Solutions to Weekly Questions

Mathematics Society

10-11-2019

Remarks:

- Note that the marks allocated **is directly proportional** to the difficulty.
- Bonus questions are **not necessarily more difficult**.
- The solution will be released on the following Wednesday.

1. Given $\frac{1}{x^4+1} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{dx^2+ex+f}$.

   (a) (½ point) Factorize x^4+1.

Solution:

$$\begin{aligned} x^4+1 &= (x^2)^2 + 1^2 \\ &= (x^2)^2 + 2x^2 + 1^2 - 2x^2 \\ &= (x^2+1)^2 - 2x^2 \\ &= (x^2+1)^2 - (\sqrt{2}x)^2 \\ &= (x^2+1-\sqrt{2}x)(x^2+1+\sqrt{2}x) \\ &= (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1) \end{aligned}$$

(b) (1 ½ points) Find solutions to a, b, c, d, e, f, A, B, C, D.
Hint: partial fractions decomposition.

Solution:
Let

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}$$

$$\begin{aligned}
(x^4+1)\left(\frac{1}{x^4+1}\right) =& (x^4+1)\left(\frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}\right) \\
1 =& (Ax+B)(x^2+\sqrt{2}x+1) \\
& (Cx+D)(x^2-\sqrt{2}x+1) \\
1 =& Ax^3 + \sqrt{2}Ax^2 + Ax + Bx^2 + \sqrt{2}Bx + B \\
& + Cx^3 - \sqrt{2}Cx^2 + Cx + Dx^2 - \sqrt{2}Dx + D \\
1 =& (A+C)x^3 + (\sqrt{2}A + B - \sqrt{2}C + D)x^2 \\
& + (A + \sqrt{2}B + C - \sqrt{2}D)x + (B+D)
\end{aligned}$$

$$\begin{aligned}
A + C &= 0 \\
\sqrt{2}A + B - \sqrt{2}C + D &= 0 \\
A + \sqrt{2}B + C - \sqrt{2}D &= 0 \\
B + D &= 1
\end{aligned}$$

$$A = -\frac{1}{2\sqrt{2}},\ B = \frac{1}{2},\ C = \frac{1}{2\sqrt{2}},\ D = \frac{1}{2}$$

$$a = 1,\ b = -\sqrt{2},\ c = 1,\ d = 1,\ e = \sqrt{2}, f = 1$$

2. Evaluate the following integrals.

(a) (2 points) $\int \frac{x}{\sqrt{x^2+1}}\,dx$.

Solution: Let $u = x^2 + 1$

$$dx = \frac{du}{2x}$$

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2+1}}\,dx &= \int \frac{x}{\sqrt{u}}\,\frac{du}{2x} \\
&= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\
&= \frac{1}{2}\int u^{-\frac{1}{2}}\,du \\
&= \frac{1}{2}\cdot\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\
&= u^{\frac{1}{2}} + C \\
&= \sqrt{x^2+1} + C
\end{aligned}$$

(b) (1 point) $\int x^{\frac{x}{\ln x}}\,dx$.

Solution:

$$\begin{aligned}
\int x^{\frac{x}{\ln x}}\,dx &= \int e^{\ln(x^{\frac{x}{\ln x}})}\,dx \\
&= \int e^{\frac{x}{\ln x}\ln x}\,dx \\
&= \int e^{x}\,dx \\
&= e^{x} + C
\end{aligned}$$

3. Prove the following statements in **ZFC**.

(a) (3 points) There are arbitrarily large limit ordinals. $\forall\alpha, \exists\beta > \alpha$, where β is a limit ordinal.

Solution:

Proof. Given $\alpha_0 \in \mathbf{Ord}$, define $\alpha_{n+1} = \alpha_n + 1$. Let $\beta = \sup\{\alpha_n \mid n < \omega\} = \bigcup\{\alpha_n \mid n < \omega\} = lim_{n\to\omega}\alpha_n$. Since the union of ordinals is an ordinal, β is an ordinal. And for every $\gamma < \beta$, there exists $\alpha_n > \gamma$, otherwise $\sup\{\alpha_n \mid n < \omega\} \leq \gamma$, a contradiction. Thus $\gamma+1 < \alpha_n+1 = \alpha_{n+1} < \beta$, and so β is a limit ordinal. Therefore, there are arbitrarily large limit ordinals. □

(b) (3 points) Every normal sequence $\langle\gamma_\alpha \mid \alpha \in \mathbf{Ord}\rangle$ has arbitrarily large fixed points, α such that $\gamma_\alpha = \alpha$.

Solution:

Proof. Since $\langle\gamma_\alpha \mid \alpha \in \mathbf{Ord}\rangle$ is increasing, for every $\beta \in \mathbf{Ord}$, there exists $m \in \mathbf{Ord}$ such that $\gamma_m > \beta$. Now let $\alpha_0 = \gamma_m$, $\alpha_{n+1} = \gamma_{\alpha_n}$. Then $\langle\alpha_n \mid n \in \gamma\rangle$ is increasing. Let $\alpha = \lim_{n\to\omega}\gamma_n$. Repeating the argument in (a), α is a limit ordinal. Hence, $\alpha = \lim_{n\to\omega}\alpha_{n+1} = \lim_{n\to\omega}\gamma_{\alpha_n} = \lim_{\xi\to\alpha}\gamma_\xi = \gamma_{\lim_{\xi\to\alpha}\alpha} = \gamma_\alpha$. □