4.
Eigenvalue Problem

The dynamic behavior of a linear time invariant system is strongly related to certain characteristics of the system matrix in the governing differential equation. In fact, the dynamic response of a system can be expressed in terms of its eigenvalues and eigenvectors. The phrase eigenvalue originates from German word *eigen*, which means *characteristic*, so that eigenvalue is also known as *characteristic value* in the literature. From the knowledge of the eigenvalues of the system matrix, the systems engineer can predict expected transient behavior of the system. For this reason, design criteria for control systems are also often expressed in terms of a set of desired eigenvalues for the closed loop matrix, and sometimes in terms of both desired eigenvalues and eigenvectors. Furthermore, as we shall see in Chapter 11 and 12, certain control system design methods are also based on eigenvalues and eigenvectors of the closed loop system. This chapter introduces the concepts of eigenvalues and eigenvectors along with their computational methods.

4.1 Eigenvalues and Eigenvectors

Given a square matrix $\mathbf{A}$ *of size* $n \times n$*, if there exists a scalar* λ *and a nontrivial* n*-vector* $\mathbf{v}$ *such that*

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \tag{4.1}$$

then λ *is called an eigenvalue of the matrix* $\mathbf{A}$*, and* $\mathbf{v}$ *is an eigenvector of* $\mathbf{A}$ *corresponding to the eigenvalue* λ*.* From a geometric point of view, the above equation means that the vector $\mathbf{v}$ is invariant under the transformation $\mathbf{A}$, i.e., the vectors $\mathbf{v}$ and $\mathbf{A}\mathbf{v}$ are precisely in the same (or opposite) direction, and differ in magnitude by the factor λ.

It is obvious that equation (4.1) is satisfied if $\mathbf{v}$ is a zero vector, which, however, is not an eigenvector. For any square matrix, there always exist certain very specific scalars and corresponding nonzero vectors for which (4.1) is satisfied. These are the eigenvalues and the eigenvectors for the given matrix. In view of equation (4.1), for

λ to be an eigenvalue, it is necessary that the equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0, \tag{4.2}$$

be satisfied for a nonzero vector $\mathbf{v}$. Equation (4.2) has a nonzero solution if and only if the determinant of the coefficient matrix is zero, that is,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{4.3}$$

Equation (4.2) also shows that the vector $\mathbf{v}$ must belong to the null space of the matrix $(\mathbf{A} - \lambda\mathbf{I})$, and as such, the eigenvectors can be computed by finding the null space of the matrix $(\mathbf{A} - \lambda\mathbf{I})$. It is known that a matrix can have a nonempty null space only if it is singular, that is, if its determinant is zero. Equation (4.3) is known as the characteristic equation of the matrix $\mathbf{A}$. This equation may be expanded as

$$|\mathbf{A} - \lambda\mathbf{I}| = (-1)^n[\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0] = 0, \tag{4.4}$$

where the coefficients $c_0, c_1, \cdots, c_{n-1}$ are easily computed for any given matrix. Clearly, equation (4.4) is a polynomial of degree n so that it has exactly n solutions, and these n solutions are the n eigenvalues of the matrix $\mathbf{A}$. Also corresponding to each eigenvalue, there exists an element in the null space of $(\mathbf{A} - \lambda\mathbf{I})$. Thus a matrix of dimension $n \times n$ has exactly n eigenvalues; however, it may not have a full set of n eigenvectors. This will be further discussed in the sequel.

Note that equation (4.2) is equivalent to $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = 0$ so that the eigenvalues of the matrix $\mathbf{A}$ can also be computed by finding the roots of the equation $|\lambda\mathbf{I} - \mathbf{A}| = 0$. Likewise, eigenvectors remain unchanged since any multiple of an eigenvector is also an eigenvector.

There are several methods for numerical computation of eigenvalues and eigenvectors. The method used in the following examples is essentially based on the fundamental definition given above.

Example 4.1

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}. \tag{4.5}$$

Then we have

$$\begin{aligned}(\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\lambda - 2 & 0 & 1 \\ 0 & -\lambda + 1 & 0 \\ 0 & 1 & -\lambda + 3 \end{bmatrix},\end{aligned} \tag{4.6}$$

so that

$$
\begin{aligned}
|\mathbf{A} - \lambda\mathbf{I}| = 0 &= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \\
&= -(\lambda + 2)(\lambda - 1)(\lambda - 3).
\end{aligned}
$$

This shows that the matrix $\mathbf{A}$ given above has three eigenvalues, $\lambda = -2, 1, 3$. Next we find the eigenvectors. Since the eigenvectors are the elements of the null space of $(\mathbf{A} - \lambda\mathbf{I})$, we compute the null space by elementary row operations. From (4.6), corresponding to the eigenvalue $\lambda = -2$, we have

$$
\begin{aligned}
(\mathbf{A} - \lambda\mathbf{I})\big|_{\lambda=-2} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \\
&\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

This gives

$$
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
$$

For the eigenvalue $\lambda = 1$, we have

$$
\begin{aligned}
(\mathbf{A} - \lambda\mathbf{I})\big|_{\lambda=1} &= \begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

This gives the eigenvector $\mathbf{v}_2$ as

$$
\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{3} \\ 2 \\ -1 \end{bmatrix}.
$$

Since any multiple of an element in null space is also an element of null space, this implies that *any multiple of an eigenvector is also an eigenvector.* (Prove this statement using the definition (4.1).) Thus, for convenience, we may take

$$
\mathbf{v}_2 = \begin{bmatrix} 1 \\ -6 \\ 3 \end{bmatrix},
$$

as the eigenvector corresponding to $\lambda = 1$.

The eigenvector corresponding to $\lambda = 3$ is obtained in a similar manner, and is given by

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$

We leave the details as an exercise for the reader. ▷

Certain matrices have complex eigenvalues. However, for real matrices, *complex eigenvalues always appear in conjugate pairs*, that is, if λ is a complex eigenvalue, then its conjugate λ^* is also another eigenvalue of the same matrix. Similarly, the eigenvectors corresponding to complex eigenvalues (of real matrices) will always occur in conjugate pairs. This is shown by the following example.

Example 4.2

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 5 \\ 0 & 1 & 1 \\ -4 & 1 & -5 \end{bmatrix}. \tag{4.7}$$

The characteristic polynomial for this matrix is given by

$$\begin{aligned} 0 = |\mathbf{A} - \lambda\mathbf{I}| &= -\lambda^3 + 2\lambda - 4 \\ &= -(\lambda + 2)(\lambda - 1 - i)(\lambda - 1 + i). \end{aligned}$$

Thus the eigenvalues are $\lambda = -2, 1 + i, 1 - i$, where $i = \sqrt{-1}$. Note that one of the eigenvalues is real, and the other two are complex conjugate of each other.

Computation of eigenvectors is carried out following the same approach as in Example 6.1. The eigenvector corresponding to $\lambda = -2$ is obtained as

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 2 \\ -6 \end{bmatrix}.$$

Note that this eigenvector is real since the eigenvalue $\lambda = -2$ is. However, a complex eigenvector is expected for $\lambda = 1 + i$. In this case we carry out elementary row operations on $(\mathbf{A} - \lambda\mathbf{I})$ to obtain

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})\big|_{\lambda=1+i} &= \begin{bmatrix} 3-i & 0 & 5 \\ 0 & -i & 1 \\ -4 & 1 & -6-i \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & \frac{3+i}{2} \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

This gives the eigenvector as

$$\mathbf{v}_2 = \begin{bmatrix} 3+i \\ 2i \\ -2 \end{bmatrix}.$$

Since $\lambda = 1 - i$ is conjugate of $\lambda = 1 + i$, the eigenvector corresponding to $\lambda = 1 - i$ is obtained just by taking the conjugate of $\mathbf{v}_2$:

$$\mathbf{v}_3 = \begin{bmatrix} 3 - i \\ -2i \\ -2 \end{bmatrix}.$$

$\triangleright$

The **MATLAB** command `eig` can be used to compute the eigenvalues and the eigenvectors of a matrix. It is also noted that the eigenvectors computed by `eig` are normalized in the sense that $\mathbf{v}_{\text{MATLAB}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$, where $\|\mathbf{v}\|$ is the norm of the vector. In case there are repeated eigenvalues, the eigenvectors computed by `eig` may not be accurate. For such cases there may not exists a full set of distinct eigenvectors. This will be the subject of further discussion in Section 4.6.

4.2 Properties of Eigenvalues and Eigenvectors

There are certain fundamental properties of the eigenvalues and eigenvectors of a matrix that are very useful in analysis of linear systems.

Property 1: *If λ is an eigenvalue of a matrix $\mathbf{A}$, then λ^k is an eigenvalue of the matrix $\mathbf{A}^k$, where k is any integer, positive or negative.*

To prove this statement, first consider k as a positive integer. By definition, λ satisfies the equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}. \tag{4.8}$$

Multiplying the above equation by $\mathbf{A}$, we have

$$\mathbf{A}^2\mathbf{v} = \lambda\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v}, \qquad \text{using (4.8)}$$

which shows that λ^2 is an eigenvalue of $\mathbf{A}^2$, and $\mathbf{v}$ is the corresponding eigenvector. The above equation may be multiplied again by $\mathbf{A}$ to show that λ^3 is an eigenvalue of $\mathbf{A}^3$. This can be repeated for any positive integer.

