Sufficient LMI Conditions for H_∞ Output Feedback Stabilization of Linear Discrete-Time Systems

Kwan Ho Lee, Joon Hwa Lee, and Wook Hyun Kwon

***Abstract*—In this note, sufficient conditions for H_∞ output feedback stabilization of linear discrete-time systems are proposed via linear matrix inequalities (LMIs). In order to reduce conservatism existing in earlier LMI methods, auxiliary slack variables with structure are employed. It is shown that degree of freedoms by the introduction of auxiliary slack variables lead to more flexibility in obtaining an approximate solution of H_∞ output feedback stabilization problems. Consequently, the proposed method can yield a less conservative result than earlier LMI methods. In particular, typical output feedback control problems, such as decentralized H_∞ output feedback control and simultaneous H_∞ output feedback control, can be more efficiently solved. Numerical examples are included to illustrate the advantages of the proposed LMI method.**

***Index Terms*—Discrete-time systems, H_∞ output feedback stabilization, linear matrix inequality (LMI), sufficient condition.**

I. Introduction

It is well known that the H_∞ output feedback stabilization problem cannot be formulated in terms of a linear matrix inequality (LMI), except for some particular cases as noted in [1], [2]. In general, the problem can be represented as a biaffine matrix inequality (BMI) problem. However, the BMI problem is nonconvex and known to be NP-hard [3]. Although necessary and sufficient conditions for the existence of an H_∞ output feedback controller exist [1], [4], determining a solution satisfying such conditions leads to solving transformed nonconvex nonlinear optimization problems, which are numerically nontrivial.

To avoid solving computationally nontrivial BMI problems or transformed nonlinear optimization problems [4]–[7], recently, sufficient conditions for H_∞ output feedback stabilization were proposed via LMIs at the cost of necessity [2], [8]–[10]. In [2] and [8], sufficient conditions for continuous-time systems were presented by forcing a Lyapunov variable to have a block diagonal structure. In [9], [10], sufficient conditions for continuous-time and discrete-time systems were presented by inserting a linear matrix equality constraint on a Lyapunov variable. It should be noted that, in standpoint of the computational complexity of NP-hardness, these LMI approximation methods, which ensure computational simplicity, can be preferred for BMI optimization problems to local or global search methods, such as in [4]–[7]. However, they suffer from conservatism. In many cases, it can even lead to infeasibility of the optimization, even though there exists a solution. A common source of their conservatism is a structural restriction imposed on a Lyapunov variable [11], [12].

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In this note, sufficient conditions are suggested, which are significantly less conservative, for H_∞ output feedback stabilization of linear discrete-time systems. The structural restriction imposed on a Lyapunov variable is bypassed by employing auxiliary slack variables with structure. Due to the extra degree of freedom provided by the introduction of slack variables, we have additional flexibility in obtaining an approximate solution of H_∞ output feedback stabilization problems. The consequence of our work is that the degree of conservatism existing in earlier LMI methods can now be significantly reduced. Our method is applied to several typical output feedback control problems, such as decentralized H_∞ output feedback control and simultaneous H_∞ output feedback control. The advantages of the proposed method are illustrated by numerical comparisons with earlier LMI methods.

II. H_∞ Output Feedback Stabilization

Consider a plant described by

$$\begin{aligned} \Sigma : x(k+1) &= Ax(k) + B_w w(k) + B_u u(k) \\ z(k) &= C_z x(k) + D_{zw} w(k) + D_{zu} u(k) \\ y(k) &= C_y x(k) + D_{yw} w(k) \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^{n_x}$ is the plant state vector, $u \in \mathbf{R}^{n_u}$ is the controller input vector, $w \in \mathbf{R}^{n_w}$ is the exogenous input vector, $y \in \mathbf{R}^{n_y}$ is the measured output vector, and $z \in \mathbf{R}^{n_z}$ is the controlled output vector. Consider an output feedback controller

$$\begin{aligned} \Sigma_c : x_c(k+1) &= A_c x_c(k) + B_c y(k) \\ u(k) &= C_c x_c(k) + D_c y(k) \end{aligned} \tag{2}$$

where $x_c \in \mathbf{R}^{n_c}$ is the controller state vector and n_c is a preassigned order of the controller. If the controller order $n_c = 0$, Σ_c is a static output feedback controller. Define a system matrix $\mathcal{K}$ of the controller Σ_c as

$$\mathcal{K} := \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}. \tag{3}$$

The closed-loop system is then given by the following state equations:

$$\begin{aligned} \Sigma_{cl} : x_{cl}(k+1) &= A_{cl} x_{cl}(k) + B_{cl} w(k) \\ z(k) &= C_{cl} x_{cl}(k) + D_{cl} w(k) \end{aligned} \tag{4}$$

where $x_{cl} = (\, x^T \quad x_c^T \,)^T$ and the system matrix Θ_{cl} of the closed-loop system Σ_{cl} is given by

$$\Theta_{cl} := \begin{pmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{pmatrix} \tag{5}$$

$$= \left(\begin{array}{cc:c} A & 0 & B_w \\ 0 & 0 & 0 \\ \hdashline C_z & 0 & D_{zw} \end{array}\right) + \left(\begin{array}{cc} 0 & B_u \\ I_{n_c} & 0 \\ \hdashline 0 & D_{zu} \end{array}\right) \times \mathcal{K} \left(\begin{array}{cc:c} 0 & I_{n_c} & 0 \\ C_y & 0 & D_{yw} \end{array}\right) \tag{6}$$

$$=: \begin{pmatrix} \mathcal{A} & \mathcal{B}_w \\ \mathcal{C}_z & \mathcal{D}_{zw} \end{pmatrix} + \begin{pmatrix} \mathcal{B}_u \\ \mathcal{D}_{zu} \end{pmatrix} \mathcal{K} \,(\, \mathcal{C}_y \quad \mathcal{D}_{yw} \,) \tag{7}$$

which shows that the system matrix Θ_{cl} is an affine function of $\mathcal{K}$.

