An Introduction to Observers

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Abstract—**Observers which approximately reconstruct missing state-variable information necessary for control are presented in an introductory manner. The special topics of the identity observer, a reduced-order observer, linear functional observers, stability properties, and dual observers are discussed.**

I. Introduction

IT IS OFTEN convenient when designing feedback control systems to assume initially that the entire state vector of the system to be controlled is available through measurement. Thus for the linear time-invariant system governed by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

where x is an $n \times 1$ state vector, u is an $r \times 1$ input vector, A is an $n \times n$ system matrix, and B is an $n \times r$ distribution matrix, one might design a feedback law of the form $u(t) = \psi(x(t), t)$ which could be implemented if $x(t)$ were available. This is, for example, precisely the form of control law that results from solution of a quadratic loss optimization problem posed for the system (1), from design techniques that place poles at prespecified points, and from numerous other techniques that insure stability and in some sense improve system performance.

If the entire state vector cannot be measured, as is typical in most complex systems, the control law deduced in the form $u(t) = \psi(x(t), t)$ cannot be implemented. Thus either a new approach that directly accounts for the nonavailability of the entire state vector must be devised, or a suitable approximation to the state vector must be determined that can be substituted into the control law. In almost every situation the latter approach, that of developing and using an approximate state vector, is vastly simpler than a new direct attack on the design problem.

Adopting this point of view, that an approximate state vector will be substituted for the unavailable state, results in the decomposition of a control design problem into two phases. The first phase is design of the control law assuming that the state vector is available. This may be based on optimization or other design techniques and typically results in a control law without dynamics. The second phase is the design of a system that produces an approximation to the state vector. This system, which in a deterministic setting is called an observer, or Luenberger observer to distinguish it from the Kalman filter, has as its inputs the inputs and available outputs of the system whose state is to be approximated and has a state vector that is linearly related to the desired approximation. The observer is a dynamic system whose characteristics are somewhat free to be determined by the designer, and it is through its introduction that dynamics enter the overall two-phase design procedure when the entire state is not available.

The observer was first proposed and developed in [1] and further developed in [2]. Since these early papers, which concentrated on observers for purely deterministic continuous-time linear time-invariant systems, observer theory has been extended by several researchers to include time-varying systems, discrete systems, and stochastic systems [3]–[18]. The effect of an observer on system performance (with respect to a quadratic cost functional) has been examined [5], [19]–[22]. New insights have been obtained, and the theory has been substantially streamlined [23]–[25]. At the same time, since 1964, observers have formed an integral part of numerous control system designs of which a small percentage have been explicitly reported [26]–[31]. The simplicity of its design and its resolution of the difficulty imposed by missing measurements make the observer an attractive general design component [24], [32], [33].

In addition to their practical utility, observers offer a unique theoretical fascination. The associated theory is intimately related to the fundamental linear system concepts of controllability, observability, dynamic response, and stability, and provides a simple setting in which all of these concepts interact. This theoretical richness has made the observer an attractive area of research.

This paper discusses the basic elements of observer design from an elementary viewpoint. For simplicity attention is restricted, as in the early papers, to deterministic continuous-time linear time-invariant systems. The approach taken in this paper, however, is influenced substantially by the simplification and insights derived from the work of several other authors during the past seven years. In order to highlight the new techniques and to provide the opportunity for comparison with the old, many of the example systems presented in this paper are the same as in the earlier papers.

II. Basic Theory

A. *Almost any System is an Observer*

Initially, consider the problem of observing a free system S_1, i.e., a system with zero input. If the available outputs of this system are used as inputs to drive another system S_2, the second system will almost always serve as an observer of the first system in that its state will

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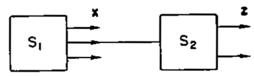

Fig. 1. A simple observer.

tend to track a linear transformation of the state of the first system (see Fig. 1). This result forms the basis of observer theory and explains why there is a great deal of freedom in the design of an observer.

