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Brief Paper

Input Observability and Input Reconstruction*

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Key Words—Inputs; observability; input estimation; inverse system; descriptor systems; linear systems; fault diagnosis.

Abstract—This paper addresses the problems of input observability and reconstruction for linear time-invariant systems. The corresponding necessary and sufficient conditions are provided. A systematic and simple approach to the design of schemes for input reconstruction is developed. The design is based on a matrix pencil decomposition form which is obtainable using numerically stable orthogonal transformations. The relations among input reconstruction, system inversion and disturbance-decoupled observers are explored. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

The problem of reconstructing inputs which may represent unknown external excitation, faults or any other uncertainties in a dynamic system is of great interest mainly from two aspects, i.e. system supervision and robust or fault-tolerant control. Input reconstruction, like the concept of system inversion, can also have applications in filtering and coding theory.

In the context of fault diagnosis, Patton *et al.* (1992) discussed the reconstruction of faults in discrete-time systems using a deconvolution approach. The faults can be uniquely reconstructed using the approach under a very restrictive condition, namely the system must have as many independent measurements as states.

The first issue arising in input reconstruction is an evaluation of observability of inputs. Roughly, input observability means that the change of inputs in a dynamic system can reflect itself in the change of measurements.

If the inputs are observable, the question remaining is how to reconstruct the inputs. It is worth noting that the initial condition of the system may be unknown, as it is in most applications. Hence, input reconstruction should not assume knowledge of initial conditions of the system in most cases. The input reconstruction problem corresponds to formulation of a reconstructor whose inputs are the measurements of the original system and whose outputs converge to the inputs of the original system. The convergence speed can be specified arbitrarily. The proposed reconstructor is a generalised system as the derivatives of the measurements of the original system are normally required in the reconstruction process.

Conceptually, input reconstruction is very close to system inversion. Having outputs of the original system as inputs, an inverse system recovers the inputs of the original system. The investigation of system inversion procedures started from examining the invertibility of the transfer function with the initial condition of the system assumed known, as discussed early, for example, by Silverman (1969), Sain and Massey (1969) and Dorato (1969). The stable inversion problem, i.e. finding stable inverse systems, was stressed by Moylan (1977) and necessary and sufficient conditions for the stable invertibility were provided. The significance of Moylan's work is that the assumption of knowledge of the initial condition is dropped. From the viewpoint of the geometry, the system inversion problem was also explored by Wonham (1985), Wyman and Sain (1986) and Soroka and Shaked (1986). Moreover, in the book of Basile and Marro (1992), many cases including the unknown-initial-state case are dealt with and constructive conditions for the problems to have solution have been provided.

In certain circumstances, this paper shows the equivalence between input observability and system invertibility. However, it is believed that addressing input observability is more natural and clearer than the system invertibility for the purpose of input reconstruction. The problem formulation in this paper is not based on any particular assumption, such as controllability or observability of the underlying system.

The problem of designing state observers for systems with unknown inputs, known as disturbance-decoupled observers or unknown-input observers, is also linked to the problem discussed in this paper. The corresponding observability concept, known as the strong* observability, was proposed by Molinari (1976) and the explicit test conditions were derived by Hautus (1983).

For simplicity, there is no loss of generality in assuming that the inputs to the system under consideration are unknown. The issue of simultaneously observing states and inputs has been investigated by Park and Stein (1988). In fact, as shown by Hou and Müller (1992), if a disturbance-decoupled observer exists, it is then straightforward to obtain an estimation of the inputs. It is also possible to modify some geometric methods, see, e.g. Basile and Marro (1969) and Bhattacharyya (1978), in order to obtain an observer estimating inputs and partial inputs.

When addressed without observing the whole state, the input reconstruction problem has a clear physical meaning. This is because it is rarely the case that an estimation of the whole state is required in practical applications of controller design as well as for the synthesis of fault diagnosis schemes.

It is remarkable that, except for some very trivial cases, derivatives of the measurements are unavoidable when input reconstructing is required. The proposed design guarantees that the lowest order of the derivatives will be involved in the input estimator.

