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Reinforcement Learning

Lecture 4. Policy-based Methods

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Content

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- Summary of Model-based and Model-free methods
- ~~Proving Policy Gradient Theorem~~
- Trust Region Policy Optimization (**TRPO**)
- Proximal Policy Optimization (**PPO**)
- Generalized Advantage Estimation (**GAE**)
- Soft Actor-Critic (**SAC**)

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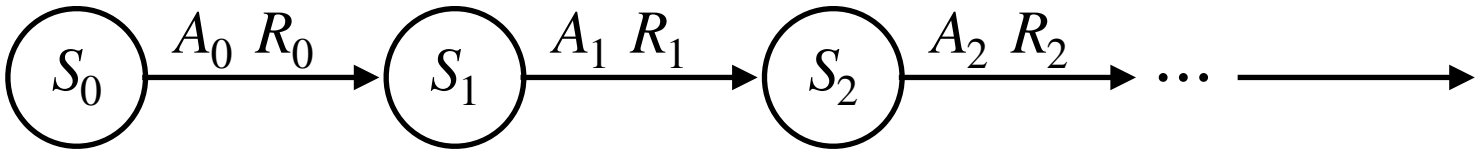
Summary of Model-based and Model-free Methods

Markov Decision Process



- Now, we have three random variables:
  - State: S_t
  - Reward: R_t
  - **Action: A_t**

Markov Decision Process

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- Formally, an MDP is a tuple (S, A, P, R, d):
  - A set of states $s \in S$.
  - A set of actions $a \in A$
  - A state transition function (or matrix)
    - $P(s' \mid s, a) = P(S_{t=1} = s \mid S_t = s, A_t = a)$
    - $P_{sas'} = P(s' \mid s, a)$
  - A reward function
    - $r(s) = \mathbb{E}[R_{t+1} \mid S_t = s]$
    - It depends on both state and action.
  - An initial state distribution d

Return

- **Return** G_t is the (discounted) sum of future rewards, and hence, also a random variable.

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+1+k}$$

- Discount factor γ:

1

Worth Now

γ

Worth Next Step

γ^2

Worth In Two Steps

(State) Value Function

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- The **state-value function** $V(s)$ is a function of a state and is the expected return starting from the state s.

$$
\begin{aligned}
V(s) &= \mathbb{E}[G_t \mid S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+1} \cdots \mid S_t = s]
\end{aligned}
$$

(State-Action) Value Function

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- The **state-action-value function** $Q(s,a)$ is a function of a state and is the expected return starting from the state s.

$$
\begin{aligned}
Q(s,a) &= \mathbb{E}[G_t \mid S_t = s, A_t = a] \\
&= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+1} \cdots \mid S_t = s, A_t = a]
\end{aligned}
$$

Policy

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- A **policy** is a **distribution over actions** given state.

$$\pi(a \mid s) = P(A_t = a \mid S_t = s)$$

  - In an MDP, a policy is a function of the current state s.
  - A policy is stationary (time-independent).
  - A deterministic policy can also be represented by a distribution.

Optimality

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What does it mean by **solving** an MDP?

Optimal Value and Policy

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- Optimal state value

$$V^*(s) = \max_{\pi} V_{\pi}(s)$$

- Optimal state-action value

$$Q^*(s,a) = \max_{\pi} Q_{\pi}(s,a)$$

- Optimal policy

$$\pi^*(a \mid s) = 1 \text{ for } a = \arg\max_{a'} Q^*(s,a')$$

Value Iteration

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Given an MDP (S, A, P, R, d), how can we **solve** an MDP?

Value Iteration

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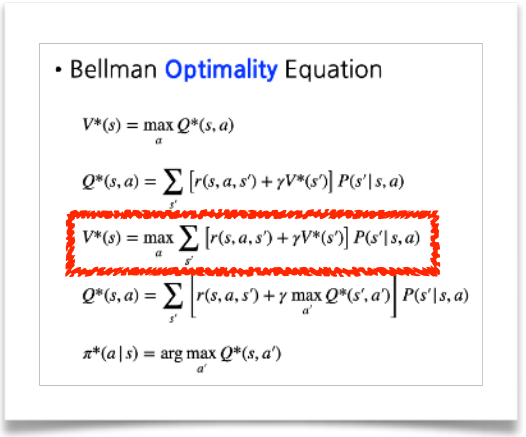
- Value iteration utilizes the **principle of optimality**.

$$V^*(s) = \max_{\pi} V_{\pi}(S)$$

- Then, the solution $V^*(s)$ can be found by one-step lookahead

$$V^*(s) = \max_{a} \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s' \mid s,a)$$



- The main idea is to apply these updates iteratively, repeat until convergence.
  - It is guaranteed to converge to the unique optimal value.
- However, there is no explicit policy.

Q-Value Iteration

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- However, it is not straightforward to come up with an **optimal policy** solely from $V^*(s)$.
- Hence, we use following relations between $V^*(s)$ and $Q^*(s,a)$ (aka the Bellman optimality equation).

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s'|s,a)$$

- Bellman Equation

$$V_\pi(s) = \sum_a \pi(a|s) Q^(s,a)$$

$$Q_\pi(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V_\pi(s') \right] P(s'|s,a)$$

$$V_\pi(s) = \sum_a \pi(a|s) \sum_{s'} \left[r(s,a,s') + \gamma V_\pi(s') \right] P(s'|s,a)$$

$$Q_\pi(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \sum_{a'} Q_\pi(s',a') \pi(a'|s') \right] P(s'|s,a)$$

- Bellman Optimality Equation

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s'|s,a)$$

$$V^*(s) = \max_a \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s'|s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \max_{a'} Q^*(s',a') \right] P(s'|s,a)$$

$$\pi^*(a|s) = \arg\max_{a'} Q^*(s,a')$$

Q-Value Iteration

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- Start from the random initial V_0
- For all states $s \in S$:

$$Q_k(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s,a)$$

$$V_{k+1}(s) = \max_{a'} Q_k(s,a')$$

- We now have an explicit form of the policy:

$$\pi(a \mid s) = 1 \text{ for } a = \arg\max_{a'} Q(s,a')$$

  - Note that this policy is **deterministic**.

Policy Iteration

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- **Step 1: Policy evaluation**: calculate the value function for a fixed policy until convergence.
  - Start from a random initial V_0.
  - Iterate until value converges:

$$V_{k+1}(s) = \sum_a \pi_i(a \mid s) \sum_{s'} [r(s, a, s') + \gamma V_k(s')] P(s' \mid s, a)$$

- **Step 2: Policy improvement**: update the policy using one-step lookahead using the converged value function.
  - One-step lookahead:

$$Q_{\pi_k}(s, a) = \sum_{s'} [r(s, a, s') + \gamma V_{\pi_i}(s')] P(s' \mid s, a)$$

$$\pi_{i+1}(a \mid s) = 1 \text{ for } a = \arg\max_{a'} Q_{\pi_i}(s, a')$$

Limitation of Model-based Methods

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Value Iteration

- Start from the random initial V_0
- For all states $s \in S$:

$$Q_k(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s'|s,a)$$

$$V_{k+1}(s) = \max_{a'} Q_k(s,a')$$

- We now have an explicit form of the policy:

$$\pi_{k=1}(a|s) = 1 \text{ for } a = \arg\max_{a'} Q_k(s,a')$$

Policy Iteration

- **Step 1: Policy evaluation**: calculate the value function for a fixed policy until convergence.
  - Start from a random initial V_0.
  - Iterate until value converges:

$$V_{k+1}(s) = \sum_{a} \pi_i(a|s) \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s'|s,a)$$

- **Step 2: Policy improvement**: update the policy using one-step lookahead using the converged value function.
  - One-step lookahead:

$$Q_{\pi_k}(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V_{\pi_i}(s') \right] P(s'|s,a)$$

$$\pi_{i+1}(a|s) = 1 \text{ for } a = \arg\max_{a'} Q_{\pi_i}(s,a')$$

What is the common **key limitation** of both methods?

Both methods require state transition probability $P(s'|s,a)$.

Model-free Methods

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- The goal of a **model-based method** can be written as

$$\text{Given } (S, A, R, P, d), \text{ find } \pi \text{ such that } \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_t \,|\, \pi\right] \text{ is maximized.}$$

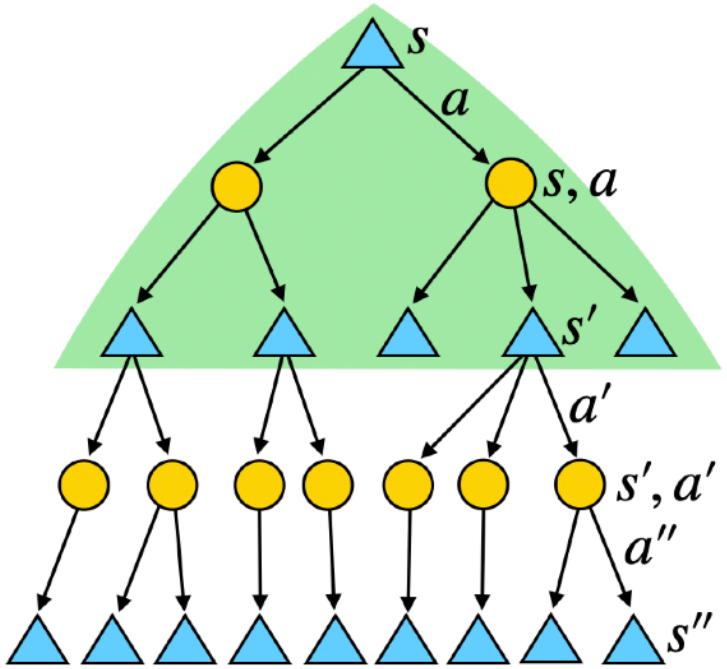
- The goal of a **model-free method** can be written as

$$\text{Given } (S, A, R), \text{ find } \pi \text{ such that } \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_t \,|\, \pi\right] \text{ is maximized.}$$

  - Note that the **state transition probability** P and the **initial state distribution** d are omitted in RL.
  - Hence, the main challenge is to approximate the **expectation** part!

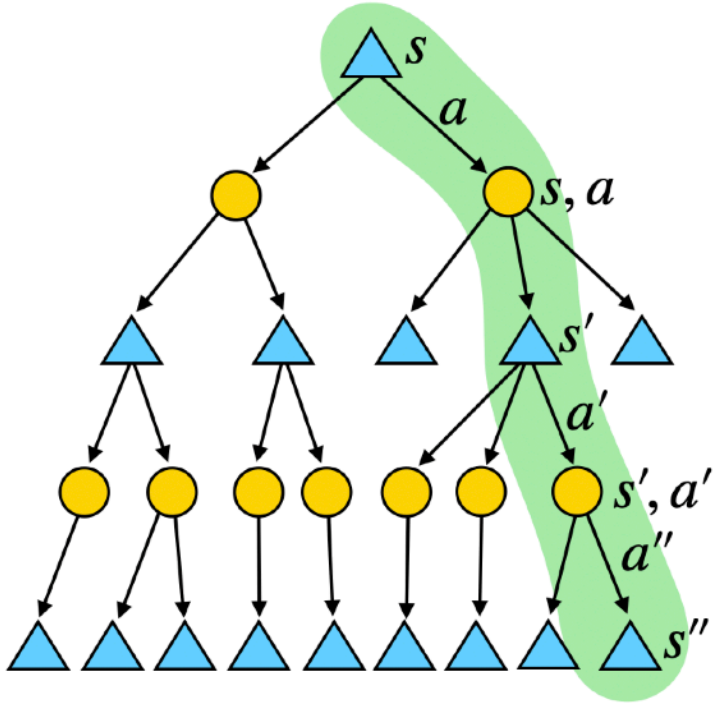
Model-free Methods

- A



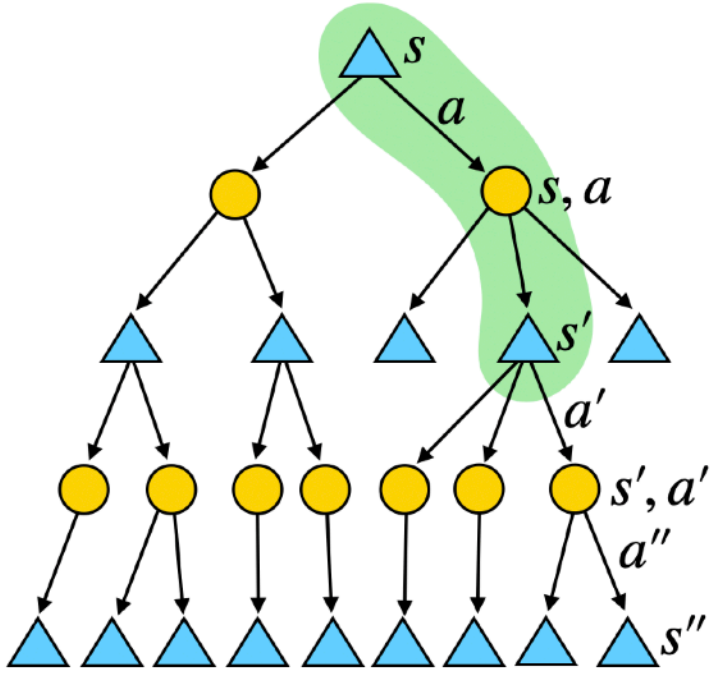
Dynamic Programming

$$V_{k+1}(s) = \sum_{a} \pi(a \mid s) \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s, a)$$



Monte-Carlo Learning

$$V_{k+1}(s) = V_k(s) + \alpha \left(G_t - V_k(s) \right)$$



Temporal Difference Learning

$$V_{k+1}(s) = V_k(s) + \alpha \left(r(s,a,s') + \gamma V_k(s') - V_k(s) \right)$$

SARSA

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- **Step 1: Policy evaluation**
  - Estimate $\widehat{Q}(s, a)$ from samples using (S, A, R, S', A')

$$\widehat{Q}(s_t, a_t) \leftarrow \widehat{Q}(s_t, a_t) + \alpha\left(R_{t+1} + \gamma \widehat{Q}(s_{t+1}, a_{t+1}) - \widehat{Q}(s_t, a_t)\right) \text{ (TD)}$$

- **Step 2: Policy improvement**
  - For the sake of better exploration, we use an ϵ-greedy policy.
    - With probability $1 - \epsilon$, choose the greedy action $a = \arg\max_{a'} Q_{\pi_i}(s, a')$.
    - With probability ϵ, choose a random action.

