

ENGR 5022 Engineering Analysis

Course Notes

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Abstract

These notes were taken during the 2012-2013 academic year for Temple University's ENGR 5022 Engineering Analysis course. The outline follows the given course notes as well as the following additional sources: "Fourier Series and Orthogonal Functions" by Harry F. Davis, "Linear Algebra Done Right" by Sheldon Axler, and "Advanced Linear Algebra" by Steven Roman. Propositions and theorems are supplied by the source material while the proofs were written by me with the aid of the source material only as a last result.

1. Linear Spaces

Proposition 1.1. *Every finite-dimensional inner-product space has an orthonormal basis.*

Proof:

□

Proposition 1.2. *Every orthonormal list of vectors in V can be extended to an orthonormal basis in V .*

Proof:

□

Proposition 1.3. *Suppose $T \in \mathcal{L}(V)$. If T has an upper-triangular matrix with respect to some basis of V then T has an upper triangular matrix with respect to some orthonormal basis of V*

Proof:

□

Proposition 1.4. *Schur's Theorem: Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper triangular matrix with respect to some orthonormal basis of V*

Proof:

□

1.1. Orthogonal Projections and Minimization Problems

1.2. Linear Functionals and Adjoint

2. Orthogonal Functions

The central idea of this section is the following. We are given a sequence of functions $f_1(x), f_2(x), \dots$ which have the following property:

$$\int_a^b f_i(x) \overline{f_j(x)} dx = 0$$

for all $i \neq j$. We desire to expand an arbitrary function $F(x)$ in an infinite series

$$F(x) = c_1 f_1(x) + c_2 f_2(x) + \dots$$

2.1. Inner Products

Definition 2.1. An inner product of a linear space is a function of pairs of elements of the space, that is, to each pair f and g in the linear space there is a number $\langle f|g \rangle$ satisfying the following axioms:

- (1) $\langle f|f \rangle \geq 0$ for every f (Positivity)
 - (2) $\langle f|f \rangle = 0 \Leftrightarrow f = 0$ (Definiteness)
 - (3) $\langle f|g \rangle = \overline{\langle g|f \rangle}$ for every pair f and g (Conjugate Symmetry)
 - (4) $\langle \alpha f + \beta g|h \rangle = \alpha \langle f|h \rangle + \beta \langle g|h \rangle$ (Additivity and Homogeneity in the First Slot)
- where α and β are arbitrary numbers.

Definition 2.2. An inner product space is a vector space V along with an inner product on V .

Definition 2.3. The norm is defined as

$$\|f\| = \langle f|f \rangle^{\frac{1}{2}} \quad (1)$$

Proposition 2.1. Properties of the norm:

- (1) $\|f\| \geq 0$
- (2) $\|f\| = 0 \Leftrightarrow f = 0$
- (3) $\|\alpha f\| = |\alpha| \|f\|$
- (4) $\|f + g\| \leq \|f\| + \|g\|$

Proof:

□

Proposition 2.2. Cauchy - Schwarz Inequality:

$$|\langle f, g \rangle| \leq \|f\| \|g\| \quad (2)$$

Proof:

□

Proposition 2.3. Minkowski Inequality:

$$\|f + g\| \leq \|f\| + \|g\| \quad (3)$$

Proof:

□

Proposition 2.4. Triangle Inequality:

$$\|f - g\| \leq \|f - h\| + \|h - g\| \quad (4)$$

Proof:

□

If we drop the requirement that $\langle f, g \rangle = 0$ only when $f = 0$ then we call $\langle f, g \rangle$ a *pseudo-inner product* and $\|f\|$ a *pseudo-norm*.

Examples of inner products:

- (1) In \mathbb{R}^n we have $\langle x, y \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$
- (2) In \mathbb{C}^n we have $\langle x, y \rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \dots + x_n\bar{y}_n$
- (3) In the space of all continuous functions of period 2π we have

$$\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx$$

- (4) Let f and g be bounded, integrable functions of x , and let r be a fixed function, defined and continuous in the interval $a \leq x \leq b$, having the property $r(x) > 0$ whenever $a < x < b$. Then the inner product with respect to the weight function r is defined as

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}r(x)dx$$

Definition 2.4. Two elements f and g of a vector space are called *orthogonal* if $\langle f, g \rangle = 0$.

