3.

Function Space

The importance of the vector space discussed in the previous chapter lies in describing the evolution of dynamic systems. From a control systems perspective, engineers are more concerned about the properties of the system response as a function of time, specifically, transients, stability, and steady state performance. For example, the state of an aircraft can be described in terms of its angular positions, roll, ϕ, pitch, θ, and yaw, ψ, angles, and the corresponding velocities, $\dot{\phi}, \dot{\theta}, \dot{\psi}$, respectively. These variables are usually expressed as a vector $\mathbf{x} = [\psi, \theta, \psi, \dot{\psi}, \dot{\theta}, \dot{\psi}]'$ in the Euclidean space $\mathcal{R}^6$. This description, however, is only partial, since it doesn't tell anything about properties of the state variables as a function of time. A more precise description could be given by denoting $\mathbf{x} \in C(I, \mathcal{R}^6)$, which says that, as a function of time, the vector $\mathbf{x}$ is continuous, and for each t, the state vector $\mathbf{x}(t)$ belongs to $\mathcal{R}^6$. Note however that not all systems can be described in terms of vectors in the conventional Euclidean space. For example, consider transverse vibration of a cable or a high voltage power line. Clearly, the displacement of the cable would be different at various points along its axial length. A model of the cable vibration considering only a few points along its length would be inaccurate, since there are actually infinitely many points along its length; and it is not possible to introduce a vector with infinitely many entries. An alternate but effective approach would be to express the vibrations in terms of a suitable function of two variables, x, and t, where x is the distance along the axial length, and t is the time. For example, one may express the displacement, y, as a square integrable function of x. The state of this as well as many other systems can be described by a function rather than vectors.

In this chapter, we extend the concepts of linear vector space to certain function spaces that are commonly used control system analysis and design. These concepts are also useful in many signal processing analysis. This chapter introduces only some of the fundamental concepts of linear space that are commonly used in control system analysis along with some applications. Interested readers should consult books on Functional Analysis, such as, [64, 65] for complete details of this subject.

3.1 Linear Space

The concept of linear space is a generalization of vector space introduced in Chapter 3. In particular a vector space is an example of linear space. Thus a linear space is defined in a similar manner as a vector space except that the elements of a linear space are not typical vectors, rather are elements that satisfy similar properties.

A linear space $\mathcal{X}$ is a nonempty set of elements satisfying the following axioms:

1) If f and g are elements in $\mathcal{X}$ then $f+g$ is in $\mathcal{X}$,
2) $f+g=g+f$,
3) $f+(g+h)=(f+g)+h$,
4) There is a null element 0 in X such that $f+0=f$,
5) For each f there is a negative f, denoted, $-f$, such that $f+(-f)=0$,
6) If f is an element in $\mathcal{X}$ then cf is also an element in f, where c is a scalar,
7) For any scalar c, $c(f+g)=cf+cg$,
8) $(c_1+c_2)f=c_1f+c_2f$, where c_1 and c_2 are scalars,
9) $c_1(c_2f)=c_1c_2f$, where c_1 and c_2 are scalars,
10) $1f=f$.

A function space is a specific type of linear space whose elements are functions. In this chapter we shall discuss various aspects of some function spaces that are of importance for engineering applications.

Let $\mathcal{X}$ be the set of all second degree polynomials, $f(x)=b_0+b_1x$, defined over the domain $a<x<b$. Define addition of functions and scalar multiplication as

$$(f+g)(x)=f(x)+g(x)$$
$$(kf)(x)=kf(x)$$

i.e., sum of two functions at any point x is the sum of their individual values, $f(x)$ and $g(x)$. Similarly, the function $(kf)(x)$ is the product of the scalar k and the function $f(x)$. A zero function is defined as $0(x)=0$ which is identically zero for all x over the domain, and the function $-f(x)$ is defined as the function with values that are negative of $f(x)$. Then one can easily verify that all axioms of linear space defined earlier are satisfied so that $\mathcal{X}$ is a linear space.

The set of all continuous real valued functions defined over the interval (a,b) forms a linear space, more specifically, $C(a,b)$. On this space, we define addition as point-by-point algebraic addition. Similarly, scalar multiplication is defined as simple multiplication for every point. To verify the axioms of linear space, sum of two continuous functions is taken by point-wise addition, and is always a continuous function, which satisfies the first axiom. A null element in $C(a,b)$ is a function which is identically zero everywhere. Using this null element, one proves the fourth axiom. Other axioms of linear space are verified in a similar manner.

Continuity: A real valued function f is said to be continuous at x_0 if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$|f(x) - f(x_0| < \varepsilon \qquad \text{whenever} \quad |x - x_0| < \delta$$

While the above is a formal mathematical definition of continuity, what it means in simple terms is that the distance between $f(x)$ and $f(x_0)$ is arbitrarily small if the distance between x and x_0 is arbitrarily small. The function f is said to be continuous if it is continuous everywhere in the given interval.

A sine function over the interval $[0, 2\pi]$ is a continuous function. In other words, given $f = \sin x$, we have $f : \mathcal{R} \to \mathcal{R}$ since it maps real numbers, x, to real numbers, $\sin(x)$, and it is easy to verify that $\sin x$ is a continuous function. A step function, $g(x)$ over the interval $[-1, 1]$, defined as, $g(x) = 0$ for $x < 0$, and $g(x) = 1$ for $x \geq 0$, is not a continuous function. A more formal mathematical definition of continuity will be introduced in the next section.

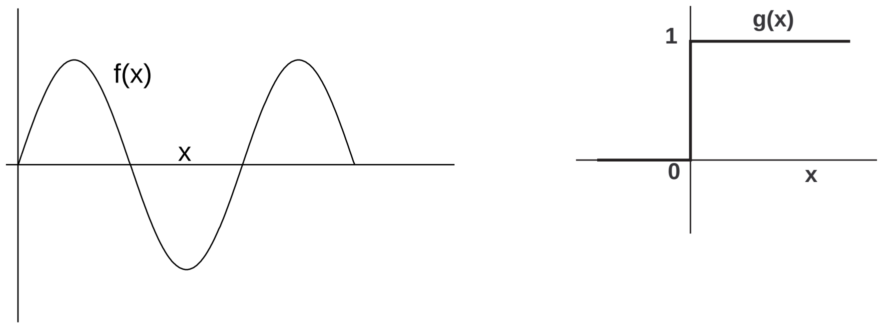

Figure 3.1 Continuity

In engineering applications, functions are defined over a set. A signal is defined over a time horizon $0 \leq t \leq T$. Displacement of a plate is defined over a region $0 \leq x \leq a$, $0 \leq y \leq b$. In general we will use the symbol Ω to denote the set over which the function space is defined. The following linear function spaces are very commonly used in engineering analysis and design:

i) $C(\Omega)$ is the set of all functions that are continuous over the set Ω.
ii) $C^1(\Omega)$ is the set of all functions that are continuous and have continuous first partial derivatives over Ω.
iii) $C^n(\Omega)$ is the set of all functions that are continuous, and have first n derivatives that are continuous.
iv) $C^\infty(\Omega)$ is the set of all functions that are infinitely differentiable.
v) $L_2(\Omega)$ is the (equivalent class) of all square integrable functions.
vi) $L_p(\Omega)$, $1 \leq p < \infty$ is the (equivalent class) of all functions over Ω whose p-th power are integrable.

We proceed with some mathematical concepts for function spaces.

3.2 Norm and Metric on Function Space

Norm of a function space can be defined much like the way it is defined for finite dimensional spaces. For R^n, the norm of a vector is a measure of its length, however, one must interpret the norm of a function as its 'magnitude' in some sense.

A norm of a function on a linear space $\mathcal{X}$ is a real valued function, $\|\cdot\| : \mathcal{X} \to R$ satisfying the following axioms:

i) $\|f\| \geq 0$ for all $f \in \mathcal{X}$

ii) $\|f\| = 0$ if and only if $f = 0$

iii) $\|\alpha f\| = |\alpha| \; \|f\|$ for all $x \in \mathcal{X}$, and all scalar α

iv) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in \mathcal{X}$

The linear space $\mathcal{X}$ with the norm $\|\cdot\|$ is called a *normed linear space*, or simply a *normed space*, denoted by the pair $\{\mathcal{X}, \|\cdot\|\}$.

Let $\mathcal{X} = C[a, b]$, the set of all continuous functions defined over the closed interval $[a, b]$. One can verify all properties if we define a norm as:

$$\|f\|_C = \max_{x \in [a,b]} |f(x)| \tag{3.1}$$

In case the functions are defined over the open interval (a, b), one has to use the supremum instead of maximum.

