Weekly Questions

Mathematics Society

November 17, 2020

Remarks:

- Note that the marks allocated **is proportional** to the difficulty.
- Bonus questions are **not necessarily more difficult**.
- The solution will be released on the following Wednesday.

1. (3 points) You have a calculator which only has the following buttons:

$\times 6$, $\times 10$ and $\times 35$	Multiplies result by 6, 10 or 35.
$\div 6$, $\div 10$ and $\div 35$	Divides result by 6, 10 or 35.
x^n	Raises the result to the power of any positive integer

Below is an example of how one might use the calculator, where "$A \xrightarrow{f} B$" represents the action of pressing button f to change the result from A to B.

$$60 \xrightarrow{\div 10} 6 \xrightarrow{\times 35} 210 \xrightarrow{\div 10} 21 \xrightarrow{x^2} 441$$

Starting with 60, can 72 be obtained on this calculator? If it can, what is the sequence of operations needed?

Solution: $\boxed{\text{72 cannot be obtained on this calculator.}}$
$72 = 2^3 \cdot 3^2$ has 5 prime factors.
$\times 6$, $\times 10$, $\times 35$, $\div 6$, $\div 10$ and $\div 35$ each add or remove 2 prime factors; Raising the result to a power of n simply multiplies the number of prime factors by n. The parity of the number of prime factors remains the same no matter what button is pressed.
Thus, starting from 60 ($= 2^2 \cdot 3 \cdot 5$) it is impossible to obtain 72.

2. (3 points) Suppose you have a strip of paper with length L cm and thickness t cm. If you fold the strip of paper in half in the same direction over and over, the paper will eventually be too thick to fold. Let n be the the number of possible folds. The relationship between L, t and n is shown below.

$$L = \frac{\pi t}{6}(2^n + 4)(2^n - 1)$$

Source: Guinness World Records

Make n the subject of the above formula.

Solution: Let $x = 2^n$.

$$\begin{aligned} L &= \frac{\pi t}{6}(x+4)(x-1) \\ \frac{6L}{\pi t} &= (x+4)(x-1) \\ x^2 + 3x - 4 &= \frac{6L}{\pi t} \\ x^2 + 3x &= \frac{6L}{\pi t} + 4 \end{aligned}$$

We complete the square by adding $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ on both sides:

$$\begin{aligned} x^2 + 3x + \frac{9}{4} &= \frac{6L}{\pi t} + 4 + \frac{9}{4} \\ \left(x + \frac{3}{2}\right)^2 &= \frac{6L}{\pi t} + \frac{25}{4} \\ \left(x + \frac{3}{2}\right)^2 &= \frac{24L + 25\pi t}{4\pi t} \end{aligned}$$

Since x is positive, we will only take the positive square root.

$$\begin{aligned} x + \frac{3}{2} &= \sqrt{\frac{24L + 25\pi t}{4\pi t}} \\ x &= \sqrt{\frac{24L + 25\pi t}{4\pi t}} - \frac{3}{2} \end{aligned}$$

Since $x = 2^n$, $\boxed{n = \log_2\left(\sqrt{\frac{24L + 25\pi t}{4\pi t}} - \frac{3}{2}\right)}$.

Other equivalent answers such as

$$\begin{aligned} n &= \log_2\left(\frac{1}{2} \cdot \left(\sqrt{\frac{24L + 25\pi t}{\pi t}} - 3\right)\right) \\ &= \log_2\left(\sqrt{\frac{24L + 25\pi t}{\pi t}} - 3\right) - 1 \end{aligned}$$

are also accepted.

3. (5 points) A point $P(x, y)$ is rotated θ radians anticlockwise around the origin to point P'. Express the coordinates of P' in terms of x, y and θ.

Solution: Suppose the point $P(x, y)$ is in the complex plane. It can thus be represented by the complex number $N = x + iy$. To rotate P by θ radians anticlockwise around the origin, we multiply N by $\cos\theta + i\sin\theta$ to get N':

$$\begin{aligned} N' &= (\cos\theta + i\sin\theta)N \\ &= (\cos\theta + i\sin\theta)(x + iy) \\ &= (\cos\theta + i\sin\theta)x + (\cos\theta + i\sin\theta)iy \\ &= x\cos\theta + ix\sin\theta + iy\cos\theta - y\sin\theta \\ &= x\cos\theta - y\sin\theta + (x\sin\theta + y\cos\theta)i \end{aligned}$$

Therefore $P' = (\mathrm{Re}(N'),\ \mathrm{Im}(N')) = \boxed{(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)}$

Alternatively, this can be done through linear algebra by multiplying the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ by the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, resulting in the new vector $\begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$.