Mathematics Society
Weekly Questions (Week 8 = 0)

February 10, 2024

Some of these are false, and some are valid. Can you tell them apart?

Proof 1 *Roots of unity*
Let's solve the equation

$$x^2 + x + 1 = 0$$

. Multiply both sides by $x - 1$ to get

$$(x-1)(x^2 + x + 1) = 0$$

$$x^3 - 1 = 0$$

$$x^3 = 1$$

The only solution is x = 1. Substitute into the original equation to get

$$0 = x^2 + x + 1 = 1^2 + 1 + 1 = 3$$

. Dividing both sides by 3, we get $1 = 0$.

Proof 2 *Horses*

This proof uses the principle of mathematical induction. If you don't know it, you can read this [1] document about induction, or ask a student of M2, or watch this video [2] . Then, you can come back to this proof.

We'll aim to prove this statement: For any positive integer n, all horses in any collection of n horses are the same color. (So, all horses are the same color.)

Since this is a statement about positive integers, we can use induction. Let's induct on n, starting with the base case.

Base case: n = 1. Clearly, if there is one horse, it is the same color as itself.

Inductive case: Assume there is some positive integer for which the statement holds. Call it k. Then the statement holds for k+1 too.

Suppose all horses in any collection of k horses are the same color.

Then, consider any arbitrary collection of k+1 horses. Put them in a line from the first to the last. Then the first k horses are the same color by our hypothesis, and the last k horses are the same color by our hypothesis. Since there are some horses who are both in the collection of the first k horses and the collection of the last k horses, all k+1 horses are the same color. This is what we wanted to prove.

By induction, since we proved the base case and the inductive case, our statement holds for all positive integers n. Therefore all horses are the same color.

[1] https://www.sydney.edu.au/content/dam/students/documents/mathematics-learning-centre/mathematical-induction.pdf

[2] https://youtu.be/GdM_iA1Zek4?si=XiDXB_6Mc2fz7c9J

Proof 3 *Students in 5C*
There are no students in 5C this year (due to there not being enough people for each class.) We just have 5A, 5B, 5D, 5E, and 5F. So, we can prove that all students in 5C are taller than 2 meters, completely rigorously.
Suppose, for the sake of contradiction, that not all the students in 5C are taller than 2m.
This is equivalent to saying that there is some student in 5C who is not taller than 2m.
So that student is a student in 5C.
But there are no students in 5C.
We have arrived at our contradiction. Therefore all students in 5C are taller than 2m.
To add to this statement, we notice that the statement "There are some students in 5C who are taller than 2m" is false, because it would imply there are some students in 5C, but there are none.

Conclusion Can you summarize the morals of each of the above proofs?