Example)

$$\begin{cases} \quad 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

$\downarrow$

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & 5 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

$$\xrightarrow{\cdots} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & -2 & 3 & 0 & -24 \\ 0 & \boxed{1} & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right] \rightarrow \begin{cases} x_1 = -24 + 2x_3 - 3x_4 \\ x_2 = -7 + 2x_3 - 2x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \\ x_5 = 4 \end{cases}$$

$$\rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4$$

- Questions of existence and uniqueness

Theorem) A linear system is consistent if and only if an echelon form of the augmented matrix has no row of the form

$$[\,0 \;\; 0 \;\; \cdots \;\; 0 \mid b\,]$$

where $b \neq 0$.

Example) For what values of h will the following system be consistent?

$$\begin{cases} 3x_1 - 9x_2 = 4 \\ -2x_1 + 6x_2 = h \end{cases}$$

$$\Rightarrow \begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & h \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{2}{3}R_1} \left[\begin{array}{cc|c} 3 & -9 & 4 \\ 0 & 0 & h + \frac{8}{3} \end{array}\right] \rightarrow h + \frac{8}{3} = 0 \qquad \therefore h = -\frac{8}{3}$$

Summary :
- Each linear system corresponds to an augmented mtx.
- From the Gaussian elimination, we can
  - read off, whether the system has no, one, or infinitely many solutions.
  - find all solutions.
- We can further row reduce to the reduced echelon form,
  - General solution in parametric form.
  - This form is unique.

- True or False:
  - There is no more than one pivot in any row. : True
  - There is no more than one pivot in any column (in echelon form). : True
  - There cannot be more free variables than pivot variable. : False

- The geometry of linear equations.

Example) $\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & -1 & 2 & 0 \\ 3 & 2 & -2 & 0 \end{bmatrix}$

column vector.

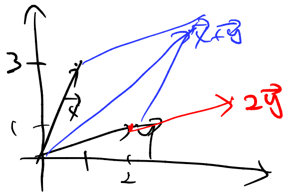
Example) The fundamental property of vectors is that vectors of the same kind, can be added and scaled.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}, \quad 7 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \\ 7x_3 \end{bmatrix}$$

Example) A vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ represents a point (x_1, x_2) in a plane.

Given $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, graph x, y, $x+y$, $2y$

$\rightarrow$



- Definition) Given vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m \in \mathbb{R}^n$ and scalars $c_1, c_2, \cdots, c_m \in \mathbb{R}$ the vector

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

is a linear combination of $\{\vec{v}_i\}_{i=1}^{m}$. The scalars $\{c_i\}_{i=1}^{m}$ are coefficient.

Example) Express $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$\Rightarrow \quad c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} 2c_1 - c_2 = 1 \\ c_1 + c_2 = 5 \end{cases}$$

$$\rightarrow \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 5 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 1.5 & 4.5 \end{array}\right] \xrightarrow{BS} \begin{array}{l} c_2 = 3 \\ 2c_1 = 4 \rightarrow c_1 = 2 \end{array}$$

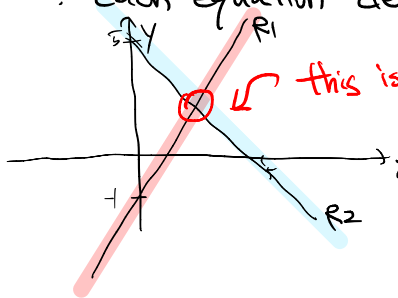
$$\therefore \; 2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \checkmark$$

- The row and column picture

example) $\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$

i) Row picture

: each equation defines a line in $\mathbb{R}^2$.

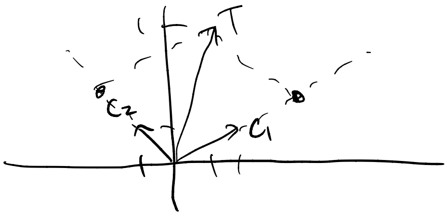


this is the solution.

ii) Column picture

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

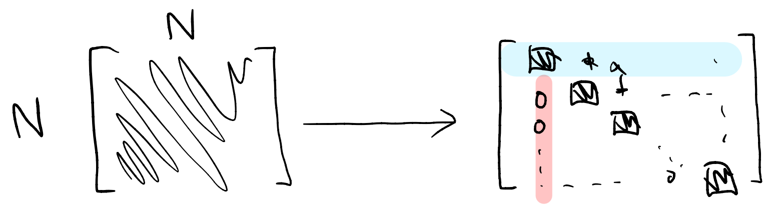
: Which linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ produces $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?



$\left[\begin{array}{l} \text{Each row of a matrix} \longrightarrow \text{a lin. equation} \\ \text{Each column of a matrix} \longrightarrow \text{a basis vector to span.} \end{array}\right.$

Question) How fast can we solve N linear equations in N unknowns?

$\Rightarrow$ Estimated cost of Gaussian elimination



$\Rightarrow$ i) To make a pivot: to create the zeros under the pivot.