Section 2 formulates the problem. Necessary and sufficient conditions for input observability are provided in Section 3, where relations among input reconstruction, system inversion and disturbance-decoupled observers are also explored. A design procedure for input reconstruction is given in Section 4. The design is further illustrated by several examples in Section 5, followed by conclusions in Section 6.

2. *Problem formulation*

Denote the time variables $u(t)$ and $y(t)$ as the input and measurement of a process, respectively. Assume that the measurement $y(t)$ is available but input $u(t)$ is unknown.

Definition 1. The input $u(t)$ is said to be *observable* if $y(t) = 0$ for $t \geq 0$ implies $u(t) = 0$ for $t > 0$.

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Definition 2. The input $u(t)$ is said to be detectable if $y(t)=0$ for $t \ge 0$ implies $u(t) \to 0$ as $t \to \infty$.

This paper exploits these concepts for the linear time-invariant systems of the form

$$\dot{x}(t) = A\,x(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t) + Du(t). \tag{2}$$

All coefficient matrices appearing in equations (1) and (2) are known and real with compatible dimensions. The initial condition $x(0_-)$ of the system is assumed *unknown*.

For the case that the initial condition is known, input observability is defined in the following way.

Definition 3. The input $u(t)$ is said to be *observable* with knowledge of $x(0_-)$ if $y(t)=0$ for $t \ge 0$ implies $u(t)=0$ for $t>0$, provided $x(0_-)=0$.

The input reconstruction problem is to find an estimator in the form

$$\dot{z}(t) = F_1\,z(t) + F_2\,y(t), \tag{3}$$

$$\hat{u}(t) = G_1 z(t) + \sum_{i=0}^{p} G_{2,i}\,y^{(i)}(t), \tag{4}$$

where $\hat{u} \to u$ for arbitrary y and $x(0_-)$ of the system (1)–(2). $y^{(i)}$ denotes the ith derivative of y. The number p is known as the *index* of the estimator. So, the existence of the derivatives of y up to order p is assumed. The convergence speed of $\hat{u} \to u$ is required to be arbitrarily assignable. Nevertheless, a remark on the case that the convergence speed cannot arbitrarily be assigned will be given at a suitable point.

The design of the estimator consists of finding all coefficient matrices in equations (3) and (4) and determining the lowest index p.

The form of the estimator (3)–(4) is not unique. It is possible to design an input estimator in which derivatives of y appear also in equation (3). This is so as will be shown, the existence of estimators does not depend on whether derivatives of y appear in equation (3) and the lowest index of possible estimator is unique. Design of an estimator in form (3)–(4) with the lowest index is of interest in this paper.

3. *Necessary and sufficient conditions*

Two matrices M and N of the same size with a complex variable λ may be used to construct a so-called *matrix pencil* denoted by $\lambda M - N$. Normally, the rank of $\lambda M - N$ depends on the value of λ. Nevertheless, the maximal rank or the *normal rank* (Van Dooren, 1979; Kailath, 1980) of a matrix pencil is unique and independent of λ. A complex number $\bar{\lambda}$ is said to be a finite eigenvalue of $\lambda M - N$, if rank $(\bar{\lambda} M - N) <$ normal-rank $(\lambda M - N)$. A matrix pencil is said to be *column unimodular*, if the pencil has full column rank for all finite λ.

Denote the set of the finite eigenvalues (single and multiple finite eigenvalues) of a matrix pencil as $\sigma(\lambda M - N)$. The necessary and sufficient condition of input observability for the system (1)–(2) is given below.

Theorem 1. The input $u(t)$ in equations (1) and (2) is observable if and only if

$$\sigma\left(\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}\right). \tag{5}$$

Proof. Set $y(t)=0$ in equation (2). The input $u(t)$ and the state $x(t)$ with initial condition $x(0_-)$ should satisfy the following equation in the frequency domain

$$\begin{bmatrix} -sI + A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -x(0_-) \\ 0 \end{bmatrix}. \tag{6}$$

The objective is to show that, whatever $x(0_-)$ and therefore $x(t)$ are, equation (6) has a unique solution for u as $u=0$ if and only if equation (5) is true.

