

# 000 001 002 003 004 005 CONFORMAL CONFIDENCE SETS FOR BIOMEDICAL 006 IMAGE SEGMENTATION 007 008 009

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## ABSTRACT

011 We develop confidence sets which provide spatial uncertainty guarantees for the  
 012 output of a black-box machine learning model designed for image segmentation.  
 013 To do so we adapt conformal inference to the imaging setting, learning thresholds  
 014 on a calibration dataset based on the distribution of the maximum of the trans-  
 015 formed logit scores within and outside of the ground truth masks. We show that  
 016 these confidence sets, when applied to new predictions of the model, are guaran-  
 017 teed to contain the true unknown segmented mask with desired probability. We  
 018 illustrate and validate our approach on a polyps tumor segmentation dataset. To  
 019 do so we obtain the logit scores from a deep neural network trained for polyps  
 020 segmentation and show that using distance transformed scores to obtain outer con-  
 021 fidence sets and the original scores for inner confidence set enables tight bounds  
 022 on tumor location whilst controlling the false coverage rate.

## 1 INTRODUCTION

027 Deep neural networks promise to significantly enhance a wide range of important tasks in biomedical  
 028 imaging. However these models, as typically used, lack formal uncertainty guarantees on their out-  
 029 put which can lead to overconfident predictions and critical errors. Misclassifications or inaccurate  
 030 segmentations can lead to serious consequences, including misdiagnosis, inappropriate treatment  
 031 decisions, or missed opportunities for early intervention. As a consequence, despite their potential  
 032 utility, medical professionals cannot yet rely on deep learning models to provide accurate informa-  
 033 tion and predictions which greatly limits their use in practical applications.

034 In order to address this problem, conformal inference, a robust framework for uncertainty quan-  
 035 tification, has become increasingly used as a means of providing prediction guarantees, offering  
 036 reliable, distribution-free confidence sets for the output of neural networks which have finite sample  
 037 validity. This approach, originally introduced in XXX, has become increasingly popular (CITE)  
 038 due to its ability to provide rigorous statistical guarantees without making strong assumptions about  
 039 the underlying data distribution or model architecture. Conformal prediction methods, in their most  
 040 commonly used form - split conformal inference - work by calibrating the predictions of the model  
 041 on a held-out dataset in order to provide sets which contain the output with a given probability, see  
 042 Angelopoulos & Bates (2021) for a good introduction.

043 In the context of image segmentation, we have a decision to make at each pixel/voxel of an im-  
 044 age which can lead to a large multiple testing problem. Traditional conformal methods, typically  
 045 designed for scalar outputs, require adaptation to handle multiple tests and their inherent spatial  
 046 dependencies. To do so Angelopoulos et al. (2021) applied conformal inference pixelwise and per-  
 047 formed multiple testing correction on the resulting  $p$ -values, however this approach does not take  
 048 into account of the complex dependence structure inherent in the images. To take advantage of this  
 049 structure, in an approach analogous to the FDR control of (Benjamini & Hochberg, 1995), Bates  
 050 et al. (2021) and Angelopoulos et al. (2022) sought to control the expected risk of a given loss func-  
 051 tion over the image and used a conformal approach to produce outer confidence sets for segmented  
 052 images which control the expected false negative rate. Other work considering conformal inference  
 053 in the context of multiple dependent hypotheses include XXX and XXX who established conformal  
 FDR control when testing for the presence of missing links in graphs. Under exchangeability  
 of the considered hypotheses XXX provides false coverage rate control over multiple conformal

054 inferences. XXX and XXX instead applied conformal inference in the imaging setting but did not  
 055 account for multiple comparisons.  
 056

057 In this work we argue that bounding the segmented outcome with guarantees in probability rather  
 058 than in expectation/proportion can be more informative, avoiding errors at the borders of potential  
 059 tumors. This is analogous to the tradeoff between FWER and FDR/FDP control in the multiple testing  
 060 literature in which there is a balance between power and coverage rate, the distinction being that  
 061 in medical image segmentation there can be a potentially serious consequence to making mistakes.  
 062 Under-segmentation might cause part of the tumor to be missed, potentially leading to inadequate  
 063 treatment. Over-segmentation, on the other hand, could result in unnecessary interventions, increasing  
 064 patient risk and healthcare costs. Unlike bounds on the proportion of discovered pixels/voxels,  
 065 confidence sets are guaranteed to contain the outcome with a given level of confidence and allow  
 066 medical practitioners to follow-up on the images where there is greater uncertainty. Since the guar-  
 067 antees are more meaningful the problem is more difficult. As we shall see, using the original scores  
 068 can lead to rather large and uninformative outer confidence sets. In order to address this, we use  
 069 a held out learning dataset to learn the score transformations which provide the most informative  
 070 confidence regions.

071 In order to obtain confidence sets we use a split-conformal inference approach in which we learn  
 072 appropriate cutoffs, with which to threshold the output of an image segmenter, from a calibration  
 073 dataset. These thresholds are obtained by considering the distribution of the maximum logit (trans-  
 074 formed) scores provided by the model within and outside of the ground truth masks. This approach  
 075 allows us to capture the spatial nature of the uncertainty in segmentation tasks, going beyond simple  
 076 pixel-wise confidence measures. By applying these learned thresholds to new predictions, we can  
 077 generate confidence sets that are guaranteed to contain the true, unknown segmented mask with a  
 078 desired probability.

## 079 2 THEORY

### 080 2.1 SET UP

082 Let  $\mathcal{V} \subset \mathbb{R}^m$ , for some dimension  $m \in \mathbb{N}$ , be a finite set corresponding to the domain which  
 083 represents the pixels/voxels at which we observe imaging data. Let  $\mathcal{X} = \{g : \mathcal{V} \rightarrow \mathbb{R}\}$  be the set  
 084 of real functions on  $\mathcal{V}$  and let  $\mathcal{Y} = \{g : \mathcal{V} \rightarrow \{0, 1\}\}$  be the set of all functions taking the values  
 085 0 or 1. We shall refer to elements of  $\mathcal{X}$  and  $\mathcal{Y}$  as images. Suppose that we observe a calibration  
 086 dataset  $(X_i, Y_i)_{i=1}^n$  of random images, where  $X_i : \mathcal{V} \rightarrow \mathbb{R}$  represents the  $i$ th observed calibration  
 087 image and  $Y_i : \mathcal{V} \rightarrow \{0, 1\}$  outputs labels at each  $v \in \mathcal{V}$  giving 1s at the true location of the objects  
 088 in the image  $X_i$  that we wish to identify and 0s elsewhere. Let  $\mathcal{P}(\mathcal{V})$  be the set of all subsets of  $\mathcal{V}$ .  
 089 Moreover, given a function  $f : \mathcal{X} \rightarrow \mathcal{X}$ , we shall write  $f(X, v)$  to denote  $f(X)(v)$  for all  $v \in \mathcal{V}$ .

090 Let  $s : \mathcal{X} \rightarrow \mathcal{X}$  be a score function - trained on an independent dataset - such that given an image  
 091 pair  $(X, Y) \in \mathcal{X} \times \mathcal{Y}$ ,  $s(X)$  is a score image in which  $s(X, v)$  is intended to be higher at the  $v \in \mathcal{V}$   
 092 for which  $Y(v) = 1$ . The score function can for instance be the logit scores obtained from a deep  
 093 neural network image segmentation method to the image  $X$  as input e.g. CITE. Given  $X \in \mathcal{X}$ ,  
 094 let  $\hat{M}(X) \in \mathcal{Y}$  be the predicted mask based on the segmentation model. I.e. let  $M(X, v) = 1$  if  
 095  $s(X, v) > 0.5$  and 0 otherwise for each  $v \in \mathcal{V}$ .