SARSA

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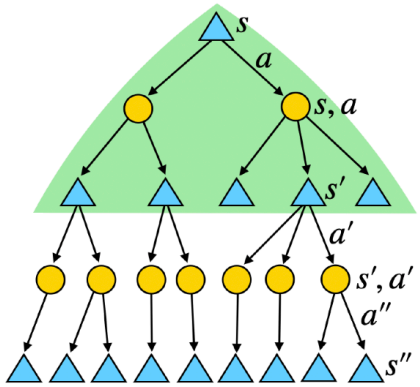
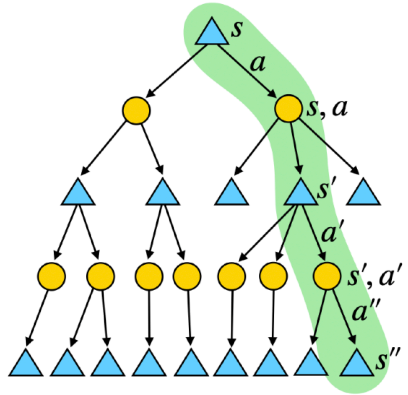
- Initialize $Q(s, a)$
- Repeat (for each episodes)
  - Sample an initial state s_0.
  - Sample a_0 from an ϵ-greedy policy π.
  - Repeat (for each time step t)
    - Get reward r_{t+1} and next state s_{t+1}.
    - Sample a_{t+1} from the ϵ-greedy policy π.
    - Update $\widehat{Q}(s, a)$ using $(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})$
      - $\widehat{Q}(s_t, a_t) \leftarrow \widehat{Q}(s_t, a_t) + \alpha \left(r_{t+1} + \gamma \widehat{Q}(s_{t+1}, a_{t+1}) - \widehat{Q}(s_t, a_t) \right)$

DP vs. MC vs. TD vs. SARSA



Dynamic Programming

$$V_{k+1}(s) = \sum_a \pi(a|s) \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s'|s,a)$$



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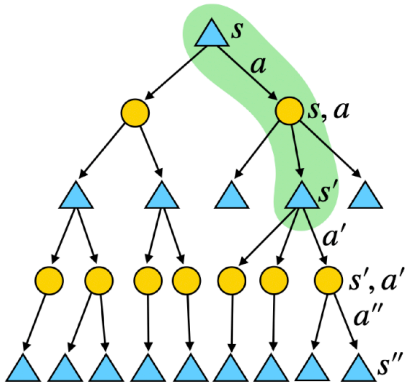
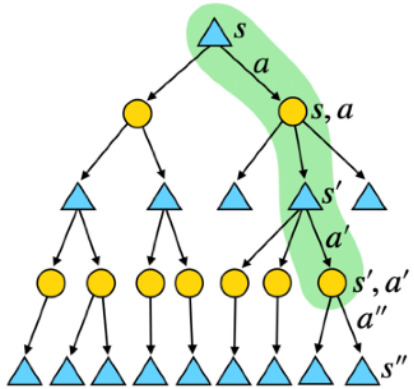


Temporal Difference Learning

$$V_{k+1}(s) = V_k(s) + \alpha \left(r(s,a,s') + \gamma V_k(s') - V_k(s) \right)$$



SARSA

$$\widehat{Q}(s_t, a_t) \leftarrow \widehat{Q}(s_t, a_t) + \alpha \left(r_{t+1} + \gamma \widehat{Q}(s_{t+1}, a_{t+1}) - \widehat{Q}(s_t, a_t) \right)$$

Q-Learning

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- Basic concepts of **Q-Learning**
  - It is a **model-free value iteration**.

$$V_{k+1}(s) = \max_a \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s'|s,a)$$

  - Q-values are more useful in terms of getting π.

$$Q_{k+1}(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \max_{a'} Q_k(s',a') \right] P(s'|s,a)$$

- But we still need the (transition) model $P(s'|s,a)$.
- Suppose that we are using **any behavior policy μ** to get a at any state s, and proceed to the next state s' following $P(s'|s,a)$,

$$Q_{k+1}(s,a) \approx r(s,a,s') + \gamma \max_{a'} Q_k(s',a')$$

Q-Learning

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- It can be regarded as a **model-free value iteration**.
- Q-Learning
  - For each time step

$$\widehat{Q}(s,a) \leftarrow \widehat{Q}(s,a) + \alpha\left(r(s,a,s') + \gamma \max_{a'} \widehat{Q}(s',a') - \widehat{Q}(s,a)\right)$$

SARSA: $\widehat{Q}(s_t,a_t) \leftarrow \widehat{Q}(s_t,a_t) + \alpha\left(R_{t+1} + \gamma \widehat{Q}(s_{t+1},a_{t+1}) - \widehat{Q}(s_t,a_t)\right)$ (TD)

- Note that $\max_{a'} \widehat{Q}(s',a')$ does not require the next action (compared to SARSA), hence we only need (s,a,s') for Q-Learning.
- We can use any **arbitrary behavior policy** to sample episodes as long as $s' \sim P(s'|s,a)$ which enables to utilize experience replays.

Q-Learning

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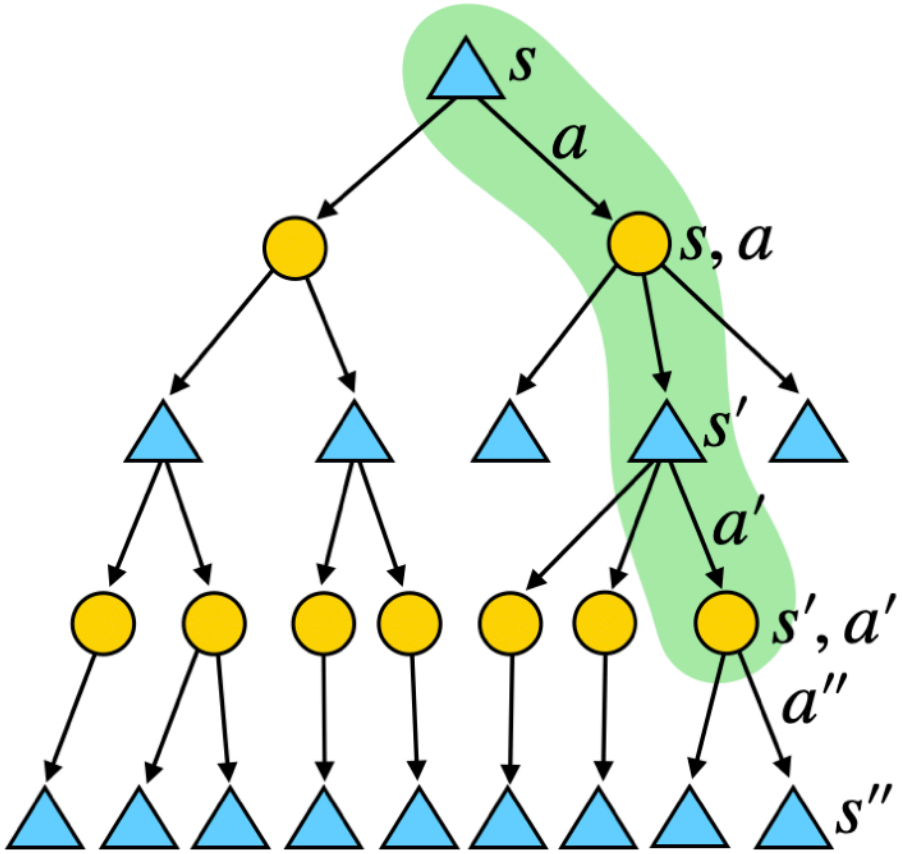
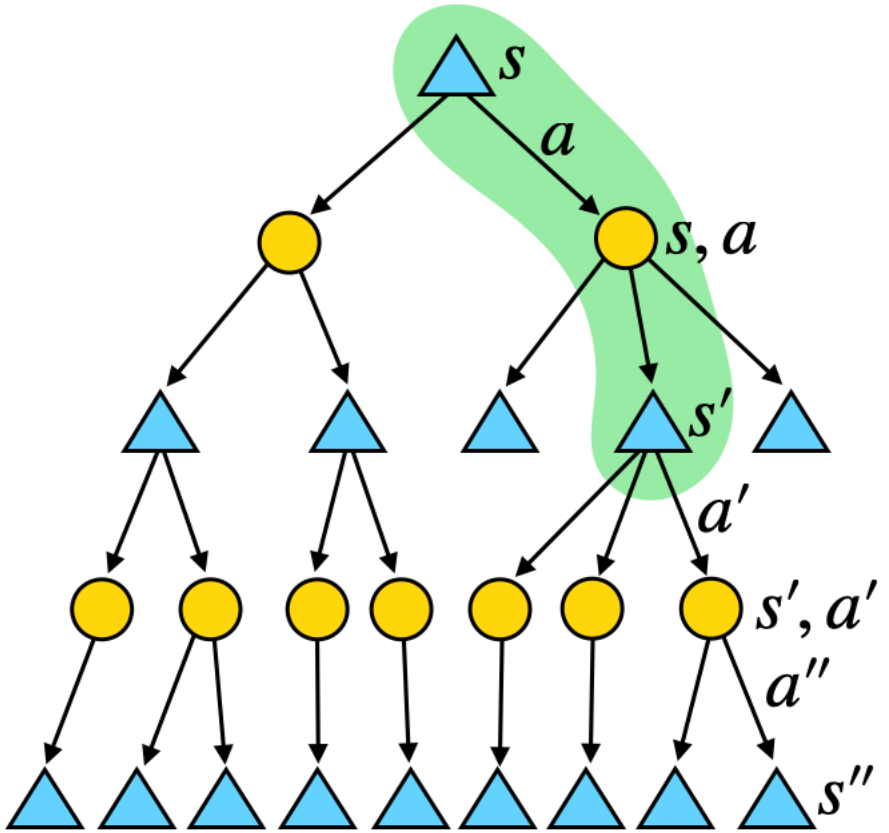
SARSA



$$\widehat{Q}(s_t, a_t) \leftarrow \widehat{Q}(s_t, a_t) + \alpha \left(r_{t+1} + \gamma \widehat{Q}(s_{t+1}, a_{t+1}) - \widehat{Q}(s_t, a_t) \right)$$

Q-Learning



$$\widehat{Q}(s, a) \leftarrow \widehat{Q}(s, a) + \alpha \left(r(s, a, s') + \gamma \max_{a'} \widehat{Q}(s', a') - \widehat{Q}(s, a) \right)$$

Deep Q Network (DQN)

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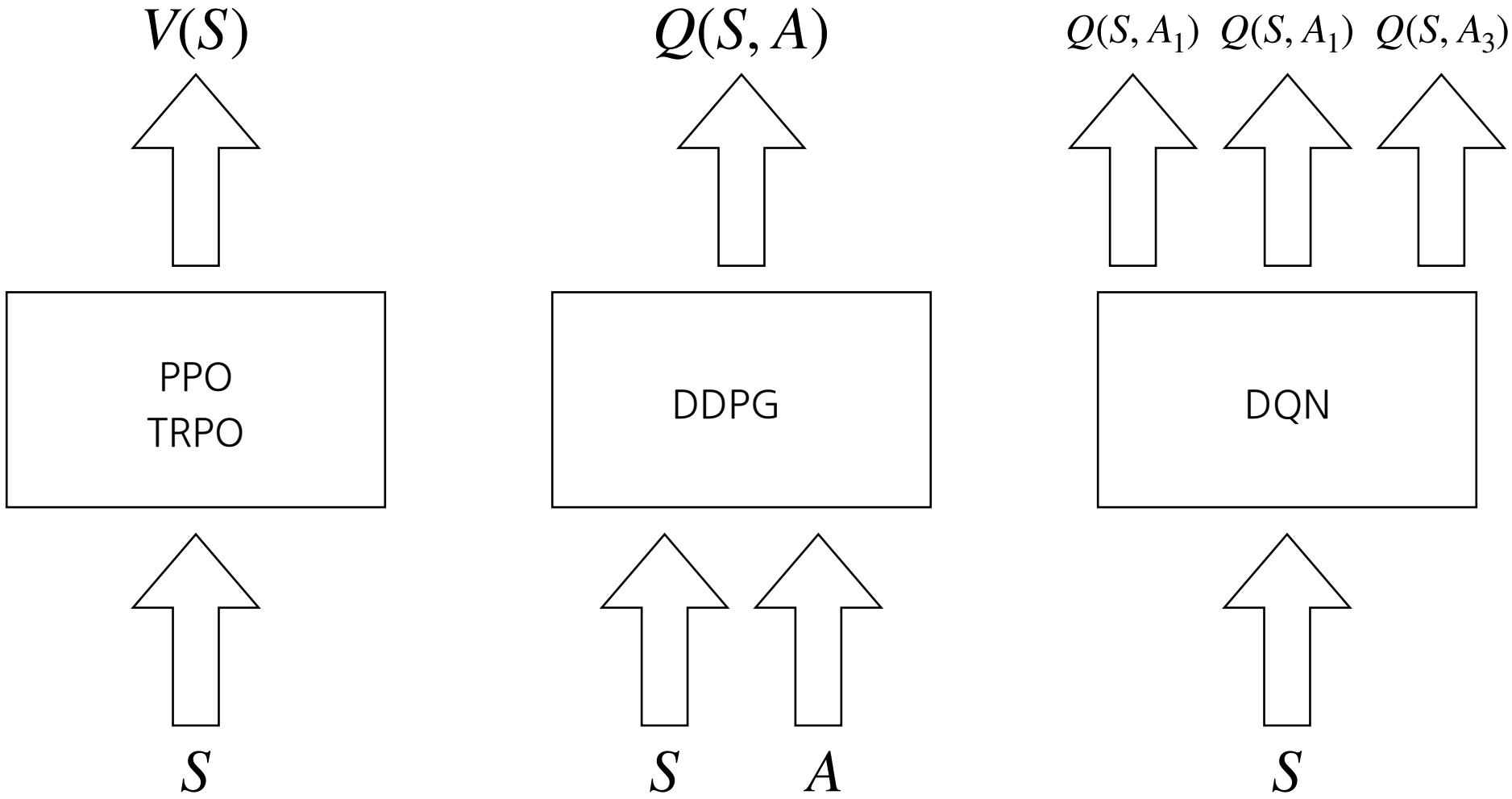
- Recall the original Q-Learning

$$Q_{k+1}(s,a) \leftarrow Q_k(s,a) + \alpha \left(\underbrace{r(s,a,s') + \gamma \max_{a'} Q_k(s',a')}_{\text{Target}} - \underbrace{Q_k(s,a)}_{\text{Prediction}} \right)$$