Let T_{wz} be the closed-loop transfer function from w to z. Then, from the bounded real lemma [13], the closed-loop system Σ_{cl} is stable and the H_∞-norm of T_{wz} is smaller than γ, i.e., $\|T_{wz}\|_\infty < \gamma$, if and only if there exist a symmetric matrix $P_\infty \in \mathbf{R}^{(n_x+n_c)\times(n_x+n_c)}$ and a controller system matrix $\mathcal{K} \in \mathbf{R}^{(n_c+n_u)\times(n_c+n_y)}$ satisfying the following inequality:

$$\begin{pmatrix} P_\infty & 0 & A_{cl}^T P_\infty & C_{cl}^T \\ 0 & \gamma I_{n_w} & B_{cl}^T P_\infty & D_{cl}^T \\ P_\infty A_{cl} & P_\infty B_{cl} & P_\infty & 0 \\ C_{cl} & D_{cl} & 0 & \gamma I_{n_z} \end{pmatrix} > 0. \tag{8}$$

Note that A_{cl}, B_{cl}, C_{cl}, and D_{cl} are affine transforms of $\mathcal{K}$ as shown in (7). Hence the matrix inequality (8) is a biaffine matrix inequality (BMI) on the variables P_∞ and $\mathcal{K}$. The BMI problem is nonconvex and known to be NP-hard to solve [3].

In the following, we will suggest sufficient LMI conditions for the BMI (8).

Theorem 1: Assume that D_{zu} is a null matrix and B_u is a full-column rank matrix. The system Σ is stabilizable by the output feedback controller Σ_c and the H_∞-norm of the corresponding closed-loop system is smaller than γ, i.e., $\|T_{wz}\|_\infty < \gamma$, if there exists a solution $\{\bar{P}_\infty, \bar{S}_1, \bar{S}_2, L_1, \gamma\}$ of the following Problem 1. If there exists a solution of Problem 1, a suitable H_∞ output feedback controller is given by $\mathcal{K} = \bar{S}_1^{-1} L_1$.

Problem 1:

$$\text{Minimize}_{\{\bar{P}_\infty, \bar{S}_1, \bar{S}_2, L_1, \gamma\}} \ \gamma$$

subject to (9), as shown at the bottom of the page where $\bar{A} = T_u \mathcal{A} T_u^{-1}$, $\bar{B}_w = T_u \mathcal{B}_w$, $\bar{C}_z = \mathcal{C}_z T_u^{-1}$, and $\bar{C}_y = \mathcal{C}_y T_u^{-1}$. T_u is a nonsingular state coordinate transformation matrix such that $\bar{B}_u = T_u \mathcal{B}_u = (\, I_{n_u+n_c} \ \ 0\,)^T$. $\bar{S}$ is a block diagonal structure variable defined as $\bar{S} := \mathrm{diag}(\bar{S}_1, \bar{S}_2)$. L is a structured variable defined as $L := (\, L_1^T \ \ 0\,)^T$.

Proof: Assume that a solution $\{\bar{P}_\infty, \bar{S}_1, \bar{S}_2, L_1, \gamma\}$ of Problem 1 exists. Substitute $\bar{A}$, $\bar{B}_w$, $\bar{C}_z$, and $\bar{C}_y$ to (9). Pre- and postmultiplying $\mathrm{diag}\left(T_u^T, I_{n_w}, T_u^T, I_{n_z}\right)$ and $\mathrm{diag}(T_u, I_{n_w}, T_u, I_{n_z})$ to (9), respectively. Then, we have (10), as shown at the bottom of the page. Define $S := T_u^T \bar{S} T_u$ and $P_\infty := T_u^T \bar{P}_\infty T_u$. By the definition of T_u, $\bar{S}$, L, and S, it follows that

$$\begin{aligned} T_u^T L &= T_u^T \begin{pmatrix} L_1 \\ 0 \end{pmatrix} = T_u^T \mathrm{diag}(\bar{S}_1, \bar{S}_2) \begin{pmatrix} I_{n_u+n_c} \\ 0 \end{pmatrix} \mathcal{K} \\ &= T_u^T \bar{S} \bar{B}_u \mathcal{K} = S \mathcal{B}_u \mathcal{K}. \end{aligned} \tag{11}$$

Therefore, (10) can be rewritten as (12), as shown at the bottom of the page. By the definition of (7), $A_{cl} = \mathcal{A} + \mathcal{B}_u \mathcal{K} \mathcal{C}_y$, $B_{cl} = \mathcal{B}_w + \mathcal{B}_u \mathcal{K} \mathcal{D}_{yw}$, $C_{cl} = \mathcal{C}_z$, and $D_{cl} = \mathcal{D}_{zw}$. Pre- and postmultiplying $\mathrm{diag}\left(I_{n_x}, I_{n_w}, S^{-1}, I_{n_z}\right)$ and $\mathrm{diag}\left(I_{n_x}, I_{n_w}, S^{-T}, I_{n_z}\right)$ to (12), respectively. Then the inequality (12) can be rewritten in the following form:

$$\begin{pmatrix} P_\infty & 0 & A_{cl}^T & C_{cl}^T \\ 0 & \gamma I_{n_w} & B_{cl}^T & D_{cl}^T \\ A_{cl} & B_{cl} & S^{-1} + S^{-T} - S^{-1} P_\infty S^{-T} & 0 \\ C_{cl} & D_{cl} & 0 & \gamma I_{n_z} \end{pmatrix} > 0. \tag{13}$$

