Weekly Questions

Mathematics Society

4-11-2020

Remarks:

- Note that the marks allocated **is proportional** to the difficulty.
- Bonus questions are **not necessarily more difficult**.
- The solution will be released on the following Wednesday.

1. (2 points) (a, b, c) is said to be a *Pythagorean triple* if it satisfies the equation $a^2 + b^2 = c^2$ where a, b, and c are distinct positive integers in ascending order. If three positive integers a, b, and c form both a Pythagorean triple and an arithmetic progression, prove that $a : b : c = 3 : 4 : 5$.

Solution:

Proof. Let their common difference be d. a and c can thus be represented as $b - d$ and $b + d$ respectively. Since they form a Pythagorean triple, we have

$$\begin{aligned} (b-d)^2 + b^2 &= (b+d)^2 \\ (b^2 - 2bd + d^2) + b^2 &= b^2 + 2bd + d^2 \\ 2b^2 - 2bd + d^2 &= b^2 + 2bd + d^2 \\ b^2 &= 4bd \end{aligned}$$

Since b is nonzero, we can divide it from both sides:

$$b = 4d$$

Therefore,

$$\begin{aligned} a &= b - d = 4d - d = 3d \\ c &= b + d = 4d + d = 5d \end{aligned}$$

Since $d \neq 0$, we can conclude

$$a : b : c = 3d : 4d : 5d = 3 : 4 : 5$$

□

2. (5 points) Find the derivative of $f(x) = e^{(x^x)}$.

Solution: Apply the chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ with $f(x)$ as e^x and $g(x)$ as x^x:

$$\frac{d}{dx}\left(e^{x^x}\right) = e^{x^x} \cdot \frac{d}{dx}\left(x^x\right)$$

Rewrite x^x as $e^{x \cdot \log(x)}$,

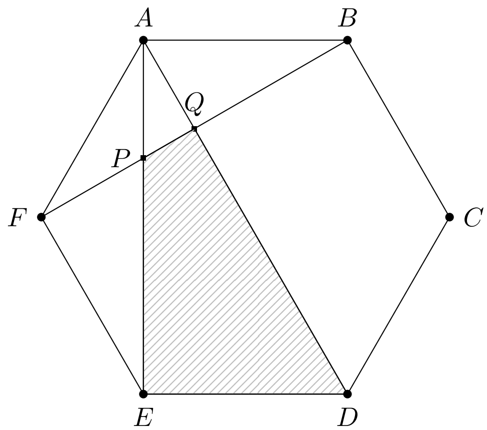
$$e^{x^x} \cdot \frac{d}{dx}\left(e^{x \cdot \log(x)}\right)$$

Apply the chain rule again with $f(x)$ as e^x and $g(x)$ as $x \cdot \log(x)$,

$$e^{x^x} \cdot e^{x \cdot \log(x)} \cdot \frac{d}{dx}\left(x \cdot \log(x)\right)$$

Apply the product rule $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x)$ with $f(x)$ as x and $g(x)$ as $\log(x)$.

$$e^{x^x} \cdot e^{x \cdot \log(x)} \cdot \left(1 \cdot \log(x) + \frac{1}{x} \cdot x\right) = \boxed{e^{x^x} \cdot e^{x \cdot \log(x)} \cdot (\log(x) + 1)}$$



3. (7 points) The above figure shows a regular hexagon $ABCDEF$ with side length 1. AQD, APE and $FPQB$ are straight lines and are diagonals of the hexagon. Find the

   (a) (3 points) area of the shaded region.

Solution:

Lemma 3.1. $\angle PAQ = 30°$

All internal angles in a hexagon is equal to $120°$.$\triangle AEF$ is isosceles, therefore $\angle FAP = \frac{180°-120°}{2} = 30°$;

AD bisects $\angle FAB$, therefore $\angle FAQ = \frac{120°}{2} = 60°$;

From the above, we can conclude $\angle PAQ = 60° - 30° = 30°$.

Lemma 3.2. Line segment AQ has length $\frac{1}{2}$.

Construct line segment EC and suppose AD and EC meet at R. Since $BCEF$ is a rectangle, $QR = EF = 1$. We know regular hexagons can be spilt into 6 equilateral triangles and it follows that $AD = 2$.

RD=AQ (symmetry). Therefore, $AQ = \frac{AD-QR}{2} = \frac{2-1}{2} = \frac{1}{2}$

Lemma 3.3. Line segment PQ has length $\dfrac{\sqrt{3}}{6}$.

It is obvious that $\angle AQF = 90°$. From lemmas 3.1 and 3.2:

$$\tan\angle PAQ = \frac{PQ}{AQ}$$
$$\tan 30° = \frac{PQ}{0.5}$$
$$PQ = \frac{\tan 30°}{2} = \frac{\sqrt{3}}{6}$$

Lemma 3.4. Line segment AP has length $\frac{\sqrt{3}}{3}$.

$$\begin{aligned}\cos 30^\circ &= \frac{0.5}{AP}\\ AP &= \frac{0.5}{\cos 30^\circ} = \frac{\sqrt{3}}{3}\end{aligned}$$

Lemma 3.5. Line segment AE has length $\sqrt{3}$.
Construct line segment FC. Suppose FC and FC meet at S. Similar to lemmas 3.1, 3.2 and 3.3, we know that $\angle EFS = 60^\circ$, $FS = 0.5$ and $\angle FSE = 90^\circ$.

$$\begin{aligned}\tan 60^\circ &= \frac{ES}{0.5}\\ ES &= \frac{\tan 60^\circ}{2} = \frac{\sqrt{3}}{2}\end{aligned}$$

By prop. of isos. $\triangle$, $AS = ES$. Therefore, $AE = \frac{\sqrt{3}}{2} \cdot 2 = \sqrt{3}$.
Using these five lemmas, we have:

$$\begin{aligned}&\text{Area of the shaded region}\\ =&\,\text{Area}(AED) - \text{Area}(APQ)\\ =&\,\frac{AE \cdot ED}{2} - \frac{AQ \cdot PQ}{2}\\ =&\,\frac{\sqrt{3} \cdot 1}{2} - \frac{0.5 \cdot \frac{\sqrt{3}}{6}}{2}\\ =&\,\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{24}\\ =&\,\boxed{\frac{11\sqrt{3}}{24}}\end{aligned}$$

(b) (4 points) perimeter of the shaded region.

Solution: Using lemmas 3.1 through 3.5, we have

$$\begin{aligned}
&\text{Perimeter of the shaded region}\\
=&PQ + QD + DE + EP\\
=&\frac{\sqrt{3}}{6} + (AD - AQ) + 1 + (AE - AP)\\
=&\frac{\sqrt{3}}{6} + (2 - \frac{1}{2}) + 1 + (\sqrt{3} - \frac{\sqrt{3}}{3})\\
=&\frac{5}{2} + \frac{5\sqrt{3}}{6}\\
=&\boxed{\frac{15 + 5\sqrt{3}}{6}}
\end{aligned}$$