MATH SOCIETY

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Math Journal

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Preface

Words from our principal

Mathematics lover,

Someone told me that this year is our 144th anniversary, which is not very special! To me, this is not true! 144 is a perfect square, it is special! The total number of positive factors of a perfect square must be odd! Do you know why? Can you find a number other than perfect squares that has the same property?

How many positive factors are there for 144?

Give you a hint $144 = 2^4 \times 3^2$,

Oh, it should be 5x3 =15. Do you know why?

So 144th anniversary is indeed very special to all of us!

Mr. Perrick Ching
SJC

It is always a good habit to take notes in your learning process.

Words from our author

After many years showing myself to be probably the nerdiest in my class when it comes to math, I encountered this problem the most frequently: Why do we even study math?

This results in headbanging for most of the cases. "You don't need calculus and complex analysis when shopping for the groceries", they said. To me, it is not the applications of mathematics that matters, but rather the beauties of the mathematics behind. If your life goal is to buy groceries in the market, and society is still forcing you to do hundreds of math problems and reciting trigonometric formulas in your head, then they are wasting such a diligent youth! But I suppose you weren't that kind of people and instead aims for high-standard education and job. The truth is, most of the subjects you study in university, including physics, engineering and sometimes even biomedical sciences. All the while math is the bedrock of all the knowledge that flourishes on top.

"Math is demanding and unsatisfying", they said. Every time I put this square □ next to a proof (which is a synonym for "this completes the proof"), it is so satisfying, as if you put in the last piece to a gigantic puzzle with a several thousand pieces. You also need to find the equations that you need slowly among thousands of research papers and then construct them together to make it a proof. This process is undoubtedly arduous to say at least, but it is just like an artist wanting to demonstrate his artistic ideas but unwilling to spend hours to draw. Math has to be demanding, and I know that I can't force you to enjoy math. But, next time when you solve a math problem, try to generalise patterns and ways to take care of them so to make your life easier. After all, math is all about sorting patterns and solving puzzles like these.

So why should we study math? My answer is: Why not? Famous mathematician Godel tells us an essential rule to math, as well as a vital philosophy on life. The number of numbers is infinite that we could not try every case of an unproved statement to prove it, as well as we could not hate every abstract math problems and calculus textbooks to make studying math meaningful. Only when you try to learn its rules and actually try to solve the puzzle will you be able to prove any conjecture and study any math in a meaningful way.

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Chapter 1: Subtract infinity from infinity

Please subtract infinity from infinity. What do you get? The most straightforward, instinctive answer would be zero. However, it could be π. It could also be -1. In fact, with some clever exploits of infinite sums of sequences, you can get any anything you want.

Introduction: the "paradox"

Bernhard Riemann (1826-1866) is responsible for numerous headaches in the world of mathematics. One of the most being the Riemann Hypothesis. When he was alive, he thought the hypothesis could be easily proven, and so he put it aside and left it unsolved for 150 years until now.

We would introduce one of the most easily understandable theorems by Riemann. This theorem is called Riemann Rearrangement Theorem. It states that if an infinite series of real numbers is conditionally convergent, then we could manipulate it to whatever we want.

But you would ask: what on earth is "conditionally convergent"? In simple terms, the sum of the series exists but if you take the absolute value of every term, it does not.

Let me show you an example:

$$A = 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} \dots$$

The series A clearly adds to 1. However, if we take out all the positive terms, or the negative terms, the resultant sequence goes to infinity.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots = \infty$ $\qquad$ $-\frac{1}{2} - \frac{1}{3} - \frac{1}{4} \dots = -\infty$

Now if we add both sequences, it would be $\infty - \infty$. I will show you in a moment how the sum could be any number. This seems paradoxical doesn't it?

Adding and subtracting in a controlled manner

π is arguably one of the most important mathematical constant right? Let's try and get π first.

First, we will start at 1. π is clearly larger than our 1. So, we will take the left sum, and add some of the terms to it.

$$1 + \frac{1}{2} = 1.5$$

This is still less than π. Let's add some more.

$$1 + \frac{1}{2} + \frac{1}{3} = 1.8333 \dots$$

Still less than our favorite dessert constant. But no worries. We know that the sum goes to infinity, so we would eventually get there right?