$\Rightarrow$ we have $(N-1)$ rows to process $\times$ per each row, (N) multiplication and (N) substitutions $\Rightarrow O(N^2)$ for a pivot.

ii) We have to make upto N pivots $\rightarrow$ $O(N^3)$

We can do better:

- Strasson algorithm (1969) : $N^{\log_2 7} = N^{2.807}$
- Coppersmith-Winograd (1990) : $N^{2.375}$
- Strothers-Williams-Le Gall (2014) : $N^{2.373}$

Example) Consider the vectors

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

Determine if $\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} x_3 = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right] \xrightarrow{R3 \leftarrow R3-3R1} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 0 & 2 & 1 & -2 \end{array}\right] \xrightarrow{R3 \leftarrow R3-R2} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & \boxed{2} & 6 & 8 \\ 0 & 0 & \boxed{-5} & -10 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2,\ -\frac{1}{5}R_3} \left[\begin{array}{ccc|c} \boxed{1} & 4 & 3 & -1 \\ 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right] \xrightarrow{R2 \leftarrow R2-3R3} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R1 \leftarrow R1-4R2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R1 \leftarrow R1-3R3} \left[\begin{array}{ccc|c} 1 & & & 1 \\ & 1 & & -2 \\ & & 1 & 2 \end{array}\right] \text{(RREF)} \rightarrow \begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 2 \end{cases}$$

Summary : A vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m = \vec{b}$$

has the same solution set as the linear system with augmented matrix:

$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m & \vec{b} \\ | & | & & | & | \end{array}\right].$$ (column picture)

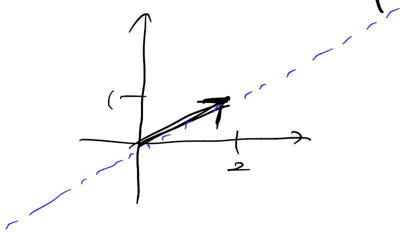
In particular, $\vec{b}$ can be generated by a linear combination of $\{\vec{a}_i\}_{i=1}^{m}$ if and only if this linear system is consistent.

- The span of a set of vectors

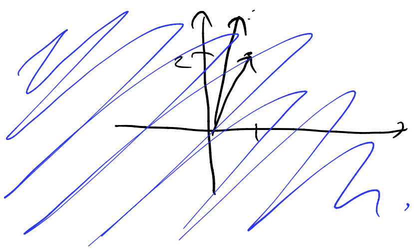
- Definition) The span of vectors $\vec{v}_1, \cdots, \vec{v}_m$ is the set of all their linear combinations. We denote it by $\text{span}\{\vec{v}_1, \cdots, \vec{v}_m\}$.

$$\Rightarrow \text{span}\{\vec{v}_1, \cdots, \vec{v}_m\} = \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \;\middle|\; c_1, \cdots, c_m \in \mathbb{R} \right\}$$

Example) (a) Describe $\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$ geometrically.



(b) Describe $\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}\right\}$, ~.



$= \mathbb{R}^2$

$\Rightarrow$ Let $\begin{bmatrix}b_1\\b_2\end{bmatrix}$ be any point in $\mathbb{R}^2$.

$$\begin{bmatrix}2\\1\end{bmatrix}c_1 + \begin{bmatrix}4\\1\end{bmatrix}c_2 = \begin{bmatrix}b_1\\b_2\end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 1 & 1 & b_2 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 0 & -1 & b_2 - \frac{1}{2}b_1 \end{array}\right]$$

this system is consistent!

$\therefore$ There exist c_1 and c_2 for every b_1 and b_2.

(c) Describe $\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}\right\}$, geometrically.

$$\rightarrow \left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 1 & 2 & b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & 4 & b_1 \\ 0 & 0 & b_2 - \frac{1}{2}b_1 \end{array}\right] \longrightarrow b_2 - \frac{1}{2}b_1 = 0 \rightarrow \boxed{b_2 = \frac{1}{2}b_1}$$

Example) Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Is $\vec{b}$ in the plane spanned by the columns of A?
($\triangleq$ column space)

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{array}\right] \xrightarrow[R3 \leftarrow R3 + R2]{R2 \leftarrow R2 - 3R1} \left[\begin{array}{cc|c} \boxed{1} & 2 & 8 \\ 0 & \boxed{-5} & -21 \\ 0 & 0 & -4 \end{array}\right]$$

"inconsistent"

Hence, $\vec{b}$ is not in the plane spanned by the columns of A.

- Summary : The span of vectors is the set of all their linear combinations. Some vector $\vec{b}$ is in span$\{\vec{a}_1, \cdots, \vec{a}_m\}$ iff there is a solution to the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m & \vec{b} \end{array}\right]$$

- Matrix operations

For an $m \times n$ matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

- Matrix times vector

  - Recall that $(x_1, \cdots, x_n)$ solves the linear system with an augmented matrix

$$\left[\begin{array}{c|c} A & b \end{array}\right] = \begin{bmatrix} | & | & & | & | \\ a_1 & a_2 & \cdots & a_n & b \\ | & | & & | & | \end{bmatrix}$$

if and only if

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$

It is therefore natural to define the product of matrix times vector as

$$A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

* $A\vec{x}$ is a linear combination of columns of A with weights given by the entries of $\vec{x}$.

Example) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 2 \\ 4 \end{bmatrix} x_2 = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\downarrow$

$\begin{cases} x_1 + 2x_2 = b_1 \\ 3x_1 + 4x_2 = b_2 \end{cases}$: A linear system can be represented as $\underline{A\vec{x} = \vec{b}}$