \mathbb{P}_3 \to \mathbb{P}_2$ be the linear map given by

$$T(\underbrace{p(t)}_{\in \mathbb{P}_3}) = \underbrace{\frac{d}{dt} p(t)}_{\in \mathbb{P}_2}$$

What is the matrix A that represents T wrt the standard bases (of $\mathbb{P}_3, \mathbb{P}_2$)

$\Rightarrow$ The bases for $\mathbb{P}_3 = \{1, t, t^2, t^3\}$ $\Rightarrow$ $A \in \mathbb{R}^{3\times4}$

$\mathbb{P}_2 = \{1, t, t^2\}$,

i) $T(1) = 0 \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to$ $A = \begin{bmatrix} 0 & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$

ii) $T(t) = 1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to$ $A = \begin{bmatrix} 0 & 1 & - & - \\ 0 & 0 & - & - \\ 0 & 0 & - & - \end{bmatrix}$

iii) $T(t^2) = 2t \to \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \to$ $A = \begin{bmatrix} 0 & 1 & 0 & \cdot \\ 0 & 0 & 2 & \cdot \\ 0 & 0 & 0 & \cdot \end{bmatrix}$

iv) $T(t^3) = 3t^2 \to \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \to$ $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

We can do differentiation with matrix multiplications!

Note, what is the nullspace of A?

$$\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{P}_3$$

What does it mean? It corresponds to $p(t) = c \in \mathbb{R}$

This makes sense in that $\frac{d}{dt} p(t) = \frac{d}{dt}(c) = 0$

Example) $p(t) = 7t^3 - t + 3$, what is $p'(t)$?

$\Rightarrow$ $p(t) : \begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 21 \end{bmatrix} \to -1 + 21t^2 = p'(t)$

- Orthognality

  - The inner product and distances.

Definition) The inner product (or dot product) of $\vec{v}, \vec{w} \in \mathbb{R}^n$

$$\vec{v}\cdot\vec{w} = \vec{v}^T\vec{w} = v_1w_1 + \cdots + v_nw_n$$

Example) $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\-2\end{bmatrix} = 1-2-6 = -7$

Definition) The norm of a vector $\vec{v} \in \mathbb{R}^n$ is

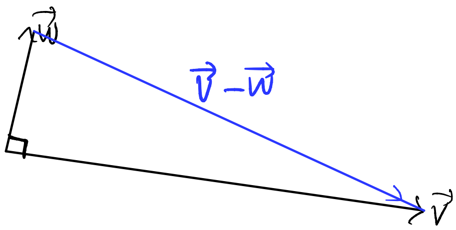
$$\|\vec{v}\| \triangleq \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$$

The distance between $\vec{v}$ and $\vec{w}$

$$\text{dist}(\vec{v}, \vec{w}) \triangleq \|\vec{v} - \vec{w}\|$$

Definition) $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$ are orthogonal if

$$\vec{v}\cdot\vec{w} = 0$$



Pythagoras: $\vec{v}$ and $\vec{w}$ are orthogonal

$$\Leftrightarrow \|v\|^2 + \|w\|^2 = \|v-w\|^2$$

$$\Leftrightarrow v\cdot v + w\cdot w = (v-w)\cdot(v-w) = v\cdot v + w\cdot w - 2v\cdot w$$

$$\Leftrightarrow v\cdot w = 0$$

Theorem. Suppose that $\vec{v}_1, \ldots, \vec{v}_m$ are nonzero and pairwise orthogonal. Then, they are Independent.

proof) Suppose that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0} \qquad -(*)$$

and we will examine $c_1, \ldots, c_m$.

i) Take the dot product of $\vec{v}_1$ on $(*)$

$$\rightarrow c_1\,\vec{v}_1\cdot\vec{v}_1 + \underbrace{\cdots}_{=0} = \vec{0}\cdot\vec{v}_1 = 0$$

$$\Leftrightarrow c_1\cdot\|v_1\|^2 = 0$$

$$\Leftrightarrow c_1 = 0$$

$\rightarrow$ Hence $(*)$ only has the trivial solution

$\rightarrow$ $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent.

Example) Let us consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$.

Find Nul(A) and Col(A^T).

i) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} x_1 = -2x_2 \\ x_2 = x_2 \end{matrix}$

$\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ "orthogonal"!

ii) $\mathrm{Col}(A^T) = \mathrm{Col}\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \right) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Example) Repeat for $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$

i) Nul(A) : $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = -2x_2 \\ x_2 = x_2 \\ x_3 = 0 \end{cases} \rightarrow \mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

ii) $\mathrm{Col}(A^T) = \mathrm{Col}\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 0 \end{bmatrix} \right) \rightarrow \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ orthogonal!

* Vectors in Nul(A) are orthogonal to the vectors in Col(A^T).

proof) Suppose that

$$A = n\begin{bmatrix} \overset{m}{—\ \vec{a}_{(1)}\ —} \\ \vdots \\ —\ \vec{a}_{(n)}\ — \end{bmatrix} \in \mathbb{R}^{n \times m}$$

where $\vec{a}_{(i)} \in \mathbb{R}^m$ is i-th row vector of A.