Theorem 1 (Observation of a Free System): Let S_1 be a free system, $\dot{x}(t) = \boldsymbol{A}x(t)$, which drives S_2, $\dot{z}(t) = \boldsymbol{F}z(t) + \boldsymbol{H}x(t)$. Suppose there is a transformation $\boldsymbol{T}$ satisfying $\boldsymbol{TA} - \boldsymbol{FT} = \boldsymbol{H}$. If $z(0) = \boldsymbol{T}x(0)$, then $z(t) = \boldsymbol{T}x(t)$ for all $t \geq 0$. Or more generally,

$$z(t) = \boldsymbol{T}x(t) + e^{\boldsymbol{F}t}[z(0) - \boldsymbol{T}x(0)]. \tag{2.1}$$

Proof: We may write immediately

$$\dot{z}(t) - \boldsymbol{T}\dot{x}(t) = \boldsymbol{F}z(t) + \boldsymbol{H}x(t) - \boldsymbol{TA}x(t).$$

Substituting $\boldsymbol{TA} - \boldsymbol{FT} = \boldsymbol{H}$ this becomes

$$\dot{z}(t) - \boldsymbol{T}\dot{x}(t) = \boldsymbol{F}[z(t) - \boldsymbol{T}x(t)]$$

which has (2.1) as a solution.

It should be noted that the two systems S_1 and S_2 need not have the same dimension. Also, it can be shown [1] that there is a unique solution $\boldsymbol{T}$ to the equation $\boldsymbol{TA} - \boldsymbol{FT} = \boldsymbol{H}$ if $\boldsymbol{A}$ and $\boldsymbol{F}$ have no common eigenvalues. Thus any system S_2 having different eigenvalues from A is an observer for S_1 in the sense of Theorem 1.

Next, we note that the result of Theorem 1 for free systems can be easily extended to forced systems by including the input in the observer as well as the original system. Thus if S_1 is governed by

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}u(t) \tag{2.2}$$

a system S_2 governed by

$$\dot{z}(t) = \boldsymbol{F}z(t) + \boldsymbol{H}x(t) + \boldsymbol{TB}u(t) \tag{2.3}$$

will satisfy (2.1). Therefore, an observer for a system can be designed by first assuming the system is free and then incorporating the inputs as in (2.3).

B. Identity Observer

An obviously convenient observer would be one in which the transformation $\boldsymbol{T}$ relating the state of the observer to the state of the original system is the identity transformation. This requires that the observer S_2 be of the same dynamic order as the original system S_1 and that (with $\boldsymbol{T} = \boldsymbol{I}$) $\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{H}$. Specification of such an observer rests therefore on specification of the matrix $\boldsymbol{H}$.

The matrix $\boldsymbol{H}$ is determined partly by the fixed output structure of the original system and partly by the input structure of the observer. If S_1, with an m-dimensional output vector $\boldsymbol{y}$, is governed by

$$\dot{x}(t) = \boldsymbol{A}x(t) \tag{2.4a}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}x(t) \tag{2.4b}$$

and S_2, the observer, is governed by

$$\dot{z}(t) = \boldsymbol{F}z(t) + \boldsymbol{G}\boldsymbol{y}(t) \tag{2.5}$$

then $\boldsymbol{H} = \boldsymbol{GC}$. In designing the observer the $m \times n$ matrix $\boldsymbol{C}$ is fixed and the $n \times m$ matrix $\boldsymbol{G}$ is arbitrary. Thus an identity observer is determined uniquely by selection of $\boldsymbol{G}$ and takes the form

$$\dot{z}(t) = (\boldsymbol{A} - \boldsymbol{GC})z(t) + \boldsymbol{G}\boldsymbol{y}(t). \tag{2.6}$$

Any $\boldsymbol{G}$ leads to an identity observer but the dynamic response of the observing process is, according to Theorem 1, determined by the matrix $\boldsymbol{A} - \boldsymbol{GC}$.

