Quantum Chaos and Random Matrix Theory: The BGS Conjecture

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Abstract

Random Matrix Theory (RMT) was initially introduced by Wigner to analyze the quantum energy level spectra of complex heavy nuclei. Several decades later, this mathematical tool found a remarkable application in describing the level statistics of quantum chaotic system. The connection between quantum chaos and RMT was solidified through the celebrated Bohigas-Giannoni-Schmit (BGS) conjecture. In this review, we explore the predictions of RMT, showing its capability to accurately characterize the spectral properties of quantum chaotic systems. In addition, we will sketch the formal proof towards the BGS conjecture in the context of semiclassical approximation ($\hbar \to 0$).

1 Chaos in Physics and Random Matrix Theory

1.1 Classical Chaos

Chaos, while not having a universally accepted rigorous definition, usually refers to the seemingly random, unpredictable behaviors in a dynamical system. A system is considered chaotic if it displays a strong sensitivity of its dynamical process to the initial condition. In other words, a small difference in the initial state of a chaotic system can cause a dramatic(exponential) difference in the future evolution paths. A famous metaphor for this chaotic behavior is the "butterfly effect", which states that a butterfly flapping its wings in Texas may cause a tornado in Brazil. It is worth mentioning that finite-dimensional systems with linear equations of motions are never chaotic since the difference in initial conditions can only grow linearly along evolution. For a system to display chaos, it must be either non-linear, or has infinite dimensions [10].

Classically, the states of a dynamical system can be described as points in the phase space. Using this language, chaotic system is generally defined as a system that exhibits an exponential sensitivity of phase space trajectories to small perturbations. The sensitive dependence on initial condition makes the prediction of long-term dynamics of chaotic systems almost impossible (due to inevitable errors in simulations) and thus the behavior of a chaotic systems would appear random at long time. Such chaotic behavior can happen even when the system has a deterministic dynamical evolution, meaning that the evolution is completely described by a fixed set of equations of motions and the initial condition, and no randomness is evolved. In contrast to chaotic systems, there is a class of systems called integrable systems, for which the dynamics are simple and predictable due to the existence of many conserved quantities (integral of motions) in the system. Such definition of chaotic and integrable systems can also be generalized to many-body systems, where the system is usually called chaotic if it does not possess an extensive number of conserved quantities, and integrable otherwise.

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To get a straightforward feeling of the difference of integrable and chaotic systems, comparison of the trajectories of a particle moving in an integrable and chaotic 2-dimensional billiards is illustrated in Fig[1]. Particle in such billiards move and bounce in the walls without losing energy (elastic collision). We can see a simple and periodic (in angular and radial directions) trajectory in the integrable circular billiard Fig[1](a), while a seemingly completely random and *ergodic* trajectory is observed in the chaotic Bunimovich stadium billiard Fig[1](b). As an example of the "butterfly effect", in the Bunimovich stadium, if we compare the two trajectories of two particles that have very close initial points in the phase space, we will find that the two particles soon become uncorrelated in terms of their momentums and positions.

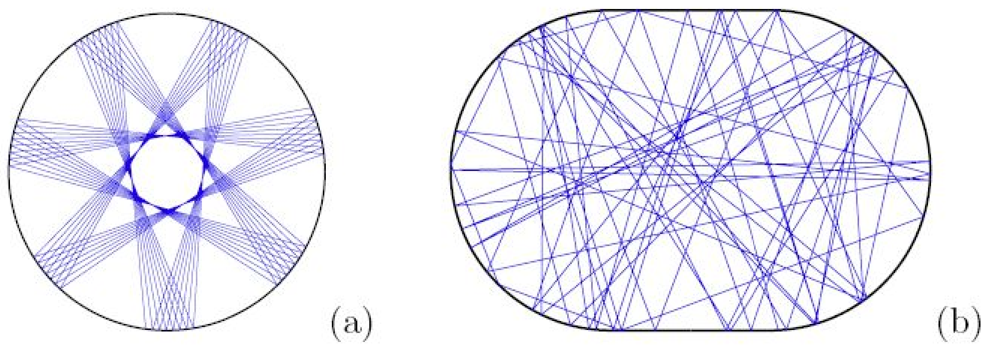


Figure 1: Examples of trajectories of a particle bouncing in a cavity:(a) non-chaotic circular and (b) chaotic Bunimovich stadium.

1.2 Quantum Chaos: Prediction from RMT

It is clear that the notion of classical chaos cannot be directly applied to quantum mechanical systems by looking at the form of equations dominating the evolution of the system. In quantum mechanic, the equation of motion is the Schrodinger's equation

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi, \tag{1}$$

which is linear and therefore cannot result in an exponential dependence of wavefunction trajectories to the initial conditions. In fact, the overlap between two different quantum states evolving under the same Hamiltonian remains constant in time. In addition, due to the Heisenberg uncertainty principle $[x, p] = i\hbar$ we do not have a good notion of trajectory in phase-space in quantum mechanics since the coordinates and momenta of particles cannot be defined as an exact number simultaneously (as a lazy notation we use a letter, e.g. x, without the hat to denote an operator in the rest of the paper when there is no confusion). Therefore, we lose the natural classical definition of sensitivity of phase-space trajectories to initial conditions in quantum world.