Now we show that the result also holds for negative integers. Multiply (4.8) by $\mathbf{A}^{-1}$ (This assumes that $\mathbf{A}$ is nonsingular). This gives

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{v} = \lambda\mathbf{A}^{-1}\mathbf{v},$$

which is equivalent to

$$\mathbf{A}^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}.$$

Again multiplying the above equation by $\mathbf{A}^{-1}$, one obtains

$$\mathbf{A}^{-2}\mathbf{v} = \lambda^{-1}\mathbf{A}^{-1}\mathbf{v} = \lambda^{-2}\mathbf{v}.$$

Thus we have the result by induction. $\triangleright$

It can be shown that Property 1 is valid also if k is a fraction, not necessarily an integer; for example, $\lambda^{\frac{1}{2}}$ is an eigenvalue of $\mathbf{A}^{\frac{1}{2}}$. However, this cannot be proved following the above approach; interested reader may consult [68].

Property 2: *If the characteristic equation of a matrix* $\mathbf{A}$ *is given by (4.4), then the determinant of the matrix is*

$$det(\mathbf{A}) = (-1)^n c_0, \tag{4.9}$$

where n *is the size of the matrix.*

From the characteristic equation we have

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n[\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0]. \tag{4.10}$$

Setting $\lambda = 0$ in the above equation, we obtain (4.9) $\triangleright$

Property 3: *The matrix* $\mathbf{A}$ *is invertible if and only if* $\lambda = 0$ *is not an eigenvalue of the matrix.*

This follows easily from (4.4). The polynomial (4.4) will have a solution $\lambda = 0$ if and only if $c_0 = 0$. Then if $c_0 = 0$, equation (4.9) shows that $\det(\mathbf{A}) = 0$, and the matrix will not be invertible. $\triangleright$

Property 4: *The eigenvectors corresponding to the distinct eigenvalues are linearly independent.*

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the distinct eigenvalues of a matrix $\mathbf{A}$, and let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be the corresponding eigenvectors. We show that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent.

We prove it by contradiction. First assume that the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, for $k < n$, are linearly independent, $\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \cdots, \mathbf{v}_n$ are linearly dependent. We show that this assumption will contradict with the basic definition of linear independence.

By assumption, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}$ is a linearly dependent set. Therefore, the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} = 0, \tag{4.11}$$

holds for some constants $c_1, c_2, \cdots, c_k, c_{k+1}$, not all zero. Multiplying (4.11) by the matrix $\mathbf{A}$, and using

$$\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad \mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \quad \cdots \quad \mathbf{A}_k\mathbf{v}_k = \lambda_k\mathbf{v}_k, \quad \mathbf{A}_{k+1}\mathbf{v}_{k+1} = \lambda_{k+1}\mathbf{v}_{k+1},$$

it follows from (4.11) that

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots c_k\lambda_k\mathbf{v}_k + c_{k+1}\lambda_{k+1}\mathbf{v}_{k+1} = 0. \tag{4.12}$$

Also multiplying (4.11) by λ_{k+1} and subtracting from (4.12), we obtain

$$c_1(\lambda_1 - \lambda_{k+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{k+1})\mathbf{v}_2 + \cdots + c_k(\lambda_k - \lambda_{k+1})\mathbf{v}_k = 0.$$

But by assumption the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent so that the coefficients in the above equation must be all zero:

$$\begin{aligned} c_1(\lambda_1 - \lambda_{k+1}) &= 0 \\ c_2(\lambda_2 - \lambda_{k+1}) &= 0 \\ \vdots \qquad & \quad \vdots \\ c_k(\lambda_k - \lambda_{k+1}) &= 0. \end{aligned}$$

Since the eigenvalues are distinct, i.e., $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots \neq \lambda_k \neq \lambda_{k+1}$, this shows that the coefficients $c_1 = c_2 = \cdots = c_k = 0$. Using this result in (4.11), yields

$$c_{k+1}\mathbf{v}_{k+1} = 0.$$

Since $\mathbf{v}_{k+1}$ is an eigenvector, it is nonzero, so that $c_{k+1} = 0$. This contradicts with the assumption that $\{c_1, c_2, \cdots, c_k, c_{k+1}\}$ are not all zero. This completes the proof.

$\triangleright$

Property actually holds also for matrices with repeated eigenvalues, however, its proof is somewhat more complex.

4.3 Similarity Transformation

As mentioned earlier, state space model of a dynamic system is not unique. It turns out that certain forms of state space models are more convenient for control system analysis and design. For example, the controllable canonical form is frequently used for control design purposes. However, the dynamic model of the actual physical system derived from the first principle may not be in the controllable canonical form. Similarity transformation provides a useful tool to find an equivalent state space representation of the system in the controllable canonical form. In general, using an appropriate similarity transformation it is possible to develop other equivalent state space models of the system, not just the controllable canonical model. In the next chapter, we shall review physical interpretation of similarity transformation from a systems point of view. The mathematical concept of similarity transformation is introduced first.

A matrix $\mathbf{B}$ *is said to be similar to the matrix* $\mathbf{A}$ *if there exists a nonsingular transformation matrix* $\mathbf{T}$ *such that*

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \tag{4.13}$$

and the matrix $\mathbf{T}$ *is called the similarity transformation.*

Example 4.3

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.14}$$

and a transformation matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

It can be verified that $\mathbf{T}$ in invertible, and its inverse is given by

$$\mathbf{T}^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}.$$

Then we compute

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & -4 & -1 \end{bmatrix}.$$

Therefore, the matrix $\mathbf{B}$ is similar to the matrix $\mathbf{A}$, and the matrix $\mathbf{T}$ is a similarity transformation.

It can also be verified that the matrix $\mathbf{C}$ defined as

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

is similar to $\mathbf{A}$ because of the similarity transformation

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

▷

The eigenvalues of a matrix are invariant under similarity transformation. In other words, if $\mathbf{A}$ and $\mathbf{B}$ are similar matrices, then they have the same exact eigenvalues. This can be proved very easily. Suppose $\mathbf{T}$ is the similarity transformation so that $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. Suppose λ_i is an eigenvalue of $\mathbf{B}$. Using the characteristic equation for $\mathbf{B}$ and substituting the above relation, we obtain

$$\begin{aligned} 0 &= |\mathbf{B} - \lambda_i \mathbf{I}| \\ &= |\mathbf{T}^{-1}\mathbf{A}\mathbf{T} - \lambda_i \mathbf{I}| \\ &= |\mathbf{T}^{-1}\mathbf{A}\mathbf{T} - \lambda_i \mathbf{T}^{-1}\mathbf{T}| \\ &= |\mathbf{T}^{-1}(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{T}| \\ &= |\mathbf{T}^{-1}| \cdot |\mathbf{A} - \lambda_i \mathbf{I}| \cdot |\mathbf{T}| \\ &= |\mathbf{A} - \lambda_i \mathbf{I}|, \end{aligned}$$

which shows that λ_i is also an eigenvalue of $\mathbf{A}$. Note that in the above analysis we have used the fact that $|\mathbf{T}^{-1}| = \frac{1}{|\mathbf{T}|}$ which can be easily proved.

It is also instructive to find a relation between the eigenvectors of similar matrices. Suppose $\mathbf{v}_i$ is an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue λ_i. Therefore

$$\mathbf{B}\mathbf{v}_i = \lambda_i \mathbf{v}_i,$$

Using $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$, this gives

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

Premultiplying both sides of above equation by $\mathbf{T}$,

$$\mathbf{A}\mathbf{T}\mathbf{v}_i = \lambda_i \mathbf{T}\mathbf{v}_i.$$

Denoting $\mathbf{u}_i = \mathbf{T}\mathbf{v}_i$, this gives

$$\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i,$$

which shows that $\mathbf{u}_i = \mathbf{T}\mathbf{v}_i$ is an eigenvector of $\mathbf{A}$.

As mentioned above, similar matrices have the same eigenvalues, but the reverse is not necessarily true, that is, *two matrices may have the same set of eigenvalues, but may not be similar.* Consider for example the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

One can easily verify that the eigenvalues of both $\mathbf{A}$ and $\mathbf{B}$ are $\{2, 2, 2\}$, but they are not similar since there is no nonsingular transformation that can transform $\mathbf{A}$ into $\mathbf{B}$.

As an example, consider the series RLC circuit with a voltage source as shown in Figure 6.1. The dynamic model of the circuit can be easily obtained using the Kirchoff's voltage law. This system has an input, $e(t)$, and let's denote the voltage across the capacitor as $v_c(t)$, and the loop current as $i(t)$.