- From the Q-learning objective, we can derive the following loss function:

$$L(\theta) = \sum_i \left(\underbrace{r_i + \gamma \max_{a'} Q(s_i', a'; \theta)}_{\text{Target}} - \underbrace{Q(s_i, a_i; \theta)}_{\text{Prediction}} \right)^2$$

Deep Q Network (DQN)



Deep Q Network (DQN)

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- (Stable) Update Rule

$$L(\theta) = \sum_i \left(r_i + \gamma \max_{a'} Q(s_i', a'; \theta^-) - Q(s_i, a_i; \theta) \right)^2$$

- Delayed update
  - For numerical stability, slowly update the target network
  - θ^-: previous parameter (for the Q estimation)
  - θ: current parameter to update
- Other tricks
  - Gradient clipping
  - Input normalization

Deep Q Network (DQN)

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- (Stable) Update Rule

$$L(\theta) = \sum_i \left(r_i + \gamma \max_{a'} Q(s_i', a'; \theta^-) - Q(s_i, a_i; \theta) \right)^2$$

- Delayed update
  - For numerical stability,
  - θ^-: previous parameter
  - θ: current parameter to
- Other tricks
  - Gradient clipping
  - Input normalization

```
def update_main_network(self, o_batch, a_batch, r_batch, o1_batch, d_batch):
    o1_q = self.target_network(o1_batch)
    max_o1_q = o1_q.max(1)[0].detach().numpy()
    d_batch = d_batch.astype(int)
    expected_q = r_batch + self.gamma*max_o1_q*(1.0-d_batch)
    expected_q = expected_q.astype(np.float64) # R + gamma*max(Q)
    expected_q = torch.from_numpy(expected_q)
    main_q = self.main_network(o_batch).max(1)[0]
    loss = F.smooth_l1_loss(main_q.float(), expected_q.float())

    self.optimizer.zero_grad()
    loss.backward()
    self.optimizer.step()

    return loss
```

Rule of Thumb

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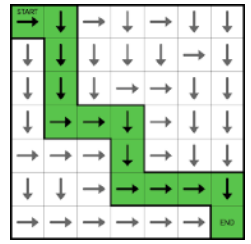
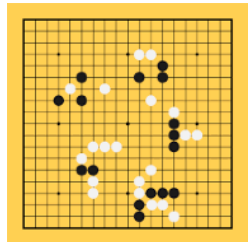
- Discrete (and small) state space & discrete action space
  - E.g., grid world
  - In this case, the state-transition model can easily be defined.
  - Use **value iteration** or **policy iteration**.
- Discrete (and large) state space & discrete action space
  - E.g., Go
  - In this case, the state-transition model is cumbersome to be defined.
  - Use **Q-learning**.
- Continuous state space & discrete action space
  - E.g., Atari games
  - In this case, the state-transition model is impossible to be defined.
  - Use **DQN**.
- Continuous state space & continuous action space
  - E.g., Robotics
  - In this case, the state-transition model is impossible to be defined.
  - Use **PPO** or **SAC**.





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Policy Gradient Theorem

Limitations of Previous Methods

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What happens if we want to model **continuous** action space?

Policy Optimization

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- Policy gradient methods cast reinforcement learning into an optimization problem.

Policy Optimization

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- Policy gradient methods cast reinforcement learning into an optimization problem.

- Find θ that maximizes the return:

$$\eta(\pi_\theta) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_t \,|\, \pi_\theta\right]$$

Policy Optimization

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- Policy gradient methods cast reinforcement learning into an optimization problem.

- Find θ that maximizes the return:

$$\eta(\pi_\theta) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_t \,|\, \pi_\theta\right]$$

- We update the parameters of the **policy** function by computing the **gradient** of the parameters of the objective function:

$$\theta \leftarrow \theta + \alpha \nabla_\theta \eta(\pi_\theta)$$

How to compute the gradients

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$$\nabla_\theta \eta(\pi_\theta) = \nabla_\theta \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_t \,|\, \pi_\theta\right]$$

- Policy Gradient Theorem:

$$\nabla_\theta \eta(\pi_\theta) = \frac{1}{(1-\gamma)} \sum_s \rho_{\pi_\theta} \sum_a \nabla_\theta \pi_\theta(a \,|\, s) Q^{\pi_\theta}(s,a)$$

$$\nabla_\theta \eta(\pi_\theta) \approx \nabla_\theta \log \pi_\theta(a_t \,|\, s_t) Q_{\pi_\theta}(s_t, a_t)$$

- Note that we only require the gradient of $\pi_\theta(\,\cdot\,)$ not $Q^{\pi_\theta}(\,\cdot\,)$!

R. Sutton et al, "Policy Gradient Methods for Reinforcement Learning with Function Approximation", NeurIPS, 2000

State Visitation

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- Stationary distribution of the state given $\pi_\theta(\cdot)$

$$\rho_{\pi_\theta}(s) = (1-\gamma)\mathbb{E}\left[\sum_{t=0}^{\infty}\gamma^t \mathbb{I}_{(St=s)}\right] = (1-\gamma)\sum_{t=0}^{\infty}\gamma^t P_{\pi_\theta}(S_t = s)$$

State Visitation

ROBOT INTELLIGENCE LAB

- Stationary distribution of the state given $\pi_\theta(\cdot)$

$$\rho_{\pi_\theta}(s) = (1-\gamma)\mathbb{E}\left[\sum_{t=0}^{\infty}\gamma^t \mathbb{I}_{(St=s)}\right] = (1-\gamma)\sum_{t=0}^{\infty}\gamma^t P_{\pi_\theta}(S_t = s)$$

- $\rho_{\pi\theta}(s)$ is a probability mass function

$$\sum_s \rho_{\pi_\theta}(s) = (1-\gamma)\mathbb{E}\left[\sum_{t=0}^{\infty}\gamma^t \sum_s \mathbb{I}_{(S_t=s)}\right] = (1-\gamma)\mathbb{E}\left[\sum_{t=0}^{\infty}\gamma^t\right] = (1-\gamma)\frac{1}{1-\gamma} = 1$$

Proof of Policy Gradient (1/10)

ROBOT INTELLIGENCE LAB

- Return $\eta(\pi_\theta)$ of a policy $\pi_\theta(\cdot)$ and its gradient:

$$\eta(\pi_\theta) = \sum_s d(s) V_{\pi_\theta}(s)$$

$$\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \nabla_\theta V_{\pi_\theta}(s)$$

Proof of Policy Gradient (1/10)

ROBOT INTELLIGENCE LAB

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$$\eta(\pi_\theta) = \sum_s d(s) V_{\pi_\theta}(s)$$

$$\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \nabla_\theta V_{\pi_\theta}(s)$$

- Let's select an arbitrary state, say s_1, and compute $\nabla_\theta V_{\pi_\theta}(s_1)$:

$$\nabla_\theta V_{\pi_\theta}(s_1) = \nabla_\theta \sum_a \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a)$$

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \nabla_\theta Q_{\pi_\theta}(s_1, a)$$

Proof of Policy Gradient (2/10)

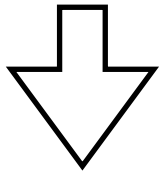
ROBOT INTELLIGENCE LAB

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \nabla_\theta Q_{\pi_\theta}(s_1, a)$$

Proof of Policy Gradient (2/10)

ROBOT INTELLIGENCE LAB

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \nabla_\theta Q_{\pi_\theta}(s_1, a)$$

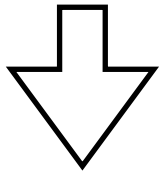


$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \nabla_\theta \left[r(s_1, a) + \gamma \sum_{s'} V_{\pi_\theta}(s') P(s' \mid s_1, a) \right]$$

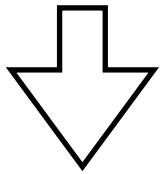
Proof of Policy Gradient (2/10)

ROBOT INTELLIGENCE LAB

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \nabla_\theta Q_{\pi_\theta}(s_1, a)$$



$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \nabla_\theta \left[r(s_1, a) + \gamma \sum_{s'} V_{\pi_\theta}(s') P(s' \mid s_1, a) \right]$$



$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \left[\gamma \sum_{s'} \nabla_\theta V_{\pi_\theta}(s') P(s' \mid s_1, a) \right]$$

Proof of Policy Gradient (3/10)

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \left[\gamma \sum_{s'} \nabla_\theta V_{\pi_\theta}(s') P(s' \mid s_1, a) \right]$$

$$\nabla_\theta V_{\pi_\theta}(s') = \sum_{a'} \nabla_\theta \pi_\theta(a' \mid s') Q_{\pi_\theta}(s', a') + \pi_\theta(a' \mid s') \left[\gamma \sum_{s''} \nabla_\theta V_{\pi_\theta}(s'') P(s'' \mid s', a') \right]$$

- By plugging in:

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \left[\gamma \sum_{s'} \left[\sum_{a'} \nabla_\theta \pi_\theta(a' \mid s') Q_{\pi_\theta}(s', a') + \pi_\theta(a' \mid s') \left[\gamma \sum_{s''} \nabla_\theta V_{\pi_\theta}(s'') P(s'' \mid s', a') \right] \right] P(s' \mid s_1, a) \right]$$

Proof of Policy Gradient (4/10)

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$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_a \left[\nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a) + \pi_\theta(a \mid s_1) \left[\gamma \sum_{s'} \left[\sum_{a'} \nabla_\theta \pi_\theta(a' \mid s') Q_{\pi_\theta}(s', a') + \pi_\theta(a' \mid s') \left[\gamma \sum_{s''} \nabla_\theta V_{\pi_\theta}(s'') P(s'' \mid s', a') \right] \right] P(s' \mid s_1, a) \right] \right]$$

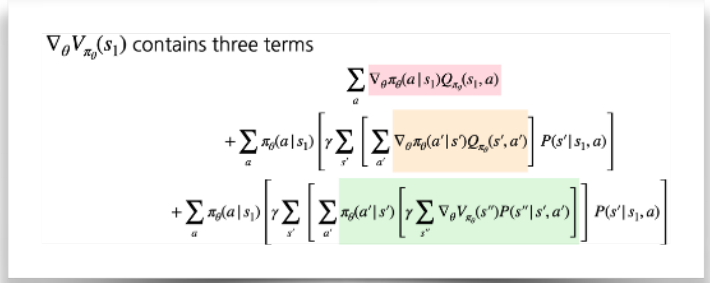
- $\nabla_\theta V_{\pi_\theta}(s_1)$ contains three terms

$$\sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a)$$

$$+ \sum_a \pi_\theta(a \mid s_1) \left[\gamma \sum_{s'} \left[\sum_{a'} \nabla_\theta \pi_\theta(a' \mid s') Q_{\pi_\theta}(s', a') \right] P(s' \mid s_1, a) \right]$$

$$+ \sum_a \pi_\theta(a \mid s_1) \left[\gamma \sum_{s'} \left[\sum_{a'} \pi_\theta(a' \mid s') \left[\gamma \sum_{s''} \nabla_\theta V_{\pi_\theta}(s'') P(s'' \mid s', a') \right] \right] P(s' \mid s_1, a) \right]$$

Proof of Policy Gradient (5/10)



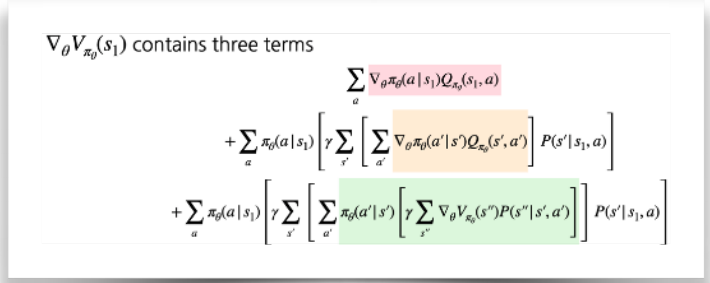
- If we focus on the **first** term

$$\sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s_1, a)$$

$$\sum_s \sum_a \nabla_\theta \pi_\theta(a \mid s_1) Q_{\pi_\theta}(s, a) P_{\pi_\theta}(S_0 = s \mid S_0 = s_1)$$

$P_{\pi_\theta}(S_0 = s' \mid S_0 = s_1)$ is nonzero only when $s' = s_1$

Proof of Policy Gradient (6/10)



- If we focus on the **second** term

$$\sum_a \pi_\theta(a|s_1)\gamma \sum_{s'}\sum_{a'} \nabla_\theta \pi_\theta(a'|s') Q_{\pi_\theta}(s', a') P(s'|s_1, a)$$

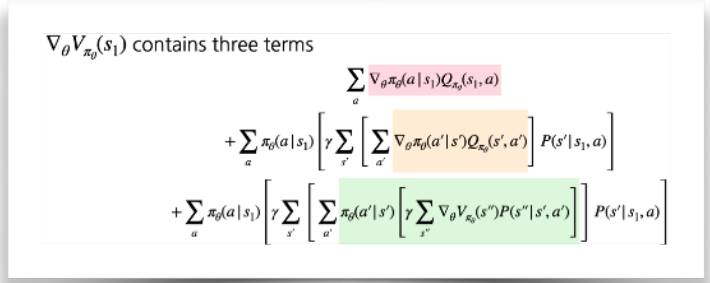
Rearrange

$$\sum_{s'}\sum_{a'} \nabla_\theta \pi_\theta(a'|s') Q_{\pi_\theta}(s', a')\gamma \sum_a P(s'|s_1, a)\pi_\theta(a|s_1)$$

State Transition Probability

$$\sum_{s'}\sum_{a'} \nabla_\theta \pi_\theta(a'|s') Q_{\pi_\theta}(s', a')\gamma P_{\pi_\theta}(S_1 = s'|S_0 = s_1)$$

Proof of Policy Gradient (7/10)



- If we focus on the **third** term

$$\sum_a \pi_\theta(a|s_1)\gamma \sum_{s'}\sum_{a'} \pi_\theta(a'|s')\gamma \sum_{s''} \nabla_\theta V_{\pi_\theta}(s'') P(s''|s', a') P(s'|s_1, a)$$

Rearrange

$$\sum_{s''} \nabla_\theta V_{\pi_\theta}(s'')\gamma^2 \sum_a \sum_{s'} \sum_{a'} \pi_\theta(a|s_1)\pi_\theta(a'|s') P(s''|s', a') P(s'|s_1, a)$$

$$\sum_{s''} \nabla_\theta V_{\pi_\theta}(s'')\gamma^2 P_{\pi_\theta}(S_2 = s''|S_0 = s_1)$$

State Transition Probability

Proof of Policy Gradient (8/10)