First, the input coefficient matrix

$$F = \begin{bmatrix} B \\ D \end{bmatrix}$$

must be of full column rank, which is also implied in the condition (5) due to the full column normal rank of

$$\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}.$$

Otherwise, there exists a $u \ne 0$ satisfying $Fu = 0$ and therefore, by letting $x(0_-)=0$, this non-zero u satisfies equation (6) as well, a contradiction to Definition 1.

Thus, pre-multiplying equation (6) by

$$\begin{bmatrix} I - FF^+ \\ F^+ \end{bmatrix}$$

with $F^+ = (F^{\mathrm{T}}F)^{-1}F^{\mathrm{T}}$ yields the equivalent equation

$$\begin{bmatrix} -sE_1 + A_1 & 0 \\ -sE_2 + A_2 & I \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -E_1 x(0_-) \\ -E_2\,x(0_-) \end{bmatrix}, \tag{7}$$

where

$$E_1 = (I - FF^+)\begin{bmatrix} E \\ 0 \end{bmatrix}, \qquad A_1 = (I - FF^+)\begin{bmatrix} A \\ C \end{bmatrix}, \tag{8}$$

$$E_2 = F^+\begin{bmatrix} E \\ 0 \end{bmatrix}, \qquad A_2 = F^+\begin{bmatrix} A \\ C \end{bmatrix}. \tag{9}$$

The matrix pencil $\lambda E_1 - A_1$ has to be of full column normal rank, which is also implied in the condition (5) for the same reason as for the full column rank of F. Otherwise, since

$$\begin{aligned} \text{normal-rank}(\lambda E_1 - A_1) &< \text{normal-rank}\begin{bmatrix} \lambda E_1 - A_1 \\ \lambda E_2 - A_2 \end{bmatrix} \\ &= \text{normal-rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \\ &= \dim(x) \end{aligned} \tag{10}$$

it is easy to show that there exists a $x(t)=0$ with initial condition $x(0_-)$ satisfying $E_1\dot{x} = A_1 x$ but $E_2\dot{x} = A_2 x$. This leads to $u(t) = E_2\dot{x} - A_2 x = 0$, a contradiction of Definition 1.

Consider the fact that $\lambda E_1 - A_1$ must be of full column normal rank. According to the Kronecker decomposition of a matrix pencil (Gantmacher, 1959), it can further be assumed, without loss of generality, that equation (7) is in the form

$$\left[\begin{array}{cc|c} -sI + A_f & & \\ & -sE_\varepsilon + A_\varepsilon & \\ \hline A_{21} & -sE_{22} + A_{22} & I \end{array}\right]\left[\begin{array}{c} x_f \\ x_\varepsilon \\ \hline u \end{array}\right] = \left[\begin{array}{c} -x_f(0_-) \\ -E_\varepsilon x_\varepsilon(0_-) \\ \hline -E_{22}x_\varepsilon(0_-) \end{array}\right], \tag{11}$$

where $-sE_\varepsilon + A_\varepsilon$ is column unimodular. Note the fact, see, e.g. Gantmacher (1959) and Dai (1989), that the descriptor system $E_\varepsilon \dot{x}_\varepsilon(t) = A_\varepsilon x(t)_\varepsilon$ with initial condition $E_\varepsilon x_\varepsilon(0_-)$ has the unique solution $x_\varepsilon(t)=0$ for $t>0$. Hence, since $x_f(t)$ can be arbitrary, $u(t)=0$ for $t>0$ if and only if $A_{21}=0$.

It is clear that $A_{21}=0$ if and only if

$$\sigma(\lambda I - A_f) = \sigma\left(\begin{bmatrix} -\lambda I + A_f \\ A_{21} \end{bmatrix}\right).$$

Consider further the relations

$$\sigma\left(\begin{bmatrix} -\lambda I + A_f \\ A_{21} \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}\right) \tag{12}$$

and

$$\sigma(-\lambda I + A_f) = \sigma\left(\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}\right). \tag{13}$$

The proof is completed. □

In the above proof, it has been shown that the full column normal rank of $\lambda E_1 - A_1$ is a necessary condition for input observability. The condition is equivalent to the full column normal rank of

$$\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}$$

which is also implied in equation (5). Under this condition, from equations (12) and (13) it is straightforward to show that

$$\sigma\left(\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}\right) \supseteq \sigma\left(\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}\right) \tag{14}$$

which leads to the following corollary, where the notation $N_\sigma(\cdot)$ represents the total number of the elements in $\sigma(\cdot)$. It is worth noting that equation (14) may not hold if

$$\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}$$

is not of full column rank.