096 In what follows we will use the calibration dataset to construct a confidence functions  $I, O : \mathcal{X} \rightarrow$   
 097  $\mathcal{P}(\mathcal{V})$  such that for a new image pair  $(X, Y) \sim \mathcal{D}$ , given error rates  $\alpha_1, \alpha_2 \in (0, 1)$  we have

$$\mathbb{P}(I(X) \subseteq \{v \in \mathcal{V} : Y(v) = 1\}) \geq 1 - \alpha_1, \quad (1)$$

$$\text{and } \mathbb{P}(\{v \in \mathcal{V} : Y(v) = 1\} \subseteq O(X)) \geq 1 - \alpha_2. \quad (2)$$

101 Here  $I(X)$  and  $O(X)$  serve as inner and outer confidence sets for the location of the true segmented  
 102 mask. Their interpretation is that, up to the guarantees provided by the probabilistic statements (1)  
 103 and (9), we can be sure that for each  $v \in I(X)$ ,  $Y(v) = 1$  or that for each  $v \notin O(X)$ ,  $Y(v) = 0$ .  
 104 See Figure 3 for an example of this in practice. Joint control over the events can also be guaranteed,  
 105 either by sensible choices of  $\alpha_1$  and  $\alpha_2$  or by using the joint distribution of the maxima of the logit  
 106 scores - see Section 2.3.

107 In order to establish conformal confidence results we shall require the following exchangeability  
 108 assumption.

108     **Assumption 1.** Given a new random image pair,  $(X_{n+1}, Y_{n+1})$ , suppose that  $(X_i, Y_i)_{i=1}^{n+1}$  is an  
 109     exchangeable sequence of random image pairs in the sense that  
 110

$$111 \quad \{(X_1, Y_1), \dots, (X_{n+1}, Y_{n+1})\} =_d \{(X_{\sigma(1)}, Y_{\sigma(1)}), \dots, (X_{\sigma(n+1)}, Y_{\sigma(n+1)})\}$$

112     for any permutation  $\sigma \in S_{n+1}$ . Here  $=_d$  denotes equality in distribution and  $S_{n+1}$  is the group of  
 113     permutations of the integers  $\{1, \dots, n+1\}$ .  
 114

115     Exchangeability or a variant is a standard assumption in the conformal inference literature (An-  
 116     gelopoulos & Bates, 2021) and facilitates coverage guarantees. It holds for instance if we assume  
 117     that the collection  $(X_i, Y_i)_{i=1}^{n+1}$  is an i.i.d. sequence of image pairs but is more general and in prin-  
 118     ciple allows for other dependence structures.  
 119

## 120     2.2 MARGINAL CONFIDENCE SETS

121     In order to construct conformal confidence sets let  $f_I, f_O : \mathcal{X} \rightarrow \mathcal{X}$  be inner and outer trans-  
 122     formation functions and for each  $1 \leq i \leq n+1$ , let  $\tau_i = \max_{v \in \mathcal{V}: Y_i(v)=0} f_I(s(X_i), v)$  and  
 123      $\gamma_i = \max_{v \in \mathcal{V}: Y_i(v)=1} f_O(-s(X_i), v)$  be the maxima of the function transformed scores over the  
 124     areas at which the true labels equal 0 and 1 respectively. We will require the following assumption  
 125     on the scores and the transformation functions.  
 126

127     **Assumption 2.** (Independence of scores)  $(X_i, Y_i)_{i=1}^{n+1}$  is independent of the functions  $s, f_O, f_I$ .  
 128

Given this we construct confidence sets as follows.

**Theorem 2.1.** (*Marginal inner set*) Under Assumptions 1 and 2, given  $\alpha_1 \in (0, 1)$ , let

$$131 \quad \lambda_I(\alpha_1) = \inf \left\{ \lambda : \frac{1}{n} \sum_{i=1}^n 1[\tau_i \leq \lambda] \geq \frac{\lceil (1 - \alpha_1)(n + 1) \rceil}{n} \right\},$$

134     and define  $I(X) = \{v \in \mathcal{V} : f_I(s(X), v) > \lambda_I(\alpha_2)\}$ . Then,

$$136 \quad \mathbb{P}(I(X_{n+1}) \subseteq \{v \in \mathcal{V} : Y_{n+1}(v) = 1\}) \geq 1 - \alpha_1. \quad (3)$$

137     *Proof.* Under Assumptions 1 and 2, exchangeability of the image pairs implies exchangeability  
 138     of the sequence  $(\tau_i)_{i=1}^{n+1}$ . In particular, as  $\lambda_I(\alpha_1)$  is the upper  $\alpha_1$  quantile of the distribution of  
 139      $(\tau_i)_{i=1}^n \cup \{\infty\}$  by Lemma 1 of Tibshirani et al. (2019), it follows that  
 140

$$141 \quad \mathbb{P}(\tau_{n+1} \leq \lambda_I(\alpha_1)) \geq 1 - \alpha_1.$$

142     Now consider the event that  $\tau_{n+1} \leq \lambda_I(\alpha)$ . On this event,  $f_I(s(X_{n+1}), v) \leq \lambda_I(\alpha)$  for all  $v \in \mathcal{V}$   
 143     such that  $Y_{n+1}(v) = 0$ . As such, given  $u \in \mathcal{V}$  such that  $f_I(s(X_{n+1}), u) > \lambda_I(\alpha)$ , we must have  
 144      $Y_{n+1}(u) = 1$  so it follows that  $I(X_{n+1}) \subseteq \{v \in \mathcal{V} : Y_{n+1}(v) = 1\}$  and in particular that  
 145

$$146 \quad \mathbb{P}(I(X_{n+1}) \subseteq \{v \in \mathcal{V} : Y_{n+1}(v) = 1\}) \geq \mathbb{P}(\tau_{n+1} \leq \lambda_I(\alpha_1)) \geq 1 - \alpha_1.$$

147                          □

149     For the outer set we have the following analogous result.

150     **Theorem 2.2.** (*Marginal outer set*) Under Assumptions 1 and 2, given  $\alpha_2 \in (0, 1)$ , let

$$152 \quad \lambda_O(\alpha_2) = \inf \left\{ \lambda : \frac{1}{n} \sum_{i=1}^n 1[\gamma_i \leq \lambda] \geq \frac{\lceil (1 - \alpha_2)(n + 1) \rceil}{n} \right\},$$

155     and define  $O(X) = \{v \in \mathcal{V} : f_O(-s(X), v) \leq \lambda_O(\alpha_2)\}$ . Then,

$$157 \quad \mathbb{P}(\{v \in \mathcal{V} : Y_{n+1}(v) = 1\} \subseteq O(X_{n+1})) \geq 1 - \alpha_2. \quad (4)$$

159     *Proof.* Arguing as in the proof of Theorem 2.1, it follows that  $\mathbb{P}(\gamma_{n+1} \leq \lambda_O(\alpha_2)) \geq 1 - \alpha_2$ .  
 160     Now on the event that  $\gamma_{n+1} \leq \lambda_O(\alpha_2)$  we have  $f_O(-s(X_{n+1}, v)) \leq \lambda_O(\alpha_2)$  for all  $v \in \mathcal{V}$  such  
 161     that  $Y_{n+1}(v) = 1$ . As such, given  $u \in \mathcal{V}$  such that  $f_O(-s(X_{n+1}, u)) > \lambda_O(\alpha_2)$ , we must have  
 162      $Y_{n+1}(u) = 0$  and so  $O(X)^C \subseteq \{v \in \mathcal{V} : Y(v) = 0\}$ . The result then follows as above.                  □

**Remark 2.3.** We have used the maximum over the transformed scores in order to combine score information on and off the ground truth masks. The maximum is a natural combination function in imaging and is commonly used in the context of multiple testing (Worsley et al., 1992). However the theory above is valid for any increasing combination function. We show this in Appendix A.1 where we establish generalized versions of these results.

**Remark 2.4.** Inner and outer coverage can also be viewed as a special case of conformal risk control with an appropriate choice of loss function. We can thus instead establish coverage results as a corollary to risk control, see Appendix A.2 for details. This amounts to an alternative proof of the results as the proof of the validity of risk control is different though still strongly relies on exchangeability.

### 2.3 JOINT CONFIDENCE SETS

Instead of focusing on marginal control one can instead spend all of the  $\alpha$  available to construct sets which have a joint probabilistic guarantees. This gain comes at the expense of a loss of precision. The simplest means of constructing jointly valid confidence sets is via the marginal sets themselves.

**Corollary 2.5.** (Joint from marginal) Assume Assumptions 1 and 2 hold and given  $\alpha \in (0, 1)$  and  $\alpha_1, \alpha_2 \in (0, 1)$  such that  $\alpha_1 + \alpha_2 \leq \alpha$ , define  $I(X)$  and  $O(X)$  as in Theorems 2.1 and 2.2. Then

$$\mathbb{P}(I(X_{n+1}) \subseteq \{v \in \mathcal{V} : Y_{n+1}(v) = 1\} \subseteq O(X_{n+1})) \geq \frac{\lceil (1 - \alpha)(n + 1) \rceil}{n}. \quad (5)$$

Alternatively joint control can be obtained using the joint distribution of the maxima of the logit scores as follows.

