Superthreshold Clusters Containing TFCE-Significant Clusters are LCE-Significant

1 Main Result

Lemma 1. *Let R' be an h_0-superthreshold cluster for a cluster defing threshold $h_0 \in \mathbb{R}$. Then for any $v \in R'$ and any $h \geq h_0$, the connected component $C_v(h)$ containing v of the excursion set $\mathcal{E}(h) = \{w \in \mathcal{B} : T_w \geq h\}$ satisfies $C_v(h) \subseteq R'$.*

Proof. Since R' is an h_0-superthreshold cluster, all voxels in $\mathcal{B} \setminus R'$ that are neighbors of R' have $T_w < h_0 \leq h$. Therefore, the connected component $C_v(h)$ cannot extend beyond R'. □

Theorem 2. *Let R be a TFCE-significant cluster. Then any h_0-superthreshold cluster $R' \supseteq R$ is LCE-significant.*

Proof. Let R be a TFCE-significant cluster. By definition, for all $v \in R$ we have $S_v(\mathbf{X}_{\mathcal{B}}) \geq t^*_{\mathcal{B}}$.

Step 1: TFCE-significant voxels lie in h_0-superthreshold clusters.

If $T_v < h_0$, then $v \notin \mathcal{E}(h)$ for all $h \geq h_0$, so $e_v(h) = 0$ and hence $S_v = 0$. Since $t^*_{\mathcal{B}} > 0$, any TFCE-significant voxel must have $T_v \geq h_0$ and therefore lies in an h_0-superthreshold cluster.

Step 2: TFCE statistics are unchanged after masking to R'.

Let R' be any h_0-superthreshold cluster containing R. For any $v \in R'$ and any $h \geq h_0$, we claim that:

$$e_v(h, \mathbf{X}_{R'}) = e_v(h, \mathbf{X}_{\mathcal{B}}).$$

By Lemma 1, $C_v(h) \subseteq R'$. Since masking to R' does not affect the cluster $C_v(h)$, we have:

$$e_v(h, \mathbf{X}_{R'}) = |C_v(h)| = e_v(h, \mathbf{X}_{\mathcal{B}}).$$

Consequently, for all $v \in R'$:

$$S_v(\mathbf{X}_{R'}) = \int_{h_0}^{\infty} f(h) g(e_v(h, \mathbf{X}_{R'}))\, dh = \int_{h_0}^{\infty} f(h) g(e_v(h, \mathbf{X}_{\mathcal{B}}))\, dh = S_v(\mathbf{X}_{\mathcal{B}}).$$

Step 3: Concluding LCE significance.

Since $R \subseteq R'$, there exists $v \in R \subseteq R'$ such that:

$$S_v(\mathbf{X}_{R'}) = S_v(\mathbf{X}_{\mathcal{B}}) \geq t^*_{\mathcal{B}}.$$

Therefore:

$$\max_{v \in R'} S_v(\mathbf{X}_{R'}) \geq t^*_{\mathcal{B}}.$$

By the definition of LCE significance, R' is LCE-significant. □

Corollary 3. *For any TFCE-significant cluster R, its support (the h_0-superthreshold cluster containing R) is LCE-significant.*

Proof. By Step 1, every voxel in R satisfies $T_v \geq h_0$, so R is contained in a unique h_0-superthreshold cluster R'. The result follows immediately from Theorem 2. □

Remark 1. The key insight of this proof is that h_0-superthreshold clusters act as "islands" in the test statistic map: when masking to such a cluster, the TFCE statistics for voxels inside the cluster remain unchanged because the cluster boundaries (where $T_v < h_0$) prevent any information from voxels outside the region from influencing the statistic. This is precisely why the threshold h_0 plays such a crucial role in determining the spatial support of valid TFCE inference.

Corollary 4. *For every TFCE-significant cluster R, there exists a unique minimal h_0-superthreshold cluster $R' \supseteq R$ that is LCE-significant. This cluster R' is precisely the connected component of $\{v \in \mathcal{B} : T_v \geq h_0\}$ containing R.*