Example) Is the given set W a vector space?

(a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x = y + 2z \right\}$

$\Rightarrow$ i) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$

ii) $\underset{\in W}{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}} + \underset{\in W}{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}} = \underset{\in W}{\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}}$

$\therefore$ Yes

(b) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x+1 = y + 2z \right\}$

$\Rightarrow$ No, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$

(c) $W = \left\{ \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

$\Rightarrow$ Yes.

$$W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} y : x, y \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Example) Find Nul(A) where $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$.

$\Rightarrow \text{Nul}(A) = \{ x \mid Ax = 0 \}$.

$$\begin{bmatrix} \boxed{1} & 3 & 5 & 0 \\ 0 & \boxed{1} & 4 & -2 \end{bmatrix} \xrightarrow{R1 \leftarrow R1 - 3R2} \begin{bmatrix} \boxed{1} & 0 & -7 & 6 \\ 0 & \boxed{1} & 4 & -2 \end{bmatrix} \rightarrow \begin{cases} x_1 = 7x_3 - 6x_4 \\ x_2 = -4x_3 + 2x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{cases}$$

$$x = \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_4 \Rightarrow \text{span}\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

• Linear independence.

Review) · span$\{v_1, v_2, \ldots, v_m\}$ is a vector space.

Example) Is span $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix} \right\}$ equal to $\mathbb{R}^3$?

$\Rightarrow$ Recall that the span is equal to

$$W = \left\{ \begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix} X \ : \ X \in \mathbb{R}^3 \right\}$$

Hence, the span is equal to $\mathbb{R}^3$ iff the system with the augmented matrix

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & b_1\\1 & 2 & 1 & b_2\\1 & 3 & 3 & b_3\end{array}\right]$$

is constent for all b_1, b_2, and b_3.

Gaussian elimination

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & b_1\\1 & 2 & 1 & b_2\\1 & 3 & 3 & b_3\end{array}\right] \xrightarrow[R3 \leftarrow R3-R1]{R2 \leftarrow R2-R1} \left[\begin{array}{ccc|c}\boxed{1} & 1 & -1 & b_1\\0 & \boxed{1} & 2 & b_2-b_1\\0 & \boxed{2} & 4 & b_3-b_1\end{array}\right] \xrightarrow{R3 \leftarrow R3-2R2} \left[\begin{array}{ccc|c}1 & 1 & -1 & b_1\\0 & 1 & 2 & b_2-b_1\\0 & 0 & 0 & b_3-2b_2+b_1\end{array}\right]$$

It is consistent iff $[0\ 0\ 0 \mid b_3-2b_2+b_1]$ is not the form of $[0\ 0\ 0 \mid b]$ $b \neq 0$.

$\Leftrightarrow (b_3 - 2b_2 + b_1 = 0)$

Hence, the span does not equal $\mathbb{R}^3$.

• What went wrong with span $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix} \right\}$?

$$\rightarrow \begin{bmatrix}-1\\1\\3\end{bmatrix} = -3\begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\text{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\}.$$

Definition) Vectors $v_1, \ldots, v_p$ are said to be linearly independent if the equation

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$$

has only the trivial solution. (i.e., $x_1 = x_2 = \cdots = x_p = 0$).

Example) Are the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\3\end{bmatrix}$ independent?

$\Rightarrow$ We need to check whether the equation

$$x_1\begin{bmatrix}1\\1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\2\\3\end{bmatrix} + x_3\begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

has more than the trivial solution.

In other words, $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has no free variables.

↳ unique solution.

To check,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} \boxed{1} & 1 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{0} \end{bmatrix} \Rightarrow \text{the vectors are linearly dep.}$$

x_3 is free.

To find the linear dependence relation.

RREF: $\begin{bmatrix} \boxed{1} & 0 & -3 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{0} \end{bmatrix} \rightarrow \begin{cases} x_1 = 3x_3 \\ x_2 = -2x_2 \\ x_3 = \text{free} \end{cases} \quad x = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

To check, $3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\cdot\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

- Linear independence of matrix columns.

→ Each linear dependence relation of columns of A corresponds to a non trivial solution to $Ax=0$.

