Reinforcement Learning

Lecture 2. Model-based Methods

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Markov Process

Introduction

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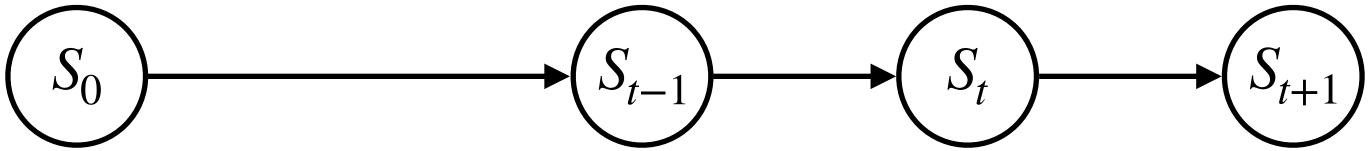
When can we formulate our problem as reinforcement learning?

Markov Process

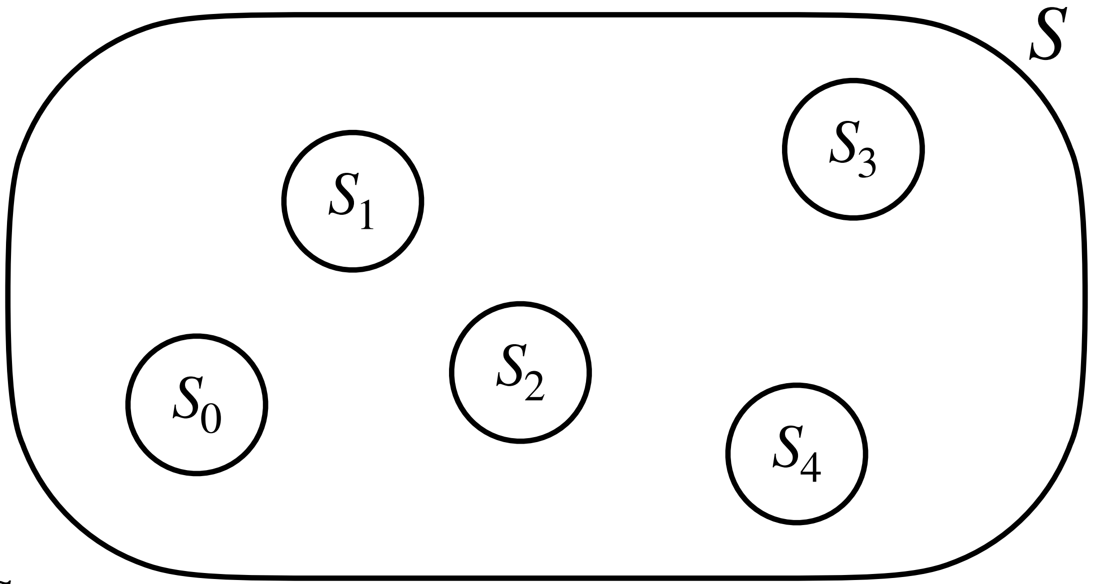
- A **Markov chain** is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event (aka **Markov** property).

- Random variable: S_t (state)



Markov Process

- **State space**
  - All possible state values



- Random variable: S_t
- Outcome: $S_t = s_t$

Markov Property

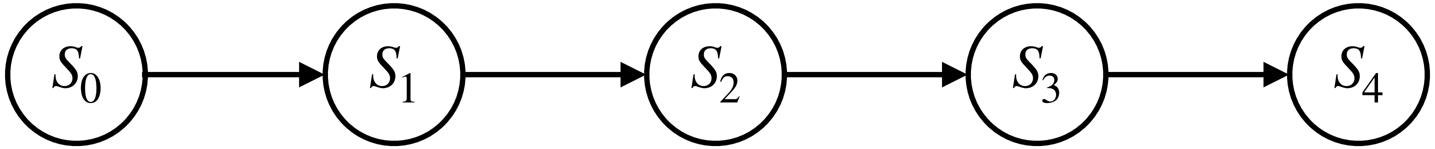
Andrey Markov (1856-1922)

- "Markov" (or Markov property) means that given the present state, the future and the past are independent.

$$P(\underbrace{S_{t+1} = s'}_{\text{future}} \mid \underbrace{S_t = s_t}_{\text{present}}, \underbrace{S_{t-1} = s_{t-1}, \ldots, S_0 = s_0}_{\text{past}}) = P(\underbrace{S_{t+1} = s'}_{\text{future}} \mid \underbrace{S_t = s_t}_{\text{present}})$$

Markov Process

- Given a state sequence $\{s_0, s_1, s_2, s_3, s_4\}$, how **plausible** is this?



$$P(S_0 = s_0, S_1 = s_1, S_2 = s_2, S_3 = s_3, S_4 = s_4)$$

$$= p(s_0, s_1, s_2, s_3, s_4)$$

Chain Rule

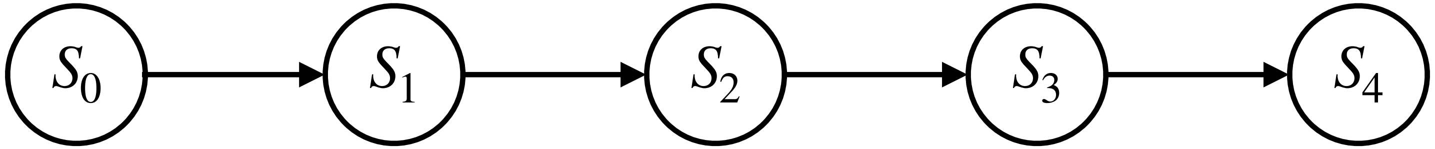
$$= p(s_4 \mid s_3, s_2, s_1, s_0)p(s_3 \mid s_2, s_1, s_0)p(s_2 \mid s_1, s_0)p(s_1 \mid s_0)p(s_0)$$

Markov Property

$$= p(s_4 \mid s_3)p(s_3 \mid s_2)p(s_2 \mid s_1)p(s_1 \mid s_0)p(s_0)$$

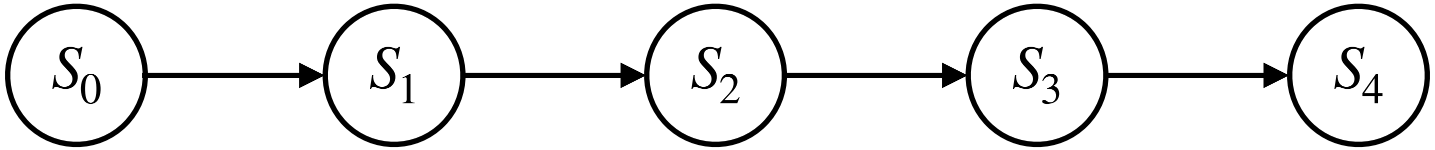
Markov Process

- Given a state sequence $\{s_0, s_1, s_2, s_3, s_4\}$, how **plausible** is this?



