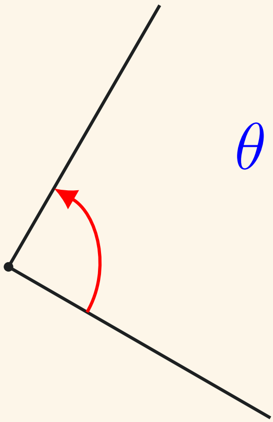
The Comprehensive List of

# References in Geometry

AN ILLUSTRATED MANUAL




**Isaac Li**

Mathematics Society

DECEMBER 2020

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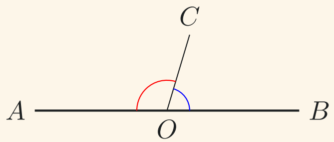
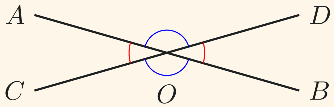
# Lines



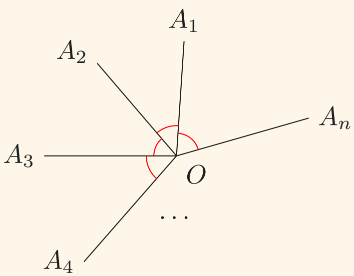

$\boxed{\textbf{\textit{Adj. }} \angle \textbf{\textit{s on st. line}}}$

$$\color{red}{\angle AOC} + \color{blue}{\angle COB} \color{black}{= 180^\circ.}$$




$\boxed{\textbf{\textit{Vert. oppo. }} \angle \textbf{\textit{s}}}$

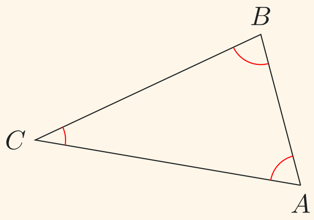
$$\color{red}{\angle AOC = DOB},$$
$$\color{blue}{\angle AOD = COB}.$$




$\boxed{\angle \textbf{\textit{s at a pt.}}}$

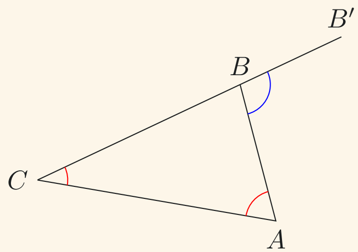
$$\color{red}{\angle A_1OA_2} + \color{red}{\angle A_2OA_3}$$
$$+ \cdots + \color{red}{\angle A_{n-1}OA_n} = 360^\circ.$$

# Triangles




**$\angle$ sum of $\triangle$s**

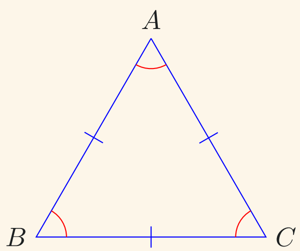
$$\angle CAB + \angle ABC + \angle BCA = 180^\circ.$$




**Ext. $\angle$ of $\triangle$s**

$$\angle CAB + \angle BCA = \angle B'BA.$$

## Equilateral Triangles

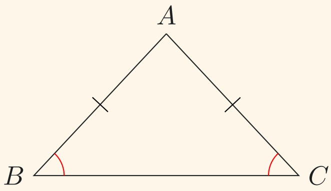



**Prop. of equil. $\triangle$s**

$$AB = BC = CA,$$

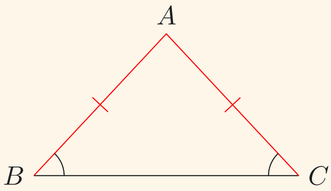
$$\angle CAB = \angle ABC = \angle BCA = 60^\circ.$$

## Isosceles Triangles



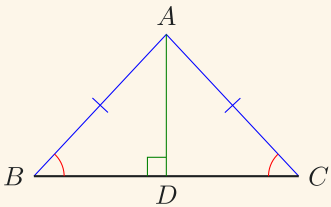

$\boxed{\textit{Base } \angle\textit{s, isos. } \triangle}$

$$AB = AC$$
$$\Downarrow$$
$$\angle ABC = \angle ACB.$$




$\boxed{\textit{Sides oppo. eq. } \angle\textit{s}}$

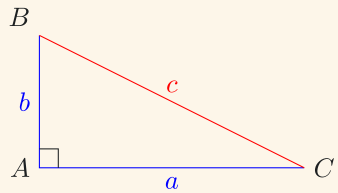
$$\angle ABC = \angle ACB$$
$$\Downarrow$$
$$AB = AC.$$




$\boxed{\textit{Prop. of isos. } \triangle\textit{s}}$

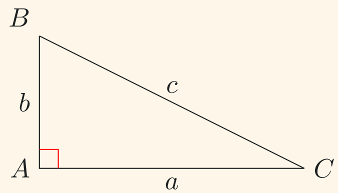
$$\angle ABC = \angle ACB$$
$$\Updownarrow$$
$$AB = AC$$
$$\Updownarrow$$
$$AD \perp BC.$$

## Pythagoras's Theorem




$\boxed{\textbf{\textit{Pyth. thrm.}}}$

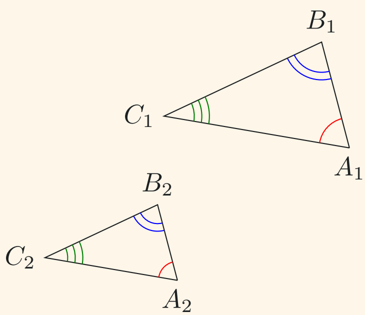
$$BA \perp AC$$
$$\Downarrow$$
$$a^2 + b^2 = c^2.$$