2.2. Orthogonal Sequences

In this section V is an inner product space.

Definition 2.5. Let ϕ be an element of V having unit norm. For an f of class V the projection of f in the direction of ϕ is denoted $\text{proj}(f|\phi)$ and is defined by

$$\text{proj}(f : \phi) = \langle f, \phi \rangle \phi$$

Theorem 2.1. The projection of f in the direction of ϕ is equal to $f \Leftrightarrow f$ is a scalar multiple of ϕ

Proof: If $\text{proj}(f : \phi) = f$ then by definition we have $f = \langle f, \phi \rangle \phi$, i.e. a scalar multiple of ϕ . If $f = \alpha\phi$ then $\text{proj}(f : \phi) = \langle f, \phi \rangle \phi = \langle \alpha\phi, \phi \rangle \phi = \alpha\langle \phi, \phi \rangle \phi = \alpha\phi = f$. □

Theorem 2.2. $\text{proj}(f : \phi)$ is orthonormal to ϕ

Proof:

□

Let us assume we are given n mutually orthogonal elements of V , all having unit norm: ϕ_1, \dots, ϕ_n , where $\langle \phi_i, \phi_j \rangle = 0$ for all $i \neq j$ and $\|\phi_j\| = 1$ for all j . These n elements of V span a subspace of V which we denote H_n .

Definition 2.6. The projection of f into the subspace H_n is defined by

$$\text{proj}(f : \phi_1, \phi_2, \dots, \phi_n) = \langle f, \phi_1 \rangle \phi_1 + \dots + \langle f, \phi_n \rangle \phi_n \quad (5)$$

2.3. Differential Operators

2.4. Integral Operators

2.5. Convolution and Dirichlet Kernel

3. Fourier Series

3.1. Definitions and Examples

3.2. Local Convergence of Fourier Series

3.3. Uniform Convergence

3.4. Convergence of Fourier Series

3.5. Divergent Series

3.6. Generalized Functions

4. Legendre Polynomials and Bessel Functions

4.1. Partial Differential Equations

4.2. The Laplacian Operator

4.3. Legendre Polynomials

4.4. Laplace's Equation in Spherical Coordinates

4.5. Spherical Harmonics

4.6. Bessel Functions

5. The Fourier Integral

6. Algebraic Concepts in Analysis

6.1. Functions on Groups

7. The Eigenvalue Problem

Given a square matrix A of size $n \times n$, if there exists a scalar λ and a non trivial n -vector \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v} \tag{6}$$

then λ is called an *eigenvalue* of the matrix \mathbf{A} and \mathbf{v} is called an *eigenvector* of \mathbf{A} corresponding to the eigenvalue λ . We note that from the above, the vector \mathbf{v} is invariant under the transformation \mathbf{A} . In view of this equation, for λ to be an eigenvalue, it is necessary that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \quad (7)$$

be satisfied for a nonzero vector \mathbf{v} . Equation (7) has a nonzero solution if and only if the determinant of the coefficient matrix is zero, i.e.,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \quad (8)$$

(7) also shows that the vector \mathbf{v} must belong to the null space of the matrix $(\mathbf{A} - \lambda\mathbf{I})$ and therefore the eigenvectors can be computed by finding the null space of $(\mathbf{A} - \lambda\mathbf{I})$.

We note that a matrix can have a nonempty null space only if it is singular, i.e. its determinant is zero. Equation (8) can be expanded as follows

$$|\mathbf{A} - \lambda\mathbf{I}| = (-1)^n [\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0] = 0, \quad (9)$$

where the coefficients c_0, c_1, \dots, c_{n-1} are computed from the matrix. We see that (9) is a polynomial of degree n and thus has n solutions. Thus, we have n eigenvalues of the matrix \mathbf{A} . Also, corresponding to each eigenvalue, there exists an element in the null space of $(\mathbf{A} - \lambda\mathbf{I})$. Therefore, we have, for an $n \times n$ matrix, exactly n eigenvalues but not necessarily a full set of n eigenvectors.