From the inequality $\left(S^{-1} - P_\infty^{-1}\right) P_\infty \left(S^{-1} - P_\infty^{-1}\right)^T \geq 0$, it is clear that $P_\infty^{-1} \geq S^{-1} + S^{-T} - S^{-1} P_\infty S^{-T}$. Hence, it holds that

$$\begin{pmatrix} P_\infty & 0 & A_{cl}^T & C_{cl}^T \\ 0 & \gamma I_{n_w} & B_{cl}^T & D_{cl}^T \\ A_{cl} & B_{cl} & P_\infty^{-1} & 0 \\ C_{cl} & D_{cl} & 0 & \gamma I_{n_z} \end{pmatrix} > 0. \tag{14}$$

Finally, by pre- and postmultiplying $\mathrm{diag}\left(I_{n_x}, I_{n_w}, P_\infty, I_{n_z}\right)$ and $\mathrm{diag}\left(I_{n_x}, I_{n_w}, P_\infty, I_{n_z}\right)$ to (14), respectively, we obtain (8). This completes the proof. ■

Theorem 2: Assume that D_{yw} is a null matrix and C_y is a full row rank matrix. The system Σ is stabilizable by the output feedback controller Σ_c and the H_∞-norm of the corresponding closed-loop system is smaller than γ, i.e., $\|T_{wz}\|_\infty < \gamma$, if there exists a solution $\{\bar{Q}_\infty, \bar{Z}_1, \bar{Z}_2, Y, \gamma\}$ of the following Problem 2. If there exists a solution of Problem 2, a suitable H_∞ output feedback controller is given by $\mathcal{K} = Y_1 \bar{Z}_1^{-1}$.

Problem 2:

$$\text{Minimize}_{\{\bar{Q}_\infty, \bar{Z}_1, \bar{Z}_2, Y_1, \gamma\}} \ \gamma$$

subject to (15), as shown at the bottom of the page, where $\bar{A} = T_y \mathcal{A} T_y^{-1}$, $\bar{B}_w = T_y \mathcal{B}_w$, $\bar{B}_u = T_y \mathcal{B}_u$, and $\bar{C}_z = \mathcal{C}_z T_y^{-1}$.

$$\begin{pmatrix} \bar{P}_\infty & 0 & \bar{A}^T \bar{S}^T + \bar{C}_y^T L^T & \bar{C}_z^T \\ 0 & \gamma I_{n_w} & \bar{B}_w^T \bar{S}^T + \mathcal{D}_{yw}^T L^T & \mathcal{D}_{zw}^T \\ \bar{S}\bar{A} + L\bar{C}_y & \bar{S}\bar{B}_w + L\mathcal{D}_{yw} & \bar{S} + \bar{S}^T - \bar{P}_\infty & 0 \\ \bar{C}_z & \mathcal{D}_{zw} & 0 & \gamma I_{n_z} \end{pmatrix} > 0 \tag{9}$$

$$\begin{pmatrix} T_u^T \bar{P}_\infty T_u & 0 & \mathcal{A}^T T_u^T \bar{S}^T T_u + \mathcal{C}_y^T L^T T_u & \mathcal{C}_z^T \\ 0 & \gamma I_{n_w} & \mathcal{B}_w^T T_u^T \bar{S}^T T_u + \mathcal{D}_{yw}^T L^T T_u & \mathcal{D}_{zw}^T \\ T_u^T \bar{S} T_u \mathcal{A} + T_u^T L \mathcal{C}_y & T_u^T \bar{S} T_u \mathcal{B}_w + T_u^T L \mathcal{D}_{yw} & T_u^T \left(\bar{S} + \bar{S}^T - \bar{P}_\infty\right) T_u & 0 \\ \mathcal{C}_z & \mathcal{D}_{zw} & 0 & \gamma I_{n_z} \end{pmatrix} > 0 \tag{10}$$

$$\begin{pmatrix} P_\infty & 0 & (\mathcal{A} + \mathcal{B}_u \mathcal{K} \mathcal{C}_y)^T S^T & \mathcal{C}_z^T \\ 0 & \gamma I_{n_w} & (\mathcal{B}_w + \mathcal{B}_u \mathcal{K} \mathcal{D}_{yw})^T S^T & \mathcal{D}_{zw}^T \\ S(\mathcal{A} + \mathcal{B}_u \mathcal{K} \mathcal{C}_y) & S(\mathcal{B}_w + \mathcal{B}_u \mathcal{K} \mathcal{D}_{yw}) & S + S^T - P_\infty & 0 \\ \mathcal{C}_z & \mathcal{D}_{zw} & 0 & \gamma I_{n_z} \end{pmatrix} > 0 \tag{12}$$

$$\begin{pmatrix} \bar{Z} + \bar{Z}^T - \bar{Q}_\infty & 0 & \bar{Z}^T \bar{A}^T + Y^T \bar{B}_u^T & \bar{Z}^T \bar{C}_z^T + Y^T \mathcal{D}_{zu}^T \\ 0 & \gamma I_{n_w} & \bar{B}_w^T & \mathcal{D}_{zw}^T \\ \bar{A}\bar{Z} + \bar{B}_u Y & \bar{B}_w & \bar{Q}_\infty & 0 \\ \bar{C}_z \bar{Z} + \mathcal{D}_{zu} Y & \mathcal{D}_{zw} & 0 & \gamma I_{n_z} \end{pmatrix} > 0 \tag{15}$$

T_y is a nonsingular state coordinate transformation matrix such that $\bar{C}_y = \mathcal{C}_y T_y^{-1} = (\, I_{n_y+n_c} \quad 0\,)$. $\bar{Z}$ is a block diagonal structure variable defined as $\bar{Z} := \mathrm{diag}(\bar{Z}_1, \bar{Z}_2)$. Y is a structured variable defined as $Y := (\, Y_1 \quad 0\,)$.