We now state a fundamental lemma for linear systems that shows that an identity observer can be designed to have arbitrary dynamics if the original system is completely observable. First recall that a system (2.4) is completely observable if the matrix

$$[\boldsymbol{C}' \,|\, \boldsymbol{A}'\boldsymbol{C}' \,|\, (\boldsymbol{A}')^2\boldsymbol{C}' \,|\, \cdots \,|\, (\boldsymbol{A}')^{n-1}\boldsymbol{C}']$$

has rank n. Generally, if an $n \times n$ matrix $\boldsymbol{A}$ and an $m \times n$ matrix $\boldsymbol{C}$ satisfy this condition we shall say $(\boldsymbol{C}, \boldsymbol{A})$ is completely observable.

Lemma 1: Corresponding to the real matrices $\boldsymbol{C}$ and $\boldsymbol{A}$, then the set of eigenvalues of $\boldsymbol{A} - \boldsymbol{GC}$ can be made to correspond to the set of eigenvalues of any $n \times n$ real matrix by suitable choice of the real matrix $\boldsymbol{C}$ if and only if $(\boldsymbol{C}, \boldsymbol{A})$ is completely observable.

This lemma, which is now a cornerstone of linear system theory, was developed in several steps over a period of nearly a decade. For the case $m = 1$, corresponding to single output systems, early statements can be found in Kalman [34] and Luenberger [1], [35]. The general result is implicitly contained in Luenberger [2], [36], and the problem is treated definitively in Wonham [37]. A nice proof is given by Gopinath [25]. (It was recently pointed out to me that Popov [38] published a proof of a result of this type in 1964.) Calculation of the appropriate G matrix to achieve given eigenvalue placement for a high-dimensional multivariable system can, however, be a difficult computational chore.

The result of this basic lemma translates directly into a result on observers.

Theorem 2: An identity observer having arbitrary dynamics can be designed for a linear time-invariant system if and only if the system is completely observable.

In practice, the eigenvalues of the observer are selected to be negative, so that the state of the observer will converge to the state of the observed system, and they are chosen to be somewhat more negative than the eigenvalues of the observed system so that convergence is faster than other system effects. Theoretically, the eigenvalues can be moved arbitrarily toward minus infinity, yielding extremely rapid convergence. This tends, however, to make the observer act like a differentiator and thereby become highly sensitive to noise, and to introduce other difficulties. The general problem of selecting good

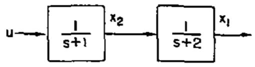

Fig. 2. A second-order system.

eigenvalues is still not completely resolved but the practice of placing them so that the observer is slightly faster than the rest of the (closed-loop) system seems to be a good one.

Example: Consider the system shown in Fig. 2. This has state-variable representation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{2.7a}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.7b}$$

An identity observer is determined by specifying the observer input vector

$$G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

The resulting observer system matrix is

$$A - GC = \begin{bmatrix} -2 - g_1 & 1 \\ -g_2 & -1 \end{bmatrix} \tag{2.8}$$

which has corresponding characteristic equation

$$\lambda^2 + (3 + g_1)\lambda + 2 + g_1 + g_2 = 0. \tag{2.9}$$

Suppose we decide to make the observer have two eigenvalues equal to -3. This would give the characteristic equation $(\lambda + 3)^2 = \lambda^2 + 6\lambda + 9 = 0$. Matching coefficients from (2.9) yields $g_1 = 3$, $g_2 = 4$. The observer is thus governed by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

III. Reduced Dimension Observer

The identity observer although possessing an ample measure of simplicity also possesses a certain degree of redundancy. The redundancy stems from the fact that while the observer constructs an estimate of the entire state, part of the state as given by the system outputs are already available by direct measurement. This redundancy can be eliminated and an observer of lower dimension but still of arbitrary dynamics can be constructed.

