Example) Apply Gram-Schmidt to $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\Rightarrow$ $\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\quad \vec{q}_1 = \vec{b}_1 / \|\vec{b}_1\| = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$$\vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right\rangle \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \qquad \vec{q}_2 = \vec{b}_2 / \|\vec{b}_2\| = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\vec{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right\rangle \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \right\rangle \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \qquad \vec{q}_3 = \vec{b}_3 / \|\vec{b}_3\| = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

* Note, $Q = \begin{bmatrix} | & | & | \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \\ | & | & | \end{bmatrix} \rightarrow Q^T Q = \begin{bmatrix} q_1 \cdot q_1 & q_1 \cdot q_2 & q_1 \cdot q_3 \\ q_2 \cdot q_1 & q_2 \cdot q_2 & \cdots \\ - & - - - & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- The QR decomposition.

  - Let $A \in \mathbb{R}^{m \times n}$ of rank n. (columns are indep)
    Then, we have the QR decomposition $A = QR$
    where $Q \in \mathbb{R}^{m \times n}$ with orthonormal vectors, and
    R is upper-triangular, $n \times n$, invertible.

Example) Find the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$.
$\begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array}$

$\Rightarrow$ First, apply GS on columns of A.

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \langle \vec{a}_2, \vec{q}_1 \rangle \vec{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{q}_2 = \vec{b}_2 / \|\vec{b}_2\| = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{b}_3 = \vec{a}_3 - \langle \vec{a}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{a}_3, \vec{q}_2 \rangle \vec{q}_2 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \quad \vec{q}_3 = \vec{b}_3 / \|\vec{b}_3\| = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow$ To find R in $A = QR$? $\quad R = Q^{-1}A = Q^T A \quad (\because Q^T Q = I)$

$$R = Q^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\therefore A = QR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Note that
$$R = Q^T A = \begin{bmatrix} q_1^T a_1 & q_1^T a_2 & q_1^T a_3 & \cdots \\ 0 & q_2^T a_2 & q_2^T a_3 & \cdots \\ 0 & 0 & q_3^T a_3 & \cdots \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{bmatrix}$$

* The QR decomposition is useful for solving LS solutions

$$\text{LS}: A^TA\hat{x} = A^T\vec{b} \overset{A=QR}{\Longleftrightarrow} (QR)^T QR\hat{x} = (QR)^T\vec{b}$$

$$\Longleftrightarrow R^TR\hat{x} = R^TQ^T\vec{b}$$

R is square & invertible

$$\Longleftrightarrow R\hat{x} = Q^T\vec{b} \quad \text{--- } (*)$$

Since, R is upper-triangular, (*) can be solved by back substitution.

To summarize,

$\hat{x}$ is a LS solution of $A\vec{x}=b$

$\Longleftrightarrow R\hat{x} = Q^T\vec{b}$ where $A=QR$

∘ Determinants

Definition) The determinant is characterized by

- the normalization.

$$\det I = 1$$

- and how it is affected by elementary row operations.
  - (replacement) $R1 \leftarrow R1 + 2R2$
    → does not change the Det.
  - (interchange) $R1 \leftrightarrow R2$
    → reverses the sign of the Det
  - (scaling) $R2 \leftarrow s R2$
    → multiplies the Det, with s.

Example) Compute $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{vmatrix} \xrightarrow{R2 \leftarrow \frac{1}{2}R2} 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{vmatrix} \xrightarrow{R3 \leftarrow \frac{1}{7}R3} 7\cdot 2 \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{=1} = 14$$

Example) compute $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{vmatrix}$

$$\rightarrow \begin{vmatrix} 1 & 2 & 3 \\ & 2 & 4 \\ & & 7 \end{vmatrix} \xrightarrow{R2 \leftarrow \frac{1}{2}R2} 2\begin{vmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 7 \end{vmatrix} \xrightarrow{R3 \leftarrow \frac{1}{7}R3} 2\cdot 7\begin{vmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{vmatrix}$$

$$\xrightarrow[R1 \leftarrow R1 - 3R3]{R2 \leftarrow R2 - 2R3} 14\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow{R1 \leftarrow R1 - 2R2} 14\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 14$$

* The determinant of a triangular matrix is the product of diagonal entries.

Example Compute $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

$$\Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow{R2 \leftarrow R2 - \frac{c}{a} R1} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix} = a\left(d - \frac{c}{a} b\right) = ad - bc$$

* Properties of the Det.

