for the case where $n = 1$, it is possible to tighten the sufficiency condition to prove the following necessary and sufficient theorem.

Theorem 3: Consider the following memoryless finite communication system with state dimension 1:

$$\begin{cases} \dfrac{dx}{dt} = ax + bu, \quad x(0) = x_0 \\ y(t) = cx(t) \end{cases} \tag{32}$$

with $a > 0$ and b, c nonzero.

Assume that the same set of codewords are used for coding and feedback control. Then the system is containable if and only if

$$\tau^2 \le D. \tag{33}$$

Proof: For a system with state dimension 1, $\tau = \tau_\infty = e^\delta$. Hence, the necessary part follows from Corollary 1.

For the sufficiency part, partition the interval $[0, 1)$ so that there are $D - 1$ first level subintervals each with length $(\tau^2 - 1)/(D - 1)(\tau^2)$; there are $D - 1$ second level subintervals each with length $\big((\tau^2 - 1)/(D - 1)(\tau^2)\big)\big(1/\tau^2\big)$; there are $D - 1$ third level subintervals each with length $\big((\tau^2 - 1)/(D - 1)(\tau^2)\big)\big(1/\tau^4\big)$, and so on. The endpoints of these intervals are assumed to be closed on the left and open on the right. Let $1, 2, \cdots, D$ denote the D symbols used in the codeword. Define a coding scheme by mapping the jth subintervals in the ith level by a codeword with length i where the first $i - 1$ symbols are D and the last symbol is j. For example, if $i = 3$, then the codeword is (D, D, j) for the jth subinterval. Define the coding scheme for the feedback law so that if the codeword c_i is received, the intended feedback law is coded by the same codeword. Since the system is clearly reachable, there exists feedback control for each subinterval to ensure it stays within the interval $[0, 1)$ at the next sampling instant provided

$$\tau^{2(i+1)} \frac{\tau^2 - 1}{(D - 1)(\tau^2)} \frac{1}{\tau^{2i}} \le 1. \tag{34}$$

This holds if $\tau^2 \le D$. The rest of the argument follows as in Theorem 2. □

VI. Conclusion

In this paper, the issue of feedback control of a system with finite communication constraint is considered. The concept of containability is introduced and simple necessary and sufficient conditions for containability are derived. The problems introduced in this paper are relatively new and much more effort is needed in the future to provide deeper insight into this class of systems. An interesting geometric question central to the issue is the following:

Problem: Consider a unit cube or sphere in $\Re^n$, I. Given a function f that maps I into $\Re^n$, what are the necessary and sufficient conditions for the existence of a partition of I into I_n's and a corresponding sequence $\{m_i\}$ satisfying $\sum 1/2^{m_i} \le 1$, such that

$$f^{m_i}(I_n) + v_i \subset I \tag{35}$$

for some vector v_i for all i?

The results in this paper provide only a partial answer to this question. There is a possibility that weaker sufficiency conditions can be obtained for the case when A has some stable eigenvalues. Another interesting question concerns the behavior of the trajectory set defined by $x(r_i)$. It is likely under suitable conditions that this sequence may exhibit chaotic motion behavior.

References

[1] D. Bertsekas and R. Gallager, *Data Networks.* Englewood Cliffs, NJ: Prentice-Hall, 1992.

[2] V. S. Borkar and S. K. Mitter, "LQG control with communication constraints," in *Communications, Computation, Control and Signal Processing, a Tribute to Thomas Kailath,* A. Paulraj, V. Roychowdhury, and C. D. Schaper, Eds. New York: Kluwer, 1997, pp. 365–373.

[3] T. M. Cover and J. A. Thomas, *Elements of Information Theory.* New York: Wiley-Interscience, 1991.

[4] D. F. Delchamps, "Stabilizing a linear system with quantized state feedback," *IEEE Trans. Automat. Contr.,* vol. 35, pp. 916–924, 1990.

[5] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities,* 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1952.

[6] P. T. Kabamba and S. Hara, "Worst-case analysis and design of sampled-data control systems," *IEEE Trans. Automat. Contr.,* vol. 38, pp. 1337–1357, Sept. 1993.

[7] V. Lakshmikantham, S. Leela, and A. A. Martynyuk, *Practical Stability of Nonlinear Systems.* Singapore: World Scientific, 1990.