One natural thought towards the study of quantum chaos is to consider the quantization of classical systems. In the early days of quantum physics, quantization of a system can be done by either substituting the Poisson bracket in classical mechanics with the operator commutator, or by introducing the Bohr–Sommerfeld quantization condition for classical phase-space trajectories. The first approach leads to the research of quantum chaos using out-of-time-correlators (OTOC), which relates to the study of quantum analog of Lyapunov exponent used in describing the exponential divergence of phase-space trajectories in classical chaos. We will mainly focus on the second approach of quantization for describing quantum chaos in this review. Precisely speaking, in integrable systems where the phase-space trajectories are closed, the Sommerfeld quantization condition requires the classical reduced action to give:

$$\oint pdq = 2\pi n\hbar, n \in \mathbb{Z}. \tag{2}$$

This condition is formalized in the semiclassical theory of quantum mechanics known as the WKB approximation. Intuitively, the WKB approximation implies that in the semiclassical limit of quantum systems, the trajectories in the phase space are discretized (quantized). However, the quantization theory of classical chaotic systems has been unclear for a long time, mainly because in such chaotic systems the phase-space trajectories are not closed and thus one does not obtain a simple loop integral form of the quantization. The attempts to address this issue became the focus of research under the term quantum chaos. Until today many problems in quantum chaos remain unaddressed, including the precise definition of quantum chaos. In this review we aim to describe characteristics one expect to observe in a quantum system which has a chaotic classical counterpart. To properly and accurately describe the properties of such systems, one would need the tools of random matrix theory (RMT).

RMT seems to be developed and introduced in physics in a context independent of quantum chaos. Wigner[23, 24, 25], followed by Dyson and others[8], developed a theory as a statistical approach in nuclear physics to study systems with many degrees of freedom. Specifically, the theory is used to describe the spectra statistics of complex atomic nuclei, which is a complicated quantum-mechanical system. This theory is latered known as RMT. Wigner's original idea is that for complicated quantum-mechanical many-body system, like a large nuclei, one should focus on the statistical properties of the spectrum instead of each exact eigenvalue and eigenfunction. He suggested that if we look at a narrow window of energy levels far from the edges of the spectrum of a complex quantum system, in a *non fine-tuned* state basis, the Hamiltonian would essentially look like a random matrix (subject to certain symmetries). Under such assumption, he found an unreasonably effective description of the observed energy level statistics of complex nuclei systems using RMT. More specifically, he studied the statistics of energy levels and nearest-neighbor energy level spacing in the nuclei system, and found consistency with RMT predictions. In comparison of the prediction of RMT and experiment results, an ergodic hypothesis is used implicitly. This hypothesis is one of the fundamental hypothesis in statistical physics, which stats that the ensemble average is equal to the running average, taken over a large section of the spectrum of any matrix member in the ensemble.[11]. This hypothesis has been proved for specific systems[6].

The prediction of RMT can be demonstrated using a very simple 2-level example. Consider a 2×2 Hamiltonian system with independent real (time-reversal invariant) entries (up to symmetries) drawn from standard Gaussian distribution:

$$H = \begin{bmatrix} \epsilon_1 & V/\sqrt{2} \\ V^*/\sqrt{2} & \epsilon_2 \end{bmatrix} \tag{3}$$

where $\epsilon_1, \epsilon_2, V \sim N(0,1)$ are independent standard Gaussian random variables. The joint probability distribution of the matrix elements can be written as:

$$P(H) \propto \exp{(-\frac{1}{2}\mathrm{tr}H^2)} \tag{4}$$

where we have suppressed the normalization factor. Another equivalent definition is $H = \frac{A+A^\dagger}{2}$, where all entries of matrix A satisfy $A_{ij} \sim N(0,1)$. This is exactly the GOE matrix we learned in the RMT course. The eigenvalues of such 2×2 real symmetric matrix can be easily derived from orthogonal diagonalization:

$$E_{1,2} = \frac{\epsilon_1 + \epsilon_2}{2} \pm \frac{1}{2}\sqrt{(\epsilon_1 - \epsilon_2)^2 + 2|V|^2}. \tag{5}$$

What we want to compute is the energy level spacing distribution, which has the probability density $P(\omega) = P(E_1 - E_2 = \omega)$. It can be calculated from Eq.4 and Eq.5:

$$P(\omega) = \frac{1}{(2\pi)^{3/2}} \int d\epsilon_1 \int d\epsilon_2 \int dV \delta(\sqrt{(\epsilon_1 - \epsilon_2)^2 + 2|V|^2} - \omega) \exp\left(-\frac{\epsilon_1^2 + \epsilon_2^2 + V^2}{2}\right) \tag{6}$$

$$= \frac{\omega}{2} \exp\left(\frac{\omega^2}{4}\right) \tag{7}$$

When there is no time-reversal symmetry, we should consider a GUE matrix and similarly obtain the level spacing distribution for a $N = 2$ complex Hamiltonian:

$$P(\omega) = \frac{\omega^2}{2\sqrt{\pi}} \exp\left(\frac{\omega^2}{4}\right) \tag{8}$$

