$\boxed{B \in \mathcal{G} \cap A}$

$$\cancel{= \mathbb{E}\left[ \frac{\mathbb{E}[X\mathbb{1}[A] \mid \mathcal{G} \cap A]}{\mathbb{P}(A)} \mathbb{1}[B] \right] \Big/ \mathbb{P}(A)}$$

$f_n^+|_A \uparrow (X|A)^+$

$$\cancel{\Rightarrow \mathbb{E}[X|A,\mathcal{G}] = \frac{\mathbb{E}[X\mathbb{1}[A] \mid \mathcal{G}\cap A]}{\mathbb{P}(A)}}$$

---

~~Note~~ $\forall C \in \mathcal{G}$,

$\cancel{\mathbb{E}\left[\mathbb{E}[X\mathbb{1}[A] \mid \mathcal{G}\cap A]\,\mathbb{1}[C]\right]}$ is $\mathcal{G}$ msble

$\mathcal{G}\cap A$ msble $\Rightarrow$ $\mathcal{G}$-msble as $\mathcal{G}\cap A \subseteq \mathcal{G}$.

~~eg $X$ $\mathcal{G}\cap A$ msble then for $K \in \mathcal{B}(\mathbb{R})$~~

$X^{-1}(K) \in \mathcal{G}\cap A \subseteq \mathcal{G}$

so ~~$\in \mathcal{G}$ as $\mathcal{G}$ closed~~ under $\cap$.

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$$= \frac{1}{\mathbb{P}(A)} \mathbb{E}\left[ \mathbb{E}[X\mathbb{1}[A] \mid \mathcal{G}]\ \mathbb{1}[B] \right] \quad \left(\text{as } B \in \mathcal{G}\cap A \subseteq \mathcal{G}.\right)$$

$$= \mathbb{E}\left[ \frac{\mathbb{E}[X\mathbb{1}[A] \mid \mathcal{G}]}{\mathbb{P}(A)} \mathbb{1}[B] \right]$$

$$\Rightarrow \boxed{\mathbb{E}[X|A,\mathcal{G}] = \frac{\mathbb{E}[X\mathbb{1}[A] \mid \mathcal{G}]}{\mathbb{P}(A)}.}$$

$\square$