$\boxed{\textbf{\textit{Conv. of Pyth. thrm.}}}$

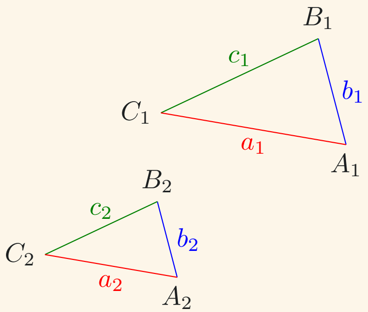
$$a^2 + b^2 = c^2$$
$$\Downarrow$$
$$BA \perp AC.$$

## Similar Triangles




$\boxed{\textbf{\textit{Corr. } }\angle \boldsymbol{s} \mid \sim \triangle \boldsymbol{s}}$

$$\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$$
$$\Downarrow$$
$$\angle C_1A_1B_1 = \angle C_2A_2B_2,$$
$$\angle A_1B_1C_1 = \angle A_2B_2C_2,$$
$$\angle B_1C_1A_1 = \angle B_2C_2A_2.$$



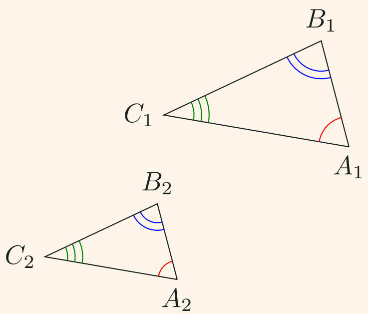

$\boxed{\textit{\textbf{Corr. sides}} \mid \sim \triangle \textit{\textbf{s}}}$

$$\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$$

$$\Downarrow$$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2},$$
$$\frac{b_1}{c_1} = \frac{b_2}{c_2},$$
$$\frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

## Conditions for Proving Similar Triangles



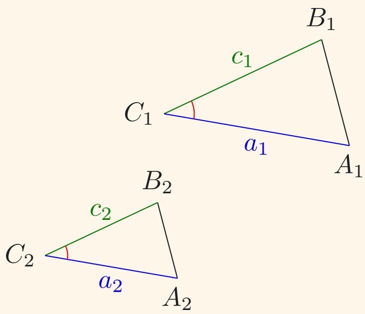

$\boxed{\textit{\textbf{AAA}}}$

$$\angle C_1A_1B_1 = \angle C_2A_2B_2,$$
$$\angle A_1B_1C_1 = \angle A_2B_2C_2,$$
$$\angle B_1C_1A_1 = \angle B_2C_2A_2$$

◀ Any two of the three is sufficient.

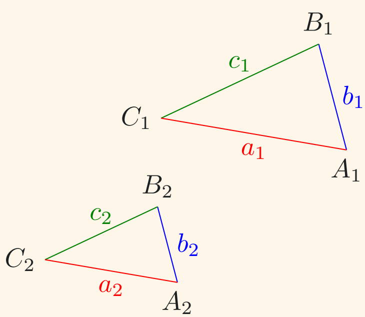
$$\Downarrow$$

$$\underline{\underline{\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2}}.$$




**Ratio of two sides, incl. $\angle$**

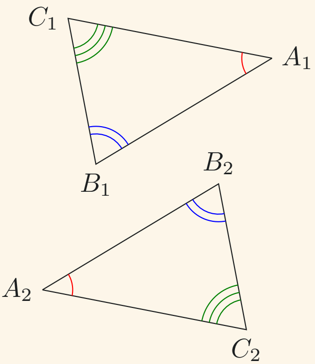
$$\frac{a_1}{c_1} = \frac{a_2}{c_2},$$
$$\angle B_1C_1A_1 = \angle B_2C_2A_2$$
$$\Downarrow$$
$$\underline{\underline{\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2.}}$$




**Three sides proportional**

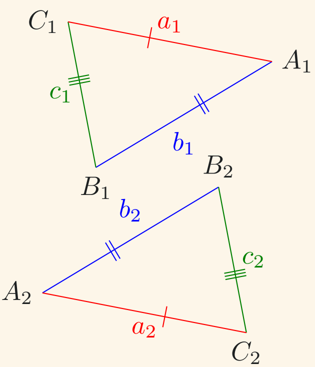
$$\frac{a_1}{b_1} = \frac{a_2}{b_2},$$
$$\frac{b_1}{c_1} = \frac{b_2}{c_2},$$
$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$
$$\Downarrow$$
$$\underline{\underline{\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2.}}$$

## Congruent Triangles




$$\boxed{\textbf{\textit{Corr. }} \angle \textbf{\textit{s}} \mid \cong \triangle \textbf{\textit{s}}}$$

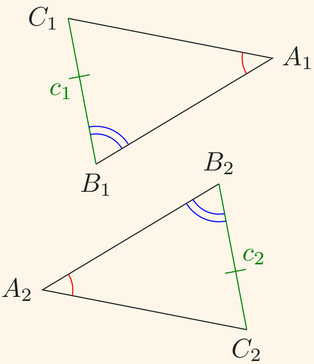
$$\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2$$
$$\Downarrow$$
$$\angle C_1A_1B_1 = \angle C_2A_2B_2,$$
$$\angle A_1B_1C_1 = \angle A_2B_2C_2,$$
$$\angle B_1C_1A_1 = \angle B_2C_2A_2.$$




$$\boxed{\textbf{\textit{Corr. sides}} \mid \cong \triangle \textbf{\textit{s}}}$$