Sadly, this would take a while.

$$1 + \frac{1}{2} + \cdots + \frac{1}{13} = 3.1801 \dots$$

Finally, we got something larger than π. Let's add the negative terms from the right-hand side now.

$$1 + \frac{1}{2} + \cdots + \frac{1}{13} - \frac{1}{2} = 2.6801 \dots$$

Smaller than π. Let's add the positive terms again.

$$1+\frac{1}{2}+\cdots+\frac{1}{13}-\frac{1}{2}+\frac{1}{14}+\cdots+\frac{1}{21}=3.1453\ldots=S$$

We are officially two decimal places accurate now. I hope you see where this is going, but let's just do it one more time. Also, I will call the above sum S, just for simplicity.

$$S-\frac{1}{3}=2.8120\ldots$$

Smaller than π. Fantastic. Let's add some more positive terms.

$$S-\frac{1}{3}+\frac{1}{22}+\cdots+\frac{1}{30}=3.16\ldots$$

Note that every term of our sum still comes from A. After infinitely many times of manipulation like this, we could get to exactly π. In fact, we could get any constants we want. Isn't that amazing?

Chapter 2: Are there infinitely many primes

Short answer: yes. A mathematician, however, won't believe it until it is proven true. As complicated as this problem might seem, it was already solved two thousand years ago by Euclid.

Introduction: The Elements by Euclid

Euclid was a great mathematician and philosopher back in his era, that we know surprisingly little about, but whose work had an indescribable impact on mathematics, even until today. For almost 2000 years, this book would stand as the pinnacle of logical rigor and human achievements, standing as the second-most republished book after the Bible. It is arguably the root of all modern mathematics. Geometry, algebra and even prototypes of calculus are found in this book.

However, what makes this book truly amazing is that it only included 5 postulates, based on common sense or proven theories. Then, everything else in the book was based on these postulates, demonstrating how far we could achieve by these simple rules. They are listed on the next page.

I would add a quick note here, that the 5^{th} postulate was neither proven nor disproven. In fact, later mathematicians, like Gauss and Bolyai would have discovered an entirely new branch of geometry called "non-Euclidian geometry", where the 5^{th} postulate does not hold.

1. A straight line segment can be drawn joining any two points.

2. Any straight line segment can be extended indefinitely in a straight line.

3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

4. All right angles are congruent.

5. If two lines intersects a third line in a way that the sum of the interior angles does not equal to 180 degrees, then, extended enough, it must meet at some point. (Postulate of parallel lines)

I hope you can see why the 5th postulate is so troublesome. It “assumed” that geometry all take place on a perfectly flat surface, in which the postulate would hold. However, on a curved surface, the postulate would not hold in every case. So, mathematician called this idealized geometry the “Euclidian geometry” and those where the postulate does not hold the “non-Euclidian geometry”.

Let’s go back to prime numbers. The proof is proposition 20 in The Elements, in which Euclid set up a proof by contradiction by first assuming otherwise, then showing which would lead to an obviously wrong statement, thus showing that the assumption itself is wrong and finally proving the statement.

Proof by contradiction

First, we will assume that there are only finitely many prime numbers. Let's say that there are precisely n prime numbers. We will label all the prime numbers as $p_1, p_2, p_3 \dots p_n$.

We will now do the following:

$$p_1 \times p_2 \times p_3 \times \dots \times p_n = some\ number$$

Clearly this "some number" is the product of all primes, and it should contain all primes as a factor.

Now let's add 1 to the number.

Since the smallest prime number is 2, we know that "Some number + 1", when dividing to any prime number, would have a remainder 1. Since every composite number could be written as a product of some prime numbers (i.e. has a unique prime factorization), then it should not divide any composite numbers as well. "Some number + 1" should be prime.

But we assumed that there are only $p_1, p_2, p_3 \dots p_n$ that are prime numbers. Clearly "Some number + 1" is bigger than all of those and would not equal to any of them.