For $C^1(\Omega)$, we define the norm as

$$\|f\|_{C^1} = \sup_{x \in \Omega} |f(x)| + \sup_{x \in \Omega} |\frac{\partial f}{\partial x}(x)| \tag{3.2}$$

In general, a norm for C^k can be defined using higher order derivatives. Clearly C^1 is a smaller space than C. In general,

$$C^\infty \cdots \subset C^k \cdots \subset C^1 \subset C$$

For the $L_p, 1 \leq p < \infty$ spaces, a norm can be defined as

$$\|f\|_{L_p} = \left[\int_\Omega |f|^p \, dx \right]^{\frac{1}{p}} \tag{3.3}$$

In particular, if $p = 2$, we have the L_2 norm

$$\|f\|_{L_2} = \left[\int_\Omega |f|^2 \, dx \right]^{\frac{1}{2}} \tag{3.4}$$

Consider the function $f(x) = \sin x$ over the interval $[0, 2\pi]$. In light of the above definition, we find that $\|f\|_C = 1$. On the other hand, in terms of L_2 norm, we have

$$\|f\|_{L_2} = \left(\int_0^{2\pi} \sin^2 x \, dx \right)^{\frac{1}{2}} = \sqrt{\pi}$$

Notice that all functions that are square integrable are not necessarily continuous. A unit step function defined over the interval $[-1, 1]$ is not a continuous function, but is a member of L_2. Thus $C \subset L_2$.

Metric

Let $\mathcal{X}$ be a linear space. Then a mapping $\rho : \mathcal{X} \times \mathcal{X} \to R$ is said to be a metric on $\mathcal{X}$ if it satisfies following properties:

a) $\rho(f, g) \geq 0$,
b) $\rho(f, f) = 0$,
c) $\rho(f, g) = \rho(g, f)$,
d) $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$.

The pair $\{\mathcal{X}, \rho(\cdot, \cdot)\}$ is called a metric space. It can be verified that a metric for the various function spaces defined above can be expressed as

$$\rho(f, g)_{\mathcal{X}} = \|f - g\|_{\mathcal{X}}$$

Suppose $f, g \in C$. Then the metric, $\rho(f, g)$, can be defined as

$$\rho(f, g) = \sup_{x \in (a,b)} \{|f(x) - g(x)|\}, \tag{3.5}$$

which is the largest absolute difference between the two functions. The first two properties given above can be easily verified using the definition (3.5). Here we prove only the triangle inequality. Let $f, g, h \in C$. Then

$$\begin{aligned} \rho(f, g) &= \sup_x \{|f(x) - g(x)|\} \\ &= \sup_x \{|f(x) - h(x) + h(x) - g(x)|\} \\ &\leq \sup_x \{|f(x) - h(x)| + |h(x) - g(x)|\} \\ &\leq \rho(f, h) + \rho(h, g). \end{aligned}$$

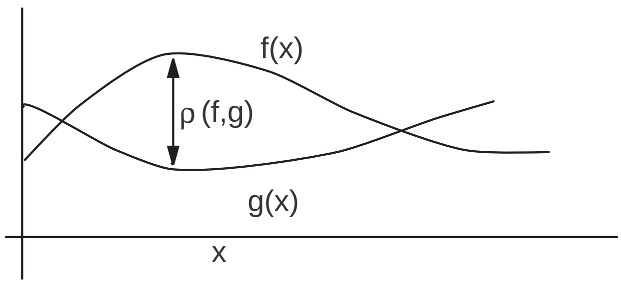

Figure 3.2 Metric on $C(a, b)$

Consider the functions $f(x) = \sin x$, and $g(x) = \cos(x)$ over the interval $[0, 2\pi]$. Then it is observed that the largest difference between the two functions occurs when $x = \frac{3\pi}{4}$ so that the distance $\rho(f, g) = |\sin(\frac{3\pi}{4}) - \cos(\frac{3\pi}{4})| = \sqrt{2}$.

For L_2, we can define the metric as

$$\rho(f,g) = \left[\int_I |f(x) - g(x)|^2 \, dx \right]^{\frac{1}{2}}. \tag{3.6}$$

so that for $f(x) = \sin x$ and $g(x) = \cos(x)$ over the interval $[0, 2\pi]$. we have

$$\rho(f,g) = \left[\int_0^{2\pi} |\sin x - \cos x|^2 \, dx \right]^{\frac{1}{2}} = \sqrt{2\pi}.$$

Let $\mathcal{X}$ denote a set of binary numbers, and define a mapping $\rho : \mathcal{X} \times \mathcal{X} \to R$ by

$$\rho(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Then it can be shown that ρ satisfies all properties of a metric. This is called the Hamming distance in communication literature.

Continuity

Using the notion of metric, continuity can be defined for a broader range of functions. Let $\mathcal{X}$ and $\mathcal{Y}$ be two linear spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be continuous at $x_0 \in \mathcal{X}$ if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$\rho(f(x), f(x_0)) < \varepsilon \qquad \text{whenever} \qquad \rho(x, x_0) < \delta$$

where ρ denotes the distance measure. The function $f : \mathcal{X} \to \mathcal{Y}$ is said to be continuous if it is continuous everywhere in $\mathcal{X}$.

3.3 Normed Linear Space

A linear space equipped with a norm is called a normed linear space. The Euclidean space R^n equipped with the standard Euclidean norm is a normed linear space. The space of continuous functions with the sup norm is a normed linear space.

Convergence: A sequence of functions $\{\phi_i, i = 1, 2, \cdots\}$ in a normed linear space $\mathcal{X}$ is said to converge to an element $\phi \in \mathcal{X}$ if for every $\varepsilon > 0$, there exists an integer N, which may depend on ε, such that

$$\|\phi_n - \phi\| < \varepsilon \qquad \text{whenever } n > N$$

If the sequence converges,

$$\lim_{n \to \infty} \|\phi_n - \phi\| \to 0$$

or equivalently

$$\lim_{n \to \infty} \phi_n = \phi$$

Note that in functions spaces, convergence is not always guaranteed. For example, consider the space $\mathcal{X}$ of continuous functions $C[-1,1]$ with the L_2 norm, and the sequence

$$\phi_n(x) = \begin{cases} 0, & \text{for } -1 \le x \le 0 \\ nx, & \text{for } \ 0 < x \le \frac{1}{n} \\ 1 & \text{for } \ \frac{1}{n} < x \le 1 \end{cases}$$

Clearly ϕ_n is a continuous function. As we take the limit $n \to \infty$, observe that the sequence converges pointwise to a step function at the origin, which is however not a continuous function. On the other hand, let's define $\mathcal{X} = L_2$ the space of square integrable functions. Then we see that as $n \to \infty$, $\{\phi_n\}$ converges to a unit step function ϕ^* at the origin, and that $\lim_{n\to\infty} \|\phi_n - \phi^*\|_{L_2} = 0$, which shows that ϕ_n converges ϕ in $\mathcal{X}$.

Cauchy Sequence: A sequence of functions $\{\phi_n\}$ in a normed linear space $\mathcal{X}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists an integer N, which may depend on ε, such that

$$\|\phi_n - \phi_m\| < \varepsilon$$

whenever $n > N$ and $m > N$. This means that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{X}$ if and only if

$$\|\phi_n - \phi_m\| \to 0 \text{ as } n, m \to \infty$$

Consider the space $C(-1,1)$ with L_1 norm, and the sequence $\{\phi_n\}$ defined above. Then one can verify that

$$\begin{aligned} \|\phi_n - \phi_m\|_{L_1} &= \int_{-1}^{1} |\phi_n(x) - \phi_m(x)|\, dx \\ &= \frac{1}{m} - \frac{1}{n} \\ &\to 0 \text{ as } n, m \to \infty \end{aligned}$$

Thus $\{\phi_n\}$ is a Cauchy sequence. However, ϕ_n does not converge to a function which is continuous. Thus a Cauchy sequence in a linear normed space need not be convergent. In other words, all convergent sequences are Cauchy sequence, but all Cauchy sequences need not converge to a point in the space.

A normed linear space $\mathcal{X}$ is said to be complete if every Cauchy sequence in $\mathcal{X}$ has a limit in $\mathcal{X}$. That is, if every sequence $\{\phi_n\} \in \mathcal{X}$ is a Cauchy sequence, and $\{\phi_n\}$ converges to a limit, then there exists a limit $\phi^* \in \mathcal{X}$ such that $\lim_{n\to\infty} \|\phi_n - \phi^*\| = 0$. A complete normed linear space is called a Banach space, i.e., a normed linear space is a Banach space if every Cauchy sequence has a limit.

Every finite dimensional space is a complete normed space. $C[a,b]$ under the sup norm is complete, and hence is a Banach space. Based on the example given above, $C(-1,1)$ equipped with the L_1 norm is not a Banach space. In general, $C[a,b]$ is complete with the sup norm, but is not complete with the $L_p, 1 \le p < \infty$ norm.

3.4 Space of Square Integrable Functions, L_2 Space

One of the most commonly used function spaces in engineering analysis is the L_2 space, the space of all square integrable functions. We have already introduced the concepts of norm and distance metric for functions in L_2. Nevertheless, because of its importance, we shall review the L_2 space is more details.

First of all, as defined earlier, a norm of L_2 can be defined as

$$L_2(I) \equiv \{f : \int_I |f(x)|^2\, dx < \infty\}, \tag{3.7}$$

where I is an interval. For example, the function $f(x) = \sin x$ belongs to $L_2(0, 2\pi)$ since $\sin^2 x$ integrated over the interval $[0, 2\pi]$ is finite. Similarly, the step function defined in Figure 5.1 is a member of $L_2(-1, 1)$, but not a member of $C(-1, 1)$, which shows that $L_2(-1, 1)$ is a larger space than $C(-1, 1)$, i.e., $C(a, b) \subset L_2(a, b)$. In engineering terms, this is the class of functions that have finite energy.