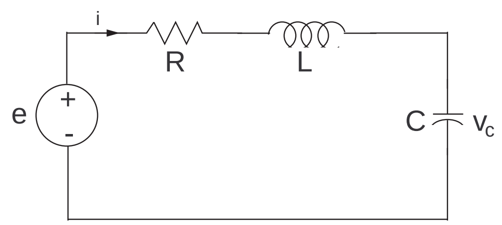

Fig. 4.1 A Simple Electrical Circuit

Then we have

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\, d\tau = e, \tag{4.15}$$

The state space method of system modeling requires that the model be given in the form of a set of first order differential equations. From the fundamentals of electric circuit theory, the capacitor voltage is given by

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i\, d\tau, \tag{4.16}$$

so that equation (4.15) becomes

$$Ri + L\frac{di}{dt} + v_c = e, \tag{4.17}$$

which is a first order differential equation for i. A differential equation for v_c is easily obtained from (4.16) as

$$\frac{dv_c}{dt} = \frac{1}{C}i. \tag{4.18}$$

Equations (4.17) and (4.18) are then written in the form of a matrix differential equation as

$$\frac{d}{dt}\begin{bmatrix} i \\ v_c \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} i \\ v_c \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} e. \tag{4.19}$$

This equation is a complete mathematical representation of the transient behavior of the system. For notational simplicity, we express the model as

$$\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1 u \tag{4.20}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \qquad \mathbf{B}_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \tag{4.21}$$

Let's look at the system in a different way. Define

$$i = C\frac{dv_c}{dt}$$

Then from (4.15), we obtain

$$RC\frac{dv_c}{dt} + LC\frac{d^2v_c}{dt^2} + v_c = e$$

To rewrite the equation in the state space form, define

$$z_1 = v_c, \quad z_2 = \frac{dv_c}{dt}$$

This gives

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} e. \tag{4.22}$$

which we rewrite as

$$\dot{\mathbf{z}} = \mathbf{A}_2\mathbf{z} + \mathbf{B}_2 u \tag{4.23}$$

where

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, \qquad \mathbf{B}_2 = \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} \tag{4.24}$$

Note that the two models, i.e., equation (4.19) and (4.22) are similar in every aspect in that they represent the same system in two different ways. In fact, the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are similar, and they are related by the transformation

$$\mathbf{A}_2 = \mathbf{T}^{-1}\mathbf{A}_1\mathbf{T}$$

where

$$\mathbf{T} = \begin{bmatrix} 0 & C \\ 1 & 0 \end{bmatrix}$$

By direct calculation, one can easily verify that the two matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ have the same pair of eigenvalues. From an engineering perspective, the two models cannot the different since they represent the same system. System characteristics do not change if they are represented in different ways.

The choice of a similarity transformation for a given system is not arbitrary, rather is very specific to the required form of the desired similar matrix. There are several types of similarity transformations that are routinely used in control system design. The diagonal matrix is a useful form of similar matrix. This is introduced in the next section.

4.4 Diagonalization of Matrices

Given an arbitrary matrix $\mathbf{A}$, we would like to find a similarity transformation $\mathbf{T}$ so that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}, \tag{4.25}$$

where $\mathbf{D}$ is a diagonal matrix. This raises two important questions: a) does there exist any such transformation matrix, and b) if there exists one, how to find it. If there exists a transformation matrix $\mathbf{T}$ for which (4.25) holds, we say that the matrix $\mathbf{A}$ is diagonalizable. It turns out that not all matrices are diagonalizable.

Suppose the eigenvalues of the $n \times n$ matrix $\mathbf{A}$ are all distinct. Then $\mathbf{A}$ is diagonalizable using the transformation matrix

$$\mathbf{T} = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,], \tag{4.26}$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ are the eigenvectors of $\mathbf{A}$. Furthermore, the diagonal entries of the matrix $\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ are the eigenvalues of the matrix $\mathbf{A}$.

First of all we note that $\mathbf{T}$ is a nonsingular matrix since the eigenvectors are linearly independent. Next we show that the transformation matrix $\mathbf{T}$ defined in (4.26) diagnoalizes the matrix $\mathbf{A}$.

Let the eigenvalues of the matrix $\mathbf{A}$ be $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, which are all distinct, and let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be the corresponding eigenvectors satisfying $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i, i = 1, 2, \cdots, n$. Then we have

$$\begin{aligned}
\mathbf{A}\mathbf{T} &= \mathbf{A}\,[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,] \\
&= [\,\mathbf{A}\mathbf{v}_1 \quad \mathbf{A}\mathbf{v}_2 \quad \cdots \quad \mathbf{A}\mathbf{v}_n\,] \\
&= [\,\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n\,] \\
&= [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,] \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\
&= \mathbf{T}\mathbf{D}.
\end{aligned}$$

This shows that $\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{D}$. Premultiplying both sides by $\mathbf{T}^{-1}$, we have $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$ as required. Recall that $\mathbf{T}$ is nonsingular so that $\mathbf{T}^{-1}$ exists. We illustrate the result using the following example.

Example 4.4

Consider the matrix given in Example 4.1:

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}. \tag{4.27}$$

The matrix $\mathbf{A}$ has all distinct eigenvalues, $\lambda = -2, 1, 3$. Using the corresponding eigenvectors computed in Example 6.1, we take the transformation matrix $\mathbf{T}$ as

$$\mathbf{T} = \begin{bmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 5 \end{bmatrix}.$$

Then it is easily verified that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

▷

The ordering of the eigenvectors is not important in diagonalizing a matrix. In the above example, if we interchange the first two columns of $\mathbf{T}$, the corresponding eigenvalues in the diagonal matrix will switch their places.

In case the eigenvalues of a matrix are complex, the transformation matrix $\mathbf{T}$ will also be complex, and the corresponding diagonal matrix $\mathbf{D}$ will have complex entries on its diagonal.

In case the eigenvalues of a matrix are repeated, there may not exist a full set of linearly independent eigenvectors. These matrices cannot be diagonalized. However, if there exist a full set of linearly independent eigenvectors for repeated eigenvalues, then the matrix is diagonalizable.

If a matrix is in the controllable canonical form and has distinct eigenvalues, a simple alternative to the transformation matrix for diagonalization is the following:

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}. \tag{4.28}$$

The matrix (4.28) is known as the **Vandermond matrix**. This transformation matrix is simple to construct since only the eigenvalues of the matrix are needed; however, use of the method is restricted to the matrices in the controllable form only.

If there are complex eigenvalues then the diagonal form may not be very much useful because of complex numbers (eigenvalues) that will appear in the diagonal matrix. For this case, a block diagonal matrix may be more useful. Consider a matrix with one real eigenvalue and two pairs of complex eigenvalues, such as, $\{\lambda_1, \sigma_2 + i\omega_2, \sigma_2 - i\omega_2, \sigma_3 + i\omega_3, \sigma_3 - i\omega_3\}$. Let the corresponding eigenvectors are, $\mathbf{v}_1, \mathbf{v}_{2R} + i\mathbf{v}_{2I}, \mathbf{v}_{2R} - i\mathbf{v}_{2I}, \mathbf{v}_{3R} + i\mathbf{v}_{3I}, \mathbf{v}_{3R} - i\mathbf{v}_{3I}$. Note that the eigenvectors of a complex eigenvalue are also complex. For this case, we consider a transformation matrix

$$\mathbf{T} = [\, \mathbf{v}_1 \quad \mathbf{v}_{2R} \quad \mathbf{v}_{2I} \quad \mathbf{v}_{3R} \quad \mathbf{v}_{3I} \,].$$

Then the transformed matrix will have the structure

$$\hat{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & \omega_2 & 0 & 0 \\ 0 & -\omega_2 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_3 & \omega_3 \\ 0 & 0 & 0 & -\omega_3 & \sigma_3 \end{bmatrix}.$$

The transformed matrix has the block diagonal structure, and has no complex numbers.

4.5 Generalized Eigenvectors

It is possible that a matrix has repeated eigenvalues rather than all distinct eigenvalues. Computation of these repeated eigenvalues is also carried out using (4.3). However, for a repeated eigenvalue, there may not exist a full set of eigenvectors. In such cases one can still find a set of linearly independent vectors that have 'eigenvector'-like properties. These vectors are known as the generalized eigenvectors. Specifically, a generalized eigenvector of rank k is defined as a nonzero vector satisfying

$$\begin{aligned} (\mathbf{A}-\lambda_r\mathbf{I})^k\mathbf{v}_k &= 0 \\ (\mathbf{A}-\lambda_r\mathbf{I})^{k-1}\mathbf{v}_k &\neq 0. \end{aligned} \tag{4.29}$$

A regular eigenvector is a special case of the above definition with $k = 1$.