$$
\begin{aligned}
P(S_0 = s_0, S_1 = s_1, S_2 = s_2, S_3 = s_3, S_4 = s_4)& \\
&= p(s_0, s_1, s_2, s_3, s_4) \\
&= p(s_4 \mid s_3, s_2, s_1, s_0) p(s_3 \mid s_2, s_1, s_0) p(s_2 \mid s_1, s_0) p(s_1 \mid s_0) p(s_0) \\
&= p(s_4 \mid s_3) p(s_3 \mid s_2) p(s_2 \mid s_1) \underbrace{p(s_1 \mid s_0)}_{\text{Transition Probability}} \underbrace{p(s_0)}_{\text{Initial Probability}}
\end{aligned}
$$

Markov Process

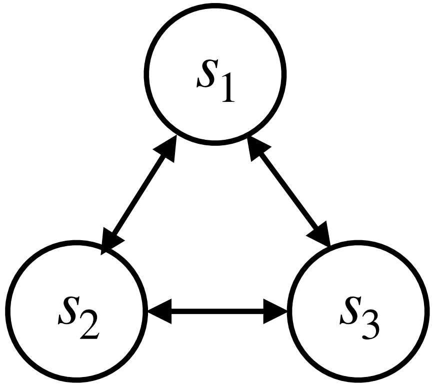


- A **Markov process** is specified by the **initial probability** and the **transition probabilities**.
  - Initial probability
    - $P(S_0 = s)$
  - Transition probability
    - $P(S_{t+1} = s' | S_t = s)$

Markov Process

- Initial distribution vector
  - $d_i = P(S_0 = s_i)$

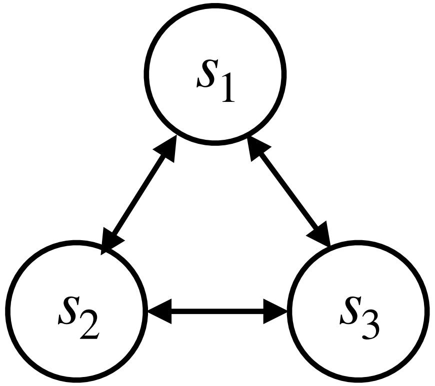
$$
\mathbf{d} = \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \begin{bmatrix} P(S_0 = s_1) \\ P(S_0 = s_2) \\ p(S_0 = s_3) \end{bmatrix}
$$



Markov Process

- Transition probability matrix
  - $P_{(i,j)} = P(S_{t+1} = s_j | S_t = s_i)$

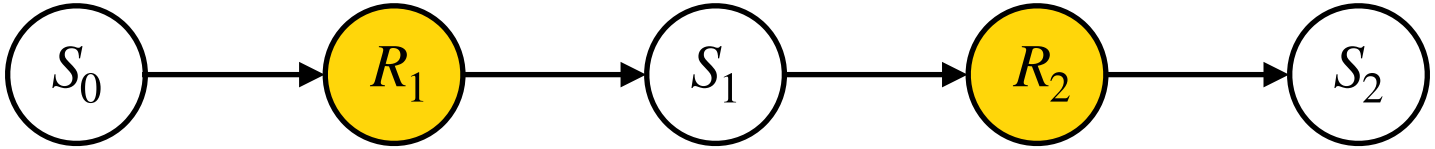
$$P = \begin{array}{c|ccc|} & S_1 & S_2 & S_3 \\ \hline S_1 & p(s_1|s_1) & p(s_2|s_1) & p(s_3|s_1) \\ S_2 & p(s_1|s_2) & p(s_2|s_2) & p(s_3|s_2) \\ S_3 & p(s_1|s_3) & p(s_2|s_3) & p(s_3|s_3) \\ \hline \end{array}$$



Markov Reward Process

Markov Reward Process

- Now, we obtain rewards as we move to states.



- We have two random variables.
  - State: S_t
  - Reward: R_t
- Since the reward is a random variable, we take expectation to compute the reward function.
  - Reward function: $r(s) = \mathbb{E}[R_{t+1} \mid S_t = s]$

Return

- **Return** G_t is the (discounted) sum of future rewards, and hence, also a random variable.

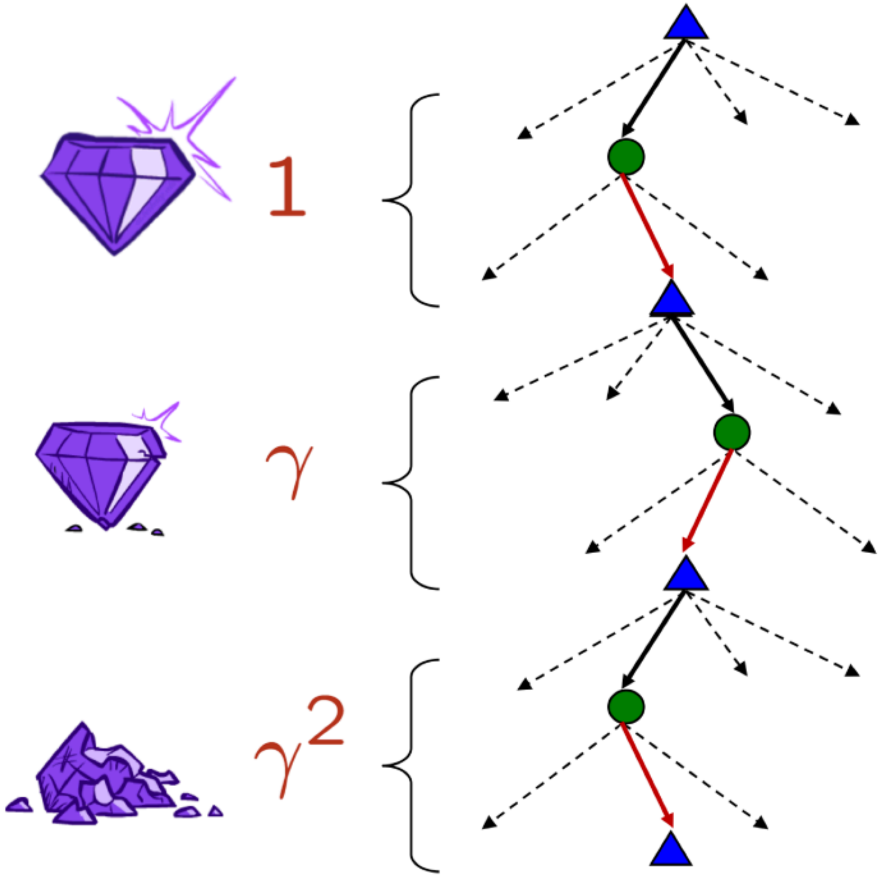
$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+1+k}$$

- Discount factor γ:

Discount Factor

- How to **discount**?
  - Each time we descend a level, we multiply the d

- Why **discount**?
  - Sooner rewards may have higher utility that late
  - It also helps the algorithms to converge.