Note also that L_2 functions could have finite discontinuities at finite number of points in the domain since (squared) integration of these functions remain unchanged in the presence of such discontinuities. In this context, a zero function in L_2 is not necessarily a function that is identically zero everywhere, but could also be a function that is zero everywhere except at some finite number of points where the function has finite values. Presence of such discontinuities in electrical signals is very common in engineering applications, because of which L_2 class plays a significant role in engineering analysis.

Scalar Product and Norm for L_2

We shall use the notation $(\cdot, \cdot)$ to define scalar product in L_2, and in case there is ambiguity, we shall use $(\cdot, \cdot)_{L_2}$ explicitly. For $L_2(I)$, we can define a scalar product as

$$(f, g) = \int_I f(x)g(x)\, dx. \tag{3.8}$$

For example, on $L_2(-1, 1)$, consider $f(x) = \sin \pi x$, and g is the step function defined earlier. Then we compute $(f, g) = \frac{2}{\pi}$.

Various properties of the scalar product can be proved using the definition (3.8). The scalar product (3.8) also induces a norm for $L_2(I)$ defined by

$$\|f\|_{L_2} = \left\{\int_I |f(x)|^2\, dx\right\}^{\frac{1}{2}}. \tag{3.9}$$

For example, on $L_2(0, 1)$, consider the function $h(x) = \sin 2\pi x$. Then we have

$$\|h\|_{L_2} = \left(\int_0^1 \sin^2 2\pi x\, dx\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

A scalar product space is called a Hilbert space if it is a Banach space with respect to the norm induced by the scalar product. In other words, a scalar product space $\mathcal{H}$ is a Hilbert space, it every Cauchy sequence in $\mathcal{H}$ convergence in norm to a limit in $\mathcal{H}$. With respect to the scalar product, and the corresponding norm defined above, the L_2 space is a Hilbert space. A Hilbert space is a special Banach space, but all Banach spaces are not necessarily Hilbert spaces. One of the characteristics of a Hilbert space, H, is that the parallelogram law holds:

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

for all f and g in H.

The space C equipped with the sup norm is a Banach space, but is not a Hilbert space. A L_p space is not a Hilbert space except when $p = 2$.

Orthogonality

The concept of orthogonality of functions in L_2 can be introduced using the definition of the scalar product (3.8). The functions f and g are said to be orthogonal if

$$(f, g) = 0. \tag{3.10}$$

Furthermore, the functions f and g are orthonormal if

$$(f, g) = \begin{cases} 1 & \text{for } f = g, \\ 0 & \text{for } f \neq g. \end{cases} \tag{3.11}$$

On the space $L_2(0,1)$, the trigonometric functions $\sin n\pi x$ and $\cos m\pi x$ are orthogonal since

$$\begin{aligned}(\sin n\pi x, \sin m\pi x) &= \int_0^1 \sin n\pi x \sin m\pi x \, dx \\ &= 0\end{aligned}$$

whenever $n \neq m$, however, this is not an orthonormal sequence since norm of the function is not 1. In fact,

$$\|\sin n\pi x\|^2 = \int_0^1 \sin^2 n\pi x \, dx = \frac{1}{2}$$

Thus normalizing the sequence, we define

$$\phi_n(x) = \{\sqrt{2} \sin n\pi x\}$$

to form an orthonormal sequence in $L_2(0,1)$ since $\int_0^1 \phi_n^2(x)\, dx = 1$, and $\int_0^1 \phi_n(x)\phi_m(x)\, dx = 0$ for $n \neq m$.

Metric on L_2

The norm (3.9) induces a metric on L_2 defined by

$$\rho(f, g) = \left[\int_I |f(x) - g(x)|^2 \, dx \right]^{\frac{1}{2}}. \tag{3.12}$$

Consider $f(x) = \sin x$ and $g(x) = \cos(x)$ over the interval $[0, 2\pi]$. Then it is easily verified that

$$\rho(f, g) = \left[\int_0^{2\pi} |\sin x - \cos x|^2 \, dx \right]^{\frac{1}{2}} = \sqrt{2\pi}.$$

Although the above discussion is based on scalar valued functions, the concepts easily extend to vector valued functions. Consider the following example: Let $x \in \Omega = \{x_1, x_2 : -1 \leq x_1 \leq 1, \ -1 \leq x_2 \leq 1\}$, and the functions $\mathbf{f}(x)$ and $\mathbf{g}(x)$ be given as

$$\mathbf{f}(x) = \begin{pmatrix} x_1 \\ \sin \pi x_2 \end{pmatrix}, \qquad \mathbf{g}(x) = \begin{pmatrix} x_1 + x_2 \\ \cos \pi x_2 \end{pmatrix}$$

Then the norm $\|\mathbf{f}\|$ is obtained from

$$\begin{aligned} \|\mathbf{f}\|_{L_2}^2 &= \int_{-1}^{1} \int_{-1}^{1} \langle \mathbf{f}(x), \mathbf{f}(x) \rangle_{R^2} \, dx_1 \, dx_2 \\ &= \int_{-1}^{1} \int_{-1}^{1} (x_1^2 + \sin^2 \pi x_2) \, dx_1 \, dx_2 \\ &= \frac{10}{3} \end{aligned}$$

so that $\|\mathbf{f}\| = \sqrt{\frac{10}{3}}$. Note that we have first taken the scalar product in R^2 sense since the function $\mathbf{f}$ is a vector values function.

We can also find the scalar product between $\mathbf{f}$ and $\mathbf{g}$ in a similar way.

$$\begin{aligned} (\mathbf{f}, \mathbf{g})_{L_2} &= \int_{-1}^{1} \int_{-1}^{1} \langle \mathbf{f}(x), \mathbf{g}(x) \rangle_{R^2} \, dx_1 \, dx_2 \\ &= \int_{-1}^{1} \int_{-1}^{1} (x_1^2 + x_1 x_2 + \sin \pi x_2 \, \cos \pi x_2) \, dx_1 \, dx_2 \\ &= \frac{4}{3} \end{aligned}$$

where for each value of x we first take the scalar product in the R^2 sense, and then integrate over Ω.

Schwarz Inequality: For real valued functions $f(x)$ and $g(x)$ over the domain Ω, Schwarz inequality is given by

$$|\langle f, g\rangle| \leq \|f\|\,\|g\| \tag{3.13}$$

where equality occurs only if f and g are linearly independent. ■

Schwarz inequality can be easily proved. Suppose $g \neq 0$ is arbitrary, and λ is any scalar. Then we have

$$\begin{aligned} 0 &\leq \|f - \lambda g\|^2 \\ &= \langle f - \lambda g, f - \lambda g\rangle \\ &= \|f\|^2 - 2\lambda\langle f, g\rangle + \lambda^2\|g\|^2 \end{aligned}$$

Suppose we choose λ such that

$$\lambda = \frac{\langle f, g\rangle}{\|g\|^2}$$

Substituting in the above equation, we have

$$0 \leq \|f\|^2 - \frac{|\langle f, g\rangle|^2}{\|g\|^2}$$

which simplifies to the inequality (3.13).

In terms of L_2 scalar product and norm, Schwarz inequality is explicitly written as

$$\left|\int_\Omega f(x)\,g(x)\,dx\right| \leq \left[\int_\Omega |f(x)|^2 dx\right]^{\frac{1}{2}} \left[\int_\Omega |g(x)|^2 dx\right]^{\frac{1}{2}}$$

Consider $\Omega = \{x : -1 \leq x \leq 1\}$, and $f(x) = x^2$ and $g(x) = x + 1$. Then we have

$$(f, g)_\Omega = \int_{-1}^{1} f(x)\,g(x)\,dx = \int_{-1}^{1} (x^3 + x^2)\,dx = \frac{2}{3}$$

and

$$\begin{aligned} \|f\|^2 &= \int_{-1}^{1} |x^2|^2 dx = \frac{2}{5} \\ \|g\|^2 &= \int_{-1}^{1} |x + 1|^2 dx = \frac{8}{3} \end{aligned}$$

which satisfies the Schwarz inequality.

Triangle Inequality: For real valued functions $f(x)$ and $g(x)$ over the domain Ω, the triangle inequality is given by

$$\|f + g\| \leq \|f\| + \|g\| \tag{3.14}$$

or equivalently,

$$\left[\int_\Omega |f(x)+g(x)|^2\,dx\right]^{\frac{1}{2}} \le \left[\int_\Omega |f(x)|^2\,dx\right]^{\frac{1}{2}} + \left[\int_\Omega |g(x)|^2\,dx\right]^{\frac{1}{2}}$$

The proof follows from Schwarz inequality. We have

$$\begin{aligned}
\|f+g\|^2 &= \langle f+g, f+g\rangle \\
&= \|f\|^2 + 2\langle f,g\rangle + \|g\|^2 \\
&\le \|f\|^2 + 2|\langle f,g\rangle| + \|g\|^2 \\
&\le \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 \qquad \text{by Schwartz inequality} \\
&= \big(\|f\| + \|g\|\big)^2
\end{aligned}$$

For the above example, we have

$$\begin{aligned}
\|f+g\| &= \big[\int_{-1}^{1} (x^2+x+1)^2\,dx\big]^{\frac{1}{2}} = \sqrt{\frac{22}{5}} \\
\|f\| &= \sqrt{\frac{2}{5}} \\
\|g\| &= \sqrt{\frac{8}{3}}
\end{aligned}$$

and we verify that triangle inequality is satisfied.

