Example) Let $A=\begin{bmatrix}1&3&3&2\\2&6&9&7\\-1&-3&3&4\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}1\\5\\5\end{bmatrix}$. Find the parametric description of $A\vec{x}=\vec{b}$.

$$\to \left[\begin{array}{cccc|c}1&3&3&2&1\\2&6&9&7&5\\-1&-3&3&4&5\end{array}\right] \xrightarrow{\text{Gauss-Jordan}} \left[\begin{array}{cccc|c}\boxed{1}&3&0&-1&-2\\0&0&\boxed{1}&1&1\\0&0&0&0&0\end{array}\right] \to \begin{cases}x_1=-2-3x_2+x_4\\x_2=\text{free}\\x_3=1-x_4\\x_4=\text{free}\end{cases}$$

$$\vec{x}=\underbrace{\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}}_{\vec{x}_p}+\underbrace{\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}x_2+\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}x_4}_{\text{span}\left(\begin{bmatrix}-3\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}\right)=\text{Nul}(A)}$$

$*\ \vec{x}=\vec{x}_p+\text{Nul}(A)$

Example) Find Nul(A) where $A=\begin{bmatrix}1&3&5&0\\0&1&4&-2\end{bmatrix}$.

$$\to \begin{bmatrix}\boxed{1}&3&5&0\\0&\boxed{1}&4&-2\end{bmatrix}\xrightarrow{R1\leftarrow R1-3R2}\begin{bmatrix}\boxed{1}&0&-7&6\\0&\boxed{1}&4&-2\end{bmatrix}\to\begin{cases}x_1=7x_3-6x_4\\x_2=-4x_3+2x_4\\x_3=\text{free}\\x_4=\text{free}\end{cases}$$

$$\vec{x}=\begin{bmatrix}7\\-4\\1\\0\end{bmatrix}x_3+\begin{bmatrix}-6\\2\\0\\1\end{bmatrix}x_4 \qquad \therefore \text{Nul}(A)=\text{span}\left\{\begin{bmatrix}7\\-4\\1\\0\end{bmatrix},\begin{bmatrix}-6\\2\\0\\1\end{bmatrix}\right\}.$$

Example) Is $\text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}-1\\1\\3\end{bmatrix}\right\}$ equal to $\mathbb{R}^3$?

$$\to \left[\begin{array}{ccc|c}\boxed{1}&1&-1&b_1\\1&2&1&b_2\\1&3&3&b_3\end{array}\right]\xrightarrow[R3\leftarrow R3-R1]{R2\leftarrow R2-R1}\left[\begin{array}{ccc|c}\boxed{1}&1&-1&b_1\\0&\boxed{1}&2&b_2-b_1\\0&2&4&b_3-b_1\end{array}\right]\xrightarrow{R3\leftarrow R3-2R2}\left[\begin{array}{ccc|c}1&1&-1&b_1\\0&1&2&b_2-b_1\\0&0&0&b_3-b_1-2b_2+2b_1\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c}1&1&-1&b_1\\0&1&2&b_2-b_1\\0&0&0&b_1-2b_2+b_3\end{array}\right] \quad -(*)$$

$\to$ $(*)$ is consistent iff $(b_1-2b_2+b_3=0)$

hence $\text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}-1\\1\\3\end{bmatrix}\right\}$ is not equal to $\mathbb{R}^3$.

- Linear Independence.

Definition) Vectors $\vec{v}_1, \dots, \vec{v}_p$ are said to be linearly independent if the equation :

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution, i.e., $x_1 = x_2 = \cdots = x_p = 0$.

Example) Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}$ independent?

⇒ we need to check whether the equation

$$x_1\begin{bmatrix}1\\1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\2\\3\end{bmatrix} + x_3\begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

has more than the trivial solution.

↓

In other words, the three vectors are independent iff

$$\begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix}\vec{x} = \vec{0}$$

has no free variable! → ∵ no free variable → unique solution!

To check, $\begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix} \xrightarrow{G.E} \begin{bmatrix}\boxed{1} & 1 & -1\\0 & \boxed{1} & 2\\0 & 0 & 0\end{bmatrix}$, hence, the vectors are linearly dependent!