We arrived a contradiction. So, we can conclude that the assumption is wrong and there are infinitely many primes.

□

Chapter 3: Why does $e^{i\pi} + 1 = 0$?

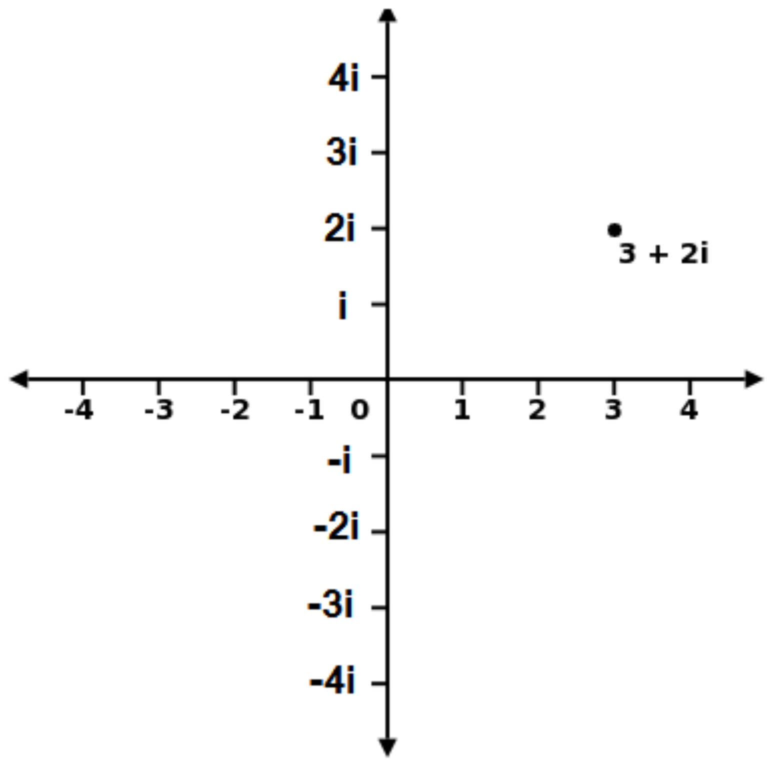
$e^{i\pi} + 1 = 0$ is perhaps the most well-known equation in the world of mathematics. The sense of elegance comes to you when you could see 5 of the most important mathematical constants could come together into a simple equation like this.

Introduction: the historical sketch

Four and a half centuries have passed since the first discoveries of complex numbers. As you might have already known, the term "complex number" refers to some entity of the form $a + bi$, where a and b are ordinary real numbers. i, on the other hand, is unlike any other real number and has the property of $i^2 = -1$. Also, don't be scared by the term "complex number". Mathematicians are just bad at naming things.

The number i is first introduced in 1702 by Leibniz. It was not well received at that time, in fact being described as "the amphibian between existence and not". Even in 1770 the situation was still confusing enough for famous mathematician as Euler to argue that $\sqrt{-2}\sqrt{-3} = \sqrt{6}$

I assume that you are all familiar with the number line. In late 18th century, famous mathematician Gauss found where the number i should lie on the number line, or at least in part. He argued that we should think of the number line as a number "plane" instead. The number i, and every other multiple of i, would lie on the new axis we have created. This way, every complex number would be represented by points on the new “complex plane” we have created.



Connection between complex plane and the formula

First, I must tell you that in fact Euler discovered a more general form of the identity:

$$e^{ix} = cosx + i\ sinx$$

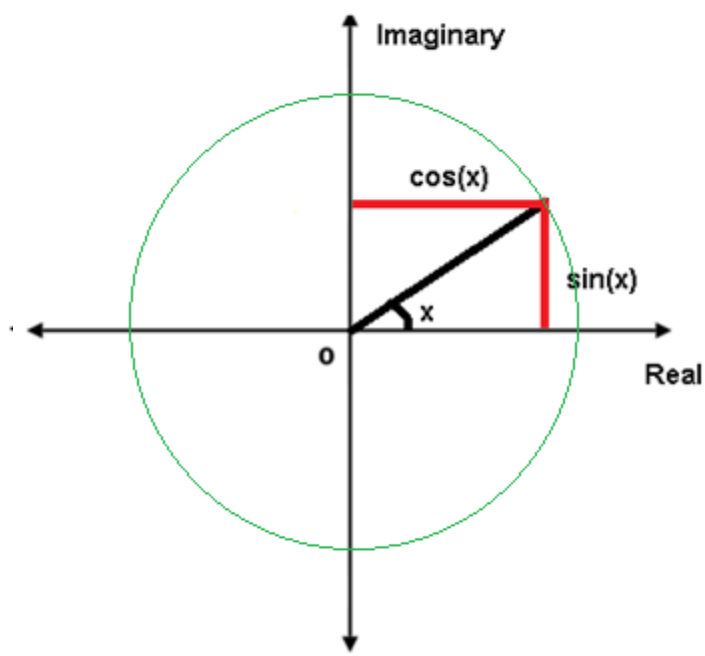
Before we look at how complex numbers and our ingenious formula would correlate, we should perhaps first look at the what every part of the formula meant.

e: It is defined as $(1+\frac{1}{n})^n$ as n approaches infinity. The most important property is that the function $y = e^x$ shares the same value with its area between the curve and the x-axis at any given x value. It has the value of around 2.718.

i: It has the property of $i^2 = -1$

π: the ratio between the diameter of a circle and its circumference. It has the value of around 3.142. In fact, π is proven to be neither possible to be written as a ratio between two rational numbers, nor is it a solution to any polynomial with rational coefficients. We will discuss this in chapter 6.

Now let's set up a unit circle around 0 on the complex plane. We will choose a point on the unit circle.



It is obvious now – $cos(x)$ is the real part of the number and $sin(x)$ is the imaginary part of the function. Scale it by the magnitude r (the distance between the origin and the point) of the number, then you can get any complex number.

Since $\cos(x) + isin(x) = e^{ix}$ we can multiply by r on both sides. This means we can represent any complex number in the form of re^{ix}! Isn't that amazing?

Proof of the identity

Now let's proof the identity:

Let $f(x) = \frac{cosx+i\ sinx}{e^{ix}}$

Then by quotient rule,

$$f'(x) = \frac{-e^{ix}\sin(x) + ie^{ix}\cos(x) - ie^{ix}\cos(x) + e^{ix}\sin(x)}{(e^{ix})^2}$$

We can see that all the terms cancel out. So, the numerator would be 0, which means $f'(x) = 0$

Now,

$$\int f'(x)\,dx = f(x) = C$$

Let $x = 0$

$$f(0) = \frac{cosx + i\ sinx}{e^{ix}} = 1 = C$$

$f(x) = \frac{cosx+i\ sinx}{e^{ix}} = 1$ implies that $\cos(x) + i\sin(x) = e^{ix}$ $\square$

Chapter 4: The sum of all inverse squares

Let me tell you that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\sum_{n=0}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

"Wait… I think you made a mistake. π is supposed to be the ratio of diameters and circumferences. How does it show up here? How is it squared? There was no circle…" You made a totally valid point. In fact, there is a circle hiding somewhere in this formula. But let us look at the background info before we get started.

History of the notorious problem

This problem is also known as the Basel problem. Proposed in 1644 by Mengoli, it was solved in 1735 by Euler. Originally, this problem brought so much headache to even the top mathematicians back in the days. So, when Euler solved the problem, it brought him immediate fame.

Later the Riemann Hypothesis was announced, which is not solved until now. In fact, anyone who can prove that there are infinitely many non-trivial roots can earn $1 million (in US dollars) reward. The Riemann Hypothesis introduces the zeta function $\zeta(z)=\frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\cdots$. Although Riemann was more intrigued by the complex inputs of the function, $\zeta(2)$ turns out to be the same sum the Basel Problem introduced. Furthermore, it led to a generalization of values of $\zeta(2),\ \zeta(4)$ etc.