(State) Value Function

- The **state-value function** $V(s)$ is a function of a state and is the expected return starting from the state s.

$$
\begin{aligned}
V(s) &= \mathbb{E}[G_t \,|\, S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+1} \cdots \,|\, S_t = s]
\end{aligned}
$$

(State) Value Function

- The (state) **value function** is the expected return (discounted sum of rewards) starting from state s.

$$
\begin{aligned}
\mathbb{E}[G_t \mid S_t = s] &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} \cdots \mid S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma (R_{t+2} + \gamma R_{t+3} + \cdots) \mid S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma G_{t+1} \mid S_t = s]
\end{aligned}
$$

Recursive equation

(State) Value Function

$$
\begin{aligned}
V(s) &= \mathbb{E}[G_t \,|\, S_t = s] = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots \,|\, S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma G_{t+1} \,|\, S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma \mathbb{E}[G_{t+1} \,|\, S_{t+1}] \,|\, S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma V(S_{t+1}) \,|\, S_t = s] \\
&= \mathbb{E}[R_{t+1} \,|\, S_t = s] + \gamma \mathbb{E}[V(S_{t+1}) \,|\, S_t = s] \\
&= r(s) + \gamma \sum_{s'} V(s') P(s' \,|\, s)
\end{aligned}
$$

Bellman Equation for Markov Reward Processes

$$V(s) = r(s) + \gamma \sum_{s'} V(s')P(s' \mid s)$$

Bellman Equation for Markov Reward Processes

$$\underbrace{V(s)}_{\text{Current Value}} = \underbrace{r(s)}_{\text{Current Reward}} + \underbrace{\gamma}_{\text{Discounted}} \sum_{s'} \underbrace{V(s')}_{\text{Next Value}} \underbrace{P(s' \mid s)}_{\text{Transition Probability}}$$

Bellman Equation for Markov Reward Processes

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$$V(s) = r(s) + \gamma \sum_{s'} V(s')P(s' \mid s)$$

For multiple states s_1, s_2, s_3

$$V(s_1) = r(s_1) + \gamma \left(V(s_1)P(s_1 \mid s_1) + V(s_2)P(s_2 \mid s_1) + V(s_3)P(s_3 \mid s_1) \right)$$

$$V(s_2) = r(s_2) + \gamma \left(V(s_1)P(s_1 \mid s_2) + V(s_2)P(s_2 \mid s_2) + V(s_3)P(s_3 \mid s_2) \right)$$

$$V(s_3) = r(s_3) + \gamma \left(V(s_1)P(s_1 \mid s_3) + V(s_2)P(s_2 \mid s_3) + V(s_3)P(s_3 \mid s_3) \right)$$

Bellman Equation for Markov Reward Processes

$$V(s) = r(s) + \gamma \sum_{s'} V(s')P(s' | s)$$

(matrix form)

$$V = R + \gamma PV$$

$$V = R + \gamma \; P(s' | s) \; V$$

Bellman Equation for Markov Reward Processes

- The **Bellman equation** is a linear equation!

$$V = R + \gamma PV$$

Solve the equation w.r.t V

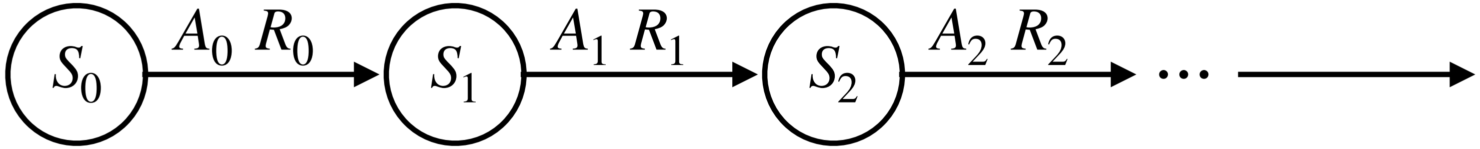
$$V = (I - \gamma P)^{-1} R$$

Markov Reward Process

- Summary
  - S is a finite set of states.
  - d is an initial state distribution.
  - P is a state transition probability matrix.
    - $P_{ss'} = P(S_{t+1} = s' | S_t = s)$
  - r is a reward function.
    - $r(s) = \mathbb{E}\left[R_{t+1} | S_t = s\right]$
  - Note that there is no notion of **action** here.

Markov Decision Process

Markov Decision Process



- Now, we have three random variables:
  - State: S_t
  - Reward: R_t
  - **Action: A_t**

Markov Decision Process

- Formally, an MDP is a tuple (S, A, P, R, d):
  - A set of states $s \in S$.
  - A set of actions $a \in A$
  - A state transition function (or matrix)
    - $P(s' | s, a) = P(S_{t=1} = s | S_t = s, A_t = a)$
    - $P_{sas'} = P(s' | s, a)$
  - A reward function
    - $r(s) = \mathbb{E}[R_{t+1} | S_t = s]$
    - It depends on both state and action.
  - An initial state distribution d

Policy

- A **policy** is a **distribution over actions** given state.

$$\pi(a \mid s) = P(A_t = a \mid S_t = s)$$

  - In an MDP, a policy is a function of the current state s.
  - A policy is stationary (time-independent).
  - A deterministic policy can also be represented by a distribution.

Policy

- Given a fixed policy, a **Markov decision process** becomes and a **Markov reward process**.
- We denote $P^{\pi}_{ss'}$ a policy-conditioned state transition probability matrix:

$$
\begin{aligned}
P^{\pi}_{ss'} &= \sum_{a} P(S_{t+1} = s' | S_t = s, A_t = a)\pi(A_t = a | S_t = s) \\
&= \sum_{a} \pi(a|s) P_{sas'}
\end{aligned}
$$

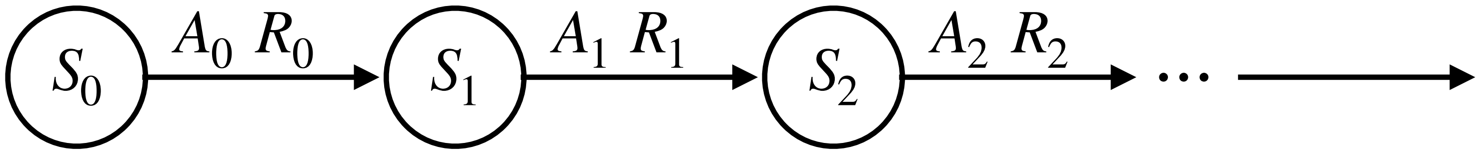
- Similarly, we can compute a policy-conditioned reward function

$$
r^{\pi}(s) = \mathbb{E}[R_{t+1} | S_t = s] = \sum_{a} r(s,a)\pi(a|s)
$$

Return

- Given a policy π , return G_t is determined.