For a matrix with repeated eigenvalues, the characteristic equation can be written as

$$|\mathbf{A}-\lambda\mathbf{I}| = (-1)^n(\lambda-\lambda_1)^{d_1}(\lambda-\lambda_2)^{d_2}\cdots(\lambda-\lambda_s)^{d_s} = 0, \tag{4.30}$$

where $d_1 + d_2 + \cdots d_s = n$. Then the integer d_i is called the *algebraic multiplicity* of the eigenvalue λ_i, i.e., algebraic multiplicity of an eigenvalue is the multiplicity of λ_i as a root of the characteristic equation. Repeated eigenvalues are not always associated with a distinct eigenvector. For an eigenvalue λ_i with an algebraic multiplicity d_i, the number of linearly independent eigenvectors is equal to the dimension of the null space of $\mathbf{A}-\lambda\mathbf{I}$, and, is given by

$$q_i = n - \text{ rank } (\mathbf{A}-\lambda_i\mathbf{I}). \tag{4.31}$$

The integer q_i is called the degeneracy or *geometric multiplicity* of λ_i, in other words, geometric multiplicity of λ_i is the dimension of the eigenspace of $\mathbf{A}$. Clearly, geometric multiplicity of an eigenvalue cannot exceed the algebraic multiplicity of the eigenvalue; i.e., $1 \leq q_i \leq d_i$. In case a repeated eigenvalue λ_i does not have a full set of distinct eigenvectors, the number of generalized eigenvectors corresponding to λ_i is $d_i - q_i$. Thus, for an eigenvalue λ_i which is repeated d_i times, there is a total of d_i eigenvectors, regular and generalized. Summarizing the above, corresponding to each eigenvalue, λ_i, we have

$$\begin{aligned} \text{Number of regular eigenvectors: } q_i &= \text{Geometric Multiplicity of } \lambda_i \\ &= \text{Dimension of null space of } (\mathbf{A}-\lambda\mathbf{I}) \\ &= n - \text{ rank } (\mathbf{A}-\lambda_i\mathbf{I}) \\ \text{Number of generalized eigenvectors: } &= d_i - q_i, \quad d_i = \text{algebraic multiplicity} \end{aligned}$$

Computation of generalized eigenvectors can be carried out by solving the following chain of equations: Suppose the eigenvalue λ_r is repeated r times, and suppose the (regular) eigenvector $\mathbf{v}_r$ is computed. Then, in view of (4.29), the generalized eigenvectors satisfy

$$\begin{aligned}(\mathbf{A}-\lambda_r\mathbf{I})\mathbf{v}_{r+1} &= \mathbf{v}_r\\ (\mathbf{A}-\lambda_r\mathbf{I})\mathbf{v}_{r+2} &= \mathbf{v}_{r+1}\\ \vdots \qquad & \quad \vdots\\ (\mathbf{A}-\lambda_r\mathbf{I})\mathbf{v}_{k} &= \mathbf{v}_{k-1},\end{aligned}, \tag{4.32}$$

where the vectors $\{\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \cdots, \mathbf{v}_k\}$ are generalized eigenvectors. Note that for each equation given above there are multiple solutions consisting of a homogeneous solution and a particular solution; only particular solution is taken as the generalized eigenvector. Also by repeated substitution of the above equations, one can easily verify that equations (4.29).

Computation of eigenvectors and generalized eigenvectors actually depend on their number. For an eigenvalues λ_i, since $1 \leq q_i \leq d_i$, one may encounter three cases of eigenvectors/generalized eigenvectors.

Case 1, $q_i = 1$: For the case of geometric multiplicity is 1, there is only one (regular) eigenvector corresponding to λ_i so that there are $d_i - 1$ generalized eigenvectors. The eigenvector is computed by finding the null space of $(\mathbf{A}-\lambda_i\mathbf{I})$, and the generalized eigenvectors are computed using the chain of equations (4.32).

Case 2, $q_i = d_i$: If the geometric multiplicity is same as the algebraic multiplicity, there exists a full set of d_i eigenvectors corresponding to the eigenvalue λ_i. These eigenvectors are easily found from null space calculation of $(\mathbf{A}-\lambda_i\mathbf{I})$.

Case 3, $1 < q_i < d_i$: In this case there are q_i regular eigenvectors, and $d_i - q_i$ generalized eigenvectors. The generalized eigenvectors may be associated with one or more regular eigenvectors; however, it is difficult to specify an exact number of generalized eigenvectors that may be associated with a certain regular eigenvector. We illustrate the results using the following examples.

Example 4.5: Case 1

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0\\ 0 & 2 & 0\\ 1 & -1 & 2\end{bmatrix}. \tag{4.33}$$

It can be easily verified that the eigenvalues of the matrix are $\lambda = 2, 2, 2$, i.e., the eigenvalue $\lambda = 2$ has algebraic multiplicity 3. Next we determine the number of eigenvectors corresponding to $\lambda = 2$, using (4.31).

$$(\mathbf{A} - \lambda\mathbf{I})\big|_{\lambda=2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

which has rank $= 2$, so that the geometric multiplicity of $\lambda = 2$ is $q = 3 - 2 = 1$. Thus the number of (regular) eigenvectors corresponding to $\lambda = 2$ is 1. This also tells us that there are two generalized eigenvectors. The (regular) eigenvector is computed as

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The two generalized eigenvectors are computed by solving

$$(\mathbf{A} - \lambda\mathbf{I})\big|_{\lambda=2}\mathbf{v}_2 = \mathbf{v}_1$$
$$(\mathbf{A} - \lambda\mathbf{I})\big|_{\lambda=2}\mathbf{v}_3 = \mathbf{v}_2.$$

These are given by

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$\triangleright$

Note that in the above example, we have taken the particular solution only in computing the solution for $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$, and $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3 = \mathbf{v}_2$. Keeping in mind that these equations have infinitely many solutions, and any of those solutions may be taken as valid solutions for the generalized eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$ as long as these vectors are linearly independent with $\mathbf{v}_1$. *Give a formal proof of this statement. Hint: Use the fact that homogeneous solution of these equations is actually the regular eigenvector* $\mathbf{v}_1$.

Example 4.6: Case 2

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{4.34}$$

Then we compute

$$|\mathbf{A} - \lambda\mathbf{I}| = -(\lambda - 1)(\lambda - 2)(\lambda - 2).$$

Clearly the matrix $\mathbf{A}$ has one distinct eigenvalue $\lambda = 1$ and two repeated eigenvalues $\lambda = 2, 2$. We also verify that corresponding to $\lambda = 2$, we have $q = 2$ so that there are two regular eigenvectors (and no generalized eigenvectors). The eigenvectors are computed from

$$(\mathbf{A} - \lambda\mathbf{I})|_{\lambda=2}\mathbf{v} = 0,$$

and are given by

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvector corresponding to $\lambda = 1$ is a regular eigenvector, and is given by

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$\triangleright$

Can you find a 3×3 matrix that has eigenvalues $\{2, 2, 2\}$ with geometric multiplicity 3? For this matrix you must have three regular eigenvectors and no generalized eigenvectors. Answer this question in light of (4.31).

Example 4.7: Case 3

We consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The eigenvalues of this matrix are clearly $\lambda = 2, 2, 2$, and the algebraic multiplicity is 3. By (4.31), since $\mathrm{rank}(\mathbf{A} - \lambda\mathbf{I}) = 1$, geometric multiplicity is 2, and there are two regular eigenvectors corresponding to $\lambda = 2$ so that there are two regular eigenvectors. These eigenvectors are obtained by elementary row operations on the matrix $(\mathbf{A} - \lambda\mathbf{I})$. Indeed,

$$(\mathbf{A} - \lambda\mathbf{I})\big|_{\lambda=2} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives the two eigenvectors as

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}. \tag{4.35}$$

For computation of the generalized eigenvector, note that

$$(\mathbf{A} - \lambda\mathbf{I})\big|_{\lambda=2}\mathbf{v}_3 = \mathbf{v}_2, \tag{4.36}$$

must hold. Using the eigenvector $\mathbf{v}_2$ given in (4.35) it is easily verified that the above equation does not have a solution. This however is not a failure of the method for computation of the eigenvectors. Note that the nomenclature of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ given in (4.35) is rather arbitrary. We could have denoted these eigenvectors as

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{4.37}$$

Using $\mathbf{v}_2$ given above and solving (4.36), the generalized eigenvector $\mathbf{v}_3$ is obtained as

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\triangleright$

Before we close this section, it is worthwhile to mention that for a given matrix, eigenvectors and generalized eigenvectors are not unique. Since a regular eigenvector is the homogeneous solution of the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$, it is clear that any multiple of an eigenvector is also an eigenvector. Likewise, for the generalized eigenvectors, one may consider a linear combination of homogeneous solution and particular solution of the chain of equations that define the generalized eigenvectors.

Note also that *the set of eigenvectors and generalized eigenvectors are linearly independent, and forms a basis for the state space, $\mathcal{R}^n$.* This can be verified from the examples given above; however, we shall not pursue any analytical proof here; interested readers may consult [68] for the proof.