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- Substituting the first, second and third terms:

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_s \sum_a \nabla_\theta \pi_\theta(a|s_1) Q_{\pi_\theta}(s,a) P_{\pi_\theta}(S_0 = s | S_0 = s_1) + \sum_{s'} \sum_{a'} \nabla_\theta \pi_\theta(a'|s') Q_{\pi_\theta}(s',a') \gamma P_{\pi_\theta}(S_1 = s' | S_0 = s_1) + \sum_{s''} \nabla_\theta V_{\pi_\theta}(s'') \gamma^2 P_{\pi_\theta}(S_2 = s'' | S_0 = s_1)$$

Mathematical Induction

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_s \sum_a \nabla_\theta \pi_\theta(a|s) Q_{\pi_\theta}(s,a) \left(P_{\pi_\theta}(S_0 = s | S_0 = s_1) + \gamma P_{\pi_\theta}(S_1 = s | S_0 = s_1) + \gamma^2 P_{\pi_\theta}(S_2 = s | S_0 = s_1) + \gamma^3 P_{\pi_\theta}(S_3 = s | S_0 = s_1) + \cdots \right)$$

$$\nabla_\theta V_{\pi_\theta}(s_1) = \sum_s \sum_a \nabla_\theta \pi_\theta(a|s) Q_{\pi_\theta}(s,a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s | S_0 = s_1)$$

Proof of Policy Gradient (9/10)

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- Plugging in $\nabla_\theta V_{\pi_\theta}(s_1) = \sum_s \sum_a \nabla_\theta \pi_\theta(a \mid s) Q_{\pi_\theta}(s,a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s \mid S_0 = s_1)$ to $\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \nabla_\theta V_{\pi_\theta}(s)$

Proof of Policy Gradient (9/10)

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- Plugging in $\nabla_\theta V_{\pi_\theta}(s_1) = \sum_s \sum_a \nabla_\theta \pi_\theta(a|s) Q_{\pi_\theta}(s,a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s | S_0 = s_1)$ to $\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \nabla_\theta V_{\pi_\theta}(s)$

$$\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s' | S_0 = s)$$

Proof of Policy Gradient (9/10)

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- Plugging in $\nabla_\theta V_{\pi_\theta}(s_1) = \sum_s \sum_a \nabla_\theta \pi_\theta(a|s) Q_{\pi_\theta}(s,a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s | S_0 = s_1)$ to $\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \nabla_\theta V_{\pi_\theta}(s)$

$$\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s' | S_0 = s)$$

$$= \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t \sum_s P_{\pi_\theta}(S_t = s' | S_0 = s) d(s)$$

Rearrange

Proof of Policy Gradient (9/10)

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- Plugging in $\nabla_\theta V_{\pi_\theta}(s_1) = \sum_s \sum_a \nabla_\theta \pi_\theta(a|s) Q_{\pi_\theta}(s,a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s | S_0 = s_1)$ to $\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \nabla_\theta V_{\pi_\theta}(s)$

$$\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s' | S_0 = s)$$

$$= \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t \sum_s P_{\pi_\theta}(S_t = s' | S_0 = s) d(s)$$

$$= \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s')$$

By definition

Proof of Policy Gradient (9/10)

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- Plugging in $\nabla_\theta V_{\pi_\theta}(s_1) = \sum_s \sum_a \nabla_\theta \pi_\theta(a|s) Q_{\pi_\theta}(s,a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s | S_0 = s_1)$ to $\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \nabla_\theta V_{\pi_\theta}(s)$

$$\nabla_\theta \eta(\pi_\theta) = \sum_s d(s) \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s' | S_0 = s)$$

$$= \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t \sum_s P_{\pi_\theta}(S_t = s' | S_0 = s) d(s)$$

$$= \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s')$$

$$= \frac{1}{1-\gamma} \sum_{s'} \sum_a \nabla_\theta \pi_\theta(a|s') Q_{\pi_\theta}(s',a) \rho_{\pi_\theta}(s')$$

$$\rho_{\pi_\theta}(s) = (1-\gamma) \sum_{t=0}^{\infty} \gamma^t P_{\pi_\theta}(S_t = s)$$

Proof of Policy Gradient (10/10)

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$$\nabla_\theta \eta(\pi_\theta) = \frac{1}{1-\gamma} \sum_{s'} \sum_{a} \nabla_\theta \pi_\theta(a \mid s') Q_{\pi_\theta}(s', a) \rho_{\pi_\theta}(s')$$

$$\propto \mathbb{E}_{s \sim \rho_{\pi_\theta}} \left[\sum_{a} \nabla_\theta \pi_\theta(a \mid s) Q_{\pi_\theta}(s, a) \right]$$

- Note that the states should be sampled from the distribution induced from the current policy (i.e., $s \sim \rho_{\pi_\theta}(s)$)
- This makes policy gradient methods **on-policy**.

Log Ratio Trick

$$\nabla_\theta \eta(\pi_\theta) = \frac{1}{1-\gamma} \sum_{s'} \sum_{a} \nabla_\theta \pi_\theta(a \mid s') Q_{\pi_\theta}(s', a) \rho_{\pi_\theta}(s')$$

- However, we should summate over all possible states and actions.

Log Ratio Trick

$$\nabla_\theta \eta(\pi_\theta) = \frac{1}{1-\gamma} \sum_{s'} \sum_{a} \nabla_\theta \pi_\theta(a \mid s') Q_{\pi_\theta}(s', a) \rho_{\pi_\theta}(s')$$

- However, we should summate over all possible states and actions.

- We can use the log ratio trick to overcome this issue.

$$\nabla_\theta \mathbb{E}\left[f(x)\right] = \sum_x f(x) \nabla_\theta p_\theta(x) = \sum_x f(x) p_\theta(x) \frac{\nabla_\theta p_\theta(x)}{p_\theta(x)} = \sum_x f(x) p_\theta(x) \nabla_\theta \log p_\theta(x) = \mathbb{E}\left[f(x) \nabla_\theta \log p_\theta(x)\right]$$

- To summarize

$$\nabla_\theta \mathbb{E}\left[f(x)\right] = \mathbb{E}\left[f(x) \nabla_\theta \log p_\theta(x)\right]$$

Log Ratio Trick

$$\nabla_\theta \eta(\pi_\theta) = \frac{1}{1-\gamma} \sum_{s'} \sum_{a} \nabla_\theta \pi_\theta(a \mid s') Q_{\pi_\theta}(s', a) \rho_{\pi_\theta}(s')$$

$$= \frac{1}{1-\gamma} \sum_{s'} \sum_{a} \pi_\theta(a \mid s') \frac{\nabla_\theta \pi_\theta(a \mid s')}{\pi_\theta(a \mid s')} Q_{\pi_\theta}(s', a) \rho_{\pi_\theta}(s')$$

$$= \frac{1}{1-\gamma} \sum_{s'} \sum_{a} \pi_\theta(a \mid s') \nabla_\theta \log \pi_\theta(a \mid s') Q_{\pi_\theta}(s', a) \rho_{\pi_\theta}(s')$$

$$= \frac{1}{1-\gamma} \mathbb{E}_{a \sim \pi_\theta, s \sim \rho_\pi(S)} \left[\nabla_\theta \log \pi_\theta(a \mid s') Q_{\pi_\theta}(s', a) \right]$$

$$\approx \nabla_\theta \log \pi_\theta(a_t \mid s_t) Q_{\pi_\theta}(s_t, a_t)$$

- To summarize

$$\nabla_\theta \eta(\pi_\theta) \approx \nabla_\theta \log \pi_\theta(a_t \mid s_t) Q_{\pi_\theta}(s_t, a_t)$$

Off-Policy vs. On-Policy Methods

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- Off-Policy Learning
  - In **Q-Learning**, do we have any assumptions on the trajectories?

$$\widehat{Q}(s,a) \leftarrow \widehat{Q}(s,a) + \alpha\left(r(s,a,s') + \gamma \max_{a'} \widehat{Q}(s',a') - \widehat{Q}(s,a)\right)$$

- On-Policy Learning
  - How about **policy gradient**?

$$\nabla_\theta \eta(\pi_\theta) \approx \nabla_\theta \log \pi_\theta(a_t \mid s_t) Q_{\pi_\theta}(s_t, a_t)$$

Trust Region Policy Optimization (TRPO)

"Trust Region Policy Optimization", 2015

PG as an optimization problem

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- Policy-based reinforcement learning is an optimization problem.

- The goal is to find θ that maximizes

$$\eta(\pi_\theta) = \mathbb{E}_{s,a}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t)\right]$$

where $s_0 \sim \rho_0(s)$, $a_t \sim \pi(a_t \mid s_t)$, $s_{t+1} \sim P(s_{t+1} \mid s_t, a_t)$.

PG as an optimization problem

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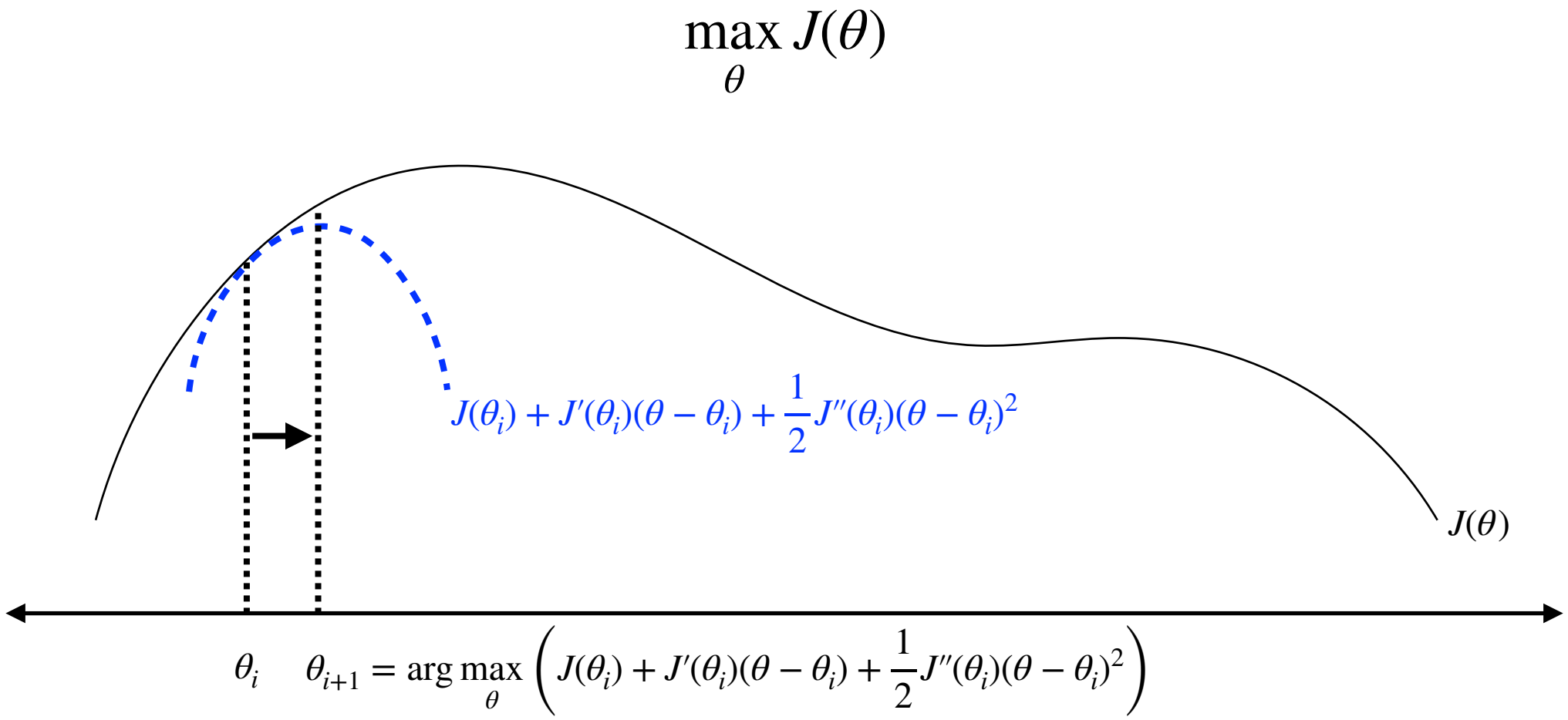
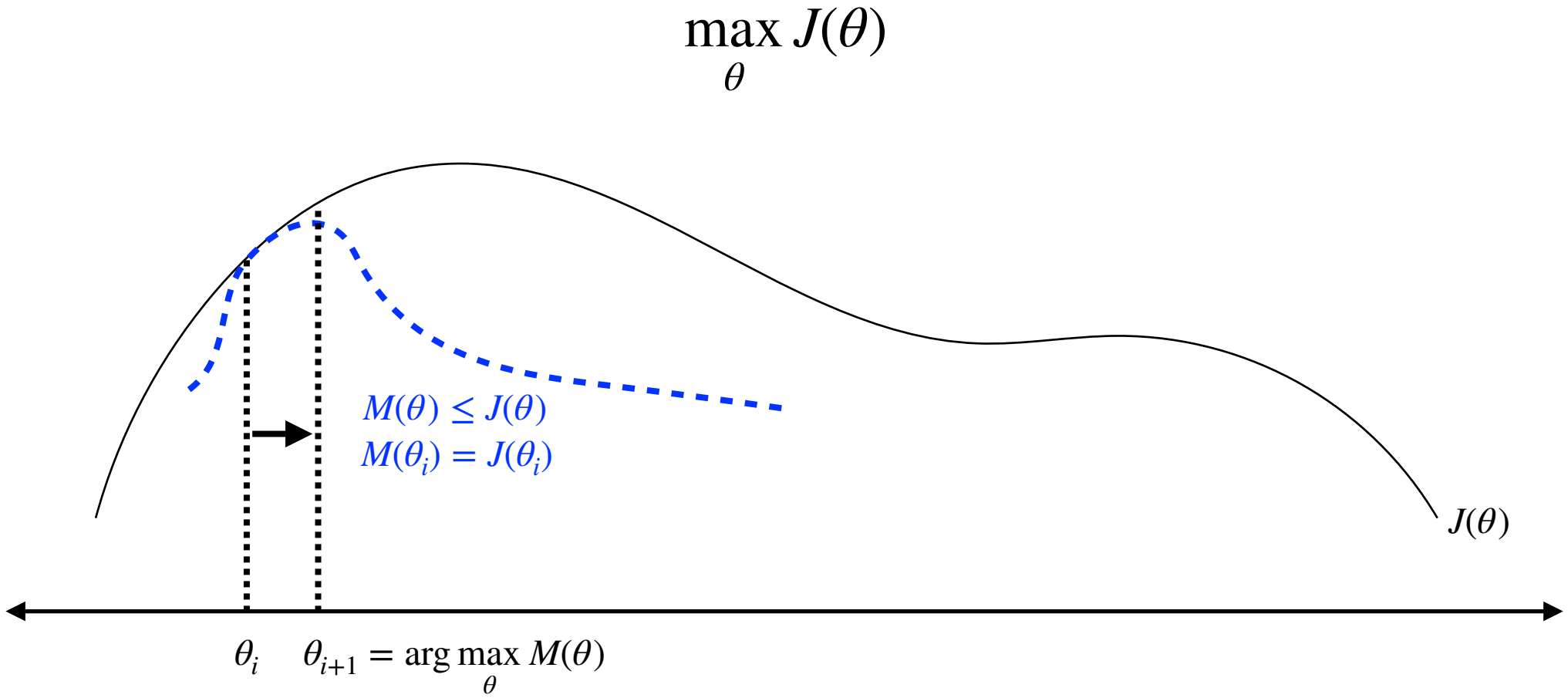
where $s_0 \sim \rho_0(s)$, $a_t \sim \pi(a_t \mid s_t)$, $s_{t+1} \sim P(s_{t+1} \mid s_t, a_t)$.