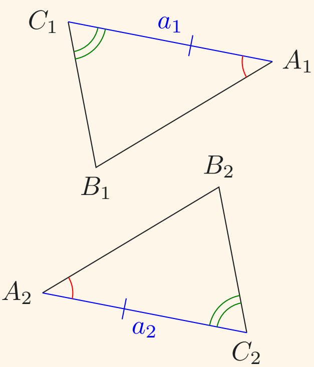
$$\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2$$
$$\Downarrow$$
$$a_1 = a_2,$$
$$b_1 = a_2,$$
$$c_1 = a_2.$$

## Conditions for Proving Congruent Triangles



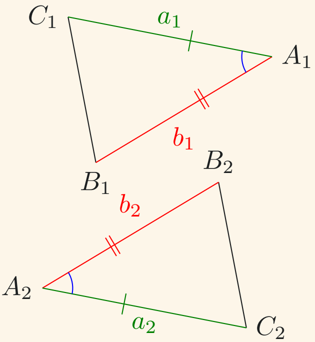

$\boxed{\textbf{\textit{AAS}}}$

$$\angle C_1A_1B_1 = \angle C_2A_2B_2,$$
$$\angle A_1B_1C_1 = \angle A_2B_2C_2,$$
$$c_1 = c_2$$
$$\Downarrow$$
$$\underline{\underline{\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2.}}$$



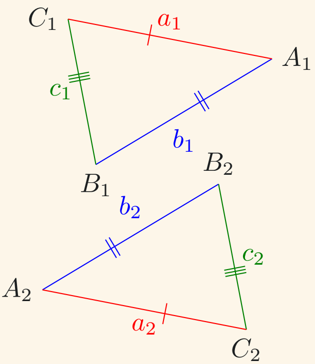

$\boxed{\textbf{\textit{ASA}}}$

$$\angle A_1B_1C_1 = \angle C_2A_2B_2,$$
$$a_1 = a_2,$$
$$\angle B_1C_1A_1 = \angle B_2C_2A_2,$$
$$\Downarrow$$
$$\underline{\underline{\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2.}}$$



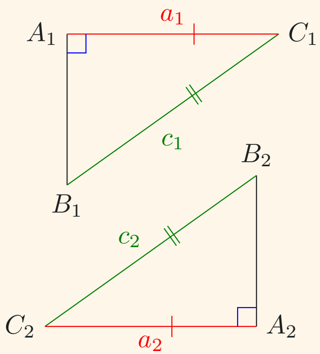

$\boxed{\boldsymbol{SAS}}$

$$a_1 = b_1,$$
$$\angle B_1A_1C_1 = \angle B_2A_2C_2,$$
$$b_1 = b_2$$
$$\Downarrow$$
$$\underline{\underline{\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2.}}$$




$\boxed{\boldsymbol{SSS}}$

$$a_1 = a_2,$$
$$b_1 = a_2,$$
$$c_1 = a_2$$
$$\Downarrow$$
$$\underline{\underline{\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2.}}$$

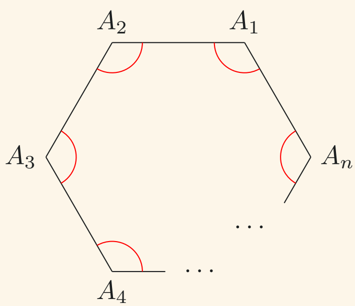



$\boxed{\textbf{\textit{RHS}}}$

$$\angle B_1A_1C_1 = \angle B_2A_2C_2$$
$$= 90^\circ,$$
$$c_1 = c_2,$$
$$a_1 = a_2$$
$$\Downarrow$$
$$\underline{\underline{\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2}}.$$

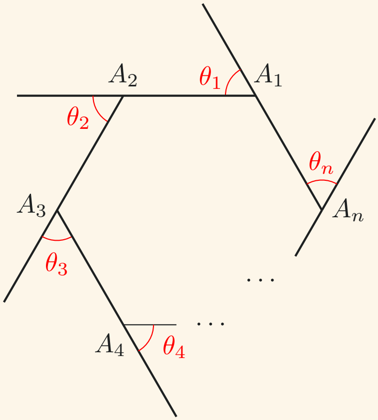
◀ One of the sides must be the hypotenuse.

# Polygons




$\boxed{\angle\ \textbf{\textit{sum of polygons}}}$

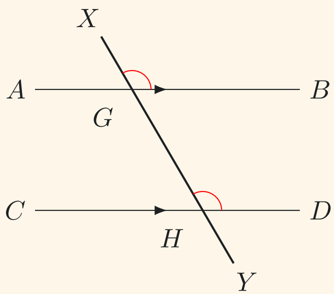
$$\angle A_nA_1A_2 + \angle A_1A_2A_3 + \cdots$$
$$+\angle A_{n-1}A_nA_1 = (n-2)\cdot 180^\circ.$$