- $\det(A) = 0 \iff A$ is not invertible
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = 1/\det(A)$
- $\det(A^T) = \det(A)$

- Eigenvectors & eigenvalues

$A \in \mathbb{R}^{n \times n}$

Definition) An eigenvector of A is a non-zero $\vec{x}$ such that $A\vec{x} = \lambda \vec{x}$ for some scalar λ.
the λ is the corresponding eigenvalue.

How to solve $A\vec{x} = \lambda\vec{x}$

$$\begin{aligned} & A\vec{x} = \lambda\vec{x} \\ \iff & A\vec{x} - \lambda\vec{x} = 0 \\ \iff & (A - \lambda I)\vec{x} = 0 \\ \iff & \underline{\det(A - \lambda I) = 0} \end{aligned}$$

↳ characteristic polynomial of A

Example Find the eigen vectors & eigenvalues of A.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0 \qquad \therefore \lambda_1 = 2, \quad \lambda_2 = 4$$

i) e.v of $\lambda_1 = 2$

$$(A - \lambda I)\vec{x} = 0 \rightarrow (A - 2I)\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{x}_i \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

ii) e.$\vec{v}$ for $\lambda_2=4$

$$[A-4I]\vec{x_2} = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\vec{x_2} = 0 \qquad \vec{x_2}\in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

- The eigen space of $\lambda_2=4$ is $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

Theorem) If $\vec{x_1}, \dots, \vec{x_m}$ are eigen vectors of A corresponding to different eigenvalues, then they are independent.

⇒ Suppose that $\vec{x_1}, \dots, \vec{x_n}$ are dependent,
By excluding some of $\vec{x_i}$, we may assume that there exists $\vec{c}$ s.t.

$$c_1\vec{x_1} + \quad \cdots \quad + c_m\vec{x_m} = \vec{0} \qquad \text{— } (*)$$

However

$$A(c_1\vec{x_1} + \cdots + c_m\vec{x_m}) = \underbrace{c_1\lambda_1}_{c_1'}\vec{x_1} + \cdots + \underbrace{c_m\lambda_m}_{c_m'}\vec{x_m} = \vec{0}$$

gives another $\{c_1\lambda_1, \dots, c_m\lambda_m\}$ that make $(*)$ true.
Hence, $\vec{x_1}, \dots, \vec{x_m}$ are independent.

∘ Diagonalization

Example) If $A = \begin{bmatrix}6 & -1\\ 2 & 3\end{bmatrix}$, what is A^{100}?

⇒ Let's find the columns of A^{100}, one by one.
For computing $A^{100}\begin{bmatrix}1\\0\end{bmatrix}$, we rely on the following fact:

If $A\vec{v_1} = \lambda_1\vec{v_1}$, then $A^2\vec{v_1} = A\cdot\lambda_1\vec{v_1} = \lambda_1^2\vec{v_1}$
↳ $A^{100}\vec{v_1} = \lambda_1^{100}\vec{v_1}$

Furthermore,

$$A^{100}\underbrace{\vec{x}}_{\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}} = A^{100}(c_1\vec{v_1} + c_2\vec{v_2}) = (c_1\lambda_1^{100})\vec{v_1} + (c_2\lambda_2^{100})\vec{v_2}$$

- EVD on A → $\begin{cases}\lambda_1=4 & \vec{v_1} = \begin{bmatrix}1\\2\end{bmatrix}\\ \lambda_2=5 & \vec{v_2} = \begin{bmatrix}1\\1\end{bmatrix}\end{cases}$

Since, $\begin{bmatrix}1\\0\end{bmatrix} = -\vec{v_1} + 2\vec{v_2}$

$$A^{100}\begin{bmatrix}1\\0\end{bmatrix} = A^{100}(-\vec{v_1} + 2\vec{v_2}) = -4^{100}\begin{bmatrix}1\\2\end{bmatrix} + 2\cdot 5^{100}\begin{bmatrix}1\\1\end{bmatrix}$$

similarly, $\begin{bmatrix}0\\1\end{bmatrix} = \vec{v_1} - \vec{v_2}$

$$A^{100}\begin{bmatrix}0\\1\end{bmatrix} = 4^{100}\begin{bmatrix}1\\2\end{bmatrix} - 5^{100}\begin{bmatrix}1\\1\end{bmatrix}$$

Definition) Suppose that $A \in \mathbb{R}^{n\times n}$ has n independent eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$.

$$A\vec{v}_i = \lambda_i \vec{v}_i \Rightarrow A \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}}_{P} = \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{D}$$

In summary $AP = PD \Rightarrow A = PDP^{-1}$ (diagonalization)

Example) Suppose $A = PDP^{-1}$, what is A^n?