**Theorem 2.6.** (Joint coverage) Assume that Assumption 1 and 2 hold. Given  $\alpha \in (0, 1)$ , define

$$\lambda(\alpha) = \inf \left\{ \lambda : \frac{1}{n} \sum_{i=1}^n 1[\max(\tau_i, \gamma_i) \leq \lambda] \geq 1 - \alpha \right\}.$$

Let  $O(X) = \{v \in \mathcal{V} : f_O(-s(X, v)) \leq \lambda(\alpha)\}$  and  $I(X) = \{v \in \mathcal{V} : f_I(s(X, v)) > \lambda(\alpha)\}$ . Then,

$$\mathbb{P}(I(X_{n+1}) \subseteq \{v \in \mathcal{V} : Y(v) = 1\} \subseteq O(X_{n+1})) \geq 1 - \alpha. \quad (6)$$

*Proof.* Exchangeability of the image pairs implies exchangeability of the sequence  $(\tau_i, \gamma_i)_{i=1}^{n+1}$ . Moreover on the event that  $\max(\tau_{n+1}, \gamma_{n+1}) \leq \lambda(\alpha)$  we have  $\tau_{n+1} \leq \lambda(\alpha)$  and  $\gamma_{n+1} \leq \lambda(\alpha)$  so the result follows via a proof similar to that of Theorem 2.1.  $\square$

**Remark 2.7.** The advantage of Corollary 2.5 is that the resulting inner and outer sets provide pivotal inference - not favouring one side or the other - which can be important when the distribution of the score function is asymmetric. Moreover the levels  $\alpha_1$  and  $\alpha_2$  can be used to provide a greater weight to either inner or outer sets whilst maintaining joint coverage. Theorem 2.6 may instead be useful when there are strong levels of dependence between  $\tau_1$  and  $\gamma_1$ . However, when this dependence is low, scale differences in the scores can lead to a lack of pivotality. This can be improved by appropriate choices of the score transformations  $f_I$  and  $f_O$  however in practice it may be simpler to construct joint sets using Corollary 2.5.

### 2.4 BETTER SEGMENTORS PROVIDE MORE PRECISE CONFORMAL CONFIDENCE SETS

Given two real random variables,  $A$  and  $B$  write  $A \succeq B$  to indicate that  $\mathbb{P}(A > t) \geq \mathbb{P}(B > t)$  for all  $t \in \mathbb{R}$ . Then we have the following result.

**Theorem 2.8.** Suppose that  $(X_i, Y_i)_{i=1}^{n+1}$  is an i.i.d. sequence, and let  $s, t : \mathcal{V} \rightarrow \mathbb{R}$  be two score functions. Assume that  $\max_{v \in \mathcal{V}: Y_1(v)=0} s_v(X_1) \succeq \max_{v \in \mathcal{V}: Y_1(v)=0} t_v(X_1)$

### 2.5 OPTIMIZING SCORE TRANSFORMATIONS

The choice of score transformations  $f_I$  and  $f_O$  is extremely important and can have a large impact on the size of the conformal confidence sets. The best choice depends on both the distribution of the data and on the nature of the output of the trained segmentor used to calculate the scores. We thus

216 recommend setting aside a learning dataset independent from both the calibration dataset, used to  
 217 compute the conformal thresholds, and the test dataset. This approach was used in Sun & Yu (2024)  
 218 to learn the best copula transformation for combining dependent data streams.

219 In order to make efficient use of the data available, the learning dataset can in fact contain some  
 220 or all of the data used to train the image segmentor. This data is assumed to be independent of  
 221 the calibration and test data and so can be used to learn the best score transformations without  
 222 compromising validity . The advantage of doing so is that less additional data needs to be set aside  
 223 or collected for the purposes of learning a score function. Moreover it allows for additional data to  
 224 be used to train the model resulting in better segmentation performance. The disadvantage is that  
 225 machine learning models typically overfit their training data meaning that certain score functions  
 226 may appear to perform better on this data than they do in practice. The choice of whether to include  
 227 training data in the learning dataset thus depends on the quantity of data available and the quality of  
 228 the segmentation model.

229 A score transformation that we will make particular use of in Section 3 is based on the distance  
 230 transformation which we define as follows. Given  $\mathcal{A} \subseteq \mathcal{V}$ , let  $E(\mathcal{A})$  be the set of points on the  
 231 boundary of  $\mathcal{A}$  obtained using the marching squares algorithm (Maple, 2003). Given a distance  
 232 metric  $\rho$  define the distance transformation  $d_\rho : \mathcal{P}(\mathcal{V}) \times \mathcal{V} \rightarrow \mathbb{R}$ , which sends  $\mathcal{A} \in \mathcal{P}(\mathcal{V})$  and  $v \in \mathcal{V}$   
 233 to

$$d_\rho(\mathcal{A}, v) = \text{sign}(\mathcal{A}, v) \min\{\rho(v, e) : e \in E(\mathcal{A})\},$$

234 where  $\text{sign}(\mathcal{A}, v) = 1$  if  $v \in \mathcal{A}$  and equals  $-1$  otherwise. The function  $d_\rho$  is an adaption of the  
 235 distance transform of Borgefors (1986) which provides positive values within the set  $\mathcal{A}$  and negative  
 236 values outside of  $\mathcal{A}$ .

## 239 2.6 CONSTRUCTING CONFIDENCE SETS FROM BOUNDING BOXES

240 Existing work on conformal confidence sets which aim to provide coverage of the entire ground  
 241 truth mask with a given probability has primarily focused on bounding boxes, see e.g. (de Grancey  
 242 et al., 2022; Andéol et al., 2023; Mukama et al., 2024). These papers adjust for multiple comparisons  
 243 over the 4 edges of the bounding box, doing so conformally by comparing the distance between the  
 244 predicted bounding box and the bounding box of the ground truth mask. These approaches aggregate  
 245 the predictions over all objects within all of the calibration images, often combining multiple  
 246 bounding boxes per image. However, as observed in Section 5 of de Grancey et al. (2022), doing so  
 247 violates exchangeability which is needed for valid conformal inference, as there is dependence be-  
 248 tween the objects within each image. These papers do not provide formal proofs and their theoretical  
 249 validity is thus unclear.

250 In order to provide a more formal justification of bounding box methods we establish the validity of  
 251 an adapted version of the max-additive bounding box method of Andéol et al. (2023) as a corollary  
 252 to our results, see Appendix A.3. We compare to this approach in our experiments below. Targetting  
 253 bounding boxes does not directly target the mask itself and so the resulting confidence sets are  
 254 typically conservative.

## 256 3 APPLICATION TO POLPPS TUMOR SEGMENTATION

257 In order to illustrate and validate our approach we consider the problem of polyps tumor segmen-  
 258 tation. To do so we use the same dataset as in Angelopoulos et al. (2022) in which 1798 polyps  
 259 images, with available ground truth masks were combined from 5 open-source datasets (published  
 260 in Pogorelov et al. (2017), Borgli et al. (2020) Bernal et al. (2012), Silva et al. (2014)). Logit scores  
 261 were obtained for this data using the PraNet model Fan et al. (2020), which is based on the Unet  
 262 architecture CITE CHECK!

### 265 3.1 CHOOSING A SCORE TRANSFORMATION

266 In order to optimize the size of our confidence sets we set aside 298 of the 1798 polyps images  
 267 to form a learning dataset on which to choose the best score transformations. Importantly as the  
 268 learning dataset is independent of the remaining 1500 images set-aside, we can study it as much as  
 269 we like without compromising the validity of the follow-up analyses in Sections 3.2,

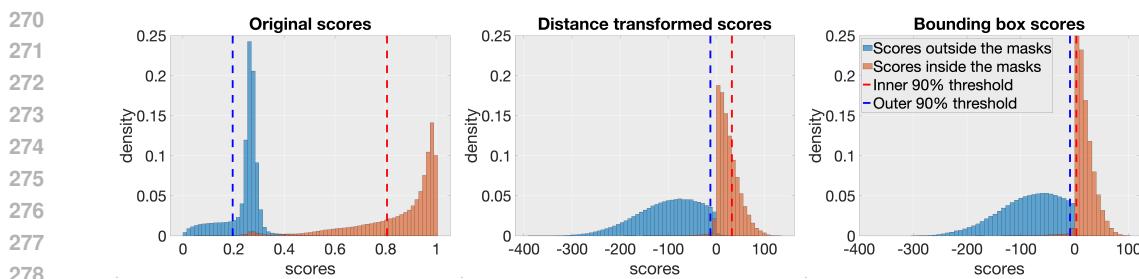


Figure 1: Histograms of the distribution of the scores on the learning dataset within and outside the ground truth masks. Thresholds obtained for the marginal 90% inner and outer confidence sets, based on the respective scores, are displayed in red and blue.