Eq.7 and Eq.8 can be written in a unified form as:

$$P(\omega) = A_\beta \omega^\beta \exp(B_\beta \omega^2), \tag{9}$$

which is exact for a 2×2 matrix model. Here β is the Dyson index labeling which Gaussian ensemble the random matrix is drawn from, with $\beta = 1, 2, 4$ for GOE, GUE and GSE, respectively. The coefficients A_β and B_β are found by normalizing $P(\omega)$ and fixing the mean level spacing. If we fix the mean level spacing to be unity, then the coefficients are explicitly given: $a_1 = \pi/2, b_1 = \pi/4$ (GOE), $a_2 = 32/\pi^2, b_2 = 4/\pi$ (GUE) and $a_4 = 262144/729\pi^3, b_4 = 64/9\pi$ (GSE). Wigner extended this result to approximately describe the spacing distribution for larger matrices, and thus the distribution Eq.9 is also known as *Wigner surmise* when it is used for larger $N \times N$ matrices. In principle, the exact level spacing distribution $p_{N,\beta}(\omega)$ of Gaussian ensemble matrices (knwon as Wigner-Dyson distribution) can be deduced analytically from the joint probability distribution of eigenvalues of a $N \times N$ Gaussian matrix:

$$\rho(x_1, ..., x_N) = \frac{1}{Z_{N,\beta}} e^{-\frac{1}{2}\sum_{i=1}^N x_i^2} \prod_{j<k} |x_j - x_k|^\beta \tag{10}$$

where

$$Z_{N,\beta} = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)} \tag{11}$$

is the normalization constant of the pdf, and $\beta = 1, 2, 4$. Eq.10 can be derived from change of variable in Eq.4 and computing the corresponding Jacobian (which gives the Vandermonde determinant term). However, it turned out that the analytical calculation of $p_N(\omega)$ is highly non-trivial and $p_N(\omega)$ do not have a closed analytic form (it involves infinite products). In practice, experimental results show that Wigner surmise $P(\omega)$ serves as an excellent approximation for $p_N(\omega)$.

Given these results at hand, we can already observe something non-trivial happening in the spectrum of Gaussian matrix. In Eq.9 and Eq.10 there are terms that both reflect the "repulsion" effect between energy levels in these Gaussian matrices. More precisely, the factor ω^β in Eq.9 and the Vandermonde determinant $\Delta_N(\mathbf{x})^\beta = \prod_{j<k} |x_j - x_k|^\beta$, prevent neighboring two levels from getting arbitrarily close since the probability $P(\omega)$ and $\rho(\mathbf{x})$ vanishes as $\omega \to 0$ and $x_j \to x_k$. On the other hand, assuming independence of the eigenvalues, we can model the probability of having n levels in an energy interval $[E, E + \delta E]$ with a Poisson distribution:

$$P_n = \frac{\lambda^n}{n!} e^{-\lambda} \tag{12}$$

where $\lambda = \rho\delta E$ is the mean number of levels in the interval, and $\rho = 1/\langle\omega\rangle$ is the mean level density per energy unit. The nearest neighbor spacing distribution of Poisson statistics then has the exponential form:

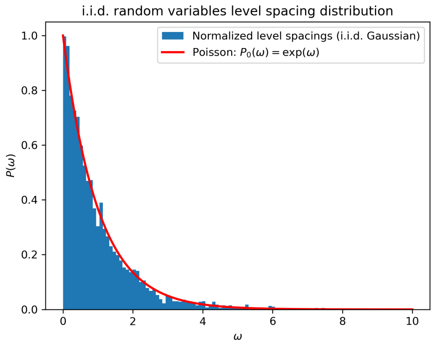
$$P_0(\omega) = \exp(-\rho\omega) \tag{13}$$

which clearly behaves differently from the Wigner surmise. We note that the above Poisson-type statistics of spacings for i.i.d. random variables should be universal and asymptotically exact when the number of random values $N \to \infty$ (at least for continuous random variables, yet I did not find a very convincing proof except one in [18]). To give a sense that Eq.9 and Eq.13 indeed correctly describe the level spacing distribution, we performed simple numerical tests on GUE matrices and i.i.d. Gaussian random variables for illustration in Fig.2. The theoretical predictions and numerical results show good consistency.

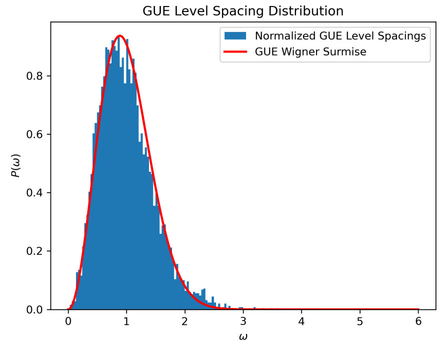
Another important prediction from RMT applies to the Dyson-Mehta statistics Δ_3, which is also known as the spectral rigidity. To introduce this concept, and to be able to compare RMT results and experimental data, we need to first briefly explain the unfolding procedure for spectral analysis. Basically, we want to rescale our data to remove the information of the system-specific mean level density. Consider a spectrum of N eigenvalues x_n on the real axis, then an ensemble of infinitely many of such spectra is defined in terms of the joint probability function $P_N(x_1, ..., x_N)$. For the volumn element, a direct choice is the Euclidean measure $\prod_i dx_i$. The k-point correlation function is defined as the unnormalized k-indices marginal distribution:

$$R_k(x_1, ..., x_k) = \frac{N!}{(N-k)!} \int_{-\infty}^{\infty} dx_{k+1} ... \int_{-\infty}^{\infty} dx_N P_N(x_1, ..., x_N) \tag{14}$$

Assume $P_N(x_1, ..., x_N)$ is invariant under permutation, then $R_k(x_1, ..., x_k)$ is independent of labeling of the levels. It is normalzied to $\frac{N!}{(N-k)!}$. The mean density of states at level x is then naturally $R_1(x)$.



