LMI Methods in Optimal and Robust Control

Matthew M. Peet (Also contributions from Sanjay Lall)
Arizona State University

Lecture 5: LMIs for Controllability and Feedback Stabilization

Recall: Stability and Solutions of the Lyapunov Equation

Lemma 1.

A is Hurwitz if and only if for any $Q > 0$, there exists a $P > 0$ such that

$$A^T P + PA = -Q < 0$$

One such solution is:

$$P = \int_0^\infty e^{A^T s} Q e^{As} ds$$

Lemma 2.

A is Schur if and only if for any $Q > 0$, there exists a $P > 0$ such that

$$A^T PA - P = -Q < 0$$

One Such solution is:

$$P = \sum_{k=0}^\infty (A^T)^k Q A^k$$

Recall: Stability and Solutions of the Lyapunov Equation

Lemma 1.
A is Hurwitz if and only if for any $Q > 0$, there exists a $P > 0$ such that
$$A^T P + PA = -Q < 0$$

One such solution is:
$$P = \int_0^\infty e^{A^T s} Q e^{A s} ds$$

Lemma 2.
A is Schur if and only if for any $Q > 0$, there exists a $P > 0$ such that
$$A^T PA - P = -Q < 0$$

One Such solution is:
$$P = \sum_{k=0}^\infty (A^T)^k Q A^k$$

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

In the previous lecture, we focused on properties of the map

$$x_0 \mapsto x(t) = e^{At} x_0$$

in terms of the eigenvalues of A.

- We proposed LMIs to constrain the eigenvalues of A.

In this lecture we look at the effect of the *input*, $u(\cdot)$, on the *state*, $x(t)$ and on the *output*, $y(t)$.

$$u(\cdot) \mapsto y(\cdot)$$

and

$$u(\cdot) \mapsto x(\cdot)$$

In this case, the map is more complicated. However, we will attempt to characterize its properties in terms of properties of the matrices (A, B, C)
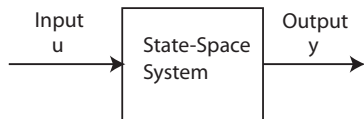
Find the output given the input
Solution for State-Space

State-Space System:
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$

Input u → State-Space System → Output y

Input-Output Map:
$$x(t) = \int_0^t e^{A(t-s)}Bu(s)ds$$
$$y(t) = Cx(t) + Du(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$$

Can we get to any desired state, $x(t)$, by using $u(t)$?

- How fast can we get there?
- What about if we use *feedback*: $u(t) = Kx(t)$?

For a discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k \qquad x_0 = 0$$

The solution is

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu_i$$

$$y_k = CA^k x_0 + \sum_{i=0}^{k-1} CA^{k-i-1} Bu_i + Du_k$$

Where, again, we usually take $x_0 = 0$.

Controllability

Definition 3.

For a given continuous-time system (A, B), the **state x_f is Reachable** if for any fixed T_f, there exists a $u(t)$ such that

$$x_f = \int_0^{T_f} e^{A(T_f - s)} B u(s) ds$$

Definition 4.

The **continuous-time system** (A, B) **is reachable** if any point $x_f \in \mathbb{R}^n$ is reachable.

Definition 5.

The **continuous-time system** (A, B) **is controllable** if for any T_f and any *initial* point $x_0 \in \mathbb{R}^n$, there exists a $u(t)$ such that $x(T_f) = 0$.

The reachable set and controllable set are Subspaces.

Definition 3.
For a given continuous-time system (A, B), the **state** x_f **is Reachable** if for any fixed T_f, there exists a $u(t)$ such that

$$x_f = \int_0^{T_f} e^{A(T_f - s)} B u(s) ds$$

Definition 4.
The **continuous-time system** (A, B) **is reachable** if any point $x_f \in \mathbb{R}^n$ is reachable.

Definition 5.
The **continuous-time system** (A, B) **is controllable** if for any T_f and any initial point $x_0 \in \mathbb{R}^n$, there exists a $u(t)$ such that $x(T_f) = 0$.

The reachable set and controllable set are Subspaces.

- The difference between controllability and reachability and stabilizability is subtle. The difference from stabilizability is the finite-time question. The difference between reachability and controllability is away from origin vs. to origin. For a linear system, we can move the origin (introducing a bias in the input), which implies there is no difference at all for these systems.