The score transformations we considered were the identity (after softmax transformation), distance transformations of the predicted masks and bounding box scores.

The PraNet scores for several typical examples are shown, after applying these transformations, in Figure XXX. From these we see that PraNet assigns a high softmax score to the polyps regions which decreases in the regions directly around the boundary of the tumor before returning to a higher level away from the polyps. This results in tight inner sets but large outer sets as the model struggles to identify where the tumor ends.

A further 10 examples are shown in Appendix XXX.

Based on the images set aside for alpha weighting we decided to use  $\alpha_1 = 0.02$  and  $\alpha_2 = 0.08$  to ensure a joint coverage of 90%. This ratio was chosen in light of the fact that in this data identifying where a given tumor ends appears to be more challenging than identifying pixels where we are sure that there is a tumor. For comparison we also present the results of an equal weighting scheme.

From the histograms in Figure 1 we can see that thresholding the scores at the inner threshold captures most of the data. However this is not the case for the outer threshold. From Figure XXX we can see that confidence sets based on the original scores struggle to identify where the tumor ends, resulting in very large sets.

### 3.2 ILLUSTRATING THE PERFORMANCE OF CONFORMAL CONFIDENCE SETS

Based on the results of the learning dataset we decided to combine the best of the approaches for the inner and outer sets respectively, taking  $f_I$  to be the softmax transformation and  $f_O$  to be the distance transformation of the predicted mask.

We divide the 1500 images at random into 500 for conformal calibration, and 1000 for validation. The resulting conformal confidence sets for this data are shown in the second row of Figure 3. For comparison we have also shown the sets obtained based on the untransformed softmax scores in the top row. From this figure we see that the method, using the transformed scores, effectively delineates polyp regions. Inner sets are plotted in red and the outer sets are shown in blue. The ground truth mask for each polyps is shown in yellow and can be compared to the original images. In each of the examples considered the ground truth mask is bounded from within by the inner set and from without by the outer set.

The inner sets are shown in red and represent regions where we can have high confidence of the presence of polyps. The outer sets are shown in blue and represent regions in which the polyps may be.

These results show that we can provide informative confidence bounds for the location of the polyps and allow us to use the PraNet segmentation model with uncertainty guarantees. They also illustrate the limitations of the model which is essential for applications. Larger uncertainty bounds may require specialist follow-up in order to be certain about the true extent of the observed tumor. Improved uncertainty quantification would require an improved segmentation model.

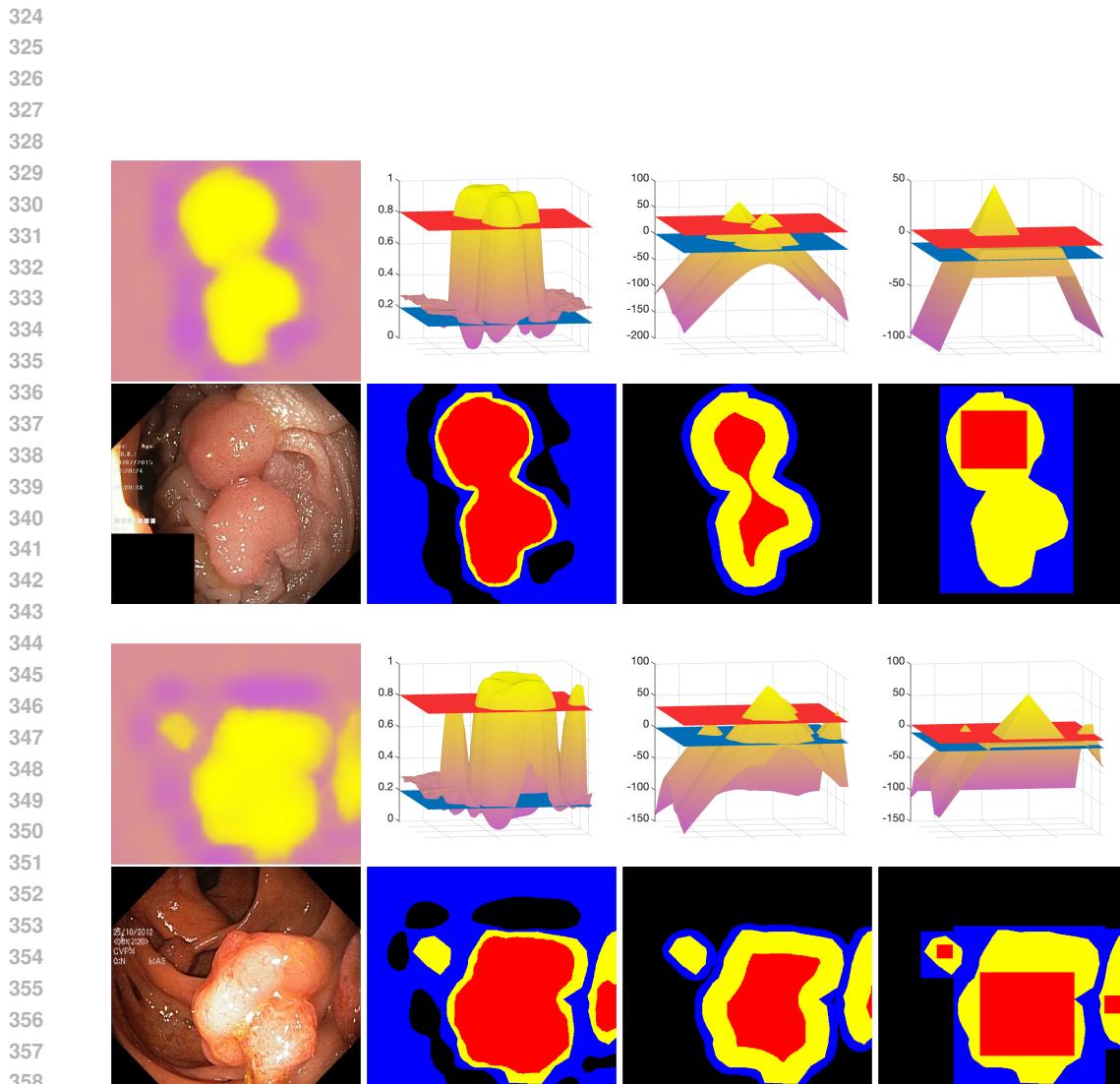


Figure 2: Illustrating the performance of the different score transformations on the learning dataset. We display 2 example tumors and present the results of each in 8 panels. These panels are as follows. Bottom right: the original image of the polyps tumor. Top Left: an intensity plot of the scores obtained from PraNet with purple/yellow indicating areas of lower/higher assigned probability. For the remaining panels, 3 different score transformations are shown which from left to right are the original scores, distance transformed scores and bounding box scores. In each of the panels on the top row a surface plot of the transformed PraNet scores is shown, along with the marginal conformal thresholds which are used to obtain the marginal 90% inner and outer sets. These thresholds are illustrated via red and blue planes respectively and are obtained over the learning dataset. The panels on the bottom show the corresponding conformal confidence sets. Here the inner set is shown in red, plotted over the ground truth mask of the polyps, shown in yellow, plotted over the outer set which is shown in blue. The outer set contains the ground truth mask which contains the inner set in all examples. From these figures we see that the original scores provide tight inner confidence sets and the distance transformed scores instead provide tight outer confidence sets. The conclusion from the learning dataset is therefore that it makes sense to combine these two score transformations.

