Introduction to systems of linear equations.

Definition) A linear equation in the variables $x_1, \cdots, x_n$ is an equation that can be written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.

Example) Which of the following equations are linear?

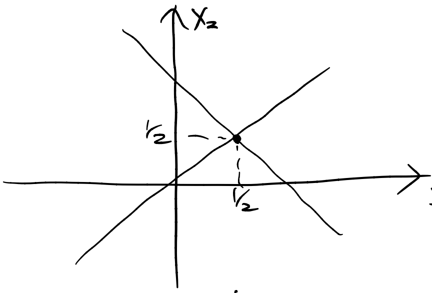
- $4x_1 - 5x_2 + 2 = x_1$
- $x_2 = 2(\sqrt{6} - x_1) + x_3$
- $4x_1 - 6x_2 = x_1x_2$
- $x_2 = 2\sqrt{x_1} - 7$

Definition) A linear system is a collection of one or more linear equations involving the same set of variables, say $x_1, x_2, \cdots, x_n$.

A solution of a linear system is a list $(s_1, s_2, \cdots, s_n)$ of numbers that make each equation in the system true when the values $(s_1, \cdots, s_n)$ are substituted for $x_1, x_2, \cdots, x_n$, respectively.

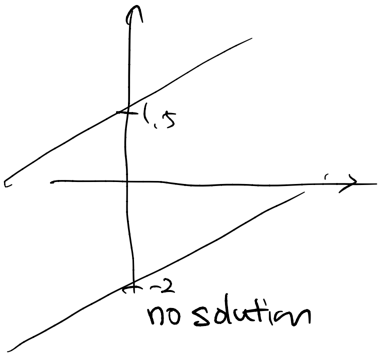
Example) Sketch the set of all solutions.

$$\begin{cases} x_1 + x_2 = 1 \\ -x_1 + x_2 = 0 \end{cases}$$



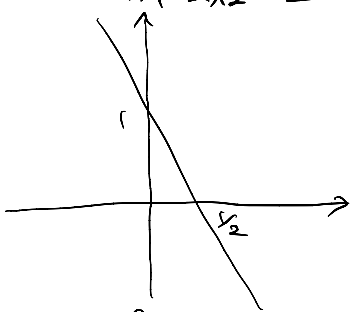
one solution

$$\begin{cases} x_1 - 2x_2 = -3 \\ 2x_1 - 4x_2 = 8 \end{cases}$$



no solution

$$\begin{cases} 2x_1 + x_2 = 1 \\ -4x_1 - 2x_2 = -2 \end{cases}$$



infinite solutions.

Theorem) A linear system has either

- no solution
- one unique solution
- infinitely many solutions.

Definition) A system is consistent if a solution exists.

How to solve a linear system.

Strategy : replace a system with an equivalent system which is easier to solve.

Definition) Linear systems are equivalent if they have the same set of solutions.

Example) $\begin{cases} x_1 + x_2 = 1 \\ -x_1 + x_2 = 0 \end{cases} \xrightarrow{R2 \leftarrow R2 + R1} \begin{cases} x_1 + x_2 = 1 \\ 0 + 2x_2 = 1 \end{cases}$ triangular form

Once in this triangular form, we find the solutions by back-substitution.

$\begin{cases} x_2 = 1/2 \\ x_1 = 1/2 \end{cases}$

Example) $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \xrightarrow{R3 \leftarrow R3 + 4R1} \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$

$\xrightarrow{R3 \leftarrow R3 + \frac{3}{2}R2} \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ x_3 = 3 \end{cases} \xrightarrow{\text{Back substitution.}} \begin{cases} x_3 = 3 \\ x_2 = 16 \\ x_1 = 29 \end{cases}$

- Matrix Notation.

$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases} \longrightarrow \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$: coefficient matrix

$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$: augmented matrix.

Definition) An elementary row operation is one of the following:
- (replacement) Add one row to a multiple of another row
- (interchange) Interchange two rows
- (scaling) Multiply all entries in a row, by a non-zero constant.

Definition) Two matrices are row equivalent, if one matrix can be transformed into the other by a sequence of elementary row operations.

Theorem) If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example) Solve,

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\ \ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + \frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \text{(Echelon form)}$$

$$\xrightarrow{R_2 \leftarrow R_2 + 8R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 \ 3 \end{cases}$$

↳ Reduced Echelon form

- Row reduction and echelon forms.

Definition) A matrix is in the echelon form if:

(1) Each leading entry of a row is in a column to the right of the leading entry of the row above it.
(2) All entries in a column below a leading entry are zero.
(3) All nonzero rows are above any rows of all zeros.

Example) Here is a representative matrix in echelon form

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & & \\ 0 & 0 & \blacksquare & * & * & \cdots & \\ 0 & 0 & 0 & 0 & \blacksquare & * & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

pivots

($*$: any value, $\blacksquare$: for any nonzero value.

pivot.

(a)
$$\begin{bmatrix} \blacksquare & * & * & \cdots \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

→ Yes

(b)
$$\begin{bmatrix} 0 & \blacksquare & * & * & * \\ \blacksquare & * & * & * & \\ 0 & 0 & \cdots & & \\ 0 & 0 & \cdots & & \end{bmatrix}$$

→ No.

(c) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$ Yes

(d) $\begin{bmatrix} \blacksquare & 0 & 0 \\ * & \blacksquare & 0 \\ * & 0 & \blacksquare \\ * & 0 & 0 \end{bmatrix}$ No →

Definition) A matrix is in reduced echelon form, if in addition to being echelon form, it also satisfies:

- Each pivot is 1.
- Each pivot is the only non-zero item in its column.

$$\underset{\text{REF}}{\begin{bmatrix} 0 & \blacksquare & * & * & \cdots & \\ 0 & 0 & 0 & \blacksquare & * & \cdots \\ 0 & 0 & 0 & 0 & \blacksquare & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \longrightarrow \underset{\text{RREF}}{\begin{bmatrix} 0 & \overset{=1}{\blacksquare} & * & 0 & 0 & 0 \\ 0 & 0 & 0 & \overset{=1}{\blacksquare} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overset{=1}{\blacksquare} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

either nonzero or zero.

Example) Are the following matrices in reduced echelon form?

(a) $\begin{bmatrix} 1 & 0 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ No $\longrightarrow \begin{bmatrix} \boxed{1} & 0 & 5 & 0 & -7 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -2 & 3 & 2 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R1 \leftarrow R1 - 2R3} \begin{bmatrix} \boxed{1} & 0 & -2 & 3 & 0 & -32 \\ 0 & \boxed{1} & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$

Theorem) Each matrix is row equivalent to one and only reduced echelon form

Homework) Find the reduced echelon form of

(a) $A = \left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$.

(b) then, suppose that A is the augmented matrix of a certain linear system, then, describe the solution of it.

- Solution of linear systems via row reduction.

Example) Suppose, we have RREF

$$\left[\begin{array}{ccccc|c} \boxed{1} & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{array}\right] \longrightarrow \begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 8x_4 = 5 \\ x_5 = 7 \end{cases}$$

- The pivots are located in columns 1, 3, 5. $\Rightarrow$ x_1, x_3, x_5 are pivot variables.
- The remaining variables, x_2, x_4 are free variables
- Then, we can write down the pivot variables in terms of free variables.

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 = \text{free} \\ x_3 = 8x_4 + 5 \\ x_4 = \text{free} \\ x_5 = 7 \end{cases}$$

general solution in parametric form.

free variables

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} x_2 + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}} x_4 + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix}$$

"basis"