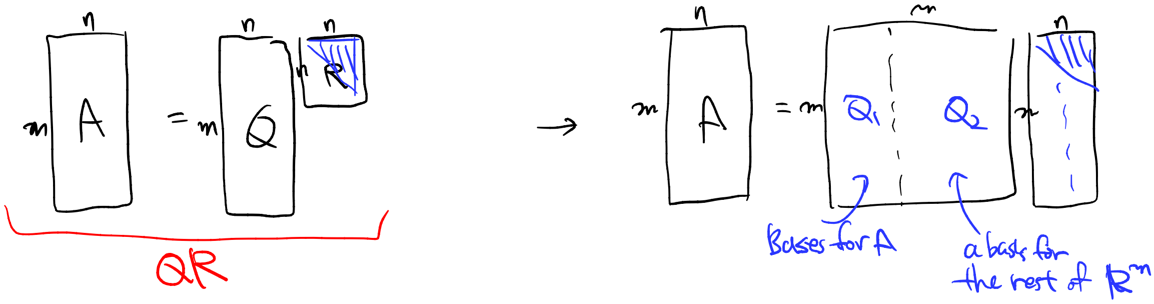
$\rightarrow A^n = P \cdot \underline{D}^n \cdot P^{-1}$

diagonal

- Singular Value Decomposition (SVD)

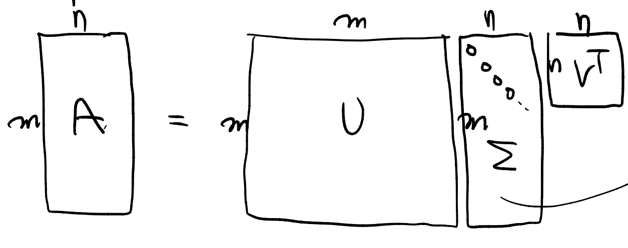
* From QR to SVD

Recall that for $A \in \mathbb{R}^{m\times n}$, $A = QR$ requires $\mathrm{rank}(A) = n$



- But, what if $\mathrm{rank}(A) < n$? SVD comes in to rescue.

$A = U\Sigma V^T$



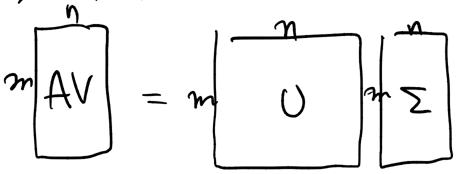
$$\Sigma = \begin{array}{c} \\ m \end{array}\overset{n}{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & \\ & & & & \end{bmatrix}}$$

there could be zero(s) in the diagonal

In this case $\mathrm{rank}(A) = 3$

where U and V are orthonormal matrices and $\Sigma \in \mathbb{R}^{m\times n}$ is a diagonal matrix.

$$A = U\Sigma V^T \iff AV = U\Sigma$$



- Key difference between QR & SVD is that R is upper triangular and Σ is diagonal.
- U & V are not unique but Σ is unique.

- Rank-deficient least squares

  - Suppose that $A = U\Sigma V^T$

    To solve $A\vec{x} = \vec{b}$

$$\downarrow$$

$$U\Sigma V^T \vec{x} = \vec{b}$$

$$\downarrow$$

$$\vec{x} = V\Sigma^{-1} U^T \vec{b}$$

Σ^{-1} may not be possible to compute.

Let's use LS.

$$\|A\vec{x} - \vec{b}\|^2 = \|U\Sigma V^T\vec{x} - \vec{b}\|^2$$

$\frac{1}{\|U\|^2}$

$$= \|\Sigma V^T \vec{x} - U^T\vec{b}\|^2$$

$$= \left\| \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \vec{x} - \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \vec{b} \right\|^2$$

$$= \underbrace{\|\Sigma_1 V_1^T \vec{x} - U_1^T\vec{b}\|^2}_{\text{this part can go to zero.}} + \underbrace{\|U_2^T \vec{b}\|^2}_{\text{residual}}$$

$$\Downarrow$$

$$\therefore \vec{x} = V_1 \Sigma_1^{-1} U_1^T \vec{b}$$