Solutions to Weekly Questions

Mathematics Society

1-12-2019

Remarks:

- Note that the marks allocated **is directly proportional** to the difficulty.
- Bonus questions are **not necessarily more difficult**.
- The solution will be released on the following Wednesday.

1. Prove by mathematical induction the following:

   (a) (1 point) $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

   Solution:

   Proof. The proof will be spilt into two parts. The first part is showing that the formula holds for 1, and the second part is showing that if the formula holds for any $k \in \mathbb{N}$, then it holds for $k + 1$ as well.
   We show that the formula holds for 1:

$$\begin{aligned} 1 &= \frac{1(1+1)}{2} \\ &= 1 \end{aligned}$$

   Then, we will prove the successor case:

$$\begin{aligned} 1 + 2 + 3 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+1)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

   Thus, we have shown that the formula holds for all natural numbers. $\square$

(b) (1 point) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution:

Proof. We will prove this also by induction.

$$\begin{aligned} 1^2 &= \frac{1(1+1)(2+1)}{6} \\ &= 1 \end{aligned}$$

Successor case:

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k^2+k+6k+1)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \end{aligned}$$

□

2. (2 points) Let $(G, \circ)$ be a group and let $(S_1, \circ)$ and $(S_2, \circ)$ be it's subgroups. Prove that $(S_1 \cap S_2, \circ)$ is a subgroup of $(G, \circ)$.

The conditions for being a subgroup (and a group) are listed below for convenience.
Let $(G, \circ)$ be a group. Assume S is a subset of G and $(S, \circ)$ is a group. Then $(S, \circ)$ is a subgroup of $(G, \circ)$.
The requirements for being a group are listed below. $(H, \circ)$ is a group (where $\circ$ is a binary operation on H) if and only if:

$$\begin{aligned}
&\text{(i).}\, \forall a, b \in H, a \circ b \in H. \\
&\text{(ii).}\, \forall a, b, c \in H, (a \circ b) \circ c = a \circ (b \circ c). \\
&\text{(iii).}\, \exists e \in H\, \forall a \in H, a \circ e = e = e \circ a. \\
&\text{(iv).}\, \forall a \in H\, \exists a', a \circ a' = a' \circ a = e.
\end{aligned}$$

Solution:

Proof. We will show that $(S_1 \cap S_2, \circ)$ satisfies the conditions for being a subgroup of $(G, \circ)$.

$S_1 \cap S_2$ is obviously a subset of G.
For every $a, b \in S_1 \cap S_2$, $a, b \in S_1$ and $a, b \in S_2$, thus by definition $a \circ b \in S_1$ and $a \circ b \in S_2$, therefore $a \circ b \in S_1 \cap S_2$.
For every $a, b, c \in S_1 \cap S_2$, $a, b, c \in S_1$ and $a, b, c \in S_2$, thus by definition $(a \circ b) \circ c = a \circ (b \circ c)$.
$e \in S_1$ and $e \in S_2$, therefore $e \in S_1 \cap S_2$.
For every $a \in S_1$, $a' \in S_1$ and similarly for any $a \in S_2$, $a' \in S_2$. Thus for any $a \in S_1 \cap S_2$, $a' \in S_1 \cap S_2$. $\square$

3. (3 points) Prove that the set of all real numbers $\mathbb{R}$ is uncountable.

Solution:

Proof. Assume that $\mathbb{R}$ is countable, and let $j_0, j_1, j_2, \ldots, j_n, \ldots n \in \mathbb{N}$ be an enumeration of $\mathbb{R}$. Let $a_0 = j_0$ and $y_0 = j_{s_0}$, where s_0 is the least s such that $a_0 < j_s$. For each n define $a_{n+1} = j_{q_n}$ where q_n is the least q such that $a_n < j_q < y_n$, and define $y_{n+1} = j_{s_n}$, where s_n is the least s such that $a_{n+1} < j_s < y_n$. Now let $a = \sup\{a_n \mid n \in \mathbb{N}\}$, it is obvious that $a \neq j_m$ for all m. Thus $\mathbb{R}$ is uncountable. $\square$