Linear Independence

A set of vector valued functions $f_1, f_2, \cdots, f_n$ is said to be linearly dependent if

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = 0 \tag{3.15}$$

holds for some constants $a_1, a_2, \cdots, a_n$, not all identically zero. The set is linearly independent if the above equation holds only for $a_1 = a_2 = \cdots = a_n = 0$.

As an example, consider

$$f_1 = \begin{pmatrix} 1 \\ x^2 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 2x \\ 1 \end{pmatrix}, \qquad f_3 = \begin{pmatrix} 2x-3 \\ 1-3x^2 \end{pmatrix}$$

Then we have

$$3f_1 - f_2 + f_3 = 0$$

for which $a_1 = 3, a_2 = -1$, and $a_3 = 1$. Thus the functions are linearly dependent.

In L_2, if a set of functions are orthogonal, then they are linearly independent. Let the functions $f_1, f_2, \cdots, f_n$ be orthogonal. Consider

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = 0$$

and take scalar product (in L_2) with respect to f_1. This gives

$$a_1(f_1, f_1) + a_2(f_2, f_1) + \cdots + a_n(f_n, f_1) = (f_1, 0) = 0$$

Since the functions are orthogonal, $(f_i, f_j) = 0$ for $i \neq j$ so that the above equation simplifies to

$$a_1(f_1, f_1) = 0$$

and hence $a_1 = 0$. Repeating the above analysis by taking L_2 scalar product with $f_2, \cdots, f_n$, we obtain $a_2 = \cdots = a_n = 0$. Therefore the functions are linearly independent.

In case, the functions are not orthogonal, linear independence of a finite number functions can be verified using its basic definition. Indeed, scalar multiplying equation (3.15) in L_2 by f_1, we obtain

$$a_1(f_1, f_1) + a_2(f_2, f_1) + \cdots + a_n(f_n, f_1) = (0, f_1) = 0$$

Likewise, we scalar multiply equation (3.15) by $f_2, f_3, \cdots$ and arrange all equations in a matrix form as

$$\begin{bmatrix} (f_1, f_1) & (f_2, f_1) & \cdots & (f_n, f_1) \\ (f_1, f_2) & (f_2, f_2) & \cdots & (f_n, f_2) \\ \vdots & \vdots & \ddots & \vdots \\ (f_1, f_1) & (f_2, f_1) & \cdots & (f_n, f_1) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.16}$$

This equation can be solved to obtain the unknown coefficients which then determines linear dependence of the set of functions.

Example Consider the functions $1, x, x^2$ over the interval $[0, 1]$. Then using (3.16), we obtain

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This equation has a unique solution $a_1 = a_2 = a_3 = 0$ so that the three functions are linearly independent.

Gram-Schmidt Orthogonalization Process

Expansion of L_2 functions using Fourier series is an important concept in engineering analysis. In the sequel we shall investigate Fourier series, and its generalization. In preparation of these concepts, we first investigate generation of a set of orthonormal functions from a given set of functions that are linearly independent but not orthogonal. The method of analysis is same as that for finite dimensional spaces discussed in Chapter 2.

Let $\{\phi_i, i = 1, 2, \cdots n\}$ be a sequence of linearly independent functions in L_2 over the domain Ω. Using these functions, we construct a orthogonal sequence, $\psi_i, i = 1, 2, \cdots n$.

Choose $\psi_1 = \phi_1$, and define ψ_2 as

$$\psi_2 = \phi_2 - c\,\psi_1$$

where the constant c is unknown. It is necessary that ψ_1 and ψ_2 are orthogonal. Thus taking scalar product of both sides of the above equation in L_2 with respect to ψ_1, and using the orthogonality condition, we set

$$0 = (\psi_1, \psi_2) = (\psi_1, \phi_2) - c(\psi_1, \psi_1)$$

solving which we obtain

$$c = \frac{(\psi_1, \phi_2)}{(\psi_1, \psi_1)}.$$

This gives the second function in the orthogonal sequence as

$$\psi_2 = \phi_2 - \frac{(\psi_1, \phi_2)}{(\psi_1, \psi_1)}\psi_1$$

To find the next function in the orthogonal sequence, we set

$$\psi_3 = \phi_3 - c_1\psi_1 - c_2\psi_2$$

where the unknown constants c_1 and c_2 are found by taking L_2 scalar product of the above equation with respect to ψ_1 and ψ_2, respectively. Indeed, taking scalar product with respect to ψ_1, we have

$$\begin{aligned} 0 = (\psi_1, \psi_3) &= (\psi_1, \phi_3) - c_1(\psi_1, \psi_1) - c_2(\psi_1, \psi_2) \\ &= (\psi_1, \phi_3) - c_1(\psi_1, \psi_1) \end{aligned}$$

since ψ_1 and ψ_2 are orthogonal, $(\psi_1, \psi_2) = 0$. This gives c_1 as

$$c_1 = \frac{(\psi_1, \phi_3)}{(\psi_1, \psi_1)}.$$

The constant c_2 is obtained in a similar way by taking scalar product with respect to c_2, and

$$c_2 = \frac{(\psi_2, \phi_3)}{(\psi_2, \psi_2)}.$$

This gives ψ_3 as

$$\psi_3 = \phi_3 - \frac{(\psi_1, \phi_3)}{(\psi_1, \psi_1)}\psi_1 - \frac{(\psi_2, \phi_3)}{(\psi_2, \psi_2)}\psi_2$$

In general the orthogonal sequence is obtained as

$$\psi_n = \phi_n - \sum_{k=1}^{n-1} \frac{\langle \psi_k, \phi_n \rangle}{\langle \psi_k, \psi_k \rangle}\psi_k \tag{3.17}$$

where, once again, the scalar products must be interpreted in the sense of the L_2 space. For obtaining the orthonormal sequence we normalize each function by dividing it with its norm, for example,

$$\hat{\psi}_i = \frac{\psi_i}{\|\psi_i\|}, \quad i = 1, 2, \cdots \tag{3.18}$$

is the desired orthonormal sequence.

As an example, consider $\Omega = (0, 1)$ and the set of functions $\{1, x, x^2, x^3 \cdots\}$ in the space $L_2(0, 1)$. This is a linearly independent but not an orthogonal set (prove it). We construct a sequence of orthonormal functions for this space using Gram-Schmidt process:

Using the above notations, we have

$$\begin{aligned}
\phi_1 &= 1 \\
\phi_2 &= x \\
\phi_3 &= x^2 \\
\phi_4 &= x^3 \\
&\vdots
\end{aligned}$$

For the orthonormal sequence, we take

$$\psi_1 = 1$$

and define

$$\psi_2 = \phi_2 - \frac{(\psi_1, \phi_2)}{(\psi_1, \psi_1)}\psi_1$$

Computing the scalar products,

$$(\psi_1, \phi_2) = \int_0^1 \psi_1\, \phi_2\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

and

$$(\psi_1, \psi_1) = \int_0^1 1\, dx = 1$$

This gives

$$\psi_2 = x - \frac{1}{2}$$

For the next term in the sequence, we set

$$\psi_3 = \phi_3 - \frac{(\psi_1, \phi_3)}{(\psi_1, \psi_1)}\psi_1 - \frac{(\psi_2, \phi_3)}{(\psi_2, \psi_2)}\psi_2$$

Computing the scalar products

$$\begin{aligned}
(\psi_1, \phi_3) &= \int_0^1 x^2\, dx = \frac{1}{3} \\
(\psi_1, \psi_1) &= \int_0^1 1\, dx = 1 \\
(\psi_2, \phi_3) &= \int_0^1 x^2(x - \frac{1}{2})\, dx = \frac{1}{12} \\
(\psi_2, \psi_2) &= \int_0^1 (x - \frac{1}{2})^2\, dx = \frac{1}{12}
\end{aligned}$$

which gives

$$\psi_3 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

The process continues for any number of terms in the sequence. For obtaining the orthonormal sequence, we normalize the sequence $\{\psi_i\}$.

$$\begin{aligned}
\hat{\psi}_1 &= \frac{\psi_1}{\|\psi_1\|} = 1 \\
\hat{\psi}_2 &= \frac{\psi_2}{\|\psi_2\|} = \frac{x - \frac{1}{2}}{\sqrt{\frac{1}{12}}} = \sqrt{3}(2x - 1) \\
\hat{\psi}_3 &= \frac{\psi_3}{\|\psi_3\|} = \sqrt{5}(6x^2 - 6x + 1)
\end{aligned}$$

Once can verify that the sequence $\{\hat{\psi}_i\}$ is orthonormal in $L_2(0, 1)$.