Since any multiple of an element in the null space is also an element of the null space, we have that *any multiple of an eigenvector is also an eigenvector*. This can be easily seen from (6).

For matrices with real entries, *complex eigenvalues always appear in conjugate pairs*. Similarly, the eigenvectors corresponding to complex eigenvalues (of real matrices) will always occur in conjugate pairs.

7.1. Properties of Eigenvalues and Eigenvectors

Proposition 7.1. *Proof by induction. Base Case ($k = 2$): If λ is an eigenvalue of a matrix \mathbf{A} , then λ^k is an eigenvalue of the matrix \mathbf{A}^k , where k is any integer positive or negative.*

Proof: First consider $k \in \mathbb{Z}^+$. By definition we have λ satisfying

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

Multiplying the above by \mathbf{A} results in

$$\mathbf{A}^2\mathbf{v} = \lambda\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v},$$

which shows that λ^2 is an eigenvalue of \mathbf{A}^2 .

Now assume the statement holds for $k = n$. Thus we have

$$\mathbf{A}^n\mathbf{v} = \lambda^n\mathbf{v}.$$

Again, multiply both sides by \mathbf{A} , which results in

$$\mathbf{A}^{n+1}\mathbf{v} = \mathbf{A}\mathbf{A}^n\mathbf{v} = \mathbf{A}\lambda^n\mathbf{v} = \lambda^n\mathbf{A}\mathbf{v} = \lambda^{n+1}\mathbf{v}.$$

The same sets can show the result holds for $k < 0$. □

Proposition 7.2. *If the characteristic equation of a matrix A is given by (9) then the determinant of the matrix is*

$$\det(A) = (-1)^n c_0 \quad (10)$$

Proof: Set $\lambda = 0$ in (9) and the result follows. \square

Proposition 7.3. *The matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of the matrix.*

Proof: If A is not invertible then $\det(A)=0$. Thus by (10) we have $c_0 = 0$. Setting $c_0 = 0$ in (9) we see that the characteristic equation equals zero when $\lambda = 0$. \square

Proposition 7.4. *The eigenvectors corresponding to the distinct eigenvalues are linearly independent.*

Proof: We prove this by contradiction. Let λ_i , for $i = 1, \dots, n$ be distinct eigenvalues of a matrix A with the corresponding eigenvectors \mathbf{v}_i . Assume that the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, with $k < n$, are linearly independent. Thus, $\mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ are linearly dependent. Therefore, the following equation holds

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k + c_{k+1} \mathbf{v}_{k+1} = 0 \quad (11)$$

for some (not all zero) constants c_1, \dots, c_{k+1} . Next we multiply the above by A and use the fact that

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \dots, A\mathbf{v}_{k+1} = \lambda_{k+1} \mathbf{v}_{k+1}.$$

This results in

$$c_1 \lambda_1 \mathbf{v}_1 + \dots + c_{k+1} \lambda_{k+1} \mathbf{v}_{k+1} = 0. \quad (12)$$

Finally, we multiply (11) by λ_{k+1} and subtract from (12) which results in

$$c_1(\lambda_1 - \lambda_{k+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{k+1})\mathbf{v}_2 + \dots + c_k(\lambda_k - \lambda_{k+1})\mathbf{v}_k = 0. \quad (13)$$

By assumption we have the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent and thus the coefficients in (13) must all be equal to zero. Since each eigenvalue is distinct it follows that the coefficients $c_1 = c_2 = \dots = c_k = 0$. From (12) it follows that

$$c_{k+1} \mathbf{v}_{k+1} = 0.$$

With \mathbf{v}_{k+1} an eigenvector, it is nonzero, and thus $c_{k+1} = 0$, which is a contradiction since we assume the coefficients c_1, \dots, c_k, c_{k+1} are not all zero. \square