(a)



(b)

Figure 2: Comparison of the level spacings statistics, with mean level spacing normalized to one. The total number of values upon which the spacings are computed are both $N = 15$, which means the number of Gaussian random variables and the dimension of the GUE matrices are both 15. The histograms are generated by averaging $T = 500$ samples. (a): Gaussian random variables. (b): GUE eigenvalues.

We can define a set of dimensionless scaled variables:

$$\xi_p = \xi_p(x_p) = \int_{-\infty}^{x_p} R_1(x)dx \tag{15}$$

We can effectively unfold the variables x_k to the system-independent counting variables ξ_k, and corresponding rescale the correlation functions to $X_k(\xi_1, ..., \xi_k)$ by maintaining $X_k(\xi_1, ..., \xi_k)d\xi_1...d\xi_k = R_k(x_1, ..., x_k)dx_1...dx_k$.

One particular quantity that will be of important use when matching RMT and real physics is the unfolded two-level correlation function $X_2(\xi_1, \xi_2)$, which for translation invariant spectra is only dependent on the energy difference $r = \xi_1 - \xi_2$ thus can be written as $X_2(r)$. Define $Y_2(r) = 1 - X_2(r)$, it is shown that for Poisson statistics one has $Y_2(r) = 0$, while for Gaussian ensembles one has:

$$Y_{2,1}(r) = s^2(r) + \frac{ds(r)}{dr}\int_t^{\infty} s(r')dr' \tag{16}$$

$$Y_{2,2}(r) = s^2(r) \tag{17}$$

$$Y_{2,4}(r) = s^2(2r) - \frac{ds(2r)}{dr}\int_0^{r} s(2r')dr' \tag{18}$$

$$s(r) = \frac{\sin \pi r}{\pi r} \tag{19}$$

for $Y_{2,\beta}$ and $\beta = 1, 2, 4$. Another important quantity predicted by RMT is the Fourier transform of Y_2, which we only list the GUE result:

$$b_{2,2}(t) = \int_{-\infty}^{+\infty} Y_{2,2}(r)\exp(i2\pi rt)dr = \begin{cases} 1 - |t| & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases} \tag{20}$$

For Poisson statistics, $b_2(t) = 0$.

In real physical systems, consider a measurement that yields an ordered sequence of energy levels $E_1, ..., E_N$, which has the spectral function:

$$S(E) = \sum_{i=1}^{N} \delta(E - E_i) \tag{21}$$

The cumulative spectral function reads:

$$\eta(E) = \int_{-\infty}^{E} S_{E'}dE' = \sum_{i=1}^{N} \Theta(E - E_n) \tag{22}$$

where $\Theta(x)$ is the Heaviside step function. Function $\eta(E)$ counts the number of energy levels below E and has a shape of staircase, so it is also referred to as the staircase function. $\eta(E)$ can be similarly defined for random matrices. It can be further decomposed into a smooth part $\xi(E)$ and fluctuating part $\eta_{fl}(E)$ as $\eta(E) = \xi(E) + \eta_{fl}(E)$. Here the smooth part $\xi(E)$ computes the average number of levels up to E, which is equivalent to Eq.15, but is a non-trivial term in real physical systems. In simple systems like two-dimensional billiards, which is a class of two-dimensional system we will mention more later, the smooth part $\xi(E)$ is given by Weyl-type formula [2]:

$$\xi(E) = \frac{\pi}{4}(SE - L\sqrt{E} + K) \tag{23}$$

where S and L are, respectively, the surface and perimeter of the billiard and K is a constant of the order of unity describing the topological properties of the billiard. Given this formula, we can map the spectrum E_i onto $\xi(E_i)$ and get rid of the system-dependent energy scale and are only left with dimensionless number $\xi_i = \xi(E_i)$.

Using these new variables, the cumulative spectral function (for both random matrices and physical spectrum) reads:

$$\hat{\eta}(\xi) = \xi + \hat{\eta}_{fl}(\xi) \tag{24}$$

which is the unfolded staircase function. Δ_3 is then defined, for a fixed interval on the unfolded scale $\xi \in [x, x+L]$, as the least-square deviation of the staircase function from its best straight line fit:

$$\Delta_3(L, x) = \frac{1}{L} \min_{a,b} \int_x^{x+L} [\hat{\eta}(\xi) - a\xi - b]^2 \, d\xi \tag{25}$$

