Superthreshold Clusters Containing TFCE-Significant Clusters are LCE-Significant

1 Preliminaries

We work within the framework established in the main paper. Let $\mathcal{B} \subset \mathbb{R}^3$ denote the set of voxels making up the brain, and for each voxel $v \in \mathcal{B}$, let T_v denote the test statistic. The excursion set at threshold h is $\mathcal{E}(h) = \{v \in \mathcal{B} : T_v \geq h\}$.

For the generalized TFCE statistic, we have:

$$S_v = \int_{h_0}^{\infty} f(h) g(e_v(h))\, dh$$

where $e_v(h) = |C_v(h)|$ is the size of the connected component of $\mathcal{E}(h)$ containing v (or zero if $v \notin \mathcal{E}(h)$). We assume g is non-decreasing with $g(0) = 0$, and $f(h) \geq 0$ for $h \geq h_0$.

Definition 1. *An h_0-**superthreshold cluster** is a maximal connected component of the set $\{v \in \mathcal{B} : T_v \geq h_0\}$.*

Definition 2. *A voxel v is **TFCE-significant** if $S_v(\mathbf{X}_\mathcal{B}) \geq t^*_\mathcal{B}$, where $t^*_\mathcal{B}$ is the permutation-based critical value. A **TFCE-significant cluster** is a connected set of TFCE-significant voxels.*

Definition 3. *A region $R \subseteq \mathcal{B}$ is **LCE-significant** if $\max_{v \in R} S_v(\mathbf{X}_R) > t^*_\mathcal{B}$, where $\mathbf{X}_R$ denotes the masked data with voxels outside R set to zero.*

2 Main Result

Theorem 4. *Let R be a TFCE-significant cluster. Then any h_0-superthreshold cluster $R' \supseteq R$ is LCE-significant.*

Proof. Let R be a TFCE-significant cluster. By definition, for all $v \in R$ we have $S_v(\mathbf{X}_\mathcal{B}) \geq t^*_\mathcal{B}$.

Step 1: TFCE-significant voxels lie in h_0-superthreshold clusters.

By the assumptions on f and g, we have $S_v \geq 0$ for all voxels and all permutations, so $t^*_\mathcal{B} > 0$.

For any voxel v with $T_v < h_0$, we have $e_v(h) = 0$ for all $h \geq h_0$ since $v \notin \mathcal{E}(h)$ when $T_v < h$. Therefore:

$$S_v(\mathbf{X}_\mathcal{B}) = \int_{h_0}^{\infty} f(h) g(e_v(h))\, dh = \int_{h_0}^{\infty} f(h) g(0)\, dh = 0$$

since $g(0) = 0$. This means $S_v(\mathbf{X}_\mathcal{B}) = 0 < t^*_\mathcal{B}$, so v cannot be TFCE-significant.

By contraposition, every TFCE-significant voxel $v \in R$ must satisfy $T_v \geq h_0$. Since R is connected, there exists an h_0-superthreshold cluster $R' \supseteq R$.

Step 2: TFCE statistics are unchanged after masking to R'.

Let R' be any h_0-superthreshold cluster containing R. For any $v \in R'$ and any $h \geq h_0$, we claim that:

$$e_v(h, \mathbf{X}_{R'}) = e_v(h, \mathbf{X}_{\mathcal{B}}).$$

To establish this, observe that $C_v(h)$, the connected component of the excursion set $\mathcal{E}(h) = \{w \in \mathcal{B} : T_w \geq h\}$ containing v, must satisfy $C_v(h) \subseteq R'$. This follows because:

- All voxels in $\mathcal{B} \setminus R'$ that are neighbors of R' have $T_w < h_0 \leq h$ (by the definition of R' being an h_0-superthreshold cluster).
- Therefore, the connected component $C_v(h)$ cannot extend beyond R'.

Since masking to R' does not affect the cluster $C_v(h)$, we have:

$$e_v(h, \mathbf{X}_{R'}) = |C_v(h)| = e_v(h, \mathbf{X}_{\mathcal{B}}).$$

Consequently, for all $v \in R'$:

$$S_v(\mathbf{X}_{R'}) = \int_{h_0}^{\infty} f(h) g(e_v(h, \mathbf{X}_{R'}))\, dh = \int_{h_0}^{\infty} f(h) g(e_v(h, \mathbf{X}_{\mathcal{B}}))\, dh = S_v(\mathbf{X}_{\mathcal{B}}).$$

Step 3: Concluding LCE significance.

Since $R \subseteq R'$, there exists $v \in R \subseteq R'$ such that:

$$S_v(\mathbf{X}_{R'}) = S_v(\mathbf{X}_{\mathcal{B}}) \geq t^*_{\mathcal{B}}.$$

Therefore:

$$\max_{v \in R'} S_v(\mathbf{X}_{R'}) \geq t^*_{\mathcal{B}}.$$

By the definition of LCE significance, R' is LCE-significant. □

Remark 1. The key insight of this proof is that h_0-superthreshold clusters act as "islands" in the test statistic map: when masking to such a cluster, the TFCE statistics for voxels inside the cluster remain unchanged because the cluster boundaries (where $T_v < h_0$) prevent any information from voxels outside the region from influencing the statistic. This is precisely why the threshold h_0 plays such a crucial role in determining the spatial support of valid TFCE inference.

Corollary 5. *For every TFCE-significant cluster R, there exists a unique minimal h_0-superthreshold cluster $R' \supseteq R$ that is LCE-significant. This cluster R' is precisely the connected component of $\{v \in \mathcal{B} : T_v \geq h_0\}$ containing R.*