- We can use optimization techniques:
  - Minorization maximization
  - Conjugate gradient descent

Newton Method



Minorization Maximization



Preliminaries

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- The goal of reinforcement learning is to find π_θ that maximizes the expected return:

$$\eta(\pi_\theta) = \mathbb{E}_{s,a}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t)\right]$$

where $s_0 \sim \rho_0(s)$, $a_t \sim \pi(a_t \mid s_t)$, $s_{t+1} \sim P(s_{t+1} \mid s_t, a_t)$.

- Basic definitions of Markov decision processes

$$Q_\pi(s_t, a_t) = \mathbb{E}_{s,a}\left[\sum_{l=0}^{\infty} \gamma^l r(s_{t+l})\right]$$

$$V_\pi(s_t) = \mathbb{E}_{s,a}\left[\sum_{l=0}^{\infty} \gamma^l r(s_{t+l})\right]$$

$$A_\pi(s, a) = Q_\pi(s, a) - V_\pi(s)$$

Useful Identity

- The improvement of the expected return:

$$\eta(\pi') = \eta(\pi) + \mathbb{E}_{s,a\sim\pi'}\left[\sum_{t=0}^{\infty}\gamma^t A_\pi(s_t, a_t)\right]$$

Improvement of π' over π

- Let $\rho_\pi(s)$ be the (unnormalized) discounted visitation frequencies

$$\rho_\pi(s) = P(s_0 = s) + \gamma P(s_1 = s) + \gamma^2 P(s_2 = s) + \cdots$$

- Then the return improvement can be written as

$$\eta(\pi') = \eta(\pi) + \sum_s \rho_{\pi'}(s) \sum_a \pi'(a \mid s) A_\pi(s, a).$$

Performance Improvement

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- The return improvement

$$\eta(\pi') = \eta(\pi) + \sum_s \rho_{\pi'}(s) \sum_a \pi'(a \mid s) A_\pi(s, a)$$

- Then, the **policy improvement** step of policy iteration will increase the policy performance if the following is guaranteed:

$$\sum_a \pi'(a \mid s) A_\pi(s, a) \geq 0$$

- Hence, $\pi'(s) = \arg\max_a A_\pi(s, a)$ will improve the policy there is at least one state-action pair with a positive value per each state s.
- However, due to the approximation of $A_\pi(s, a)$, it is not always guaranteed.

Performance Improvement

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- The following return improvement is not practical due to $\rho_{\pi'}(s)$:

$$\eta(\pi') = \eta(\pi) + \sum_s \rho_{\pi'}(s) \sum_a \pi'(a \mid s) A_\pi(s, a)$$

- Why?

Performance Improvement

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- The following return improvement is not practical due to $\rho_{\pi'}(s)$:

$$\eta(\pi') = \eta(\pi) + \sum_s \rho_{\pi'}(s) \sum_a \pi'(a \mid s) A_\pi(s, a)$$

- Why?

- The following local approximation, $\rho_{\pi'}(s) \Rightarrow \rho_\pi(s)$, is made:

$$L_\pi(\pi') \approx \eta(\pi')$$

$$L_\pi(\pi') = \eta(\pi) + \sum_s \rho_\pi(s) \sum_a \pi'(a \mid s) A_\pi(s, a)$$

Performance Improvement

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- Local approximation:

$$L_{\pi}(\pi') = \eta(\pi) + \sum_{s} \rho_{\pi}(s) \sum_{a} \pi'(a \mid s) A_{\pi}(s, a)$$

- Hence the following can be used as a learning objective:

$$\mathbb{E}_{s_t \sim P, a_t \sim \pi'} \left[\sum_{t=0}^{\infty} \gamma^t A_{\pi}(s_t, a_t) \right]$$

- If we want to use the state-action pairs collected from the current policy π, we can use importance-sampling:

$$\mathbb{E}_{s_t \sim P, a_t \sim \pi} \left[\sum_{t=0}^{\infty} \gamma^t \frac{\pi'(a_t \mid s_t)}{\pi(a_t \mid s_t)} A_{\pi}(s_t, a_t) \right]$$

$$= \mathbb{E}_{s_t \sim \rho_{\pi}, a_t \sim \pi} \left[\frac{\pi'(a_t \mid s_t)}{\pi(a_t \mid s_t)} A_{\pi}(s_t, a_t) \right]$$

Minorization for RL

$$L_\pi(\pi') = \mathbb{E}_{s\sim\rho_\pi, a\sim\pi}\left[\frac{\pi'(a_t \mid s_t)}{\pi(a_t \mid s_t)} A_\pi(s_t, a_t)\right]$$

- We can define the following **minorization** of $\eta(\pi)$ using KLD

$$M_\pi(\pi') = \eta(\pi) + L_\pi(\pi') - cD_{KL}^{max}(\pi, \pi')$$

where $D_{KL}^{max}(\pi, \pi') = \max_s D_{KL}\left(\pi(\cdot \mid s), \pi'(\cdot \mid s)\right)$

- Then the following properties hold:

$$M_\pi(\pi) = \eta(\pi)$$

$$M_\pi(\pi') \leq \eta(\pi')$$

Minorization for RL

- Now, we optimize

$$L_\pi(\pi') = \mathbb{E}_{s\sim\rho_\pi, a\sim\pi}\left[\frac{\pi'(a\,|\,s)}{\pi(a\,|\,s)}A_\pi(s,a)\right]$$

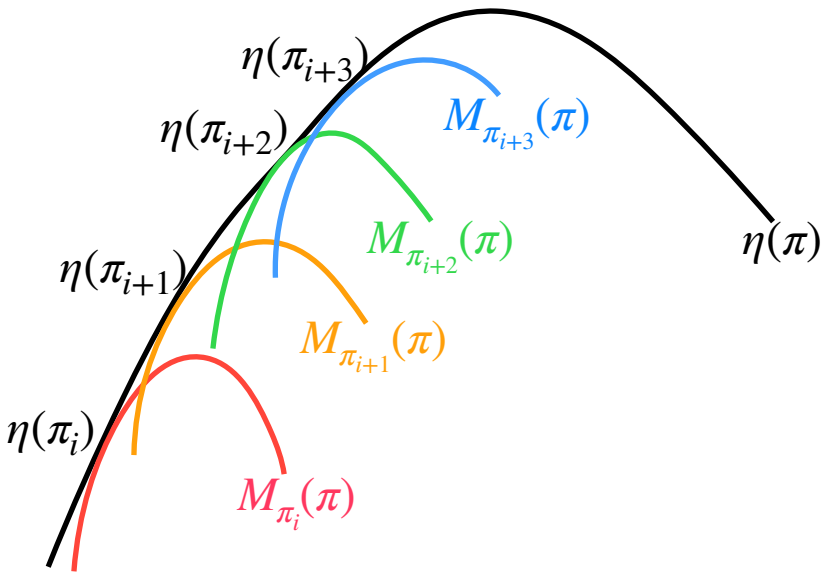
$$M_\pi(\pi') = \eta(\pi) + L_\pi(\pi') - cD_{KL}^{max}(\pi, \pi')$$

$$\max_{\theta_{i+1}} M_{\pi_{\theta_i}}(\pi_{\theta_{i+1}}) = \eta(\pi_{\theta_i}) + L_{\pi_{\theta_i}}(\pi_{\theta_{i+1}}) - cD_{KL}^{max}(\pi_{\theta_i}, \pi_{\theta_{i+1}})$$

- Lagrangian relaxation becomes

$$\max_{\theta_{i+1}} L_{\pi_{\theta_i}}(\pi_{\theta_{i+1}})$$

$$\text{subject to } D_{KL}^{max}(\pi_\theta, \pi_{\theta_{i+1}}) \le \delta$$



Trust Region Policy Optimization

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$$\max_{\theta_{i+1}} L_{\pi_{\theta_i}}(\pi_{\theta_{i+1}}) = \mathbb{E}_{s\sim\rho_{\pi_{\theta_i}}, a\sim\pi_{\theta_i}}\left[\frac{\pi_{\theta_{i+1}}(a\,|\,s)}{\pi_{\theta_i}(a\,|\,s)} A_{\pi_{\theta_i}}(s,a)\right]$$

$$\text{subject to } D_{KL}^{max}(\pi_\theta, \pi_{\theta_{i+1}}) \le \delta$$

- We approximate the KL divergence:

$$D_{KL}^{max}(\pi_\theta, \pi_{\theta_{i+1}}) = \max_s D_{KL}\left(\pi_{\theta_i}(\cdot\,|\,s), \pi_{\theta_{i+1}}(\cdot\,|\,s)\right)$$

$$\downarrow$$

$$D_{KL}^{\rho}(\pi_\theta, \pi_{\theta_{i+1}}) = \mathbb{E}_{s\sim\rho_{\pi_{\theta_i}}}\left[D_{KL}\left(\pi_{\theta_i}(\cdot\,|\,s), \pi_{\theta_{i+1}}(\cdot\,|\,s)\right)\right]$$

Trust Region Policy Optimization

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$$\max_{\theta_{i+1}} L_{\pi_{\theta_i}}(\pi_{\theta_{i+1}}) = \mathbb{E}_{s\sim\rho_{\pi_{\theta_i}}, a\sim\pi_{\theta_i}}\left[\frac{\pi_{\theta_{i+1}}(a\,|\,s)}{\pi_{\theta_i}(a\,|\,s)} A_{\pi_{\theta_i}}(s,a)\right]$$

$$\text{subject to } D_{KL}^{\rho}(\pi_{\theta}, \pi_{\theta_{i+1}}) \le \delta$$

- In summary,
  - TRPO is a minorization maximization framework for RL.
  - Interpretation of the trust region method:
    1. Update policy distribution slowly
    2. Consider the geometry of the distribution space
  - There are two approximations: 1) $\mathbb{E}_{s\sim\rho_{\pi'}} \Rightarrow \mathbb{E}_{s\sim\rho_{\pi}}$ and 2) $D_{KL}^{\max} \Rightarrow D_{KL}^{\rho}$

Trust Region Policy Optimization

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- How do we estimate $A_{\pi_{\theta_i}}$?
- We estimate $Q_{\pi_{\theta_i}}(s,a)$ instead of $A_{\pi_{\theta_i}}(s,a)$:

$$A_{\pi_{\theta_i}}(s,a) = Q_{\pi_{\theta_i}}(s,a) - V_{\pi_{\theta_i}}(s)$$

- We use the Monte Carlo Estimate of Q:

$$Q_{\pi_{\theta_i}}(s_t,a_t) \approx G_t = \sum_{k=1} \gamma^k R_{t+1+k}$$

Trust Region Policy Optimization

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$$\max_{\theta_{i+1}} L_{\pi_{\theta_i}}(\pi_{\theta_{i+1}}) = \mathbb{E}_{s\sim\rho_{\pi_{\theta_i}}, a\sim\pi_{\theta_i}}\left[\frac{\pi_{\theta_{i+1}}(a\,|\,s)}{\pi_{\theta_i}(a\,|\,s)} A_{\pi_{\theta_i}}(s,a)\right]$$

$$\text{subject to } D_{KL}^{\rho}(\pi_{\theta}, \pi_{\theta_{i+1}}) \le \delta$$

⇩ Linear approximation to the loss and quadratic approximation to the constraint

$$\max_{\theta} \nabla_{\theta} L_{\theta_{old}}(\theta)\,|_{\theta=\theta_{old}} \cdot (\theta - \theta_{old})$$

$$\text{subject to } \frac{1}{2}(\theta_{old} - \theta)^T H(\theta_{old})(\theta_{old} - \theta) \le \delta$$

$$\text{where } H(\theta_{old})_{(i,j)} = \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}\mathbb{E}_{s\sim\rho_{\pi}}\left[D_{KL}(\pi(\,\cdot\,|\,s,\theta_{old})\|\pi(\,\cdot\,|\,s,\theta))\right]|_{\theta=\theta_{old}}$$

Trust Region Policy Optimization

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- The final TRPO objective becomes:

$$\max_{\theta} g(\theta_{old})^T(\theta - \theta_{old})$$

$$\text{subject to } \frac{1}{2}(\theta_{old} - \theta)^T H(\theta_{old})(\theta_{old} - \theta) \leq \delta$$

$$\text{where } g(\theta_{old}) = \nabla_\theta L_{\theta_{old}}(\theta)\,|_{\theta=\theta_{old}} \text{ and}$$

$$H(\theta_{old})_{(i,j)} = \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}\mathbb{E}_{s\sim\rho_\pi}\left[D_{KL}(\pi(\cdot\,|\,s,\theta_{old})\|\pi(\cdot\,|\,s,\theta))\right]|_{\theta=\theta_{old}}$$

- The update rule of the above problem is

$$\theta_{new} = \theta_{old} + \frac{1}{\lambda}H(\theta_{old})^{-1}g(\theta_{old})$$

Trust Region Policy Optimization

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- The update rule of the above problem is

$$\theta_{new} = \theta_{old} + \frac{1}{\lambda} H(\theta_{old})^{-1} g(\theta_{old})$$

- However, the hessian matrix $H(\theta_{old}) \in \mathbb{R}^{n \times n}$ where n is the number of parameters and the computational complexity of the inverse becomes $O(n^3)$.

