$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \end{cases} \rightarrow \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}}_{\vec{a}_1} x_1 + \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}}_{\vec{a}_2} x_2 + \underbrace{\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}}_{\vec{a}_3} x_3 = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\vec{b}}$$

$$\rightarrow \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{b} \end{bmatrix} \rightarrow A\vec{x} = \vec{b}$$

- Matrix times matrix

$$AB \triangleq \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} \triangleq \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{bmatrix}$$

matrix times vector

$\Rightarrow$ Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B.

Remark) Show that $(AB)\vec{x} = A(B\vec{x})$ (associative property)

$$\rightarrow (AB)\vec{x} = x_1 \underbrace{[AB]_1}_{\text{first column of }(AB)} + x_2 [AB]_2 + \cdots$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \end{bmatrix}$$

$$= x_1 A\vec{b}_1 + x_2 A\vec{b}_2 + \cdots$$

$$= Ax_1\vec{b}_1 + Ax_2\vec{b}_2$$

$$= A(\underbrace{x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots}_{B\vec{x}})$$

$$= A(B\vec{x})$$

Example) Suppose that $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$

(a) Under which condition does AB makes sense?

$\rightarrow$ $n = p$

(b) What is the dim. of AB?

$\rightarrow$ $AB \in \mathbb{R}^{m \times q}$

Theorem)

- $A(BC) = (AB)C$ (associative property) HW
- $A(B+C) = AB + AC$ (left distributive property)
- $(A+B)C = AC + BC$ (right " —— ")

- Transpose of a matrix

Def) The transpose A^T of a matrix A is the matrix whose columns are formed from the corresponding row of A.

Example) $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$

A matrix is called symmetric if $A^T = A$.

Example) Show that $(AB)^T = B^T A^T$

$\Rightarrow \underbrace{[AB]_{(i,j)}}_{(i,j)\text{ th element of }(AB)} = \langle \underbrace{[A]_{(i)}}_{i\text{ th row vector of }A} , \underbrace{[B]_j}_{j\text{ th column vector of }B} \rangle$

$[(AB)^T]_{(i,j)} = [(AB)]_{(j,i)} = \langle [A]_{(j)}, [B]_i \rangle$ — $(*)$

$[B^T A^T]_{(i,j)} = \langle [B^T]_{(i)}, [A^T]_j \rangle = \langle [B]_i, [A]_{(j)} \rangle$ — $(**)$

$\therefore$ $(*) = (**)$, $(AB)^T = B^T A^T$.

Theorem)
$$(A^T)^T = A$$
$$(A+B)^T = A^T + B^T$$
$$(AB)^T = B^T A^T$$

Example) Show that $(ABC)^T = C^T B^T A^T$

$\Rightarrow (ABC)^T = ((AB)C)^T = C^T (AB)^T = C^T B^T A^T$

- LU decomposition

- Definition) An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

  → The result of an elementary row operation on A is EA. (elementary matrix)

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix}0&1\\1&0\end{bmatrix} \Rightarrow \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}c&d\\a&b\end{bmatrix}$$

$$\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix} \xrightarrow{R2 \leftarrow 2R2} \begin{bmatrix}1&&\\&2&\\&&1\end{bmatrix} \Rightarrow \begin{bmatrix}1&&\\&2&\\&&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = \begin{bmatrix}a&b&c\\2d&2e&2f\\g&h&i\end{bmatrix}$$

$$\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix} \xrightarrow{R3 \leftarrow R3+3R1} \begin{bmatrix}1&&\\&1&\\3&&1\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = \begin{bmatrix}a&b&c\\d&e&f\\g+3a&h+3b&i+3c\end{bmatrix}$$

- Elementary matrices are invertible because elementary row operations are reversible.

$$\begin{bmatrix}1&&\\&1&\\3&&1\end{bmatrix}^{-1} = \begin{bmatrix}1&&\\&1&\\-3&&1\end{bmatrix} \qquad R3 \leftarrow R3+3R1$$

$$\begin{bmatrix}1&&\\2&1&\\&&1\end{bmatrix}^{-1} = \begin{bmatrix}1&&\\-2&1&\\&&1\end{bmatrix}$$

$$\begin{bmatrix}1&&\\&2&\\&&1\end{bmatrix}^{-1} = \begin{bmatrix}1&&\\&\frac{1}{2}&\\&&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}^{-1} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \quad \text{(interchange)}$$

- Gaussian elimination revisited.

$$A = \begin{bmatrix}2&1\\4&-6\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2&1\\4&-6\end{bmatrix} \xrightarrow{R2 \leftarrow R2-2R1} \begin{bmatrix}2&1\\0&-8\end{bmatrix} \quad \text{(echelon form)}$$

$$\begin{bmatrix}1&0\\-2&1\end{bmatrix}\begin{bmatrix}2&1\\4&-6\end{bmatrix} = \begin{bmatrix}2&1\\0&-8\end{bmatrix}$$

$$E \quad A \quad = B$$

lower triangular form, upper triangular form

$$A = E^{-1}B \Rightarrow \underbrace{\begin{bmatrix}2&1\\4&-6\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}1&0\\2&1\end{bmatrix}}_{E^{-1}}\underbrace{\begin{bmatrix}2&1\\0&-8\end{bmatrix}}_{B}$$

* $A = LU$ is know as the LU decomposition of A.

Example) Factor $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ as $A = LU$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \xrightarrow[E_1]{R2 \leftarrow R2-2R1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \xrightarrow[E_2]{R3 \leftarrow R3+R1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \xrightarrow[E_3]{R3 \leftarrow R3+R2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$E_3 E_2 E_1 A = U \Rightarrow A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$$

where $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_1^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix}$

$E_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}$

, $L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$$\therefore \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

- Remark) Once we have $A = LU$, it is simple to solve $A\vec{x} = \vec{b}$

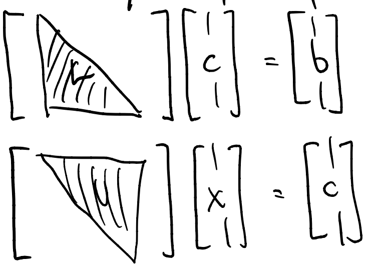
$$A\vec{x} = \vec{b}$$

$$\Leftrightarrow L(\underbrace{U\vec{x}}_{\vec{c}}) = \vec{b}$$

$$\Leftrightarrow L\vec{c} = \vec{b}, \quad U\vec{x} = \vec{c}$$

* Both of $(L\vec{c} = \vec{b})$ and $(U\vec{x} = \vec{c})$ are triangular and hence easily solved.
    - $L\vec{c} = \vec{b}$ by forward substitution
    - $U\vec{x} = \vec{c}$ by back substitution

- This can be quickly repeated for different $\vec{b}$!



* $O(N)$

Example) Solve $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 10 \\ -3 \end{bmatrix}$ from $\underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix}}$

$\Rightarrow$ HW

Definition) A permutation matrix is one that is obtained by performing row exchanges on an identity matrix

Theorem) For any matrix A, there is a permutation matrix P such that

$$PA = LU$$

Example) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$, do LU decomposition.

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E]{R2 \leftarrow R2 - \frac{1}{2}R1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E(PA) = U$$

$$PA = E^{-1}U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=L} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=U}$$

$$\therefore PA = LU$$