Data-driven input reconstruction

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***Abstract*— This paper addresses a data-driven input reconstruction problem based on Willems' Fundamental Lemma. The corresponding unknown input estimator (UIE) is constructed directly from historical data. Given the output measurement, the estimated input by the UIE asymptotically converges to the actual input without knowing the initial conditions. The necessary and sufficient existence conditions of such a UIE are provided. Two design methods are elaborated based on the Lyapunov condition and generalized inverse of Hankel matrices. Finally, the proposed data-driven UIE is validated by both simulation and experiment.**

I. INTRODUCTION

The input reconstruction problem has intrigued the control community for decades with applications in system supervision, sensor fault detection, and robust control [1], [2], [3], [4]. In such a problem, the output of a so-called unknown input estimator (UIE) converges to the original system's unknown input while the original output measurement is given, but the initial system condition is unknown. In [1], input observability is defined to check the existence condition of the estimator for LTI system, and a systematic design approach is given based on a matrix pencil decomposition form. [5] propose to find the stable inverse system for input reconstruction and define the necessary and sufficient conditions for stable invertibility.

More recently, simultaneous unknown input reconstruction and state estimation are achieved by unknown input observer (UIO) [6], [7], optimal filters [8], [9], generalized inverse approach [10] and sliding mode observers [11]. In [6], an elegant design procedure is proposed to design a full-order UIO. Given the estimated state by UIO, the input estimate can be computed by matrix inversion and converge to the actual input along with the state convergence. [10] utilizes the generalized matrix for simultaneous state estimation and input reconstruction, and the estimation is achieved in a finite number of steps.

All the above work assumes the system model is given. Therefore a step of system identification should be performed before designing the input reconstruction. As an alternative to the above two-step workflow, we utilize the behavioral theory to directly design an unknown input estimator from data in this work. The Williems' Fundamental Lemma is such a method that describes the trajectory of the LTI system based on historical measured input and output signals [12]. This description provides a convenient interface to data-driven output prediction [13] and different controller design [14], [15], [16], [17], [18], [19].

In [20], the authors build a data-driven UIO based on historical state, unknown input, and output data. Our work is distinguished from it with a specific focus on input reconstruction and no state knowledge required. Furthermore, we claim some freedom to design the data-driven UIE, and two specific design procedures will be presented.

Occupancy estimation by the indoor carbon dioxide (CO2) levels is the motivation of this work. In the building community, the occupant number is a critical factor for human comfort satisfaction and building energy simulation and control [21]. However, some occupancy detection tools, such as cameras and Wifi, suffer from costs and privacy issues, while CO2 sensors serve as an alternative choice [22]. Thus, we assume that for the original system, output y_t is always measurable, but only a limited number of inputs u_t can be measured offline. Measurable inputs are omitted for simplicity, which does not affect the generality of the discussion. This situaltion is also common in the industry, as laboratory experiments can measure some input offline. Nevertheless, it is unrealistic or costly to measure online, such as the machine tool's cutting force [11].

In the following, the output prediction based on Willems' Fundamental Lemma is reviewed, and the target unknown input estimator is defined in Section II. In Section III and IV, a open-loop UIE and a closed-loop UIE are proposed, respectively, with their sufficient and necessary existence conditions. In addition, the design and operation process of the UIEs is elaborated. Section V shows the results of a simple simulation and an experimental application. We conduct a case study of occupancy estimation based on indoor CO2 levels in an actual building. It concludes the paper and provides some future potentials in Section VI.

Notation

$\mathbf{I}_n \in \mathbb{R}^{n\times n}$ denotes an identity matrix. Regarding a matrix M, its numbers of columns and rows are respectively denoted by n_M and m_M such that $M \in \mathbb{R}^{n_M \times m_M}$. Accordingly, Null(M) denotes its null space. Similarly, the dimension of a vector s denoted by n_s. Given an ordered sequence of vector $\{s_t, s_{t+1}, \dots, s_{t+L}\}$, its vectorization is denoted by $s_{t:t+L} = [s_t^\top, \dots, s_{t+L}^\top]^\top$.

some notation to add: null space. Done

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You might need to include the funding with Lothar?

II. PRELIMINARIES

Consider a discrete-time linear time-invariant (LTI) system, dubbed $\mathcal{B}(A,B,C,D)$, is defined as

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \ , \\ y_t &= Cx_t + Du_t \ , \end{aligned} \tag{1}$$

whose states, inputs and outputs are denoted by $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ separately. The order of the system is defined as $\mathbf{n}(\mathcal{B}(A,B,C,D)) := n_x$. The lag of the system $\mathbf{l}(\mathcal{B}(A,B,C,D))$ is defined as the smallest integer ℓ such that the observability matrix $\mathcal{O}_\ell$

$$\mathcal{O}_\ell := \left[C^\top, (CA)^\top, \dots, (CA^{\ell-1})^\top\right]^\top$$

has rank n_x. An L-step I/O trajectory generated by this system concatenates I/O sequence by $[u_{1:L}; y_{1:L}]$, and the set of all possible L-step trajectories generated by $\mathfrak{B}(A,B,C,D)$ is denoted by $\mathfrak{B}_L(A,B,C,D)$.