Theorem) Let A be an $m\times n$ matrix, the columns of A are linearly independent

$\iff$ $Ax=0$ has only the solution $x=0$

$\iff$ $\mathrm{Nul}(A) = \{0\}$

$\iff$ A has n pivots (A has no free variable).

(practical)

Example) Are the vectors independent?

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\}$

HW

$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} \boxed{1} & 1 & -1 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$ ∴ linearly independent.

(b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$

HW

$\rightarrow \begin{bmatrix} \boxed{1} & 1 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$ ∴ linearly dependent

Example) Are the following statements true?

(a) A set of a single vector is linearly independent?
$\rightarrow$ No, $\{\vec{0}\}$ $\because$ $c_1 \vec{0} = \vec{0}$ for all $c_1 \in \mathbb{R}$

(b) A set of a single nonzero vector is linearly indep.?
$\rightarrow$ Yes, $c_1 V = 0$ only for $c_1 = 0$, hence $\{V\}$ is linearly indep.
$V \neq \vec{0}$.

(c) A set of two vectors $\{v_1, v_2\}$ is linearly indep. iff neither of the vectors is the multiple of others.
$\rightarrow$ Yes,
$\because$ If $c_1 v_1 + c_2 v_2 = 0$, where $c_1 \neq 0$ or $c_2 \neq 0$. then $v_2 = -\frac{c_1}{c_2} v_1$.

(d) A set of vectors $\{v_1, \ldots, v_p\}$ containing the zero vector is linearly dependent.
$\rightarrow$ Yes
$\because$ Suppose $v_1 = 0$ (WLOG), then $\underline{1 \cdot} v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 v_p = \vec{0}$

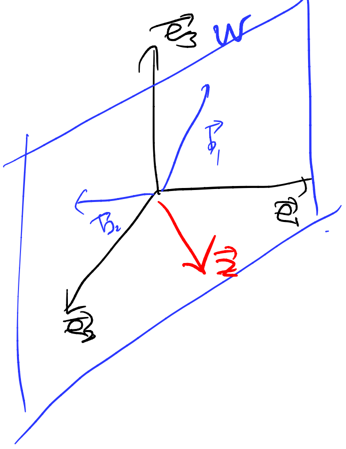
- A basis of a vector space.

Definition) A set of vectors $\{v_1, \ldots, v_p\}$ in V is a basis of V if
1) $V = \text{span}\{v_1, \ldots, v_p\}$

2) the vectors $v_1, \ldots, v_p$ are linearly indep.

Example) Let $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, show that $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ is a basis of $\mathbb{R}^3$?

$\Rightarrow$ "HW" (TA: submit next week)



$$\vec{z} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in W$$

$$\vec{z} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \cdot \vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$$

$$= \alpha \cdot \vec{b}_1 + \beta \cdot \vec{b}_2 \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Theorem) If S is a basis of a vector space V, then every vector in V has exactly one representation as a linear combination of elements of S.

$\Rightarrow$ Every vector $\vec{v} \in V$ is equal to the linear combination of the vectors from S.
(By contradiction), suppose that there exist a vector $\vec{a} \in V$ which is equal to two different combinations of vectors from S:

$$\vec{a} = x_1 \vec{s}_1 + \cdots + x_n \vec{s}_n$$
$$\vec{a} = y_1 \vec{s}_1 + \cdots + y_n \vec{s}_n$$

Substracting the first from the second:

$$\vec{0} = (x_1 - y_1)\vec{s}_1 + \cdots + (x_n - y_n)\vec{s}_n$$

Thus, the zero vector $\vec{0}$ is equal to a linear combination of elements of S.
Since, $\vec{x} \neq \vec{y}$, some of the coefficient, $(x_i - y_i)$, are not zero. But this contradicts the assumption that S is a basis ($\rightarrow$ linearly indep.).
This completes the proof.

- Theorem. If V has a basis with n elements, then every set of m vectors ($m > n$), is linearly dependent.

$\Rightarrow$ Let $\{v_1, \ldots, v_n\}$ be a basis of V and let S be a subset of V with $m > n$. We need to prove that S is linearly dependent.
basis vectors.