The reachable set is defined by the map from input to solution at time T

$$u(\cdot) \mapsto x(T)$$

The reachable set is the image space of this map, $\Gamma_T : u \mapsto x$, where

$$x(T) = \underbrace{\int_0^T e^{A(T-s)} B u(s) ds}_{\Gamma_T}$$

Review: Subspaces, Image, and Kernel

Definition 6.

A set, C, is a **Vector Space** if it is closed under addition and scalar multiplication.

1. $\alpha(u + v) = \alpha u + \alpha v \in C$ for all $\alpha \in \mathbb{R}$ and $u, v \in C$. (vector distributivity)
2. $(\alpha + \beta)u = \alpha u + \beta u \in C$ for all $\alpha, \beta \in \mathbb{R}$ and $u \in C$. (scalar distributivity)

Definition 7.

A **subspace** is a subset of a vector space which is also a vector space using the same definitions of addition and multiplication.

For right now, the most important subspaces are the image and kernel of a matrix ($M \in \mathbb{R}^{n \times m}$)/function/operator.

Definition 8 (Image and Kernel of a Matrix, M).

$$\operatorname{Im} M := \{x \in \mathbb{R}^n \ : \ x = My \ \text{for some} \ y \in \mathbb{R}^m\}$$
$$\ker M := \{x \in \mathbb{R}^m \ : \ Mx = 0\}$$

Controllability

For a fixed t, the set of reachable states is defined as
$$R_t := \{x \ : \ x = \int_0^t e^{A(t-s)} Bu(s)ds \text{ for some function } u.\}$$
Note: The mapping $\Gamma_t : u \mapsto x(t)$ is *linear*.

- Hence $R_t = \operatorname{Im} \Gamma_t$ is a subspace of $\mathbb{R}^n$

Definition 9.

For a given system (A, B), the **Controllability Matrix** is
$$C(A, B) := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$
where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

Definition 10.

For a given (A, B), the **Controllable Subspace** is
$$C_{AB} = \operatorname{Image} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Fact: The system (A, B) is **Controllable** if $C_{AB} = \operatorname{Im} C(A, B) = \mathbb{R}^n$.

For a fixed t, the set of reachable states is defined as

$$R_t := \{x \; : \; x = \int_0^t e^{A(t-s)} B u(s) ds \text{ for some function } u.\}$$

Note: The mapping $\Gamma_t : u \mapsto x(t)$ is *linear*.
  • Hence $R_t = \operatorname{Im} \Gamma_t$ is a subspace of $\mathbb{R}^n$

Definition 9.

For a given system (A, B), the **Controllability Matrix** is
$$C(A, B) := [B \quad AB \quad A^2 B \quad \cdots \quad A^{n-1} B]$$
where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

Definition 10.

For a given (A, B), the **Controllable Subspace** is
$$C_{AB} = \operatorname{Image} [B \quad AB \quad A^2 B \quad \cdots \quad A^{n-1} B]$$

Fact: The system (A, B) is **Controllable** if $C_{AB} = \operatorname{Im} C(A, B) = \mathbb{R}^n$.

$C(A, B)$ is a matrix and C_{AB} is the image of this matrix.
Controllability is a property of the *system*

  • Thus we characterize a property of the system map $u(\cdot) \mapsto x(\cdot)$ using properties of the matrices B and A.

There is a more physical interpretation:

  • B is the set of states input u affects directly. AB is the set of states u affects by affecting one state and then that state affects another. etc. until we achieve $n - 1$ degrees of separation, which by Cayley-Hamilton, means we can stop looking.

Controllability

Definition 11.

The finite-time **Controllability Grammian** of pair (A, B) is

$$W_t := \int_0^t e^{As} BB^T e^{A^T s} ds$$

W_t is a positive semidefinite matrix.
The following relates these three concepts of controllability

Theorem 12.

For any $t \geq 0$,

$$R_t = C_{AB} = \textit{Image } (W_t)$$

or

$$\textsf{Image } \Gamma_t = \textsf{Image } C(A, B) = \textsf{Image } (W_t)$$

W_t is positive **Definite** if and only if (A, B) is controllable.
Note the reachable set does not depend on t!