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+1+k}$$



State Value and State Action Value of Policy

- Given a fixed policy π
  - State value function, $V_\pi(s)$, is the expected return starting from state s, and then following policy

$$V_\pi(s) = \mathbb{E}[G_t | S_t = s]$$

  - State-action value function, $Q_\pi(s, a)$, is the expected return starting from state s and action a, and then following policy

$$Q_\pi(s, a) = \mathbb{E}[G_t | S_t = s, A_t = a]$$

Bellman Equation (1)

- Given a fixed policy π

$$V_\pi(s) = \sum_a \pi(a \mid s) Q_\pi(s, a)$$

Relation between a **state value function**, $V_\pi(s) = \mathbb{E}[G_t \mid S_t = s]$, and a **state-action value function**. $Q_\pi(s, a) = \mathbb{E}[G_t \mid S_t = s, A_t = a]$.

Bellman Equation (2)

$$
\begin{aligned}
V(s) &= \mathbb{E}[G_t \mid S_t = s] = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots \mid S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma \mathbb{E}[G_{t+1} \mid S_{t+1}] \mid S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma V(S_{t+1}) \mid S_t = s] \\
&= \mathbb{E}[R_{t+1} \mid S_t = s] + \gamma \mathbb{E}[V(S_{t+1}) \mid S_t = s] \\
&= r(s) + \gamma \sum_{s'} V(s') P(s' \mid s)
\end{aligned}
$$

$$V_\pi(s) = \mathbb{E}[R_{t+1} + \gamma V(S_{t+1}) \mid S_t = s]$$

Bellman equation of a Markov reward process

$$= r(s) + \sum_{s'} V_\pi(s') P_\pi(s' \mid s)$$

Definition of reward and transition probability

$$= \sum_a r(s,a)\pi(a \mid s) + \sum_{s'} V_\pi(s') \sum_a \pi(a \mid s) P(s' \mid s, a)$$

Simple rearrange

$$= \sum_a \left[r(s,a) + \sum_{s'} V_\pi(s') P(s' \mid s, a) \right] \pi(a \mid s)$$

Bellman Equation (3)

- Similarly, for $Q_\pi(s,a)$,

$$
\begin{aligned}
Q_\pi(s,a) &= \mathbb{E}[R_{t+1} + \gamma G_{t+1} \,|\, S_t = s, A_t = a] \\
&= \mathbb{E}[R_{t+1} \,|\, S_t = s, A_t = a] + \gamma \mathbb{E}[G_{t+1} \,|\, S_t = s, A_t = a] \\
&= r(s,a) + \gamma \mathbb{E}[V(S_{t+1}) \,|\, S_t = s, A_t = a] \\
&= r(s,a) + \gamma \sum_{s'} V(s') P(s' \,|\, s, a)
\end{aligned}
$$

Bellman Equations of an MDP

- Summary

$$V_\pi(s) = \sum_a \pi(a|s) Q_\pi(s,a)$$

$$V_\pi(s) = \sum_a \left[r(s,a) + \sum_{s'} V_\pi(s') P(s'|s,a) \right] \pi(a|s)$$

$$Q_\pi(s,a) = r(s,a) + \gamma \sum_{s'} V(s') P(s'|s,a)$$

Bellman (Expectation) Equation of Q_π

- State-action value function $Q_\pi(s, a) = \mathbb{E}[G_t | S_t = s, A_t = a, \pi]$
  - The expected return starting from state s, taking action a, and then follows the policy π

$$Q_\pi(s, a) = \sum_{s'} \left[r(s, a, s') + \gamma \sum_{a'} Q_\pi(s', a')\pi(a' | s') \right] P(s' | s, a)$$

Optimality

Optimality

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What does it mean by **solving** an MDP?

Optimal Value and Policy

- Optimal state value

$$V^*(s) = \max_{\pi} V_\pi(s)$$

- Optimal state-action value

$$Q^*(s, a) = \max_{\pi} Q_\pi(s, a)$$

- Optimal policy

$$\pi^*(a \mid s) = 1 \text{ for } a = \arg\max_{a'} Q^*(s, a')$$

Bellman Optimality Equation

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s' \,|\, s,a)$$

$$V^*(s) = \max_a \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s' \,|\, s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \max_{a'} Q^*(s',a') \right] P(s' \,|\, s,a)$$

Summary

- Bellman **Optimality** Equation

- Bellman Equation

$$V_\pi(s) = \sum_a \pi(a|s) Q^{(}s, a)$$

$$Q_\pi(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V_\pi(s') \right] P(s'|s,a)$$

$$V_\pi(s) = \sum_a \pi(a|s) \sum_{s'} \left[r(s,a,s') + \gamma V_\pi(s') \right] P(s'|s,a)$$

$$Q_\pi(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \sum_{a'} Q_\pi(s',a')\pi(a'|s') \right] P(s'|s,a)$$

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s'|s,a)$$

$$V^*(s) = \max_a \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s'|s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \max_{a'} Q^*(s',a') \right] P(s'|s,a)$$

$$\pi^*(a|s) = \arg\max_{a'} Q^*(s,a')$$

Value Iteration

Value Iteration

Given an MDP (S, A, P, R, d), how can we **solve** an MDP?

Value Iteration

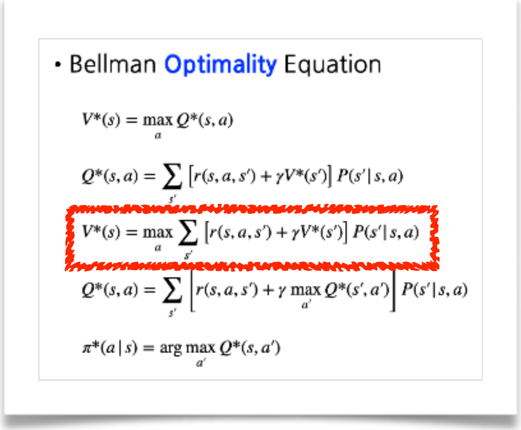
- Value iteration utilizes the **principle of optimality**.

$$V^*(s) = \max_{\pi} V_{\pi}(S)$$

- Then, the solution $V^*(s)$ can be found by one-step lookahead

$$V^*(s) = \max_{a} \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s' \mid s, a)$$

- The main idea is to apply these updates iteratively, repeat until convergence.
  - It is guaranteed to converge to the unique optimal value.
- However, there is no explicit policy.