[8] X. Li and W. S. Wong, "State estimation with communication constraint," *Syst. Contr. Lett.,* vol. 28, pp. 49–54, 1996.

[9] ——, "Coder-estimator sequence for power control," in *Proc. Amer. Control Conf.,* Albuquerque, NM, June 4–6, 1997, pp. 220–221.

[10] D. Williamson, "Finite wordlength design of digital Kalman filters for state estimation," *IEEE Trans. Automat. Contr.,* vol. 30, pp. 930–939, 1990.

[11] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints—I: State estimation problems," *IEEE Trans. Automat. Contr.,* vol. 42, pp. 1294–1298, Sept. 1997.

Sufficient LMI Conditions for Output Feedback Control Problems

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***Abstract*—In this paper the authors present linear matrix inequality (LMI) conditions for output feedback control problems. The results are based on sufficient conditions because they are dependent on the particular state-space representation used for describing the system. Nevertheless, the conditions are not sensitive to a certain class of state-space transformations, and if the control problem is feasible then there exists some state-space transformation leading the conditions to be necessary and sufficient for the problem. The authors approach can be used for designing decentralized controllers and is easily extended to $\mathcal{H}_2, \mathcal{H}_\infty$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems via standard LMI techniques. The continuous- and discrete-time cases are considered and numerical examples are given to illustrate the results.**

***Index Terms*—LMI, robustness, static output feedback.**

I. Introduction

The static output feedback stabilization is among the most important control problems for which a complete solution is not available yet. During the last decades various approaches have been proposed to deal with the problem [1]–[9].

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The output feedback control problem turns out to be much more difficult to solve when compared to state feedback control problems. Many important control problems using state feedback can be solved via linear matrix inequality (LMI) techniques [10]. Unfortunately, this is not the case for the available design techniques for output feedback controllers: if on one hand the existing convex approaches are based on sufficient conditions that may be restrictive, on the other hand the approaches based on necessary and sufficient conditions are not numerically tractable.

In this paper we present sufficient LMI conditions for output feedback control problems. Three important features of the conditions are of interest: 1) if the control problem is feasible, then there exists some similarity transformation leading the conditions to be necessary and sufficient; 2) the conditions are not sensitive to a certain class of state-space transformations that include, for instance, orthogonal transformations; and 3) they are easily extended for more complex control problems via standard LMI techniques.

The paper is organized as follows: in Section II we present the main results concerning the output feedback stabilization problem for the continuous-time case. In Section III we extend the results for the discrete-time case, for systems having polytopic uncertainties, for decentralized output feedback and to the $\mathcal{H}_\infty$ control problem. In Section IV the sufficiency is analyzed in order to show the tightness of the approach, and a numerical example is given in Section V to illustrate the proposed results. Finally we draw some conclusions in Section VI.

The notation is standard: $M > 0$ means that $M = M'$ and that M is positive definite.

II. Stabilization Problem

Let us consider the continuous-time system represented by the equations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times n_u}$, and $C \in \mathbf{R}^{n_y \times n}$. It is well known that system (1) is stabilizable via state feedback if and only if there exist matrices $P > 0$ and F, of compatible dimensions, such that

$$P(A - BF) + (A - BF)'P < 0. \tag{2}$$

Multiplying (2) from both sides by $W = P^{-1}$ we get

$$(A - BF)W + W(A - BF)' < 0. \tag{3}$$

Now, defining $L = FW$ the last equation becomes

$$AW + WA' - BL - L'B' < 0. \tag{4}$$

In fact, it is a well-known result [10] that the LMI (4) is feasible in the variables (W, L) if and only if the pair (A, B) is stabilizable, and in this case the feedback $u = -LW^{-1}x$ stabilizes system (1). To find a solution to this problem or to declare the problem unfeasible, if solutions do not exist within a given precision, is a simple task that can be easily carried out with efficient algorithms [10].

Now let us consider the static output feedback case, that is, the desired control law has now the structure $u = -F_o y$ or equivalently, $u = -Fx$ with the constraint $F = F_o C$. From (3) we get

$$(A - BF_oC)W + W(A - BF_oC)' < 0, \qquad W > 0. \tag{5}$$

The problem of numerically solving the above matrices inequalities, for W and F_o, is in fact a very difficult one because it is not convex in general. Related to this nonconvex problem is the following convex one.

