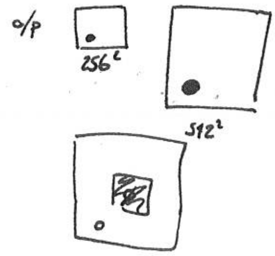
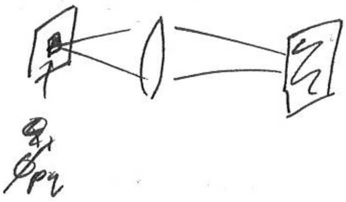
Input: ϕ_{pq}

$$E^{(in)}_{pq} = A_0 S_{pq} \exp(i\phi_{pq})$$ 256x256

T_{nm} : 512 x 512



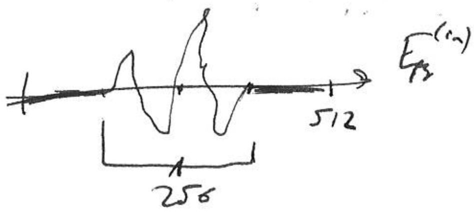
Then output $E^{(out)}_{nm} = \frac{1}{N}\sum_{pq} E^{(in)}_{pq} \exp\left(-\frac{2\pi i}{N}(pn+qm)\right)$



$$E^{(out)}_{n} = \frac{1}{N}\sum_p E^{(in)}_p e^{-\frac{2\pi i}{N}pn}, \quad E_p = A_0 S_p \exp(i\phi_p)$$

$$\therefore \frac{\partial E^{(out)}_n}{\partial \phi_{p'}} = \frac{1}{N}\sum_p \frac{\partial E^{(in)}_p}{\partial \phi_{p'}} e^{-\frac{2\pi i}{N}pn}$$

$\hookrightarrow iE^{(in)}_p \delta_{pp'}$



$$= \frac{1}{N}\sum_p iE_p^{(in)} e^{-\frac{2\pi i}{N}pn} \delta_{pp'}$$

$$\underbrace{\frac{\partial E^{(out)}_n}{\partial \phi_{p'}}}_{\Lambda_{np'}} = \frac{1}{N} iE^{(in)}_{p'} e^{-\frac{2\pi i}{N}p'n}$$

$$\left(e^{-\frac{2\pi i p'}{N}}\right)^n$$

$$E^{(in)}_{p'} e^{-\frac{2\pi i p'}{N}}$$

$E^{(in)}_{pq} = A_0 S_{pq} \exp(i\phi_{pq}), \ (p,q : 0 \to 255) \quad n_{pixel} = 256$

$\cos\phi_{pq} + i\sin\phi_{pq}$

$S^{(r)}_{pq} + iS^{(i)}_{pq}$

$$= A_0\left(S^{(r)}_{pq}\cos\phi_{pq} - S^{(i)}_{pq}\sin\phi_{pq}\right) + iA_0\left(S^{(i)}_{pq}\cos\phi_{pq} + S^{(r)}_{pq}\sin\phi_{pq}\right)$$

$$\sum_n \frac{\partial F_n}{\partial x_j} z_n = \frac{1}{N} iE_{j} e^{-\frac{2\pi i}{N}jn} z_n$$

$$\underline{\underline{J}} = \frac{dF}{d\underline{x}} = \left[\frac{\partial F}{\partial x_1} \cdots \frac{\partial F}{\partial x_n}\right] = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

$$F = \begin{pmatrix} \vdots \end{pmatrix}, \text{ then need } \underline{\underline{J}}^T \underline{z} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}z_1 + \frac{\partial F_2}{\partial x_1}z_2 + \cdots \frac{\partial F_m}{\partial x_1}z_m \\ \text{---} \\ \text{---} \end{pmatrix} = \begin{pmatrix} \sum_i \frac{\partial F_i}{\partial x_1} z_i \\ \sum_i \frac{\partial F_i}{\partial x_2} z_i \\ \vdots \end{pmatrix}$$

Need $\underline{\underline{J}}^T \underline{z} = \begin{pmatrix} \sum_i \frac{\partial F_i}{\partial x_1} z_i \\ \sum_i \frac{\partial F_i}{\partial x_2} z_i \\ \vdots \end{pmatrix}$ in 1D