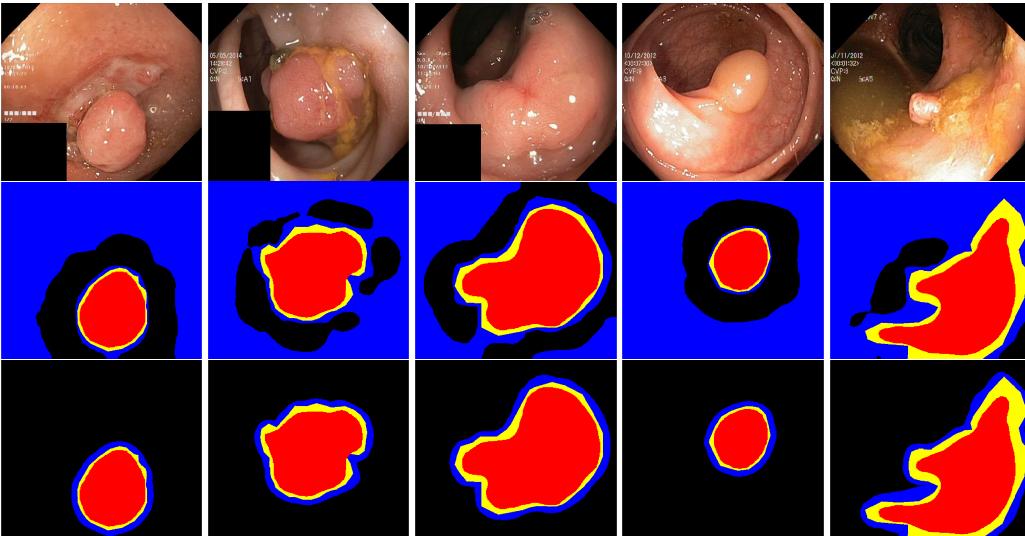


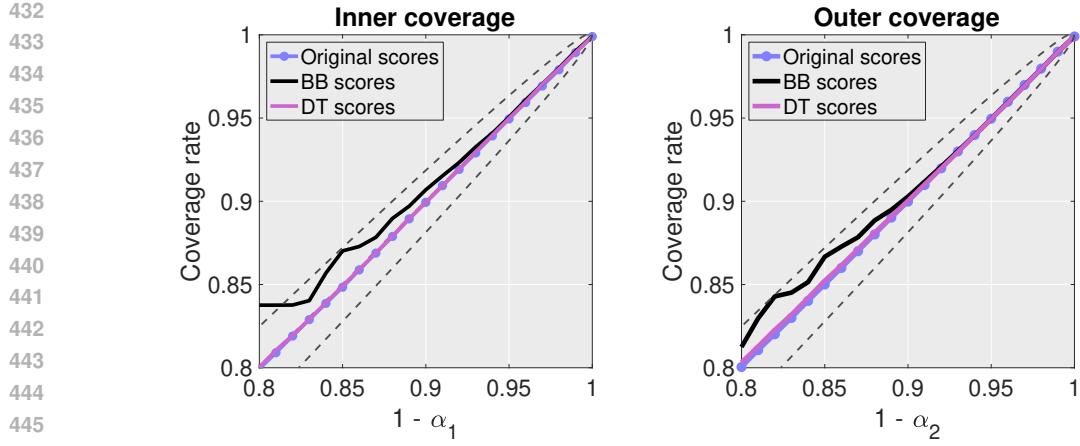
Figure 3: Conformal confidence sets for the polyps data. The bottom row shows the original endoscopic images with visible polyps. The top two rows present the conformal confidence sets, with the ground truth masks shown in yellow. The inner sets and outer sets are shown in red and blue respectively. The top row illustrates the sets which arise when using the original scores. Instead the middle show the resulting sets when  $f_O$  is given by the distance transformation of the predicted polyps mask. The figure shows the benefit of transforming the score function and illustrates the method’s effectiveness in accurately identifying polyp regions whilst providing informative spatial uncertainty bounds.

More precise results can be obtained at the expense of probabilistic guarantees, see Figure XXX. A trade off must be made between precision and confidence and this can also be determined in advance based on the learning dataset. The approach of CITE controls the empirical false negative risk yielding additional precision but at the cost of coverage as shown in Figure XXX.

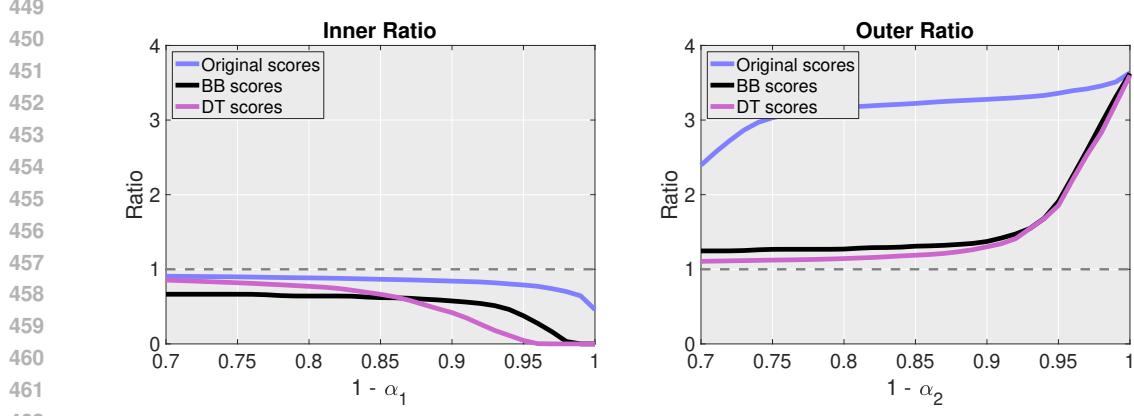
### 3.3 MEASURING THE COVERGE RATE

In this section we run validations to evaluate the false coverage rate of our approach. To do so we take the set aside 1500 images and run 1000 validations, in each validation dividing the data into 1000 calibration and 500 test images. In each division we calculate the conformal confidence sets using the above approaches, based on thresholds derived from the calibration dataset, and evaluate the coverage rate on the test dataset. We average over all 1000 validations and present the results in Figure 4. Histograms for the 90% coverage obtained over each validation run are shown in Figure 6. From these results we can see that all the approaches the coverage rate is controlled at or above the nominal level as desired. The coverage for the bounding box scores slightly over cover at lower levels. This is likely due to the discontinuities in the score functions.

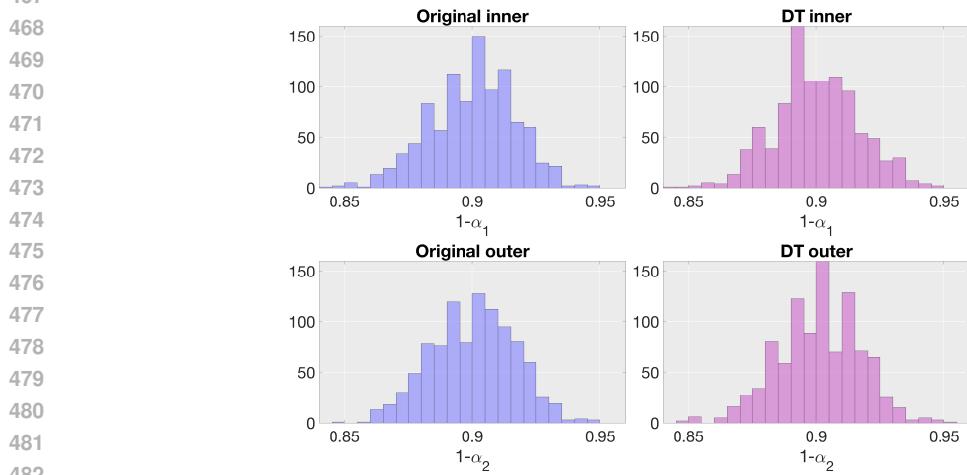
XXX In this Figure we also compare to the coverage attained by using Conformal Risk control . We can see that conformal risk control can have highly inflated error rates - this is because it is designed to control the expected proportion of discoveries not cover the tumors. The results indicate the trade-off that must be made when choosing between the methodss, i.e. whilst risk control can provide meaningful inference CITE it comes with a cost in terms of under coverage. Instead, in this setting, conformal confidence sets provide informative segmentation bounds (as illustrated in Section 3.2) and come with strong coverage guarantees.