Corollary 1. The input $u(t)$ in equations (1)–(2) is observable if and only if

$$\text{normal-rank}\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix} = \dim(x) + \dim(u) \tag{15}$$

and

$$N_\sigma\left(\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}\right) = N_\sigma\left(\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}\right). \tag{16}$$

This corollary is important because no eigenvalue calculation is required in checking equations (15) and (16). Numerically effective tests for input observability will be detailed in Section 4.

From the above analysis and the proof of Theorem 1, there is no difficulty in deriving the following necessary and sufficient condition for input detectability.

Theorem 2. The input $u(t)$ in equations (1)–(2) is detectable if and only if

$$\sigma\left(\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}\right) - \sigma\left(\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}\right) \in \mathbf{C}^- \tag{17}$$

where $\mathbf{C}^-$ stands for the open left half complex plane.

This theorem indicates that u is detectable if and only if all the finite eigenvalues of

$$\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}$$

which are different from those of

$$\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}$$

must have negative real parts.

If the initial condition $x(0_-)$ is known, the condition for input observability becomes very weak as shown below.

Theorem 3. The input $u(t)$ in equations (1)–(2) is observable with knowledge of initial condition $x(0_-)$ if and only if equation (15) holds.

Proof. Using the same arguments as in the proof of Theorem 1, it can be shown that both F and $-\lambda E_1 + A_1$ have to be of full column rank and full column normal rank, respectively. These full rank properties of F and $-\lambda E_1 + A_1$ are equivalent to condition (15). This proves the necessity of equation (14). Now, assume that equation (15) is true. It then follows that $-\lambda E_1 + A_1$ has full column normal rank. According to Definition 3, set $x(0_-) = 0$ in the equation (7). Thus, $E_1\dot{x} = A_1 x$ with $x(0_-) = 0$ gives the solution $x(t) = 0$ for $t > 0$ and hence, $u(t) = 0$ for $t > 0$. □

Remark 1. The condition (15) is the well-known necessary and sufficient condition for system invertibility with known initial conditions. The physical meaning of this condition under the context of input observability becomes particularly clear.

Remark 2. Under the observability and controllability assumptions, the necessary and sufficient condition for system invertibility with unknown initial conditions is

$$\text{rank}\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix} = \dim(x) + \dim(u), \quad \forall \lambda \in \mathbf{C} \tag{18}$$

as given by Moylan (1977). Since the observability of the pair $\{A, C\}$ means that no finite eigenvalues of

$$\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}$$

exist, it is clear that condition (18) is only a special case of equation (5). Furthermore, the proof of sufficiency of equation (5) in this paper has been based directly on the input observability definition, no complicated procedure of reconstructing inputs has been involved in the proof.

Note that equation (15) is implied by equation (18) and, under the condition (15), equation (14) is always true. Thus, the observability of $\{A, C\}$ is implied by (18). This suggests that the observability assumption is redundant for the development of the condition (18) for system invertibility. It can also be proved that the controllability of the pair $\{A, B\}$ has no connection with the input observability and system invertibility problems.

Remark 3. In the design of state observers for systems with unknown inputs, condition (18) or its detectability version is necessary but not sufficient (Hutus, 1983; Hou and Müller, 1994). Nevertheless, if a generalised form, similar to equations (3)–(4), is allowed in the unknown-input observer design, condition (18) or its detectability version also becomes sufficient.

As a by-product of the existing unknown-input observer theory, an input estimator can also be provided as a part of the design. The estimation needs only the first order derivatives of each measurement. By contrast, input reconstruction in the context of input observability or system inversion generally needs high order derivatives of the measurements. This will be discussed further in Section 4.

4. *Input reconstruction*

This section shows that input observability is a necessary and sufficient condition for the existence of an estimator reconstructing inputs. A systematic and simple design procedure for input reconstruction will be provided.