***Ext.* $\angle$*s of polygons***

$$\theta_1 + \theta_2 + \cdots + \theta_n = 360^\circ.$$

# Parallel Lines

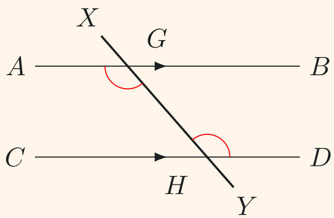



***Corr.* $\angle$*s* | $AB /\!/ CD$**

$$AB /\!/ CD$$

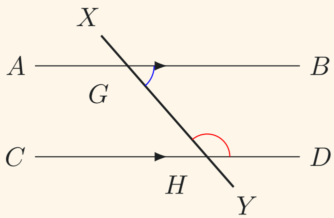
$$\Downarrow$$

$$\angle XGB = \angle XHD.$$




$\boxed{\textit{\textbf{Alt.}}\ \angle s \mid AB \mathbin{/\!/} CD}$

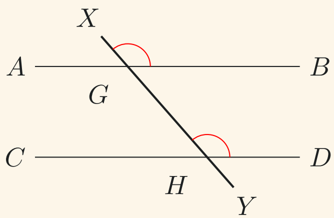
$$AB \mathbin{/\!/} CD$$
$$\Downarrow$$
$$\angle YGA = \angle XHD.$$




$\boxed{\textit{\textbf{Int.}}\ \angle s \mid AB \mathbin{/\!/} CD}$

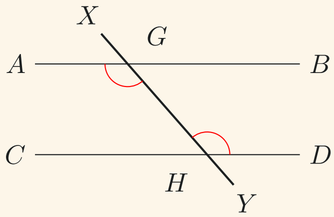
$$AB \mathbin{/\!/} CD$$
$$\Downarrow$$
$$\angle YGB + \angle XHD = 180^\circ.$$

## Conditions for Proving Parallelism



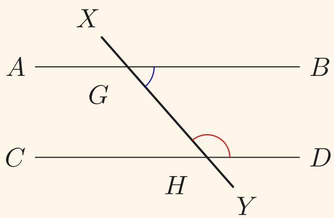

$\boxed{\textit{\textbf{Int.}}\ \angle s\ \textit{\textbf{eq.}}}$

$$\angle YGB + \angle XHD = 180^\circ$$
$$\Downarrow$$
$$\underline{\underline{AB \mathbin{/\!/} CD}}.$$




$\boxed{\textbf{\textit{Alt. }}\angle \textbf{\textit{s eq.}}}$

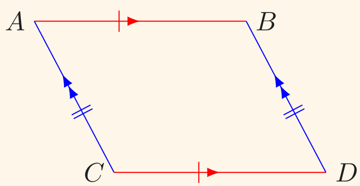
$$\angle YGA = \angle XHD$$
$$\Downarrow$$
$$\underline{\underline{AB \parallel CD}}.$$




$\boxed{\textbf{\textit{Int. }}\angle \textbf{\textit{s eq}}}$

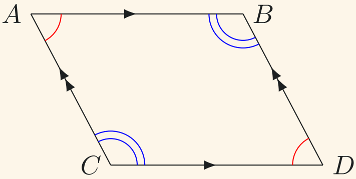
$$\angle YGB + \angle XHD = 180^\circ$$
$$\Downarrow$$
$$\underline{\underline{AB \parallel CD}}.$$

# Parallelograms



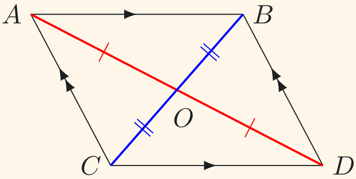

$\boxed{\textbf{\textit{Oppo. sides of }}\parallel\textbf{\textit{-gram}}}$

$$AB = CD,$$
$$AC = BD.$$




**Oppo. $\angle$s of $/\!/$-gram**

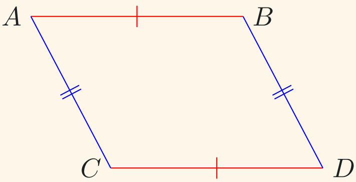
$\angle BAC = \angle BDC$,
$\angle ACD = \angle ABD$.




**Diags. of $/\!/$-gram**

$OA = OD$,
$OB = OC$.

## Conditions for Identifying Parallelograms

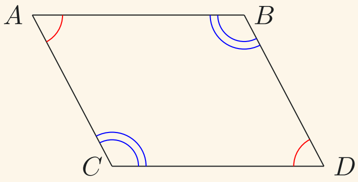



**Oppo. sides eq.**

$AB = CD$,
$AC = BD$

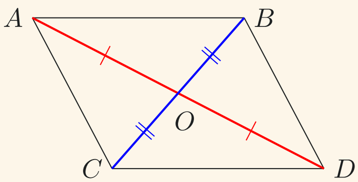
$\Downarrow$

$ABCD$ is a $/\!/$-gram.



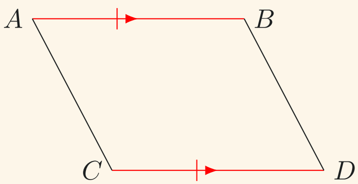

$\boxed{\textbf{\textit{Oppo. $\angle$s eq.}}}$

$$\angle BAC = \angle BDC,$$
$$\angle ACD = \angle ABD$$
$$\Downarrow$$
$$\underline{\underline{ABCD \text{ is a } /\!/\text{-gram.}}}$$




$\boxed{\textbf{\textit{Diags. bisect each other}}}$

$$OA = OD,$$
$$OB = OC$$
$$\Downarrow$$
$$\underline{\underline{ABCD \text{ is a } /\!/\text{-gram.}}}$$