- Instead of computing H^{-1}, we solve the linear equation $Hx = g$ using a conjugate gradient method.

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Proximal Policy Optimization (PPO)

"Proximal Policy Optimization Algorithms", 2017

Preliminaries

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- Policy gradient method
  - The gradient estimate of the policy w.r.t. the return is

$$\hat{g} = \mathbb{E}_{s_t, a_t}\left[\nabla_\theta \log \pi_\theta(a_t \mid s_t)\hat{A}_t\right]$$

where $\hat{A}_t$ is an estimator of the advantage function.

- Trust region method
  - The TRPO objective is

$$\max_\theta \mathbb{E}\left[\frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)}\hat{A}_t\right]$$

$$\text{s.t. } D_{KL}^{\rho}\left[\pi_{old}(\cdot \mid s_t), \pi_\theta(\cdot \mid s_t)\right] \leq \delta$$

Clipped Surrogate Objective

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- The objective of the TRPO is:

$$L(\theta) = \mathbb{E}\left[\frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)}\hat{A}_t\right] = \mathbb{E}\left[r_t(\theta)\hat{A}_t\right]$$

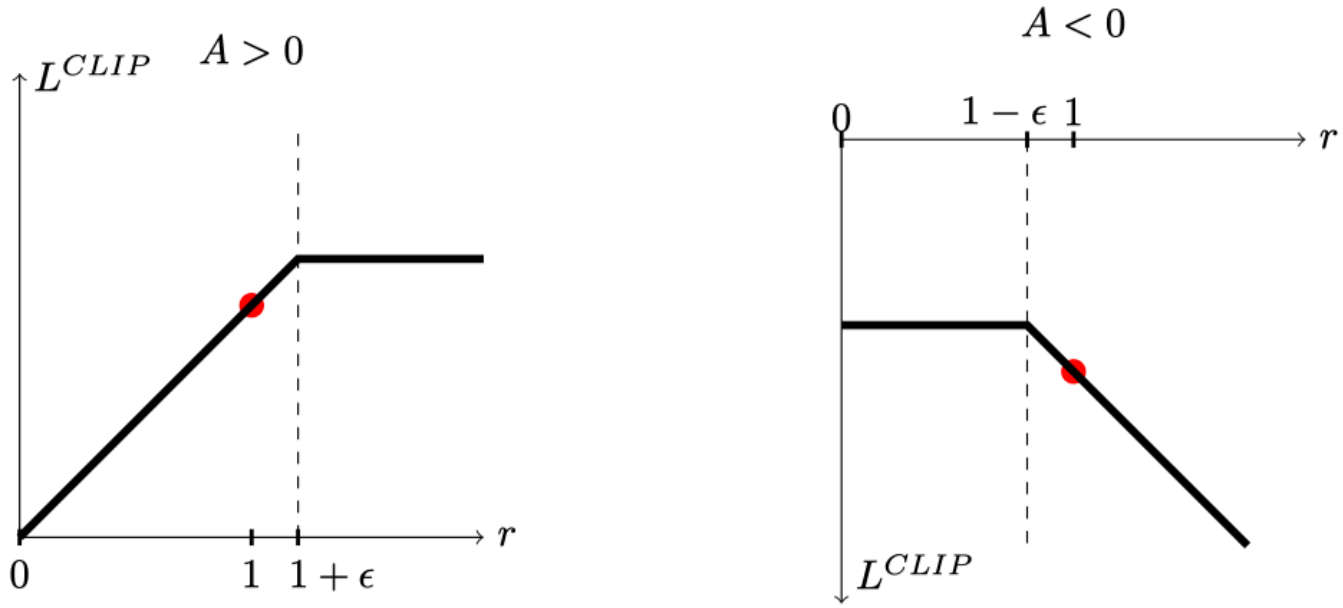
- The main objective of clipped surrogate is:

$$L^{\text{CLIP}}(\theta) = \mathbb{E}\left[\min\left(r_t(\theta)\hat{A}_t,\ \text{clip}(r_t(\theta), 1-\epsilon, 1+\epsilon)\hat{A}_t\right)\right]$$

  - The first term $r_t(\theta)\hat{A}_t$ is identical to the TRPO objective.
  - The second term clips the probability ratio $r_t(\theta)$, which removes the incentive for moving $r_t(\theta)$ outside of the interval $[1-\epsilon, 1+\epsilon]$.

Clipped Surrogate Objective

$$L(\theta) = \mathbb{E}\left[\frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)}\hat{A}_t\right] = \mathbb{E}\left[r_t(\theta)\hat{A}_t\right] \text{ and } L^{\text{CLIP}}(\theta) = \mathbb{E}\left[\min\left(r_t(\theta)\hat{A}_t,\ \text{clip}(r_t(\theta), 1-\epsilon, 1+\epsilon)\hat{A}_t\right)\right]$$



- When $A_t > 0$, we have to worry about increasing $L(\theta)$ by increasing $r_t(\theta)$, and vice versa. Hence, we clip the objective when $r_t(\theta)$ exceeds $1+\epsilon$ when $A_t > 0$.

Proximal Policy Optimization (Adaptive KL Penalty)

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- The TRPO objective is:

$$\max_{\theta} \mathbb{E}\left[\frac{\pi_{\theta}(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)}\hat{A}_t\right] \text{ s.t. } D_{KL}^{\rho}\left[\pi_{old}(\cdot \mid s_t), \pi_{\theta}(\cdot \mid s_t)\right] \leq \delta$$

- The unconstrained objective of TRPO is:

$$L(\theta) = \max_{\theta} \mathbb{E}\left[\frac{\pi_{\theta}(a_t \mid s_t)}{\pi_{\theta_{old}}(a_t \mid s_t)}\hat{A}_t - \beta D_{KL}^{\rho}\left[\pi_{\theta_{old}}(\cdot \mid s_t), \pi_{\theta}(\cdot \mid s_t)\right]\right]$$

- The adaptive KL penalty method for PPO is to adaptively change β by checking $d = \mathbb{E}_t\left[D_{KL}[\pi_{\theta_{old}}, \pi_{\theta}]\right]$:
  - If $d < d_{targ}/1.5$, $\beta \leftarrow \beta/2$
  - If $d > d_{targ} \times 1.5$, $\beta \leftarrow \beta \times 2$

PPO Implementation

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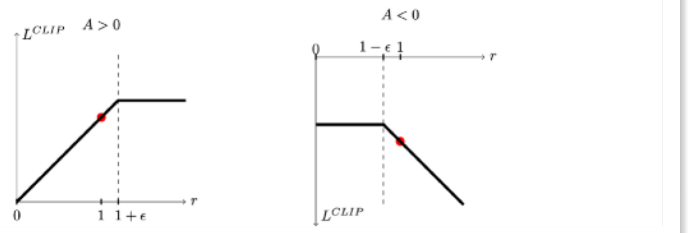
```python
# Update policy
for _ in range(self.train_pi_iters):

    # Cipped Surrogate Obejctive
    _, logp_a, _, _ = self.actor_critic.policy(obs, act)
    ratio = torch.exp(logp_a - logp_a_old)  # pi(a|s) / pi_old(a|s)
    min_adv = torch.where(adv > 0, (1 + self.clip_ratio) * adv, (1 - self.clip_ratio) * adv)
    pi_loss = -torch.mean(torch.minimum(ratio * adv, min_adv))

    # Gradient clipping
    self.train_pi.zero_grad()
    pi_loss.backward()
    self.train_pi.step()

    # KL divergence upper-bound (trust region)
    kl = torch.mean(logp_a_old - logp_a)
    if kl > 1.5 * self.target_kl:
        break
```

$$L(\theta) = \mathbb{E}\left[\frac{\pi_\theta(a_t|s_t)}{\pi_{\theta_{old}}(a_t|s_t)}\hat{A}_t\right] = \mathbb{E}\left[r_t(\theta)\hat{A}_t\right] \text{ and } L^{\text{CLIP}}(\theta) = \mathbb{E}\left[\min\left(r_t(\theta)\hat{A}_t, \text{clip}(r_t(\theta), 1-\epsilon, 1+\epsilon)\hat{A}_t\right)\right]$$



- The adaptive KL penalty method for PPO is to adaptively change β by checking $d = \mathbb{E}_t\left[D_{KL}[\pi_{\theta_{old}}, \pi_\theta]\right]$:
  - If $d < d_{targ}/1.5$, $\beta \leftarrow \beta/2$
  - If $d > d_{targ} \times 1.5$, $\beta \leftarrow \beta \times 2$

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Generalized Advantage Estimation (GAE)

"HIGH-DIMENSIONAL CONTINUOUS CONTROL USING GENERALIZED ADVANTAGE ESTIMATION," 2018

Advantage Function Estimation

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- Let V be an approximate value function. Then define

$$\delta_t^V = r_t + \gamma V(s_{t+1}) - V(s_t)$$

i.e., the TD residual of V with discount γ.

- Note that δ_t^V can be considered as an estimate of the advantage of the action a_t, i.e., $\hat{A}_t$. Now, let's define the following series:
  - $\hat{A}_t^{(1)} = \delta_t^V$
  - $\hat{A}_t^{(2)} = \delta_t^V + \gamma\delta_{t+1}^V$
  - $\hat{A}_t^{(3)} = \delta_t^V + \gamma\delta_{t+1}^V + \gamma^2\delta_{t+2}^V$
- Finally, we define the λ-exponentially-weighted average of $\hat{A}_t$:

$$\hat{A}_t^{\mathrm{GAE}(\gamma,\lambda)} = (1-\lambda)\left(\hat{A}_t^{(1)} + \lambda\hat{A}_t^{(2)} + \lambda^2\hat{A}_t^{(3)} + \cdots\right) = \sum_{t=0}^{\infty}(\gamma\lambda)^l\delta_{t+1}^V$$

Advantage Function Estimation

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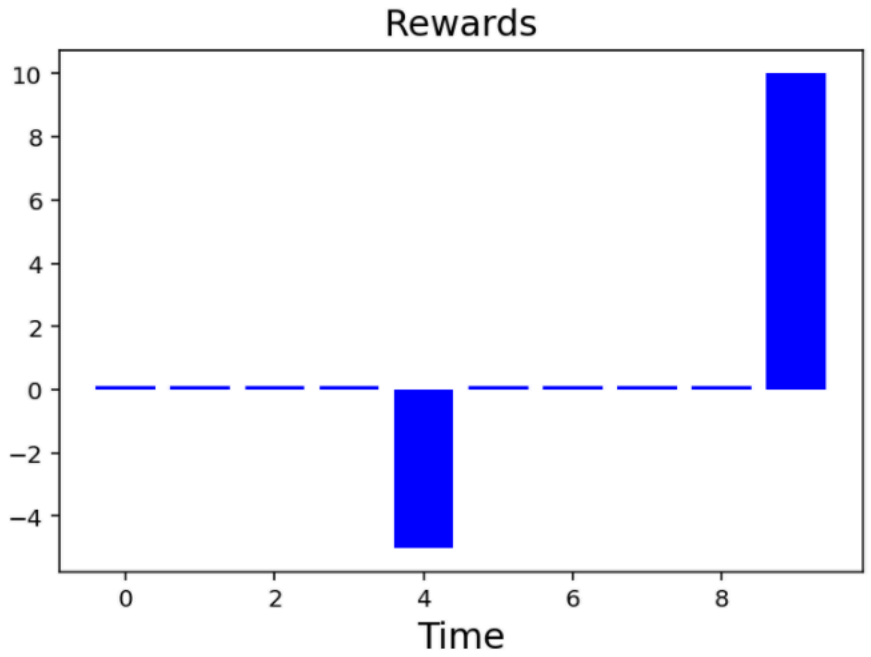
Rewards

```
1  # Plot rewards
2  plt.bar(times,rewards,color='b')
3  plt.title("Rewards",fontsize=15)
4  plt.xlabel("Time",fontsize=15)
5  plt.show()
```



Advantage Function Estimation

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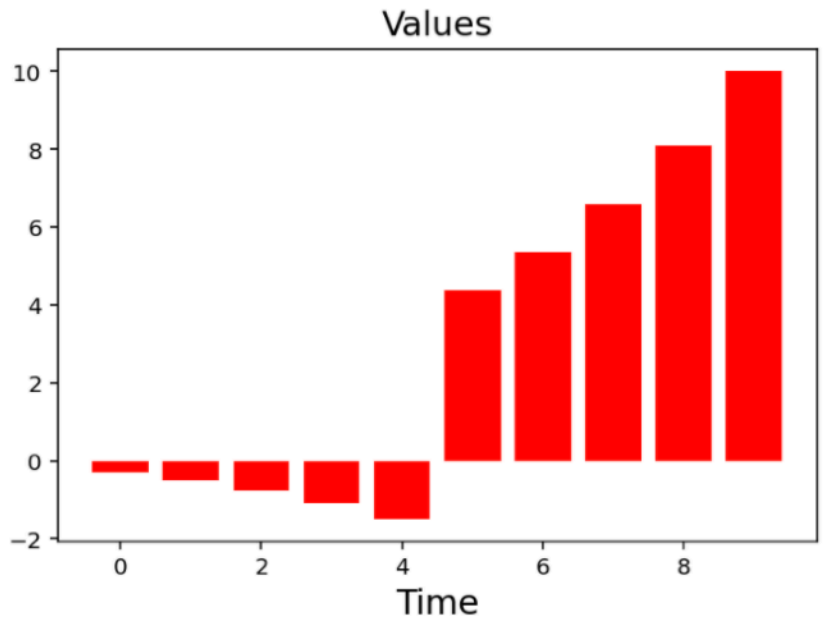
Values

$V(s_t) = \sum_{l=0}^{\infty} \gamma^l r(s_{t+l})$ and $V(s_t) = r(s_t) + \gamma V(s_{t+1})$

```
values = np.zeros(L); values[L-1] = rewards[L-1]
for t in reversed(range(L-1)):
    values[t] = rewards[t] + gamma*values[t+1]
# Plot values
plt.bar(times,values,color='r')
plt.title("Values",fontsize=15)
plt.xlabel("Time",fontsize=15)
plt.show()
```