Controllability

Definition 11.
The finite-time **Controllability Grammian** of pair (A, B) is
$$W_t := \int_0^t e^{As} BB^T e^{A^T s} ds$$

W_t is a positive semidefinite matrix.
The following relates these three concepts of controllability

Theorem 12.
For any $t \geq 0$,
$$R_t = C_{AB} = \text{Image}(W_t)$$

or
$$\text{Image } \Gamma_t = \text{Image } C(A, B) = \text{Image}(W_t)$$

W_t is positive **Definite** if and only if (A, B) is controllable.
Note the reachable set does not depend on t!

This result connects

1. Existence of positive matrix (not yet an LMI, however)

2. Properties of A, B

3. Properties of the map $u \mapsto x$

Controllability: $Im(W_t) \subset R_t$

Proposition 1.

$$Im(W_t) \subset R_t$$

Proof.

First, suppose that $x \in Im(W_t)$ for some $t > 0$. Then $x = W_t z$ for some z.

- Now let $u(s) = B^T e^{A^T(t-s)} z$. Then

$$\Gamma_t u = \int_0^t e^{A(t-s)} B u(s) ds$$

$$= \int_0^t e^{A(t-s)} B B^T e^{A^T(t-s)} z \, ds = W_t z = x$$

- Thus $x \in Im(\Gamma_t) = R_t$. Hence $Im(W_t) \subset R_t$. $\square$

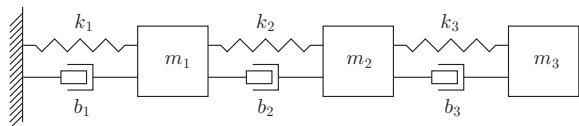This proof is useful, since for any $x_d \in R_{T_f}$, it gives us the input

- Let $u(t) = B^T e^{A^T(T_f - t)} W_{T_f}^{-1} x_d$.
- Then for $\dot{x}(t) = Ax(t) + B(t)$, $x(0) = 0$, we have $x(T_f) = x_d$.

Lecture 5
└─ Controllability

    └─ Controllability: $\text{Im}(W_t) \subset R_t$

This tells us a specific $u(\cdot)$ where

$$u(\cdot) \mapsto x(T_f) = x_d$$
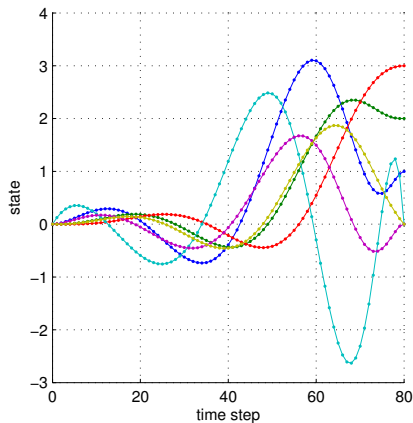
Numerical Example

masses $m_i = 1$, spring constants $k = 1$, damping constants $b = 0.8$

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -1.6 & 0.8 & 0 \\ 1 & -2 & 1 & 0.8 & -1.6 & 0.8 \\ 0 & 1 & -1 & 0 & 0.8 & -0.8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$u(t)$ is force applied to mass 1

$x_{\mathsf{des}} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$ at time step $T = 80$.

Numerical Example
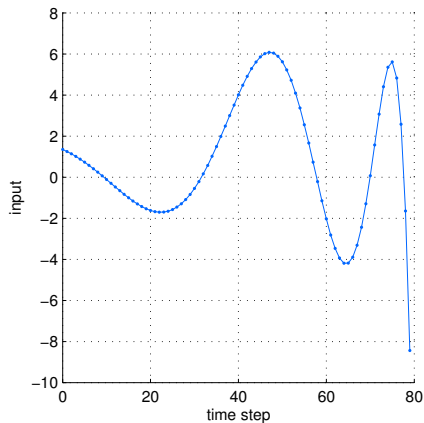
sampling period $h = 0.1$, optimal input achieves desired state

$$x(80) = x_{\text{des}} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$$

The Controllability Gramian and Ellipsoid

Minimum Energy Ellipsoid: The set of states reachable in time T with an input u of size $\|u\|_{L_2}^2 = \int_0^T \|u(t)\|^2 dt = 1$ is

$$\{x \in \mathbb{R}^n \ : \ x^T W_T^{-1} x \leq 1\}$$

- Ellipsoid with semiaxis lengths $\lambda_i(W_T)$
- Ellipsoid with semiaxis directions given by eigenvectors of W_T

Extend this to infinite time:

Definition 13.