Two other orthonormal sequence that are frequently used in engineering analysis are Hermite functions, and the Laguerre functions. Hermite polynomials over the interval $(-\infty, \infty)$ are obtained by applying Gram-Schmidt procedure to the sequence, $\{e^{-\frac{t^2}{2}}, te^{-\frac{t^2}{2}}, t^2e^{-\frac{t^2}{2}}, \cdots\}$. Likewise the Laguerre polynomials over $[0, \infty)$ are obtained by Gram-Schmidt orthogonalization of the sequence

$\{e^{-\frac{t}{2}}, te^{-\frac{t}{2}}, t^2e^{-\frac{t}{2}}, \cdots\}$. These functions are orthonormal with respect to the scalar product

$$(f, g)_w = \int_a^b w(x)f(x)g(x)\, dx \tag{3.19}$$

where for Hermite functions we take $w(x) = e^{-t^2}$, and for Laguerre functions $w(t) = e^{-t}$. (Show that (3.19) satisfies all axioms of a scalar product.)

Orthonormal functions are frequently used as basis functions in engineering analysis. Orthonormality is not a requirement for a set of functions to be a basis set. However, determination of Fourier coefficients becomes a simple job if the basis functions are orthonormal. If the basis functions are not orthonormal, one can use the Gram-Schmidt method to generate an orthonormal set. First we discuss the concept of basis in the next section.

3.5 Basis and Least Square Approximation

In Chapter 2 we have seen that an arbitrary vector in an Euclidean space can be expressed in terms of its basis. Similar concept also holds for function spaces such as L_2 space. Fundamentally one defines the basis set for the purpose of expressing (or approximating) an arbitrary function in terms of the basis. For example, it is known that a piecewise continuous function that has at most finite number of finite discontinuities can be represented in terms of a Fourier series. In this context, the function that is being approximated is a member of L_2 over an appropriate domain, and the functions $\{\frac{1}{\sqrt{2}}, \sqrt{2}\sin n\pi x, \sqrt{2}\cos n\pi x, n = 1, 2, \cdots\}$ represent a basis for L_2. These functions are linearly independent, orthonormal, and span the entire space $L_2(-1, 1)$.

It is also interesting to note that there are infinitely many functions in the basis set of L_2 defined above. Recall that the dimension of a vector space is equal to the number of vectors in the basis set. Since the number of functions in the basis set of L_2 is infinite, this space is correctly known as an **infinite dimensional space**.

Next we consider least square approximation of functions in L_2 using a linear combination of an orthonormal sequence $\{\phi_i, i = 1, 2, \cdots, n, \cdots\}$. Let $f(x)$ be a square integrable function over the domain Ω, and express

$$f = a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n + \cdots \tag{3.20}$$

We wish to find the coefficients $a_1, a_2, \cdots a_n, \cdots$ such that the error between the function f and its approximation is minimum. Define the error as

$$\begin{aligned} E &= \frac{1}{2}\|f - \sum_i a_i\phi_i\|_{L_2}^2 \\ &= \frac{1}{2}\int_\Omega |f(x) - \sum_i a_i\, \phi(x)|^2\, dx \end{aligned} \tag{3.21}$$

To minimize the error with respect to the unknown constants, we have

$$\begin{aligned} 0 = \frac{\partial E}{\partial a_j} &= -\int_\Omega (f(x) - \sum_i a_i\ \phi_i(x))\phi_j(x)\,dx \\ &= -\int_\Omega f(x)\ \phi_j(x)\,dx + \sum_i a_i \int_\Omega \phi_i(x)\,\phi_j(x)\,dx \\ &= -\int_\Omega f(x)\ \phi_j(x)\,dx + a_j \end{aligned}$$

where we have used the orthonormality property of $\{\phi_i\}$ in evaluating the second integral:

$$\int_\Omega \phi_i(x)\ \phi_j(x)\,dx = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

This gives the unknown coefficients for the least square approximation for f in L_2 sense as

$$a_i = \int_\Omega f(x)\phi_i(x)\,dx. \tag{3.22}$$

Note that the above formulation does not require that the basis functions be taken as the conventional trigonometric functions, such as, sine and cosine functions. It suffices to choose any orthonormal sequence that is complete in L_2.

Fourier Series

The Fourier series is a special case of expansion of a L_2 function using the trigonometric functions, sine, and cosine. Lets consider the space $L_2(-T,T)$, and choose the basis as

$$\{1,\ \cos\frac{n\pi}{T}x,\ \sin\frac{n\pi}{T}x, \quad n = 1,2,3,\cdots\}$$

Note that these functions are not orthonormal, but satisfy the orthogonality condition:

$$\int_{-T}^{T} \sin \frac{n\pi}{T} x \, dx = 0 \qquad \text{for all } n$$

$$\int_{-T}^{T} \cos \frac{n\pi}{T} x \, dx = 0 \qquad \text{for all } n$$

$$\int_{-T}^{T} \sin \frac{n\pi}{T} x \quad \cos \frac{m\pi}{T} x \, dx = 0 \qquad \text{for all } n, \, m$$

$$\int_{-T}^{T} \sin \frac{n\pi}{T} x \quad \sin \frac{m\pi}{T} x \, dx = \begin{cases} 0 & \text{if } n \neq m \\ T & \text{if } n = m \end{cases}$$

$$\int_{-T}^{T} \cos \frac{n\pi}{T} x \quad \cos \frac{m\pi}{T} x \, dx = \begin{cases} 0 & \text{if } n \neq m \\ T & \text{if } n = m \end{cases}$$

Consider the Fourier expansion of $f \in L_2(-T, T)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{T} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{T} x$$

Then we define the error

$$E = \int_{-T}^{T} \| \left(a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{T} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{T} x \right) - f(x) \|^2 \, dx$$

and minimize it with respect to the unknown coefficients:

$$0 = \frac{\partial E}{\partial a_0} = \int_{-T}^{T} \left(a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{T} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{T} x - f(x) \right) dx$$

which simplifies to

$$a_0 = \frac{1}{2T} \int_{-T}^{T} f(x) \, dx$$

Likewise for the coefficient a_n,

$$0 = \frac{\partial E}{\partial a_n} = \int_{-T}^{T} \left(a_0 + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi}{T} x + \sum_{m=1}^{\infty} b_m \sin \frac{m\pi}{T} x - f(x) \right) \cos \frac{n\pi}{T} x \, dx$$

Using orthogonality properties, this equation simplifies to

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \, \cos \frac{n\pi}{T} x \, dx$$

For the coefficient b_n, we minimize the error with respect to b_n leading to

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\,\sin\frac{n\pi}{T}x\,dx$$

This completes the derivation of all terms of the conventional Fourier analysis.

As an example, consider Fourier expansion of the function $f(x) = \sin\frac{\pi}{2}x$ over the domain $-1 \le x \le 1$. First we consider the standard trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Using the results derived above, we obtain

$$\begin{aligned} a_0 &= 0 \\ a_n &= 0, \quad \text{for all} \quad n \\ b_n &= \begin{cases} \frac{8n}{(4n^2-1)\pi}, & \text{for } n \text{ odd} \\ -\frac{8n}{(4n^2-1)\pi}, & \text{for } n \text{ even} \end{cases} \end{aligned}$$

Using the first three terms of the series, we then obtain

$$f(x) = \frac{8}{3\pi}\sin\pi x - \frac{16}{15\pi}\sin 2\pi x + \frac{24}{35\pi}\sin 3\pi x + \cdots$$

Next we consider an expansion using the Legendre basis over the same space. First of all, an orthonormal sequence in $L_2(-1,1)$ is given by (see exercise)

$$\begin{aligned} \phi_1(x) &= \frac{1}{\sqrt{2}} \\ \phi_2(x) &= \sqrt{\frac{3}{2}}x \\ \phi_3(x) &= \sqrt{\frac{5}{8}}(3x^2-1) \\ &\vdots \end{aligned}$$

To simplify the analysis, we choose

$$f(x) \simeq a_1\phi_1(x) + a_2\phi_2(x)$$

where the constants are obtained by minimizing the error:

$$\begin{aligned} a_1 &= \int_{-1}^{1} \sin\frac{\pi}{2}x\,\phi_1(x)\,dx = 0 \\ a_2 &= \int_{-1}^{1} \sin\frac{\pi}{2}x\,\phi_2(x)\,dx = \int_{-1}^{1} \sin\frac{\pi}{2}x\sqrt{\frac{3}{2}}x\,dx = \frac{8}{\pi^2}\sqrt{\frac{3}{2}} \end{aligned}$$

This gives

$$f(x) = \sin \frac{\pi}{2} x \simeq \frac{8}{\pi^2} \sqrt{\frac{3}{2}}\, \phi_2(x) = \frac{12}{\pi^2} x.$$

One may certainly obtain a better approximation by finding additional terms in the expansion. We compare the standard (four sine terms) Fourier expansion with that using the Legendre basis (two terms shown as dotted line) in Figure 3.3.