Theorem 4. There exists an estimator for input reconstruction in the form (3)–(4) if and only if the system (1)–(2) is input observable.

Proof. By using arguments which are similar to those leading to the requirement for equation (5), the necessity can be verified. Sufficiency will be proved through completing a design procedure for input reconstruction, given later in this section, under the condition (5) or equivalently under equations (15) and (16). □

It is convenient to use the Kronecker form for proof of input observability conditions. But, due to the numerical unreliability of the computation (Van Dooren, 1981), this form is not suitable for developing the input reconstruction design procedure. The staircase form (Van Dooren, 1979) which will be used in the input reconstruction exhibits the structure of the Kronecker form and can be implemented via numerically stable orthogonal transformations.

According to Van Dooren (1979), by using orthogonal constant matrices U and V, i.e. $U^{\mathrm{T}}U = I$ and $V^{\mathrm{T}}V = I$, the matrix pencil

$$\begin{bmatrix} -\lambda I + A \\ C \end{bmatrix}$$

of full column normal rank can be transformed into the form

$$U\begin{bmatrix} -\lambda I + AC \end{bmatrix} V = \begin{bmatrix} \lambda E_{fAC} - A_{fAC} & \times & \times \\ & \lambda E_{\infty AC} - A_{\infty AC} & \times \\ & & \lambda E_{rAC} - A_{rAC} \end{bmatrix}, \tag{19}$$

where

(a) $\lambda E_{fAC} - A_{fAC}$ is square and $|E_{fAC}| \neq 0$;
(b) $\lambda E_{\infty AC} - A_{\infty AC}$ is square, $|A_{\infty AC}| \neq 0$ and $A_{\infty AC}^{-1} E_{\infty AC}$ is a nilpotent matrix with index l. That is, $(A_{\infty AC}^{-1} E_{\infty AC})^l = 0$ but $(A_{\infty AC}^{-1} E_{\infty AC})^k \neq 0$ for $0 < k < l$, where $l \leq \dim E_{\infty AC}$.
(c) $\lambda E_{rAC} - A_{rAC}$ is column unimodular and both E_{rAC} and A_{rAC} are of full column rank;
(d) '$\times$' denotes any pencils with appropriate dimensions, but of no particular interest.

Similarly, the matrix pencil

$$\begin{bmatrix} -\lambda I + A & B \\ C & D \end{bmatrix}$$

which is generally not of full column normal rank can be transformed via orthogonal transformations into the form

$$\begin{bmatrix} \lambda E_{cSYS} - A_{cSYS} & \times & \times & \times \\ & \lambda E_{fSYS} - A_{fSYS} & \times & \times \\ & & \lambda E_{\infty SYS} - A_{\infty SYS} & \times \\ & & & \lambda E_{rSYS} - A_{rSYS} \end{bmatrix}, \tag{20}$$

where $\lambda E_{fSYS} - A_{fSYS}$, $\lambda E_{\infty SYS} - A_{\infty SYS}$ and $\lambda E_{rSYS} - A_{rSYS}$ have the same structures as their correspondences in equation (19). $\lambda E_{cSYS}^{\mathrm{T}} - A_{cSYS}^{\mathrm{T}}$ is column unimodular and has the same structure as $\lambda E_{rSYS} - A_{rSYS}$.

By virtue of Corollary 1, the following theorem containing a numerically effective test for input observability is immediate.

Theorem 5. The system (1)–(2) is input observable if and only if the block $\lambda E_{cSYS} - A_{cSYS}$ in equation (20) vanishes and $\dim(E_{fAC}) = \dim(E_{fSYS})$, where E_{fAC} and E_{fSYS} are defined in equations (19) and (20), respectively.