Proof: Assume that a solution $\{\bar{Q}_\infty, \bar{Z}_1, \bar{Z}_2, Y_1, \gamma\}$ of Problem 2 exists. Substitute $\bar{A}$, $\bar{B}_w$, $\bar{B}_u$, and $\bar{C}_z$ to (15). Pre- and postmultiply $\mathrm{diag}\left(T_y^{-1}, I_{n_w}, T_y^{-1}, I_{n_z}\right)$ and $\mathrm{diag}\left(T_y^{-T}, I_{n_w}, T_y^{-T}, I_{n_z}\right)$ to (15), respectively. Then define $Z := T_y^{-1}\bar{Z}T_y^{-T}$ and $Q_\infty := T_y^{-1}\bar{Q}_\infty T_y^{-T}$. By the definition of T_y, $\bar{Z}$, Y, and Z given in Problem 2, it follows that $YT_y^{-T} = \mathcal{K}\mathcal{C}_y Z$. The proof is now done similarly to that of Theorem 1. Finally, by defining $P_\infty := Q_\infty^{-1}$, we obtain (8). This completes the proof. ■

Remark 1: Problem 2 is dual to Problem 1. If we replace the constant matrices and the variables $\bar{P}_\infty$, $\bar{S}$, L in Problem 1 as follows: $(\bar{A}, \bar{B}_w, \bar{B}_u, \bar{C}_z, D_{zw}, \bar{C}_y, D_{yw}, \bar{P}_\infty, \bar{S}, L) \longrightarrow (\bar{A}^T, \bar{C}_z^T, \bar{C}_y^T, \bar{B}_w^T, D_{zw}^T, \bar{B}_u^T, D_{zu}^T, \bar{Q}_\infty, \bar{Z}^T, Y^T)$, then we have Problem 2 with the dual variables $\bar{Q}_\infty, \bar{Z}, Y$ and a controller is given by $\mathcal{K} \longrightarrow \mathcal{K}^T$. Output feedback stabilization problems can also be solved from the dual formulations. ◇

The sufficient conditions (9) and (15), which employed auxiliary slack variables with structure, namely, $\bar{S}$ and $\bar{Z}$, respectively, are new and less conservative representations of the earlier results in [2], [8]–[10]. The slack variables introduced in this note provide an additional flexibility in solving the H_∞ output feedback control problem. As a result, the degree of conservatism can be reduced. Also, notice that the earlier sufficient conditions in [2], [8]–[10] are derived from (9) and (15) by simple manipulations. That is, let $\bar{P}_\infty$ (or $\bar{Q}_\infty$) have the same structure as the slack variable $\bar{S}$ (or $\bar{Z}$) and impose the additional constraints $\bar{P}_\infty = \bar{S} = \bar{S}^T$ (or $\bar{Q}_\infty = \bar{Z} = \bar{Z}^T$). Then the proposed sufficient conditions (9) and (15) are converted into sufficient LMI conditions of [2], [8]. They are also readily converted into sufficient LMI conditions with a linear matrix equality constraint of [9], [10]. In this respect, the sufficient conditions of this note encompass the earlier ones in [2], [8]–[10].

It is noted that the LMI-based design framework allows efficient techniques in dealing with auxiliary variables or degree of freedoms. Similar ideas on introducing an auxiliary variable have been presented in the literature [14]–[18].

In [15] and [16], slack variables are introduced for the robust control of continuous-time systems and discrete-time systems, respectively. In [14], an auxiliary dummy variable plays an important role in deriving an iterative LMI algorithm of the static output feedback control. In [17] and [18], by using an additional auxiliary variable a less conservative inequality condition is introduced for the delay-dependent control of time-delay systems. It is noticed that the contribution of this note is to introduce a structured slack variable to reduce conservatism existing in earlier LMI methods derived for solving the H_∞ output feedback problem. Numerical examples will illustrate the efficiency of the proposed method.

III. Applications

In this section, we will consider several applications of the H_∞ output feedback stabilization results presented in Section II to typical output feedback control problems, such as decentralized H_∞ output feedback control and simultaneous H_∞ output feedback control. For convenience, we will refer to the type of Problem 1 as $\bar{P}_\infty$-problem and the type of Problem 2 as $\bar{Q}_\infty$-problem.

A. Decentralized H_∞ Output Feedback Control

Consider the following large-scale systems with r control agents:

$$\begin{aligned}\Sigma_d : x(k+1) &= Ax(k) + B_w w(k) + \sum_{j=1}^{r} B_{u_j} u_j(k)\\ z(k) &= C_z x(k) + D_{zw} w(k) + \sum_{j=1}^{r} D_{zu_j} u_j(k)\\ y_j(k) &= C_{y_j} x(k) + D_{y_j w} w(k)\\ &\forall\, j = 1, \ldots, r\end{aligned} \tag{16}$$

where Σ_d can be composed of interconnected subsystems and y_j is the measurement available to the jth control agent. The decentralized stabilization problem is defined as the problem of designing a decentralized controller for a large-scale system Σ_d such that the resultant closed-loop system is stable. For the system Σ_d, let us consider $\bar{Q}_\infty$-problem of finding a decentralized H_∞ output feedback controller