- Bellman Optimality Equation

$$V^*(s) = \max_{a} Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s' \mid s, a)$$

$$\boxed{V^*(s) = \max_{a} \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s' \mid s, a)}$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \max_{a'} Q^*(s',a') \right] P(s' \mid s, a)$$

$$\pi^*(a \mid s) = \arg\max_{a'} Q^*(s,a')$$

Value Iteration

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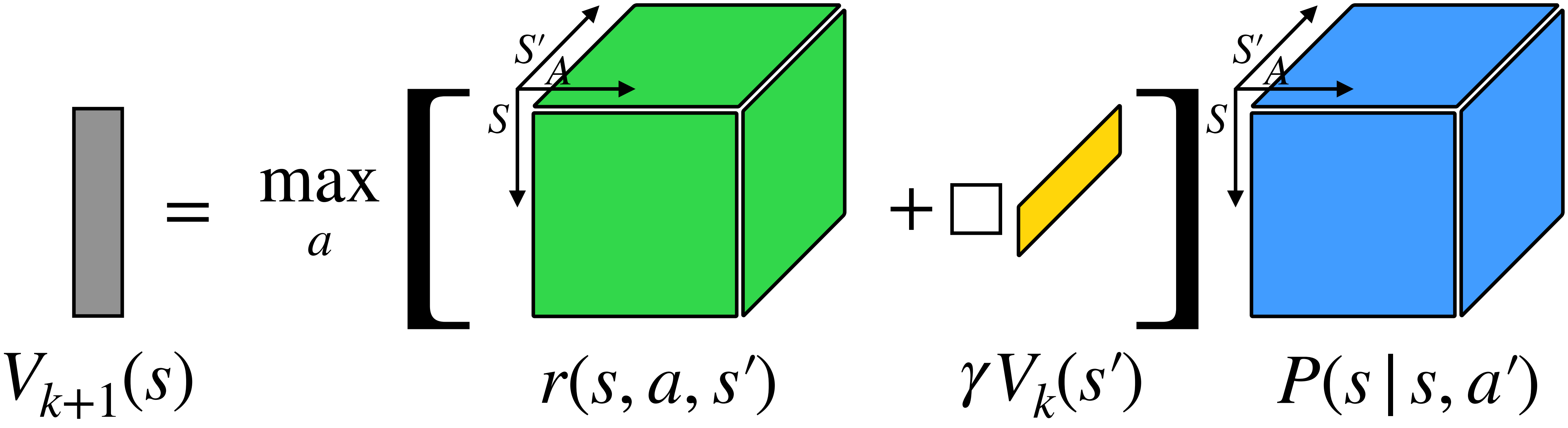
- Start from a random initial V_0
- Initialize V_{k+1} with $-\infty$
- Loop
  - For all states $s \in S$:
    - For all actions $a \in A$:
      - For all next states $s' \in S$:
        - If $V_{k+1}(s) \leq [r(s,a,s') + \gamma V_k(s')]P(s' \mid s,a)$
          - $V_{k+1}(s) = [r(s,a,s') + \gamma V_k(s')]P(s' \mid s,a)$
  - If $\| V_{k+1} - V_k \|_\infty = \max_s | V_{k+1}(s) - V_k(s) | \leq \epsilon$
    - stop

Bellman Optimality Equation

$$\boxed{V_{k+1}(s) = \max_a \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s,a)}$$

Value Iteration

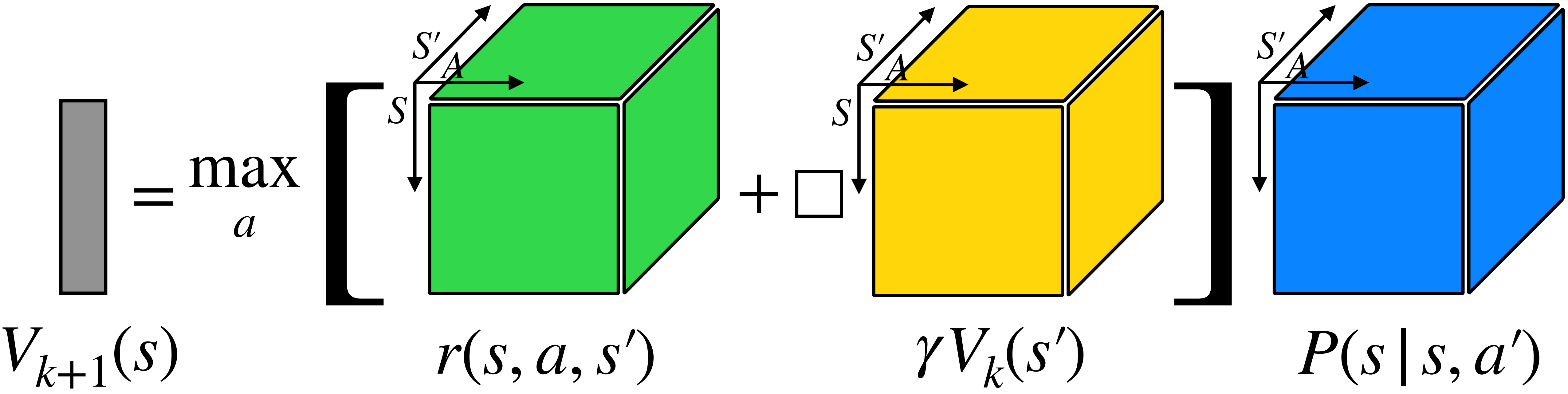
$$V_{k+1}(s) = \max_{a} \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s, a)$$



Value Iteration

$$V_{k+1}(s) = \max_{a} \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s'|s,a)$$

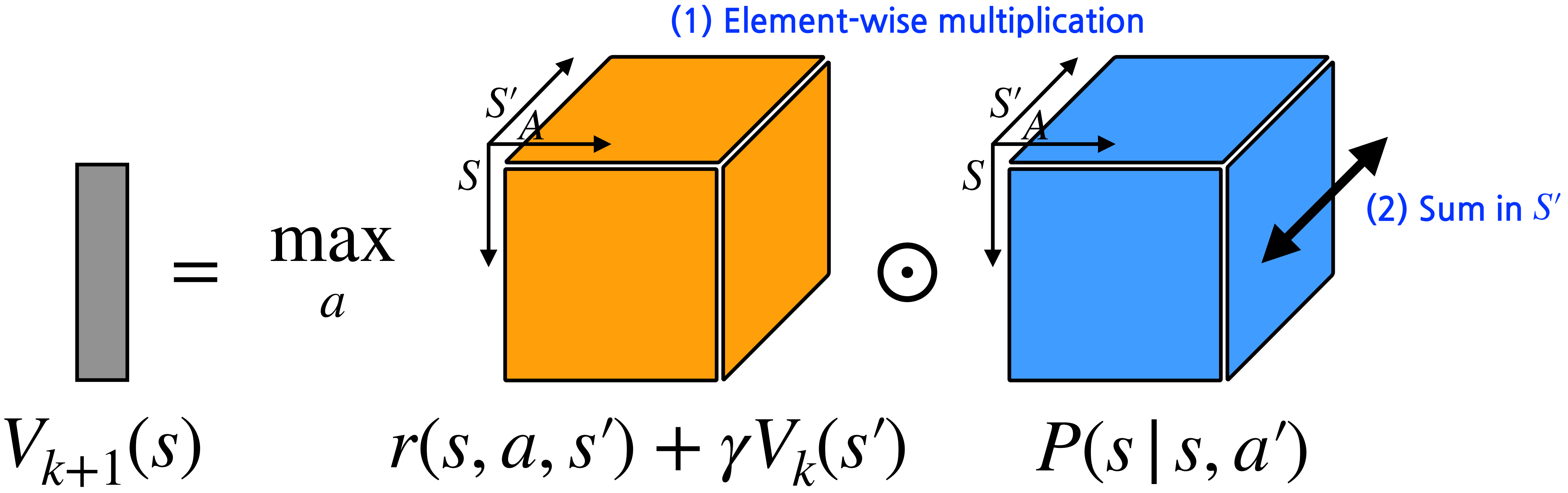
(1) Tile (duplicate) to match dimensions



