Introduction to systems of linear equation.

Definition) A linear equation in the variables $x_1, \dots, x_n$ is an equation that can be written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

Example) Which of the following equations are linear?

$4x_1 - 5x_2 + 2 = x_1$ ✓

$x_2 = 2(\sqrt{6} - x_1) + x_3$ ✓

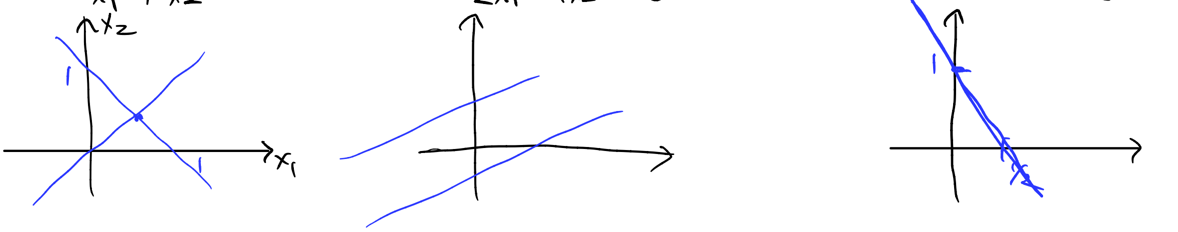
$4x_1 - 6x_2 = x_1x_2$

$x_2 = 2\sqrt{x_1} - 7$

Definition) A linear system is a collection of one or more linear equations involving the same set of variables say $x_1, \dots, x_n$.
A solution of a linear system is a list $(s_1, \dots, s_n)$ of numbers that make each equation in the system true, when the values are substituted for $x_1, x_2, \dots, x_n$, respectively.

Example) Sketch the set of all solutions.

$$\begin{cases} x_1 + x_2 = 1 \\ -x_1 + x_2 = 0 \end{cases} \qquad \begin{cases} x_1 - 2x_2 = -3 \\ 2x_1 - 4x_2 = 8 \end{cases} \qquad \begin{cases} 2x_1 + x_2 = 1 \\ -4x_1 - 2x_2 = -2 \end{cases}$$



Theorem) A linear system has either
- no solution
- one unique solution
- Infinitely many solutions

Definition) A system is consistent if a solution exists.

- How to solve a linear system

  - Strategy: replace a system with an equivalent system which is easier to solve.

Definition) Linear systems are equivalent, if they have the same set of solutions.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 0 \end{array}\right] \quad \text{augmented matrix} \qquad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 1 \end{array}\right]$$

Example)
$$\begin{cases} x_1 + x_2 = 1 \\ -x_1 + x_2 = 0 \end{cases} \xrightarrow{R2 \leftarrow R2 + R1} \begin{cases} x_1 + x_2 = 1 \\ 0 + 2x_2 = 1 \end{cases}$$

Once in this triangular form, we find the solutions by back-substitution.

$x_1 = 1/2$, $x_2 = 1/2$

Definition) An elementary row operation is one of the following:

- (replacement) Add one row to a multiple of another row
- (interchange) Interchange two rows
- (scaling) Multiply all entries in a row by a non-zero constant.

Theorem) If the augmented matrices of two linear systems are row equivalent then, the two systems have the same solution set.

Definition) Two matrices are row equivalent, if one matrix can be transformed into the other by a sequence of elementary row operations

Example) Solve,
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\;\; 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\xrightarrow{R3 \leftarrow R3 + 4R1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R3 \leftarrow R3 + \frac{3}{2}R2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

REF (Echelon form)

Gauss-Elimination

$$\xrightarrow{R2 \leftarrow R2 + 8R3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R1 \leftarrow R1 - R3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R1 \leftarrow R1 + R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R2 \leftarrow \frac{1}{2}R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

RREF (Reduced Echelon form)

Gauss-Jordan Elimination

$\therefore$ $x_1 = 29$, $x_2 = 16$, $x_3 = 3$

• Row reduction & Echelon forms

Definition) A matrix is in Echelon form if:

(1) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
(2) All entries in a column below a leading entry are zero
(3) All nonzero rows are above any rows of all zeros.

Example) Here is a representative matrix in echelon form:

pivot

non-zero

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & \cdots & \\ 0 & 0 & \blacksquare & * & * & * & & \\ 0 & 0 & 0 & 0 & \blacksquare & * & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & & 0 \end{bmatrix}$$

(a) $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ yes

(b) $\begin{bmatrix} 0 & \blacksquare & * & * \\ \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ No,

(c) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$ Yes

(d) $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$ Yes

Definition) A matrix is in reduced echelon form, if in addition to being echelon form, it also satisfies:

1. Each pivot is 1.
2. Each pivot is the only nonzero item in its column.

Example)

$$\underset{\text{REF}}{\begin{bmatrix} 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \longrightarrow \underset{\text{RREF}}{\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}}$$

Theorem) Each matrix is row equivalent to one and only one reduced echelon form.

Example)

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{R2 \leftarrow R2 - R1} \begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{R3 \leftarrow R3 - \frac{3}{2} R2} \begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix} \text{ REF}$$

$$\xrightarrow{R1 \leftarrow \frac{1}{3} R1,\ R2 \leftarrow \frac{1}{2} R2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 - R3} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R1 \leftarrow R1 - 2R3} \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R1 \leftarrow R1 + 3R2} \begin{bmatrix} \boxed{1} & 0 & -2 & 3 & 0 & -24 \\ 0 & \boxed{1} & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix} \text{ RREF}$$