Assuming the translation invariance of spectrum fluctuations on ξ axis, Δ_3 can be computed independent of the initial point x. The dependence of Δ_3 on L serves as an indicator between Poisson statistics and Wigner-Dyson statistics, where the minimization of Eq.25 can be done analytically from the two-level correlation functions (Eq.16-Eq.20) [9, 20, 19], and one expects:

$$\Delta_3(L) = \frac{L}{15} \tag{26}$$

for Poisson statistics, and

$$\Delta_{3,1}(L) = \frac{1}{\pi^2}\left(\ln 2\pi L + \gamma - \frac{5}{4} - \frac{\pi^2}{8}\right) \tag{27}$$

$$\Delta_{3,2}(L) = \frac{1}{2\pi^2}\left(\ln 2\pi L + \gamma - \frac{5}{4}\right) \tag{28}$$

$$\Delta_{3,4}(L) = \frac{1}{4\pi^2}\left(\ln 2\pi L + \gamma - \frac{5}{4} + \frac{\pi^2}{8}\right) \tag{29}$$

for Gaussian ensembles. Here $\gamma = 0.5772...$ is the Euler's constant, and the second index in the subscripts of Δ_3 is the Dyson index β. The dependence of $\Delta_{3,\beta}$ on L for the Wigner-Dyson statistics is logarithmic, while Δ_3 depends linearly on L for Poisson statistics.

Historically, Wigner found that the energy level spacing distribution of heavy nuclei systems can be accurately described by Wigner-Dyson statistics. Following Wigner's original ideas, by studying the statistical properties of random matrices, one can gain insights on the spectral properties of complex quantum systems. However, in quantum system spectrum both Wigner-Dyson statistics and Poisson statistics are possible and it was not clear to which "complex quantum systems" is RMT applicable. To figure out under what condition would one of the statistics appears in the spectrum comprises the main component of the two conjectures trying to link the chaotic property of the system's classical counterpart, and its quantum spectrum statistics: the Bohigas-Giannoni-Schmit (BGS) conjecture and Berry-Tabor conjecture.

1.3 Bohigas-Giannoni-Schmit Conjecture

In their 1984 paper, Bohigas, Giannoni and Schmit found that the level fluctuations of a quantum particle placed in a classically chaotic Sinai's billiard are consistent with the GOE random matrices[5]. More precisely, they found that provided that first lowest levels of the Sinai's billiard are omitted, and when one is looking at a sufficiently narrow energy window, the level statistics (nearest-neighbor spacing distribution $p(x)$ and the Dyson-Mehta statistics Δ_3, which will be explained later) are fully consistent with Wigner-Dyson statistics of the GOE predictions, see Fig.3. Before their discovery, when applying hypothesis leading to RMT, people had always assumed that one is dealing with a complex system with many particles. Yet their results showed that this is by no means a necessary condition, since a single-particle quantum system with chaotic classical counterpart with two degrees of freedom (one particle in two-dimensional billiard) shows also the GOE fluctuations. In this way, the previously disconnected areas—random matrix theory and the study of chaotic systems—were put in close contact. This finding led to their famous conjecture: *Spectra of time-reversal-invariant systems whose classical analogs are K systems (strongly chaotic systems) show the same fluctuation properties as predicted by GOE.* To date, only non-generic counterexamples are found to violate this conjecture. In the BGS paper, they also first introduced the term "quantum chaotic system" to refer to the quantum Sinai's billiard. Therefore, the emergence of Wigner-Dyson statistics of the level spacings is now often regarded as one of the defining property of quantum chaotic systems, whether it has a classical counterpart or not (e.g. a many-body system does not have an obvious classical counterpart but can still display GOE statistics).

1.4 Berry-Tabor Conjecture

Nowadays, the indicator of whether a quantum system is chaotic or integrable is the energy level statistics. For quantum chaotic systems, as discussed above, the energy level statistics is described by Wigner-Dyson statistics. The question of level statistics in quantum integrable systems was first addressed by Berry and Tabor[4], earlier than the BGS conjecture. Briefly speaking, in quantum integrable systems, each component of the Hamiltonian can be addressed (diagonalized) independently due to the existence of many symmetries, which makes the many-body energy levels E being effectively independent with each other and can be treated as random numbers. This leads to the previously discussed Poisson-type energy level statistics $[P_0(\omega) = \exp(-\omega)]$, which is very different from Wigner surmise. It is now known as Berry-Tabor conjecture: *For quantum systems whose corresponding classical counterpart is integrable, the energy eigenvalues generically behave like a sequence of independent random variables and exhibit Poisson statistics.*[7, 4] There are examples where the conjecture fails, which are usually results of extra symmetries in the Hamiltonian that lead to commensurability of the spectra [7].

2 Experimental and Numerical Results in Physical Systems

In this section, we will briefly review some experimental and numerical results of the energy level statistics in realistic quantum physical systems, comparing with either Wigner-Dyson statistics or Poisson statistics.