The basic construction of a reduced-order observer is shown in Fig. 3. If $\boldsymbol{y}(t)$ is of dimension m, an observer of order $n - m$ is constructed with state $z(t)$ that approximates $\boldsymbol{T}x(t)$ for some $m \times n$ matrix $\boldsymbol{T}$, as in Theorem 1. Then an estimate $\hat{x}(t)$ of $x(t)$ can be determined through

$$\hat{x}(t) = \begin{bmatrix} \boldsymbol{T} \\ \hdashline \boldsymbol{C} \end{bmatrix}^{-1} \begin{bmatrix} z(t) \\ \boldsymbol{y}(t) \end{bmatrix} \tag{3.1}$$

provided that the indicated partitioned matrix is invertible. Thus the $\boldsymbol{T}$ associated with the observer must have $n - m$ rows that are linearly independent of the rows of $\boldsymbol{C}$.

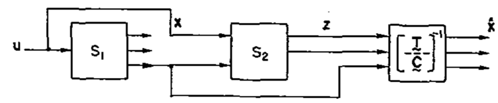

Fig. 3. Structure of reduced-order observer.

The reduced-order observer was first introduced in [1]. The simple development presented in this section is due to Gopinath [25].

We again consider the system

$$\dot{x}(t) = \boldsymbol{A}x(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{3.2a}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}x(t) \tag{3.2b}$$

and assume without loss of generality that the m outputs of the system are linearly independent—or equivalently that the output distribution matrix $\boldsymbol{C}$ has rank m. In this case it can also be assumed, by possibly introducing a change of coordinates, that the matrix $\boldsymbol{C}$ takes the form $\boldsymbol{C} = [I \,\vdots\, 0]$, i.e., C is partitioned into an $m \times m$ identity matrix and an $m \times (n - m)$ zero matrix. (An appropriate change of coordinates is obtained by selecting an $(n - m) \times n$ matrix $\boldsymbol{D}$ in such a way that

$$M = \begin{bmatrix} \boldsymbol{C} \\ \hdashline \boldsymbol{D} \end{bmatrix}$$

is nonsingular and using the variables $\bar{x} = \boldsymbol{M}x$.) It is then convenient to partition the state vector as

$$x = \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{w} \end{bmatrix}$$

and accordingly write the system in the form

$$\dot{\boldsymbol{y}}(t) = A_{11}\boldsymbol{y}(t) + A_{12}\boldsymbol{w}(t) + \boldsymbol{B}_1\boldsymbol{u}(t) \tag{3.3a}$$

$$\dot{\boldsymbol{w}}(t) = A_{21}\boldsymbol{y}(t) + A_{22}\boldsymbol{w}(t) + \boldsymbol{B}_2\boldsymbol{u}(t). \tag{3.3b}$$

The idea of the construction is then as follows. The vector $\boldsymbol{y}(t)$ is available for measurement, and if we differentiate it, so is $\dot{\boldsymbol{y}}(t)$. Since $\boldsymbol{u}(t)$ is also measureable (3.3a) provides the measurement $A_{12}\boldsymbol{w}(t)$ for the system (3.3b) which has state vector $\boldsymbol{w}(t)$ and input $A_{21}\boldsymbol{y}(t) + \boldsymbol{B}_2\boldsymbol{u}(t)$. An identity observer of order $n - m$ is constructed for (3.3b) using this measurement. Later the necessity to differentiate $\boldsymbol{y}$ is circumvented.

The justification of the construction is based on the following lemma [25].

Lemma 2: If $(\boldsymbol{C}, A)$ is completely observable, then so is (A_{12}, A_{22}).

The validity of this statement is, in view of the preceding discussion, intuitively clear. It can be easily proved directly by applying the definition of complete observability.