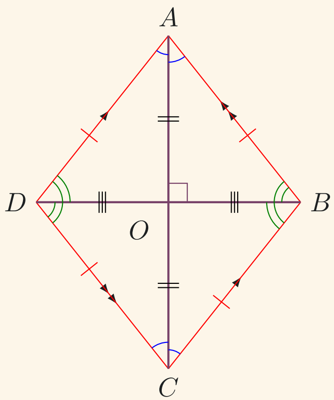



$\boxed{\textbf{\textit{Two sides eq. and $/\!/$}}}$

$$AB = CD,$$
$$AB \;/\!/\; BD$$
$$\Downarrow$$
$$\underline{\underline{ABCD \text{ is a } /\!/\text{-gram.}}}$$

# Other Quadrilaterals

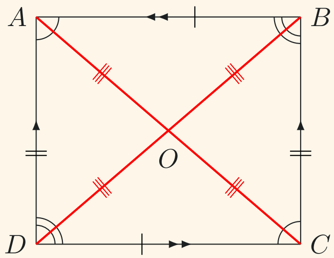
## Rhombus - a quadrilateral with four equal sides




$\boxed{\textbf{\textit{Prop. of rhombuses}}}$

$$AB \parallel CD \qquad AD \parallel BC,$$
$$OA = OC \qquad OB = OD,$$
$$AB = BC = CD = DA,$$
$$AC \perp BD,$$
$$\angle OAD = \angle OAB$$
$$= \angle OCD = \angle OCB,$$
$$\angle ODA = \angle ODC$$
$$= \angle OBA = \angle OBC.$$

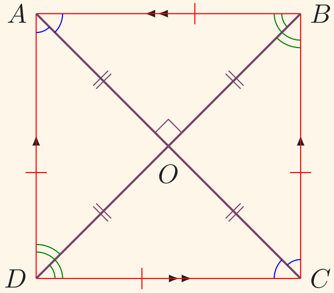
## Rectangle - a quadrilateral with four equal interior angles




$\boxed{\textbf{\textit{Prop. of rectangles}}}$

$$AB \parallel CD \qquad AD \parallel BC,$$
$$AB = CD \qquad AD = BC,$$
$$\angle BAD = \angle BCD,$$
$$\angle ABC = \angle ADC,$$
$$OA = OB = OC = OD.$$

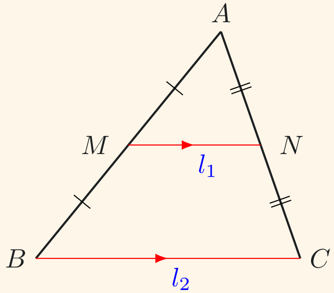
### Square - a quadrilateral with four equal sides and interior angles




**Prop. of squares**

$$AB \parallel CD \qquad AD \parallel BC,$$
$$AB = BC = CD = DA,$$
$$OA = OB = OC = OD,$$
$$AC \perp BD,$$
$$\angle OAD = \angle OAB = 90^\circ$$
$$= \angle OCD = \angle OCB,$$
$$\angle ODA = \angle ODC = 90^\circ$$
$$= \angle OBA = \angle OBC.$$

## Miscellaneous Results

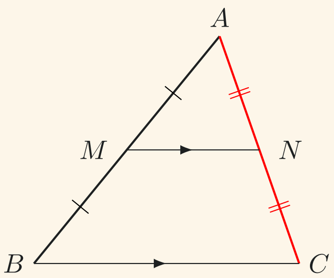



**Mid-pt. thrm.**

$$MA = MA \quad NA = NC$$

$$\Downarrow$$

$$2l_1 = l_2,$$
$$MN \parallel BC.$$

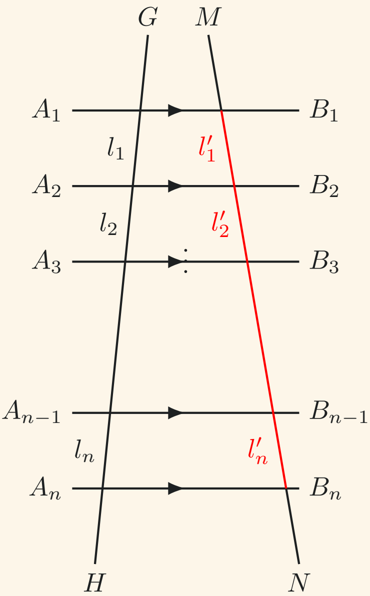



**Intercept thrm.** variant i

$$MA = MA \quad MN \parallel BC$$

$$\Downarrow$$

$$NA = NC.$$




**Intercept thrm.** variant ii

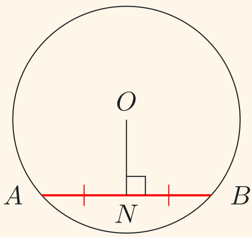
$$A_1B_1 \parallel A_2B_2 \parallel \cdots \parallel A_nB_n$$