447 Figure 4: False coverage levels of the inner and outer sets averaged over 1000 validations for the  
448 original, distance transformed (DT) and bounding box (BB) scores.



463 Figure 5: Measuring the efficiency of the bound using the ratio of the diameter of the coverage set to  
464 the diameter of the true tumor mask The closer the ratio is to one the better. Higher coverage rates  
465 lead to a lower efficiency. The original scores provide the most efficient inner sets and the distance  
466 transformed scores provide the most efficient outer sets.



483 Figure 6: Histograms of the coverage rates obtained across each of the validation resamples for  
484 90% inner and outer marginal confidence sets. We plot the results for the original scores, distance  
485 transformed scores (DT) and boundary box scores (BB) from left to right.

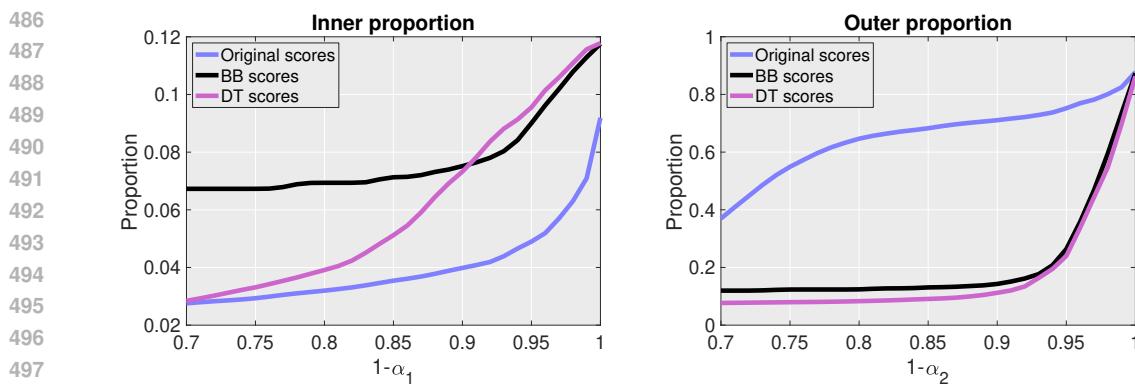


Figure 7: Measuring the proportion of the entire image which is under/over covered by the respective confidence sets. Left: proportion of the image which lies within the true mask but outside of the inner set. Middle: proportion of the image which lies within the confidence set but outside of the true mask. For both a lower proportion corresponds to increased precision.

### 3.4 COMPARING THE EFFICIENCY OF THE BOUNDS

### 3.5 IMPROVING RISK CONTROL USING TRANSFORMED SCORES

Risk control can also benefit

## 4 DISCUSSION

In this work, we have developed conformal confidence sets which offer probabilistic guarantees for the output of a image segmentation model. Our work helps to address the lack of formal uncertainty quantification in the application of deep neural networks to medical imaging which has limited the reliability and adoption of these models in practice.

Discuss how the method is very fast

The confidence sets we develop in this paper are related in spirit to work on uncertainty quantification for spatial excursion sets (Bowring et al. (2019), ?, CHEN). These approaches instead assume that multiple observations from a signal plus noise model are observed and perform inference on the underlying signal rather than prediction, obtaining confidence regions with asymptotic coverage guarantees.

One of the key strengths of our method is its ability to provide spatially resolved uncertainty estimates. Unlike global uncertainty measures, our approach allows for the identification of specific regions within an image where the model’s predictions are less certain.

Future work could explore more efficient algorithms or approximations that maintain the statistical guarantees while reducing computational cost. Second, while our method provides valid coverage guarantees, the tightness of the confidence sets may vary depending on the underlying model’s performance and the complexity of the segmentation task. In some cases, the confidence sets may be conservatively large, potentially limiting their practical utility. Investigating ways to produce tighter confidence sets while maintaining coverage guarantees is an important direction for future research.

Third, our current approach treats each pixel or voxel independently when constructing confidence sets. This may not fully capture the spatial correlations inherent in many biological structures. Developing methods that incorporate spatial dependencies and prior anatomical knowledge could lead to more informative and biologically plausible uncertainty estimates.

The implications of our work extend beyond the immediate technical contributions. By providing a rigorous framework for uncertainty quantification, we address a critical need in the deployment of AI systems in high-stakes applications like medical diagnosis. Our method can enhance the trust-

worthiness of AI-assisted image analysis by clearly communicating the limits of model certainty. This transparency is crucial for responsible AI deployment and could help mitigate risks associated with overreliance on automated systems.

Moreover, the insights gained from our uncertainty estimates could feed back into the development of improved segmentation models. By identifying consistent patterns of uncertainty, researchers may uncover systematic limitations in current architectures or training approaches, guiding future innovations in the field.

In conclusion, our work represents a significant step forward in bringing the power of conformal prediction to the domain of image segmentation. By providing spatial uncertainty guarantees with finite sample validity, we offer a valuable tool for researchers and clinicians alike. As AI continues to play an increasingly prominent role in medical imaging and beyond, methods like ours will be essential in ensuring that these powerful technologies are deployed responsibly and effectively.

Additionally, investigating the relationship between model calibration, uncertainty estimates, and out-of-distribution detection could further enhance the robustness of AI systems in real-world deployment scenarios.

Our approach has the potential to help enhance the overall reliability and trustworthiness of AI-assisted image analysis systems. By clearly delineating the limits of model certainty, we can help prevent overconfidence in automated predictions and promote a more nuanced integration of AI tools into professional workflows.

## AVAILABILITY OF CODE

Matlab code to reproduce the results of the paper is available in the supplementary material.

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## A APPENDIX

### A.1 OBTAINING CONFORMAL CONFIDENCE SETS WITH INCREASING COMBINATION FUNCTIONS

639 As discussed in Remark 2.3 the results of Sections 2.2 and 2.3 can be generalized to a wider class  
 640 of combination functions.

642 **Definition A.1.** We define a suitable combination function to be a function  $C : \mathcal{P}(\mathcal{V}) \times \mathcal{X} \rightarrow \mathbb{R}$   
 643 which is increasing in the sense that for all sets  $\mathcal{A} \subseteq \mathcal{V}$  and each  $v \in \mathcal{A}$ ,  $C(v, X) \leq C(\mathcal{A}, X)$  for  
 644 all  $X \in \mathcal{X}$ .

645 The maximum is a suitable combination function since  $X(v) = \max_{v \in \{v\}} X(v) \leq \max_{v \in \mathcal{A}} X(v)$ .  
 646 As such this framework directly generalizes the results of the main text.

647 We can construct generalized marginal confidence sets as follows.

648   **Theorem A.2.** (*Marginal inner set*) Under Assumptions 1 and 2, given  $\alpha_1 \in (0, 1)$ , define  
 649

$$650 \quad \lambda_I(\alpha_1) = \inf \left\{ \lambda : \frac{1}{n} \sum_{i=1}^n 1[C(\{v \in \mathcal{V} : Y_i(v) = 1\}, f_I(s(X_i))) \leq \lambda] \geq 1 - \alpha_1 \right\},$$

653   for a suitable combination function  $C$ , and define  $I(X) = \{v \in \mathcal{V} : C(v, f_I(s(X))) > \lambda_I(\alpha_1)\}$ .  
 654   Then,

$$655 \quad \mathbb{P}(I(X_{n+1}) \subseteq \{v \in \mathcal{V} : Y_{n+1} = 1\}) \geq 1 - \alpha_1. \quad (7)$$

656   The proof follows that of Theorem 2.1. The key observation is that for any suitable combination  
 657   function  $C$ , given  $\lambda \in \mathbb{R}$ ,  $\mathcal{A} \subseteq \mathcal{V}$  and  $X \in \mathcal{X}$ , we have that  $C(\mathcal{A}, X) \leq \lambda$  implies that  $C(v, X) \leq \lambda$ .  
 658   This is the relevant property of the maximum which we used for the results in the main text. For the  
 659   outer set we similarly have the following.