Then, every element $\{s_1, \ldots, s_m\}$ of S is a linear combination of $\{v_1, \ldots, v_n\}$.

$$\begin{cases} \vec{s}_1 = a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n \\ \vdots \\ \vec{s}_m = a_{m1}\vec{v}_1 + \cdots + a_{mn}\vec{v}_n \end{cases}$$

Consider the following n-dim vectors

$$\begin{cases} \vec{t}_1 = (a_{11}, a_{12}, \ldots, a_{1n}) \in \mathbb{R}^n \\ \vdots \\ \vec{t}_m = (a_{m1}, a_{m2}, \ldots, a_{mn}) \in \mathbb{R}^n \end{cases}$$

$\swarrow$ coefficient vector

We will show that $\{\vec{t}_1, \cdots, \vec{t}_m\}$ are linearly dependent. We can show this by constructing

$$T = \begin{array}{c} \\ n \\ \\ \end{array} \underbrace{\begin{bmatrix} | & | & & | \\ \vec{t}_1 & \vec{t}_2 & \cdots & \vec{t}_m \\ | & | & & | \end{bmatrix}}_{m} \in \mathbb{R}^{n \times m}$$

where $m > n$.

Since T can have at most n pivots, $\{\vec{t}_i\}_{i=1}^{m}$ are linearly dependent.
It means that there exists non-trivial $\{x_i\}_{i=1}^{m}$ such that

$$x_1\vec{t}_1 + \cdots + x_m\vec{t}_m = 0$$

$$x_1\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} + x_2\begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix} + \cdots + x_m\begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Recall that

$$\begin{cases} \vec{s}_1 = a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n \\ \vdots \\ \vec{s}_m = a_{m1}\vec{v}_1 + \cdots + a_{mn}\vec{v}_n \end{cases}$$

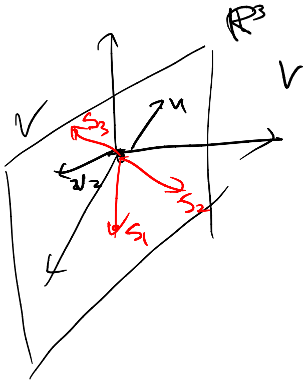
multiply i-th entry with x_i

$$\begin{bmatrix} x_1\vec{s}_1 = x_1 a_{11}\vec{v}_1 + \cdots + x_1 a_{1n}\vec{v}_n \\ \vdots \\ x_m\vec{s}_m = x_m a_{m1}\vec{v}_1 + \cdots + x_m a_{mn}\vec{v}_n \end{bmatrix}$$

Then, add all elements

$$\begin{aligned} x_1\vec{s}_1 + \cdots + x_m\vec{s}_m &= (x_1a_{11} + x_2a_{21} + \cdots + x_ma_{m1})\vec{v}_1 + \\ &\quad (x_1a_{12} + x_2a_{22} + \cdots + x_ma_{m2})\vec{v}_2 + \\ &\quad \vdots \\ &\quad (x_1a_{1m} + x_2a_{2m} + \cdots + x_ma_{mm})\vec{v}_n \\ &= p_1\vec{v}_1 + p_2\vec{v}_2 + \cdots + p_n\vec{v}_n \\ &= 0 \end{aligned}$$

where x is nontrivial.
We have shown that $\{\vec{s}_1, \cdots, \vec{s}_n\}$ are linearly dependent.



V has two basis vector in $\mathbb{R}^3$.
$S = \{s_1, s_2, s_3\} \leftarrow$ Is S lin. dep.?

Theorem) If V has a basis with n elements, then every set of vectors fewer than n elements, say $m<n$, does not span V.

$\Rightarrow$ Suppose that S is a basis $S=\{\vec{s}_1, \cdots, \vec{s}_n\}$ of V and T is a subset of V with m elements $(m<n)$.
By contradiction, assume that span$(T)=V$. Then S has n elements and from the previous theorem, the vectors of S are linearly dependent.
But this contradicts the fact that S is linearly independent ($\because$ basis)
This completes the proof.

Theorem) If V has a basis with n elements, then all bases of V have the same number of elements.

$\Rightarrow$ If V has a basis with n elements
 $\rightarrow$ every set of $(m>n)$ elements are linearly dep.
 $\rightarrow$ every set of $(m<n)$ elements does not span V.

Hence, V has a basis with n element,

Definition: dimension $\triangleq$ number of elements in a basis.