Theorem) Suppose that V has dimension d.

a) A set of d vectors in V are a basis if they span V.

b) A set of d vectors in V are a basis if they are linearly independent.

Example) Are the following sets a basis for $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$: No

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \right\}$: No

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$

→ To check lin. indep.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix} \quad \therefore \text{LI} \to \text{Yes}$$

Example) Let $\mathbb{P}_2$ be the space of polynomials of degree at most 2.

$\mathbb{P}_2 = \{ a + bt + ct^2 \mid a, b, c \in \mathbb{R} \}$

- What is the dimension of $\mathbb{P}_2$?
  → 3, $\because \{1, t, t^2\}$ is a basis of $\mathbb{P}_2$
- Is $\{t, 1-t, 1+t-t^2\}$ a basis of $\mathbb{P}_2$?

$$x_1 t + x_2(1-t) + x_3(1+t-t^2) = 0$$

$$\underbrace{(x_2+x_3)}_{=0} + \underbrace{(x_1-x_2+x_3)}_{=0}t \underbrace{-x_3}_{=0}t^2 = 0$$

$$\begin{cases} x_2+x_3=0 \\ x_1-x_2+x_3=0 \\ x_3=0 \end{cases} \to \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \because \text{trivial solution!}$$

Hence, $\{t, (1-t), (1+t-t^2)\}$ is linearly independent → a basis of $\mathbb{P}_2$.

Example) Subspaces of $\mathbb{R}^3$ can have dimension, 0, 1, 2, 3.

- The only 0-dimensional subspace is $\{\vec{0}\}$.
- The 1-dimensional subspace is of the form span$\{\vec{v}\}$ where $\vec{v} \neq \vec{0}$.
- The 2-dim. subspace is of the form span$\{\vec{v}, \vec{w}\}$ where $\vec{v} \neq \vec{0}$, $\vec{w} \neq \vec{0}$ and $\vec{v}$ and $\vec{w}$ are not multiples of each other.
- The only 3-dim. subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

True or False:

1) Suppose that V has dimension n. Then any set in V containing more than n vectors are linearly dependent.
   $\rightarrow$ True
2) $\mathbb{P}_n$ of polynomials of degree at most n has dimension $n+1$.
   $\rightarrow$ True. $\{1, t, t^2, \dots, t^n\}$
3) The vector space of functions $f: \mathbb{R} \rightarrow \mathbb{R}$ is infinite-dimensional.
   $\rightarrow$ True
4) Consider $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$. If one of the vectors, say $\vec{v}_k$, in the spanning set is a linear combination of the remaining ones, then the remaining ones still span V.
   $\rightarrow$ True

- Bases for column and null spaces

To find a basis for Nul (A)

- find the parametric form of the solutions to $A\vec{x}=0$.
- express solutions $\vec{x}$ as a linear combination of vectors with free variables as coefficients.
- These vectors form a basis of Nul(A).

- Example) Find the basis of Nul (A) where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 - 2R1} \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 0 & 0 & 3 & -6 & -15 \end{bmatrix} \xrightarrow{R1 \leftarrow R1 - 2R2} \begin{bmatrix} \boxed{3} & 6 & 0 & 15 & 39 \\ 0 & 0 & \boxed{3} & -6 & -15 \end{bmatrix}$$

$$\xrightarrow{R1/3,\ R2/3} \begin{bmatrix} \boxed{1} & 2 & 0 & 5 & 13 \\ 0 & 0 & \boxed{1} & -2 & -5 \end{bmatrix} \longrightarrow \begin{cases} x_1 = -2x_2 - 5x_4 - 13x_5 \\ x_2 = x_2 \\ x_3 = 2x_4 + 5x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} x_5 \qquad \therefore \text{Nul}(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\Rightarrow$ Are the vectors linearly indep.?

$$\begin{array}{c} \begin{bmatrix} \boxed{1} & 2 & 0 & 5 & 13 \\ 0 & 0 & \boxed{1} & -2 & -5 \end{bmatrix} \\ \begin{array}{ccccc} \quad & x_2 & \quad & x_4 & x_5 \end{array} \end{array} \longrightarrow \left\{ \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=\vec{v}_1}, \underbrace{\begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}}_{=\vec{v}_2}, \underbrace{\begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}}_{=\vec{v}_3} \right\}$$

Then, $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ only when $c_1 = c_2 = c_3 = 0$

- Bases for column spaces

→ A basis of Col(A) is given by the pivot columns of A.