Proof of the problem

By the Parseval's identity:

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

Where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Now, put in $f(x) = x$, we get

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx$$

Where

$c_n = \frac{n\pi \cos(n\pi) - \sin(n\pi)}{n^2 \pi} = \frac{\cos(n\pi)}{n} i = \frac{(-1)^n}{n} i$

for $n \neq 0$ and $c_0 = 0$. We can conclude that

$$|c_n|^2 = \begin{cases} \frac{1}{n^2} \; for \; n \neq 0 \\ 0 \; for \; n = 0 \end{cases}$$

And,

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{-1} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} + |c_0|^2$$

Since $\frac{1}{n^2}$ is an even function, we can conclude that,

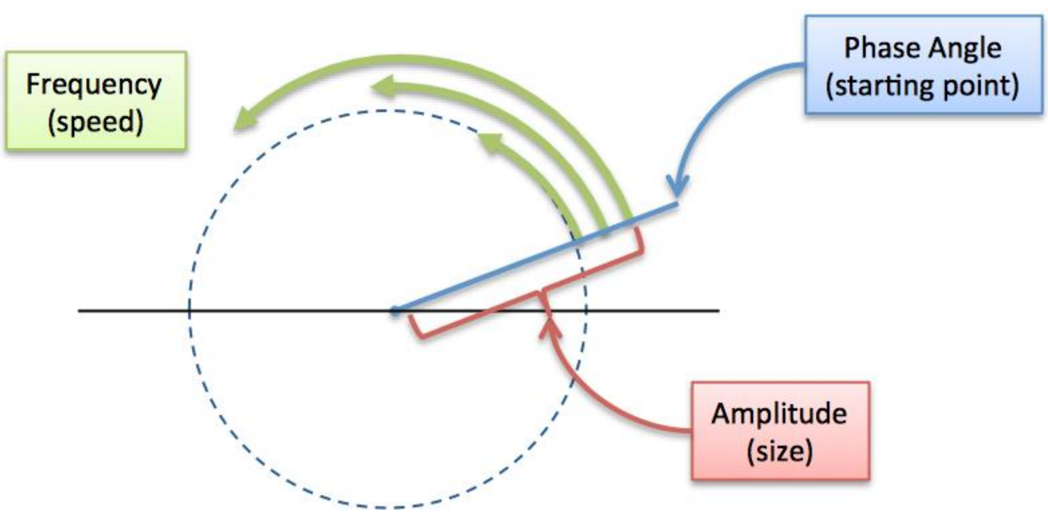
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{6}$$

□

Then… where is the circle?

First, I must confess that the proof I provided above was not from Euler. In fact, Euler used Taylor expansions of $\sin(x)$ and compare it for coefficients for every degree. But please do look at the hidden circle in this proof. Originally, $f(x)$ can only be used for periodic functions of 2π. We can, however, force the function to go around the unit circle, as it has a circumference of 2π. Now take a look at c_n again. The e^{-inx} part, if you could relate that to Euler's identity mentioned in Chapter 3, does exactly it.

The method used above is called Fourier Transformation. Applied in many different aspects such as sound wave editing, it is a more advanced topic of Complex Analysis, whereas Euler's identity, $\cos(x) + i\sin(x) = e^{ix}$, is arguably the most important building block of Complex Analysis.



The basic idea of Fourier Transform

Chapter 5: Trigonometric Functions in the complex world

First, let's refresh our memory on the definition of trigonometric functions. The following would be the Taylor expansion of sin and cos, and we will define the rest of the functions in terms of sin and cos.

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \sec(z) = \frac{1}{\cos(z)}$$

$$\csc(z) = \frac{1}{\sin(z)} \cot(z) = \frac{1}{\tan(z)}$$

Now, by changing the terms of the Euler's identity mentioned in Chapter 1, which states that $\cos(x) + i\sin(x) = e^{ix}$, we can see that:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

This introduces us the hyperbolic trigonometric functions, which their relationship is stated by the following identities:

$$\sin(z) = -isinh(iz)$$

$$\cos(z) = cosh(iz)$$

Trigonometry beyond the boundaries.

