let n have prime factorization

$$n = \prod_{i=1}^{r} {p_i}^{k_i}$$

let

$$F(n) = \sum\nolimits_{d|n} d\varphi(d)$$

let M represent the set of all multiplicative functions

$$d, \varphi(d) \in M \Rightarrow d\varphi(d) \in M$$
$$f(d) \in M \Rightarrow \sum\nolimits_{d|n} f(d) \in M \Rightarrow F(n) \in M$$

let $\mathbb{Z}'$ represent the set of all prime numbers

consider $p^k \ni p \in \mathbb{Z}', k \in \mathbb{Z}'$

the divisors of p^k are $1, p, p^2, \dots, p^k$

therefore,

$$\begin{aligned}\sum\nolimits_{d|p^k} d\varphi(d) &= (1)(1) + (p)(p-1) + (p^2)(p^2 - p) \\ &\quad + (p^3)(p^3 - p^2) + \cdots + (p^k)(p^k - p^{k-1}) \\ &= 1 + p^2 - p + p^4 - p^3 + p^{2k} - p^{2k-1} \\ &= \sum_{i=0}^{2k} (-1)^i (p)^i\end{aligned}$$

we will now establish that

$$\sum_{i=0}^{2k} (-1)^i (p)^i = \frac{p^{2k+1} + 1}{p + 1}$$

when $2k = 0$,

$$\sum_{i=0}^{2k} (-1)^i (p)^i = 1, \qquad \frac{p^{2k+1} + 1}{p + 1} = 1$$

assume that

$$\sum_{i=0}^{2m}(-1)^i(p)^i = \frac{p^{2m+1}+1}{p+1}$$

for some arbitrary integer, m

consider the next term,

$$\sum_{i=0}^{2(m+1)}(-1)^i(p)^i$$
$$= \sum_{i=0}^{2m}(-1)^i(p)^i + (-1)^{2m+1}(p^{2m+1}) + (-1)^{2m+2}(p^{2m+2})$$

consider the expression

$$\frac{p^{2m+1}+1}{p+1} + (-1)^{2m+1}(p^{2m+1}) + (-1)^{2m+2}(p^{2m+2})$$
$$= \frac{p^{2m+1}+1}{p+1} + (-1)(-1)^{2^m}(p^{2m+1}) + (-1)^{2^{m+1}}(p^{2m+2})$$
$$= \frac{p^{2m+1}+1}{p+1} - (p^{2m+1}) + (p^{2m+2})$$
$$= \frac{p^{2m+1}+1}{p+1} - \frac{(p^{2m+1})(p+1)}{(p+1)} + \frac{(p^{2m+2})(p+1)}{(p+1)}$$
$$= \frac{p^{2m+1}+1-p^{2m+2}-p^{2m+1}+p^{2m+3}+p^{2m+2}}{p+1}$$
$$= \frac{p^{2m+3}}{p+1} = \frac{p^{2(m+1)+1}}{p+1},$$

the next term

we have established by mathematical induction that

$$\sum_{i=0}^{2k}(-1)^i(p)^i = \frac{p^{2k+1}+1}{p+1}$$

therefore,

$$F(n) = \prod_{i=1}^{r} F({p_i}^{k_i}) \Rightarrow \sum\nolimits_{d|n} d\varphi(d) = \prod_{i=1}^{r}\frac{{p_i}^{2k_i+1}+1}{p+1}$$