Example) Find a basis for Col(A) with

$$A=\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

- The four fundamental subspaces.

- Definition
  - The row space of A is the column space of A^T
  - The left nullspace of A is the nullspace of A^T.

Let $A \in \mathbb{R}^{m \times n}$ of rank r.

- $\dim \operatorname{Col}(A) = r$
- $\dim \operatorname{Col}(A^T) = r$
- $\dim \operatorname{Nul}(A) = n - r$
- $\dim \operatorname{Nul}(A^T) = m - r$

Linear tranformations

- Let V and W be vector spaces.

Definition) A map $T: V \to W$ is a linear transformation if
$T(c\vec{x}+d\vec{y}) = c\cdot T(\vec{x}) + d\cdot T(\vec{y}) \in W$, $\forall \vec{x},\vec{y} \in V$ and $c,d \in \mathbb{R}$

$\downarrow$

In other words, a linear transformation respects addition & scaling
1) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$
2) $T(c\cdot\vec{x}) = c\cdot T(\vec{x})$
3) $T(\vec{0}) = \vec{0}$

Example) Let A be an $m \times n$ matrix
Then, $T(\vec{x}) = A\vec{x}$ is a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$.

$$\because A(c\vec{x}+d\vec{y}) = c\,A\vec{x} + d\,A\vec{y}$$

Example) Let $\mathbb{P}_n$ be the vector space of polynomials of degree at most n.
Consider the map $T: \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by

$$T(p(t)) = \frac{d}{dt}\,p(t)$$

This map is linear. $\because$ $\frac{d}{dt}\left[a\cdot p(t) + b\,q(t)\right] = a\cdot\frac{d}{dt}p(t) + b\cdot\frac{d}{dt}q(t)$

where $p(t), q(t) \in \mathbb{P}_n$, $a,b \in \mathbb{R}$

Representing linear maps by matrices.

- Let $\vec{x}_1, \dots, \vec{x}_n$ be a basis for V.
A linear map $T: V \to W$ is determined by the values $T(\vec{x}_1), \dots, T(\vec{x}_n)$.
Why?
Take any $\vec{v} \in V$, then $\vec{v} = c_1\vec{x}_1 + \dots + c_n\vec{x}_n$
$\hookrightarrow$ Again, it means that we can represent $\vec{x}$ as $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

By the linearity of the map T,
$T(\vec{v}) = T(c_1\vec{x}_1 + \dots + c_n\vec{x}_n) = c_1T(\vec{x}_1) + \dots + c_nT(\vec{x}_n) \in W$

Definition) From linear maps to matrices
Let $\vec{x}_1, \dots, \vec{x}_n$ be a basis for V, $\vec{y}_1, \dots, \vec{y}_m$ a basis for W.
The matrix representing T with respect to these bases
- has n columns (one per $\vec{x}_j$)
- the jth column has m entries $a_{1j}, a_{2j}, \dots, a_{mj}$ determined by

$$T(\vec{x}_j) = a_{1j}\vec{y}_1 + a_{2j}\vec{y}_2 + \dots + a_{mj}\vec{y}_m$$

$$T = \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ \dim(W) = m \end{array} \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & \bullet & \\ & & & & \\ & & & & \end{bmatrix} \in \mathbb{R}^{m \times n} \qquad \bullet \longrightarrow T(\vec{x}_j \in \mathbb{R}^n) = a_{1j}\vec{y}_1 + \cdots + a_{mj}\vec{y}_m$$

$$\begin{array}{cccccc} 1 & 2 & \cdots & j & \cdots & n = \dim(V) \end{array}$$

Example) $V = \mathbb{R}^2$, $W = \mathbb{R}^3$. Let T be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}$$

What is the matrix A representing T w.r.t. the standard bases.

$\Rightarrow$ The standard bases are

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $\mathbb{R}^2$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for $\mathbb{R}^3$.

Then,

$$T(\vec{x}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \underset{a_{11}}{1} \cdot \vec{y}_1 + \underset{a_{21}}{2} \cdot \vec{y}_2 + \underset{a_{31}}{3}\vec{y}_3$$

$$A = \begin{bmatrix} 1 & \square \\ 2 & \square \\ 3 & \square \end{bmatrix}$$

$$T(\vec{x}_2) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= 4\vec{y}_1 + 0\vec{y}_2 + 7\vec{y}_3$$

$$\therefore A = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 7 \end{bmatrix}$$