Definition 1—W-Problem: Given matrices A, B, C, with C full row rank, the W-problem consists of finding, if possible, matrices W, M, N such that

$$\begin{cases} AW + WA' - BNC - C'N'B' < 0 \\ W > 0 \\ MC = CW. \end{cases} \tag{6}$$

The interest of the W-problem is twofold: it is convex, and hence it may be solved with efficient algorithms [10]; in addition, if it is feasible then the original static output feedback stabilization problem (5), which is not convex, is feasible too, as it is shown in the sequel.

Theorem 1: Let W, M, N be solutions of the W-problem. Then the feedback

$$u = -NM^{-1}y = -F_o y$$

stabilizes system (1).

Proof: If C is full row rank, then it follows from $MC = CW$ that M is also full rank, and thus invertible, yielding $C = M^{-1}CW$. Using this fact and defining $F_o = NM^{-1}$ we get (5) from (6), completing the proof. □

If the starting point is (2) instead of (3), we get the following additional results.

Definition 2—P-Problem: Given matrices A, B, C, with B full column rank, the P-problem consists of finding, if possible, matrices P, M, N such that

$$\begin{cases} PA + A'P - C'N'B' - BNC < 0 \\ P > 0 \\ BM = PB. \end{cases} \tag{7}$$

Corollary 1: Let P, M, and N be solutions of the P-problem. Then the feedback

$$u = -M^{-1}Ny$$

stabilizes system (1).

The feasibility of either the P-problem or the W-problem is a sufficient condition for the static output feedback problem that has the advantage of being convex and as such can be tested with efficient and reliable algorithms. Moreover, it is shown in the next sections that it is possible to extend the above results to the cases of discrete-time systems, decentralized output feedback, H_∞ control, and robust control for systems with polytopic uncertainties. In Section IV we analyze the sufficiency of the above results and it is shown that there exist similarity transformations such that the P-problem and W-problem are feasible if and only if there exists an output feedback that stabilizes system (1).

When $C = I$ the W-problem is reduced to the standard LMI technique (4) for solving the state feedback stabilization problem because the equality constraint $MC = CW$ becomes redundant. In this case the P-problem furnishes an alternative way for computing stabilizing state feedback gains that may be of interest in some cases, as for example, in the circle criteria synthesis, where the sector of nonlinearities appears affinely in the equations [11].

III. Extensions

A. Discrete-Time Case

Let us consider now the discrete-time system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k). \end{cases} \tag{8}$$

The discrete-time counterpart of (5) is

$$\begin{pmatrix} W & WA' - WC'F_o'B' \\ AW - BF_oCW & W \end{pmatrix} > 0. \tag{9}$$

Based on this matrix inequality we get the following results.

Definition 3—Discrete P-Problem: Given matrices A, B, C, with B full column rank, the discrete P-problem consists of finding, if possible, matrices P, M, N such that

$$\begin{cases} \begin{pmatrix} -P & A'P - C'N'B' \\ PA - BNC & -P \end{pmatrix} < 0 \\ P > 0 \\ BM = PB. \end{cases}$$

Definition 4—Discrete W-Problem: Given matrices A, B, C, with C full row rank, the discrete W-problem consists of finding, if possible, matrices W, M, N such that

$$\begin{cases} \begin{pmatrix} -W & WA' - C'N'B' \\ AW - BNC & -W \end{pmatrix} < 0 \\ W > 0 \\ MC = CW. \end{cases}$$

Similarly to the continuous-time case it is not difficult to verify that if the discrete P-problem is feasible then the control law $u(k) = -M^{-1}Ny(k) = -F_oy(k)$ stabilizes system (8), and if the discrete W-problem is feasible the control law $u(k) = -NM^{-1}y(k) = -F_oy(k)$ stabilizes system (8).