660   **Theorem A.3.** (*Marginal outer set*) Under Assumptions 1 and 2, given  $\alpha_2 \in (0, 1)$ , define  
 661

$$662 \quad \lambda_O(\alpha_2) = \inf \left\{ \lambda : \frac{1}{n} \sum_{i=1}^n 1[C(\{v \in \mathcal{V} : Y_i(v) = 0\}, f_O(-s(X_i))) \leq \lambda] \geq 1 - \alpha_2 \right\}.$$

665   for a suitable combination function  $C$ , and let  $O(X) = \{v \in \mathcal{V} : C(v, f_O(-s(X))) \leq \lambda_O(\alpha_2)\}$ .  
 666   Then,

$$667 \quad \mathbb{P}(\{v \in \mathcal{V} : Y_{n+1}(v) = 1\} \subseteq O(X_{n+1})) \geq 1 - \alpha_2. \quad (8)$$

668   Joint results can be analogously obtained.  
 669

## 670   A.2 OBTAINING CONFIDENCE SETS FROM RISK CONTROL 671

672   We can alternatively establish Theorems 2.1 and A.2 using an argument from risk control (Angelopoulos et al., 2022). In particular, given an image pair  $(X, Y)$  and  $\lambda \in \mathbb{R}$ , let

$$674 \quad I_\lambda(X) = \{v \in \mathcal{V} : C(v, f_I(s(X))) > \lambda\}.$$

676   Define a loss function,  $L : \mathcal{P}(\mathcal{V}) \times \mathcal{Y} \rightarrow \mathbb{R}$  which sends  $(X, Y)$  to

$$677 \quad L(I_\lambda(X), Y) = 1[I_\lambda(X) \not\subseteq \{v \in \mathcal{V} : Y_{n+1} = 1\}].$$

678   For  $i = 1, \dots, n+1$ , let  $L_i(\lambda) = L(I_\lambda(X_i), Y_i)$ . Then applying Theorem 1 of Angelopoulos et al.  
 679   (2022) it follows that

$$680 \quad \mathbb{E}[L_{n+1}(\hat{\lambda})] \leq \alpha_1$$

682   where  $\hat{\lambda} = \inf \left\{ \lambda : \frac{1}{n} \sum_{i=1}^n L_i(\lambda) \leq \alpha_1 - \frac{1-\alpha_1}{n} \right\}$ . Arguing as in Appendix A of (Angelopoulos  
 683   et al., 2022) it in fact follows that  $\hat{\lambda} = \lambda_I(\alpha_1)$  and so  $I(X) = I_{\hat{\lambda}}(X)$ . As such

$$685 \quad \mathbb{P}(I(X_{n+1}) \subseteq \{v \in \mathcal{V} : Y_{n+1} = 1\}) = 1 - \mathbb{E}[L_{n+1}(\hat{\lambda})] \geq 1 - \alpha_1, \quad (9)$$

687   and we recover the desired result. Arguing similarly it is possible to establish proofs of Theorems  
 688   2.2 and A.3.

## 689   A.3 PROVIDING THEORY FOR DERIVING CONFIDENCE SETS FROM BOUNDING BOXES 690

692   We can use our results in order to provide valid inference for bounding boxes. In what follows we  
 693   adapt the approach of Andéol et al. (2023) in order to ensure validity. In particular given  $Z \in \mathcal{Y}$ , let  
 694    $B_{I,\max}(Z)$  be the largest box which can be contained within the set  $\{v \in \mathcal{V} : Z(v) = 1\}$  and let  
 695    $B_{O,\min}(Z)$  be the smallest box which contains it. Given  $Y \in \mathcal{Y}$ , let  $cc(Y) \subseteq \mathcal{P}(\mathcal{V})$  denote the set  
 696   of connected components of the set  $\{v \in \mathcal{V} : Y(v) = 1\}$  for a given connectivity criterion (which  
 697   we take to be 4 in our examples), and note that these can themselves be identified as elements of  $\mathcal{Y}$ .  
 698   Define

$$699 \quad B_I(Y) = \cup_{c \in cc(Y)} B_{I,\max}(c) \text{ and } B_O(Y) = \cup_{c \in cc(Y)} B_{O,\min}(c)$$

700   to be the unions of the largest inner and smallest outer boxes of the connected components of the  
 701   image  $Y$ , respectively. Then define

$$702 \quad \hat{B}_I(s(X)) = \cup_{c \in cc(\hat{M}(X))} B_{I,\max}(c) \text{ and } \hat{B}_O(s(X)) = \cup_{c \in cc(\hat{M}(X))} B_{O,\min}(c)$$

702 to be the unions of the largest inner and smallest outer boxes of the connected components of the  
 703 predicted mask  $\hat{M}(X)$ , respectively. Note that this is well-defined as  $\hat{M}(X)$  is a function of  $s(X)$ .  
 704

705 For the remainder of this section we shall assume that  $\mathcal{V} \subset \mathbb{R}^2$ , this is not strictly necessary but  
 706 will help to simplify notation. Given  $u, v \in \mathcal{V}$ , write  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  and let  
 707  $\rho(u, v) = \max(|u_1 - v_1|, |u_2 - v_2|)$  be the chessboard metric.

708 **Definition A.4.** (Bounding box scores) For each  $X \in \mathcal{X}$  and  $v \in \mathcal{V}$ , let

$$709 \quad b_I(s(X), v) = d_\rho(\hat{B}_I(s(X)), v) \text{ and } b_O(s(X), v) = d_\rho(\hat{B}_O(s(X)), v)$$

710 be the distance transformed scores based on the chessboard distance to the predicted inner and outer  
 711 box collections  $\hat{B}_I(s(X))$  and  $\hat{B}_O(s(X))$ , respectively. We also define a combination of these  $b_M$ ,  
 712 primarily for the purposes of plotting in Figure 2, as follows. Let  $b_M(s(X), v) = b_O(s(X), v)$  for  
 713 each  $v \notin \hat{B}_O$  and let  $b_M(s(X), v) = \max(b_I(s(X), v), 0)$  for  $v \in \hat{B}_O$ . We shall write  $b_I(s(X)) \in$   
 714  $\mathcal{X}$  to denote the image which has  $b_I(s(X))(v) = b_I(s(X), v)$  and similarly for  $b_O(s(X))$  and  
 715  $b_M(s(X))$ . An illustration of these scores for two example tumors is shown in Figure XXX.

716 Now consider the sequences of image pairs  $(X_i, B_i^I)_{i=1}^n$  and  $(X_i, B_i^O)_{i=1}^n$ . These both satisfy ex-  
 717 changeability and so, applying Theorems 2.1 and 2.2 we obtain the following bounding box validity  
 718 results.

719 **Corollary A.5. (Marginal inner bounding boxes)** Suppose Assumption 1 holds and that  $(X_i, Y_i)_{i=1}^{n+1}$   
 720 is independent of the functions  $s$  and  $b_I$ . Given  $\alpha_1 \in (0, 1)$ , define

$$721 \quad \lambda_I(\alpha_1) = \inf \left\{ \lambda : \frac{1}{n} \sum_{i=1}^n \mathbf{1}[C(B_i^I, b_I(s(X_i))) \leq \lambda] \geq \frac{\lceil (1 - \alpha_1)(n + 1) \rceil}{n} \right\}, \quad (10)$$

722 for a suitable combination function  $C$ , and define  $I(X) = \{v \in \mathcal{V} : C(v, b_I(s(X))) > \lambda_I(\alpha_1)\}$ .  
 723 Then,

$$724 \quad \mathbb{P}(I(X_{n+1}) \subseteq B_{n+1}^I \subseteq \{v \in \mathcal{V} : Y_{n+1}(v) = 1\}) \geq 1 - \alpha_1.$$

725 **Corollary A.6. (Marginal outer bounding boxes)** Suppose Assumption 1 holds and that  $(X_i, Y_i)_{i=1}^{n+1}$   
 726 is independent of the functions  $s$  and  $b_O$ . Given  $\alpha_2 \in (0, 1)$ , define

$$727 \quad \lambda_O(\alpha_2) = \inf \left\{ \lambda : \frac{1}{n} \sum_{i=1}^n \mathbf{1}[C(B_i^O, -b_O(s(X_i))) \leq \lambda] \geq \frac{\lceil (1 - \alpha_2)(n + 1) \rceil}{n} \right\}. \quad (11)$$

728 for a suitable combination function  $C$ , and let  $O(X) = \{v \in \mathcal{V} : C(v, -b_O(s(X))) \leq \lambda_O(\alpha_2)\}$ .  
 729 Then,

$$730 \quad \mathbb{P}(\{v \in \mathcal{V} : Y_{n+1}(v) = 1\} \subseteq B_{n+1}^O \subseteq O(X_{n+1})) \geq 1 - \alpha_2.$$

