• Subspaces

- Definition : A subset W of a vector space V is a subspace, if W is itself a vector space.
  Since the rules like associativity, commutativity, and distributivity still hold,
  we only need to check the followings.

  $W \subseteq V$ is a subspace of V if
  1. W contains the zero vector $\vec{0}$.
  2. W is closed under addition
  3. W is closed under scaling

Example) Is $W = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ a subspace of $\mathbb{R}^2$?

$\Rightarrow \begin{bmatrix}a\\a\end{bmatrix} \in W$, for $a \in \mathbb{R}$

1) $\vec{0} = \begin{bmatrix}0\\0\end{bmatrix} \in W$
2) $\begin{bmatrix}a\\a\end{bmatrix} + \begin{bmatrix}b\\b\end{bmatrix} = \begin{bmatrix}a+b\\a+b\end{bmatrix} \in W$
3) $\gamma\begin{bmatrix}a\\a\end{bmatrix} = \begin{bmatrix}\gamma a\\\gamma a\end{bmatrix} \in W$

$\therefore W$ is a subspace of $\mathbb{R}^2$.

Example) Is $W = \left\{\begin{bmatrix}a\\1\\b\end{bmatrix} : a, b \in \mathbb{R}\right\}$ a subspace of $\mathbb{R}^3$?

$\Rightarrow$ 1) $\begin{bmatrix}0\\0\\0\end{bmatrix} \notin W$

2) $\begin{bmatrix}a_1\\1\\b_1\end{bmatrix} + \begin{bmatrix}a_2\\1\\b_2\end{bmatrix} = \begin{bmatrix}a_1+a_2\\2\\b_1+b_2\end{bmatrix} \notin W$

$\therefore W$ is not a subspace of $\mathbb{R}^3$.

Example) Is $W = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ a subspace of $\mathbb{R}^2$?

$\Rightarrow$ 1) $\begin{bmatrix}0\\0\end{bmatrix} \in W$

2) $\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \in W$

3) $r\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \in W$

Example) Is $W = \left\{\begin{bmatrix}x\\x+1\end{bmatrix} : x \in \mathbb{R}\right\}$ a subspace of $\mathbb{R}^2$?

$\Rightarrow$ 1) $\begin{bmatrix}0\\0\end{bmatrix} \notin W$ $\rightarrow$ W is not a subspace of $\mathbb{R}^2$.

Example) Is $W = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} \cup \left\{\begin{bmatrix}x\\x+1\end{bmatrix} : x \in \mathbb{R}\right\}$ a subspace of $\mathbb{R}^2$

$\Rightarrow$ 1) $\begin{bmatrix}0\\0\end{bmatrix} \in W$

2) $\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}q\\q+1\end{bmatrix} = \begin{bmatrix}q\\q+1\end{bmatrix} \in W$, $\begin{bmatrix}p\\p+1\end{bmatrix} + \begin{bmatrix}q\\q+1\end{bmatrix} = \begin{bmatrix}p+q\\p+q+2\end{bmatrix} \notin W$

$\rightarrow$ W is not a subspace of $\mathbb{R}^2$.

- Span of vectors are subspaces

- Review : $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \;\middle|\; c_1, c_2, \dots, c_m \in \mathbb{R} \right\}$

Theorem) If $\vec{v}_1, \dots, \vec{v}_m$ are in a vectorspace V, then $\text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$ is a subspace of V.

$\Rightarrow$ 1. $\vec{0} \in \text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$

2. $[c_1\vec{v}_1 + \cdots + c_m\vec{v}_m] + [d_1\vec{v}_1 + \cdots + d_m\vec{v}_m]$
$= [(c_1+d_1)\vec{v}_1 + \cdots + (c_m+d_m)\vec{v}_m] \in \text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$

3. $r[c_1\vec{v}_1 + \cdots + c_m\vec{v}_m] = (rc_1)\vec{v}_1 + \cdots + (rc_m)\vec{v}_m \in \text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$.

Example) Is $W = \left\{ \begin{bmatrix} a+3b \\ 2a-b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$?

$\Rightarrow W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} a + \begin{bmatrix} 3 \\ -1 \end{bmatrix} b : a, b \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$ $\therefore$ W is subspace of $\mathbb{R}^2$.

Example) Is $W = \left\{ \begin{bmatrix} -a & 2b \\ a+b & 3a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a subspace of $M_{2\times2}$?

$\Rightarrow W = \text{span}\left\{ \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right\}$ $\therefore$ W is a subspace of $M_{2\times2}$.

Example) Are the following sets vector spaces?

(a) $W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+3b=0, \; 2a-c=1 \right\}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W_1$ $\rightarrow$ W_1 is not a vectorspace.

(b) $W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : ab \geq 0 \right\}$

$\Rightarrow$ 1) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$

2) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}^{\in W} + \begin{bmatrix} -2 \\ -4 \end{bmatrix}^{\in W} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \notin W$

3) $r\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra \\ rb \end{bmatrix} \rightarrow r^2 ab \geq 0 ? \in W$

W is not a vectorspace!

(c) W is the set of all polynomials $p(t)$ such that $p'(2) = 1$.

$\Rightarrow \vec{0} = P_0(t) = 0 + 0t + 0t^2 + \cdots \notin W$

$\because P_0'(t) = 0 + 0 + \cdots = 0$

$\therefore$ W is not a vectorspace.