4.6 Jordan Canonical Form

In case a matrix cannot be diagonalized, we can at least transform it to a *nearly diagonal* form. For matrices with repeated eigenvalues, one such nearly diagonal form is known as the Jordan Canonical form. A Jordan canonical matrix has 1's in its upper diagonal along with the repeated eigenvalue along the main diagonal, such as

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}. \tag{4.38}$$

The actual structure of the Jordan canonical matrix depends on the algebraic multiplicity and the geometric multiplicity of the repeated eigenvalue. In general, for a matrix $\mathbf{A}$ with repeated eigenvalues there exists a nonsingular transformation $\mathbf{T}$ that transforms $\mathbf{A}$ to the Jordan canonical form with an appropriate number of Jordan blocks. For a matrix with eigenvalues $\lambda_i, i = 1, 2, \cdots, s$, with the corresponding algebraic multiplicities d_i, the Jordan canonical matrix has the structure

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{J}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{J}_s \end{bmatrix}, \tag{4.39}$$

where each $\mathbf{J}_i$ is a Jordan block of size $d_i \times d_i$. The actual structure of the various Jordan blocks depends on the geometric multiplicities of the various eigenvalues. As in the previous section, we shall consider three cases:

Case 1, $q_i = 1$: In this case there is one regular eigenvector corresponding to λ_i and the remaining $d_i - 1$ are generalized eigenvectors. Thus the Jordan block $\mathbf{J}_i$ is of size $d_i \times d_i$ and has the structure of (4.38).

Case 2, $q_i = d_i$: In this case there are d_i regular eigenvectors corresponding to λ_i and no generalized eigenvectors, so that $\mathbf{J}_i$ will have a complete diagonal structure of size $d_i \times d_i$. In this case, $\mathbf{J}_i$ is actually a diagonal matrix.

Case 3, $1 < q_i < d_i$: In this case there are q_i regular eigenvectors so that there will be q_i Jordan blocks which have the same eigenvalue λ_i. The actual size of these blocks will depend on the association of generalized eigenvectors with the regular eigenvectors.

We illustrate the results using the following examples:

Example 4.8: Case 1

Consider the matrix given in Example 4.5:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}. \tag{4.40}$$

As derived before, the eigenvalues are $\lambda = 2, 2, 2$ with algebraic multiplicity 3 and geometric multiplicity 1. Thus, there is only one (regular) eigenvector, $\mathbf{v}_1$, and two generalized eigenvectors, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then we form the transformation matrix

$$\mathbf{T} = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,] = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

and verify that

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}. \qquad \triangleright$$

It is important to note that the order of eigenvectors in the construction of the transformation matrix $\mathbf{T}$ is very critical. In this example, $\mathbf{v}_1$ is the regular eigenvector, and the generalized eigenvector $\mathbf{v}_2$ is associated with the eigenvector $\mathbf{v}_1$ through $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$, and the generalized eigenvector $\mathbf{v}_3$ is associated with $\mathbf{v}_2$ through $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3 = \mathbf{v}_2$. In order to obtain the Jordan canonical matrix, the transformation matrix $\mathbf{T}$ must be formed following the same sequence of dependence of eigenvectors, that is, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Transformation of the matrix $\mathbf{A}$ to the Jordan canonical form can be observed from the chain of equations that are used to compute the eigenvectors and the generalized eigenvectors. Indeed, the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ satisfy

$$\begin{aligned}(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 &= 0\\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1\\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3 &= \mathbf{v}_2,\end{aligned}$$

which can be rearranged as

$$\begin{aligned}\mathbf{A}\mathbf{v}_1 &= \lambda\mathbf{v}_1\\ A\mathbf{v}_2 &= \mathbf{v}_1 + \lambda\mathbf{v}_2\\ A\mathbf{v}_3 &= \mathbf{v}_2 + \lambda\mathbf{v}_3,\end{aligned}$$

or equivalently,

$$\mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}.$$

This gives $\mathbf{AT} = \mathbf{TJ}$ from which $\mathbf{J} = \mathbf{T}^{-1}\mathbf{AT}$ follows.

Example 4.9: Case 2

Consider the matrix given in Example 4.6.

$$\mathbf{A} = \begin{bmatrix}1 & 2 & 2\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}. \tag{4.41}$$

Here we have corresponding to $\lambda = 2$, algebraic multiplicity is 2 and geometric multiplicity is also 2 so that we have a complete set of two eigenvectors as shown in Example 6.6. The third eigenvector corresponding to $\lambda = 1$ is also easily computed. The transformation matrix is obtained as

$$\mathbf{T} = \begin{bmatrix}1 & 2 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The corresponding Jordan canonical matrix is

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}.$$

▷

Example 4.10: Case 3

Next we consider the matrix given in Example 4.7:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

In this case although the eigenvalues are $\lambda = 2, 2, 2$ with algebraic multiplicity 3 and geometric multiplicity 2. Thus, there are two (regular) eigenvectors, $\mathbf{v}_1$, and $\mathbf{v}_2$, and one generalized eigenvector, $\mathbf{v}_3$, as given in Example 6.7. Although the eigenvalues are same, in effect, we have two groups of eigenvalues, $\{\lambda = 2, 2\}$, and $\{\lambda = 2\}$, each group being associated with one regular eigenvector. Form the transformation matrix using $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ while keeping their natural order,

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then the Jordan canonical matrix has the form

$$\mathbf{J} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

The structure of the resulting Jordan canonical form depends on the arrangement of eigenvectors and generalized eigenvectors in the transformation matrix. In fact, the order in which the (regular) eigenvectors are listed is rather arbitrary, but care must be taken to list the generalized eigenvectors according to their their natural order. If we take the transformation matrix as

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

which gives rise to

$$\mathbf{J} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \qquad \triangleright$$

For matrices in the controllable canonical form, a simple alternative to the transformation matrix for finding the Jordan canonical form is the following:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ \lambda_1 & 1 & 0 & \cdots & \lambda_n \\ \lambda_1^2 & 2\lambda_1 & 2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & (n-1)\lambda_1^{n-2} & (n-1)(n-2)\lambda_1^{n-3} & \cdots & \lambda_n^{n-1} \end{bmatrix}.$$

Note that the second column is the differential of the first column, the third column is the differential of the second column, and so on. If there are distinct eigenvalues, the corresponding columns of $\mathbf{T}$ will have the same form as the first column. This is the modified Vandermonde matrix.

MATLAB uses a macro `jordan` in its symbolic toolbox to compute the generalized eigenvectors and the Jordan canonical matrix.

4.7 Left Eigenvector

Given a square matrix $\mathbf{A}$*, if there exists a scalar* λ *and a non-trivial vector* $\mathbf{w}$ *such that*

$$\mathbf{w}'\mathbf{A} = \lambda\mathbf{w}' \tag{4.42}$$

then λ *is an eigenvalue of* $\mathbf{A}$ *and* $\mathbf{w}$ *is the corresponding left eigenvector.* The reason for using the terminology *left* is quite clear once we recall that eigenvectors discussed in Chapter 6 multiplies the matrix $\mathbf{A}$ in the right side so that they may be more correctly called *right eigenvectors.* Right eigenvectors are commonly known as the eigenvectors of a matrix. It is also worthwhile to note that there are no left eigenvalues or right eigenvalues of a matrix; the eigenvalues are unique irrespective of whether they are associated with a right eigenvector or a left eigenvector.

Left eigenvectors can be easily computed with the help of right eigenvectors. Suppose the matrix $\mathbf{A}$ has all distinct eigenvalues. Denoting $\mathbf{T}$ as the matrix of (right) eigenvectors, $\mathbf{T} = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,]$, where $\mathbf{v}_i$ are the eigenvectors, we have

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \Lambda$$

where Λ is the diagonal matrix of distinct eigenvalues. Then we have

$$\mathbf{T}^{-1}\mathbf{A} = \Lambda\mathbf{T}^{-1}$$

Denote

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{w}_1' \\ \mathbf{w}_2' \\ \vdots \\ \mathbf{w}_n' \end{bmatrix}$$

Then it follows from the above that

$$\mathbf{w}_i'\mathbf{A} = \lambda_i\mathbf{w}_i', \qquad i = 1, 2, \cdots n$$

so that $\mathbf{w}_i$ is a left eigenvector of the matrix $\mathbf{A}$. In other words, left eigenvectors of a matrix are the rows of the inverse of the right eigenvector matrix. In addition, since $\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$, it is clear that

$$\langle \mathbf{v}_i, \mathbf{w}_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{4.43}$$

It is also easily seen that left eigenvectors of $\mathbf{A}$ are the (right) eigenvectors of the matrix $\mathbf{A}'$. In fact this clearly follows by taking transpose of the above equation. Also recall that eigenvalues of a matrix are invariant under transposition.

For matrices that have repeated eigenvalues, left generalized eigenvectors of a matrix can be introduced in a similar manner using generalized eigenvectors. Eigenstructure assignment method of control system design is based on left eigenvectors of the system matrix.

4.8 Eigenvalue Sensitivity

In engineering applications, the matrix $\mathbf{A}$ often represents the system structure, and its elements are functions of various physical parameters of the system. The system analyst then designs the system to ensure its stability. It is known that the eigenvalues of the system matrix determines the stability of the system. Variations in system parameters are not uncommon in practical systems, and can occur due to aging, imprecise fabrication, inaccuracies in parameter measurement, and various other reasons. Thus systems must be designed so that in addition to the requirement that the eigenvalues be at certain locations in the left half plane, they should be least sensitive to variations in system parameters. In what follows, we show that the sensitivity of eigenvalues is related to a certain property of the system eigenvector, commonly known as the condition number.