(2) Element-wise sum of $r(s,a,s')$ and $\gamma V_k(s,a,s')$

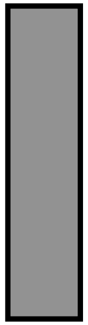
Value Iteration

$$V_{k+1}(s) = \max_{a} \sum_{s'} \left[r(s, a, s') + \gamma V_k(s') \right] P(s' \mid s, a)$$

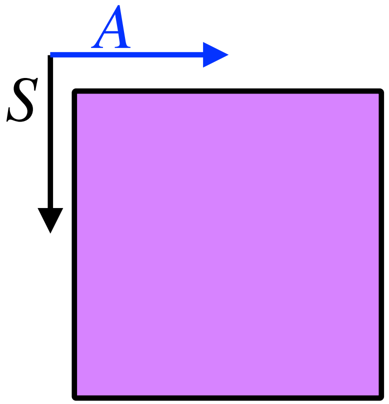


Value Iteration

$$\boxed{V_{k+1}(s) = \max_{a} \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s,a)}$$



$=$ $\max_{a}$

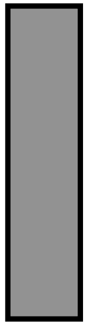


$V_{k+1}(s)$

$$\sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s,a)$$

Value Iteration

$$V_{k+1}(s) = \max_a \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s, a)$$



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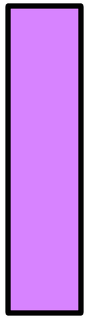

$$V_{k+1}(s) \qquad \max_a \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s, a)$$

Value Iteration

- Does **value iteration** converge?
  - Yes.
- **Why** does it converge?
  - The Bellman (Optimality) Backup operation is a **contraction** mapping.

Bellman Optimality Equation

$$V_{k+1}(s) = \max_{a} \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s, a)$$

Value Iteration

Bellman Optimality Backup Operator

$$\boxed{TX(s) = \max_a \sum_{s'} \left[r(s,a,s') + \gamma X(s') \right] P(s' | s, a)}$$

$$X \in \mathbb{R}^{|S|}$$

- Value iteration can be represented as $V_{k+1} = TV_k$.
- Bellman optimality equation becomes $V^* = TV^*$.
- Then, we can show that T is a γ-contraction mapping:
  - $|TU - TV|_\infty \leq \gamma |U - V|_\infty$

Value Iteration

- Value iteration can be represented as $V_{k+1} = TV_k$.
- Bellman optimality equation becomes $V^* = TV^*$.
- Then, we can show that T is a γ-contraction mapping:
  - $|TU - TV|_\infty \leq \gamma |U - V|_\infty$
  - $|TU - TV|_\infty = \max_s |TU(s) - TV(s)|$

$$\leq \max_s \left| \max_a \sum_{s'} \gamma |U(s') - V(s')| P(s'|s, a) \right|$$

$$\leq \max_s \left| \max_a \max_{s'} \gamma |U(s') - V(s')| \right|$$

$$= \gamma \max_{s'} |U(s') - V(s)| = \gamma |U - V|_\infty$$

Q-Value Iteration

- However, it is not straightforward to come up with an **optimal policy** solely from $V^*(s)$.
- Hence, we use following relations between $V^*(s)$ and $Q^*(s,a)$ (aka the Bellman optimality equation).

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s' | s,a)$$

- Bellman Equation

$$V_\pi(s) = \sum_a \pi(a|s) Q_\pi(s,a)$$

$$Q_\pi(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V_\pi(s') \right] P(s'|s,a)$$

$$V_\pi(s) = \sum_a \pi(a|s) \sum_{s'} \left[r(s,a,s') + \gamma V_\pi(s') \right] P(s'|s,a)$$

$$Q_\pi(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \sum_{a'} Q_\pi(s',a') \pi(a'|s') \right] P(s'|s,a)$$

- Bellman Optimality Equation

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s'|s,a)$$

$$V^*(s) = \max_a \sum_{s'} \left[r(s,a,s') + \gamma V^*(s') \right] P(s'|s,a)$$

$$Q^*(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma \max_{a'} Q^*(s',a') \right] P(s'|s,a)$$

$$\pi^*(a|s) = \arg\max_{a'} Q^*(s,a')$$

Q-Value Iteration

- Start from the random initial V_0
- For all states $s \in S$:

$$Q_k(s,a) = \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s,a)$$

$$V_{k+1}(s) = \max_{a'} Q_k(s,a')$$

- We now have an explicit form of the policy:

$$\pi_{k=1}(a \mid s) = 1 \text{ for } a = \arg\max_{a'} Q_k(s,a')$$

  - Note that this policy is **deterministic**.

Policy Iteration

Policy Iteration

- Basic concept
  - Step1) Predict the value of the policy
    - Evaluate the expected return of the current policy
  - Step2) Improve the policy
    - Update the current policy to get a better expected return
  - Iterate

Policy Iteration

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- **Step 1: Policy evaluation**: calculate the value function for a fixed policy until convergence.
- **Step 2: Policy improvement**: update the policy using one-step lookahead using the converged value function.
- Repeat steps until the **policy** converges.
- It is guaranteed to converge to the optimal policy.
- It often converges much faster than value iteration.