Denote the system (1)–(2) as

$$\begin{bmatrix} -\rho I + A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} y, \tag{21}$$

where ρ denotes the differential operator $\mathrm{d}/\mathrm{d}t$. According to equation (19), through orthogonal transformations, the following form, equivalent to equation (21), can be obtained as

$$\left[\begin{array}{ccc|c} -\rho E_f + A_f & -\rho E_{12} + A_{12} & -\rho E_{13} + A_{13} & \\ & -\rho E_\infty + A_\infty & -\rho E_{23} + A_{23} & \\ & & -\rho E_r + A_r & \\ \hline -\rho E_{41} + A_{41} & -\rho E_{42} + A_{42} & -\rho E_{43} + A_{43} & \bar{F} \end{array}\right] \times \begin{bmatrix} x_f \\ x_\infty \\ x_r \\ \hline u \end{bmatrix} = \begin{bmatrix} V_f \\ V_\infty \\ V_r \\ \hline V_4 \end{bmatrix} y, \tag{22}$$

where $\bar{F}$ is non-singular, matrices E_f, A_f, E_∞, A_∞, E_r and A_r have the same structures as those in equation (19), the rest matrices are not important.

Eliminating E_{41} in equation (21) yields

$$\left[\begin{array}{ccc|c} -\rho E_f + A_f & -\rho E_{12} + A_{12} & -\rho E_{13} + A_{13} & \\ & -\rho E_\infty + A_\infty & -\rho E_{23} + A_{23} & \\ & & -\rho E_r + A_r & \\ \hline \bar{A}_{41} & -\rho \bar{E}_{42} + \bar{A}_{42} & -\rho \bar{E}_{43} + \bar{A}_{43} & \bar{F} \end{array}\right] \times \begin{bmatrix} x_f \\ x_\infty \\ x_r \\ \hline u \end{bmatrix} = \begin{bmatrix} V_f \\ V_\infty \\ V_r \\ \hline V_4 \end{bmatrix} y, \tag{23}$$

where

$$\bar{A}_{41} = A_{41} - E_f^{-1} A_f, \qquad \bar{E}_{42} = E_{42} - E_f^{-1} E_{12},$$
$$\bar{A}_{42} = A_{42} - E_f^{-1} A_{12}, \tag{24}$$
$$\bar{E}_{43} = E_{43} - E_f^{-1} E_{13}, \qquad \bar{A}_{43} = A_{43} - E_f^{-1} A_{13},$$
$$\bar{V}_4 = V_4 - E_f^{-1} V_f. \tag{25}$$

According to Theorem 4 and the proof of Theorem 1, it is easy to verify that the input reconstruction problem is solvable if and only if $\bar{A}_{41} = 0$. Furthermore, note that, see, e.g. Dai (1989), the descriptor system $E_\infty \dot{x}_\infty(t) = A_\infty x_\infty(t) + B_\infty \bar{u}(t)$ with initial condition $E_\infty x_\infty(0_-)$ has a unique solution

$$x_\infty(t) = -\sum_{i=0}^{l-1} (A_\infty^{-1} E_\infty)^i A_\infty^{-1} B_\infty \bar{u}^{(i)}(t), \quad t > 0. \tag{26}$$

It is then straightforward to provide a design procedure for input reconstruction as follows.

Input reconstruction design procedure:

Step 1. Test the solvability of the input reconstruction using Theorem 5;

Step 2. Decompose the extended matrix pencil

$$\begin{bmatrix} -\rho I + A & B & 0 \\ C & D & I \end{bmatrix}$$

into the form shown in equation (22) using orthogonal transformations;

Step 3. Perform simple row elementary transformation on the block

$$[-\rho E_r + A_r V_r]$$

to obtain

$$\begin{bmatrix} -\rho I + \bar{A} & V_{r1} \\ \bar{C} & V_{r2} \end{bmatrix},$$

where the pair $\{\bar{A}, \bar{C}\}$ is observable; based on this expression, design an observer for x_r defined in equation (22);

Step 4. Using the form (26), derive the explicit expression for $\hat{x}_\infty$ in terms of $\hat{x}_r$ (obtained from Step 3) and y (including its derivatives);

Step 5. Using the last line block in equation (23), obtain the explicit expression for $\hat{u}$ in terms of $\hat{x}_\infty$, $\hat{x}_r$ and y, which ends up the design procedure for input reconstruction with the form (3)–(4).

