Full-Order Observers for Linear Systems with Unknown Inputs

M. Darouach, M. Zasadzinski, and S. J. Xu

***Abstract*—This note presents a simple method to design a full-order observer for linear systems with unknown inputs. The necessary and sufficient conditions for the existence of the observer are given.**

I. Introduction

The problem of observing the state vector of a linear-time invariant multivariable system, subjected to unknown inputs, has received considerable attention in the last two decades [1]–[8]. One approach developed consists of modeling the unknown inputs by the response of a suitably chosen dynamical system [1]. This method, however, increases the dimension of the observer considerably. More interesting is the approach developed by Wang *et al.* [2], which propose a procedure to design reduced-order observers without any knowledge of these inputs. The existence conditions for this observer were given by Kudva *et al.* [3]. Bhattacharyya [4] uses a geometric approach, while Miller and Mukundan [5] use the generalized inverse matrix. Kobayashi and Nakamizo [6] propose a procedure based on the Silverman's inverse method. Fairman *et al.* [7] suggest an approach using the singular value decomposition. Recently, a simple design method of reduced-order observer was proposed by Hou and Müller [8], the existence conditions of this observer were given. On the other hand, Yang and Wilde [9] propose a direct design procedure of full-order observer. However, no mention is made on the existence conditions of such observer.

This note presents a simple full-order observer design, its derivation is direct and essentially follows [9] and extends their results. It will be shown that the problem of full-order observers for linear systems with unknown inputs can be reduced to a standard one, this fact is implied in [11] and in [8] for the reduced observer. The existence conditions for the obtained observer are given.

II. Design of the Observer

Consider a linear time-invariant system described by

$$\dot{x} = Ax + Bu + Dv \tag{1a}$$

$$y = Cx \tag{1b}$$

where $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}^k$, $v \in \boldsymbol{R}^m$, and $y \in \boldsymbol{R}^p$ are the state vector, the known input vector, the unknown input vector and the output vector of the system, respectively. A, B, C, and D are known constant matrices of appropriate dimensions. We assume that $p \geq m$ and, without loss of generality, $\operatorname{rank} D = m$ and $\operatorname{rank} C = p$.

Following [9], the full-order observer is described as

$$\dot{z} = Nz + Ly + Gu \tag{2a}$$

$$\hat{x} = z - Ey \tag{2b}$$

where $z \in \boldsymbol{R}^n$, $\hat{x} \in \boldsymbol{R}^n$.

Manuscript received September 2, 1992; revised January 21, 1993.
The authors are with C.R.A.N. (C.N.R.S. UA 821) Earal Université de Nancy I, 186 Rue de lorraine 54400 Cosnes Et Romain, France.
IEEE Log Number 9214010.

N, L, G, and E are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}$ will asymptotically converge to x.

Define the observer reconstruction error by

$$e = \hat{x} - x = z - x - Ey \tag{3}$$

then, the dynamic of this observer error is

$$\dot{e} = Ne + (NP + LC - PA)x + (G - PB)u - PDv \tag{4}$$

with $P = I_n + EC$. (5)

If

$$PD = 0 \quad \text{or} \quad (I_n + EC)D = 0 \tag{6}$$

$$G = PB \tag{7}$$

and

$$NP + LC - PA = 0. \tag{8}$$

Equation (4) reduces to the homogeneous equation

$$\dot{e} = Ne. \tag{9}$$

The conditions for $\hat{x}$ to be an asymptotic state observer of x are (5)–(8), and N must be a stability matrix, i.e., has all its eigenvalues in the left-hand side of the complex plane.

In order to use the well-known results obtained for the classical full-order observer without unknown inputs [10], (8) can be written as

$$N = PA - KC \tag{10}$$

where

$$K = L + NE. \tag{11}$$

Substituting (10) into (11), we find

$$L = K(I_p + CE) - PAE. \tag{12}$$

Then the observer dynamical equation (2a) becomes

$$\dot{z} = (PA - KC)z + Ly + Gu \tag{13}$$

where matrices E, P, G, and L are obtained from (6), (5), (7), and (12), respectively.

Therefore the problem of designing the full-order observer with unknown inputs is reduced to find a matrix E satisfying (6), and a matrix K such that $(PA - KC)$ is a stability matrix. This problem is equivalent to the standard problem of the observers design when all inputs are known.