$$l_1 = l_2 = \cdots = l_n$$

$$\Downarrow$$

$$l'_1 = l'_2 = \cdots = l'_n.$$

# Circles

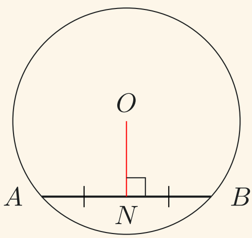



**Line from cen. $\perp$ chord bisects chord**

$$ON \perp AB$$

$$\Downarrow$$

$$NA = NB.$$

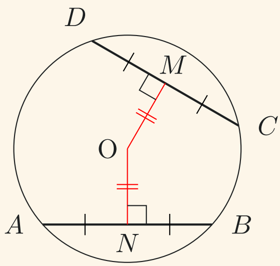



**Line joining cen. to mid-pt. of chord $\perp$ chord**

$$NA = NB$$

$$\Downarrow$$

$$ON \perp AB.$$



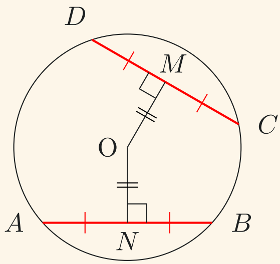

**_Eq. chords equidis. from cen._**

$$ON \perp AB \quad OM \perp CD,$$

$$AB = CD$$

$$\Downarrow$$

$$ON = OM.$$



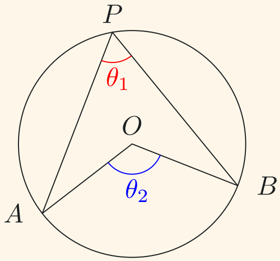

**_Chords equidis. from cen. are eq._**

$$ON \perp AB \quad OM \perp CD,$$

$$ON = OM$$

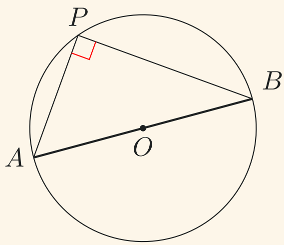
$$\Downarrow$$

$$AB = CD.$$




**$\angle$ at cen. twice $\angle$ at $\odot^{ce}$**

$$2\theta_1 = \theta_2.$$

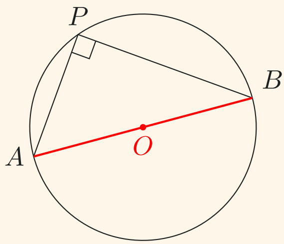



**$\angle$ in semi-circ.**

$AB$ is a diameter

$$\Downarrow$$

$$\angle APB = 90^\circ.$$

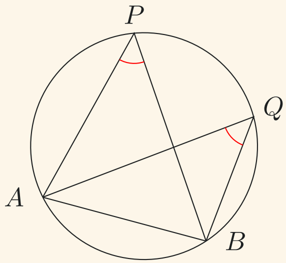



**Converse of $\angle$ in semi-circ.**

$$\angle APB = 90^\circ$$

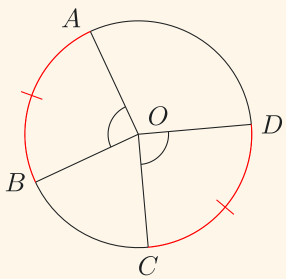
$$\Downarrow$$

$AB$ is a diameter.