We begin with a simple example. Suppose the closed loop matrix of a dynamic system is given by

$$\mathbf{A} = \begin{bmatrix} 48 & -49 & -10 \\ 51 & -52 & -10 \\ 5 & -5 & -2 \end{bmatrix}$$

Clearly, the system is stable since its eigenvalues are $\lambda = \{-3, -2, -1\}$. Let's suppose that the parameter $\mathbf{A}(1,1) = 39$ has changed to $\mathbf{A}(1,1) = 49$, which is less than one percent variation of one of the entries of the matrix. One would expect that the eigenvalues of the new matrix should be close to the original set, i.e., $\{-3, -2-1\}$. A simple calculation quickly reveals that the eigenvalues of the new matrix are $\{-9.1328, 5.1528, -1.0200\}$, which is a drastic variation from the original values. If the matrix above represents a real physical system, it would become unstable. This is the reason why engineers should consider sensitivity of eigenvalues in system design.

With this introduction, we consider the sensitivity issues of the eigenvalues. Consider the matrix $\mathbf{A}$, which has an eigenvalue λ_i, the corresponding (right) eigenvector $\mathbf{v}_i$ and the left eigenvector $\mathbf{w}_i$. Suppose that there is a small perturbation in $\mathbf{A}$ by the amount $\Delta\mathbf{A}$, and let the eigenvalue and eigenvector of the perturbed matrix be $\lambda_i + \Delta\lambda_i$ and $\mathbf{v}_i + \Delta\mathbf{v}_i$, respectively. Then by definition, we have the following equations:

$$\begin{aligned} \mathbf{A}\mathbf{v}_i &= \lambda_i \mathbf{v}_i \\ \mathbf{w}_i'\mathbf{A} &= \lambda \mathbf{w}_i' \\ (\mathbf{A} + \Delta\mathbf{A})(\mathbf{v}_i + \Delta\mathbf{v}_i) &= (\lambda_i + \Delta\lambda_i)(\mathbf{v}_i + \Delta\mathbf{v}_i) \end{aligned} \tag{4.44}$$

Considering the first order perturbations in the last equation above, we have

$$\Delta\mathbf{A}\mathbf{v}_i + \mathbf{A}\Delta\mathbf{v}_i = \Delta\lambda\mathbf{v}_i + \lambda_i\Delta\mathbf{v}_i$$

Premultiplying by $\mathbf{w}_i'$ and simplifying, we obtain

$$\Delta\lambda_i = \frac{\mathbf{w}_i'(\Delta\mathbf{A})\mathbf{v}_i}{\mathbf{w}_i'\mathbf{v}_i} \tag{4.45}$$

Equation (4.45) shows the variation in the eigenvalue λ_i due to a variation in the matrix $\mathbf{A}$. This gives

$$\frac{\partial\lambda_i}{\partial\mathbf{A}} = \frac{\mathbf{w}_i\mathbf{v}_i'}{\mathbf{w}_i'\mathbf{v}_i} \tag{4.46}$$

which shows that variations of the eigenvalues λ_i with respect to variations of the elements of the matrix $\mathbf{A}$ can be described in terms of the right and left eigenvectors of the matrix $\mathbf{A}$. The above equation is also equivalent to

$$\left\|\frac{\Delta\lambda_i}{\Delta\mathbf{A}}\right\| \leq \frac{\|\mathbf{w}_i\|\|\mathbf{v}_i\|}{|\mathbf{w}_i'\mathbf{v}_i|} \tag{4.47}$$

The right hand side of the above equation is defined [50] as the eigenvalue sensitivity or the condition number for the eigenvalue λ_i:

$$\kappa_i = \frac{\|\mathbf{w}_i\|\|\mathbf{v}_i\|}{|\mathbf{w}_i'\mathbf{v}_i|} \tag{4.48}$$

The smallest condition number for any eigenvalue is clearly 1, which means that the eigenvalue will remain unchanged irrespective of any variations in the system parameters. It can be shown that

$$\max_i \ \text{cond}\ (\lambda_i) \leq \|\mathbf{V}\|\|\mathbf{V}^{-1}\| = \ \text{cond}\ (\mathbf{V}) \tag{4.49}$$

where $\mathbf{V}$ is the matrix of eigenvectors of the matrix $\mathbf{A}$. This Condition number of the eigenvector matrix provides a simple way of measuring sensitivity of the closed loop eigenvalues of the system.

Returning to the example given above, the condition number of the individual eigenvalues are computed as $\kappa_1 = 100.2547$, $\kappa_2 = 101.2472$, and $\kappa_3 = 14.1774$. This clearly shows why the eigenvalues $\lambda = -2$ and $\lambda = -3$ changed so severely for a small change in one of the parameters of the matrix $\mathbf{A}$. The eigenvalue $\lambda = -1$ didn't change since it has a relatively small condition number. Overall condition number of the eigenvector matrix is obtained as $\kappa = 246.05$, which is actually an estimated upper bound of all individual eigenvalue sensitivities.

Condition number of the eigenvector matrix can be interpreted in a different way. Note that the eigenvectors of a matrix are linearly independent, and form a basis set. Eigenvector matrices with a large condition number are characterized by nearly parallel eigenvectors. On the other hand if the eigenvectors are orthogonal to each other, the condition number will be one.

The best condition number of an eigenvector matrix is unity for which all eigenvalues of the system will remain unchanged for small variations in the system parameters. From a design perspective, this is what a designer should look for, although in practice it can seldom be achieved. Nevertheless, the designer should attempt to design a system so that the closed loop eigenvector matrix has a condition number as closed as possible to unity.

4.9 Spectral Decomposition

Let $\mathbf{A}$ be a $n \times n$ matrix and has distinct eigenvalues. Then as shown above the eigenvectors are linearly independent and span the entire space $\mathcal{R}^n$. Thus the eigenvectors can be conveniently taken as a basis for $\mathcal{R}^n$. Let the eigenvectors be denoted as $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. Then, as shown in equation (2.21), for an arbitrary vector $\mathbf{y} \in \mathcal{R}^n$, we have

$$\mathbf{y} = \sum_{i=1}^{n} \langle \mathbf{y}, \mathbf{r}_i \rangle \ \mathbf{v}_i,$$

where $\mathbf{r}_i$ is the reciprocal basis satisfying

$$\langle \mathbf{v}_i, \mathbf{r}_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{4.50}$$

Using the above equation, we then have

$$\begin{aligned} \mathbf{A}\mathbf{y} &= \sum_{i=1}^{n} \langle \mathbf{y}, \mathbf{r}_i \rangle \ \mathbf{A}\mathbf{v}_i \\ &= \sum_{i=1}^{n} \lambda_i \langle \mathbf{y}, \mathbf{r}_i \rangle \ \mathbf{v}_i \\ &= \sum_{i=1}^{n} \lambda_i \ \mathbf{v}_i \ \mathbf{r}_i' \ \mathbf{y}. \end{aligned}$$

This shows that the matrix $\mathbf{A}$ can be represented as

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \ \mathbf{v}_i \ \mathbf{r}_i', \tag{4.51}$$

which is known as the spectral representation of $\mathbf{A}$.

Calculation of reciprocal basis was discussed in Chapter 2. As shown in equation (2.23), the reciprocal basis $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_n$ can be computed by finding the inverse of the eigenvector matrix $\mathbf{V} = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n \,]$. In light of discussions in the previous section, we note that left eigenvectors of the matrix $\mathbf{A}$ form a reciprocal basis for the eigenvectors $\mathbf{v}_i$ so that the matrix $\mathbf{A}$ has the representation

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \ \mathbf{v}_i \ \mathbf{w}_i', \tag{4.52}$$

We shall see use spectral representation of $\mathbf{A}$ is analyzing the response of dynamic systems in the next chapter.

4.10 Functions of Matrices

It can be shown that the response of dynamic systems is described in terms of the exponential of the system matrix, $\mathbf{A}$, however, its computation is not a trivial task. Specifically, *the exponential of a matrix is not the matrix of exponential of individual entries of the matrix.* Computation of matrix exponential as well as many other functions of matrices can be carried out using a very powerful result, known as the Cayley-Hamilton theorem [68].

CAYLEY-HAMILTON THEOREM. *Every matrix satisfies its own characteristic equation, i.e., if the characteristic equation of the matrix* $\mathbf{A}$ *is given by*

$$\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0 = 0. \tag{4.53}$$

then

$$\mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + c_{n-2}\mathbf{A}^{n-2} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = 0. \tag{4.54}$$

We shall not discuss the proof of the Cayley-Hamilton theorem, rather, in what follows, we shall investigate its application for solving various functions of matrices.