The **Controllability Gramian** of pair (A, B) is

$$W := \int_0^\infty e^{As} B B^T e^{A^T s} ds$$

The Controllability Gramian tells us which directions are *easily* controllable.

Lemma 14 (An LMI for the Controllability Gramian).

If (A, B) is controllable, then $W > 0$ is the unique solution to

$$AW + WA^T + BB^T = 0$$

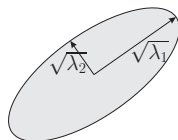Of course, one could also solve this as a set of linear equations.

The Controllability Gramian and Ellipsoid

Minimum Energy Ellipsoid: The set of states reachable in time T with an input u of size $\|u\|_{L_2}^2 = \int_0^T \|u(t)\|^2 dt = 1$ is
$$\{x \in \mathbb{R}^n : x^T W_T^{-1} x \leq 1\}$$
- Ellipsoid with semiaxis lengths $\lambda_i(W_T)$
- Ellipsoid with semiaxis directions given by eigenvectors of W_T

Extend this to infinite time:

Definition 13.

The **Controllability Gramian** of pair (A, B) is
$$W := \int_0^\infty e^{As} BB^T e^{A^T s} ds$$

The Controllability Gramian tells us which directions are easily controllable.

Lemma 14 (An LMI for the Controllability Gramian).

If (A, B) is controllable, then $W > 0$ is the unique solution to
$$AW + WA^T + BB^T = 0$$

Of course, one could also solve this as a set of linear equations.
Also, it doesn't give us the controller.

- To get from $x(0) = 0$ to $x(T_f) = x_d$ we may use
  $$u(t) = B^T e^{A^T (T_f - t)} W_{T_f}^{-1} x_d$$

- We can see that the input u which achieves x_d has magnitude

$$\|u\|_{L_2}^2 = \int_0^t \|u(s)\|^2 ds = \int_0^t x_d^T W_{T_f}^{-1} e^{A(T_f - s)} BB^T e^{A^T(T_f - s)} W_{T_f}^{-1} x_d ds$$
$$= x_d^T W_{T_f}^{-1} \left(\int_0^t e^{A(T_f - s)} BB^T e^{A^T(T_f - s)} ds \right) W_{T_f}^{-1} x_d$$
$$= x_d^T W_{T_f}^{-1} W_{T_f} W_{T_f}^{-1} x_d = x_d^T W_{T_f}^{-1} x_d$$

Hence if $x^T W_t^{-1} x \leq 1$, x is reachable at time t with input of size $\|u\| \leq 1$.

- The u defined in this way is actually *optimal*.

- Eigenvalues and SVD are the same here: $W_t = U \Sigma U^T$ so $W_t^{-1} = U^T \Sigma^{-1} U$. Hence if x_d is a unit eigenvector/singular vector v_i with eigenvalue/singular value σ_i, $\|u\|_{L_2} = \sqrt{\sigma_i^{-1}}$.

Stabilizability

Stabilizability is weaker than controllability

Definition 15.

The pair (A, B) is stabilizable if for any $x(0) = x_0$, there exists a $u(t)$ such that $x(t) = \Gamma_t u$ satisfies
$$\lim_{t \to \infty} x(t) = 0$$

- Again, no restriction on $u(t)$.
- Weaker than controllability
    - **Controllability:** Can we drive the system to $x(T_f) = 0$?
    - **Stabilizability:** Only need to *Approach* $x = 0$.
- Stabilizable if uncontrollable subspace is naturally stable.

Lemma 16.

(A, B) is stabilizable if and only if there exists a $X > 0$, $\gamma > 0$ such that

$$AX + XA^T - \gamma BB^T < 0$$

where the stabilizing controller is $u(t) = -\frac{1}{2} B^T X^{-1} x(t)$

Stabilizability

Stabilizability is weaker than controllability

Definition 15.

The pair (A, B) is stabilizable if for any $x(0) = x_0$, there exists a $u(t)$ such that $x(t) = \Gamma_t u$ satisfies

$$\lim_{t \to \infty} x(t) = 0$$

• Again, no restriction on $u(t)$.
• Weaker than controllability
  ▶ **Controllability:** Can we drive the system to $x(T_f) = 0$?
  ▶ **Stabilizability:** Only need to *Approach* $x = 0$.
• Stabilizable if uncontrollable subspace is naturally stable.