B. Polytopic Uncertainties

Let us consider system (1) and suppose that $(A, C) \in \Omega$, where

$$\Omega = \left\{ (A, C) \colon (A, C) = \sum_{i=1}^{J} \alpha_i(A_i, C_i), \quad \alpha_i \geq 0, \sum_{i=1}^{J} \alpha_i = 1, \qquad i = 1 \cdots J \right\}.$$

It can be seen that Ω describes a polytope with J vertices determined by the pairs (A_i, C_i). Let us define

$$\Phi_i(P, N) = A_i'P + PA_i - C_i'N'B' - BNC_i.$$

Corollary 2: Consider (1) with B full column rank and above notation. If the following problem in P, M, and N:

$$\begin{cases} \Phi_i(P, N) < 0, & i = 1 \cdots J \\ BM = PB, & P > 0 \end{cases}$$

is feasible, then (1) with $(A, C) \in \Omega$ is quadratically stabilizable via static output feedback. Moreover, a stabilizing gain is given by $F_o = M^{-1}N$.

A similar result can be derived for the W-problem with C full row rank and polytopic uncertainties in the matrices A and B.

C. Decentralized Output Feedback

The problem of designing decentralized output feedback control laws is equivalent to a standard output feedback control problem with additional structure constraints on the control gains [12].

When solving the P-problem or the W-problem it is possible to choose a desired structure for the matrix gain F_o by imposing the same structure on matrix N and a corresponding block diagonal structure on matrix M.

As an example, suppose we want the gain F_o to have the following structure:

$$F_o = \begin{pmatrix} \blacksquare & 0 \\ \blacksquare & \blacksquare \end{pmatrix}$$

where $\blacksquare$ means no restrictions on the particular matrix entry. Then, if we add the convex constraints

$$N = \begin{pmatrix} \blacksquare & 0 \\ \blacksquare & \blacksquare \end{pmatrix}, \qquad M = \begin{pmatrix} \blacksquare & 0 \\ 0 & \blacksquare \end{pmatrix}$$

we get $F_o = M^{-1}N$ with the desired structure in the P-problem and $F_o = NM^{-1}$ with the desired structure in the W-problem.

D. $\mathcal{H}_\infty$ Problem

Now we show how the P-problem and W-problem can be extended to solve output feedback $\mathcal{H}_\infty$ control problems. The same ideas can be used to derive extensions to other problems such as $\mathcal{H}_2$ and related ones.

Let us consider a system described by

$$\begin{cases} \dot{x} = Ax + B_u u + B_w w \\ z = C_z x + D_{uz} u + D_{wz} w \\ y = C_y x + D_{uy} u + D_{wy} w. \end{cases} \tag{10}$$

Theorem 2: Let the system be described by (10), with D_{uy} and D_{wy} null matrices. Consider the notation

$$\Phi(W, N) = WA' + AW - B_uNC_y - C_y'N'B_u'$$

and suppose that the following LMI condition:

$$\begin{cases} \begin{pmatrix} \Phi(W, N) & B_w & WC_z' - C_y'N'D_{uz}' \\ B_w' & -\gamma^2 I & D_{wz}' \\ C_zW - D_{uz}NC_y & D_{wz} & -I \end{pmatrix} < 0, \\ MC_y = C_yW, \quad W > 0 \end{cases} \tag{11}$$

is feasible in W, M, and N. Then, for $u = -NM^{-1}y$, the $\mathcal{H}_\infty$ norm of the transfer from w to z in closed loop satisfies $\|G_{wz}\|_\infty < \gamma$.

Proof: It is a well-known result [10] that the $\mathcal{H}_\infty$ norm of a system is less than γ if and only if there exists a matrix $W > 0$ such that

$$\begin{pmatrix} WA_f' + A_fW & B_f & WC_f' \\ B_f' & -\gamma^2 I & D_f' \\ C_fW & D_f & -I \end{pmatrix} < 0 \tag{12}$$

where A_f, B_f, C_f, and D_f are the system state-space representation matrices. Applying the feedback $u = -F_oy$ to system (10) we have, from w to z, the system matrices $A_f = A - B_uF_oC_y$, $B_f = B_w$, $C_f = C_z - D_{uz}F_oC_y$, and $D_f = D_{wz}$. Following the same guidelines as for the proof of the W-problem we can show that, in fact, the feasibility of (11) implies that (12) is satisfied for $A_f = A - B_uF_oC_y$ with $F_o = NM^{-1}$. Hence, the closed-loop system satisfies $\|G_{wz}\|_\infty < \gamma$. □

A similar result can be derived for the P-problem.