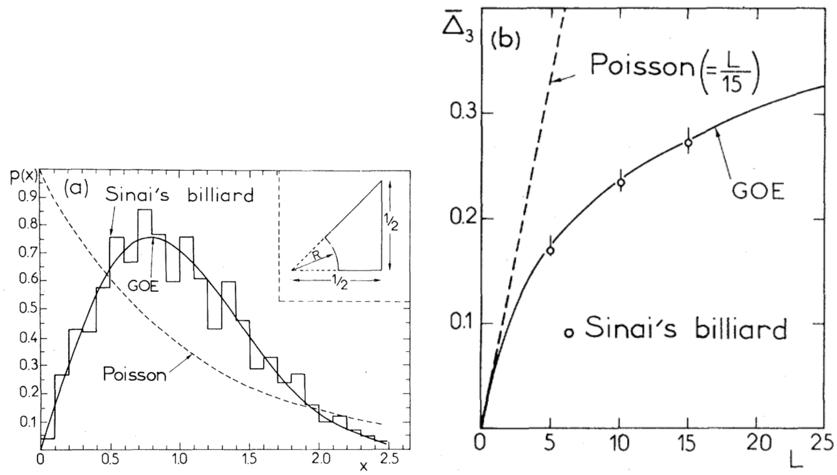
2.1 Billiards

The original system where BGS found the relation between quantum chaos and RMT is the two-dimensional Sinai's billiard. The setup is to compute the quantum energy level statistics when one single particle is placed in the cavity. In Fig.3 we show the original numerical data obtained by BGS, comparing the level spacing statistics and spectral rigidity (the Dyson-Mehta statistics) to the prediction of GOE matrices, which exhibits a good consistency.

In Fig.4, we show the distribution of level spacings for two different billiards: an integrable rectangle billiard and a chaotic billiard constructed from two circular arcs and two line segments (see Fig.4(c)). The two distinguished results in Fig.4 confirm perfectly the two conjectures mentioned above.

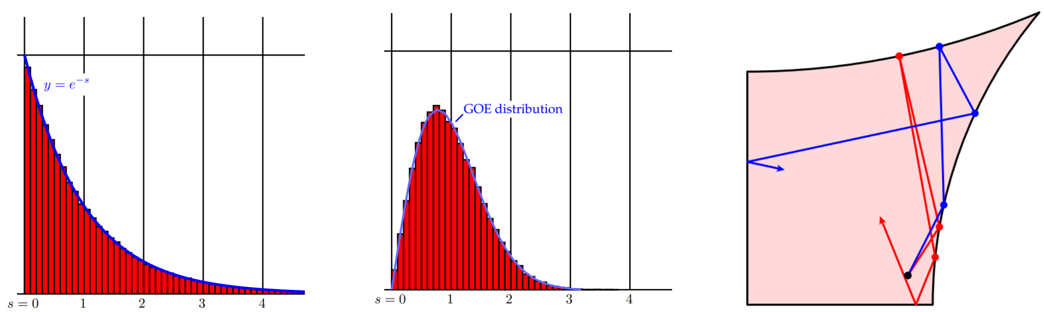
2.2 Beyond classical counterparts: quantum many-body systems

As previously mentioned, based on the energy level spacing statistics, we can define quantum chaos even for systems that do not have a classical counterpart, namely the many-body systems. Historically, these complex quantum systems are the first cases where RMT is introduced for explaining the spectrum statistics, before the concept of quantum chaos. For the completeness of the review we briefly list some results in quantum many-body systems. In fact, understanding the examples and counterexamples of quantum chaos in many-body systems has been an active research field in the past decade and is still drawing much attention, especially in many-body lattice systems [1].



(a) Level spacing statistics (b) Dyson-Mehta statistics

Figure 3: Results of energy-level fluctuations for desymrgetrized Sinai's billiards as specified in the upper right-hand corner of (a), compared to GOE predictions. Taken from [5].



(a) Rectangle billiard (b) Chaotic billiard (c) Shape of the chaotic billiard

Figure 4: Level spacing statistics of a single particle moving in two-dimensional billiards with Dirichlet boundary condition. Taken from [22]. (a): Statistics for the first 250,000 eigenvalues of a non-chaotic rectangle with side/bottom ratio $\sqrt[4]{5}$ and area 4π, compared to the expected probability density e^{-s}. (b): Statistics for the gaps between roughly 50,000 sorted eigenvalues of the chaotic billiard in (c), compared to the GOE spacing distribution. (c): One class of billiard that is proved by Sinai to be classically chaotic. The trajectories give a sense of ergodicity in the region.

Heavy Nuclei - One of the most famous example exhibiting Wigner-Dyson statistics is heavy nuclei and the data is shown in Fig.5. The figure depicts the data of level spacing distribution obtained from slow neutron resonance data and proton resonance data of about 30 different heavy nuclei [17, 5], where all spacings are normalized by the average level spacing. The Wigner-Dyson statistics matches well with the experimental data, confirming Wigner's original idea.

3 Towards a Proof of the BGS Conjecture: A Semiclassical Approximation

After the formulation of the BGS conjecture, people put efforts into establishing an analytical connection linking classical chaotic dynamics with the quantum spectral statistics described by RMT. To establish the link, it is natural to consider the semiclassical limit ($\hbar \to 0$) of quantum mechanics which links quantum and classical behaviors. We will introduce one of the earliest approach towards the proof of the BGS conjecture by Berry[3] that made use of Gutzwiller's periodic orbit theory [14, 15, 12, 13, 21], which is a saddle-point approximation to Feynman's path integral formalism of quantum mechanics for $\hbar \to 0$ and is one of the means towards semiclassical approximation.