Lemma 16.

(A, B) is stabilizable if and only if there exists a $X > 0$, $\gamma > 0$ such that

$$AX + XA^T - \gamma BB^T < 0$$

where the stabilizing controller is $u(t) = -\frac{1}{2}B^T X^{-1} x(t)$

• Note that feasibility of the stabilizability LMI does NOT require A to be stable.

$$AX + XA^T < \gamma BB^T$$

means that $AX + XA^T < 0$ only for those x in the perp of the image space of B.

• The stabilizing controller is a feedback gain!

For this LMI, the solution used in the proof is

$$X = \hat{W}_t = \gamma \int_0^t e^{-As} BB^T e^{-A^T s} ds$$

The reverse-time controllability grammian...

Eigenvalue Assignment
Static Full-State Feedback

Recall our result on Reachability:

- To reach $x(T_f) = z_f$
  - $u(t) = B^T e^{A(T_f - t)} W_T^{-1} z_f$
  - This controller is <u>open-loop</u>
- It assumes perfect knowledge of system and state.

Problems

- Prone to Errors, Disturbances, Errors in the Model

Solution

- Use continuous measurements of state to generate control

Static Full-State Feedback Assumes:

- We can directly and continuously measure the state $x(t)$
- Controller is a static linear function of the measurement

$$u(t) = Kx(t), \qquad K \in \mathbb{R}^{m \times n}$$

Previously the problem was *Analysis*:

• Determine properties of the maps $x_0 \mapsto x(\cdot)$ and $u(\cdot) \mapsto x(t)$

Now the problem becomes *Synthesis*:

• Alter the dynamics

• Change the matrix, A, so that it has desired properties

Synthesis is always a two-part, non-convex (typically bilinear) problem:

• Modify $A \mapsto A + BK$

• Ensure $A + BK$ has desired properties

These must be done *simultaneously*!

Eigenvalue Assignment
Static Full-State Feedback

State Equations: $u(t) = Kx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$= Ax(t) + BKx(t)$$
$$= (A + BK)x(t)$$

Stabilization: Find a matrix $K \in \mathbb{R}^{m \times n}$ such that

$$A + BK$$

is Hurwitz.

Eigenvalue Assignment: Given $\{\lambda_1, \cdots, \lambda_n\}$, find $K \in \mathbb{R}^{m \times n}$ such that

$$\lambda_i \in \text{eig}(A + BK) \qquad \text{for } i = 1, \cdots, n.$$

Note: A solution to the eigenvalue assignment problem can also solve the stabilization problem.

Question: Is eigenvalue assignment actually harder?

Eigenvalue Assignment
Single-Input Case

Theorem 17.

Suppose $B \in \mathbb{R}^{n \times 1}$. Eigenvalues of $A + BK$ are freely assignable if and only if (A, B) is controllable.

Use `place` to assign eigenvalues.
But this is a course on LMIs, so we take a different approach.

The Static State-Feedback Problem

Lets start with the problem of stabilization.

Definition 18.

The **Static State-Feedback Problem** is to find a feedback matrix K such that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t)$$

is stable

- Find K such that $A + BK$ is Hurwitz.

Can also be put in LMI format:

$$\textbf{Find } X > 0, K :$$
$$X(A + BK) + (A + BK)^T X < 0$$

Problem: Bilinear in K and X.

The Static State-Feedback Problem

Lets start with the problem of stabilization.

Definition 18.
The **Static State-Feedback Problem** is to find a feedback matrix K such that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t)$$

is stable

• Find K such that $A + BK$ is Hurwitz.

Can also be put in LMI format:

Find $X > 0$, K :
$$X(A + BK) + (A + BK)^T X < 0$$

Problem: Bilinear in K and X.

State-feedback refers to the fact that $u(t) = Kx(t)$ is a function of all the states, which we assume are all individually measurable. Static refers to the fact that the linear function Kx does not vary in time.

- Resolving this bilinearity is a quintessential step in the controller synthesis process.

- Carries over throughout the course in various generalizations

- The resolution is quite simple and elegant.

An Equivalent LMI for Static State-Feedback

- The bilinear problem in K and X is a common paradigm.
- Bilinear optimization is not convex.
- To convexify the problem, we use a change of variables.