IV. Sufficiency Analysis

In this section we show that the feasibility of the P-problem and W-problem are both dependent of the particular state-space representation we use for describing the system. Moreover, it will be shown that there exist similarity transformations leading the P-problem and W-problem to be feasible for the transformed systems if and only if the original system is stabilizable via static output feedback.

Let x denote the state vector of the system (1) and $x_0 = T_0^{-1}x$ the state of the transformed system. Defining $T = T_0T_0'$ and replacing (A, B, C, P, W) by $(T_0^{-1}AT_0, T_0^{-1}B, CT_0, T_0'PT_0, T_0^{-1}WT_0^{-1'})$ we may rewrite (6) and (7) as

$$\begin{cases} AW + WA' - BNCT - TC'N'B' < 0 \\ W > 0, T > 0 \\ MCT = CW \end{cases} \tag{13}$$

$$\begin{cases} PA + A'P - C'N'B'T^{-1} - T^{-1}BNC < 0 \\ P > 0, T^{-1} > 0 \\ T^{-1}BM = PB. \end{cases} \tag{14}$$

From the above matrix inequalities it is clear that both the P-problem and the W-problem are dependent of the particular state-space representation we adopt for the system, i.e., they may be unfeasible for a given state-space representation and feasible for another one. Unfortunately, (13) is not convex in the variables (W, N, M, T) and (14) is not convex in the variables (P, N, M, T^{-1}). Hence the problem of finding a state-space transformation that would lead the P-problem and W-problem to be feasible, if it is possible, is yet an open problem that we are investigating.

Notice, however, that the W-problem is insensitive to state-space transformations T_0 that satisfy the condition $CT = RC$ for some matrix R and $T = T_0T_0'$. This fact follows easily from (13) with a simple change of variables in matrices N and M. Similarly, the P-problem is insensitive to state-space transformations T_0 that satisfy the condition $T^{-1}B = BR$. For example, orthogonal transformations $(T_0T_0' = I)$ satisfy the above conditions and hence the W-problem and the P-problem are insensitive to these types of transformations.

Now let us show that there exist transformations that lead the W-problem to be feasible if and only if the original system is stabilizable via static output feedback.

The following result was taken from [2].

Lemma 1: System (1) is stabilizable via static output feedback if and only if there exists a similarity transformation that makes $C = (I \quad 0)$ and is such that (4) is feasible with W and L subject to the structural constraints

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}, \qquad L = (L_1 \quad 0)$$

where $W_1 \in \mathbf{R}^{n_y \times n_y}$ and $L_1 \in \mathbf{R}^{n_u \times n_y}$.

Notice that when $C = (I \quad 0)$, the equality constraint $MC = CW$ in the W-problem is equivalent to the same block diagonal structure on the W matrix. Now, with $L = (N \quad 0)$, we have shown that system (1) is stabilizable via static output feedback if and only if there exists a similarity transformation that makes the W-problem feasible, which is the desired result. Thus, any similarity transformation that makes the problem stated in [2] feasible will make the W-problem feasible too.[1]

Using the above results it is not difficult to show that there exist transformations leading the P-problem to be feasible if and only if the original system is stabilizable via static output feedback.

We emphasize that the problems in (13), (14), and Lemma 1 are difficult ones because the similarity transformation renders these problems not convex. In [13] may be found an interesting procedure for choosing candidates for the similarity transformation in (13) and (14).

V. Example

The numerical examples in this section were solved with the aid of the software Scilab (developed at INRIA, France). Let us consider the stabilization of a continuous-time system described[2] by (1), with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 & -3 & 3 & 0 \\ -2 & 2 & -1 & 2 \end{pmatrix}.$$

The open-loop poles of the system are located at $\{-1, 0, 0, 3\}$. Using the W-problem formulation we find that a stabilizing gain is given by

$$F_o = \begin{pmatrix} 4.6497 & 6.2001 \\ 5.7551 & 8.2946 \end{pmatrix}.$$

[1] Actually the inverse is also true, and if the W-problem is feasible then the problem proposed in [2] is also feasible for a transformation matrix to be determined.

[2] The example was borrowed from [5].