731 Joint results can be obtained in a similar manner to those in Section 2.3.

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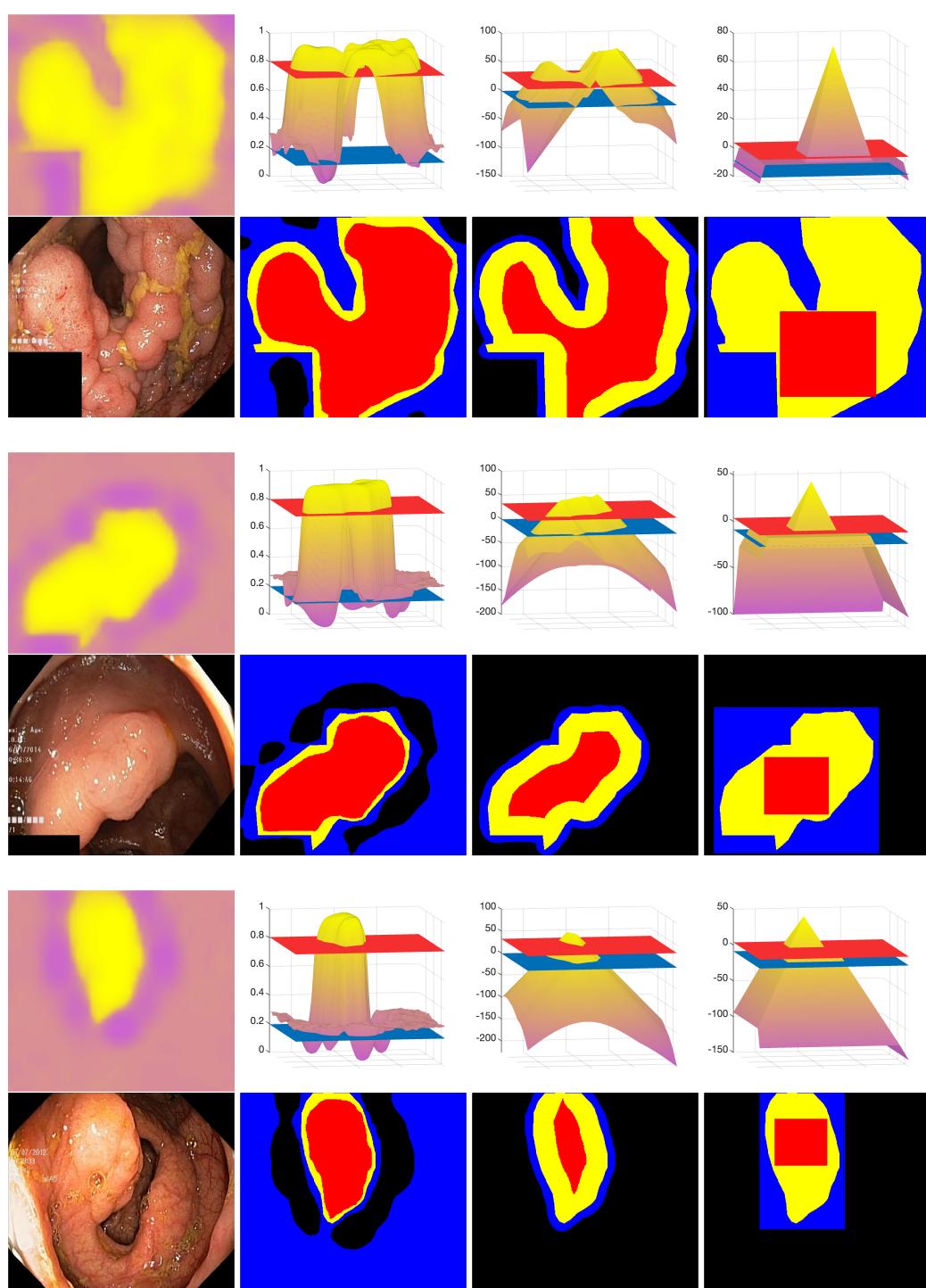
756 A.4 ADDITIONAL EXAMPLES FROM THE LEARNING DATASET  
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Figure 8: Additional examples from the learning dataset. The layout of these figures is the same as for Figure 2.

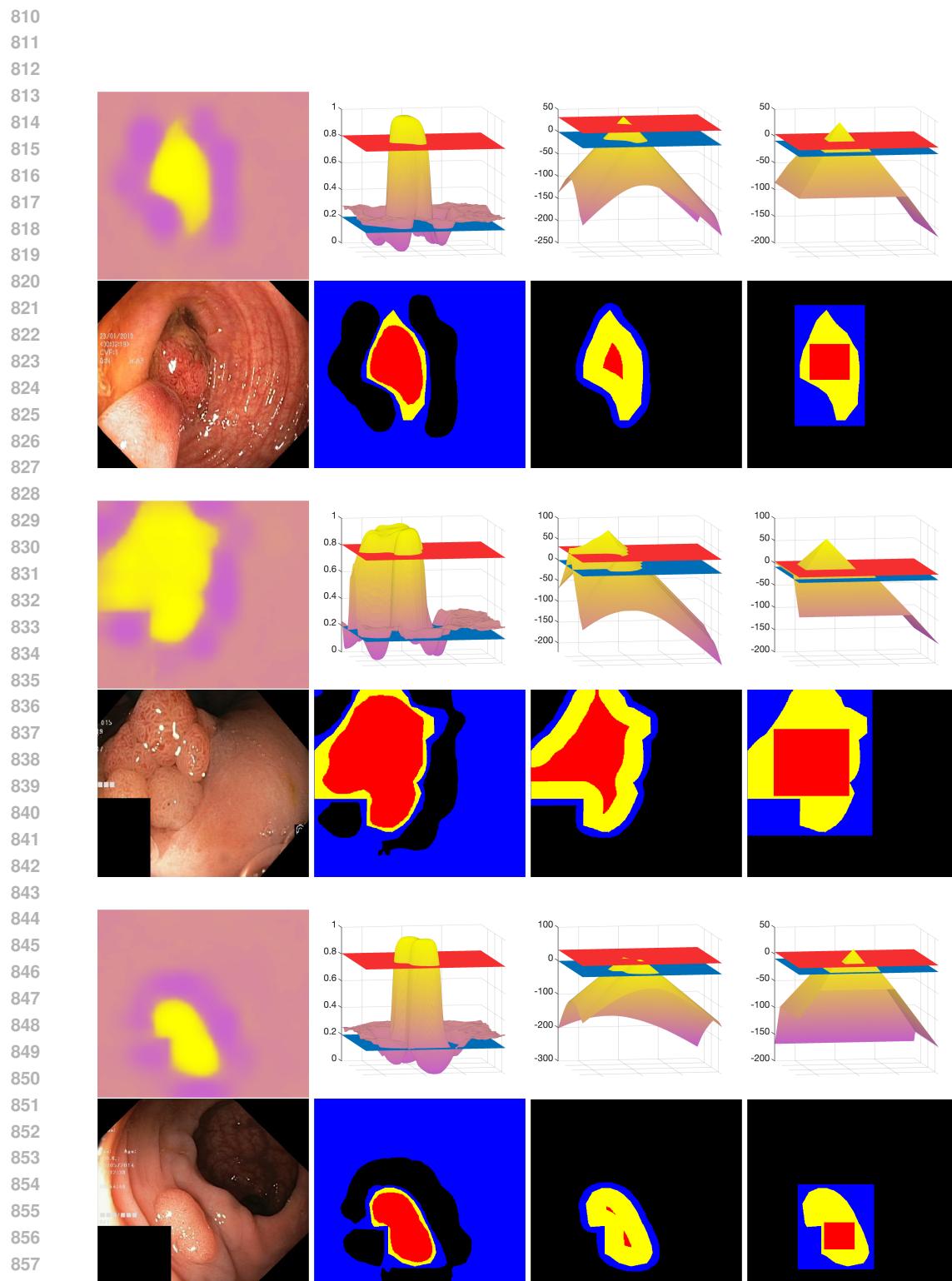


Figure 9: Futher examples from the learning dataset. The layout of these figures is the same as for Figure 2.

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### A.5 ADDITIONAL EXAMPLES FROM THE VALIDATION SET

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