Problem 1:

$$\textbf{Find } X > 0, K :$$
$$X(A + BK) + (A + BK)^T X < 0$$

Problem 2:

$$\textbf{Find } P > 0, Z :$$
$$AP + BZ + PA^T + Z^T B^T < 0$$

Definition 19.

Two optimization problems are equivalent if a solution to one will provide a solution to the other.

Theorem 20.

Problem 1 is equivalent to Problem 2.

The Dual Lyapunov LMI

Problem 1:

> **Find** $X > 0$, :
>
> $XA + A^TX < 0$

Problem 2:

> **Find** $Y > 0$, :
>
> $YA^T + AY < 0$

Lemma 21.

Problem 1 is equivalent to problem 2.

Proof.

First we show 1) solves 2). Suppose $X > 0$ is a solution to Problem 1. Let $Y = X^{-1} > 0$.

- If $XA + A^TX < 0$, then

$$X^{-1}(XA + A^TX)X^{-1} < 0$$

- Hence

$$X^{-1}(XA + A^TX)X^{-1} = AX^{-1} + X^{-1}A^T = AY + YA^T < 0$$

- Therefore, Problem 2 is feasible with solution $Y = X^{-1}$.

The Dual Lyapunov LMI

Problem 1:

$$\text{Find } X > 0, :$$
$$XA + A^TX < 0$$

Problem 2:

$$\text{Find } Y > 0, :$$
$$YA^T + AY < 0$$

Proof.

Now we show 2) solves 1) in a similar manner. Suppose $Y > 0$ is a solution to Problem 1. Let $X = Y^{-1} > 0$.

- Then

$$XA + A^TX = X(AX^{-1} + X^{-1}A^T)X$$
$$= X(AY + YA^T)X < 0$$

$\square$

Conclusion: If $V(x) = x^TPx$ proves stability of $\dot{x} = Ax$,

- Then $V(x) = x^TP^{-1}x$ proves stability of $\dot{x} = A^Tx$.

A Stabilization LMI using the Variable Substitution Trick

Thus we rephrase Problem 1

Problem 1:

    Find $P > 0, K$:

$$(A + BK)P + P(A + BK)^T < 0$$

Problem 2:

    Find $X > 0, Z$:

$$AX + BZ + XA^T + Z^T B^T < 0$$

Theorem 22.

Problem 1 is equivalent to Problem 2.

Proof.

We will show that 2) Solves 1). Suppose $X > 0$, Z solves 2). Let $P = X > 0$ and $K = ZP^{-1}$. Then

$$
\begin{aligned}
(A + BK)P + P(A + BK)^T &= AP + PA^T + BKP + PK^T B^T \\
&= AP + PA^T + BZ + Z^T B^T < 0
\end{aligned}
$$

Now suppose that $P > 0$ and K solve 1). Let $X = P > 0$ and $Z = KP$. Then

$$AP + PA^T + BZ + Z^T B^T = (A + BK)P + P(A + BK)^T < 0$$

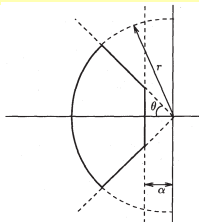# The Stabilization Problem

The result can be summarized more succinctly

Theorem 23.

(A, B) is static-state-feedback stabilizable if and only if there exists some $P > 0$ and Z such that

$$AP + PA^T + BZ + Z^T B^T < 0$$

with $u(t) = ZP^{-1}x(t)$.

Controllers for D-stability

Recall the LMI for D-stability where we add in the controller $u = Kx$.

Theorem 24.

The pole locations, $z \in \mathbb{C}$ of $A + BK$ satisfy $|z| \leq r$, $\mathrm{Re}\, x \leq -\alpha$ and $z + z^* \leq -c|z - z^*|$ if and only if there exists some $P > 0$ such that

$$\begin{bmatrix} -rP & (A+BK)P \\ ((A+BK)P)^T & -rP \end{bmatrix} < 0,$$

$$(A+BK)P + ((A+BK)P)^T + 2\alpha P < 0, \quad and$$

$$\begin{bmatrix} (A+BK)P + ((A+BK)P)^T & c((A+BK)P - ((A+BK)P)^T) \\ c(((A+BK)P)^T - (A+BK)P) & (A+BK)P + ((A+BK)P)^T \end{bmatrix} < 0$$

Note the D-stability LMI already appears in "Dual" Form

- LMIs are particularly useful in that they allow one to directly and sequentially impose constraints on the variables by combining different LMI constraints into a single LMI.