Using results from Section III we are able to consider the problem of designing feedback gains with particular structures. Suppose that we do not allow the first component of the control vector u to use information from the second component of the output vector y. This is equivalent to imposing the following structure on the feedback gain:

$$F_o = \begin{pmatrix} \blacksquare & 0 \\ \blacksquare & \blacksquare \end{pmatrix}.$$

Applying the results from Section III we get

$$F_o = \begin{pmatrix} 4.8655 & 0 \\ 7.2273 & 12.7680 \end{pmatrix}.$$

If we restrict even further the gain structure, we can find a totally decentralized feedback gain

$$F_o = \begin{pmatrix} 22.8430 & 0 \\ 0 & 23.7647 \end{pmatrix}.$$

We emphasize that these gains were obtained without any constraints on the control effort. Smaller gains can be obtained, for example, by considering an $\mathcal{H}_2$ cost function.

VI. Conclusion

We have presented in this paper LMI conditions for solving static output feedback control problems. The stabilization results presented in Section II were extended in Section III to the discrete-time case, polytopic uncertainties, decentralized control, and H_∞ control. The extensions for the H_2 and related control problems were not considered here, but they can be easily done using standard LMI techniques.

The design approach is based on the P-problem and the W-problem, which are sufficient LMI conditions for the stabilization control problem. The sufficiency was analyzed and it was shown that there exists a state-space transformation leading the P-problem and the W-problem to be feasible if and only if the original system is stabilizable via static output feedback.

An important problem that remains open consists of finding state-space transformations that will lead the P-problem and the W-problem to be feasible, if it is possible. Preliminary results for this open problem may be found in [13]. Numerical examples confirm the tightness of our approach.

The well-known LMI condition for state feedback is recovered, in the continuous- and discrete-time cases, from W-problem when $C = I$. In this case the P-problem may be viewed as an alternative condition for state feedback that can be useful, for example, in the circle criteria synthesis, where the sector of uncertainties can be used as an optimization variable when the P-problem formulation is used [11]. In [9] we present slightly different but equivalent versions of the W-problem and the P-problem; see also [8] for related problems.

References

[1] A. R. Galimidi and B. R. Barmish, "The constrained Lyapunov problem and its application to robust output feedback stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 410–419, 1986.

[2] P. L. D. Peres, J. C. Geromel, and S. R. Souza, "Optimal $\mathcal{H}_2$ control by output feedback," in *Proc. Conf. Decision and Control*, Dec. 1993, pp. 102–107.

[3] ——, "$\mathcal{H}_2$ output feedback control for discrete-time systems," in *Proc. American Control Conf.*, June 1994, pp. 2429–2433.

[4] L. El-Ghaoui and P. Gahinet, "Rank minimization under LMI constraints: A framework for output feedback problems," in *Proc. ECC*, 1993, pp. 1176–1179.

[5] G. Gu, "On the existence of linear optimal control with output feedback," *SIAM J. Contr.*, vol. 28, no. 3, pp. 771–719, 1990.

[6] R. E. Skelton and M. Ikeda, "Covariance controllers for linear continuous-time systems," *Int. J. Contr.*, vol. 49, no. 5, pp. 1773–1785, 1989.

[7] A. Trofino-Neto and V. Kucera, "Stabilization via static output feedback," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 764–765, 1993.
[8] C. A. R. Crusius and A. Trofino, "Output feedback stabilization and the constrained Lyapunov problem: A convex approach," in *Proc. Conf. Decision and Control*, Dec. 1996.
[9] C. A. R. Crusius and A. Trofino-Neto, "A convex approach to the output feedback stabilization problem," in *Proc. American Control Conf.*, 1997.
[10] S. Boyd, L. El-Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Studies in Applied Mathematics. Philadelphia, PA: SIAM, 1994.
[11] C. A. R. Crusius, "Formulacão LMI para problemas de performance e robustez," Master's dissertation, Universidade Federal de Santa Catarina, EEL/UFSC Florianópolis, Brasil, 1996.
[12] J. C. Geromel, J. Bernussou, and P. L. D. Peres, "Decentralized control through parameter space optimization," *Automatica*, vol. 30, no. 10, pp. 1565–1578, 1994.
[13] A. Trofino, A. S. Bazanella, and A. Fischman, "Designing robust controllers with operating point tracking," in *Proc. IFAC Conf. Systems Structure and Control*, Nantes, France, June 1998.