To construct the observer we initially define it in the form

$$\begin{aligned} \dot{\hat{\boldsymbol{w}}}(t) = (A_{22} - \boldsymbol{L}A_{12})\hat{\boldsymbol{w}}(t) + A_{21}\boldsymbol{y}(t) + \boldsymbol{B}_2\boldsymbol{u}(t) \\ + \boldsymbol{L}(\dot{\boldsymbol{y}}(t) - A_{11}\boldsymbol{y}(t)) - \boldsymbol{L}\boldsymbol{B}_1\boldsymbol{u}(t). \end{aligned} \tag{3.4}$$

In view of Lemmas 1 and 2, $\boldsymbol{L}$ can be selected so that

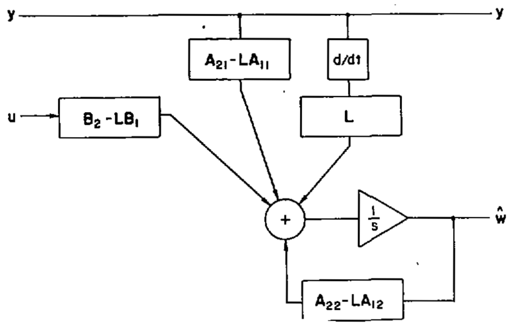

Fig. 4. Reduced-order observer using derivative.

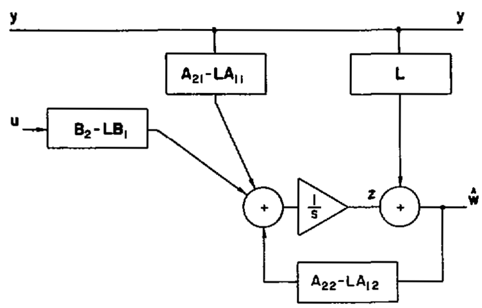

Fig. 5. Reduced-order observer.

$A_{22} - EA_{21}$ has arbitrary eigenvalues. The configuration of this observer is shown in Fig. 4.

The required differentiation of $\boldsymbol{y}$ can be avoided by modifying the block diagram of Fig. 4 to that of Fig. 5, which is equivalent at the point $\hat{\boldsymbol{w}}$. This yields the desired final form of the observer, which can be written

$$\dot{z}(t) = (A_{22} - LA_{12})z(t) + (A_{22} - LA_{12})Ly(t) + (A_{21} - LA_{11})y(t) + (B_2 - LB_1)u(t) \quad (3.5)$$

with

$$z(t) = \hat{w}(t) - Ly(t). \quad (3.6)$$

For this observer $T = [-L \mid I]$. This construction enables us to state the following theorem.

Theorem 3: Corresponding to an nth-order completely controllable linear time-invariant system having m linearly independent outputs a state observer of order $n - m$ can be constructed having arbitrary eigenvalues.

It is important to understand that the explicit form of the reduced-order observer given here, obtained by partitioning the system, is only one way to find the observer. In any specific instance, other techniques such as transforming to canonical form or simply hypothesizing the general structure and solving for the unknown parameters may be algebraically simpler. Theorem 3 guarantees that such methods will always yield an appropriate result. The preceding method used in the derivation is, of course, often a convenient one.

Example: Consider the system shown in Fig. 2 and treated in the example of Section II. This is a second-

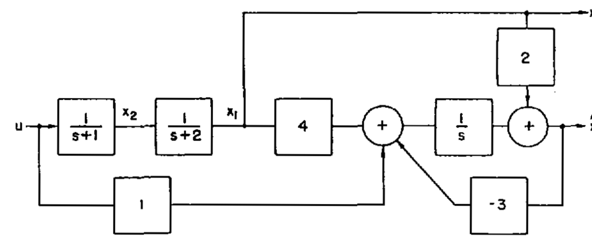

Fig. 6. Observer for second-order system.

order system with a single output so a first-order observer with an arbitrary eigenvalue can be constructed. The $\boldsymbol{C}$ matrix already has the required form, $\boldsymbol{C} = [1 \; 0]$. In this case $A_{22} - GA_{12} = -1 - G$, which gives the eigenvalue of the observer. Let us select $G = 2$ so that the observer will have its eigenvalue equal to -3. The resulting observer attached to the system is shown in Fig. 6.

