$$\sum_{i=0}^{2m} (-1)^i (p)^i = \frac{p^{2m+1}+1}{p+1}$$

for some arbitrary integer, m

consider the next term,

$$\sum_{i=0}^{2(m+1)} (-1)^i (p)^i$$

$$= \sum_{i=0}^{2m} (-1)^i (p)^i + (-1)^{2m+1}(p^{2m+1}) + (-1)^{2m+2}(p^{2m+2})$$

consider the expression

$$\frac{p^{2m+1}+1}{p+1} + (-1)^{2m+1}(p^{2m+1}) + (-1)^{2m+2}(p^{2m+2})$$
$$= \frac{p^{2m+1}+1}{p+1} + (-1)(-1)^{2^m}(p^{2m+1}) + (-1)^{2^{m+1}}(p^{2m+2})$$
$$= \frac{p^{2m+1}+1}{p+1} - (p^{2m+1}) + (p^{2m+2})$$
$$= \frac{p^{2m+1}+1}{p+1} - \frac{(p^{2m+1})(p+1)}{(p+1)} + \frac{(p^{2m+2})(p+1)}{(p+1)}$$
$$= \frac{p^{2m+1}+1-p^{2m+2}-p^{2m+1}+p^{2m+3}+p^{2m+2}}{p+1}$$
$$= \frac{p^{2m+3}}{p+1} = \frac{p^{2(m+1)+1}}{p+1},$$

the next term

we have established by mathematical induction that

$$\sum_{i=0}^{2k} (-1)^i (p)^i = \frac{p^{2k+1}+1}{p+1}$$

therefore,

$$F(n) = \prod_{i=1}^{r} F(p_i{}^{k_i}) \Rightarrow \sum\nolimits_{d|n} d\varphi(d) = \prod_{i=1}^{r} \frac{p_i{}^{2k_i+1}+1}{p+1}$$