**$\angle$s in the same segment**

$$\angle APB = \angle AQB.$$

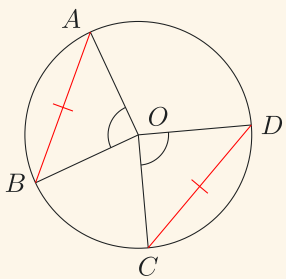



**Eq. $\angle$s, eq. arcs**

$$\angle AOB = \angle COD$$

$$\Downarrow$$

$$\overset{\frown}{AB} = \overset{\frown}{CD}.$$

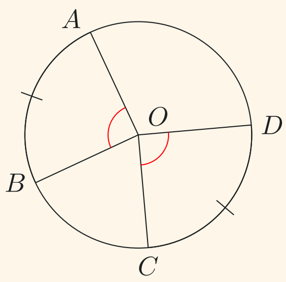



**Eq. $\angle$s, eq. chords**

$$\angle AOB = \angle COD$$

$$\Downarrow$$

$$AB = CD.$$

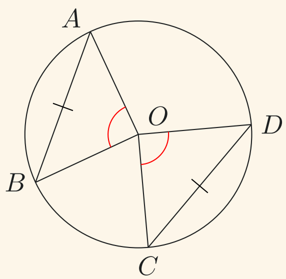



$\boxed{\textbf{\textit{Eq. arcs, eq. }}\angle\textbf{\textit{s}}}$

$$\overset{\frown}{AB} = \overset{\frown}{CD}$$

$$\Downarrow$$

$$\angle AOB = \angle COD.$$

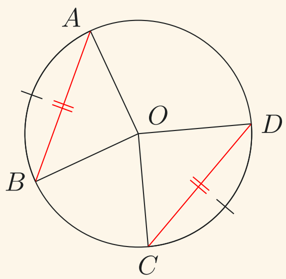



$\boxed{\textbf{\textit{Eq. chords, eq. }}\angle\textbf{\textit{s}}}$

$$AB = CD$$

$$\Downarrow$$

$$\angle AOB = \angle COD.$$

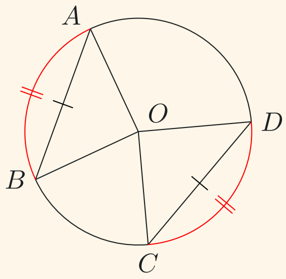



$\boxed{\textbf{\textit{Eq. arcs, eq. chords}}}$

$$\overset{\frown}{AB} = \overset{\frown}{CD}$$

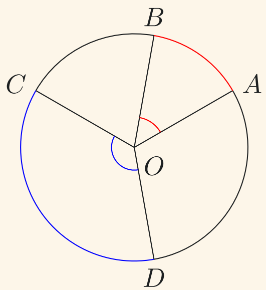
$$\Downarrow$$

$$AB = CD.$$



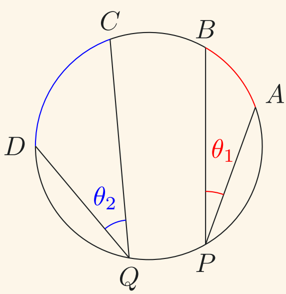

**Eq. chords, eq. arcs**

$$AB = CD$$
$$\Downarrow$$
$$\overset{\frown}{AB} = \overset{\frown}{CD}.$$




**Arcs prop. to ∠s at cen.**

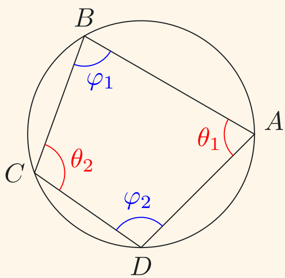
$$\frac{\overset{\frown}{AB}}{\overset{\frown}{CD}} = \frac{\theta_1}{\theta_2}.$$




**Arcs prop. to ∠s at $\odot^{ce}$**

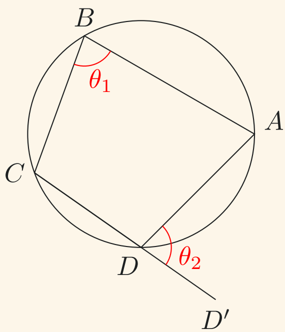
$$\frac{\overset{\frown}{AB}}{\overset{\frown}{CD}} = \frac{\theta_1}{\theta_2}.$$

## Cyclic Quadrilaterals




***Oppo. ∠s, cyclic quad.***

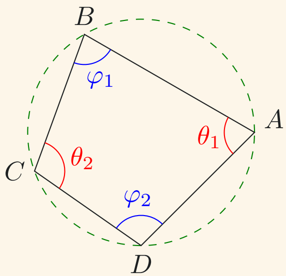
$$\theta_1 + \theta_2 = 180^\circ,$$
$$\varphi_1 + \varphi_2 = 180^\circ.$$




***Ext. ∠s, cyclic quad.***

$$\theta_1 = \theta_2.$$

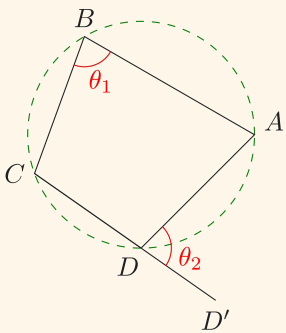
## Conditions for Identifying Cyclic Quadrilaterals




$\boxed{\textbf{\textit{Oppo. }}\angle\textbf{\textit{s supp.}}}$

$$\theta_1 + \theta_2 = 180^\circ,$$
$$\varphi_1 + \varphi_2 = 180^\circ$$
$$\Downarrow$$

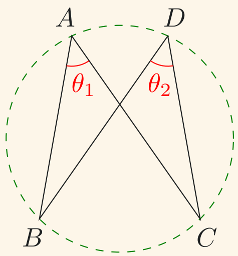
$\underline{\underline{A, B, C \text{ and } D \text{ are concylic.}}}$




$\boxed{\textbf{\textit{Ext. }}\angle = \textbf{\textit{ int. oppo. }}\angle}$

$$\theta_1 = \theta_2$$
$$\Downarrow$$

$\underline{\underline{A, B, C \text{ and } D \text{ are concylic.}}}$



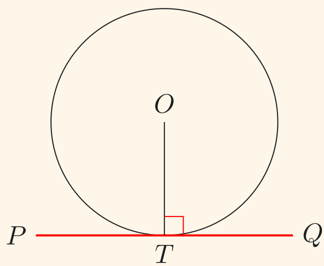

***Converse of $\angle$s in the same segment***

$\theta_1 = \theta_2$

$\Downarrow$

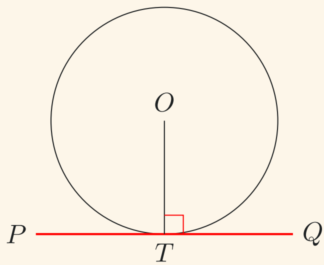
$A, B, C$ and $D$ are concylic.

## Tangents




***Tan. $\perp$ radius***

$OT \perp PQ$.

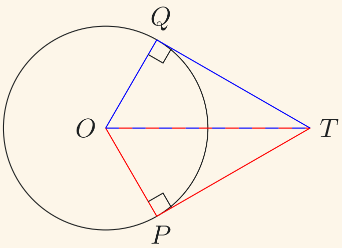



***Converse of tan. $\perp$ radius***

$OT \perp PQ$.

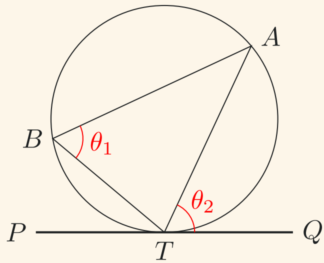
$\Downarrow$

$PQ$ is tangent to the circle at $T$.



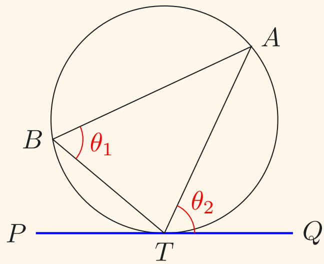

**Tan. prop.**

$\triangle OPT \cong \triangle OQT$.




**$\angle$ in alt. segment**

$\theta_1 = \theta_2$.