Definition 1: A Hankel matrix of depth L constructed by a vector-valued signal sequence $s := \{s_i\}_{i=1}^T,\ s_i \in \mathbb{R}^{n_s}$ is

$$\mathfrak{H}_L(s) := \begin{bmatrix} s_1 & s_2 & \dots & s_{T-L+1} \\ s_2 & s_3 & \dots & s_{T-L+2} \\ \vdots & \vdots & & \vdots \\ s_L & s_{L+1} & \dots & s_T \end{bmatrix} .$$

Given a sequence of input-output measurements $\{u_{\mathbf{d},i}, y_{\mathbf{d},i}\}_{i=1}^T$, the input sequence is called *persistently exciting* of order L if $\mathfrak{H}_L(u_\mathbf{d})$ is full row rank. By building the following n_c-column stacked Hankel matrix

$$\mathfrak{H}_L(u_\mathbf{d}, y_\mathbf{d}) := \begin{bmatrix} \mathfrak{H}_L(u_\mathbf{d}) \\ \mathfrak{H}_L(y_\mathbf{d}) \end{bmatrix} ,$$

we state **Willems' Fundamental Lemma** as

Lemma 1: [12, Theorem 1] Consider a controllable linear system $\mathfrak{B}(A,B,C,D)$ and assume $\{u_\mathbf{d}\}_{i=1}^T$ is persistently exciting of order $L + \mathbf{n}(\mathfrak{B}(A,B,C,D))$. The condition $\text{colspan}(\mathfrak{H}_L(u_\mathbf{d}, y_\mathbf{d})) = \mathfrak{B}_L(A,B,C,D)$ holds.

In the rest of the paper, the subscript $_\mathbf{d}$ marks a data point from the training dataset collected offline, and L is reserved for the length of the system response.

A. Output simulation and prediction

Willems' Fundamental Lemma has been used to develop data-driven output prediction methods [13], [14]. In [13], the N_{pred}-step output prediction $y_{t+1:t+N_{pred}}$ driven by an N_{pred}-step predicted output $u_{t+1:t+N_{pred}}$ is given by the solution to the following equations at time t:

$$\begin{bmatrix} \mathfrak{H}_{L,init}(u_\mathbf{d}) \\ \mathfrak{H}_{L,init}(y_\mathbf{d}) \\ \mathfrak{H}_{L,pred}(u_\mathbf{d}) \end{bmatrix} g = \begin{bmatrix} u_{t-N_{init}+1:t} \\ y_{t-N_{init}+1:t} \\ u_{t+1:t+N_{pred}} \end{bmatrix} \tag{2a}$$

$$\mathfrak{H}_{L,pred}(y_\mathbf{d}) =: y_{t+1:t+N_{pred}} \ . \tag{2b}$$

Two sub-Hankel matrices of output are defined by

$$\mathfrak{H}_L(y_\mathbf{d}) = \begin{bmatrix} \mathfrak{H}_{L,init}(y_\mathbf{d}) \\ \mathfrak{H}_{L,pred}(y_\mathbf{d}) \end{bmatrix} , \tag{3}$$

and each of them of depth N_{init} and N_{pred} respectively, such that $N_{init} + N_{pred} = L$. Similarly, the Hankel matrices $\mathfrak{H}_{L,init}(u_\mathbf{d})$ and $\mathfrak{H}_{L,pred}(u_\mathbf{d})$ are constructed. Last but not least, the solution to (2) is well-defined if $N_{init} \geq \mathbf{l}(\mathcal{B}(A,B,C,D))$. Specifically, this condition implies that $[u_{t-N_{init}+1:t}, y_{t-N_{init}+1:t}]$ the N_{init}-step input output sequences preceding the current point of time can uniquely determine the underlying state x_t. Readers are referred to [13] for more details.

B. Problem formulation

The motivation below can be moved to the introduction. Done

In this work, we aim to construct a data-driven method to estimate input by measured output based on the Fundamental *Lemma* 1. Here we give the definition of UIE as

Definition 2: An unknown input estimator (UIE) for the system (1) is defined in a form of

$$\begin{aligned} z_{t+1} &= A_{UIE} z_t + B_{UIE} d_t \ , \\ \hat{u}_t &= [\mathbf{0}\ I_{n_u}] z_t \end{aligned} \tag{4}$$

with the UIE state $z_t = [\hat{u}_{t-N_{init}+1}^\top, \dots, \hat{u}_t^\top]^\top$ which is a vectorized N_{init}-step input estimate, such that $\hat{u}_t - u_t \to 0$ as $t \to \infty$ for any initial state z_0. Moreover, if it starts from a known initial state $z_0 = u_{-N_{init}+1:0}$, one gets $\hat{u}_t = u_t$, $\forall t \geq 0$.