$F_i \equiv E_n^{(out)}$

$x_j \equiv \phi_p$

$\therefore$

$$\underline{\underline{J}}^T \underline{z} = \begin{pmatrix} \sum_n \frac{\partial F_n}{\partial \phi_1} z_n \\ \sum_n \frac{\partial F_n}{\partial \phi_2} z_n \\ \vdots \end{pmatrix}$$

$$\sum_n \frac{\partial E_n^{(out)}}{\partial \phi_p} z_n = \sum_n \frac{1}{N} i E_p^{(in)} e^{\frac{-2\pi i}{N} pn} z_n$$

$$= i \frac{1}{N} E_p^{(in)} \underbrace{\sum_n z_n e^{\frac{-2\pi i}{N} pn}}_{\tilde{z}_p \text{ , i.e. } \mathcal{F}(z_n)}$$

$$E_n^{(out)} = \frac{1}{N} \sum_p E_p^{(in)} e^{-\frac{2\pi i}{N} pn}$$

$$\therefore \frac{\partial E_n^{(out)}}{\partial E_q^{(in)}} = \frac{1}{N} \sum_p \frac{\partial E_p^{(in)}}{\partial E_q^{(in)}} e^{\frac{-2\pi i}{N} pn}$$

$$= \frac{1}{N} e^{\frac{-2\pi i}{N} qn}$$

Elem of $\underline{\underline{J}}^T \underline{z}$: $\sum_n \frac{\partial E_n^{(out)}}{\partial E_p^{(in)}} z_n$

$$= \frac{1}{N} \sum_n e^{\frac{-2\pi i}{N} pn} z_n$$

Let $y_p = \sum_n x_n e^{\frac{-2\pi i}{N} pn}$

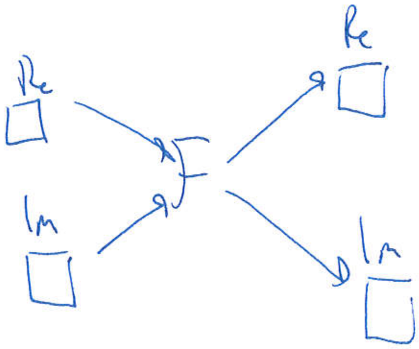
$$= \sum_n \Big(\mathrm{Re}(x_n) \cos\left(-\tfrac{2\pi i}{N} pn\right) - \mathrm{Im}(x_n) \sin\left(-\tfrac{2\pi i}{N} pn\right)$$

$$+ \mathrm{Re}(x_n) \sin(\quad) + \mathrm{Im}(\quad) \cos(\quad) \Big)$$

$$y_p = \mathrm{Re}(y_p) + i\mathrm{Im}(y_p)$$

$$\frac{\partial y_p}{\partial x_n} = e^{-\frac{2\pi i}{N} pn}$$

$$\frac{\partial \mathrm{Re}(y_p)}{\partial \mathrm{Re}(x_n)} = \cos\left(-\frac{2\pi i}{N} pn\right)$$



Grad: returns $[\underset{grad_{\mathrm{Re}(x_n)}}{q}, \underset{\text{ignore.}}{\cancel{\ }}]$

Funs on **Re → Re**:

$\underline{z}_i = \frac{\partial \mathcal{L}}{\partial y_i}$

outputs: list of Op: $[f_1, f_2]$,

$\therefore$ output_gradients = $[grad_{f_1}(C), grad_{f_2}(C)]$

$$grad_{\mathrm{Re}(x_n)}(C) = \frac{\partial \mathcal{L}}{\partial \mathrm{Re}(x_n)} = \sum_i \frac{\partial \mathcal{L}}{\partial f_i} \frac{\partial f_i}{\partial \mathrm{Re}(x_n)}$$

calc!

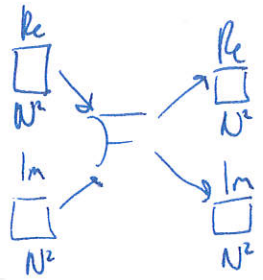
FourierOp:

Action: $y_p = \sum_n x_n e^{-\frac{2\pi i}{N} pn}$,

w/ $y_p \to \mathrm{Re}(y_p) + i\,\mathrm{Im}(y_p)$,

$x_n \to \mathrm{Re}(x_n) + i\,\mathrm{Im}(x_n)$

$$y_{pq} = \sum_{nm} x_{nm} e^{-\frac{2\pi i}{N}(pn+qm)}$$



$$y^{(R)}_{pq} + i y^{(I)}_{pq} = \sum_{nm} \left(x^{(R)}_{nm} + i x^{(I)}_{nm} \right) e^{-\frac{2\pi i}{N}(pn+qm)}$$

$$\therefore y^{(R)}_{pq} = \sum_{nm} \left(x^{(R)}_{nm} \cos\left(-\tfrac{2\pi i}{N} \cdots\right) - x^{(I)}_{nm} \sin(\cdots) \right)$$

$$y^{(I)}_{pq} = \sum_{nm} \left(x^{(I)}_{nm} \cos(\cdots) + x^{(R)}_{nm} \sin(\cdots) \right)$$

If inputs are $[x_1, \dots x_m]$, grad should return $[\mathrm{grad}_{x_1}(C), \dots \mathrm{grad}_{x_m}(C)]$,

w/ $(\mathrm{grad}_y(Z))_i = \dfrac{\partial Z}{\partial y_i}$.