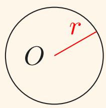
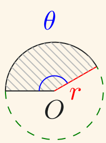
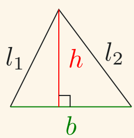
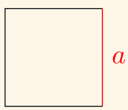
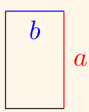

**Converse of $\angle$ in alt. segment**

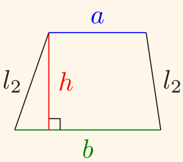
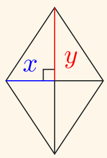
$\theta_1 = \theta_2$

$\Downarrow$

$PQ$ is tangent to the circle at $T$.

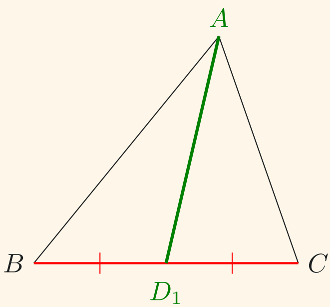
# Appendix A: Area and Perimeters of Common Plane Figures

| Figure | Area | Perimeter | |
|---|---|---|---|
|   | $\pi r^2$ | $2\pi r$ | ◀ Circles |
|   | $\dfrac{\pi\theta r^2}{360°}$ | $\dfrac{2\pi\theta r}{360°}+2r$ | ◀ Sectors |
|   | $\dfrac{hb}{2}$ | $b+l_1+l_2$ | ◀ Triangles |
|   | $a^2$ | $4a$ | ◀ Squares |
|   | $ab$ | $2\,(a+b)$ | ◀ Rectangles |

| | | | |
|---|---|---|---|
|  | $\frac{h\,(a+b)}{2}$ | $a+b+l_1+l_2$ | ◀ Trapeziums |
|  | $\frac{xy}{2}$ | $4\sqrt{x^2+y^2}$ | ◀ Rhombuses |

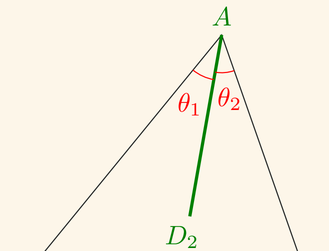
# Appendix B: Special Points and Lines in $\triangle$s

$\boldsymbol{AD_1}$ is the *median* of $BC$



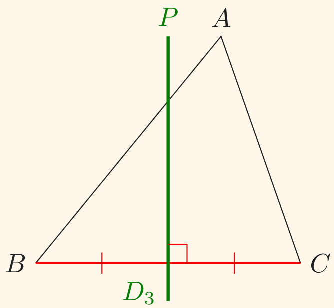

$BD_1 = CD_1$.

$\boldsymbol{AD_2}$ is an *angle bisector* of $\angle BAC$




$\theta_1 = \theta_2$.

$\boldsymbol{PD_3}$ is a *perpendicular bisector* of $BC$

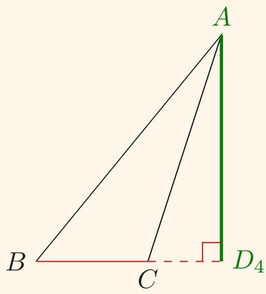



$BD_3 = CD_3$,

$PD_3 \perp BC$.

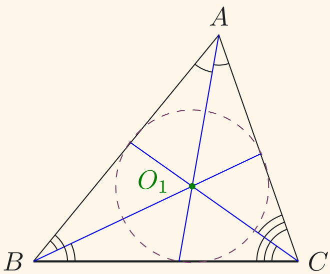
$\boldsymbol{AD_2}$ is the *altitude* of $BC$

◀ May not necessarily lie in $\triangle$.




$\theta_1 = \theta_2$.

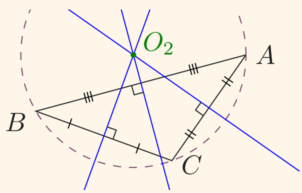
$\boldsymbol{O_1}$ is the *in-centre* of $\triangle ABC$




The *in-centre* is the point of intersection of the three angle bisectors (of a triangle).
It is the center of the triangle's inscribed circle.

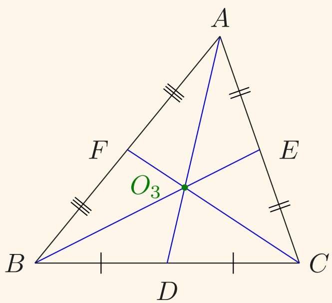
$\boldsymbol{O_2}$ is the *circumcentre* of $\angle BAC$

◀ May not necessarily lie in $\triangle$.




The *circumcentre* is the point of intersection of the three perpendicular bisectors.
It is the center of the triangle's circumcircle.

$\boldsymbol{O_1}$ is the *centroid* of $\triangle ABC$

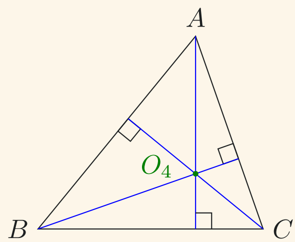



The *centroid* is the point of intersection of the three medians (of a triangle).
It divides each median in the ratio of 2 : 1, i.e.,

$$\frac{O_3A}{O_3D} = \frac{O_3B}{O_3E} = \frac{O_3C}{O_3F} = 2.$$

$\boldsymbol{O_4}$ is the *orthocentre* of $\triangle ABC$

◀ May not necessarily lie in △.




The *orthocentre* is the point of intersection of the three altitudes.

These four points are generally distinct from one another.