Advantage Function Estimation

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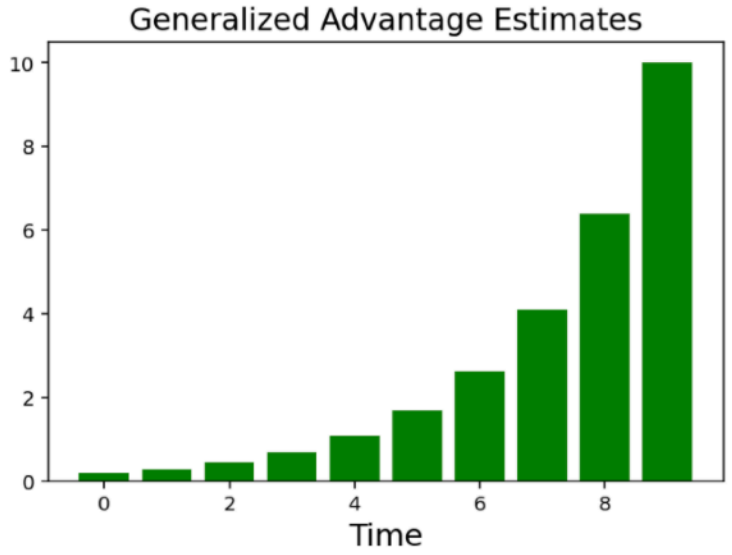
Generalized Advantage Estimates

$$\delta_t^V = r_t + \gamma V(s_{t+1}) - V(s_t) \text{ - (9)}$$

$$\hat{A}_t^{\text{GAE}(\gamma,\lambda)} = \sum_{l=0}^{\infty} (\gamma\lambda)^l \delta_{t+l}^V \text{ - (16)}$$

```
1  gaes = np.zeros(L); gaes[L-1] = rewards[L-1]
2  for t in reversed(range(L-1)):
3      delta = rewards[t] + (gamma*values[t+1]) - values[t]
4      gaes[t] = delta + (gamma*lamda*gaes[t+1])
5  # Plot GAEs
6  plt.bar(times,gaes,color='g')
7  plt.title("Generalized Advantage Estimates",fontsize=15)
8  plt.xlabel("Time",fontsize=15)
9  plt.show()
```



Advantage Function Estimation

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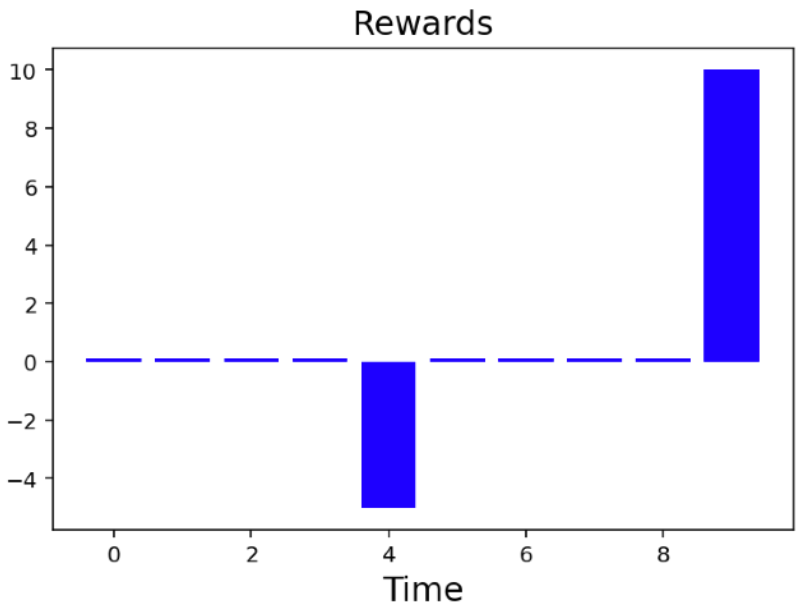
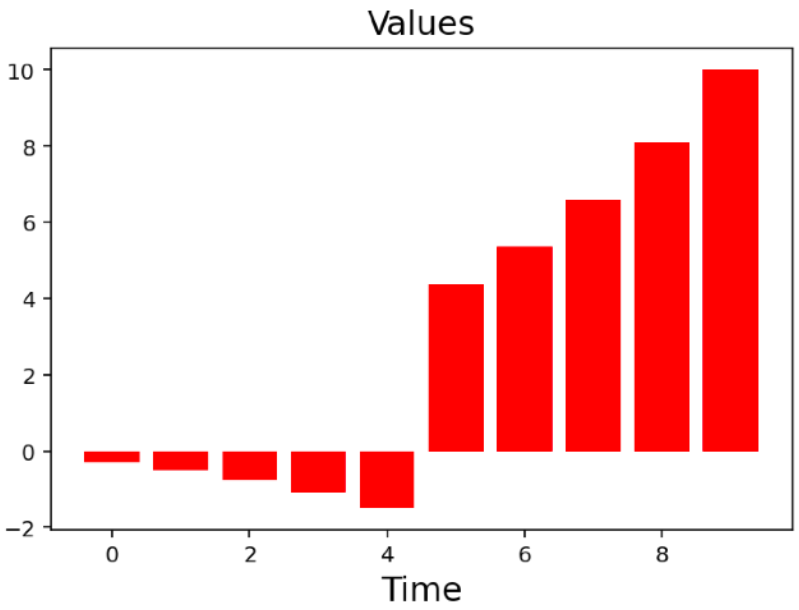
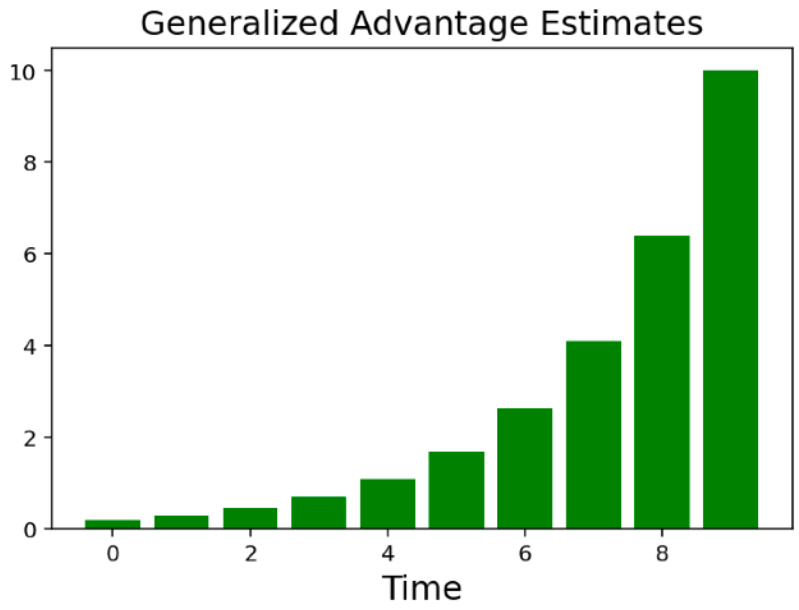
$$\hat{A}_t^{\mathrm{GAE}(\gamma,\lambda)} = (1-\lambda)\left(\hat{A}_t^{(1)} + \lambda\hat{A}_t^{(2)} + \lambda^2\hat{A}_t^{(3)} + \cdots\right) = \sum_{t=0}^{\infty}(\gamma\lambda)^l\delta_{t+1}^V$$