- So we can add closed-loop eigenvalue constraints.

- Or robustness constraints.

- However, this is limited by the variable substitution process $Z = KQ$ and $P = Q^{-1}$.

- Old variables K, P must not appear anywhere in the LMI.

Controllers for D-stability

Then we have an LMI which gives us a controller for D-stabilization

Lemma 25 (An LMI for D-Stabilization).

Suppose there exists $X > 0$ and Z such that
$$\begin{bmatrix} -rP & AP + BZ \\ (AP + BZ)^T & -rP \end{bmatrix} < 0,$$
$$AP + BZ + (AP + BZ)^T + 2\alpha P < 0, \quad and$$
$$\begin{bmatrix} AP + BZ + (AP + BZ)^T & c(AP + BZ - (AP + BZ)^T) \\ c((AP + BZ)^T - (AP + BZ)) & AP + BZ + (AP + BZ)^T \end{bmatrix} < 0$$

Then if $K = ZP^{-1}$, the pole locations, $z \in \mathbb{C}$ of $A + BK$ satisfy $|x| \leq r$,
$\operatorname{Re} x \leq -\alpha$ *and* $z + z^* \leq -c|z - z^*|$.

The Discrete-Time Case

Now consider the discrete-time system:

$$x_{k+1} = Ax_k + Bu_k$$

For discrete-time, controllability and reachability are not equivalent!

- Consider $x_{k+1} = 0$. Controllable, but not reachable.
- Lets ignore these pathological cases

Definition 26.

The Discrete-Time **Controllability Gramian** of pair (A, B) is

$$W := \sum_0^\infty A^k BB^T (A^T)^k$$

Lemma 27 (An LMI for the Controllability Gramian).

If (A, B) is controllable, then $W > 0$ is the unique solution to

$$A^TWA - W = -BB^T$$

The Discrete-Time Stabilization Problem

Again, we seek a feedback controller $u_k = Kx_k$ for which the closed-loop is Schur.

State Equations: $u_k = Kx_k$

$$x_{k+1} = Ax_k + Bu_k$$
$$= Ax_k + BKx_k$$
$$= (A + BK)x_k$$

Stabilization: Find a matrix $F \in \mathbb{R}^{m \times n}$ such that

$$A + BK$$

is Schur. Recall that $A + BK$ is Schur if and only if there exists a $P > 0$ such that $(A + BK)^T P(A + BK) - P < 0$. Hence the following non-LMI problem

Find $P > 0$, K :
$$(A + BK)^T P(A + BK) - P < 0$$

The Discrete-Time Stabilization Problem

Now consider the Schur Stability condition:

$$(A + BK)^T P(A + BK) - P < 0$$

Pre- and Post-multiplying by P^{-1} shows this matrix inequality is equivalent to

$$P^{-1} - P^{-1}(A + BK)^T P(A + BK)P^{-1} > 0$$

Applying the Schur Complement, this matrix inequality is equivalent to

$$\begin{bmatrix} P^{-1} & (A + BK)P^{-1} \\ P^{-1}(A + BK)^T & P^{-1} \end{bmatrix} > 0$$

The Discrete-Time Stabilization Problem

We now have the following two equivalent problems:

Problem 1:
Find $P > 0$, K such that

$$P - (A + BK)^T P(A + BK) > 0$$

Problem 2:
Find $X > 0$, K such that

$$\begin{bmatrix} X & (A + BK)X \\ X(A + BK)^T & X \end{bmatrix} > 0$$

Taking Problem 2 and using the change of variables $Z = KX$, we get an LMI:

Lemma 28.

Suppose there exists some $X > 0$ and Z such that

$$\begin{bmatrix} X & AX + BZ \\ (AX + BZ)^T & X \end{bmatrix} > 0$$

then if $K = ZX^{-1}$, the closed-loop system matrix $(A + BK)$ is Schur.

The Discrete-Time Stabilization Problem

Final Note: An Alternative LMI condition for stabilizability is as follows

Lemma 29.

The pair (A, B) is discrete-time stabilizable if and only if there there exists some $P > 0$ such that
$$APA^T - P < BB^T.$$