$$u_j(k) = K_j y_j(k) \qquad \forall\, j = 1, \ldots, r \tag{17}$$

where $K_j \in \mathbf{R}^{p_j \times q_j}$ for all $j = 1, \ldots, r$. Denote the matrices B_u, D_{zu}, C_y, and D_{yw} by

$$\begin{aligned}B_u &= (\, B_{u_1} \quad \ldots \quad B_{u_r}\,) \quad D_{zu} = (\, D_{zu_1} \quad \ldots \quad D_{zu_r}\,)\\ C_y &= (\, C_{y_1}^T \quad \ldots \quad C_{y_r}^T\,)^T \quad D_{yw} = (\, D_{y_1 w}^T \quad \ldots D_{y_r w}^T\,)^T\end{aligned}$$

and $n_y = \sum_{j=1}^{r} q_j$. Applying (15) to the system Σ_d with the controller (17), we have the following corollary.

Corollary 1: Assume that $D_{y_j w}$ is a null matrix and C_{y_j} is a full-row rank matrix in the system Σ_d. The system Σ_d is stabilizable by the decentralized output feedback controller (17) and the H_∞-norm of the corresponding closed-loop system is smaller than γ, i.e., $\|T_{wz}\|_\infty < \gamma$, if there exists a solution $\{\bar{Q}_\infty, \bar{Z}_{1j}, \bar{Z}_2, Y_j, \gamma\}$ of the following Problem 3. If there exists a solution of Problem 3, a suitable decentralized H_∞ output feedback controller is given by $K_j = Y_j \bar{Z}_{1j}^{-1}$ for all $j = 1, \ldots, r$.

Problem 3:

$$\mathrm{minimize}_{\{\bar{Q}_\infty, \bar{Z}_{1j}, \bar{Z}_2, Y_j, \gamma\ \forall j=1,\ldots,r\}}\ \gamma$$

subject to (18), as shown at the bottom of the next page, where $\bar{A} = T_y A T_y^{-1}$, $\bar{B}_w = T_y B_w$, $\bar{B}_u = T_y B_u$, and $\bar{C}_z = C_z T_y^{-1}$. T_y is a nonsingular state coordinate transformation matrix such that $\bar{C}_y = C_y T_y^{-1} = (\, I_{n_y} \quad 0\,)$. $\bar{Z}_d$ is a block diagonal structure variable defined as $\bar{Z}_d := \mathrm{diag}(\bar{Z}_{11}, \bar{Z}_{12}, \ldots, \bar{Z}_{1r}, \bar{Z}_2)$. Y_d is a structured variable defined as $Y_d := (\, \mathrm{diag}(Y_1, Y_2, \ldots, Y_r) \quad 0\,)$.

A similar result can be derived for $\bar{P}_\infty$-problem.

B. Simultaneous H_∞ Output Feedback Control

Consider the following set of plants Σ_j:

$$\begin{aligned}\Sigma_j : x_j(k+1) &= A_j x_j(k) + B_{w_j} w_j(k) + B_{u_j} u_j(k)\\ z_j(k) &= C_{z_j} x_j(k) + D_{zw_j} w_j(k) + D_{zu_j} u_j(k)\\ y_j(k) &= C_{y_j} x_j(k) + D_{yw_j} w_j(k)\\ &\forall\, j = 1, \ldots, r.\end{aligned} \tag{19}$$

$$\begin{pmatrix} \bar{Z}_d + \bar{Z}_d^T - \bar{Q}_\infty & 0 & \bar{Z}_d^T \bar{A}^T + Y_d^T \bar{B}_u^T & \bar{Z}_d^T \bar{C}_z^T + Y_d^T D_{zu}^T \\ 0 & \gamma I_{n_w} & \bar{B}_w^T & D_{zw}^T \\ \bar{A}\bar{Z}_d + \bar{B}_u Y_d & \bar{B}_w & \bar{Q}_\infty & 0 \\ \bar{C}_z \bar{Z}_d + D_{zu} Y_d & D_{zw} & 0 & \gamma I_{n_z} \end{pmatrix} > 0 \tag{18}$$

The simultaneous stabilization problem is defined as the problem of finding of a single controller that simultaneously stabilizes a finite collection of plants Σ_j, for example, a set of plants characterized by different operation modes or linearized by different operating points. For the set of plants Σ_j, let us consider $\bar{Q}_\infty$-problem of finding a simultaneous H_∞ output feedback stabilizer

$$u_j(k) = Ky_j(k) \qquad \forall\, j = 1, \ldots, r \tag{20}$$

where $K \in \mathbf{R}^{n_u \times n_y}$ for all $j = 1, \ldots, r$. Applying the condition (15) to the set of plants Σ_j with the stabilizer (20), we have the following corollary.

Corollary 2: Assume that D_{yw_j} is a null matrix and C_{y_j} is a full-row rank matrix in the set of plants Σ_j. The systems Σ_j are stabilizable by the simultaneous output feedback stabilizer (20) and the H_∞-norms of the corresponding closed-loop systems are smaller than γ_j, i.e., $\|T_{w_j z_j}\|_\infty < \gamma_j$, if there exists a solution $\{\bar{Q}_{\infty j}, \bar{Z}_1, \bar{Z}_2, Y_1, \gamma_j\}$ of the following Problem 4. If there exists a solution of Problem 4, a suitable simultaneous H_∞ output feedback stabilizer is given by $K = Y_1 \bar{Z}_1^{-1}$.