Also receives output-gradients $= [\mathrm{grad}_{f_1}(C), \dots \mathrm{grad}_{f_n}(C)]$, given function outputs $[f_1, \dots f_n]$

Chain: $\mathrm{grad}_{x_i}(C) = \dfrac{\partial C}{\partial x_i} = \sum_n \underbrace{\dfrac{\partial C}{\partial f_n}}_{\text{all given.}} \dfrac{\partial f_n}{\partial x_i}$

Our case, inputs $= [\underline{x}^{(R)}, \underline{x}^{(I)}]$, grad returns $[\mathrm{grad}_{\underline{x}^{(R)}}(C), \mathrm{grad}_{\underline{x}^{(I)}}(C)]$

outputs $= [\underline{y}^{(R)}, \underline{y}^{(I)}]$

Then $\left(\mathrm{grad}_{x^{(R)}}(C)\right)_{nm} = \dfrac{\partial C}{\partial x^{(R)}_{nm}}$

$$= \sum_{pq} \underbrace{\frac{\partial C}{\partial y^{(R)}_{pq}}}_{\text{given}} \frac{\partial y^{(R)}_{pq}}{\partial x^{(R)}_{nm}} + \sum_{pq} \underbrace{\frac{\partial C}{\partial y^{(I)}_{pq}}}_{\text{given}} \frac{\partial y^{(I)}_{pq}}{\partial x^{(R)}_{nm}}$$

given outputs gradients $= [\mathrm{grad}_{\underline{y}^{(R)}}(C), \mathrm{grad}_{\underline{y}^{(I)}}(C)]$

Then $\dfrac{\partial y^{(R)}_{pq}}{\partial x^{(R)}_{nm}} = \cos\left(-\dfrac{2\pi i}{N}(pn+qm)\right)$

$$= \sum_{pq} \frac{\partial C}{\partial y^{(R)}_{pq}} \cos\left(-\frac{2\pi i}{N}(pn+qm)\right) + \sum_{pq} \frac{\partial C}{\partial y^{(I)}_{pq}} \sin(-)$$

$$= \mathrm{Re}\left\{ \sum_{pq} \left(\frac{\partial C}{\partial y^{(R)}_{pq}} + i \frac{\partial C}{\partial y^{(I)}_{pq}} \right) \left(\cos(\ ) - i \sin(\ ) \right) \right\}$$

$$= \mathrm{Re}\left\{ \sum_{pq} \left(\frac{\partial C}{\partial y^{(R)}_{pq}} + i \frac{\partial C}{\partial y^{(I)}_{pq}} \right) e^{+\frac{2\pi i}{N}(pn+qm)} \right\}$$

Also, $$\left(\mathrm{grad}_{x^{(I)}}(C) \right)_{nm} = \sum_{pq} \frac{\partial C}{\partial y^{(R)}_{pq}} \underbrace{\frac{\partial y^{(R)}_{pq}}{\partial x^{(I)}_{nm}}}_{-\sin(\ldots)} + \frac{\partial C}{\partial y^{(I)}_{pq}} \underbrace{\frac{\partial y^{(I)}_{pq}}{\partial x^{(I)}_{nm}}}_{\cos(\ )}$$

$$= \sum_{pq} \left(\frac{\partial C}{\partial y^{(R)}_{pq}} \sin\left(+ \frac{2\pi i}{N}(pn+qm) \right) + \frac{\partial C}{\partial y^{(I)}_{pq}} \cos\left(\frac{2\pi i}{N}(pn+qm) \right) \right)$$

$$= \mathrm{Im}\left\{ \sum_{pq} \underbrace{\left(\frac{\partial C}{\partial y^{(R)}_{pq}} + i \frac{\partial C}{\partial y^{(I)}_{pq}} \right)}_{\substack{\text{construct from} \\ \text{output-gradients}}} e^{+\frac{2\pi i}{N}(pn+qm)} \right\}$$