- Bases for column and nullspaces

Bases for null spaces

To find a basis for Nul(A):

- Find the parametric form of the solutions to $A\vec{x}=\vec{0}$.
- Express solutions $\vec{x}$ as a linear combination of vectors with free variables as coefficients.
- These vectors form a basis of Nul(A).

Example) Find Nul(A) with

$$A=\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix} \xrightarrow{R2 \leftarrow R2-2R1} \begin{bmatrix} \boxed{3} & 6 & 6 & 3 & 9 \\ 0 & 0 & \boxed{3} & -6 & -15 \end{bmatrix} \xrightarrow{R1 \leftarrow R1-2R2} \begin{bmatrix} \boxed{3} & 6 & 0 & 15 & 39 \\ 0 & 0 & \boxed{3} & -6 & -15 \end{bmatrix}$$

$$\xrightarrow{R1/3,\ R2/3} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 5 & 13 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix}}_{\text{RREF}} \longrightarrow \begin{cases} x_1 = -2x_2-5x_4-13x_5 \\ x_2 = x_2 \\ x_3 = 2x_4+5x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases} \longrightarrow \vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} x_5$$

$$\therefore \text{Nul}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

* Are these vectors always linearly independent?

⇒ Yes,

- Bases for column spaces

→ A basis of Col(A) is given by the pivot columns of A.

Example) Find a basis for Col(A) with

$$A=\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \underbrace{\begin{bmatrix} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{RREF}}$$

$$\therefore \text{A basis of Col(A) is } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

- The four fundamental subspaces.

Definition : The row space of A is $\mathrm{Col}(A^T)$.
The left null space of A is $\mathrm{Nul}(A^T)$.

Theorem) Fundamental theorem of linear algebra. (FTLA)

Let $A \in \mathbb{R}^{m\times n}$ of rank r.

- $\dim \mathrm{Col}(A) = r$
- $\dim \mathrm{Col}(A^T) = r$ — # of columns
- $\dim \mathrm{Nul}(A) = n - r$
- $\dim \mathrm{Nul}(A^T) = m - r$ — # of rows

- Linear transformations.

Definition) A map $T: V \to W$ is a linear transformation if
$\forall \vec{x}, \vec{y} \in V$, $\forall c, d \in \mathbb{R}$

$$T(c\vec{x} + d\vec{y}) = cT(\vec{x}) + dT(\vec{y})$$

In other words, a linear tranformation respects addition & scaling.

1) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$
2) $T(c\vec{x}) = c \cdot T(\vec{x})$
3) $T(\vec{0}) = \vec{0}$

Example) Let A be an $m\times n$ matrix, then $T(\vec{x}) = A\vec{x}$ is a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$.

$$\to T(a\vec{x} + b\vec{y}) = A(a\vec{x} + b\vec{y}) = a\underbrace{A\vec{x}}_{T(\vec{x})} + b\underbrace{A\vec{y}}_{T(\vec{y})} = aT(\vec{x}) + bT(\vec{y})$$

Example) Let $\mathbb{P}_n$ be the vector space of polynomials of degree at most n.
Consider the map $T: \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by

$$T(p(t)) = \frac{d}{dt} p(t)$$

This map is linear!

$\forall p(t), q(t) \in \mathbb{P}_n$

$$\Rightarrow T(ap(t) + bq(t)) = \frac{d}{dt}(ap(t) + bq(t)) = a\frac{d}{dt}p(t) + b\frac{d}{dt}q(t)$$

$$= aT(p(t)) + bT(q(t))$$

- Representing linear maps by matrices.

  - Let $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ be a basis for V.
    A linear map $T: V \to W$ is determined by the values $T(\vec{x}_1), T(\vec{x}_2), \cdots, T(\vec{x}_n)$.

Why?
Take any $\vec{v} \in V$, then $\vec{v} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$.
Again, it means that we can represent $\vec{v}$ as $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$

By the linearity of T,
$$T(\vec{v}) = T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n) \in W$$

Definition) Let $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ be a basis for V and $\vec{y}_1, \vec{y}_2, \cdots, \vec{y}_m$ be a basis for W.
The matrix representing T with respect to these bases (of V and W)

- has n columns (one for each of the $\vec{x}_j$)
- the jth column has m entries, $a_{1j}, a_{2j}, \cdots, a_{mj}$ determined by

$$T(\vec{x}_j) = a_{1j}\vec{y}_1 + a_{2j}\vec{y}_2 + \cdots + a_{mj}\vec{y}_m$$

$$T = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \end{array} \begin{array}{c} 1\ 2 \cdots \quad j \quad \cdots n \\ \begin{bmatrix} & a_{1j} & \\ & a_{2j} & \\ & \vdots & \\ & a_{mj} & \end{bmatrix} \end{array}$$

Example) Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the linear map such that
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}$$
What is the matrix A representing T, w.r.t. standard bases.

$\Rightarrow$ $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$T(\vec{x}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a_{11}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_{31}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$\therefore a_{11} = 1, \quad a_{21} = 2, \quad a_{31} = 3$
$$T(\vec{x}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} = a_{12}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_{32}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$\therefore a_{12} = 4 \quad a_{22} = 0 \quad a_{32} = 7$

$$\therefore A = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 7 \end{bmatrix}$$

Example) Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the linear map s.t.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\0\\7\end{bmatrix}$$

What is the matrix B representing $T(\cdot)$ with respect to

$$\vec{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \vec{x}_2 = \begin{bmatrix}-1\\2\end{bmatrix} \text{ for } V, \quad \vec{y}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{y}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{y}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ for } W.$$

$$\rightarrow T(\vec{x}_1) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\0\\7\end{bmatrix} = \boxed{\begin{bmatrix}5\\2\\10\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

$c_1 = 5, \quad c_2 = -3, \quad c_3 = 5$

$$B = \begin{bmatrix}5 & ?\\-3 & ?\\5 & ?\end{bmatrix}$$

$$T(\vec{x}_2) = -1\cdot T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 2\cdot T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \boxed{\begin{bmatrix}7\\-2\\11\end{bmatrix} = d_1\begin{bmatrix}1\\1\\1\end{bmatrix} + d_2\begin{bmatrix}0\\1\\0\end{bmatrix} + d_3\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

$d_1 = 7, \quad d_2 = -9, \quad d_3 = 4$

$$\therefore B = \begin{bmatrix}5 & 7\\-3 & -9\\5 & 4\end{bmatrix} \text{ represents } T(\cdot).$$