To find the linear dependence relation

$$\begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix} \longrightarrow \begin{bmatrix}\boxed{1} & 1 & -1\\0 & \boxed{1} & 2\\0 & 0 & 0\end{bmatrix} \xrightarrow{R1 \leftarrow R1 - R2} \begin{bmatrix}1 & 0 & -3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{cases} X_1 = 3X_3 \\ X_2 = -2X_3 \\ X_3 = X_3 \end{cases} \longrightarrow X_3 = 1, \quad \vec{x} = \begin{bmatrix}3\\-2\\1\end{bmatrix} \quad \therefore \begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix}\begin{bmatrix}3\\-2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

- Linear independence of matrix columns

· Each linear dependence relation of columns of A corresponds to a nontrivial solution to $A\vec{x}=\vec{0}$.

Theorem) Let A be an $m\times n$ matrix. The columns of A are linearly independent

$\iff$ $A\vec{x}=\vec{0}$ has only the solution $\vec{x}=\vec{0}$.

$\iff$ $\mathrm{Nul}\{A\}=\{\vec{0}\}$

$\iff$ A has n pivots ($\leftrightarrow$ no free variables)

this one is practical!

Example) Are the vectors independent?

(a) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\2\\3\end{bmatrix}\right\}$

$$\rightarrow \begin{bmatrix}1&1&4\\1&2&2\\1&3&3\end{bmatrix}\xrightarrow[R3\leftarrow R3-R1]{R2\leftarrow R2-R1}\begin{bmatrix}1&1&4\\0&1&3\\0&2&4\end{bmatrix}\xrightarrow{R3\leftarrow R3-2R2}\begin{bmatrix}\boxed{1}&1&4\\0&\boxed{1}&3\\0&0&\boxed{-2}\end{bmatrix}$$

$\therefore$ Yes

Example) Are the following statements true?

(a) A set of a single vector is always linearly independent.

$\rightarrow$ No.

$\because$ $\{\vec{0}\}$ is not linearly independent

$c_1\vec{0}=\vec{0}$ for all $c_1\in\mathbb{R}$

(b) A set of a single nonzero vector is independent?

$\rightarrow$ Yes.

(c) A set of two vectors $\{\vec{v}_1,\vec{v}_2\}$ is independent iff neither of the vectors is the multiple of others.

$\rightarrow$ Yes.

(d) A set of vectors $\{\vec{v}_1,\dots,\vec{v}_p\}$ containing the zero vector is linearly dependent.

$\rightarrow$ Yes

$\because$ Say, $\vec{v}_1=\vec{0}$, then $\underset{c_1}{1}\cdot\vec{v}_1+\underset{c_2}{0}\cdot\vec{v}_2+\cdots+\underset{c_p}{0}\vec{v}_p=\vec{0}$

Review) The columns of A are linearly independent

$\iff$ each column of A contains a pivot.

- A basis of a vector space

Definition) A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in a vector space V is a basis of V if

1) $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$
2) the vectors $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent.

Example) Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

show that $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis of $\mathbb{R}^3$.