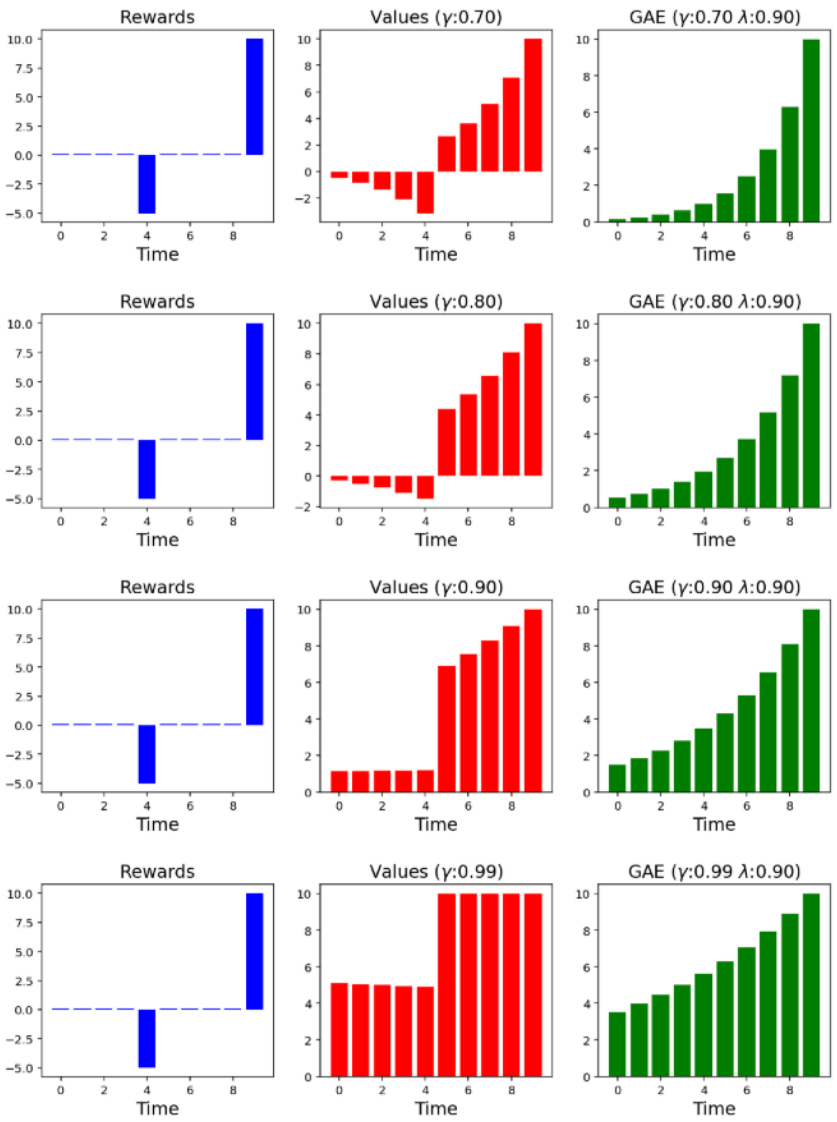


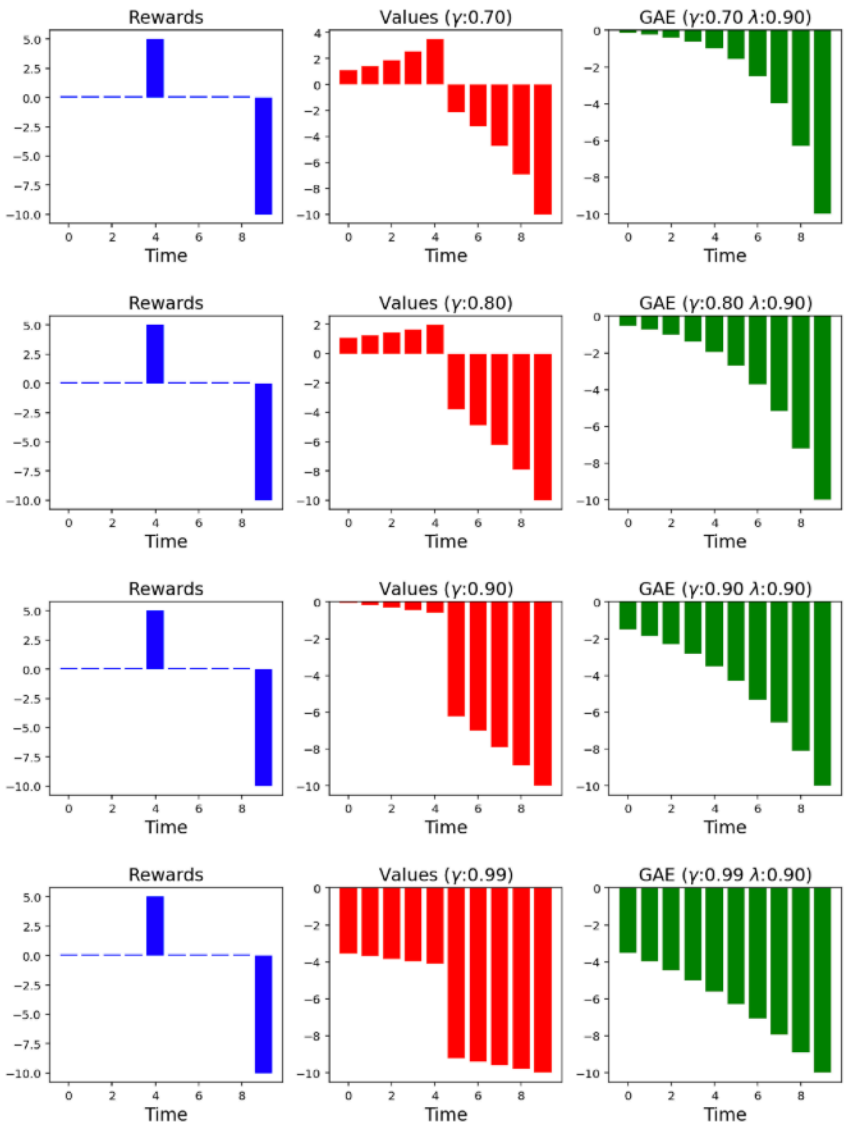




https://gist.github.com/sjchoi86/38c7a378cfa482a1cde5630e5dde937e

Advantage Function Estimation





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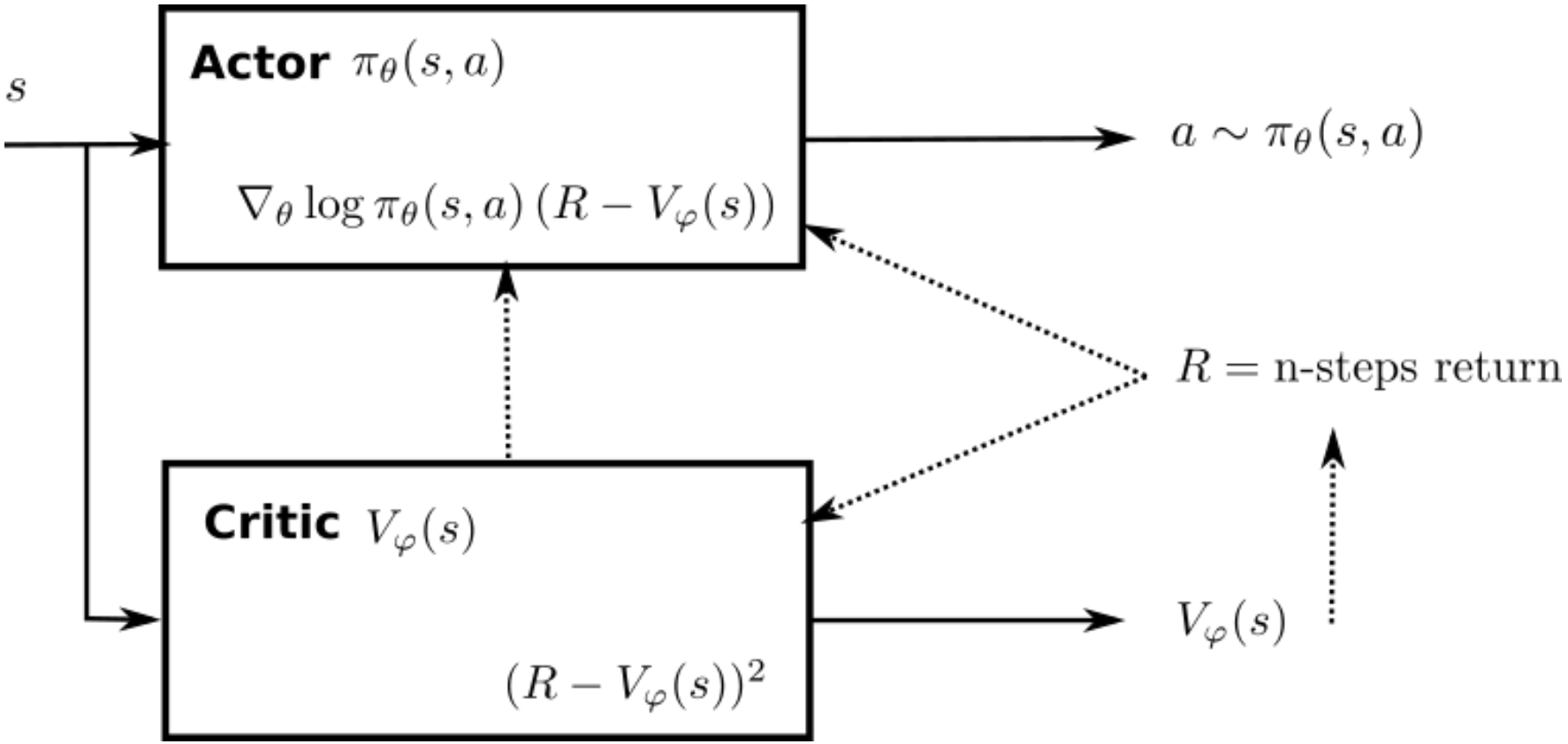
Soft Actor-Critic (SAC)

"Soft Actor-Critic: Off-Policy Maximum Entropy Deep Reinforcement Learning with a Stochastic Actor," 2018

Actor-Critic Method



Actor-Critic Method

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1. Acquire a batch of transitions (s, a, r, s') using the current policy π_θ.
2. For each state encountered, compute the discounted sum of the next n rewards and use the critic to estimate the value of the state:

$$R_t = \sum_{k=0}^{n-1} \gamma^k r_{t+k+1} V_\phi(s_{t+n+1})$$

3. Update the actor using:

$$\nabla_\theta J(\theta) = \sum_i \nabla_\theta \log \pi_\theta(s_t, a_t)(R_t - V_\phi(s_t))$$

4. Update the critic using:

$$L(\phi) = \sum_t (R_t - V_\phi(s_t))^2$$

5. Repeat

Maximum Entropy RL

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- Standard RL objective

$$J(\pi) = \sum_{t=0}^{T} \mathbb{E}_{(s_t,a_t)\sim\rho_\pi} \left[r(s_t, a_t) \right]$$

- Maximum Entropy RL objective

$$J(\pi) = \sum_{t=0}^{T} \mathbb{E}_{(s_t,a_t)\sim\rho_\pi} \left[r(s_t, a_t) + \alpha \mathscr{H} \left(\pi(\cdot \,|\, s_t) \right) \right]$$

Maximum Entropy RL

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- Maximum Entropy RL objective

$$J(\pi) = \sum_{t=0}^{T} \mathbb{E}_{(s_t,a_t)\sim\rho_\pi}\left[r(s_t,a_t) + \alpha\mathscr{H}\left(\pi(\cdot\,|\,s_t)\right)\right]$$

- **Policy evaluation step**
  - The Bellman backup operator for Max-Ent RL is:

$$T^\pi Q(s_t,a_t) \triangleq r(s_t,a_t) + \gamma\mathbb{E}_{s_{t+1}}\left[V(s_{t+1})\right]$$

$$\text{where}\quad V(s_{t+1}) = \mathbb{E}_{a_t\sim\pi}\left[Q(s_t,a_t) - \log\pi(a_t\,|\,s_t)\right].$$

- **Policy improvement step**

$$\pi_{new} = \arg\min_{\pi'} D_{KL}\left(\pi'(\cdot\,|\,s_t)\|\frac{\exp(Q^{\pi_{old}}(s_t,\cdot))}{Z^{\pi_{old}}(s_t)}\right)$$

Soft Actor-Critic

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- SAC learns three functions: $V_\psi(s)$, $Q_\theta(s,a)$, and $\pi_\phi(a\,|\,s)$.
- For learning $V_\psi(s)$:

$$J_V(\psi) = \mathbb{E}_{s_t\sim\mathcal{D}}\left[\frac{1}{2}\left(V_\psi(s_t) - \mathbb{E}_{a_t\sim\pi_\phi}\left[Q_\theta(s_t,a_t) - \log\pi_\phi(a_t\,|\,s_t)\right]\right)^2\right]$$

where actions are being sampled from the current policy $\pi_\phi(a\,|\,s)$ not from the replay.

- For learning $Q_\theta(s,a)$:

$$J_Q(\theta) = \mathbb{E}_{(s_t,a_t)\sim\mathcal{D}}\left[\frac{1}{2}\left(Q_\theta(s_t,a_t) - \hat{Q}(s_t,a_t)\right)^2\right] \text{ where } \hat{Q}(s_t,a_t) = r(s_t,a_t) + \gamma\mathbb{E}_{s_{t+1}}\left[V_\psi(s_{t+1})\right]$$

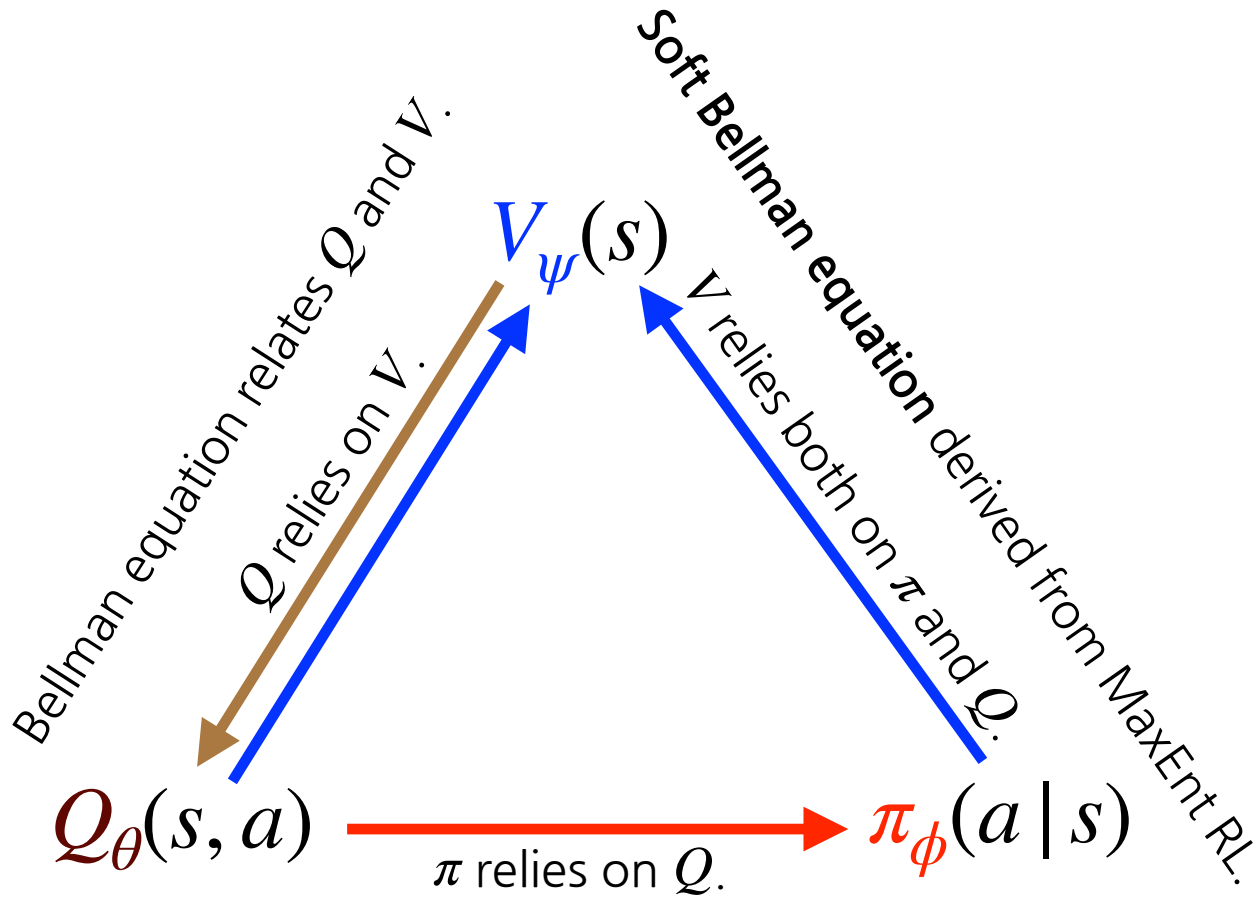
- For learning $\pi_\phi(a\,|\,s)$:

$$J_\pi(\phi) = \mathbb{E}_{s_t\sim\mathcal{D}}\left[D_{KL}\left(\pi_\phi(\,\cdot\,|\,s_t)\|\frac{\exp\left(Q_\theta(s_t,\,\cdot\,)\right)}{Z_\theta(s_t)}\right)\right]$$

If we re-parameterize the stochastic policy $a_t = f_\phi(\epsilon_t;s_t)$ where ϵ_t is sampled from some distribution,

$$J_\pi(\phi) = \mathbb{E}_{s_t\sim D,\epsilon_t\sim\mathcal{N}}\left[\log\pi_\phi\left(f_\phi(\epsilon_t;s_t)\,|\,s_t\right) - Q_\theta\left(s_t, f_\phi(\epsilon_t;s_t)\right)\right]$$

Soft Actor-Critic



Policy improvement with KL control

SAC Implementation

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```
def update_policy(self, data):
    o = data['obs1']
    _, pi, logp_pi = self.policy(o)
    q1_pi = self.q1(o, pi)
    q2_pi = self.q2(o, pi)
    min_q_pi = torch.minimum(q1_pi, q2_pi)
    pi_loss = torch.mean(self.alpha_pi*logp_pi - min_q_pi)

    self.train_pi.zero_grad()
    pi_loss.backward()
    self.train_pi.step()

    return pi_loss, logp_pi, min_q_pi

def update_Q(self, target, data):
    o,a,r,o2,d = data['obs1'],data['acts'],data['rews'],data['obs2'],data['done']
    _, pi_next, logp_pi_next = self.policy(o2)
    q1_targ = target.q1(o2, pi_next)
    q2_targ = target.q2(o2, pi_next)
    min_q_targ = torch.minimum(q1_targ, q2_targ)
    q_backup =  r + self.gamma*(1-d)*(min_q_targ - self.alpha_q*logp_pi_next
    q1 = self.q1(o, a)
    q2 = self.q2(o, a)
    q1_loss = 0.5*F.mse_loss(q1,q_backup.detach())
    q2_loss = 0.5*F.mse_loss(q2,q_backup.detach())
    value_loss = q1_loss + q2_loss

    self.train_q1.zero_grad()
    q1_loss.backward()
    self.train_q1.step()

    self.train_q2.zero_grad()
    q2_loss.backward()
    self.train_q2.step()

    return value_loss, q1, q2, logp_pi_next, q_backup, q1_targ, q2_targ
```

- SAC learns three functions: $V_\psi(s)$, $Q_\theta(s,a)$, and $\pi_\phi(a\,|\,s)$.
- For learning $V_\psi(s)$:

$$J_V(\psi) = \mathbb{E}_{s_t\sim\mathcal{D}}\left[\frac{1}{2}\left(V_\psi(s_t) - \mathbb{E}_{a_t\sim\pi_\phi}\left[Q_\theta(s_t,a_t) - \log\pi_\phi(a_t\,|\,s_t)\right]\right)^2\right]$$

where actions are being sampled from the current policy $\pi_\phi(a\,|\,s)$ not from the replay.

- For learning $Q_\theta(s,a)$:

$$J_Q(\theta) = \mathbb{E}_{(s_t,a_t)\sim\mathcal{D}}\left[\frac{1}{2}\left(Q_\theta(s_t,a_t) - \hat{Q}(s_t,a_t)\right)^2\right] \text{ where } \hat{Q}(s_t,a_t) = r(s_t,a_t) + \gamma\mathbb{E}_{s_{t+1}}\left[V_\psi(s_{t+1})\right]$$

- For learning $\pi_\phi(a\,|\,s)$:

$$J_\pi(\phi) = \mathbb{E}_{s_t\sim\mathcal{D}}\left[D_{KL}\left(\pi_\phi(\cdot\,|\,s_t)\|\frac{\exp\left(Q_\theta(s_t,\cdot)\right)}{Z_\theta(s_t)}\right)\right]$$

If we re-parameterize the stochastic policy $a_t = f_\phi(\epsilon_t;s_t)$ where ϵ_t is sampled from some distribution,

$$J_\pi(\phi) = \mathbb{E}_{s_t\sim D,\epsilon_t\sim\mathcal{N}}\left[\log\pi_\phi\left(f_\phi(\epsilon_t;s_t)\,|\,s_t\right) - Q_\theta\left(s_t, f_\phi(\epsilon_t;s_t)\right)\right]$$

Summary

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- Policy Gradient Theorem
  - Optimize the policy directly via $\nabla_\theta \eta(\pi_\theta) \approx \nabla_\theta \log \pi_\theta(a_t \mid s_t) Q_{\pi_\theta}(s_t, a_t)$
- Trust Region Policy Optimization (**TRPO**)
  - From policy improvements using minorization maximization to a trust-region method.
- Proximal Policy Optimization (**PPO**)
  - Approximate TRPO with policy ratio clipping and adaptive KL weights.
- Generalized Advantage Estimation (**GAE**)
  - More robust than the value estimate, similar to TD(λ).
- Soft Actor-Critic (**SAC**)
  - Entropy-regularized RL with an actor-critic method.

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