The eigenvalues of $(PA - KC)$ can be arbitrarily located, by choosing matrix K suitably, if and only if the pair (PA, C) is observable. If (PA, C) is not observable, then a matrix K can be found such that the observer is asymptotically stable if and only if (PA, C) is detectable.

From (6) we have

$$ECD = -D \tag{14}$$

the solution of this equation depends on the rank of matrix CD, E exists if $\operatorname{rank}(CD) = m$.

The general solution of (14) can be written as

$$E = -D(CD)^+ + Y(I_p - (CD)(CD)^+) \tag{15}$$

where $(CD)^+$ is the generalized inverse matrix of CD, given by $(CD)^+ = ((CD)^T(CD))^{-1}(CD)^T$, since CD is of full column

0018-9286/94$04.00 © 1994 IEEE

rank, and Y is an arbitrary matrix of appropriate dimension. The choice of this matrix is important in the design of the observer as can be shown below.

The observability of (PA, C) is given by the rank of the matrix

$$\boldsymbol{O} = \begin{bmatrix} C \\ CPA \\ \vdots \\ C(PA)^{n-1} \end{bmatrix}$$

where $C(PA)^k$ can easily be obtained from the sequence $CA^i (i = 1, k)$ by the following recursion.

Lemma: The matrix $C(PA)^k$ is given by

$$C(PA)^k = (I_p + CE) \sum_{j=1}^{k} M_j C A^{k-j+1} \tag{16}$$

with

$$M_j = \begin{cases} 1, & \text{if } j = 1 \\ \sum_{i=1}^{j-1} M_i C A^{j-i} E, & \text{if } j > 1. \end{cases} \tag{17}$$

Remarks: One can see from this lemma that the rank $\boldsymbol{O}$ depends on the matrix $I_p + CE = (I_p + CY)(I_p - (CD)(CD)^+)$. If $I_p + CE = 0$, then rank $\boldsymbol{O}$ = rank C and the pair (PA, C) is unobservable. This case is obtained for example when $m = p$ and CD is nonsingular [9]. In this case, the eigenvalues of $(PA - KC)$ can not be located arbitrarily, and from (12), we have $L = -PAE$.

From (5) and (15), we obtain

$$P = (I_n + YC)(I_n - D(CD)^+ C)$$

the maximal rank of P, i.e., $n - m$, is obtained when $(I_n + YC)$, or equivalently $(I_p + CY)$, is nonsingular. In this case the observability matrix $\boldsymbol{O}$ is of maximal rank.

To design a stable observer (13), the necessary and sufficient condition is given by the following theorem.

Theorem 1: For the system (1), the observer (13) exists if and only if

1) $\operatorname{rank} CD = \operatorname{rank} D = m$;

2) $\operatorname{rank} \begin{bmatrix} sP - PA \\ C \end{bmatrix} = n \ \forall s \in \boldsymbol{C}, \operatorname{Re}(s) \geq 0.$

Proof: The condition 1) is necessary for the existence of the observer as can be seen from [3], [7], and from (14).

Now, since (13) is the form of a standard observer equation, then a matrix K can be found such that the observer is asymptotically stable if and only if the pair (PA, C) is detectable, that is

$$\operatorname{rank} \begin{bmatrix} sI_n - PA \\ C \end{bmatrix} = n \qquad \forall s \in \boldsymbol{C}, \operatorname{Re}(s) \geq 0$$

or equivalently

$$\operatorname{rank} \begin{bmatrix} I_n & sE \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sI_n - PA \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} sP - PA \\ C \end{bmatrix} = n \qquad \forall s \in \boldsymbol{C}, \operatorname{Re}(s) \geq 0.$$

The relation between condition 2 and that generally adopted for the observer with unknown inputs is given in the following theorem.

Theorem 2: Assume that $\operatorname{rank} CD = \operatorname{rank} D = m$ and $\operatorname{rank} P = n - m$. Then the following conditions are equivalent:

i) the pair (PA, C) is detectable (observable);

ii) $\operatorname{rank} \begin{bmatrix} sP - PA \\ C \end{bmatrix} = n \qquad \forall s \in \boldsymbol{C} \operatorname{Re}(s) \geq 0 \ (\forall s \in \boldsymbol{C})$;

iii) $\operatorname{rank} \begin{bmatrix} sI_n - A & D \\ C & 0 \end{bmatrix} = n + m, \qquad \forall s \in \boldsymbol{C}, \operatorname{Re}(s) \geq 0 \ (\forall s \in \boldsymbol{C})$.