In the definition, we do not fix the formulation of d_t. In Section III and IV, we will present some approaches to construct a open-loop and a closed-loop UIE using two different d_t.

III. OPEN-LOOP UNKNOWN INPUT ESTIMATOR

In this section, we present a open-loop unknown input estimator (op-UIE) considering a specific d_t. We start by formulating the input estimation by Fundamental Lemma 1. Its explicit solution is analyzed to build the op-UIE in the form of (4). Then we present a sufficient and necessary existence condition for the op-UIE in Theorem 1. Finally, we show how to design such an op-UIE by Lyapunov condition and its sufficient LMI condition.

A. Existence of op-UIE

Given some measured N_{init}-step initial trajectory and the N_{est}-step output measurements, the input u_t can be computed by

$$\begin{bmatrix} \mathfrak{H}_{L,init}(u_\mathbf{d}) \\ \mathfrak{H}_{L,init}(y_\mathbf{d}) \\ \mathfrak{H}_{L,est}(y_\mathbf{d}) \end{bmatrix} g = \begin{bmatrix} u_{t-N_{init}:t-1} \\ y_{t-N_{init}+1:t-1} \\ y_{t:t+N_{est}-1} \end{bmatrix} \tag{5a}$$

$$\mathfrak{H}_{L,est(1)}(u_\mathbf{d}) g =: u_t \tag{5b}$$

The sub-Hankel matrices come from the separation similar to (3) so that $N_{ini} + N_{est} = L$. But $\mathfrak{H}_{L,est(1)}(u_\mathbf{d})$ is first n_u rows of $\mathfrak{H}_{L,est}(u_\mathbf{d})$ and only one input is estimated in (5b). Apart from $N_{init} \geq \mathbf{l}(\mathcal{B}(A,B,C,D))$, the choices of N_{est} for output measurement depends on the properties of matrices $\{B,C,D\}$ in the system (1). For example, if D is full column rank, the single input y_t will determine a unique input u_t after the behind state x_{t+1} is uniquely estimated by N_{init} steps. If state-space model is known, [6] and [23] propose checking conditions for different N_{pred} required.

Image of different u, y

However, (5) can not be directly used for input estimation because $u_{-N_{init}:-1}$ is not measured at the beginning. Next, we analyze its explicit solution and explain how to use it to build the op-UIE.

For simplicity, we rewrite (5) by

$$Hg = b, H := \begin{bmatrix} \mathfrak{H}_{L,init}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,init}(y_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \end{bmatrix}, b := \begin{bmatrix} u_{t-N_{init}:t-1} \\ y_{t-N_{init}+1:t-1} \\ y_{t:t+N_{est}-1} \end{bmatrix}$$
$$H_u g =: u_t, H_u := \mathfrak{H}_{L,est(1)}(u_{\mathbf{d}})$$

Assumption 1: The historical input signals $\{u_{\mathbf{d}}\}_{i=1}^T$ are persistently exciting of order $N_{init} + N_{est} + \mathbf{n}(\mathfrak{B}(A,B,C,D))$.

Under Assumption 1, Lemma 1 guarantees that the existence of g such that $Hg = b$. All solutions of g can be represented by

$$g = Gb + \nu \tag{6}$$

where the matrix G is the generalized inverse of H such that $HGH = H$ and the vector $\nu \in \text{Null}(H)$. There are infinite combinations of $\{G, \nu\}$. For each specific choice, we can compute u_t by (5b):

$$u_t = H_u(Gb + \nu)$$

A_{UIE}, B_{UIE} not in the proof. Done

We partition $G = [G_u \; G_y]$ where G_u and G_y respectively contain $N_{init}n_u$ and $(N_{init} + N_{pred})n_y$ columns. Then the components in the UIE form (4) are defined as:

$$\begin{aligned} &d_t = y_{t-N_{init}+1:t+N_{est}} \\ &A_{UIE} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ H_u G_u \end{bmatrix}, B_{UIE} = \begin{bmatrix} \mathbf{0} \\ H_u G_y \end{bmatrix}. \end{aligned} \tag{7}$$

lemma 2 (old) and theorem 1 together. Done

Theorem 1 (Existence of Data-driven op-UIE): Given N_{init} and N_{est}, and under Assumption 1, an op-UIE in the form (4) with matrices defined in (7) and $d_t = y_{t-N_{init}+1:t+N_{est}}$ exists if and only if the condition

$$\text{Null}(H) \subseteq \text{Null}(H_u) \tag{8}$$

holds and A_{UIE} is Schur stable.