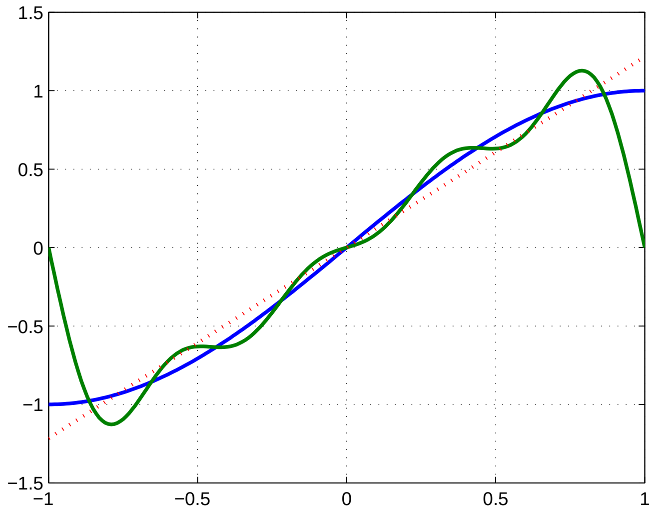

Figure 3.3 Least Square Approximation in $L_2(-1, 1)$

This result also shows that trigonometric functions are not necessarily the best choice in Fourier-type expansion. Selection of basis functions plays a significant role in mathematical analysis of engineering problems. Nevertheless, trigonometric functions are commonly used in filtering analysis. From a systems point of view, basis functions are not arbitrarily chosen. As is well known, a linear finite dimensional system is described by a matrix differential equation, and the evolution of the system can be described by the eigenvectors of the system matrix. Likewise, evolution of distributed systems can be described by partial differential equations involving certain operators, and the eigenfunctions of the operator form the basis function for the response. This will be further explored in the next chapter.

Bessel's Inequality: Let $\{\phi_i\}$ be an orthonormal sequence in L_2, and consider the approximation

$$f(x) \simeq \sum_{i=1}^{n} a_i \phi_i(x)$$

Then the error of approximation is computed as

$$\begin{aligned}
0 \le E &= \|f - \sum_{i=1}^{n} a_i\phi_i\|_{L_2}^2 \\
&= (f - \sum_{i=1}^{n} a_i\phi_i, f - \sum_{i=1}^{n} a_i\phi_i)_{L_2} \\
&= (f, f)_{L_2} - 2\sum_{i=1}^{n} a_i(f, \phi_i)_{L_2} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j(\phi_i, \phi_j)_{L_2} \\
&= \|f\|_{L_2}^2 - \sum_{i=1}^{n} a_i^2
\end{aligned}$$

where we have used the orthogonality of the basis functions, and that $a_i = (f, \phi_i)_{L_2}$. Thus we have

$$E = \|f\|_{L_2}^2 - \sum_{i=1}^{n} a_i^2 \ge 0$$

which is a nonincreasing function of n. Taking the limit as $n \to \infty$, we have

$$\|f\|_{L_2}^2 - \sum_{i=1}^{\infty} a_i^2 \ge 0 \tag{3.23}$$

or equivalently

$$\sum_{i=1}^{\infty} a_i^2 \le \|f\|_{L_2}^2 \tag{3.24}$$

which is known as the **Bessel's inequality**. Note that the limit (3.23) does not necessarily converge to zero as $n \to \infty$. In case for a set of orthonormal functions $\{\phi_i\}$, the equation (3.23) converges to zero for arbitrary functions $f \in L_2$, then the sequence ϕ_i is said to be a **complete set** for L_2, i.e., a basis for L_2. In other words, for arbitrary $f \in L_2$, if

$$\lim_{n\to\infty} \|f - \sum_{i=1}^{n} a_i\phi_i\|_{L_2}^2 \to 0$$

which is equivalent to

$$\|f\|_{L_2} = \left[\sum_{i=1}^{\infty} a_i^2\right]^{\frac{1}{2}} \tag{3.25}$$

then the sequence $\{\phi_i, i = 1, 2, \cdot, \infty\}$ is said to be a basis for L_2. Equation (3.25) is known as the **Parseval identity**, and the sequence $\sum_{i=1}^{\infty} a_i\phi_i$ converges to f in the mean square sense in L_2. Note that this convergence does not mean pointwise convergence, rather convergence of integral of squared differences. Convergence in L_2 sense is weaker than pointwise convergence, however, it is suitable for most engineering applications.

Recaping the discussion above, we say that *an orthonormal set $\mathcal{X}$ is a basis for L_2 if it is complete in L_2*. Completion of the sequence means that an arbitrary function in L_2 can be expressed as closely as necessary in terms of a Fourier series. Then the fundamental question is how to find whether a sequence of functions is complete in L_2? One way of testing whether the set $\mathcal{X}$ is a basis for L_2 is to see if the only vector that is orthogonal to each element in $\mathcal{X}$ is a zero element. As a simple example, consider the sequence $\{\sin n\pi x,\ n = 1, 2, \cdots\}$. Is this sequence complete in $L_2(0, 1)$? The answer is actually NO, since it is known that an arbitrary L_2 function cannot be expressed in terms of a Fourier series using only sine functions. To see it mathematically, we note that zero function is not the only function that is orthogonal to sine function. For example, cosine functions are also orthonormal to sine functions. This means that the sequence of sine functions is not a complete set in L_2. The result is given by the following:

A sequence of orthonormal functions $\{\varphi_i\}$ is complete in L_2 if and only if the only function that is orthogonal to all functions in the set $\{\varphi_i\}$ is the null function.

We prove the statement in two steps: First assume that $\{\varphi_i\}$ is a complete set; we show that the zero function in L_2 is orthonormal to all $\{\varphi_i\}$.

We prove it by contradiction. Suppose that there exists a nonzero function $f \in L_2$ that is orthonormal to all $\{\varphi_i\}$. Then by orthonormality

$$a_i = \langle f, \varphi_i \rangle = 0$$

for all i. Again, since $\{\varphi_i\}$ is complete, we can express f in terms of a Fourier series, so that

$$f = \sum_i a_i \varphi_i$$

where

$$a_i = \langle f, \varphi_i \rangle$$

Then

$$\text{Error } = \lim_{N\to\infty} \|f - \sum_i^N a_i \varphi_i\|^2 = \|f\|^2 \neq 0$$

where we have used the fact that $c_i = 0$ for each i. This means that the error between the function f and its Fourier expansion does not reduce to zero as $N \to \infty$. This contradicts with the assumption that the set φ is complete. Therefore f must be a zero function in L_2.

To prove the converse, let the zero function be the only function that is orthonormal to all functions φ; we show that $\{\varphi_i\}$ is complete.

Again we prove it by contradiction. Let $\{\varphi_i\}$ is not complete. This means that there exists a function f that cannot be expressed in terms of a Fourier series so that (by Bessel's inequality)

$$\int |f|^2\, dx - \sum_{i=1}^{\infty} a_i^2 > 0$$

holds as a strict inequality, where

$$a_i = \langle \varphi_i, f \rangle$$

Since $\{a_i\}$ is a converging sequence in ℓ_2, the sequence

$$g_N(x) = \sum_{i=1}^{N} a_i \varphi_i(x)$$

is a Cauchy sequence in L_2, which converges to the function $g(x)$ in L_2. Thus

$$a_i = \langle \varphi_i, g \rangle = \varphi_i, f \rangle$$

so that

$$\langle \varphi_i, f - g \rangle = 0$$

Thus the function $h = f - g$ is orthogonal to all φ_i. This gives

$$\|h\| = \|f - g\| \geq |\, \|f\| - \|g\| \,| = \big| \|f\| - \big(\sum_i^{\infty} a_i^2\big)^{\frac{1}{2}} \big| > 0$$

which shows that h is not the zero function. This contradicts with the assumption that only zero function is orthogonal to all $\{\varphi_i\}$. This completes the proof.

Generalized Fourier Series

The concept derived above is essentially a generalization of the classical Fourier series. Here we summarize it again. Let a function $f \in L_2(\Omega)$ be expressed in terms of a generalized Fourier series. For notational simplicity, we denote $\{\phi_n(x), n = 1, 2, \cdots\}$ as the basis function, which are not necessarily the set of trigonometric functions. We also assume that these functions are orthonormal. Then we express

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \tag{3.26}$$

where $\{a_n\}$ are the Fourier coefficients of f. Taking scalar product on both sides of the above equation by $\phi_m(x)$, we obtain

$$(f, \phi_m) = (\sum_{n=1}^{\infty} a_n \phi_n, \phi_m) = \sum_{n=1}^{\infty} a_n (\phi_n, \phi_m),$$

where the scalar product must be interpreted in the sense of (3.8). Since the sequence $\{\phi_n\}$ is orthonormal, i.e., $(\phi_n, \phi_m) = 0$ for $n \neq m$ and $(\phi_n, \phi_n) = 1$, the above equation reduces to

$$(f, \phi_m) = a_m.$$

Thus we obtain the generalized Fourier series for the function g as

$$f(x) = \sum_{n=1}^{\infty} (f, \phi_n) \phi_n. \tag{3.27}$$

It is not necessary that the basis functions $\{\phi_n\}$ be an orthonormal set. In case, $\{\phi_n\}$ is an orthogonal set, then following the derivation discussed above, one can show that

$$f(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{(\phi_n, \phi_n)} \phi_n.$$

Note that the basis functions $\{\phi_n\}$ need not be the usual trigonometric functions, i.e., sine or cosine functions commonly used in Fourier analysis. The only requirement is that $\{\phi_n\}$ must be a basis for L_2, which means that they must be linearly independent and must span the entire space L_2. In case one takes the functions $\{1, \sin n\pi x, \cos n\pi x, n = 1, 2, \cdots\}$ as the basis, the resulting expansion is the standard Fourier series.

