Example) Let us consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$.

Find Nul(A) and $Col(A^T)$

$\Rightarrow$ i) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \therefore \begin{array}{l} x_1 = -2x_2 \\ x_2 = x_2 \end{array}$

$\therefore$ $Nul(A) = span\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

ii) $Col(A^T) = Col\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}\right) = span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$\therefore$ $Col(A^T) = span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

* Vectors in Nul(A) are orthogonal to vectors in $Col(A^T)$.

$\Rightarrow$ Suppose that

$A \in \mathbb{R}^{n \times m}$ $\begin{bmatrix} \text{---}\ \vec{a}_{(1)}\ \text{---} \\ \vdots \\ \text{---}\ \vec{a}_{(n)}\ \text{---} \end{bmatrix}$, $\vec{a}_{(i)} \in \mathbb{R}^m$ is the i-th row vector of A.

Then, $\forall \vec{x} \in Nul(A)$, $\vec{a}_{(i)} \cdot \vec{x} = 0$ for all $i = 1, \cdots, n$

Since, $\forall \vec{y} \in Col(A^T)$, $\vec{y} = c_1 \cdot \vec{a}_{(1)} + c_2 \cdot \vec{a}_{(2)} + \cdots + c_n \vec{a}_{(n)}$, $c_i \in \mathbb{R}$

$$\begin{aligned} \vec{x} \cdot \vec{y} &= \vec{x} \cdot (c_1 \cdot \vec{a}_{(1)} + \cdots + c_n \vec{a}_{(n)}) \\ &= c_1 \cancelto{0}{\vec{x} \cdot \vec{a}_{(1)}} + \cdots + c_n \cancelto{0}{\vec{x} \cdot \vec{a}_{(n)}} \\ &= 0 \end{aligned}$$

$\therefore \forall \vec{x} \in Nul(A), \forall \vec{y} \in Col(A^T), \vec{x} \cdot \vec{y} = 0$

Definition. Let W be a subspace of $\mathbb{R}^n$ and $\vec{v} \in \mathbb{R}^n$

- $\vec{v}$ is orthogonal to W, if $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$
- Another subspace V is orthogonal to W, if every vector in $\vec{V}$ is orthogonal to W.
- The orthogonal complement of W is the space $W^\perp$ of all vectors that are orthogonal to W.

$\rightarrow W^\perp = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$

perp : perpendicular.

Example) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.

$\Rightarrow$ i) $\vec{0} \in W^\perp$ $\because$ $\vec{0} \cdot \vec{w} = \vec{0}$

ii) Let $\vec{u}, \vec{v} \in W^\perp$, then $(\vec{u} + \vec{v}) \in W^\perp$

$\because$ $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = 0$

iii) Let $\vec{u} \in W^\perp$, then $c\vec{u} \in W^\perp$.

Theorem) $Nul(A)$ is orthogonal to $Col(A^T)$. * orthogonal complement
$Nul(A^T)$ is orthogonal to $Col(A)$.

Example) Show that $\dim(W) + \dim(W^\perp) = n$, $W \subseteq \mathbb{R}^n$
where W is a subspace of $\mathbb{R}^n$

$\Rightarrow$ Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r$ be a basis for W, $\dim(W) = r$
Let $\vec{v}_{r+1}, \vec{v}_{r+2}, \cdots, \vec{v}_k$ be a basis for $W^\perp$. Thus we need to show that $k = n$.

i) First, we claim that $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r, \vec{v}_{r+1}, \cdots, \vec{v}_k\}$ is linearly independent.
Suppose that

$$\underbrace{c_1\vec{v}_1 + \cdots + c_r\vec{v}_r}_{\vec{w} \in W} + \underbrace{c_{r+1}\vec{v}_{r+1} + \cdots + c_k\vec{v}_k}_{\vec{w}' \in W^\perp} = \vec{0} \quad \text{— } (*)$$

Then, $\underbrace{\vec{w} + \vec{w}' = \vec{0}}_{\hookrightarrow\ \vec{w} = -\vec{w}'}$ and $\vec{w} \in W$, $\vec{w}' \in W^\perp$