Proof: ($\Leftarrow$) Firstly, under Assumption 1, if the condition (8) holds, we prove that given the initial $z_0 = u_{-N_{init}+1:0}$, we will have $z_t = u_{t-N_{init}+1:t}$, $\forall t \geq 0$. For one specific choice $\{G, \nu\}$, split G as $G = [G_u \; G_y]$ so that $m_{G_u} = N_{init}n_u, m_{G_y} = (N_{init} + N_{pred})n_y$. Because $\nu_1 \in \text{Null}(H) \subseteq \text{Null}(H_u)$, we can compute u_t by (5b):

$$\begin{aligned} u_t =& H_u(Gb + \nu) \\ =& H_u G_u u_{t-N_{init}:t-1} + H_u G_y y_{t-N_{init}:t+N_{est}-1} \end{aligned} \tag{9}$$

It's not difficult to prove that the above equation is equivalent to (4) with components (7) by using the augmented state $z_t = u_{t-N_{init}+1:t}$.

Consider any two choices $\{G_1, \nu_1\}$ and $\{G_2, \nu_2\}$ Because of $Hg = b$, $H(G_1 b + \nu_1 - G_2 b - \nu_2) = 0$. Then $(G_1 b + \nu_1 - G_2 b - \nu_2) \in \text{Null}(H) \subseteq \text{Null}(H_u)$, which means

$$H_u(G_1 b + \nu_1 - G_2 b - \nu_2) = 0$$

As a result, all u_t compted by different $\{G, \nu\}$ are the same. Because $[u_{t-N_{init}:t}(= z_t); y_{t-N_{init}:t}]$ is represented by linear combination of the Hankel matrices, Fundamental Lemma 1 guarantees it is the actual trajectory of (1). Finally, since initial z_0 is given, the statement holds by induction.

Secondly, we prove that the convergence of the estimated input by the op-UIE, that is $\hat{u}_t \to u_t$ as $t \to \infty$, if A_{UIE} is Schur stable. In the UIE, we can choose any arbitrary z_0' constructed by initial estimate $z_0' = [\hat{u}_{-N_{init}}^\top, \ldots, \hat{u}_{0-1}^\top]^\top$ and use (4) to output estimated $\hat{u}_t$. Denote the error between $N_{est} - step$ actual and estimated input as $e_t = [u_{t-N_{init+1}}^\top, \ldots, u_t^\top]^\top - [\hat{u}_{t-N_{init}}^\top, \ldots, u_t^\top]$. By (4) and the result in the first proof part, we have $e_{t+1} = A_{UIE}e_t$. If A_{UIE} is Schur stable, e_t converges to zero and u_t converges to $\hat{u}_t$. Together, the above two proof parts prove the existence of the op-UIE.

($\Rightarrow$) Firstly, as the op-UIE exists, because each the i-th columns of $\mathfrak{H}_{L,init}(u_{\mathbf{d}})$, $\mathfrak{H}_{L,init}(y_{\mathbf{d}})$, $\mathfrak{H}_{L,est}(y_{\mathbf{d}})$ and $\mathfrak{H}_{L,est(1)}(u_{\mathbf{d}})$ satisfy the system dynamics in the form (4), we have

$$\begin{aligned} \mathfrak{H}_{L,est(1)}(u_{\mathbf{d}}) = [\mathbf{0} \; \mathbf{I}_{n_u}](&A_{UIE} \, \mathfrak{H}_{L,init}(u_{\mathbf{d}}) \\ &+ B_{UIE} \begin{bmatrix} \mathfrak{H}_{L,est}(y_{\mathbf{d}}) \\ \mathfrak{H}_{L,init}(y_{\mathbf{d}}) \end{bmatrix}) \\ = [\mathbf{0} \; \mathbf{I}_{n_u}]&[A_{UIE} \; B_{UIE}]H \end{aligned}$$

which implies (8) holds. Secondly, $u_t - \hat{u}_t \to 0$ indicates $e_t \to 0$ as $t \to \infty$. Together with $e_{t+1} = A_{UIE}e_t$, it indicates that A_{UIE} is Schur stable. ■

B. Design of op-UIE

We firstly show how to compute the generalized inverse G of H by *Singular Value Decomposition* (SVD). As a result, it shows some freedom to build different A_{UIO} from the same H. Next, we use the Lyapunov condition and sufficient tractable LMI condition to design the op-UIE.

lemma 3 (now 2) here, not unique A_{UIE}; first proof then conclude lemma. Done

Consider the singular value decomposition (SVD) of H: $H = U \begin{bmatrix} S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V^\top$, where $S \in \mathbb{R}^{n_S, n_S,}$ is a square matrix that contains the all n_S positive singular values. Separate $V = [V_1 V_2]$ so that V_2 contains the orthonormal basis for $\text{Null}(H)$ corresponding to the all zero singular values.

