Solutions to Weekly Questions

Mathematics Society

3-11-2019

Remarks:

- Note that the marks allocated **is directly proportional** to the difficulty.
- Bonus questions are **not necessarily more difficult**.
- The solution will be released on the following Wednesday.

1. Given $ax^2 + bx + c = 0$ and $b^2 > 4ac$, derive the quadratic equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

   (a) (2 ½ points) Find e, f, g expressed by a, b, c such that $ax^2 + bx + c = e(x + f)^2 + g$.
   Hint: $g < 0$.

   Solution:

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + 2x\left(\frac{b}{2a}\right) + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 + \frac{b^2 - 4ac}{4a^2} &= 0
\end{aligned}
$$

$$e = 1,\ f = \frac{b}{2a},\ g = \frac{b^2 - 4ac}{4a^2}$$

(b) (½ point) Using (a), derive the quadratic equation.

Solution:

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 + \frac{b^2 - 4ac}{4a^2} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

2. Given two coordinates (a, b) and (c, d).

   (a) (1 ½ points) Show that if $ad - bc \neq 0$, then there **does not exist** $e, f \neq 0$ such that $(ea, eb) + (fc, fd) = (0, 0)$.

Solution:

Proof. Assume there exists such e, f pair. Then,

$$\begin{aligned} ea + fc &= 0 \\ ea &= -fc \\ a &= -\frac{f}{e}c \end{aligned}$$

$$\begin{aligned} eb + fd &= 0 \\ eb &= -fd \\ b &= -\frac{f}{e}d \end{aligned}$$

As such,

$$\begin{aligned} ad - bc &= -\frac{f}{e}cd + \frac{f}{e}dc \\ &= 0 \end{aligned}$$

Which contradicts the fact that $ad - bd \neq 0$. Hence such pairs do not exist. $\square$

(b) (1 ½ points) Show that if $ad-bc=0$, then there **exists** $e, f \neq 0$ such that $(ea, eb)+(fc, fd)=(0,0)$.

Solution:

Proof.

$$\begin{aligned} ad - bc &= 0 \\ ad &= bc \end{aligned}$$

If a or $d = 0$ then either b or $c = 0$. It could be easily seen that the solution is trivial. As such, let us assume $a, b, c, d \neq 0$

$$\begin{aligned} ea + fc &= 0 \\ ea &= -fc \\ \frac{a}{c} &= -\frac{f}{e} \end{aligned}$$

$$\begin{aligned} eb + fd &= 0 \\ eb &= -fd \\ \frac{b}{d} &= -\frac{f}{e} \end{aligned}$$

$e \neq 0 \because \frac{a}{c}$ is defined. $f \neq 0 \because \frac{a}{c} \neq 0$ As such, there exists $e, f \neq 0$ that satisfy the equation. $\square$

(c) (½ point (bonus)) Proof that if (a, b) and (c, d) are colinear with $(0,0)$, then $ad - bc = 0$.

Solution:

Proof. If (a, b) and (c, d) are colinear, the slope of the line (a, b) to $(0, 0)$ and the slope of the line (c, d) to $(0, 0)$ is equivalent. Hence,

$$\begin{aligned} \frac{d}{b} &= \frac{b}{a} \\ da &= bc \\ ad - bc &= 0 \end{aligned}$$

$\square$

3. **Prove or disprove** (in **ZFC**) the following properties of ordinal addition.
The definitions of ordinal addition are listed below for convinence.

$$\forall \alpha \in \mathbf{Ord}$$

$$\begin{aligned}
&(\text{Def. i}).\ \alpha + 0 = \alpha \\
&(\text{Def. ii}).\ \alpha + 1 = \alpha \cup \{\alpha\} \\
&(\text{Def. iii}).\ \alpha + (\beta + 1) = (\alpha + \beta) + 1, \text{ for all } \beta \\
&(\text{Def. iv}).\ \alpha + \beta = \sup\{\alpha + \xi \mid \xi < \beta\}, \text{ for all limit ordinal } \beta > 0
\end{aligned}$$

Note: **Ord** denotes the class of all ordinals, while sup denotes the supremum of a set.

(a) (3 points) $\forall \alpha, \beta, \gamma \in \mathbf{Ord}$, $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

Solution:

Proof. Let $A = \{\gamma \in \mathbf{Ord} \mid \forall \alpha, \beta \in \mathbf{Ord}, (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)\}$
We will prove by transfinite induction that $A = \mathbf{Ord}$
0 case:

$$(\alpha + \beta) + 0 = \alpha + \beta$$
$$\alpha + (\beta + 0) = \alpha + \beta$$

Successor case:

$$\begin{aligned}
(\alpha + \beta) + (\gamma + 1) &= ((\alpha + \beta) + \gamma) + 1 \\
&= (\alpha + (\beta + \gamma)) + 1 \\
&= \alpha + (\beta + (\gamma + 1))
\end{aligned}$$

Limit case:

$$\begin{aligned}
(\alpha + \beta) + \gamma &= \sup\{(\alpha + \beta) + \xi \mid \xi < \gamma\} \\
&= \sup\{\alpha + (\beta + \xi) \mid \xi < \gamma\} \\
&= \alpha + (\beta + \gamma)
\end{aligned}$$

□

(b) (2 points) $\forall \alpha, \beta \in \mathbf{Ord}$, $\alpha + \beta = \beta + \alpha$

Solution: Counterexample: $\omega + 1 > 1 + \omega = \omega$