Remark 4. Step 3 can be done in many ways, for instance, through pre-multiplying

$$[-\rho E_r + A_r V_r]$$

by the full-column-rank matrix

$$\begin{bmatrix} E_r^+ \\ I - E_r E_r^+ \end{bmatrix}$$

with $E^+ = (E_r^{\mathrm{T}} E_r)^{-1} E_r^{\mathrm{T}}$. This means,

$$\bar{A} = E_r^+ A_r,\ V_{r1} = E_r^+ V_r,\ \bar{C} = (I - E_r E_r^+) A_r$$

and

$$V_{r2} = (I - E_r E_r^+) V_r.$$

Because $-\rho E_r + A_r$ is column unimodular, the pair $\{\bar{A}, \bar{C}\}$ must be an observable pair.

Remark 5. If the system (1)–(2) is not input observable but input detectable, it can be proved that although $\bar{A}_{41} \neq 0$, $\bar{A}_{41} x_f(t) \to 0$ as $t \to \infty$. Hence, by setting $\bar{A}_{41} = 0$ formally at Step 5, the above procedure offers also an input detector in the same form as equations (3) and (4), where $\hat{u} \to u$ as $t \to \infty$. Nevertheless, the convergence speed cannot be assigned arbitrarily because no influence can be made on the convergence of $\bar{A}_{41} x_f(t) \to 0$.

Finally, there is a need to verify that the design procedure above yields indeed an input estimator with the minimal index.

Consider the decomposition form in equation (23). From equation (26) and the above design procedure, the index of the input estimator is determined through the form of the solution to x_∞. It is easy to show that the solution of x_∞ takes the form $x_\infty(t) = -\sum_{i=0}^{l-1}(A_\infty^{-1}E_\infty)^i A_\infty^{-1}B_1 y^{(i)}(t) + B_2 x_r(t)$ for $t > 0$, where B_1 and B_2 are some matrices which can easily been determined. Although the matrices in the 3-by-3 block of equation (23) are not unique, the structure of the block is unique. Assume that the 3-by-3 block along with the coefficient matrix of y is further transformed into the Kronecker form

$$\left[\begin{array}{ccc|c} -\rho I + J_f & & & \bar{V}_f \\ & -\rho J_\infty + I & & \bar{V}_\infty \\ & & -\rho \bar{E}_r + \bar{A}_r & \bar{V}_r \end{array}\right]. \tag{27}$$

It is not hard to show that $(A_\infty^{-1}E_\infty)^i A_\infty^{-1}B_1 = 0$ if and only if $J_\infty^i \bar{V}_\infty = 0$. Due to uniqueness of the Kronecker form, this means that the above design procedure does give the minimal index of the input reconstruction process.

Remark 6. The index p in equations (3) and (4) is the minimal number of p satisfying $\bar{E}_{42}(A_\infty^{-1}E_\infty)^p A_\infty^{-1}B_1 = 0$. It is clear that $p \le l$, where l is the index of the nilpotent matrix $A_\infty^{-1}E_\infty$.

5. *Illustrative examples*

This section uses a system with various measurements resulting into three examples to illustrate the design procedure of input reconstruction. In order to avoid involvement of numerical calculations and to simplify the illustration, instead of orthogonal transformations, non-singular transformations are used in the design for these examples.

Example 1. Consider a linear system of the form (1)–(2) with the following coefficient matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = 0.$$

Using a non-singular matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

yields, compare with equation (21),

$$P \left[\begin{array}{c|c|c} -\rho E + A & B & 0 \\ C & D & I \end{array}\right]$$

$$= \left[\begin{array}{cc|c|c} -\rho E_\infty + A_\infty & -\rho E_{23} + A_{23} & & P_\infty \\ & -\rho E_r + A_r & & P_r \\ \hline -\rho E_{42} + A_{42} & -\rho E_{43} + A_{43} & \bar{F} & P_4 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c|cc} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\rho & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -\rho + 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\rho + 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline -\rho + 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]. \tag{28}$$