Furthermore, we can use matrices from SVD to compute a specific kind of $G(F)$ by:

$$G(F) := V \begin{bmatrix} S^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} U^\top + VFU^\top \tag{10}$$
$$F := \begin{bmatrix} \mathbf{0} & M_1 \\ M_2 & M_3 \end{bmatrix}$$

, in which F has specific structure with arbitrary M_1, M_2, M_3. Using (10) will simply lead to $HG(F)H = H$. We can compute the corresponding $A_{UIE}(F)$ by (7):

$$\begin{aligned} A_{UIE}(F) &= \begin{bmatrix} \mathbf{0} & \mathbf{I}_{(N_{init}-1)*n_u} \\ \multicolumn{2}{c}{H_u V(\begin{bmatrix} S^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + F)U^\top \begin{bmatrix} \mathbf{I_{N_{init}n_u}} \\ \mathbf{0} \end{bmatrix}} \end{bmatrix} \\ &= N_1 + N_2 F N_3, \\ N_2 &:= \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_u} \end{bmatrix} H_u V, \quad N_3 = U^\top \begin{bmatrix} \mathbf{I_{N_{init}n_u}} \\ \mathbf{0} \end{bmatrix} \\ N1 &:= \begin{bmatrix} \mathbf{0} & \mathbf{I}_{(N_{init}-1)*n_u} \\ & \mathbf{0} \end{bmatrix} + N_2 \begin{bmatrix} S^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} N_3 \end{aligned} \tag{11}$$

, where A_{UIE} is represented by an affine equation of F by the constant matrices N_1, N_2, N_3.

Remark 1: Execute the SVD of H_u: $H_u = U_u \begin{bmatrix} S_u & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V_u^\top$. Differently, let $S_u \in \mathbb{R}^{n_S, n_S,}$ be the same shape of S. Because (8) holds, we can construct $V_u = [V_{u,1} V_{u,2}]$ such that $V_2 = V_{u,2}$. Then we can compute:

$$H_u V \begin{bmatrix} \mathbf{0} & M_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} U^\top = U_u \begin{bmatrix} \mathbf{0} & S_u V_{u,1}' V_1 M_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} U^\top$$
$$H_u V \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ M_2 & M_3 \end{bmatrix} U^\top = 0$$

, which indicates different M_1 will lead to different A_{UIE} where M_2, M_3 have no influence.

Lemma 2: Under condition (8), for different generalized inverse G such that $HGH = H$, the corresponding A_{UIE} by (7) can be different.

M_1 results in the degrees of freedom to modify A_{UIE}. Therefore based on Lyapnov condition for a stable A_{UIE}, the problem is to find $W > 0$ and G such that:

$$\begin{cases} \begin{bmatrix} W & A_{UIE} W \\ W A_{UIE}' & W \end{bmatrix} > 0 & \text{(12a)} \\ A_{UIE} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \multicolumn{2}{c}{H_u G \begin{bmatrix} \mathbf{I_{N_{init}n_u}} \\ \mathbf{0} \end{bmatrix}} \end{bmatrix} & \text{(12b)} \\ HGH = H & \text{(12c)} \end{cases}$$

However, solving the above bilinear matrix inequalities is NP-hard. In the following, we propose a sufficient solution of (12) by LMI.

Define 2 matrices $T_1 \in \mathbb{R}^{m_H, n_H}$ and $T_2 \in \mathbb{R}^{r, m_H}$ with $r = \text{rank}(T_1 N_3)$ such that

$$T_1 = [\mathbf{0} \;\; \mathbf{I}_{n_s}], \; \text{rank}(T_2 T_1 N_3) = r \tag{13}$$

Infinite number of T_2 exist and one possible solution is $T_2 = [\mathbf{I}_r 0]E$, where E is the multiplication of elementary operations to execute Gauss-Jordan Elimination for $T_1 N_3$.

Lemma 3: Using constant matrices N_1, N_2, N_3 in (11) and T_1, T_2 in (13), find $N \in \mathbb{R}^{m_H \times r}$,$M \in \mathbb{R}^{r \times r}$ and $W \in \mathbb{R}^{N_{init}nu \times N_{init}nu}$ so that the LMI

$$\begin{cases} \begin{bmatrix} W & \begin{matrix} N_1 W + \\ N_2 N T_2 T_1 N_3 \end{matrix} \\ \begin{matrix} W N_1' + \\ (N_2 N T_2 T_1 N_3)' \end{matrix} & W \end{bmatrix} > 0 & \text{(14a)} \\ T_2 T_1 N_3 W = M T_2 T_1 N_3 & \text{(14b)} \\ W > 0 & \text{(14c)} \end{cases}$$

holds. Compute $F := NM^{-1}T_2T_1$, $G(F)$ by (10) and $A_{UIE}(F)$ by (11). Then the above W, $G(F)$ and $A_{UIE}(F)$ make (12) holds.

