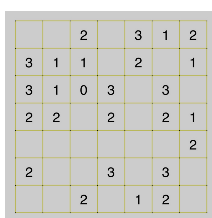
Mathematics Society
Weekly Questions (Week 5)

December 21, 2023

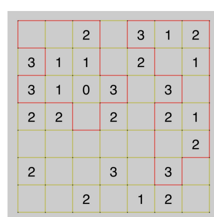
Question 1 (Puzzle) *Loopy*[1]
You are given a grid of dots, marked with yellow lines to indicate which dots you are allowed to connect directly together.
Your aim is to use some subset of those yellow lines to draw a single unbroken loop from dot to dot within the grid.

Some of the spaces between the lines contain numbers.
These numbers indicate how many of the lines around that space form part of the loop.
The loop you draw must correctly satisfy all of these clues to be considered a correct solution.

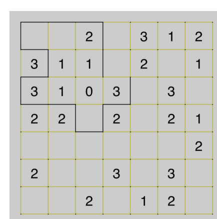
Examples:



Example Grid

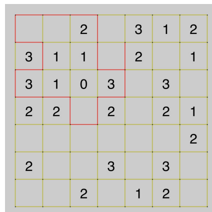


Must be only 1 loop

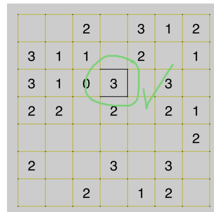


Not allowed (loop is broken)

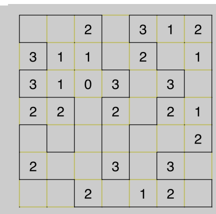
[1]Puzzle idea due to Nikoli



Also broken loop



What the numbers do



Correctly Solved

Problem Set

Easy:

	2	2	1	2	2	
	3		2			2
2				1		
3	2	2	2			
3	0					
2	2					1
1	2	2	2		3	3

	3		3			
			1	1		3
	3		1	1	0	
2	1		3		3	2
2				2		
2	2		1		2	
				3	3	

		2				
2	3	0		1		
			3			
3	1		2			3
						3
2	3		1			2
		2	2		3	

Hard:

3				2	3		3		3
	3			2	2	3		3	
3			1		1				2
	2		2	3		2		3	2
			0		3	2	2		3
3		3	2						2
2	3		2	2	2	2			
1			2		2		1	2	3
2			0	2	1				
	2	2	3	3					3

2	2		3	1					
	3			2	2				2
		3			1	2		2	3
				2			2		
3	0	1		1	1		2	1	3
3			3			3		1	
2	1	1				2		2	1
2			1	0	2	2		2	
1	2	2		1		2			
3			3						

3			3				2	3	
			0	2	0				1
3		3					3		
		3	1		1			0	3
	1		1			1	2		2
	2	3	2		1		3		2
		1		3					
1			3	2	0	1	3		
		1		2					2
2	2	2	3		1		3		3

Question 2 (Brainteasers and Problem-solving Techniques) *Counting Vertically and Horizontally*

These two problems are related by a common solution method, though neither of them require it. A more thorough description of it is given at the end of the PDF, since I don't want to spoil it right here.

a A builder must hang up 10000 signs for 10000 houses in numerical order, starting 0000, 0001, 0002, ..., 9999. So, he has to buy digits to stick on the signs. How many 3's must he buy? How many of each digit must he buy?(For example, if there were only 10 houses, he would need to buy one 0, one 1, ..., and one 9) Follow-up question: he decides to remove the leading 0's, making the signs 1, 2, 3, ..., 10, 11, ..., 100, 101, ... 9999. How many 0's does he need now?

b How many factors of 3 does 1000! (defined as $1 \times 2 \times 3 \times \cdots 1000$) have?

Question 3 (Olympiad) *Functional equations*

Watch this if you don't know what a function is or what it does.
https://youtu.be/kvGsIo1TmsM
Solutions to a functional equation will always be functions, not numbers. For example, a solution to the functional equation "For all real numbers x and y, $f(x+y) = f(x) + f(y)$" would be $f(z) = \frac{95}{2}z$.

Let's call a function *nice* (not an official name) if it takes only rational numbers as input and outputs only rational numbers, and satisfies this equation for all rationals x and y:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

a Verify that $f(x) = x^2$ is *nice*.

b Find all *nice* functions.
Hint: Substitute x = 0. Then you'll get a true equation, because the above equation is true for all rationals. If you continue substituting other values, like y = 1 and x = 0, or y = x and x = y (swapping the two), then you'll eventually be able to conclude that f must be a certain type of function.

As promised, here are the hints for question 2:
Think about lining up all the signs in a vertical column:
0000
0001
0002
0003
...
9999
How many of each digit are there in each column?
Now for the second part. Think of the factors of 3 in 1000! as being laid out in a column, like in the next row:
1
1
3
1
1
3
1
1
3*3
1
...
1
Instead of counting like 3, 3, 9, 3, 3, 9, 3, 3, 27, count the 3's in the columns: this corresponds to adding up the number of multiples of 3 to the number of multiples of 9, and then adding the number of multiples of 27, and so on. The over-counting perfectly accounts for the numbers having multiple factors of 3.