Mathematics Society
Weekly Questions (Week 2)

October 30, 2023

Question 1 (Puzzle) *Magnets*[1]

Parts a to c are intended to help you solve part d. You can ignore them if you like, or treat them as hints.

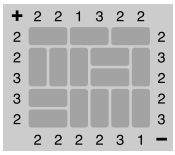
Fill the grid with magnets(or dominoes with a positive end and a negative end) and blank dominoes(which have neutral poles). Initially, you can only see their shapes. Around the grid, clues indicate how many positive and negative poles are in each row or column. Mark a square with "+" if it's a positive pole, "-" if it's a negative pole, and "x" if it belongs to one end of a domino. The correct solution must satisfy the following:

1. All the clues are satisfied, that is, the number of pluses/positive poles in each column matches the number above the column, and the number of minuses/negative poles in each column matches the number below the column. The same applies for each row.

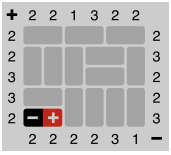
2. Like repels like. *(You can't place a positive pole right next to a positive pole(horizontally, or vertically), but you* can *place a positive pole diagonally next to a positive pole. The same applies for negative poles.)*

3. Neutral poles don't repel anything. You can place them next to anything.

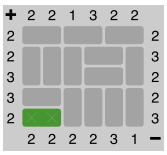
Puzzle Components:



A blank grid



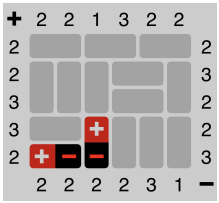
A magnet



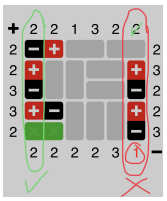
A neutral domino

[1]Origin: https://www.janko.at/Raetsel/Magnete/index.htm

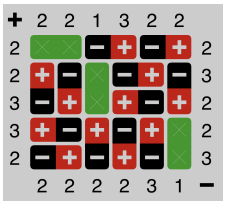
Question 1 Cont'd.
Examples:



Like repels like: putting 2 minuses together is not allowed.



How the numbers work



Correctly Solved

a If a column has 3 pluses, where must they be?

b If a column has 0 pluses or 0 minuses, what can you say about the vertical slots in it? Can you find a similar pattern for rows?

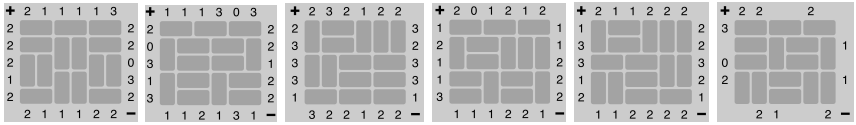
c If a column has two vertical slots and one half of a horizontal slot, what can you say about the horizontal slot if:

i the numbers at the top and at the bottom are equal

ii they differ by 1?

iii Prove that the numbers at the top and at the bottom cannot differ by more than 1.

d Using the above parts, can you solve the following 6 puzzles?



Question 2 (Logic Puzzle) *Execution*
After accidentally bringing your phone to the underground city where clocks are banned, you are caught by the government. This is their decision:

You are given 5 minutes to make a statement. If the statement is true, you are to be drowned; if the statement is false, you are to be hanged.

a What statement do you make to confuse your executioners?

b What statement do you make for the executioners to be able to choose whether or not you die by hanging or by drowning?

Question 3 (Olympiad) *Geometry*
Original question:

Let M be the midpoint of the side BC of a non-isosceles ΔABC, and let D be the foot of perpendicular from C to AM. The circumcircle of ΔABD intersects the side BC again at $E \neq B$. Suppose F is a point on the segment AE such that $FB = FC$. Prove that F is the midpoint of AE.

The following result may be useful in the proof.

Theorem 0.1. *Power of a point with respect to a circle*

a Let $ABCD$ be a cyclic quadrilateral. Suppose that AC and BD, the diagonals, intersect at point E. By considering similar triangles, one can prove that $EA \cdot EC = EB \cdot ED$.

b In geometry, $EA \cdot EC$ is denoted as (magnitude of) power of point E with respect to circle $ABCD$. Note that the power of the point **does NOT** depend on the choice of the line, i.e. any line passing through point E can be used for calculating power of E.

c You can use this result directly without proof.

The following sub-questions serve as guidance to the above question, if you have no idea on what's going on. You can treat these as hints.

Set $\mathbf{A} = (\mathbf{a}, \mathbf{b}), \mathbf{B} = (-\mathbf{c}, \mathbf{0}), \mathbf{C} = (\mathbf{c}, \mathbf{0})$
WLOG, $\mathbf{c > 0}, \mathbf{a < 0}$.

a By finding the slope of line AM, prove that $D = (d, \frac{bd}{a})$ for some d.

b Using the fact that $MD \perp DC$, express d in terms of a, b, c.

c Considering power of point E with respect to circle ABD, find the coordinate of E. Be careful of the positive and negative signs. (Note that the y-coordinate of E is 0.)

d Find the midpoint of AE and the perpendicular bisector of BC. Beware of your logic here.

e Conclude by explaining mathematically why the above step completes the proof.