Proof: From Theorem 1, i) and ii) are equivalent.

To prove that iii) is equivalent to ii), let D^+ be the left inverse of matrix D, i.e., $D^+ D = I_m$, then $\ker D^+ \cap \ker P = \{0\}$ and

$$\operatorname{rank} \begin{bmatrix} P \\ D^+ \end{bmatrix} = n.$$

Let

$$S = \begin{bmatrix} P & 0 \\ D^+ & 0 \\ 0 & I_p \end{bmatrix}$$

be an $(n + p + m)(n + p)$ matrix of full column rank, i.e., rank $S = n + p$ and

$$T = \begin{bmatrix} I_n & 0 \\ -(sD^+ - D^+ A) & I_m \end{bmatrix},$$

then the following rank conditions are satisfied

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} sI_n - A & D \\ C & 0 \end{bmatrix} &= \operatorname{rank} S \begin{bmatrix} sI_n - A & D \\ C & 0 \end{bmatrix} T \\ &= \operatorname{rank} \begin{bmatrix} sP - PA & 0 \\ 0 & I_m \\ C & 0 \end{bmatrix} \\ &= m + \operatorname{rank} \begin{bmatrix} sP - PA \\ C \end{bmatrix} \end{aligned}$$

which is equivalent to $\operatorname{rank} \begin{bmatrix} sP - PA \\ C \end{bmatrix} = n.$

Remarks: We can see that if rank $P = q < n - m$, then ii) is only a sufficient condition for iii) to hold. In fact, there exists an $(n-m-q) \cdot n$ matrix P_1 such that $P_1 D = 0$ and $\operatorname{rank} \begin{bmatrix} P \\ P_1 \end{bmatrix} = n - m$, and, using the proof of Theorem 2, we obtain

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} sI_n - A & D \\ C & 0 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} sP - PA & 0 \\ sP_1 - P_1 A & 0 \\ 0 & I_m \\ C & 0 \end{bmatrix} \\ &= m + \operatorname{rank} \begin{bmatrix} sP - PA \\ sP_1 - P_1 A \\ C \end{bmatrix}, \end{aligned}$$

and iii) is verified if ii) is satisfied.

Since rank $C = p$, we can always find Y such that $(I_p + CY)$ is nonsingular, in this case, we have rank $P = n - m$. The obvious choice of matrix Y is $Y = 0$, this yields $P = I_n - D(CD)^+ C$.

III. Examples

Consider the two examples of [9].

Example 1:

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case $CE = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is nonsingular. Thus, matrix $I_p + CE = 0$ and the pair (PA, C) is only detectable.

Matrices E, P, and PA are

$$E = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and}$$

$$PA = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & -3 & -4 \end{bmatrix}.$$

The pair (PA, C) is detectable, then we can find K such that $\mathrm{Re}(\lambda) < 0$ for all λ, where λ is an eigenvalue of $(PA - KC)$. Let $N = PA - KC$, we have

$$N = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & -3 & -4 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \\ k_5 & k_6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -k_1 & 3 - k_2 & 4 - k_1 \\ -k_3 & -k_4 & -k_3 \\ -k_5 & -3 - k_6 & -4 - k_5 \end{bmatrix}.$$

If we choose $k_1 = 4$, $k_2 = 3$, $k_3 = 0$, $k_5 = 0$, and $k_6 = -3$, we have

$$N = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -k_4 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

The matrix L can be obtained from (12)

$$L = K(I_p + CE) - PAE = -PAE = \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 0 & -3 \end{bmatrix}$$

which is the result obtained by Yang and Wilde [9].