IV. Observing a Single Linear Functional

For some applications an estimate of a single linear functional $\epsilon = \boldsymbol{a}'\boldsymbol{x}$ of the state is all that is required. For example, a linear time-invariant control law for a single input system is by definition determined simply by a linear functional of the system state. The question arises then as to whether a less complex observer can be constructed to yield an estimate of a given linear functional than an observer that estimates the entire state. Of course, again, it is desired to have freedom in the selection of the eigenvalues of the observer.

A major result for this problem [2] is that any given linear functional of the state, say, $\epsilon = \boldsymbol{a}'\boldsymbol{x}$, can be estimated with an observer having $\nu - 1$ arbitrary eigenvalues. Here ν is the *observability index* [2] defined as the least positive integer for which the matrix

$$[C' \mid A'C' \mid (A')^2C' \mid \cdots \mid (A')^{\nu-1}C']$$

has rank n. Since for any completely observable system $\nu - 1 \leq n - m$ and for many systems $\nu - 1$ is far less than $n - m$, observing a single linear functional of the state may be far simpler than observing the entire state vector.

The general form of the observer is exactly analogous to a reduced-order observer for the entire state vector. The estimate of $\epsilon = \boldsymbol{a}'\boldsymbol{x}$ is defined by

$$\hat{\epsilon}(t) = b'y(t) + c'z(t) \quad (4.1)$$

$$\dot{z}(t) = Fz(t) + Hx(t) + TBu(t) \quad (4.2)$$

where $\boldsymbol{F}$, $\boldsymbol{H}$, $\boldsymbol{T}$, $\boldsymbol{B}$ are as in Section II-A and where $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors satisfying $b'C + c'T = a'$.

Again the important result is that the observer need only have order $\nu - 1$. The precise design technique is dictated by considerations of convenience.

We illustrate the general result with a single example. The method used in this example can, however, be applied to any multivariable system.

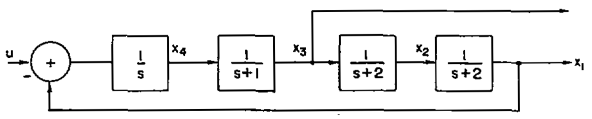

Fig. 7. A fourth-order system.

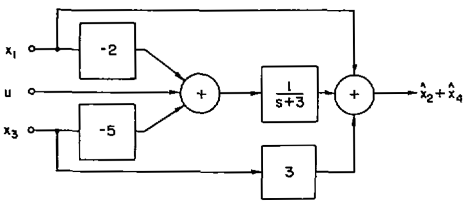

Fig. 8. Functional observer.

Example: Consider the fourth-order system shown in Fig. 7. This system with available measurements x_1 and x_3 has observability index 2. Thus any linear functional can be observed with a first-order observer. Let us decide to construct an observer with a single eigenvalue equal to -3 to observe the functional $x_2 + x_4$.

Initially neglecting the input u we hypothesize an observer of the form

$$\dot{z} = -3z + g_1x_1 + g_3x_3.$$

According to Theorem 1 this has an associated T satisfying

$$T\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} + 3T = \underline{\lfloor g_1 \quad 0 \quad g_3 \quad 0 \rfloor} \qquad (4.3)$$

If $T = \lfloor t_1 \quad t_2 \quad t_3 \quad t_4 \rfloor$, we would like $t_2 = 1$, $t_4 = 1$. Substituting these values in (4.3) we obtain the equation

$$\lfloor t_1 \quad 1 \quad t_3 \quad 1 \rfloor \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 3 \end{bmatrix} = \lfloor g_1 \quad 0 \quad g_3 \quad 0 \rfloor$$

that can be solved for the four unknowns t_1, g_1, t_3, g_3. This results in $t_1 = -1$, $t_3 = -3$, $g_1 = -2$, $g_3 = -5$. From this the final observer shown in Fig. 8 is deduced by inspection.