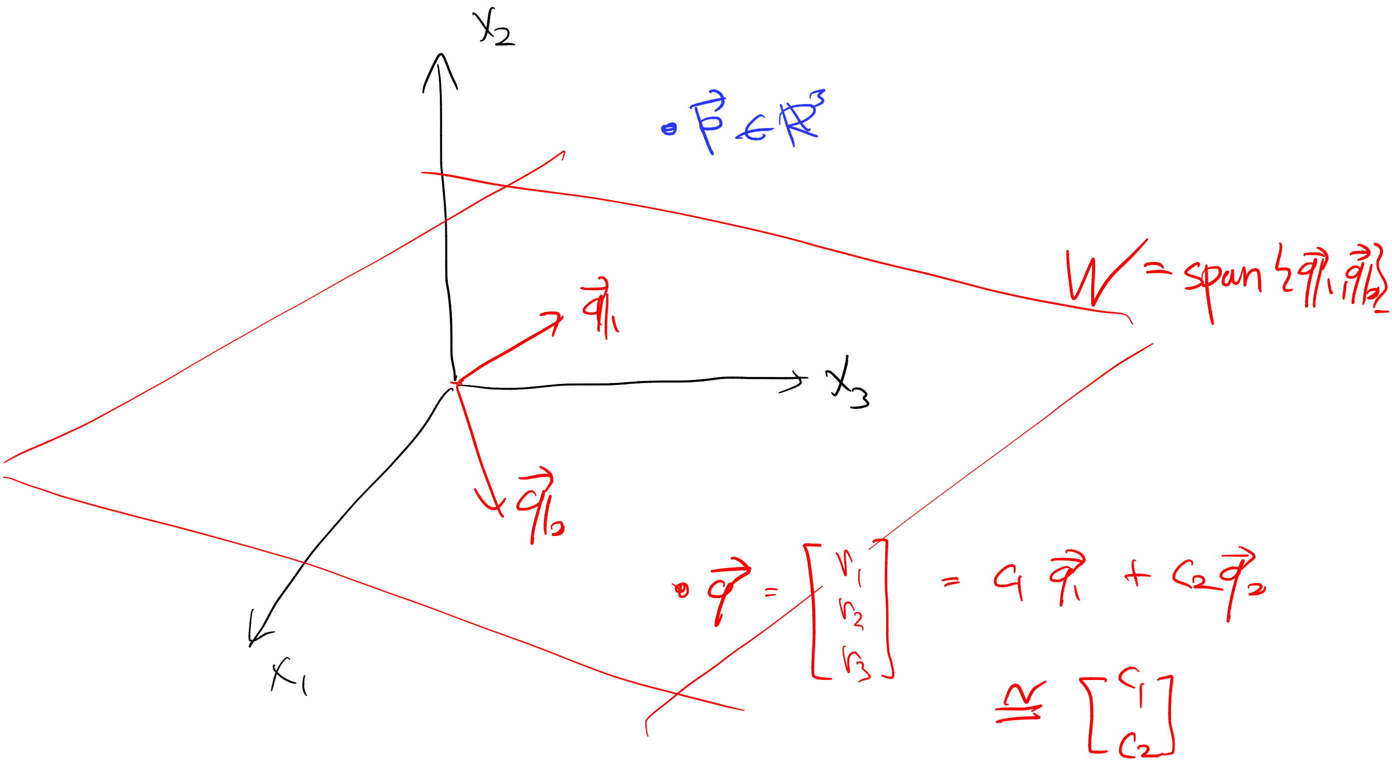
$\Rightarrow$ i) show that $\text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \mathbb{R}^3$.

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$ $\because$ $x_1 = b_1$, $x_2 = b_2$, $x_3 = b_3$ $\therefore \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \mathbb{R}^3$

ii) $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is L.I.

$\rightarrow$ $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$ has a pivot at each column $\rightarrow$ L.I.

$\therefore$ $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis of $\mathbb{R}^3$.



$\cdot\ \vec{q} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = c_1 \vec{q}_1 + c_2 \vec{q}_2$

$\cong \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

• Theorem) If S is a basis of a vector space V, then every vector in V has exactly one representation as a linear combination of elements of S.

→ Every vector $\vec{v} \in V$ is equal to the linear combination of the vectors from S.
(By contradiction), suppose that there exist a vector $\vec{a} \in V$ which is equal to two different combinations of vectors from S.
contradiction!

$$\begin{cases} \vec{a} = x_1 \vec{s}_1 + \cdots + x_n \vec{s}_n \\ \vec{a} = y_1 \vec{s}_1 + \cdots + y_n \vec{s}_n \end{cases}$$

Subtracting two, we get

$$\vec{0} = (x_1 - y_1)\vec{s}_1 + \cdots + (x_n - y_n)\vec{s}_n$$

→ Thus, the zero vector $\vec{0}$ is equal to a linear combination of elements of S.
Since $\vec{x} \neq \vec{y}$, some coefficients (e.g., $x_1 - y_1$) are not zero. But, this contradicts the assumption that S is a basis.
→ linearly independent.

This completes the proof.

* C_1 = space of continuous functions. $f(t)$
1) basis 1: $\{1, t, t^2, t^3, \cdots, t^\infty\}$.
2) basis 2: $\{\sin w_1 t, \sin w_2 t, \cdots \sin w_\infty t\}$.