Matrix Inverse

Suppose the inverse of the matrix $\mathbf{A}$ exists. Then premultiplying (4.54) by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{A}^{n-1} + c_{n-1}\mathbf{A}^{n-2} + c_{n-2}\mathbf{A}^{n-3} + \cdots + c_1\mathbf{I} + c_0\mathbf{A}^{-1} = 0.$$

Solving this equation we obtain the inverse

$$\mathbf{A}^{-1} = -\frac{1}{c_0}[\mathbf{A}^{n-1} + c_{n-1}\mathbf{A}^{n-2} + c_{n-2}\mathbf{A}^{n-3} + \cdots + c_1\mathbf{I}]. \tag{4.55}$$

This result is very significant in the sense that the inverse of a matrix is expressed in terms of various powers of the matrix. In general, matrix multiplication is a much simpler job compared to finding its inverse using Gaussian elimination or other methods.

Example 4.11

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which has the characteristic equation given by

$$\lambda^3 - 4\lambda^2 + 4\lambda + 1 = 0.$$

By Cayley-Hamilton theorem, we have

$$\mathbf{A}^3 - 4\mathbf{A}^2 + 4\mathbf{A} + \mathbf{I} = 0,$$

from which we obtain

$$\begin{aligned} \mathbf{A}^{-1} &= -\mathbf{A}^2 + 4\mathbf{A} - 4\mathbf{I} \\ &= -\begin{bmatrix} 1 & -2 & -4 \\ 3 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & -2 \\ 1 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 & -4 \\ 1 & -1 & 3 \\ -1 & 1 & -2 \end{bmatrix}. \end{aligned}$$

The result can be verified using any other method of computation of matrix inverse.

▷

Matrix Polynomial

It is clear from (4.54) that

$$\mathbf{A}^n = -[c_{n-1}\mathbf{A}^{n-1} + c_{n-2}\mathbf{A}^{n-2} + \cdots + c_1 A + c_0\mathbf{I}], \tag{4.56}$$

which shows that the matrix $\mathbf{A}^n$ can be expressed in terms of matrix powers up to $(n-1)$. Premultiplying the above equation by $\mathbf{A}$, we compute $\mathbf{A}^{n+1}$ as

$$\mathbf{A}^{n+1} = -[c_{n-1}\mathbf{A}^n + c_{n-2}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A}^2 + c_0\mathbf{A}].$$

Equation (4.56) can be substituted into the above equation to eliminate $\mathbf{A}^n$ giving

$$\mathbf{A}^{n+1} = (c_{n-1}^2 - c_{n-2})\mathbf{A}^{n-1} + (c_{n-1}c_{n-2} - c_{n-3})\mathbf{A}^{n-2} + \cdots + (c_1c_{n-1} - c_0)\mathbf{A} + c_0c_{n-1}\mathbf{I},$$

which again shows that $\mathbf{A}^{n+1}$ can also expressed in terms of matrix powers up to order $(n-1)$. We write the above equation as

$$\mathbf{A}^{n+1} = \beta_{n-1}\mathbf{A}^{n-1} + \beta_{n-2}\mathbf{A}^{n-2} + \cdots + \beta_1\mathbf{A} + \beta_0\mathbf{I}.$$

The same line of analysis can be carried out starting with the characteristic polynomial (4.53) so that the coefficients in the above equation will also satisfy

$$\lambda^{n+1} = \beta_{n-1}\lambda^{n-1} + \beta_{n-2}\lambda^{n-2} + \cdots + \beta_1\lambda + \beta_0.$$

In light of above discussion, it is clear that *all higher powers of the matrix* $\mathbf{A}$*, i.e.,* $\mathbf{A}^n$*,* $\mathbf{A}^{n+1}, \mathbf{A}^{n+2}, \cdots$ *can be expressed in terms of matrices* $\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \cdots, \mathbf{A}, \mathbf{I}$. Thus if $f(\mathbf{A})$ is a matrix polynomial, finite or convergent infinite, then it can be expressed as

$$f(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_{n-1}\mathbf{A}^{n-1}, \tag{4.57}$$

where the coefficients $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ satisfy

$$f(\lambda) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{n-1}\lambda^{n-1}. \tag{4.58}$$

Since the $n \times n$ matrix $\mathbf{A}$ has exactly n eigenvalues, equation (4.58) represents a set of n algebraic equations which can be solved for the coefficients $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$. Substitution of these coefficients into (4.57) then evaluates the polynomial $f(\mathbf{A})$. We illustrate the result by the following example:

Example 4.12

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix},$$

and suppose that the polynomial

$$f(\mathbf{A}) = \mathbf{A}^{10} + \mathbf{I},$$

needs to be evaluated. A simple (but time consuming approach) to solve this problem is to multiply the matrix $\mathbf{A}$ ten times to evaluate $\mathbf{A}^{10}$ and add it with the rest of the terms of the polynomial. Alternately, since in this problem, $n = 3$, the size of the matrix, using the result presented above we write

$$f(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2, \tag{4.59}$$

where the coefficients satisfy

$$f(\lambda) = \lambda^{10} + 1 = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2. \tag{4.60}$$

The eigenvalues of the matrix $\mathbf{A}$ are $\lambda = -1, 0, 1$. Using these eigenvalues in (4.60) we obtain

$$\begin{aligned} \alpha_0 - \alpha_1 + \alpha_2 &= 2 \\ \alpha_0 &= 1 \\ \alpha_0 + \alpha_1 + \alpha_2 &= 2. \end{aligned}$$

These equations are solved simultaneously to obtain

$$\alpha_0 = 1$$
$$\alpha_1 = 0$$
$$\alpha_2 = 1.$$

Using these values in (4.59) we obtain

$$\mathbf{A}^{10} + \mathbf{I} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} + \alpha_2 \mathbf{A}^2$$
$$= \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}.$$

▷

Matrix Exponential

It is known that the exponential of a scalar can be expressed as an infinite series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

Similarly the matrix exponential can also be expressed as an infinite but convergent series given by

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots.$$

Since all powers of $\mathbf{A}^k, k \geq n$, can be expressed in terms of powers of $\mathbf{A}$ up to $n-1$, it follows from the above equation that

$$e^{\mathbf{A}} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} + \alpha_2 \mathbf{A}^2 + \cdots + \alpha_{n-1} \mathbf{A}^{n-1}, \tag{4.61}$$

where the coefficients satisfy

$$e^{\lambda} = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots + \alpha_{n-1} \lambda^{n-1}. \tag{4.62}$$

As in the previous case, these coefficients can be computed using the eigenvalues of the matrix $\mathbf{A}$.

Example 4.13

In order to compute the exponential of the matrix given in Example 6.12, we write

$$e^{\mathbf{A}} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} + \alpha_2 \mathbf{A}^2, \tag{4.63}$$

where

$$e^{\lambda} = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2. \tag{4.64}$$

Using the eigenvalues of $\mathbf{A}$, we have

$$\begin{aligned} \alpha_0 - \alpha_1 + \alpha_2 &= e^{-1} \\ \alpha_0 &= 1 \\ \alpha_0 + \alpha_1 + \alpha_2 &= e. \end{aligned}$$

These equations are solved simultaneously to obtain

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= -\frac{1}{2}e^{-1} + \frac{1}{2}e^1 \\ \alpha_2 &= \frac{1}{2}e^{-1} + \frac{1}{2}e^1 - 1. \end{aligned}$$

Using these values in (4.61) we obtain

$$\begin{aligned} e^{\mathbf{A}} &= \begin{bmatrix} \alpha_0 + \alpha_1 + \alpha_2 & \alpha_1 + \alpha_2 & -2\alpha_1 + 3\alpha_2 \\ 0 & \alpha_0 & 3\alpha_1 - 3\alpha_2 \\ 0 & 0 & \alpha_0 - \alpha_1 + \alpha_2 \end{bmatrix} \\ &= \begin{bmatrix} e & e-1 & -3 + \frac{5}{2}e^{-1} + \frac{1}{2}e \\ 0 & 1 & 3 - 3e^{-1} \\ 0 & 0 & e^{-1} \end{bmatrix}. \end{aligned}$$

Note that the exponential of a matrix is not necessarily same as the term by term exponential of its entries. However equality holds for diagonal matrices. ▷

If a matrix has repeated eigenvalues, equations (4.64) will not represent n independent equations. In that case additional equations are obtained by repeated differentiation of (4.64) with respect to the repeated eigenvalue. Suppose the eigenvalue λ_i is repeated m times. Then m algebraic equations are obtained from

$$\frac{d^k}{d\lambda_i^k} e_i^{\lambda} = \frac{d^k}{d\lambda_i^k}\left[\alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \cdots + \alpha_{n-1}\lambda_i^{n-1}\right], \qquad k = 0, 1, 2, \cdots m-1.$$

This is illustrated by the following example.