Policy Iteration

- **Step 1: Policy evaluation**: calculate the value function for a fixed policy until convergence.
  - Start from a random initial V_0.
  - Iterate until value converges:

$$V_{k+1}(s) = \sum_a \pi_i(a \mid s) \sum_{s'} [r(s, a, s') + \gamma V_k(s')] P(s' \mid s, a)$$

- **Step 2: Policy improvement**: update the policy using one-step lookahead using the converged value function.
  - One-step lookahead:

$$Q_{\pi_k}(s, a) = \sum_{s'} [r(s, a, s') + \gamma V_{\pi_i}(s')] P(s' \mid s, a)$$

$$\pi_{i+1}(a \mid s) = 1 \text{ for } a = \arg\max_{a'} Q_{\pi_i}(s, a')$$

Policy Evaluation

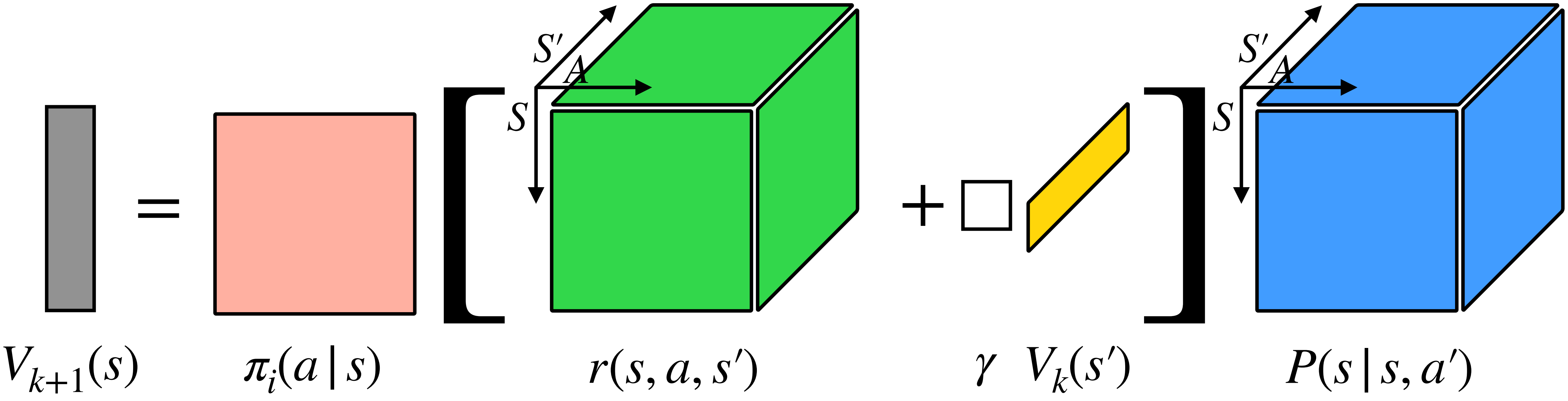
- Start from a random initial V_0
- Initialize V_{k+1} with a zero vector.
- Loop
  - For all states $s \in S$:
    - For all actions $a \in A$:
      - For all next states $s' \in S$:
        - $V_{k+1}(s) = V_k(s) + [r(s, a, s') + \gamma V_k(s')]P(s' | s, a)\pi_i(a | s)$
  - If $|V_{k+1} - V_k|_\infty = \max_s |V_{k+1}(s) - V_k(s)| \leq \epsilon$
    - stop

Bellman Equation

$$\boxed{V_{k+1}(s) = \sum_a \pi_i(a | s) \sum_{s'} \left[r(s, a, s') + \gamma V_k(s')\right] P(s' | s, a)}$$

Policy Evaluation

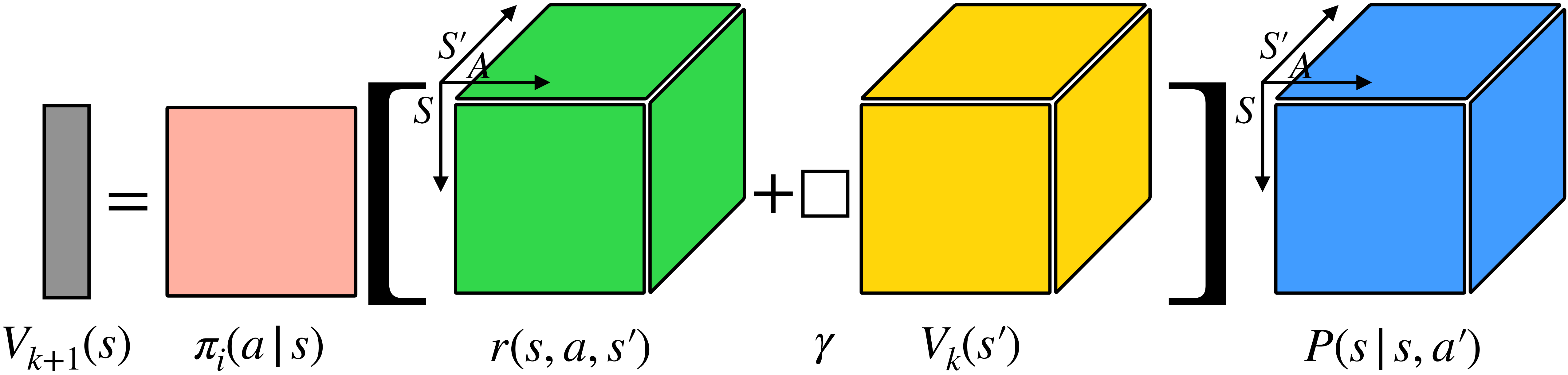
$$V_{k+1}(s) = \sum_{a} \pi_i(a \mid s) \sum_{s'} \left[r(s, a, s') + \gamma V_k(s') \right] P(s' \mid s, a)$$



Policy Evaluation

$$V_{k+1}(s) = \sum_{a} \pi_i(a|s) \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s'|s,a)$$

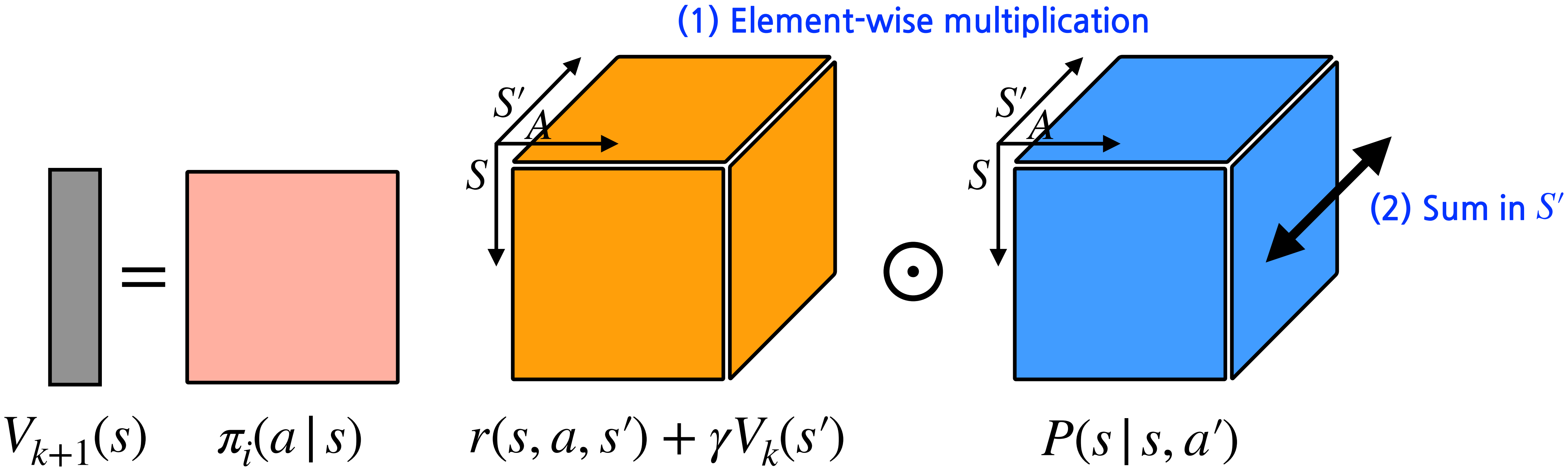
(1) Tile (duplicate) to match dimensions