The above decomposition leads to the solution to the block $-\rho E_\infty + A_\infty$ as $x_1 = y_1$ and a reduced-order observer corresponding to the block $-\rho E_r + A_r$ as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\alpha_1 \\ 1 & 1 - \alpha_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} y_1 + \begin{bmatrix} -\alpha_1 - \alpha_1\alpha_2 \\ \alpha_1 - \alpha_2^2 \end{bmatrix} y_2,$$

$$\begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} z_1 + \alpha_1 y_2 \\ z_2 + \alpha_2 y_2 \end{bmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 > 0, \alpha_2 > 1,$$

$$x_4 = y_2.$$

The last line in equation (28) leads to

$$\hat{u} = \dot{y}_1 - y_1 - \hat{x}_2 - \hat{x}_3 - y_2. \tag{29}$$

Example 2. Consider the same system as in Example 1 but with a new matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By using a non-singular matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

equation becomes now (28)

$$P \left[\begin{array}{c|c|c} -\rho E + A & B & 0 \\ C & D & I \end{array}\right] =$$

$$= \left[\begin{array}{cccc|c|cc} 1 & -\rho & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -\rho + 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\rho + 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline -\rho + 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]. \tag{30}$$

The above decomposition leads to the solution to the block $-\rho E_\infty + A_\infty$ as $x_2 = y_1$, $x_1 = \dot{y}_1$ and a reduced-order observer corresponding to the block $-\rho E_r + A_r$ as

$$\dot{z} = (1 - \alpha)z - y_1 - \alpha^2 y_2, \quad \alpha \in \mathbb{R}, \alpha > 1,$$

$$\hat{x}_3 = z + \alpha y_2,$$

$$x_4 = y_2.$$

The last line in equation (30) leads to

$$\hat{u} = \ddot{y}_1 - \dot{y}_1 - y_1 - \hat{x}_3 - y_2. \tag{31}$$

Example 3. Consider the same system as in Examples 1 and 2 but with a new matrix

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By using a non-singular matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

equation (28) takes the form

$$P\left[\begin{array}{c|c|c} -\rho E + A & B & 0 \\ C & D & I \end{array}\right] =$$

$$= \left[\begin{array}{cccc|c|cc} 1 & -\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\rho+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\rho+1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline -\rho+1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]. \quad (32)$$

The above decomposition leads to the solution to the block $-\rho E_\infty + A_\infty$ as

$$x_3 = y_1, \qquad x_2 = \dot{y}_1 - y_1, \qquad x_1 = \ddot{y}_1 - \dot{y}_1$$

and a trivial observer corresponding to the block $-\rho E_r + A_r$ as

$$x_4 = y_2.$$

The last line in equation (32) leads to

$$u = y_1^{(3)} - 2\ddot{y}_1 - y_2.$$

It is clear that, for Example 1, there also exists a disturbance-decoupled state observer. In Example 3 the input u is exactly recovered, which is also obtainable by using some known system inversion algorithms. Through some manipulations, it is also possible to reduce equations (29) and (31) as

$$u = \dot{y}_1 - y_1 - \ddot{y}_2 + \dot{y}_2 - y_2 \quad (33)$$

and

$$u = \ddot{y}_1 - \dot{y}_1 - y_1 - \dot{y}_2, \quad (34)$$

respectively, which is less in favour in practice, owing to the difficulty in realising error-free derivatives.

6. *Conclusions*

This paper has considered the problem of input reconstruction from the viewpoint of input observability. This problem has a tight relation with system inversion and state reconstruction for systems with unknown inputs. It has been shown that it is quite natural to address the input observability problem and the analysis can be performed in a very similar manner as in the standard state observability analysis.

A fairly complete theory of input observability has been developed. This is reflected by establishing the necessary and sufficient conditions which are explicit and testable, as well as through the approval of a systematic and numerical reliable design for input reconstruction. Moreover, the relationship among input observability, system invertibility and unknown-input state observation has been clarified.

It has been shown that observability assumption on the underlying system is not necessary and that the controllability assumption on the system has no connection with reconstructing inputs.

In the context of estimating inputs, the method of input reconstruction needs weaker conditions than the unknown-input observer methods, but more derivatives of the measurements are normally involved in the input reconstruction problem.

For the purpose of input reconstruction, addressing input observability is more natural and clearer than the system invertibility. The proposed input estimator design can be implemented by using numerically effective algorithms such as those in Van Dooren (1981).

The concept of input observability and design of input estimators have evident applications in system supervision, fault diagnosis and fault-tolerant control. Like the concept of system inversion, input observability and reconstruction should also have applications in filtering and coding theory.

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