Example 4.14

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix},$$

whose eigenvalues are $\lambda = 0, 1, 1$. In order to compute the matrix exponential, $e^{\mathbf{A}}$, we use the equations (4.63) and (4.64). Substituting $\lambda = 0$ and $\lambda = 1$ into (4.64), we obtain

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_0 + \alpha_1 + \alpha_2 &= e, \end{aligned} \tag{4.65}$$

Differentiating (4.64) with respect to λ we have

$$e^{\lambda} = \alpha_1 + 2\alpha_2\lambda,$$

in which we again substitute $\lambda = 1$ corresponding to the repeated eigenvalue. This gives

$$e = \alpha_1 + 2\alpha_2. \tag{4.66}$$

Solving equations (4.65) and (4.66), we obtain

$$\begin{aligned}\alpha_0 &= 1\\ \alpha_1 &= e - 2\\ \alpha_2 &= 1.\end{aligned}$$

Substituting these results into (4.63),

$$e^{\mathbf{A}} = \begin{bmatrix} e & e-1 & -2e+3 \\ 0 & 1 & 3(e-1) \\ 0 & 0 & e \end{bmatrix}.$$

▷

Computation of matrix exponential can be done using the macro `expm` under MATLAB. Note that there is a similar macro `exp` which computes the exponential of a scalar. In this regard if $\mathbf{A}$ is a matrix, the `exp(A)` will compute the exponential of individual entries of the matrix which is not equal to $e^{\mathbf{A}}$. However the exponential of a diagonal matrix is the diagonal matrix of exponential of its diagonal entries.

Computation of matrix functions is not necessarily limited to the examples given above. In fact, in the next chapter we shall we that the response of a dynamic system can be expressed using the matrix $e^{\mathbf{A}t}$ where $\mathbf{A}$ is the system matrix and t is time. This will be further explored in the next chapter.

Additional Examples

Example 4.15

Find the eigenvalues, eigenvectors, and generalized eigenvectors (if any) of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Clearly the eigenvalues are $\lambda = 1, 1, 1$. Also we have

$$(\mathbf{A} - \lambda\mathbf{I})\Big|_{\lambda=1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This shows that the two regular eigenvectors can be taken as

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Now to find the generalized eigenvector, we solve

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3 = \mathbf{v}_2$$

This equation is easily seen to be inconsistent so that there is no solution. We also observe that there is no solution of the above equation even if we interchange the nomenclature of $\mathbf{v}_1$ and $\mathbf{v}_2$. This is however not a failure of the method of computation of generalized eigenvectors. The method discussed earlier in this chapter is just one of the many methods that are available for computation of eigenvectors and generalized eigenvectors. What is needed is find any two vectors that satisfy the following equations:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = 0$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3 = \mathbf{v}_2$$

Rewriting these equations as

$$\begin{bmatrix} (\mathbf{A} - \lambda\mathbf{I}) & 0 \\ -\mathbf{I} & (\mathbf{A} - \lambda\mathbf{I}) \end{bmatrix} \begin{bmatrix} \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = 0$$

The eigenvector and generalized eigenvector pair is then obtained by finding the null space of the coefficient matrix. This gives

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

One can easily obtain the Jordan canonical matrices using

$$\mathbf{T} = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,]$$

or

$$\mathbf{T} = [\,\mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_1\,]$$

4.11 References

1. Gantmacher, F.R., *Theory of Matrices*, Chelsea, 1959.
2. Gloub, Gene and C. F. van Loan, *Matrix Computation*, Johns Hopkins Press, 1989.
3. Strang, G., *Linear Algebra and its Applications*, Academic Press, 1980.
4. Halmos, P. R., *Finite Dimensional Vector Spaces*, van Nostrand, 1958.
5. Kailath T., *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.
6. Brogan, W., *Modern Control Theory*, Prentice Hall, 1991.
7. Bellman, R., *Introduction to Matrix Analysis*, McGraw Hill, New York, 1960.

4.12 Exercise

4.1 Show that the eigenvectors of a symmetric matrix are orthogonal. The vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ are said to be orthogonal if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$.

4.2 Consider the matrix in the controllable canonical form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix}$$

Show that the characteristic equation of this matrix is

$$\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0 = 0 \tag{4.67}$$

4.3 Consider the matrix in the observable canonical form

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ \vdots & \vdots & \vdots & \ddots & 0 & -c_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

Show that the characteristic equation of this matrix is

$$\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0 = 0 \tag{4.68}$$

4.4 Consider the matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & 0 \\ -\mathbf{Q} & -\mathbf{A}' \end{bmatrix}$$

where $\mathbf{A}$ and $\mathbf{Q}$ are square matrices. This matrix is known as the Hamiltonian matrix in control system theory. Show that the eigenvalues of $\mathbf{M}$ are symmetrically located with respect to the imaginary axis.

4.5 Find the eigenvalues and the corresponding eigenvectors for the following matrices. Also find a transformation matrix for diagonalization of the given matrices.

$$\begin{bmatrix} -1 & -3 & 3 \\ -2 & -1 & 2 \\ -2 & -3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} -2 & -1 & -3 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2.5 & -0.5 & 0.5 \\ 1 & 2.5 & -1 \\ 1 & -0.5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -3 & -2 \\ 8 & -2 & -8 \\ 4 & -3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ 1 & 2 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

4.6 Find the eigenvalues, eigenvectors, and the generalized eigenvectors of the following matrices. Express these matrices in the Jordan canonical forms. Does there exist any alternative Jordan canonical forms? If there exists one, find the new Jordan canonical matrix. If there does not exist one, explain why.

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & -1 & 2 \\ 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 & -2 \\ 3 & 3 & 1 & -2 \\ -1 & 1 & 3 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix},$$

4.7 Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Using Cayley-Hamilton theorem, find the matrix $\mathbf{B} = \sqrt{\mathbf{A}}$. Also compute the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{B}$, and verify that $\lambda_{\mathbf{B}} = \sqrt{\lambda_{\mathbf{A}}}$.

4.8 For the matrix given above, find the function $\mathbf{B} = \sin(\mathbf{A})$. Also verify that $\lambda_{\mathbf{B}} = \sin(\lambda_{\mathbf{A}})$.

4.9 Compute matrix exponential $e^{\mathbf{A}}$ for the following matrices:

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.10 Using the Cayley-Hamilton theorem, compute the inverse of the following matrices:

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.11 Show that the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{A}'$ are identical, but the eigenvectors are different. Find a relation between the two sets of eigenvectors.

4.12 Suppose $\mathbf{T}$ is a transformation matrices that diagonalizes the matrix $\mathbf{A}$, i.e., $\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. Show that $\mathbf{A}^k = \mathbf{T}\mathbf{D}^k\mathbf{T}^{-1}$. Note that in this method the computation of matrix power becomes relatively simple since $\mathbf{D}^k$ is easily computed. Note: The same concept holds for any other matrix function as well.

4.13 Find the eigenvalues and the eigenvectors of the matrix $\mathbf{A}^n$ where n is arbitrary.

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

4.14 Show that if λ is an eigenvalue of $\mathbf{A}$, then $f(\lambda)$ is an eigenvalue of the matrix $f(\mathbf{A})$, where f is an analytic function of its argument. Hint: Use Cayley-Hamilton theorem and the basic definition of eigenvalue problem.

4.15 Show that for a matrix $\mathbf{A}$ with eigenvalues $\lambda_i, i = 1, 2, \cdots n$, show that

$$|\mathbf{A}| = \Pi_{i=1}^{n} \lambda_i$$

$$\mathrm{Tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$$

4.16 Suppose $\mathbf{A}$ be an $n \times n$ matrix whose eigenvalues are not necessarily distinct. Show that the set of eigenvectors and generalized eigenvectors forms a basis for the space $\mathcal{R}^n$.

4.17 Find the eigenvalues and eigenvectors of the following matrices:

$$\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

4.18 Suppose $\mathbf{A}$ be a block diagonal matrix given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{M}_1 & 0 & 0 \\ 0 & \mathbf{M}_2 & 0 \\ 0 & 0 & \mathbf{M}_3 \end{bmatrix}$$

where the diagonal blocks are square matrices. Show that the eigenvalues of $\mathbf{A}$ are those the matrices $\mathbf{M}_1, \mathbf{M}_2$ and $\mathbf{M}_3$.

4.19 Suppose the eigenvalues of a matrix $\mathbf{A}$ are $\sigma_1, \sigma_2 + i\,\omega_2, \sigma_2 - i\,\omega_2$, and $\mathbf{v}_1, \mathbf{v}_{2R} + i\mathbf{v}_{2I}, \mathbf{v}_2 - i\mathbf{v}_{2I}$ are the corresponding eigenvectors. Show that the transformation matrix

$$\mathbf{T} = [\,\mathbf{v}_1 \quad \mathbf{v}_{2R} \quad \mathbf{v}_{2I}\,]$$

transforms the given matrix to

$$\mathbf{B} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & \omega_2 \\ 0 & -\omega_2 & \sigma_2 \end{bmatrix}$$

Hint: Note that

$$[\,\mathbf{v}_1 \quad \mathbf{v}_{2R} + i\mathbf{v}_{2I} \quad \mathbf{v}_{2R} - i\mathbf{v}_{2I}\,] = [\,\mathbf{v}_1 \quad \mathbf{v}_{2R} \quad \mathbf{v}_{2I}\,] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix}$$

4.20 Show that transfer function of a system is invariant under similarity transformation.

4.21 Let λ be an eigenvalue of the matrix A. Then show that λ^γ is an eigenvalue of the matrix A^γ, where γ is not necessarily an integer.