Example 2:

$$A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this example, we have

$$CD = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Now let

$$Y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \\ y_5 & y_6 \end{bmatrix}$$

be an arbitrary matrix, then from (15) we have

$$E = \begin{bmatrix} -1 & y_2 \\ 0 & y_4 \\ 0 & y_6 \end{bmatrix}.$$

Case 1: $I_p + CE = 0$. This case corresponds to

$$E = \begin{bmatrix} -1 & 0 \\ 0 & y_4 \\ 0 & -1 \end{bmatrix},$$

$$P = I_n + EC = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & y_4 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$PA = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -y_4 & -y_4 \\ 0 & 0 & 0 \end{bmatrix}$$

and we can see that (PA, C) is only detectable, then we can find K such that $\mathrm{Re}\,\lambda_i(PA - KC) < 0$, where λ_i is an eigenvalue of $(PA - KC)$.

Let $N = PA - KC$

$$N = \begin{bmatrix} -k_1 & 0 & -k_2 \\ -1 - k_3 & -y_4 & -y_4 - k_4 \\ -k_5 & 0 & -k_6 \end{bmatrix}.$$

For $k_2 = 0$, $k_3 = -1$, $k_4 = -y_4$, $k_5 = 0$, $k_1 = y_4 = k_6 = -h$, we obtain

$$N = \begin{bmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{bmatrix},$$

$$E = \begin{bmatrix} -1 & 0 \\ 0 & -h \\ 0 & -1 \end{bmatrix},$$

and

$$L = -PAE = \begin{bmatrix} 0 & 0 \\ -1 & h + h^2 \\ 0 & 0 \end{bmatrix}.$$

This result is the same as that obtained by Yang and Wilde [9].

Case 2: $I_p + CE \neq 0$ and $\mathrm{rank}\,P = n - m = 2$.

It is obvious that this case is obtained if $Y = 0$ then matrices E, P, and PA are

$$E = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and}$$

$$PA = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}.$$

The pair (PA, C) is observable, then a matrix K can be chosen to obtain a desired characteristic polynomial for the observer.

In this case matrix $PA - KC$ is

$$PA - KC = \begin{bmatrix} -k_1 & 0 & -k_2 \\ -1 - k_3 & 0 & -k_4 \\ -k_5 & -1 & -1 - k_6 \end{bmatrix}.$$

If we choose $k_2 = 0$, $k_3 = -1$, and $k_5 = 0$, then the characteristic polynomial of $PA - KC$ is

$$p(\lambda) = (k_1 + \lambda)(\lambda^2 + (1 + k_6)\lambda - k_4).$$

The eigenvalues of the observer are given by

$$\lambda_1 = -k_1$$

$$\lambda_2 + \lambda_3 = -(1 + k_6)$$

$$\lambda_2 \lambda_3 = -k_4.$$

From (12) we obtain

$$L = \begin{bmatrix} 0 & 0 \\ -1 & -\lambda_2\lambda_3 \\ 0 & -1-\lambda_2-\lambda_3 \end{bmatrix}.$$

Then the observer is

$$\dot{z}_1 = \lambda_1 z_1$$

$$\dot{z}_2 = \lambda_2\lambda_3 z_3 - y_1 - \lambda_2\lambda_3 y_2$$

$$\dot{z}_3 = -z_2 + (\lambda_2+\lambda_3)z_3 - (1+\lambda_2+\lambda_3)y_2$$

and

$$\hat{x}_1 = y_1 + z_1$$

$$\hat{x}_2 = z_2$$

$$\hat{x}_3 = z_3.$$

This observer can be reduced to a second-order one, since $\mathrm{Re}(\lambda_1) < 0$.

IV. Conclusion

In this note, we have presented a simple method to design a full-order observer for a linear system with unknown inputs. This method reduces the design procedure of full-order observers with unknown inputs to a standard one where the inputs are known. The existence conditions are given, and it was shown that these conditions are generally adopted for unknown inputs observer problem.

References

[1] G. Hostetter and J. S. Meditch, "Observing systems with unmeasurable inputs," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 307–308, 1973.

[2] S. H. Wang, E. J. Davison, and P. Dorato "Observing the states of systems with unmeasurable disturbances," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 716–717, 1975.

[3] P. Kudva, N. Viswanadham, and A. Ramakrishna, "Observers for linear systems with unknown inputs," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 113–115, 1980.

[4] S. P. Bhattacharyya, "Observer design for linear systems with unknown inputs," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 483–484, 1978.