V. Closed-Loop Properties

Once an observer has been constructed for a linear system which produces an estimate of the state vector or of a linear transformation of the state vector it is important to consider the effect induced by using this estimate in place of the true value called for by a control law. Of paramount importance in this respect is the effect of an observer on the closed-loop stability properties of the system. It would be undesirable, for example, if an otherwise stable control design became unstable when it was realized by introduction of an observer. Observers, fortunately, do not disturb stability properties when they are introduced. In this section we show that if a linear time-invariant control law is realized with an observer, the resulting eigenvalues of the system are those of the observer itself and those that would be obtained if the control law could be directly implemented. Thus an observer does not change the closed-loop eigenvalues of a design but merely adjoins its own eigenvalues. Similar results hold for systems with nonlinear control laws [2].

Suppose we have the system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (5.1a)$$

$$y(t) = Cx(t) \qquad (5.1b)$$

and the control law

$$u(t) = Kx(t). \qquad (5.2)$$

If it were possible to realize this control law by use of available measurements (which would be possible if $K = RC$ for some R), then the closed-loop system would be governed by

$$\dot{x}(t) = (A + BK)x(t) \qquad (5.3)$$

and hence its eigenvalues would be the eigenvalues of $A + BK$.

Now if the control cannot be realized directly, an observer of the form

$$\dot{z}(t) = Fz(t) + Gy(t) + TBu(t) \qquad (5.4a)$$

$$u(t) = K\hat{x}(t) = Ez(t) + Dy(t) \qquad (5.4b)$$

where

$$TA - FT = GC \qquad (5.5a)$$

$$K = ET + DC \qquad (5.5b)$$

can be constructed. From our previous theory (C, A) completely observable is sufficient for there to be G, E, D, F, T satisfying (5.5) with F having arbitrary eigenvalues. Setting $u(t) = K\hat{x}(t)$ leads to the composite system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + BDC & BE \\ GC + TBDC & F + TBE \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}. \qquad (5.6)$$

This whole structure can be simplified by introducing $\xi = z - Tx$ and using x and ξ as coordinates. Then (5.6) becomes, using (5.5)

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A + BK & BE \\ 0 & F \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix}. \qquad (5.7)$$

Thus the eigenvalues of the composite system are those of $A + BK$ and of F.

We note that in view of Lemma 1 (applied in its dual form) if the system (5.1) is completely controllable it is possible to select K to place the closed-loop eigenvalues arbitrarily. If this control law is not realizable but the system is completely observable, an observer (of some order no greater than $n - m$) can be constructed so that the control law can be estimated. Since the eigenvalues of the observer are also arbitrary the eigenvalues of the complete composite system may be selected arbitrarily. We therefore state the following important result of linear system theory [1], [2].

Theorem 4: Corresponding to an nth order completely

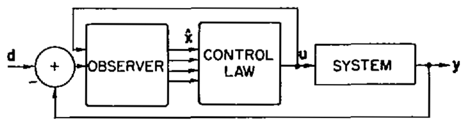

Fig. 9. A general servodesign.

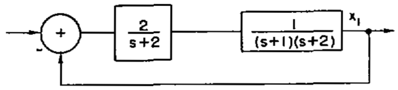

Fig. 10. Compensator for example.

controllable and completely observable system (5.1) having m linearly independent outputs, a dynamic feedback system of order $n - m$ can be constructed such that the $2n - m$ eigenvalues of the composite system take any preassigned values.

Although this eigenvalue result for linear time-invariant systems is of great theoretical interest, it should be kept in mind that the more general key result is that stability is not affected by a (stable) observer. Thus even for nonlinear or time-varying control laws an observer can supply a suitable estimate.