(d) W is the set of all polynomials $p(t)$ s.t. $p'(t=2) = 0$

$\Rightarrow$ 1) $P_0(t) = 0 \in W$ $\quad \because P_0'(t) \Big|_{t=2} = 0$

2) $\underbrace{P_a(t)}_{W} + \underbrace{P_b(t)}_{W} \in W$ $\quad \because \frac{d}{dt}(\bar{P}(t) = P_a(t) + P_b(t))\Big|_{t=2} = 0$

3) $r\underbrace{P(t)}_{W} = \bar{P}(t) \in W$ $\quad \because \frac{d}{dt}(rp(t))\Big|_{t=2} = 0$

$\therefore$ W is a vectorspace.

Example) Are the following sets vector space?

(a) $W_1 = \left\{ \begin{bmatrix} a+c & -2b \\ b+3c & c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

$\rightarrow W_1 = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}$ $\quad \therefore$ Yes

(b) $W_1 = \left\{ \begin{bmatrix} a+c & -2b \\ b+3c & c+7 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

$\rightarrow W_2 = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}}_{\text{bias}} + \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}$

- Solving $A\vec{x}=\vec{0}$ and $A\vec{x}=\vec{b}$

  - Column space

Definition) The column space, $Col(A)$, of a matrix A is the span of the columns of A.
↳ vector space

ex) $A=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}$, $Col(A) = span\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$

- Given $A\vec{x}=\vec{b}$, $\quad \vec{b}\in Col(A) \overset{iff}{\Longleftrightarrow} A\vec{x}=\vec{b}$ has a solution.

  Why? Because $A\vec{x} = \begin{bmatrix}A\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$

  is a linear combination of columns of A with coefficients given by $\vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$.

- If $A\in\mathbb{R}^{m\times n}$, then $Col(A)$ is a subspace of $\mathbb{R}^m$.

Example) Find a matrix A such that $W=Col(A)$ where

$$W=\left\{\begin{bmatrix}2x-y\\3y\\7x+y\end{bmatrix} : x,y\in\mathbb{R}\right\}$$

$\Rightarrow W = span\left\{\begin{bmatrix}2\\0\\7\end{bmatrix},\begin{bmatrix}-1\\3\\1\end{bmatrix}\right\}$

$\therefore A=\begin{bmatrix}2&-1\\0&3\\7&1\end{bmatrix}$, $\because Col(A)=W$

- Nullspaces

Definition) The nullspace of a matrix A is

$$Nul(A)=\{\vec{x} : A\vec{x}=\vec{0}\}$$

Theorem) If $A\in\mathbb{R}^{m\times n}$, then $Nul(A)$ is a subspace of $\mathbb{R}^n$.
($Col(A)\subseteq\mathbb{R}^m$, $Nul(A)\subseteq\mathbb{R}^n$)

$\Rightarrow$ 1) $\vec{0}\in Nul(A)$ $\quad\because A\vec{0}=\vec{0}$

2) $\vec{x}\in Nul(A)$ & $\vec{y}\in Nul(A)$ $\quad (A\vec{x}=A\vec{y}=\vec{0})$

then $A(\vec{x}+\vec{y}) = A\vec{x}+A\vec{y} = \vec{0}$, $\quad \vec{x}+\vec{y}\in Nul(A)$

(closed under addition)

3) $\vec{x}\in Nul(A)$, then $r\vec{x}\in Nul(A)$ $\quad\because Ar\vec{x} = rA\vec{x}=\vec{0}$

(closed under scaling)

Example) Find an explicit description of Nul(A) where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 - 2R1} \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \xrightarrow{R1 \leftarrow \frac{1}{3}R1} \begin{bmatrix} \boxed{1} & 2 & 2 & 1 & 3 \\ 0 & 0 & \boxed{1} & -6 & -15 \end{bmatrix}$$

$$\xrightarrow{R1 \leftarrow R1 - 2R2} \begin{bmatrix} \boxed{1} & 2 & 0 & 13 & 33 \\ 0 & 0 & \boxed{1} & -6 & -15 \end{bmatrix} \Rightarrow \begin{cases} \boxed{x_1} + 2x_2 + 13x_4 + 33x_5 = 0 \\ \boxed{x_3} - 6x_4 - 15x_5 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = -2x_2 - 13x_4 - 33x_5 \\ x_2 = x_2 \\ x_3 = 6x_4 + 15x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases} \rightarrow \vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} x_5$$

$$\therefore \ \mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}$$

◉ Another look at solutions to $A\vec{x} = \vec{b}$

Theorem) Let $\vec{x}_p$ be a solution of $A\vec{x} = \vec{b}$.
Then, every solution to $A\vec{x} = \vec{b}$ is of the form $\vec{x} = \vec{x}_p + \vec{x}_h$
where $\vec{x}_h$ is a solution to the homogeneous equation $A\vec{x} = \vec{0}$.

$\rightarrow$ In other words, $\{\vec{x} : A\vec{x} = \vec{b}\} = \vec{x}_p + \mathrm{Nul}(A)$

particular solution — nullspace of A

$\Rightarrow$ Let $\bar{x}$ be another solution to $A\bar{x} = \vec{b}$
then, $(\bar{x} - \vec{x}_p)$ satisfies $(\bar{x} - \vec{x}_p) \in \mathrm{Nul}(A)$ since

$$A(\bar{x} - \vec{x}_p) = A\bar{x} - A\vec{x}_p = \vec{b} - \vec{b} = \vec{0}$$