

Mathematics Society

Weekly Questions (Week 6)

December 31, 2023

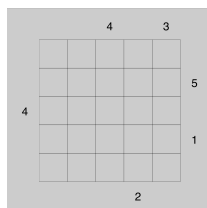
Question 1 (Puzzle) *Skyscrapers*¹

You have a square grid. On each square of the grid you can build a tower, with its height ranging from 1 to the size of the grid. Around the edge of the grid are some numeric clues.

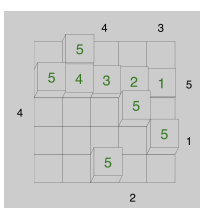
Your task is to build a tower on every square, in such a way that:

1. Each row contains every possible height of tower once
2. Each column contains every possible height of tower once
3. Each numeric clue describes the number of towers that can be seen if you look into the square from that direction, assuming that **shorter towers are hidden behind taller ones**. For example, in a 5×5 grid, a clue marked '5' indicates that the five tower heights must appear in increasing order (otherwise you would not be able to see all five towers), whereas a clue marked '1' indicates that the tallest tower (the one marked 5) must come first.

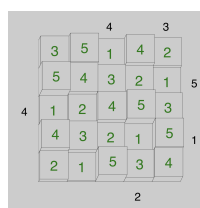
Examples:



A blank puzzle



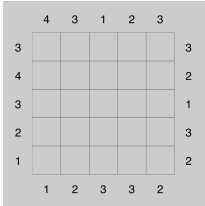
When solving



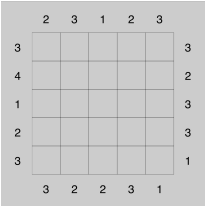
Correctly Solved

¹Origin: Invented by Masanori Natsuhara from Japan in 1992 in the Puzzler magazine. Originally entitled " " (Building City Puzzle).

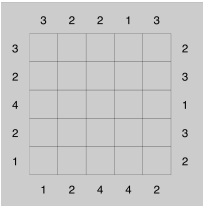
Problem Set



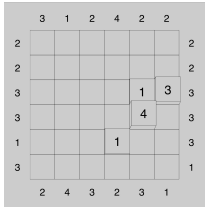
Easy



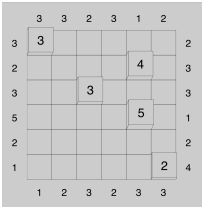
Easy



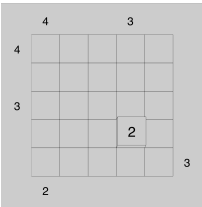
Easy



Medium



Medium



Hard

Question 2 (Brainteasers and Problem-solving Techniques) *Colored Hats Game*² A demon, who always tells the truth, guards the only door to the valuable lost library you have been looking for. When you reach it, you are told that your 23 other friends have already entered through this door, after solving the demon's logic puzzle, because they're all perfect logicians, who are very good at logic and can crack any solvable logic puzzle. He describes it to you:

The demon puts colored masks on all 23 of your friends' faces, and arranges them in a circle, so that all of them can see everyone else's masks, but not their own. After this, a bell rings every two minutes. At each bell, anyone who has logically deduced their own mask color must immediately tell the demon their color and gain entry to the library. The demon tells them that the masks are distributed in such a way that it will eventually be possible for everyone to figure out their mask color through logic alone, and everyone is aware of this.

What happened afterwards was this:

4 of your friends gained entry at the first bell.

Some of them, all in red masks, gained entry at the second bell.

Nobody gained entry at the third bell.

Some of your friends, in total wearing 2 different colors, gained entry at the fourth bell.

The demon's riddle for you is this: How many logicians gained entry at the fifth bell?

Be reminded that: (a) none of them are allowed to talk about their masks, and there is no way anyone can glimpse a reflection of themselves. (b) They aren't told anything about the possible mask colors, only that it is eventually possible to figure out their own color. (So, for example, there can't be just one silver mask, since the person wearing the silver mask would be unable to deduce that there even *were* any silver masks purely by looking everyone else. They would only see different colors. But there could be *two* silver masks, since silver would always be a possibility, even to the people wearing silver masks.) (c) All of your friends played perfectly, since they are perfect logicians.

Hints: Suppose there are exactly two people wearing the same color mask. When must they leave? What about 3?

(This relies on the fact that there can't be only one person wearing a mask of a given color.)

²Source: <https://www.youtube.com/watch?v=KT-d8MIT27A> (contains spoilers and solution)

Question 3 (Olympiad) There are $2n + 1$ people at a party. It is given that for every group of n people at the party, there is a person not in the group who knows all of the n people in the group. Explain why there must exist a person who knows all the people at the party.
(It will be useful to try small cases first.)