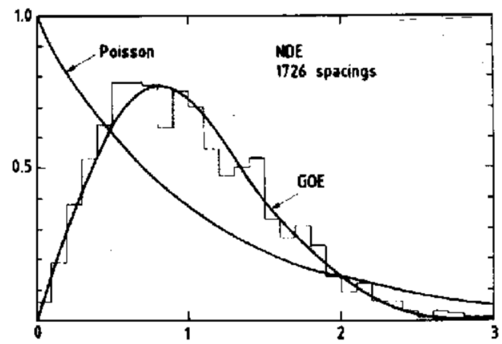


Figure 5: Nearest neighbor spacing distribution for the "Nuclear Data Ensemble" comprising 1726 spacings (histogram) versus normalized level spacing, comparing to predictions of the random matrix GOE ensemble and the Poisson distribution. Taken from [17, 11].

Berry[3] used periodic orbit theory to study universal properties of the Δ_3 statistics. Δ_3 is defined as the least-square deviation of linear fitting to the unfolded staircase function $\hat{\eta}(\xi)$. The staircase function 22 can be viewed as the integral over the level density (density of state) $\rho(E)$ over an unfolded energy interval of length L. As in 24, $\hat{\eta}(E)$ is written as the sum of a smooth part and a fluctuating part. Berry further decomposed $\rho(E)$ into two corresponding terms:

$$\rho(E) = \langle \rho(E) \rangle + \rho_{fl}(E) \tag{30}$$

where the average is taken over an properly chosen energy window $[\xi, \xi + L]$ (in the unfolded scale). Gutzwiller's periodic orbit theory (trace formula [16]) relates the density of states of the quantum system to its classical analog:

$$\rho_{fl}(E) = \frac{1}{\hbar^{\mu+1}} \sum_j A_j(E) \exp\left(\frac{i}{\hbar} S_j(E)\right) \tag{31}$$

where j is summed over all periodic orbit of the classical counterpart. Eq.31 has a similar form with Feynman's path integral. $S_j(E)$ in the phase factor is the classical action $S_j = \oint_{\text{orbit } j} \mathbf{p}\, d\mathbf{q}$ of orbit j. The amplitude $A_j(E) \propto \frac{1}{\sqrt{\det(M_j - 1)}}$, where M_j is the monodromy matrix of the orbit. The periodic orbit structures show major difference between the integrable systems and chaotic systems. In integrable system, the orbits only explore part of the $2d$-dimensional phase space by forming low-dimensional parameter families. Chaotic systems, however, are defined to be ergodic (cover the entire phase space) where all periodic orbits are isolated and unstable. It can be shown in Eq.31 that this implies $\mu = (d-1)/2$ for integrable systems and $\mu = 0$ for chaotic systems.

For semiclassical approximation Eq.31 to work, we have to deal with a proper energy scale. The energy window over which we take the average must be "small" classically, but "large" quantumly, for the theory to be in semiclassical regime. Usually the two edges are defined by system's typical energy scale: the quantum lower bound is the mean level spacing $D = \langle \omega \rangle$, and the classical upper bound is usually determined by the shortest orbit as $h/T_{\min}$ (using the energy-frequency relation $E = \hbar \times \frac{2\pi}{T}$), where $T_j = dS_j(E)/dE$ is the orbit period. Given Eq.31 and make approximation to ignore the energy dependence of the amplitude A_j and of the mean energy level D, the $\Delta_3(L)$ in Eq.25 can be written as energy integral over ρ_{fl}^2, which is a double sum over periodic orbits. Berry further performed a change of variables and simplified the expression in terms of integral over all orbit periods:

$$\Delta_3(L) = \frac{2}{\hbar^{2\mu}} \int_0^\infty \frac{dT}{T^2} \phi(T) G(\frac{DT}{2\hbar} L) \tag{32}$$

where

$$\phi(T) = \left\langle \sum_i \sum_{j,+} A_i A_j \cos\left(\frac{S_i - S_j}{\hbar}\right) \delta(T - \frac{1}{2}(T_i + T_j) \right\rangle_E \tag{33}$$

is the double sum over periodic orbit averaged over the energy interval. The plus sign in the summation of j restricts the summation to positive periods $T_j > 0$. The function $G(x) = 1 -$

$(\frac{\sin(x)}{x})^2 - 3\left(\frac{d}{dx}\frac{\sin(x)}{x}\right)^2$ is independent of orbits j. The shape of $G(x)$ is drawn in Fig.6, which shows that it selects only $x > 1$ and correspondingly pairs of orbits with longer period $T > 2\hbar/DL$ in $\phi(T)$. Longer period means more quantum since the energy scale of the orbit is smaller. Therefore it can be deduced that the deviation from the linear behavior in the staircase function (which is exactly what Δ_3 measures) is mainly contributed from "quantum" orbits (T exceeds $2\hbar/DL$).

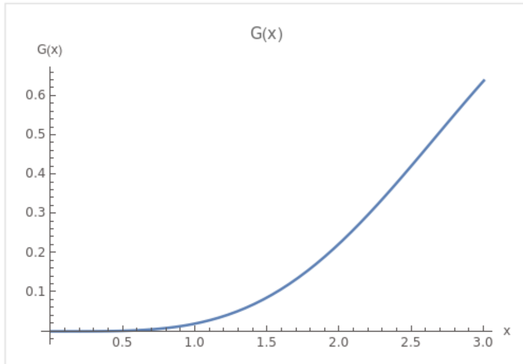


Figure 6: $G(x)$ in $(0, 3]$

Berry then argues that the double sum function $\phi(T)$, which can be viewed as some kind of Fourier transform of the two-orbit correlation function, can be written in terms of the semiclassical approximation of two-level form factor defined in Eq.20 as the Fourier transform of the two-level correlation function:

$$\phi(T) = \frac{\hbar^{2\mu+1}}{2\pi D}\left[1 - b_2^{sc}(\frac{DT}{h})\right] \tag{34}$$