Consider the set $\{\phi_n(x) = \sin \frac{(2n-1)\pi}{2} x, \quad n = 1, 2, \cdots\}$ over the interval $[0, 1]$. These trigonometric functions do not form complete cycles over the given interval, yet one can verify that they are orthogonal. These functions can be conveniently used to express all functions $f(x)$ with $f(0) = 0$, and $\frac{\partial f}{\partial x}(1) = 0$. The temperature distribution of a steel rod of length 1 unit with one end placed in a ice bath and the other end insulated belong to this class of functions.

The displacement of a cantilever rod of length L can be described by the Fourier expansion

$$y(x) = \sum_{i}^{\infty} c_i \, \varphi_i(x)$$

where

$$\varphi_n(x) = \frac{1}{\sqrt{L}}\big[(\cosh \beta_n x - \cos \beta_n x) - \gamma_n(\sinh \beta_n x - \sin \beta_n x)\big]$$

where c_i are Fourier coefficients, and

$$\gamma_n = \frac{\cosh \beta_n + \cos \beta_n}{\sinh \beta_n + \sin \beta_n}$$

with β_n satisfying the condition $\cos \beta_n \cosh \beta_n + 1 = 0$. Clearly the functions φ_n are not typical sine or cosine functions, but one can verify that φ_n is an orthonormal sequence which can be used to generate the Fourier expansion of the deflection of the cantilever rod.

Notice that the concept easily extends to a Fourier series in multiple dimensions. Consider a function $f \in L_2((0,1) \times (0,1))$ on a square $0 \le x_1 \le 1, \quad 0 \le x_2 \le 1$. Then for an orthonormal sequence, we define a two-parameter family of functions $\{\phi_{i,j}, i = 1, 2, \cdots, \quad j = 1, 2, \cdots\}$, and consider the sum

$$f(x_1, x_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} \, \phi_{i,j}(x_1, x_2)$$

leading to

$$a_{i,j} = (f, \phi_{i,j}) = \int_0^1 \int_0^1 f(x_1, x_2) \, \phi_{i,j}(x_1, x_2) \, dx_1 \, dx_2$$

There are many advantages of using basis functions that are different from traditional trigonometric functions. For example, in wavelet analysis used in signal processing, wavelets are chosen as the basis functions for expanding the signal in terms of an infinite series. In analysis of structural vibrations, one uses the system mode shape functions as the basis functions. These concepts will be further discussed in the sequel.

3.7 Wavelet Analysis

Fourier transform is a commonly used tool in various signal processing applications. Note however that Fourier transform can be applied only for stationary signals, i.e., the frequency content of the signal does not change with time. On the other hand, there are many examples of practical interest in which the signal is actually non-stationary. For example, consider a sine wave that is pure except that it has a small irregularity at a certain small time interval. Its Fourier spectrum essentially spreads out over the entire frequency range giving no indication at what point of time the irregularity actually occurred. By observing the Fourier coefficients of a signal, one can find the frequency contents of a signal, however, cannot conclude if there has been a change in the frequency at a certain time point. Fundamentally, the time scale of a signal is completely lost through the Fourier transform. This is where wavelet transform has an upper edge. A wavelet representation is a two dimensional representation of a signal, and as such, it is able to provide information both in both time and frequency. While the Fourier analysis is appropriate for periodic signals, wavelet analysis is more suitable for transient signals.

A wave is a time varying function with a certain periodicity, for example a sine wave with a certain amplitude and frequency. A wavelet is a small wave that is concentrated over a small time interval. Since a wave extends over the entire horizon, it can represent periodic and stationary phenomena. In contrast, since a wavelet is concentrated over a small portion of the time horizon, it is more useful is analyzing transient and non-stationary phenomena.

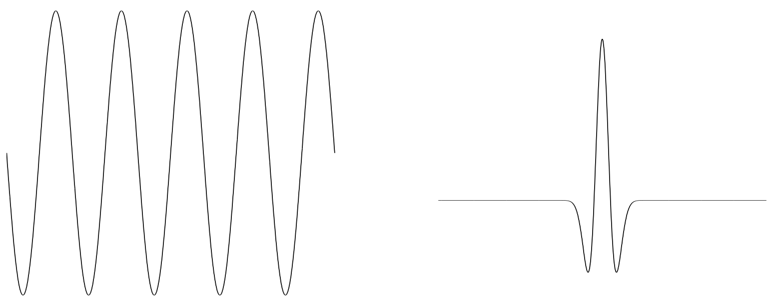

Figure 3.4 Wave and Wavelet

Wavelet Transform

A wavelet expansion is a two-parameter expansion of a signal:

$$f(t) = \sum_j \sum_k a_{j,k} \psi_{j,k}(t) \tag{3.28}$$

where $\psi_{j,k}$ are the wavelet expansion functions, which are usually orthonormal. The expansion coefficients $a_{j,k}$ are called the discrete wavelet transform of f

$$a_{j,k} = \langle f, \psi_{j,k} \rangle \tag{3.29}$$

Equation (3.28) is called the inverse wavelet transform, and equation (3.29) easily follows from orthonormality property of the wavelets. Like the Fourier coefficients, the coefficients of the wavelet expansion are expected to give useful information about the signal. One may call equation (3.28) as a wavelet series, however, it is customary to call it a wavelet transform.

The wavelet functions are generated using a 'generating wavelet' or a 'mother wavelet' by simple scaling and translation. Using a mother wavelet, $\psi(t)$, one generates a wavelet system

$$\psi_{j,k}(t) = 2^{\frac{j}{2}}\psi(2^j t - k) \tag{3.30}$$

where the factor $2^{\frac{j}{2}}$ is a normalization constant that is used to maintain a constant norm independent of the index j. As the second index k changes, the wavelet translates in time allowing detection of variations in time. An increase of j allows representation of signal at higher frequencies.

Multiresolution Analysis

In general, wavelet expansion of signals is usually done using two closely related functions, and is given by

$$f(t) = \sum_{k=1}^{\infty} c_k\phi_k(t) + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} d_{j,k}\psi_{j,k}(t) \tag{3.31}$$

where the functions ϕ_k are called the scaling functions, and $\psi_{j,k}$ are called the wavelet functions. The first summation gives a low resolution or coarse approximation of the signal. The second summation adds finer details by adding higher or finer resolution functions. The scaling functions and the wavelet functions satisfy the orthonormality properties:

$$\begin{aligned}
\langle \phi_i, \phi_j \rangle &= 0 \qquad \text{for } i \neq j \\
\langle \psi_{j,k}, \psi_{m,n} \rangle &= 0 \qquad \text{for } j \neq m \text{ and } k \neq n \\
\langle \phi_k, \psi_{m,n} \rangle &= 0
\end{aligned}$$

The expansion coefficients are computed using using orthonormality condition property of the functions, and are given by

$$\begin{aligned}
c_k &= \langle f(t), \phi_k(t) \rangle \\
d_{j,k} &= \langle f(t), \psi_{j,k}(t) \rangle
\end{aligned}$$

Using concept of multiresolution property, the scaling functions are defined as

$$\phi_k(t) = \phi(t - k)$$

where $\phi(t)$ is the basic scaling function, and $\phi_k(t)$ provides translation. Similarly, wavelet functions are defined using a 'mother wavelet' as

$$\psi_{j,k}(t) = 2^{\frac{j}{2}}\psi(2^j t - k)$$

which provides increased resolution through the index j and translation through the index k.

Selection of the mother wavelet depends on the signal characteristics of the problem. Simplest among all wavelets is the Haar wavelet defined by

$$\phi(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{elsewhere} \end{cases}$$
$$\psi(t) = \begin{cases} 1, & \text{if } 0 < t \leq 0.5 \\ -1 & \text{if } 0.5 < t < 1 \end{cases}$$

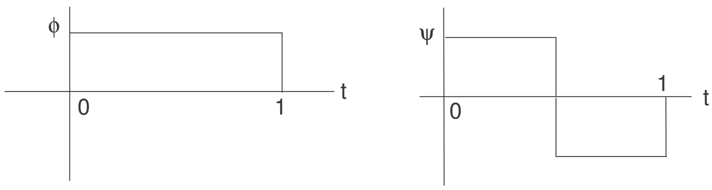

Figure 3.5 Haar Wavelet

Other wavelets used widely in the literature include Mortlet wavelet, Maxican hat function, Gaussian function etc. In what follows, we use Haar wavelets for filtering a noisy signal. Given a function $y \in L_2(0,1)$, consider the expansion

$$y = \sum_{k=1}^{\infty} c_k h_k$$

where the coefficients are computed as

$$c_j = \int_0^1 \varphi(t) h_j(t)\, dt \Big/ \int_0^1 h_j^2(t)\, dt$$

and $\{h_k, k = 1, 2, \cdots\}$ is a sequence of functions that include the Haar scaling function and wavelet functions at different resolutions. Without any loss of generality, the time domain is normalized to $[0, 1]$. The first eight functions in the sequence are shown in the figure below which are easily seen to be orthogonal.

