If ~~f~~ f is a bounded measurable fn and ~~If~~ X, Y are independent, then,

$$\mathbb{E}[f(X,Y) \mid X] = r(X) \text{ a.s.}$$

Proof? ~~E[~~ For a bounded Borel g,

$$\mathbb{E}[\cancel{\mathbb{E}}\, f(X,Y)\, g(X)] \overset{\text{need}}{=} \mathbb{E}[r(X)\, g(X)]$$

$$\mathbb{1}[X \in \pi]$$

$$\sigma(X, Y)$$

$$\mathbb{1}[X \in A, Y \in B]$$

$$\mathbb{E}[\mathbb{1}[X \in A, Y \in B]\, \mathbb{1}[X \in C]]$$

$$= \mathbb{E}[X \in A \cap C]\, \mathbb{E}[Y \in B] \quad \text{if } X, Y \text{ are indep.}$$

$? \overset{?}{=} \cancel{\mathbb{E}[\mathbb{E}[X \in A, Y} \in B$

$$\cancel{r(X)} = r(x) = \mathbb{E}[f(x, Y)]$$

$$= \mathbb{E}[\mathbb{1}[x \in A, Y \in B]]$$

$$= \mathbb{E}[\mathbb{1}[Y \in B]]\, \mathbb{1}[x \in A]$$