Introducing the dimensionless time $t = \frac{DT}{2\pi\hbar} = \frac{DT}{h}$, we can thus write the expression for Δ_3 as:

$$\Delta_3(L) = \frac{1}{2\pi^2}\int_0^\infty \frac{dt}{t^2}[1 - b_2^{sc}(t)]G(\pi L t) \tag{35}$$

Behaviors of $\phi(T)$ as the double sum can be discussed in different regions. Classical upper bound of the energy of the orbit is given by the shortest orbit with $t_{\min}$. And in the semiclassical approximation, the microscopic quantum energy scale is the mean level spacing D, which corresponds to an orbit period $\sim \frac{h}{D}$. Thus our semiclassical energy interval yields a period range $t_{\min} \ll t \ll 1$. Meanwhile, the behavior of $\phi(t)$ at $t \gg 1$ limit can be asymptotically obtained. The analysis from Berry continues by interpolating the expression of $\phi(t)$ between these regimes.

More precisely, in the clssical limit $\hbar \to 0$, we can employ the saddle-point approximation in Eq.33 and only those pairs of orbits with $\frac{d(S_i - S_j)}{dE} = T_i - T_j \to 0$ would survive in the summation (due to destructive interference of the summation over cosine function). The semiclassical approximation also falls in this regime. Thus for $t_{\min} \ll t \ll 1$, we can employ the diagonal approximation to the double sum and obtain:

$$\phi(T) \approx \phi_{\text{dig}}(T) = \left\langle \sum_{j,+} A_j^2 \delta(T - T_j) \right\rangle, \;\; T_{\min} \ll T \ll h/D, \;\; (t_{\min} \ll t \ll 1) \tag{36}$$

Meanwhile, the asymptotic behavior of $\phi(T)$ in Eq.33 at $t \gg 1$ is obtained by Berry (using sum rule) [3]:

$$\phi(T) \to \frac{\hbar^{2\mu+1}}{2\pi D} = \frac{1}{2\pi D}\begin{cases} \hbar^d & \text{if integrable }, \mu = (d-1)/2 \\ \hbar & \text{if chaotic }, \mu = 0 \end{cases} \tag{37}$$

We can also directly analyze the asymptotic behavior of $\phi_{\text{dig}}(T)$ at relatively large T ($t \gg t_{\min}$) using sum rule due to Hannay and Ozorio de Almeida and obtain different behavior for integrable and chaotic systems:

$$\phi_{\text{dig}}(T) \to \frac{1}{2\pi D}\begin{cases} \hbar^d & \text{if integrable} \\ \hbar t & \text{if chaotic} \end{cases} \tag{38}$$

Comparing Eq.37 and Eq.38, we immediately see that the diagonal approximation is still valid at the quantum limit $t \gg 1$ for integrable systems, but would break down at some point for chaotic systems. Looking at the expression of $\phi(T)$ in terms of $b_2^{sc}(T)$ in Eq.34, we see for integrable systems $b_2^{sc}(T) = 0$ is a good approximation from quantum limit all the way down to classical scale $t_{\min}$. For sufficiently small $t_{\min}$ (semiclassical approximation), we can approximate the $\Delta_3(L)$ integral in Eq.35 for integrable system as:

$$\Delta_3(L) \approx \frac{1}{2\pi^2}\int_0^\infty \frac{dt}{t^2} G(\pi L t) = \frac{L}{15}, \text{ integrable} \tag{39}$$

which coincides with the analytical prediction of Poisson statistics Eq.26. For chaotic systems, when t gets larger, the off-diagonal terms come into plays and cancel with the diagonal term, yielding the correct limit behavior in Eq.37. Berry employs a simple linear interpolation to link the two regimes (note its similarity with the GUE two-level form factor Eq.20):

$$b_2^{sc}(t) = 1 - t, \;\; t_{\min} < t < 1 \tag{40}$$

which implies

$$\Delta_3(L) = \frac{1}{\beta\pi^2}\ln(L) + c_\beta \tag{41}$$

in the region of $1 \ll L \ll L_{\max}$ (which corresponds the semiclassical orbital regime $t_{\min} \ll t \ll 1$). Here β is the usual Dyson index, where $\beta = 1$ for time-reversal invariant systems and $\beta = 2$ for non-invariant systems. c_β is a constant dependent on β. The logarithmic dependence on L in Eq.41 coincides with the RMT prediction of Gaussian matrix ensemble Eq.27-29.

In this framework of semiclassical approximation, periodic orbit theory sucessfully establishes the link between integrability of the system and its spectral statistics, showing agreement with the RMT predictions, thus validating the correctness of the BGS conjecture. There exists some other more elaborated approach towards the proof of the BGS conjecture which involves supersymmetric field theory [11], which will not be introduced here.

4 Conclusion

The proposal of BGS conjecture has found a remarkable connection between the classical integrability of the quantum system with the spectral statistics through RMT. Berry's proof formally establishes such connection in the semiclassical limit and displays the profound and rich mathematical structure underlying our real physical world.

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