Then, $\forall \vec{x} \in \mathrm{Nul}(A)$, $\vec{a}_{(i)} \cdot \vec{x} = 0$ for all $i = 1, \ldots, n$

Since, $\forall \vec{y} \in \mathrm{Col}(A^T)$, $y = c_1 \cdot \vec{a}_{(1)} + c_2 \cdot \vec{a}_{(2)} + \cdots + c_n \vec{a}_{(n)}$

$\hookrightarrow \vec{x} \cdot \vec{y} = \vec{x} \cdot (c_1 \cdot \vec{a}_{(1)} + \cdots + c_n \vec{a}_{(n)})$
$= c_1 \cdot (\vec{x} \cdot \vec{a}_{(1)}) + \cdots + c_n (\vec{x} \cdot \vec{a}_{(n)}) = 0$

$\therefore \forall \vec{x} \in \mathrm{Nul}(A)$, $\forall \vec{y} \in \mathrm{Col}(A^T)$
$\vec{x} \cdot \vec{y} = 0$

Definition) Let W be a subspace of $\mathbb{R}^n$ and $\vec{v} \in \mathbb{R}^n$

- $\vec{v}$ is orthogonal to W, if $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$
- Another subspace V is orthogonal to W, if every vector in V is orthogonal to W.
- The orthogonal complement of W is the space $W^\perp$ of all vectors that are orthogonal to W.

$\Rightarrow$ $W^\perp = \{ \vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \ \forall \vec{w} \in W \}$

Example) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$. (W is a subspace of $\mathbb{R}^n$)

$\Rightarrow$ i) $\vec{0} \in W^\perp$ $\because$ $\vec{0} \cdot \vec{w} = 0$

ii) Let $\vec{u}, \vec{v} \in W^\perp$.

$(\vec{u} + \vec{v}) \in W^\perp$ because

$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = 0 \quad \forall \vec{w} \in W$

iii) $(c \cdot \vec{u}) \in W^\perp$ because

$c \cdot \vec{u} \cdot \vec{w} = c \cdot 0 = 0$

$\therefore$ $W^\perp$ is a subspace of $\mathbb{R}^n$.

Theorem) $\mathrm{Nul}(A)$ is orthogonal to $\mathrm{Col}(A^T)$

$\mathrm{Nul}(A^T)$ is orthogonal to $\mathrm{Col}(A)$

- Example) Show that $\dim(W) + \dim(W^\perp) = n$ for $\underset{\text{subspace}}{W} \subseteq \mathbb{R}^n$

$\Rightarrow$ Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r$ be a basis for W $(\dim(W) = r)$

Let $\vec{v}_{r+1}, \vec{v}_{r+2}, \cdots, \vec{v}_k$ be a basis for $W^\perp$

$\rightarrow$ Thus, we need to show that $k = n$.

i) First, claim that $\{ \vec{v}_1, \cdots, \vec{v}_r, \vec{v}_{r+1}, \cdots, \vec{v}_k \}$ is linearly indep.

Suppose that

$$\underbrace{c_1\vec{v}_1 + \cdots + c_r\vec{v}_r}_{\vec{w} \in W} + \underbrace{c_{r+1}\vec{v}_{r+1} + \cdots + c_k\vec{v}_k}_{\vec{w}' \in W^\perp} = \vec{0} \qquad \text{— } (*)$$

$\vec{w} + \vec{w}' = \vec{0} \rightarrow \vec{w} = -\vec{w}' \rightarrow \vec{w} \in W$ and $\vec{w} \in W^\perp$

This implies that $\vec{w}$ is perpendicular to itself.

$\rightarrow \vec{w} \cdot \vec{w} = 0 \rightarrow \vec{w} = \vec{0}$, $\vec{w}' = \vec{0}$

$\rightarrow (c_1, \cdots, c_r = 0)$ & $(c_{r+1}, \cdots, c_k = 0)$

ii) We showed that $(k \leq n)$ as $\{ \vec{v}_1 \cdots, \vec{v}_k \}$ are linearly indep.

Suppose that $(k<n)$. Then the matrix

$$A = \begin{array}{c} 1 \\ 2 \\ \vdots \\ k \end{array} \begin{bmatrix} — & v_1^T & — \\ & \vdots & \\ — & v_k^T & — \end{bmatrix} \in \mathbb{R}^{k \times n}$$

Thus, $Nul(A)$ is non zero $\rightarrow$ $\exists \vec{x} \in Nul(A)$ s.t. $\vec{x} \neq \vec{0}$.

$$\rightarrow \quad \vec{0} = A\vec{x} = \begin{bmatrix} — & \vec{v}_1^T & — \\ & \vdots & \\ — & \vec{v}_k^T & — \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \\ \vdots \\ \vec{v}_k \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad — (**)$$

(**) implies that $\vec{x} \in W^\perp$ and $\vec{x} \in (W^\perp)^\perp$ $\longrightarrow$ $\vec{x} = \vec{0}$

However, it contradicts that $\vec{x} \neq \vec{0}$ hence k cannot be smaller than n.

$\therefore (k=n)$