Problem 4:

$$\text{Minimize}_{\{\bar{Q}_{\infty j}, \bar{Z}_1, \bar{Z}_2, Y_1, \gamma_j\}} \sum_{j=1}^{r} \gamma_j$$

subject to (21), as shown at the bottom of the page, where $\bar{A}_j = T_{y_j} A_j T_{y_j}^{-1}$, $\bar{B}_{w_j} = T_{y_j} B_{w_j}$, $\bar{B}_{u_j} = T_{y_j} B_{u_j}$, and $\bar{C}_{z_j} = C_{z_j} T_{y_j}^{-1}$. T_{y_j} is a nonsingular state coordinate transformation matrix such that $\bar{C}_{y_j} = C_{y_j} T_{y_j}^{-1} = (\, I_{n_y} \quad 0 \,)$. $\bar{Z}$ is a block diagonal structure variable defined as $\bar{Z} := \operatorname{diag}(\bar{Z}_1, \bar{Z}_2)$. Y is a structured variable defined as $Y := (\, Y_1 \quad 0 \,)$.

A similar result can be derived for $\bar{P}_\infty$-problem.

IV. Illustrative Examples

The LMI conditions of this note and of [2], [9] can be easily solved using semidefinite programming algorithms or existing LMI packages [19]–[21].

Example 1 H_∞ Output Feedback Stabilization: Consider an unstable plant given by

$$\begin{aligned} x(k+1) &= \begin{pmatrix} \alpha & 0.3 & 2 \\ 1 & 0 & 1 \\ 0.3 & 0.6 & -0.6 \end{pmatrix} x(k) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} w(k) \\ &\quad + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} u(k) \\ y(k) &= (\, 1 \quad 1 \quad 0 \,)\, x(k) \end{aligned} \tag{22}$$

and a performance output $z = (\, x^T \quad u^T \,)^T$, where a positive number α will be chosen to be 1.9, 2.7, 2.8, and 2.9. The larger α is, the more unstable the system (22) become. When α is 1.9, the open-loop poles of the plant are $\{2.4383, -1.0034, -0.1349\}$. For simplicity, a static output feedback H_∞ controller is to be designed for the system (22) with a fixed α. A state transformation matrix T_y is taken as $T_y = (\, C_y^T \quad C_y^{T\perp T} \,)^T$. Table I shows numerical results obtained by using Problem 2, namely, H_∞ static output feedback gains K, eigenvalues of the closed-loop systems, and actual H_∞ performance $\|T_{wz}\|_\infty$ achieved by K. For comparison, numerical results obtained using the earlier LMI methods [2], [9] are also included in Table II. In the case of $\alpha \le 2.7$, our method shows less conservative results than the LMI methods of [2], [9]. It is noticed that for $\alpha = 2.7$, the earlier LMI methods show quite conservative results. Also, in the case of $\alpha \ge 2.8$, they fail to find a feasible solution. In contrast, our LMI method provides suitable output feedback H_∞ controllers, even for larger values of α, e.g., 3.1. It is seen that compared to the earlier LMI methods [2], [9], the proposed LMI method shows less conservative results and provides better numerical performance in finding a feasible solution of the BMI (8).

We will now consider a performance output $z = (\, x^T \quad w^T \,)^T$ for the system (22) with $\alpha = 3.2$. It is noted that we can try to find a solution by using both Problem 1 and Problem 2, and the dual relation of Remark 1 makes it easy. Notice that Problem 2 and the LMI methods of [2], [9] fail to produce a feasible solution. Using Problem 1, however, we obtain a suitable H_∞ controller $u = \begin{bmatrix} -1.0850 & -0.2508 \end{bmatrix}^T y$, for which we have $\|T_{wz}\|_\infty = 10.3940$. It is seen that the less conservative dual formulation of this note can be used as an alternative to find a numerical solution efficiently.

Example 2: Decentralized H_∞ Output Feedback Control:: Consider a plant given by

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 1 & 0 & -1.6 \\ -1 & 1 & -0.3 \\ 0 & 0.4 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} w(k) \\ &\quad + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_1(k) + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u_2(k) \\ y_1(k) &= (\, 1 \quad 0 \quad 0 \,)\, x(k) \\ y_2(k) &= (\, 0 \quad 1 \quad 1 \,)\, x(k) \end{aligned} \tag{23}$$

and a performance output $z = (\, x^T \quad u_1 \quad u_2 \,)^T$, where the open-loop poles of the plant are $\{1.8154, 0.5923 \pm 0.7866i\}$. A decentralized H_∞ output feedback controller is to be designed for the plant (23). Using Problem 3, we obtain a suitable decentralized controller $u_1 = -1.4149 y_1$, $u_2 = -0.3364 y_2$, for which the actual H_∞ performance is 12.0127 and the actual H_2 performance is 4.6045, while the LMI methods of [2], [9] fails to produce a feasible solution. For another example, consider a longitudinal dynamic of a VTOL helicopter given in [22]. Using Problem 3, we have a decentralized controller $u_1 = 0.0044 y_1$, $u_2 = -0.6121 y_2$ with $\|T_{wz}\|_\infty = 63.6785$, while, using the LMI method of [2], we get $\|T_{wz}\|_\infty = 74.9665$. It is clearly seen from examples that decentralized H_∞ output feedback control problems can be more efficiently solved by the proposed LMI method.

