Mathematics Society
Weekly Questions (Week 9)

Duodecember, 2023

We all know the four basic operations of arithmetic: $+, -, \times, \div$. In these problems, we'll explore how $+$ and $\times$ are, in some sense, *nicer* than their counterparts, $-$ and $\div$, and then, next week, use this to generalize the notion of number.

Question 1 *Associativity*

We say that a *binary operation*[1] $\circ$ is **associative** if $(a \circ b) \circ c = a \circ (b \circ c)$. For example, let $\circ$ be defined by $a \circ b := a + b$. Then, since

$$(a + b) + c = a + (b + c)$$

, we know that $a \circ (b \circ c) = (a \circ b) \circ c$, so $+$ is associative.
Here's an example of an operation that is **not associative**. It is also very familiar: Let $a \circ b := a \div b$. This is not associative:

$$a \div (b \div c) = (a \div b) \times c \neq (a \div b) \div c$$

so division is not associative! This is another operation that is not associative: If $a \circ b := 2a + 3b$ this time, then

$$(a \circ b) \circ c = (2a + 3b) \circ c = 2(2a + 3b) + 3c = 4a + 6b + 3c$$

and

$$a \circ (b \circ c) = a \circ (2b + 3c) = 2a + 3(2b + 3c) = 2a + 6b + 9c \neq 4a + 6b + 3c$$

A simple way to see that an operation is not associative is to substitute some numbers in and see what happens. Consider: $1 \circ 1 = 5$, yet $1 \circ 5 = 17$ and $5 \circ 1 = 13$, so $1 \circ (1 \circ 1) = 1 \circ 5 \neq 5 \circ 1 = (1 \circ 1) \circ 1$. Since they may not be equal,

[1] $+$, $-$, $\times$, and $\div$ are all examples of "binary operations". What that means is they take in two things of the same kind, and output another of the same kind. For example, $+, -, \times$ can take in integers and output integers.

this operation is not associative. Are the following operations associative? If you think they are, then prove it algebraically with variables and equations, and if you think they aren't, then provide a counterexample.

a Multiplication: $a \circ b := a \times b$

b Subtraction: $a \circ b := a - b$

c Taking powers: $a \circ b := a^b$

d $a \circ b := ab - a - b$

e $a \circ b := \frac{a+b}{5}$

f $a \circ b := \frac{ab}{5}$

g $a \circ b := ab - 5$

h $a \circ b := a + b - 5$

i $a \circ b := a$

j $a \circ b := b$

k $a \circ b := 5$

Question 2 *Commutativity*
A binary operation is **commutative** if you can switch the order and get the same thing. For example, addition is commutative, because $a + b = b + a$, but subtraction is not commutative because $a - b \neq b - a$ in general. Another way to disprove that subtraction is commutative is to provide a counterexample, e.g. $0 - 1 = -1$, but $1 - 0 = 1$, so $0 - 1 \neq 1 - 0$.
Check whether each of the operations defined in Question 1 are commutative.

Question 3 *Identity*
0 is called the "additive identity", because if you add it to anything, it doesn't change:

$$x + 0 = 0 + x = x$$

Similarly, 1 is called the "multiplicative identity", since $1x = x1 = x$.
Does subtraction have a similar identity?
You may think so, because $x - 0 = x$, but $0 - x = -x$. So 0 isn't the identity of subtraction! Instead, it is only a *right identity.* 0 and 1 are both "two-sided" identities, so we call them identities for short.

a Is there some fixed number n that is a *left* identity for subtraction?
b Which of the operations in Question 1 have left or right identities?
c If there are both a left identity A and a right identity B for a certain operation (not necessarily associative or commutative), then prove that they must actually be equal, that is, $A = B$.

Conclusion In the above, we found that addition and multiplication are both associative, commutative, and have an identity, while subtraction and division are not associative or commutative. We also found that some operations I made up, like $a \circ b = 5$, $a \circ b = \frac{ab}{5}$, and $a \circ b = a + b - 5$, are associative, commutative, and have identity. SO they're "nicer" than subtraction and division. Next week, we'll find out why