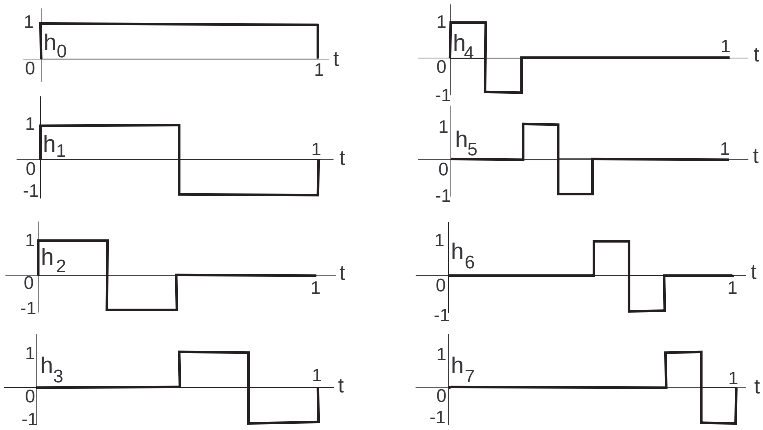

Figure 3.6 Haar Wavelets

The process of translation and dilation can be continued to generate additional wavelets as required. Here we shall use unnormalized Haar wavelets although one may use normalized wavelet functions if appropriate.

Computationally it is more convenient to use discrete wavelet transform. For example, consider a wavelet expansion using only the first four Haar wavelets as:

$$
\begin{aligned}
y &= c_0 h_0 + c_1 h_1 + c_2 h_2 + c_3 h_3 \\
&= \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} \\
&= C_4 H_4
\end{aligned}
$$

where the discrete time representation of the first four wavelets is given by

$$
H_4 = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{3.32}
$$

Then following least square expansion, we easily obtain the estimation of the coefficients of wavelet expansion as

$$
\hat{C}_4 = y H_4^{-1}
$$

and then the signal is reconstructed from

$$
y_{\text{estimated}} = \hat{C}_4 H_4
$$

Higher order wavelet transformation can be continued to the desired level of accuracy. In fact the Haar transformation matrix can be recursively generated using H_4 as shown in equation (3.32). The figure below shows the noisy signal and the filtered signal for a wavelet transform of order 64.

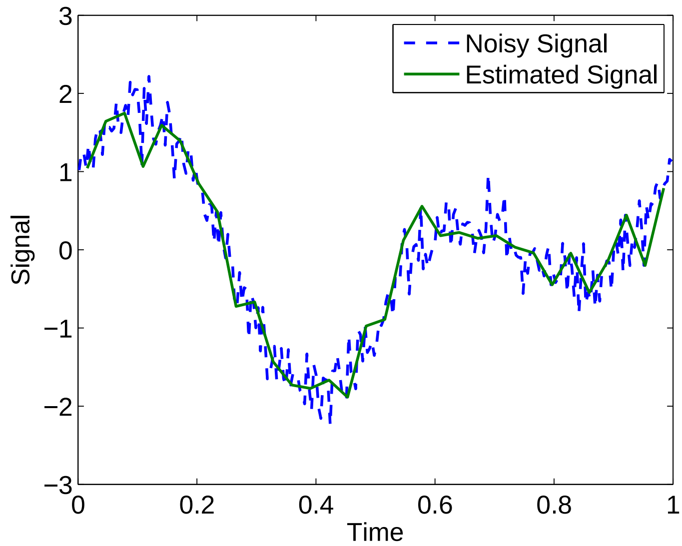

Figure 3.7 Filtering using Haar Wavelets

In summary, wavelet analysis provides information in both the time domain and the frequency domain which is often not possible in conventional signal processing methods, and because of this reason, wavelets have been successfully used in various signal processing applications, such as voice recognition, image processing, pattern recognition, medical diagnosis, and many more. Interested may find further details of the concepts and applications in standard texts on wavelets.

3.8 References

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3.9 Exercise

3.1 Consider the functions $f(x) = e^{-x}$ and its truncated series expansion $g(x) = 1 - x + \frac{x^2}{2}$. Find the norm of f and g in the sense of $L_2(0,1)$ and $C(0,1)$. Also compute the distance $\rho(f,g)$.

3.2 Suppose f and g are L_2 functions over a finite interval (a,b). Show that

$$\langle f, g\rangle = \int \alpha(x) f(x) g(x)\, dx$$

satisfies all properties of scalar product for L_2. Assume that α is a continuous function, and $\alpha > 0$ for all x in the domain.

3.3 Consider the space $L_2(-1,1)$ and the basis $\{1, x, x^2, x^3, \cdots\}$. Using the Gram-Schmidt method derive an orthonormal sequence for this functions. Compute only the first four elements of this infinite sequence.

3.4 Consider the function described by

$$f(x) = \begin{cases} 1 + x, & \text{if} \quad -1 \le x \le 0 \\ 1 - x, & \text{if} \quad 0 \le x \le 1 \end{cases}$$

a) Show that this function is an element of $L_2(-1,1)$. Find the norm of this function.

b) Express the above function in terms of generalized Fourier series using the following bases of $L_2(-1,1)$. For each case, find only the first four terms of the Fourier series.

Basis I: $\{1, x, x^2, x^3, \cdots\}$

Basis II: Legendre basis derived in Question 3.3.

Basis III: Standard Fourier series,

c) Compare the accuracy of Fourier expansion by computing the error (distance) between the given function and the approximate Fourier series. Also show your result graphically.

3.5 Show that all the axioms of linear space are satisfied for the space C^k.

5.6 Show that the trigonometric sequence

$$\{\frac{1}{\sqrt{2}\pi}, \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \cos nx, n = 1, 2, \cdots\}$$

is an orthonormal sequence in $L_2(-\pi, \pi)$.

3.7 Show that the functions

$$\phi_n(x) = \sin nx, \quad n = 1, 2, \cdots$$

are linearly independent on the interval $-\pi \le x \le \pi$.

3.8 Find the norm of the function in the sense of C and L_2

$$f(x,y) = \sin \pi x \, \cos \pi y$$

over the domain $\{0 < x < 1,\, 0 < y < 1\}$

3.9 Find the distance in the sense of C and L_2 between the following functions:

$$f(x_1,x_2) = \sin \pi x_1 \cos \pi x_2, \quad \text{and} \quad g(x_1,x_2) = 1$$

over the domain $\Omega = \{\mathbf{x} \in R^2 : -1 \le x_1 \le 1,\, -1 \le x_2 \le 1\}$.

3.10 Find the Fourier series of the function

$$f(x) = x^3, \qquad -1 \le x \le 1$$

using a) conventional trigonometric functions, and b) Legendre functions.

3.11 Show that the sequence $\{\sin \frac{2n-1}{2}\pi x,\, n = 1,2,\cdots\}$ is an orthogonal sequence in $L_2(0,1)$. Using this sequence, find the generalized Fourier series for the function, $f(x) = x$ over the interval $0 \le x \le 1$.

3.12 Using the Legendre basis over the interval $0 \le x \le 1$, express $f(x) = \sin \frac{\pi}{2} x$ in terms of a Fourier series.

3.13 Consider the function $f(x) = x^3$ over the interval $[-1,1]$. Express this function in terms of an expansion using a) Legendre basis, b) Fourier expansion using sine and cosine functions. Compare the two expansions.

3.14 Let $I = \{x : -1 < x < 1\}$, and let

$$f(x) = \sin \frac{\pi}{2} x$$

$$g(x) = \begin{cases} 1+x, & \text{for } -1 \le x \le 0 \\ 1-x & \text{for } 0 \le x \le 1 \end{cases}$$

Find the distance between f and g in the sense of L_2 and C.

3.15 Show that the functions $\{1, x, x^2, x^3, \cdots\}$ are linearly independent in $L_2(0,1)$, but are not orthogonal.

3.16 Find a function of the form $f(x) = ax^2 + bx + c$ that is orthonormal to both $g(x) = 1$ and $h(x) = 1 + x$.

5.17 Show that an arbitrary function $f \in L_2(0,T)$ can be expressed in terms a Fourier series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

where $\omega = \frac{2\pi}{T}$. Find equations for the Fourier coefficients a_0, and a_n, b_n, $n = 1, 2, \cdots$.

3.18 Using the result derived in question 17, express the hat function

$$f(x) = \begin{cases} 2x, & \text{for } 0 \le x \le 0.5 \\ 2-2x & \text{for } 0.5 \le x \le 1 \end{cases}$$

in terms of a Fourier series.

3.19 Consider the space $L_2(-\infty, \infty)$ with the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x) g(x) \, dx$$

Find an orthogonal basis for the set $\{1, x, x^2, x^3, \cdots\}$.

3.20 Consider the space $L_2(0, \infty)$ with the scalar product

$$\langle f, g \rangle = \int_{0}^{\infty} e^{-x} f(x) g(x) \, dx$$

Find an orthogonal basis for the set $\{1, x, x^2, x^3, \cdots\}$.

3.21 Consider the function

$$f(x) = e^x$$

in $L_2[0, 1]$. Find the least square approximation of $f(x)$ using the basis $\phi_1 = 1$, and $\phi_2 = x$. How does your approximation compare with another approximation taking only the first two terms of infinite series expansion of the exponential function? Explain if there are any differences in the two approximations.