Example 3: Simultaneous H_∞ Output Feedback Control:: Consider the F4E fighter aircraft models corresponding to four different cruising conditions given by

$$\begin{aligned} x_j(k+1) &= \begin{pmatrix} a_j^{11} & a_j^{12} & a_j^{13} \\ a_j^{21} & a_j^{22} & a_j^{23} \\ 0 & 0 & 8.6071e\text{-}1 \end{pmatrix} x_j(k) \\ &\quad + \begin{pmatrix} b_j^1 \\ b_j^2 \\ 0 \end{pmatrix} u_j(k) \qquad \forall\, j = 1, \ldots, 4. \end{aligned} \tag{24}$$

The above models $\{a_j, b_j\}$ are taken by discretizing the continuous models in [23] and [24] with a sample rate $T_s = 0.001$ s and a zero order hold on the input. A stabilizer is to be designed to simultaneously

$$\begin{pmatrix} \bar{Z} + \bar{Z}^T - \bar{Q}_{\infty j} & 0 & \bar{Z}^T \bar{A}_j{}^T + Y^T \bar{B}_{u_j}^T & \bar{Z}^T \bar{C}_{z_j}^T + Y^T D_{zu_j}^T \\ 0 & \gamma_j I_{n_w} & \bar{B}_{w_j}^T & D_{zw_j}^T \\ \bar{A}_j \bar{Z} + \bar{B}_{u_j} Y & \bar{B}_{w_j} & \bar{Q}_{\infty j} & 0 \\ \bar{C}_{z_j} \bar{Z} + D_{zu_j} Y & D_{zw_j} & 0 & \gamma_j I_{n_z} \end{pmatrix} > 0 \qquad \forall\, j = 1, \ldots, r \tag{21}$$

TABLE I
NUMERICAL RESULTS OF THE PROPOSED METHOD FOR EXAMPLE 1

α	K	Eigenvalues	γ	$\|T_{wz}\|_\infty$
1.9	$[-0.6637 \quad -0.4965]^T$	$0.4918, -0.1760 \pm 0.2424i$	2.3814	1.9894
2.7	$[-0.9353 \quad -0.4686]^T$	$0.7996, -0.0518 \pm 0.6473i$	8.8970	8.1654
2.8	$[-0.9654 \quad -0.4402]^T$	$0.8322, -0.0189 \pm 0.6748i$	12.3620	10.8751
2.9	$[-0.9932 \quad -0.4089]^T$	$0.8675, \ 0.0152 \pm 0.6961i$	18.7556	15.3291

TABLE II
ACTUAL H_∞ PERFORMANCE FOR DIFFERENT VALUES OF α

Design methods	$\alpha = 1.9$	$\alpha = 2.7$	$\alpha = 2.8$	$\alpha = 2.9$
LMI Method[2]	3.6438	75.2799	$\times$	$\times$
LMI Method[5]	3.6438	75.2801	$\times$	$\times$
Proposed LMI method	1.9894	8.1654	10.8751	15.3291

stabilize these four models and to guarantee an H_∞ performance on each corresponding closed-loop system. For this, let us consider the previous systems with

$$B_{w_j} = \begin{pmatrix} 1 \\ 0_{2\times 1} \end{pmatrix} \quad C_{z_j} = \begin{pmatrix} I_3 \\ 0_{1\times 3} \end{pmatrix}$$
$$D_{zw_j} = 0_{1\times 4} \quad D_{zu_j} = \begin{pmatrix} 0_{3\times 1} \\ 1 \end{pmatrix}. \tag{25}$$

A state feedback H_∞ simultaneous stabilizer

$$u_j(k) = [\,3.7054 \quad 5.7610 \quad -2.6018\,]\, x_j(k) \quad \forall\, j = 1, \ldots, 4 \tag{26}$$

is obtained using Problem 4. The sum of actual H_∞ performance values of the plants Σ_j achieved by the state feedback stabilizer (26), namely, $\sum_{j=1}^{4} \|T_{w_j z_j}\|_\infty = 32.5630$. Let us now consider the previous systems with $C_{y_j} = [\,1 \quad 0 \quad 0\,]$. Using Problem 4, we obtain an output feedback H_∞ simultaneous stabilizer

$$u_j(k) = 3.6764 y_j(k) \qquad \forall\, j = 1, \ldots, 4 \tag{27}$$

for which we get $\sum_{j=1}^{4} \|T_{w_j z_j}\|_\infty = 32.9633$, while, using earlier LMI methods [2], [9], we get $\sum_{j=1}^{4} \|T_{w_j z_j}\|_\infty = 40.4076$. It is seen that the proposed LMI method is less conservative that the earlier LMI method [2], [9].

We shall next consider the ship-steering problem, where $r = 2$, $n_x = 3$, and

$$x_j(k+1) = \begin{pmatrix} a_j^{11} & a_j^{12} & 0 \\ a_j^{21} & a_j^{22} & 0 \\ a_j^{31} & a_j^{32} & 1 \end{pmatrix} x_j(k) + \begin{pmatrix} b_j^1 \\ b_j^2 \\ b_j^2 \end{pmatrix} u_j(k) \quad \forall\, j = 1, 2. \tag{28}$$

The aforementioned models $\{a_j, b_j\}$ are taken by discretizing the continuous models in [23] and [24] with a sample rate $T_s = 0.005$ s and a zero-order hold on the input. Let us consider the previous systems with

$$B_{w_j} = \begin{pmatrix} 0 \\ 0.1 \\ 0 \end{pmatrix} \quad C_{z_j} = \begin{pmatrix} I_3 \\ 0_{1\times 3} \end{pmatrix} \quad D_{zw_j} = 0_{4\times 1}$$
$$D_{zu_j} = \begin{pmatrix} 0_{3\times 1} \\ 1 \end{pmatrix} \quad C_{y_j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{29}$$

A state transformation matrix T_{y_j} is taken as $T_{y_j} = (\, C_{y_j}^T \quad C_{y_j}^{T\perp T}\,)^T$. Using Problem 4, we obtain a simultaneous H_∞ output feedback stabilizer

$$u_j(k) = [\,-2003 \quad 12\,259\,]\, y_j(k) \qquad \forall\, j = 1, 2 \tag{30}$$

for which we get actual H_∞ performance values $\|T_{w_1 z_1}\|_\infty = 25.5588$ and $\|T_{w_2 z_2}\|_\infty = 19.4450$. It is worth noting that if we try to obtain a controller (20) by applying the iterative LMI algorithm of [14] or the earlier LMI conditions of [2], [9], we do not find any feasible solution to this example.