Proof: Because $T_1 = [\mathbf{0} \;\; \mathbf{I}_{n_s}]$, by computing $F = NM^{-1}T_2T_1$, we force it have the following structure:

$$F := \begin{bmatrix} \mathbf{0} & M_1 \\ \mathbf{0} & M_3 \end{bmatrix} \tag{15}$$

Under such F, $G(F)$ and $A_{UIE(F)}$ from (10) (11) and will automatically satisfy (12b) and (12c). By Remark 1, the degree of freedom for $A_{UIE}(F)$ is kept by M_1.

By (13), $T_2 T_1 N_3$ is full row rank, then it follows from (14b) that M is also full rank. Therefore M is invertible and $T_2 T_1 N_3 = M^{-1} T_2 T_1 N_3 W$. Then we get (12a) from (14a) by

$$\begin{aligned} N_1 W + N_2 N T_2 T_1 N_3 &= N_1 W + N_2 N M^{-1} T_2 T_1 N_3 W \\ &= N_1 W + N_2 F N_3 W = A_{UIE}(F) W \end{aligned}$$

■

IV. CLOSED-LOOP UNKNOWN INPUT ESTIMATOR

cl-UIE, why, explain it by Luen observer. Done

In this section, we formulate a closed-loop unknown input estimator (cl-UIE) using a new d_T.The principle is to add an estimation error in the dynamics (4), similar to Luenberger observer style [24], to stabilize the system. When the Hankel matrix is poisoned by the disturbance, one might have no freedom to compute a stable op-UIE, which motivates the design of a cl-UIE.

First of all, we build a data-driven equation for the estimation error. Similar to (2), we can use Fundamental Lemma 1 to predict y_t by

$$\hat{H} g_y = \hat{b}, \; \hat{H} := \begin{bmatrix} \mathfrak{H}_{L,init}(u_\mathbf{d}) \\ \mathfrak{H}_{L,pred}(u_\mathbf{d}) \\ \mathfrak{H}_{L,init}(y_\mathbf{d}) \end{bmatrix}, \; \hat{b} := \begin{bmatrix} u_{t-\hat{N}_{init}:t-1} \\ u_{t:t+\hat{N}_{pred}-1} \\ y_{t-\hat{N}_{init}:t-1} \end{bmatrix}$$
$$H_y =: y_t, \; H_y := \mathfrak{H}_{L,pred}(y_\mathbf{d})$$

where $\hat{N}_{init} = N_{init} - 1$ and $\hat{N}_{pred} = 1$

Under Assumption 1, Fundamental Lemma 1 guarantees the existence of g_y with a solution:

$$\hat{g} = \hat{G}\hat{b} + \hat{\nu}$$

where $\hat{G}$ comes from $\hat{H}\hat{G}\hat{H} = \hat{H}$ and $\hat{\nu} \in \text{Null}(\hat{H})$. There are infinite combinations of $\{\hat{G}, \hat{\nu}\}$. Furthermore, by choosing $\hat{N}_{init} = N_{init} - 1 \geq \mathbf{l}(\mathcal{B}(A, B, C, D))$, different $\hat{g}$ result in the same prediction [13]. Partition $\hat{G} = [\hat{G}_u \ \hat{G}_y]$ where $\hat{G}_u$ and $\hat{G}_y$ respectively contain $N_{init}n_u$ and $(N_{init}-1)n_y$ columns. Choosing $\hat{\nu} = 0$, y_t can represented by

$$\begin{aligned} y_t =& H_y \hat{G}\hat{b} \\ =& H_y \hat{G}_u u_{t-N_{init}+1:t} + H_y \hat{G}_y y_{t-N_{init}+1:t-1} \end{aligned} \tag{17}$$

As a result, $y_t - H_y \hat{G}_y y_{t-N_{init}+1:t-1} - H_y \hat{G}_u z_t = H_y \hat{G}_u (u_{t-N_{init}+1:t} - z_t)$, which represents the estimation error. We construct the new components for the UIE form (4) as:

$$\begin{aligned} \hat{d}_t &= \begin{bmatrix} d_t \\ y_t - H_y \hat{G}_y y_{t-N_{init}+1:t-1} - H_y \hat{G}_u z_t \end{bmatrix}, \\ \hat{A}_{UIE} &= A_{UIE}, \hat{B}_{UIE} = [B_{UIE} \ L] \end{aligned} \tag{18}$$

, where an arbitrary matrix $L \in \mathbb{R}^{N_{init}nu \times ny}$.

Theorem 2 (Design of cl-UIE): Under Assumption 1 and $N_{init} \geq \mathbf{l}(\mathcal{B}(A,B,C,D)) + 1$, a UIE in the form (4) with components defined in (18) exists if and only if (8) holds and $A_{UIE} - LH_y\hat{G}_u$ is Schur stable

Proof: Similar to proof of Theorem 1. ■

Remark 2: The new term in $\bar{d}_t$ actually add a N_{init}-step-u-error feedback component. We can choose some random G and compute a possible unstable A_{UIE}. Then use different design methods for the gain L to stabilize A_{UIE}, such as using Riccati equations.