Design Techniques for the Control of Discrete-Time Systems Subject to State and Control Constraints

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***Abstract*—In this paper, the regulation problem of linear discrete-time systems under state and control constraints is investigated. In the first part of the paper, necessary and sufficient conditions for the existence of a solution to the constrained regulation problem are established. The constructive form of the proof of this result also provides a method for the derivation of a control law transferring to the origin any state belonging to a given set of initial states, while respecting the state and control constraints. In the second part of the paper, design techniques are proposed in which the determination of the control law is reduced to simple linear programming problems.**

***Index Terms*—Constrained control, discrete-time systems, linear systems, stabilization.**

I. Introduction

The majority of realistic control problems involve both state and control constraints defined by linear algebraic inequalities [12], [14]–[16]. Two different classes of constrained control problems have been the object of most of the papers. The first class of problems is related to the determination of a control law and of a set of admissible states such that all initial states belonging to this set are transferred to the origin asymptotically, while state and control constraints are satisfied [11]. This problem is trivial because any stabilizing control law, together with a sufficiently "small" positively invariant set of the resulting controlled system, is a solution to this problem. A more interesting version of this problem is the determination of the maximal admissible set that is included in the domain of constraints [1], [6]–[8], [11].

The second class of constrained control problems is the determination of a control law with which all states belonging to a given set of initial states are transferred asymptotically to the origin while state and control constraints are respected [10], [12], [13], [15]. Most of the approaches to this problem are based on the properties of positively invariant polyhedral sets [2], [3], [5], [13]. With this approach, it is possible to obtain linear [4], [9], [13], [15], or nonlinear [10] state-feedback controllers or design open-loop control systems [10]. This approach, however, has an important disadvantage, namely that there does not always exist a control strategy such that a given subset of the state space is a positively invariant set of the controlled system. This fact reduces the class of systems for which this approach can be applied considerably.

In this paper, the second class of constrained control problems is investigated. In Section III, necessary and sufficient conditions are established for the existence of a control strategy so that any state belonging to a given set of initial states is transferred asymptotically to the origin without violating the state and control constraints. The constructive form of the proof of this result provides a method for the determination of such a control strategy. Practical methods, however, to obtain a control law which is a solution to the constrained control problem are developed in Section IV. It is shown that this problem can be reduced to a series of simple linear programming problems. A dual-mode control strategy is also presented. Since no use of the positive invariance properties is made, the proposed design approaches overcome the difficulties of the approaches established in the mentioned papers.

II. Problem Statement

Throughout this paper, capital letters denote real matrices and lower case letters denote column vectors or scalars. For two vectors $x = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ and $y = [y_1 \quad y_2 \quad \cdots \quad y_n]^T$ $x < y \quad (x \le y)$ is equivalent to $x_i < y_i \quad (x_i \le y_i)$ for $i = 1, 2, \cdots, n$.

Consider a linear discrete-time system

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the control input, t is the time variable belonging to the time set $I = \{0, 1, 2 \cdots\}$, and A and B are real matrices of appropriate dimensions.

It is known that in most of the practical control problems, the control and state variables are constrained to satisfy hard constraints. Thus, the control variables are assumed to satisfy inequality constraints of the form

$$Du(t) \le \rho \tag{2}$$

where $D \in \Re^{q\times m}$ and $\rho \in \Re^q$ is a vector with nonnegative components. This inequality defines a convex polyhedral subset of the space $\Re^m$ which contains the null control $u = 0$.

Further, the state vector is assumed to satisfy inequality constraints of the form

$$Cx(t) \le d \tag{3}$$

where $C \in \Re^{r\times n}$ and $d \in \Re^r$ is a vector with positive components. This inequality defines a convex polyhedral subset X of $\Re^n$ containing the origin $x = 0$ in its interior.

In most papers dealing with the control problem under state and control constraints, the problem considered is one of determining a set of initial states $X_0 \subset \Re^n$ and of a control law $\{u(t; x_0)\}_{t=0}^{\infty}$ so that all initial states $x_0 \in X_0$ are transferred to the origin asymptotically without violating the control and state constraints [5], [8], [11]. Such a problem, however, is trivial because a solution

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