V. CONCLUSION

In this note, sufficient conditions have been suggested for H_∞ output feedback stabilization of linear discrete-time systems via LMIs. Auxiliary slack variables with structure are employed in order to provide additional flexibility in the H_∞ output feedback control problem. The consequence of our work is a significant reduction of conservatism. Applications to typical output feedback control problems, such as decentralized H_∞ output feedback control and simultaneous H_∞ output feedback control, have been discussed. Numerical examples illustrate the efficiency of the proposed method with respect to earlier LMI methods. Extensions of this method to continuous-time systems still remain open and will be considered in a future research.

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H_∞ Control of Linear Uncertain Time-Delay Systems—A Projection Approach

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***Abstract*—The issues of stability and H_∞ control of linear systems with time-varying delays are considered. Based on the Lyapunov–Krasovskii approach and on Finsler's projection lemma, delay-dependent sufficient conditions are obtained, in terms of linear matrix inequalities (LMIs), for the stability of these systems. These conditions generalize previous results that were derived using either the descriptor approach or the first and the third model transformations. The obtained criteria are extended to deal with: stabilizability, the bounded real lemma and the H_∞ state-feedback control.**

***Index Terms*—Lyapunov–Krasovskii approach, neutral systems, robust H_∞ control, time-delay systems.**

I. Introduction

During the last decade, a considerable amount of attention has been paid to stability and control of linear systems with uncertain delays (either constant or time-varying) lying in the given segment $[0, h]$ (see, e.g., [1]–[6] and the references therein). The so-called delay-dependent sufficient stability conditions in terms of linear matrix inequalities (LMIs) have been derived by using Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions (the latter is usually more conservative). Delay-dependent conditions via Lyapunov–Krasovskii functionals are based on different model transformations. Each model transformation leads to a corresponding form of Lyapunov–Krasovskii functional. The third model transformation (according to the classification of [2]), which was applied in [7] and [8], and the most recent and less conservative one, the descriptor representation of the system [4]–[6], lead to the same Lyapunov–Krasovskii functional depending on the derivatives of the state. The derivative of this functional is, however, different in the two approaches, where in the descriptor approach both the state vector and its derivative appear in the expression for the derivative of the Lyapunov–Krasovskii functional along the trajectories of the system.

For systems without delays the LMI stability conditions are obtained either by directly differentiating the quadratic Lyapunov function along the system trajectory [9] or by applying Finsler's lemma [10]. It turns out that the LMIs that are obtained by the latter two methods are equivalent in the case without uncertainty, but since the LMIs that are based on Finsler's lemma possess more degrees of freedom they provide better results in the case where parameter uncertainty is encountered [10].

It is shown in the present note that similar improvement is achieved when applying Finsler's lemma to the robust analysis and design of retarded and neutral systems. The sufficient conditions that are obtained for testing stability and for the bounded real lemma (BRL) are more general than the results achieved based on the descriptor approach or on the first and the third model transformations (see the classification of [2]). The latter results are obtained as a special case of the new conditions by taking few of the additional free matrix parameters to be zero. Moreover, for the first time, it is theoretically proved that the descriptor approach conditions are generalization of the conditions based on the first and the third model transformations and, thus, less conservative. Utilizing the geometric structure of the resulting inequalities, for these special cases, the results of [4]–[6] are obtained by solving LMIs with fewer decision variables, where there is no longer a need to find all the matrix blocks of the Lyapunov's kernel matrix explicitly.

A new effective method for state-feedback design is introduced. The merit of the new results lies not only in the fact that it provides another geometric approach to the analysis and the synthesis of retarded systems and that it reduces the complexity of the resulting LMIs. The main merit of the proposed method is the fact that it provides additional degrees of freedom which, similar to the case without delay, lead to less conservative results when uncertainty of the polytopic type is encountered. Some effort of applying Finsler's lemma to the case of systems with time delay has been recently made. A generalization of [8] was obtained in [11] where the elimination lemma was used to generalize the results of [8] that are based on the third model transformation. A delay-independent stability conditions via Finsler's lemma have been derived recently in [12].

II. Stability

Consider the following system (the system can be extended to include more delays):

$$\dot{x}(t) - F\dot{x}(t-g) = A_0x(t) + A_1x(t-\tau(t)),\ t \geq t_0$$
$$x(\theta) = \phi(\theta),\ \theta \in \mathcal{E}_{t_0} \qquad (1)$$

where $x(t) \in \mathcal{R}^n$ is the system state, A_0, A_1 and F are constant $n \times n$-matrices, t_0 is a given initial time, ϕ is a continuously differentiable initial function and $\mathcal{E}_{t_0} = \{\theta \in \mathcal{R} : \theta = \eta - \tau(\eta) \leq t_0,\ \eta \geq t_0\} \cup [t_0 - g,\ t_0]$. It is assumed that g is a known constant delay and that the delay $\tau(t)$ is a bounded differentiable function that satisfies

$$0 \leq \tau \leq h,\ \dot{\tau}(t) \leq d < 1. \qquad (2)$$

Moreover, it is assumed that all the eigenvalues of F are inside the unit circle. The latter guarantees that the difference equation $x(t) - Fx(t-$

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