Remark 3: The design methods by Lyapunov condition (12), LMI (14) and EF-UIE in Theorem 2 do not guarantee the existence of a UIE for any system. In the model-based input reconstruction methods, some limitation is given based on the system dynamic $\mathcal{B}(A,B,C,D)$ [1], [6]. Future work can include the proof of equivalence between data-driven and model-based conditions.

In conclusion, the design process and operation of op-UIE and cl-UIE are summarized in Algorithm 1.

Algorithm 1 Design a data-driven UIE

Given historical signals $\{u_{\mathbf{d},i}, y_{\mathbf{d},i}\}_{i=1}^{T}$.

1) Choose a large N_{init}, try $N_{est} = 1, 2, ..., max_{N_{est}}$ until the condition (8) holds.
2) Build the UIE in the form (4) by either:
   a) op-UIE: Compute G by either (12) or (14). Compute and use the components in (7).
   b) cl-UIE: Choose any G of H. Compute and use the components in (18) such that $(A_{UIE} - LH_y\hat{G}_u)$ is Schur stable.
3) From $t = 0$, choose arbitrary z_0 and repeatedly compute (4) to output $\hat{u}_t$.

V. Simulation and Experiment Validation

In this section, the proposed data-driven UIE is tested by simulation and experiments. In the simulation, we test both design methods for the data-driven UIE. As for the experiment, the carbon dioxide (CO2) level inside an actual building is measured to estimate the occupancy number in the building. We compare the estimation using the proposed methods and two common statistical methods.

A. Simulation

Consider a linear system (1) with the state-space description as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} 0.7 & -0.7 & 1.7 \\ -0.2 & 1.9 & -0.7 \\ 0.8 & 0.5 & -0.7 \end{bmatrix} & \begin{bmatrix} -0.9 & -0.5 \\ 1.4 & 0.3 \\ -0.9 & -1.3 \end{bmatrix} \\ \hline \begin{bmatrix} -0.5 & 0 & 0.4 \\ 1.6 & 1.4 & 0.1 \end{bmatrix} & \gamma \begin{bmatrix} 0.9 & -0.5 \\ 0.3 & 0.5 \end{bmatrix} \end{array}\right]$$

We test two cases: $\gamma = 1$ and $\gamma = 0$. Some random inputs are applied to produce the 100-step trajectories for both cases. Among them, 50-step input and output signals are used for Hankel matrix construction and the second 50-step outputs are used to estimate inputs. Follow the Algorithm 1, we choose $N_{init} = 5$, $N_{est} = 1$ for $\gamma = 1$ and $N_{est} = 2$ for $\gamma = 0$. For both cases, we build two kinds of UIE based on two proposed design methods, respectively.

We use an arbitrary initial guess $z_0 = [0 \ 0 \ \dots 0 \ 0]^\top$ in UIE (4). In the following, denote $u_t(i)$ as the i-th element in the input and $du_t(i) = u_t(i) - \hat{u}_t(i)$ as the estimate error. Figure 1 shows the comparison of $u_t(1)$ and $\hat{u}_t(1)$ in the case of $\gamma = 1$ and UIE by the LMI (14). $\hat{u}_t(1)$ quickly converges to the actual $u_t(1)$. Because all the UIEs show similar convergent performance, we plot all the results in Figure 2 and 3, which validate the proposed data-driven UIE framework.

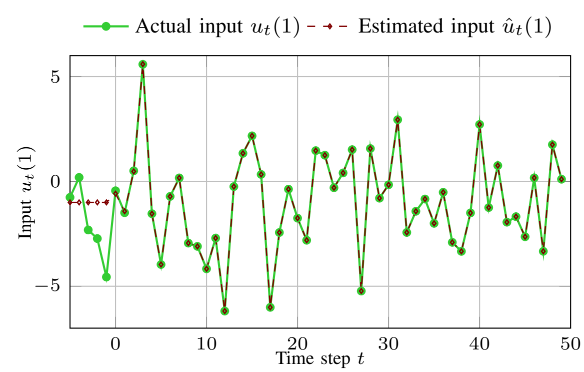

Fig. 1. Simulation with op-UIE:$\gamma = 1$, input u_1

B. Experiment

In the building community, the occupancy estimation is critical for simulating the building thermal dynamics and controlling the indoor temperature. When the ventilation flowrate is known or fixed, the dynamics of air quality can be roughly regarded as a linear model [25]. Under the assumption that the CO2 generation rate per person doing office work is relatively constant, we consider the occupant number as the input and the indoor CO2 level as the output.