You might ask: How is this useful? Normally we would think of the sine function bounded by 1 and -1. Every input would lie in between. However, with our newfound knowledge, we could make up solutions such that $\sin(x)$ equals to any number!

Let's say we will solve for $\sin(x) = 3$.

First, the identity tells us that:

$$\frac{e^{iz} - e^{-iz}}{2i} = 3$$

We can see that:

$$e^{iz} - e^{-iz} = 6i$$

We will multiply e^{iz} on both sides and rearrange the term:

$$e^{2iz} - 6ie^{iz} - 1 = 0$$

Apply the quadratic formula:

$$e^{iz} = 3 + \sqrt{10}\ or\ 3 - \sqrt{10}$$

Take the natural log on both sides and divide the answer by i.

Note that $-i = \frac{1}{i}$

$$z = -\log\left(3 \pm \sqrt{10}\right) i$$

If you were aware of the whole process, you might notice that I am attempting to take the natural log of a negative number. This shall be left as an exercise for the reader. (Hint: Since $e^{i\pi} = -1$ we can say $\log(-1) = i\pi$.)

Chapter 6: Transcendental numbers and π

Before we transcend into knowing the transcendence of numbers, let us look at the irrationality of a number first. A number is rational if it could be written as a fraction, with both its numerator and denominator an integer. Otherwise, it would be an irrational number. 3 is a rational number as well as 0.8391. They could be written as $\frac{3}{1}$ and $\frac{8391}{10000}$. Meanwhile, $\sqrt{2}$ or $\log(3)$ would be an irrational number. We will set up a proof by contradiction, which we mentioned in Chapter 4, on the irrationality of $\log(3)$ later.

Now, let's transcend in to one higher level of the hierarchy of numbers.

Established without an example

We define algebraic numbers as numbers that could be solutions to polynomials with integer exponents and rational coefficients. For instance: $f(x) = x^2 + 3$ or $f(x) = 6x^{3742} + 216x^{57} - 59$.

Both would be valid polynomials. The solutions of $f(x) = 0$ to these would all be algebraic numbers. Transcendental numbers are essentially non-algebraic numbers. They could not exist as solutions to any polynomials that satisfies the requirements.

In 1768, Lambert proved π is irrational, and conjectured that it would be transcendental. However, during the time when Lambert published his paper, transcendental numbers were not even known that if they exist or not. This was until 1844, when Liouville proved the existence of transcendental numbers.

Ironically, he could not give any example of transcendental numbers until 1851, when he invented his own constant:

$$0.110001000000000000000000100\ldots$$

It contains 1 only at 1! = 1, 2! = 2, 3! = 6… and other digits would be zero. In 1873, Hermite proved that e is transcendental, and shortly after, Lindenmann proved that transcendence of π based largely on Hermite's work.

Since Lindenmann's proof needed to first prove that e and any of its rational powers is transcendental, I would instead provide a simple proof of my own. We will not get into very technical stuff of the proof right here, but I would provide an outline of what the proof looks like. It won't be very strict and rigorous.

Proof with the Basel Problem

We will assume π is a solution to some polynomial.

Since we could convert our Basel Problem infinite sum thing (see chapter 2) into a continued fraction:

$$\cfrac{\frac{1}{2^2}}{1-\frac{1}{2^2}-\cfrac{\frac{2^2-1}{3^2}}{1+\frac{2^2-1}{3^2}-\cfrac{\frac{3^2-1}{5^2}}{1+\frac{3^2-1}{5^2}-\cfrac{\frac{5^2-1}{7^2}}{\cdots}}}}$$

Due to the distribution of primes is generally decreasing (*), $\frac{{p_n}^2-1}{{p_{n+1}}^2}$ will decrease as n increases. This means the partial sums of our continued fraction will tend to 1. However, the continued fraction $1 + \frac{1}{1+\frac{1}{1+\cdots}}$ equals to $\frac{\sqrt{5}+1}{2}$ (It's the golden ratio), which is obviously algebraic. We can see that if our sequence of partial quotient (the 1+something term in every layer of the continued fraction) would be strictly 1, then at somewhere at the k-th layer of the infinite fraction we could write the k+1-th terms onwards as some algebraic number. Then, our result would be algebraic.