(2) Element-wise sum of $r(s,a,s')$ and $\gamma V_k(s,a,s')$

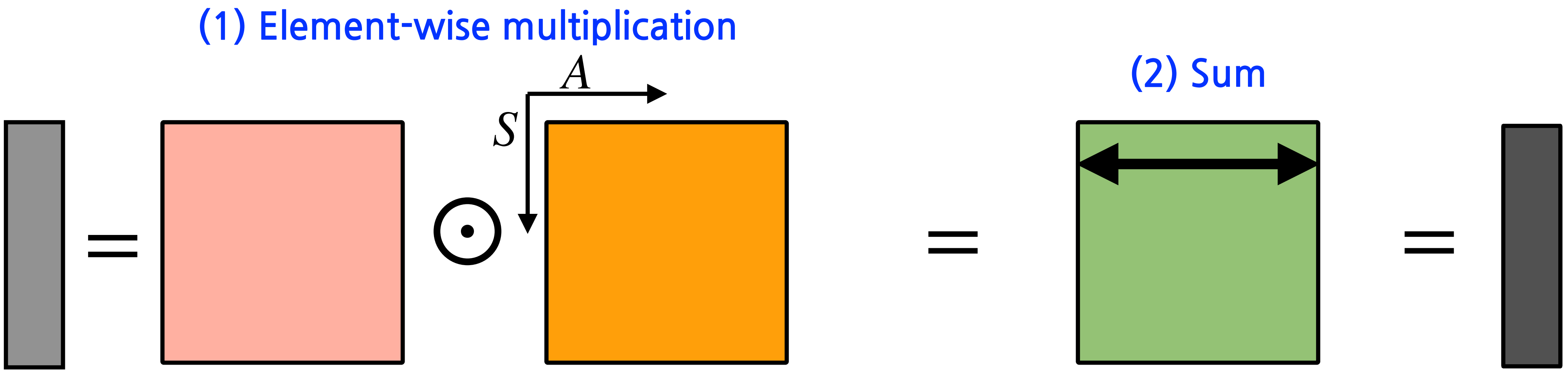
Policy Evaluation

$$V_{k+1}(s) = \sum_{a} \pi_i(a\,|\,s) \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s'\,|\,s,a)$$



Policy Evaluation

$$\boxed{V_{k+1}(s) = \sum_{a} \pi_i(a \mid s) \sum_{s'} \left[r(s,a,s') + \gamma V_k(s') \right] P(s' \mid s,a)}$$



$V_{k+1}(s)$ $\quad$ $\pi_i(a \mid s)$ $\quad$ $\sum_{s'} [r(s,a,s') + \gamma V_k(s')] P(s' \mid s,a)$

Policy Evaluation

- Does policy evaluation converge?
  - Yes.
- Why does it converge?
  - The Bellman Backup operation is a **contraction** mapping.

Policy Evaluation

Bellman Backup Operator

$$\boxed{T_\pi X(s) = \sum_a \pi(a|s) \sum_{s'} \left[r(s,a,s') + \gamma X(s')\right] P(s'|s,a)} \qquad X \in \mathbb{R}^{|S|}$$

$$\begin{aligned}
|T_\pi U - T_\pi V|_\infty &= \max_s |T_\pi U(s) - T_\pi V(s)| \\
&= \max_s \left| \sum_a \pi(a|s) \sum_{s'} [\cancel{r(s,a,s')} + \gamma U(s')] P(s'|s,a) - \sum_a \pi(a|s) \sum_{s'} [\cancel{r(s,a,s')} + \gamma V(s')] P(s'|s,a) \right| \\
&= \max_s \left| \gamma \sum_a \sum_{s'} (U(s') - V(s')) P(s'|s,a) \pi(a|s) \right| \\
&\le \max_s \left| \gamma \sum_a \max_{s'} |U(s') - V(s')| \right| \\
&= \gamma \max_{s'} |U(s') - V(s')| = \gamma |U - V|_\infty
\end{aligned}$$

Policy Improvement

- Let V_{π_i} be the value evaluated from the policy π_i.
- Our goal is to find a policy π_{i+1}.

- Compute the state-action value $Q_{\pi_i}(s, a)$ from state value $V_{\pi_i}(s)$:

$$Q_{\pi_i}(s, a) = \sum_{s'} [r(s, a, s') + \gamma V_{\pi_i}(s')]P(s' | s, a)$$

- The policy is updated via:

$$\pi_{i+1}(a | s) = 1 \text{ for } a = \arg\max_{a'} Q_{\pi_i}(s, a)$$

Policy Improvement

- It is guaranteed that the **policy improvement** improves the value.
- The value of the updated policy is

$$V_{\pi_{i+1}}(s) = \sum_a Q_{\pi_i}(s,a)\pi_{i+1}(a\,|\,s)$$

- We can show that $V_{\pi_{i+1}}(s) \geq V_{pi_i}(s)$, hence is improves the value function.

$$\begin{aligned} V_{\pi_{i+1}}(s) &= \sum_a Q_{\pi_i}(s,a)\pi_{i+1}(a\,|\,s) \\ &= \max_{a'} Q_{\pi_i}(s,a') \\ &\geq \sum_s Q_{\pi_i}(s,a)\pi_i(a\,|\,s) \\ &= V_{\pi_i}(s) \end{aligned}$$

Policy Iteration

- **Step 1: Policy evaluation**: Evaluate V_π.
- **Step 2: Policy improvement**: Generate π' where $V_{\pi'} \geq V_\pi$.

- **Policy iteration** is often more effective that **value iteration**, why?
  - It is often the case that a policy function reaches the optimal policy (policy iteration) much sooner than a value function reaches the optimal value function (value iteration).
  - In many cases, we are more interested in finding the optimal policy function rather than the optimal value function.

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