We experimented with an occupancy estimation in a 600 m^2 actual building called Polydome on the EPFL campus. It is regularly used for lectures, and up to 200 people may stay

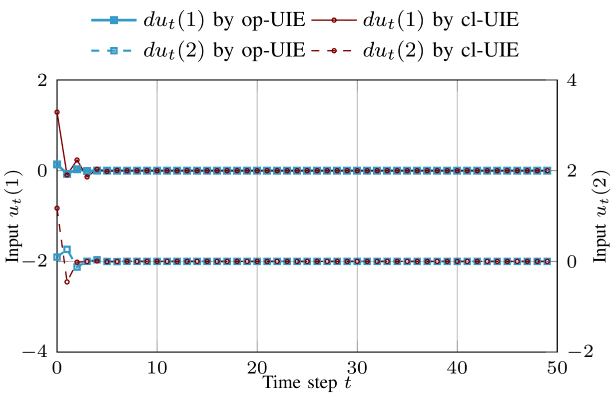

Fig. 2. Result of all input estimation by two design methods: $\gamma = 1$, $du_i = u_i - \hat{u}_i$

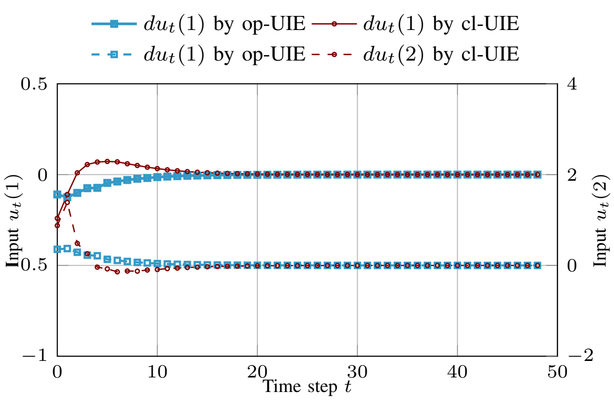

Fig. 3. Result of all input estimation by two design methods: $\gamma = 0$, $du_i = u_i - \hat{u}_i$

during the weekdays. Some data-driven predictive control scheme has been validated Polydome [18]. We collected the occupancy number by human counting and the indoor CO2 level by four air quality sensors (position shown in Figure 4) for two weeks.

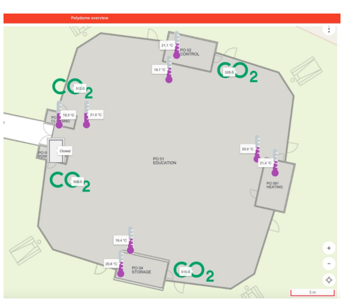

Fig. 4. Position of Air quality sensors in the Polydome

Five weekdays are used to built the Hankel matrix. We choose $N_{init} = 5$, and $N_{est} = 2$ and use cl-UIE by Theorem 2 to build UIE. As a comparison, we also use linear regression (LR) and Gaussian process regression (GPR) to estimate the occupancy. For all the methods, we force $\hat{u}_t = 0$ between 7 : 00 PM and 7 : 00 AM because there should be no occupants inside after work. Because the number of occupants is an input in the building thermal dynamics, some related data (indoor temperature, weather condition, heat pump power) are also collected and used if they can improve the accuracy.

The estimation results of another five weekdays are shown in Figure 5. The proposed data-driven UIE beats the LR and results in a little larger MAE than GR. However, UIE shows much less fluctuation than GR, which can be explained by the dynamics it learns inside the estimator. Indeed, only UIE can benefit from the thermal data while the other two methods result in bad performance due to overfitting. Admittedly, the estimated occupancy by UIE does not converge to the actual one because of the system's nonlinearity and measurement noises.

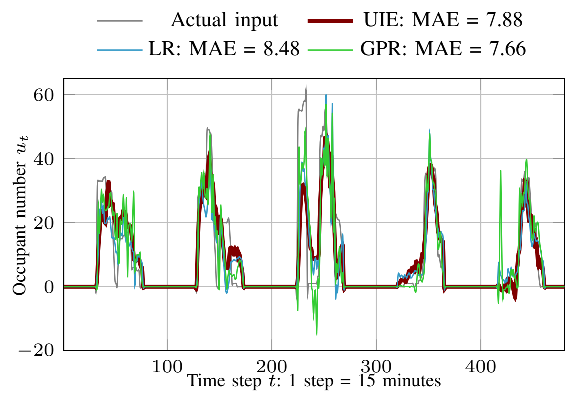

Fig. 5. Comparison of occupant number estimation by the data-driven UIE, LR and GPR. Mean absolute error (MAE) is computed for the data during the work time.

VI. Conclusion

This work elaborates on a data-driven UIE and its data-based existence condition. Two design methods are proposed based on the Lyaponov condition and generalized inverse of the Hankel matrix. Future work can be explored in the equivalence between the existence condition between data-driven and state-space methods, and extension to the stochastic system. The application in occupancy estimation also allows further experiments in occupant-included building control.

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