However, since $\frac{{p_n}^2-1}{{p_{n+1}}^2}$ only gets arbitrarily close to 0 as n increases, and algebraic numbers are countably infinite, they are only countable points of number on the number line. This implies that we could only get arbitrarily close to some algebraic number, which implies $\frac{\pi^2}{6}$ is not algebraic, but real, i.e. transcendental.

And since $\frac{x^2}{6}$ is a polynomial, for any algebraic input x it would have an algebraic output. This contradicts our result thus π itself must also be transcendental.

□

(*) – Actually, there is a theorem called PNT, which states that the distribution of primes around x tends to $\frac{x}{\ln x}$.

Chapter 7: Let's hack integration by doing differentiation

This one is a lot of fun, so let's jump straight in. The following is commonly called the Feynman technique, and normally people would think that the famous physicist Richard Feynman derived it.

This technique works on some definite integrals which antiderivative could not be expressed as an elementary function. We would introduce one more variable, usually denoted b, then we would differentiate it with respect to b, then do the integration part with respect to x, finally integrating the resultant function with respect to b and use some basic algebra to find the value of C we added after the indefinite integral.

Suppose we would like to integrate the following:

$$\int_0^\infty \frac{\sin(x)}{x} dx$$

Please do pause and ponder here: any ordinary tricks won't work. Now we will introduce our term "b" in the exponential function. We want this because we know that when we differentiate the function with respect to b, x will be a constant and by the chain rule we can cancel out the x.

$$I(b) = \int_0^\infty \frac{\sin(x)}{x} e^{-bx} dx$$

The reason we introduce a negative sign in e^{-bx} is that we want it to be convergent, which you would see why in a moment. Now, we would differentiate it:

$$I'(b) = \int_0^\infty \frac{\partial}{\partial b}(\frac{\sin(x)}{x}e^{-bx})dx = -\int_0^\infty \sin(x)\, e^{-bx}dx$$

Now we can integrate the expression with respect to x. This is a standard integration by part and we shall leave it to the reader as an exercise. We will show the answer now:

$$I'(b) = -\frac{1}{b^2+1}[-e^{-bx}(\cos(x) + bsin(x)]\begin{matrix}\infty\\0\end{matrix} = -\frac{1}{b^2+1}$$

Now you see why we put the negative sign on the exponential. If we didn't do so, the e^{bx} term would explode to infinity, which we don't want.

Now we integrate with respect to b, and we know it is the inverse tangent function $\arctan(b)$,

$$I(b) = \int_0^\infty \frac{\sin(x)}{x}e^{-bx}dx = -\arctan(b) + C$$

Put in $b \to \infty$ we get C = $\frac{\pi}{2}$

Now, we put in $b = 0$ and we get the following:

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$$

Isn't that wonderful?

Chapter 8: Factorials and the Gaussian integral

I am sure that if you made your way until here, you would most likely be familiar with what the factorial is. Denoted with the ! sign, it is the product of all integers below some number. Let's take 4 as an example: $4! = 4 \times 3 \times 2 \times 1$.

Now we will take a look at a topic that is seemingly unrelated: Integrals. Consider the following:

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

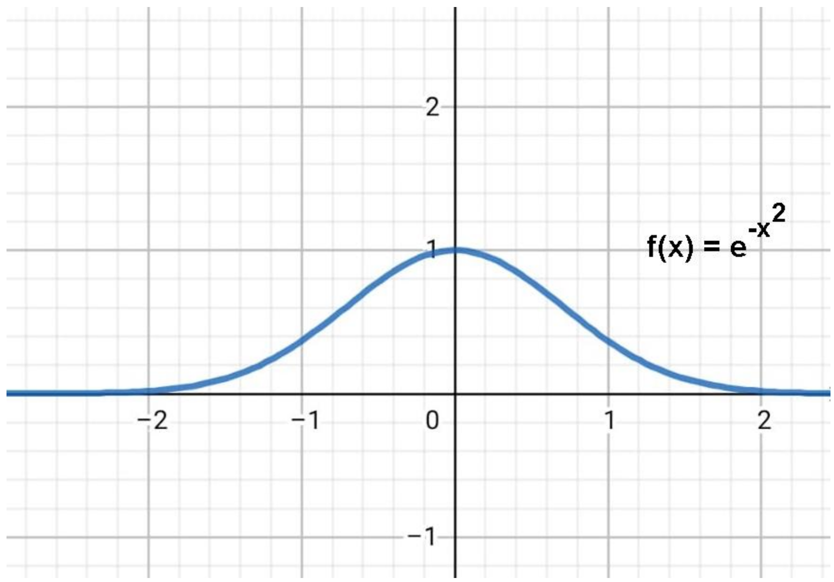
Seems easy? Try it and see what answer you would get. In fact, this thing equals $\sqrt{\pi}$. And, in the next pages, I will show you the elegant correlation between the factorials and this integral.

Introduction: the surprisingly wide applications

Let's talk about the factorials first. Normally, factorials show up in these two places: Probability and Taylor Expansions. In probability, factorials serve as a shorthand to write long expressions as a shorter one. It is used mostly to compute combinations and permutations. Meanwhile, Taylor expansions made the factorials with the power rule in differentiation. Of which the term n appears every time you differentiate a x^n term. Then if the curve is differentiated to 0, then you get a factor of n!.

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''^{(0)}x + \frac{1}{3!}f'''(0)x\ldots$$

On the other hand, the Gaussian integral is mostly useful in statistics and data analysis. Moivre discovered that integrals that would share some properties with the Gaussian integral would exist, while Gauss published this precise integral in 1809. When integrated, it could not be expressed by any elementary function (standard notation). Instead, mathematicians called the result the "Error Function", notated as $\text{erf}(x)$.



Well now we have a definition:

$$\sqrt{\pi}\,\text{erf}(x) = \int_{-x}^{x} e^{-x^2}\,dx$$

Here are some special values for $\text{erf}(x)$

When $x = 0$, $\text{erf}(x) = 0$

When $x \to \infty$, $\text{erf}(x) = 1$

When $x \to -\infty$, $\text{erf}(x) = -1$

This integral also relates to the cumulative distribution function of the normal distribution, which is arguably the most often case in any statistical results. This is also used in quantum mechanics to find probability density of ground state harmonic oscillators.

Their connection: the gamma function

Since e^{-x^2} is an even function, we can rewrite the Gaussian integral as the following:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_{0}^{\infty} e^{-x^2} dx$$

Then we will do the substitution $u = x^2$ and rearrange the terms:

$$\int_{0}^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

Let's take a look at the definition of Gamma function:

$$\Gamma(\mathrm{z}) = \int_{0}^{\infty} u^{-z} e^{-u} du$$

By comparison we can see that the Gaussian integral is $\Gamma\left(\frac{1}{2}\right)$.

Now, we will prove the relationship between gamma and factorials, in which we use a technique called induction. We first prove that for the smallest case that the statement holds true, then we prove that if the statement on some integer k holds true, then k+1 is also true. So if we know that if 1 is true, then 2 is true. If 2 is true, then 3 is true… so on and forth. But since we "proved" that 1 is true, we know that every integer is true.

Claim: $\Gamma(\mathrm{n}) = (\mathrm{n}-1)!$

$\Gamma(1) = 1 = (1-1)!$

We can see case 1 is true.

Now suppose case k is true, where k is a positive integer:

$$\Gamma(\mathrm{k}+1) = \int_0^{\infty} u^{-k-1} e^{-u} du$$

Using integration by parts we get:

$$\Gamma(\mathrm{k}+1) = \mathrm{k} \int_0^{\infty} u^{-k} e^{-u} du = k\,\Gamma(\mathrm{k})$$

Since we assumed case k is true, then we can do a substitution using our claim:

$\Gamma(\mathrm{k}+1) = k\,(\mathrm{k}-1)! = \mathrm{k}!$

We can see that case k+1 is true when case k is true.

By the principle of induction, our claim is true for any positive integer n.

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