$\rightarrow$ $\vec{w} \cdot \vec{w}' = 0 = \vec{w} \cdot (-\vec{w}) \Rightarrow \vec{w} \cdot \vec{w} = \|\vec{w}\| = 0 \quad \therefore \vec{w} = \vec{w}' = \vec{0}$

From $(*)$ $\rightarrow$ $c_1 = c_2 = \cdots = c_k = 0$!
$\therefore \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is linearly independent.

ii) We showed that $(k \le n)$.
Suppose that $(k < n)$. Then, the matrix

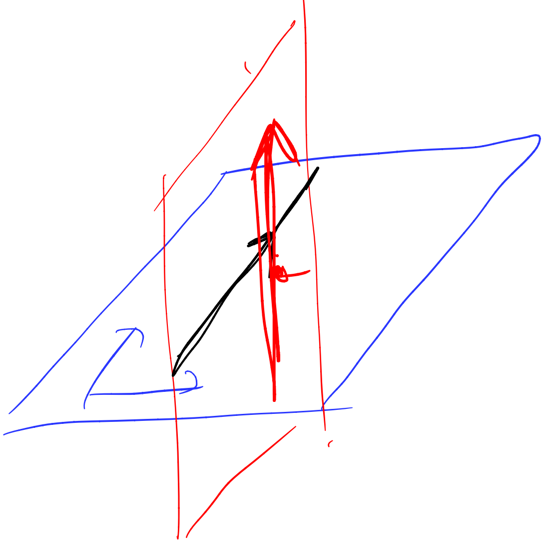
$$A = \begin{bmatrix} \text{—— } v_1^T \text{ ——} \\ \\ \text{—— } v_k^T \text{ ——} \end{bmatrix} \in \mathbb{R}^{k \times n}$$ is a fat matrix.

Thus, $Nul(A)$ is not empty $\rightarrow$ $\exists \vec{x} \in Nul(A)$ s.t., $\vec{x} \neq \vec{0}$

$$\rightarrow A\vec{x} = \begin{bmatrix} \text{—— } v_1^T \text{ ——} \\ \vdots \\ \text{—— } v_k^T \text{ ——} \end{bmatrix} \begin{bmatrix} \\ \vec{x} \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vdots \\ \vec{v}_r \cdot \vec{x} \\ \\ \vec{v}_k \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \\ 0 \end{bmatrix} \quad \text{——} (**)$$

$(**)$ implies that $\vec{x} \in W^\perp$ and $\vec{x} \in (W^\perp)^\perp$ $\rightarrow$ $\vec{x} = \vec{0}$
However, it contradicts that $\vec{x} \neq \vec{0}$.
$\longrightarrow$ $\therefore k \ge n$

$\therefore k = n$



Example) Show that $(W^\perp)^\perp = W$

$\Rightarrow$ i) $\vec{x} \in W$, then $\vec{x} \in (W^\perp)^\perp$ (HW) $\Rightarrow$ $W \subseteq (W^\perp)^\perp$

ii) Let $r = \dim(W)$

$\dim(W^\perp) = n-r$, $\dim((W^\perp)^\perp) = n-(n-r) = r$

The only r-dimensional subspace of $(W^\perp)^\perp$ is W.

Theorem) (FTLA) $A \in \mathbb{R}^{m\times n}$ of rank r.

- $\dim \mathrm{Col}(A) = r$
- $\dim \mathrm{Col}(A^T) = r$
- $\dim \mathrm{Nul}(A) = n-r$
- $\dim \mathrm{Nul}(A^T) = m-r$
- $\mathrm{Nul}(A) \perp \mathrm{Col}(A^T)$ $\leftarrow$ orthogonal complement.
- $\mathrm{Nul}(A^T) \perp \mathrm{Col}(A)$

Example) Find all vectors orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

$\rightarrow$ Orthogonal complement of $\mathrm{Col}\left(\begin{bmatrix}1&0\\1&1\\1&1\end{bmatrix}\right)$

$\rightarrow \mathrm{Nul}\left(\begin{bmatrix}\boxed{1}&1&1\\0&\boxed{1}&1\end{bmatrix}\right) \xrightarrow{R1 \leftarrow R1-R2} \begin{bmatrix}\boxed{1}&0&0\\0&\boxed{1}&1\end{bmatrix}$ $\begin{cases}x_1=0\\x_2=-x_3\\x_3=x_3\end{cases}$ $\rightarrow \mathrm{span}\left\{\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$

o A new perspective on $A\vec{x}=\vec{b}$

$A\vec{x} = \vec{b}$ is solvable.