[5] R. J. Miller and R. Mukundan, "On designing reduced-order observers for linear time-invariant systems subject to unknown inputs," *Int. J. Contr.*, vol. 35, pp. 183–188, 1982.

[6] N. Kobayashi and T. Nakamizo, "An observer design for linear systems with unknown inputs," *Int. J. Contr.*, vol. 35, pp. 605–619, 1982.

[7] F. W. Fairman, S. S. Mahil, and L. Luk, "Disturbance decoupled observer design via singular value decomposition," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 84–86, 1984.

[8] M. Hou and P. C. Müller, "Design of observers for linear systems with unknown inputs," *IEEE Trans. Automat. Contr.*, vol. AC-37, pp. 871–875, 1992.

[9] F. Yang and R. W. Wilde, "Observers for linear systems with unknown inputs," *IEEE Trans. Automat. Contr.*, vol. AC-33, pp. 677–681, 1988.

[10] J. O'Reilly, *Observers for Linear Systems.* New York: Academic, 1983.

[11] J. Kurek, "Observation of state vector of linear multivariable systems with unknown inputs," *Int. J. Contr.*, vol. 36, pp. 511–515, 1983.

Robust Motion/Force Control of Mechanical Systems with Classical Nonholonomic Constraints

Chun-Yi Su and Yury Stepanenko

***Abstract*—The position/force control of mechanical systems subject to a set of classical nonholonomic constraints represents an important class of control problems. In this note, a reduced dynamic model, suitable for simultaneous independent motion and force control, is developed. Some properties of the dynamic model are exploited to facilitate the controller design. Based on the theory of guaranteed stability of uncertain systems, a robust control algorithm is derived, which guarantees the uniform ultimate boundedness of the tracking errors. A detailed numerical example is presented to illustrate the developed method.**

I. Introduction

The control of mechanical systems with kinematic constraints has received increasing attention and is a topic of great interest. A lot of papers have been published in recent years to deal with the control problem when the kinematic constraints are holonomic constraints [1]–[4]. In contrast, if the kinematic constraints are nonholonomic, control laws developed for holonomic constraints are not applicable; only a few papers have been proposed to address these control issues. In this note, our discussions are focused on the classical nonholomonic case, and analyses are given from the Lagrangian point of view. As for the Hamiltonian case with other forms of nonholonomic constraints, the reader may refer to [12].

It is well known that in rolling or cutting motions the kinematic constraint equations are classical nonholonomic [10], and the dynamics of such systems is well understood (see, e.g. [10]). However, the literature on control with classical nonholonomic constraints is quite recent [5], [7], [8], and the discussion mainly focuses on some special examples [11], [13]–[15]. Earlier work that deals with control of nonholonomic systems is described in [9]. Bloch and McClamroch [5], Bloch *et al.* [7], and Campion *et al.* [8] demonstrated that systems with nonholonomic constraints are always controllable, but cannot be feedback stabilized to a single point with smooth feedback. By using a decomposition transformation and nonlinear feedback, conditions for smooth asymptotic stabilization to an equilibrium manifold are established. d'Andrea-Novel *et al.* [11] and Yun *et al.* [13] showed that the system is linearizable by choosing a proper set of output equations, and then applied, respectively, their results to the control of wheeled mobile robots and multiple arms. Researchers have also offered both nonsmooth feedback laws [6], [7], [14] and time-varying feedback laws [15] for stabilizing the system to a point. However, it is fair to say that the last two approaches are not yet fully general.

The above mentioned approaches, e.g. [5], [7], and [8], indeed provide a theoretic framework which can serve as a basis for the study of mechanical systems with nonholonomic constraints; however, all of those results are based on the method of a diffeomorphism and nonlinear feedback (for details, see [16]), which requires a detailed dynamic model and may be sensitive to parametric uncertainties.

In this note, a different control approach is proposed, in which the control of the constraint force due to the existence of classical constraints is also included. By assuming complete knowledge of the constraint manifold, and recognizing that the degree of freedom

Manuscript received September 28, 1992; revised January 7, 1993.

The authors are with the Department of Mechanical Engineering, University of Victoria, Victoria, B. C. V8W 3P6, Canada.

IEEE Log Number 9214009.

0018-9286/94$04.00 © 1994 IEEE