Example: Suppose a feedback control system is to be designed for the system shown in Fig. 2 so that its output closely tracks a disturbance input d. The general form of design is shown in Fig. 9.

For the particular system shown in Fig. 2 let us decide to design a control law that places the eigenvalues at $-1 \pm i$. It is easily found that $u = -2x_1 + x_2$ will accomplish this. If this law is implemented with the first-order observer constructed earlier, we obtain the overall system shown in Fig. 10, which can be verified to have eigenvalues $-3, -1 + i, -1 - i$.

VI. Dual Observers

The fundamental property of one system observing another can be applied in a reverse direction to obtain a special kind of controller. Such a controller, called a dual observer, was introduced by Brasch [33].

Suppose in Fig. 1 the system S_2 is the given system and S_1 is a system that we construct to control S_2. We have shown that the system S_2 tends to follow S_1 and hence S_1 can be considered as governing the behavior of S_2.

To make this discussion specific suppose the plant

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{6.1a}$$

$$y(t) = Cx(t) \tag{6.1b}$$

is driven by the free system

$$\dot{z}(t) = Fz(t) \tag{6.2a}$$

$$u(t) = Jz(t) \tag{6.2b}$$

where $AP - PF = BJ$ for some P. Then from Theorem 1 we see that in this case the vector $\mathfrak{n} = x + Pz$ is governed by the equation

$$\dot{\mathfrak{n}}(t) = A\mathfrak{n}(t)$$

and hence the plant follows the free system. This tracking property can be used to define a closed-loop system for the plant.

Rather than fix attention on the fact that only certain outputs of the plant are available, we concentrate on the fact that only certain inputs, as defined by B, are available. If we had complete freedom as to where inputs could be supplied, the output limitation would not much matter. Indeed, if the output $y(t) = Cx(t)$ could be fed to the system in the form

$$\dot{x}(t) = Ax(t) + Ly(t) \tag{6.3}$$

then the eigenvalues of the system would be the eigenvalues of $A + LC$. By Lemma 1, if the system is observable L can be selected to place the eigenvalues arbitrarily. The dual observer can be thought of as supplying an approximation to the desired inputs.

To achieve the desired result we construct the dual observer in the form

$$\dot{z}(t) = Fz(t) + Mw(t) \tag{6.4a}$$

$$w(t) = y(t) + CPz(t) \tag{6.4b}$$

$$u(t) = Jz(t) + Nw(t) \tag{6.4c}$$

where

$$AP - PF = BJ \tag{6.5a}$$

$$L = PM + BN. \tag{6.5b}$$

Equations (6.5) are dual to (5.5) and will have solution J, M, N, F with F having arbitrary eigenvalues if (6.1) is completely controllable.

The composite system is

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + BNC & BJ + BNCP \\ MC & F + MCP \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \tag{6.6}$$

Introducing $\mathfrak{n} = x + Pz$ and using z and $\mathfrak{n}$ for coordinates yields the composite system

$$\begin{bmatrix} \dot{\mathfrak{n}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + LC & 0 \\ MC & F \end{bmatrix} \begin{bmatrix} \mathfrak{n} \\ z \end{bmatrix} \tag{6.7}$$

which is the dual of (5.7). The eigenvalues of the composite system are thus seen to be the eigenvalues of $A + LC$ and the eigenvalues of F. We may therefore state the dual of Theorem 4.

Theorem 5: Corresponding to an nth-order completely controllable and completely observable system (6.1) having r linearly independent inputs, a dynamic feedback system of order $n - r$ can be constructed such that the $2n - r$ eigenvalues of the composite system take any preassigned values.

VII. Conclusions

It has been shown that missing state-variable information, not available for measurement, can be suitably approximated by an observer. Generally, as more output variables are made available, the required order of the observer is decreased.

Although the introductory treatment given in this paper is restricted to time-invariant deterministic continuous-time linear systems, much of the theory can be extended to more general situations. The references cited for this paper should be consulted for these extensions.

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