$\Leftrightarrow \vec{b} \in \mathrm{col}(A)$

$\Leftrightarrow \vec{b}$ is orthogonal to $\mathrm{Nul}(A^T)$

Example) Let $A = \begin{bmatrix}1&2\\3&1\\0&5\end{bmatrix}$. For which $\vec{b}$ does $A\vec{x}=\vec{b}$ have a solution?

$\Rightarrow$

i) old

$$\left[\begin{array}{cc|c}1&2&b_1\\3&1&b_2\\0&5&b_3\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}\boxed{1}&2&b_1\\0&\boxed{-5}&-3b_1+b_2\\0&0&-3b_1+b_2+b_3\end{array}\right]$$

$A\vec{x}=\vec{b}$ is consistent $\Leftrightarrow$ $-3b_1+b_2+b_3=0$

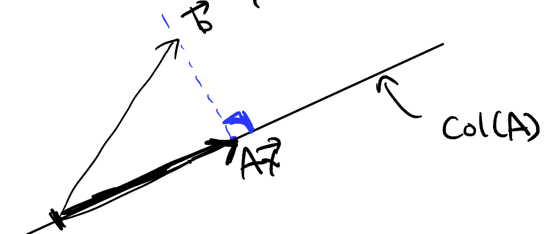
ii) New

$\mathrm{Nul}(A^T) \Rightarrow \begin{bmatrix}1&3&0\\2&1&5\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&3\\0&1&-1\end{bmatrix} \rightarrow \begin{cases}x_1=-3x_3\\x_2=x_3\\x_3=x_3\end{cases} \rightarrow \mathrm{span}\left\{\begin{bmatrix}-3\\1\\1\end{bmatrix}\right\}$

$\therefore \begin{bmatrix}-3\\1\\1\end{bmatrix}\cdot\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = 0 \rightarrow -3b_1+b_2+b_3=0$

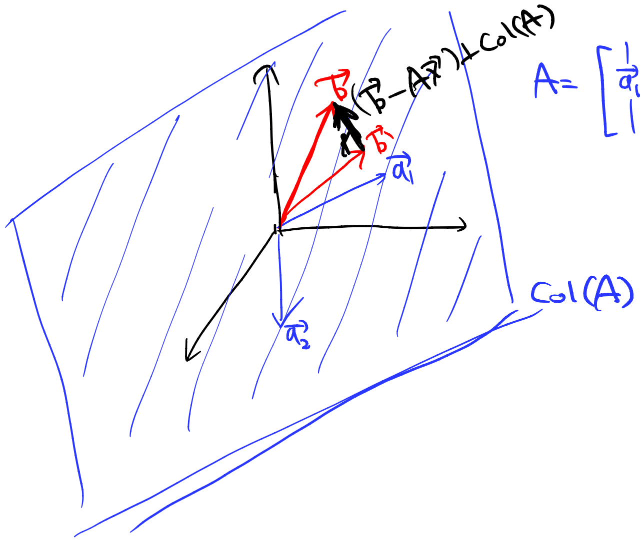
◦ Motivation

Example) Not all linear systems have solutions ($A\vec{x} = \vec{b}$)



Instead of giving up, we want $\vec{x}$ which makes $A\vec{x}$ and $\vec{b}$ as close as possible $\rightarrow$ $(\vec{b} - A\vec{x})$ minimize!

Such $\vec{x}$ is characterized by $A\vec{x}$ being orthogonal to the error $(\vec{b} - A\vec{x})$



$A = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \in \mathbb{R}^{3 \times 2}$ $\quad \vec{b} \in \mathbb{R}^3$