7.2. Similarity Transformation

Definition 7.1. *A matrix B is said to be similar to the matrix A if there exists a nonsingular transformation matrix T such that*

$$B = T^{-1}AT \quad (14)$$

and the matrix T is called the similarity transformation

Proposition 7.5. *The eigenvalues of a matrix are invariant under similarity transformation, i.e., if A and B are similar matrices then they have the same eigenvalues.*

Proof: We use the characteristic equation for \mathbf{B} :

$$\begin{aligned}
0 &= |\mathbf{B} - \lambda_i \mathbf{I}| \\
&= |\mathbf{T}^{-1} \mathbf{A} \mathbf{T} - \lambda_i \mathbf{I}| \\
&= |\mathbf{T}^{-1} \mathbf{A} \mathbf{T} - \lambda_i \mathbf{T}^{-1} \mathbf{T}| \\
&= |\mathbf{T}^{-1} (\mathbf{A} - \lambda_i \mathbf{I}) \mathbf{T}| \\
&= |\mathbf{T}^{-1}| |\mathbf{A} - \lambda_i \mathbf{I}| |\mathbf{T}| \\
&= |\mathbf{A} - \lambda_i \mathbf{I}|
\end{aligned}$$

since $|\mathbf{T}^{-1}| = \frac{1}{|\mathbf{T}|}$ □

Next we find the relation between the eigenvectors of similar matrices. Supposed \mathbf{v}_i is an eigenvector of \mathbf{B} corresponding to the eigenvalue λ_i . Thus, we have

$$\mathbf{B} \mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

Using $\mathbf{B} = \mathbf{T}^{-1} \mathbf{A} \mathbf{T}$ results in

$$\mathbf{T}^{-1} \mathbf{A} \mathbf{T} \mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

We premultiply both sides of the above equation by \mathbf{T}

$$\mathbf{A} \mathbf{T} \mathbf{v}_i = \lambda_i \mathbf{T} \mathbf{v}_i.$$

Denote $\mathbf{u}_i = \mathbf{T} \mathbf{v}_i$, which results in

$$\mathbf{A} \mathbf{u}_i = \lambda_i \mathbf{u}_i,$$

which says that $\mathbf{u}_i = \mathbf{T} \mathbf{v}_i$ is an eigenvector of \mathbf{A} .

The choice of a similarity transformation for a given system is not arbitrary. The diagonal matrix is a useful form of similar matrix, which is introduced in the next section.

7.3. Diagonalization of Matrices

Given an arbitrary matrix \mathbf{A} we would like to find a similarity transformation \mathbf{T} such that

$$\mathbf{T}^{-1} \mathbf{A} \mathbf{T} = \mathbf{D} \tag{15}$$

where \mathbf{D} is a diagonal matrix. If there exists a transformation \mathbf{T} for which (15) holds, we say that the matrix \mathbf{A} is diagonalizable.

Proposition 7.6. *Suppose the eigenvalues of the $n \times n$ matrix \mathbf{A} are all distinct. Then \mathbf{A} is diagonalizable using the transformation matrix*

$$\mathbf{T} = [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_n], \tag{16}$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are the eigenvectors of \mathbf{A} . Furthermore, the diagonal entries of the matrix $\mathbf{D} = \mathbf{T}^{-1} \mathbf{A} \mathbf{T}$ are the eigenvalues of the matrix \mathbf{A} .

Proof: First, we note that \mathbf{T} is a nonsingular matrix since the eigenvectors are linearly independent. Next let the eigenvalues of the matrix \mathbf{A} be $\{\lambda_1, \dots, \lambda_n\}$, which are all distinct, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be the corresponding eigenvectors satisfying $\mathbf{A} \mathbf{v}_i = \lambda_i \mathbf{v}_i$, for $i = 1, \dots, n$. Then we have

$$\mathbf{A} \mathbf{T} = \mathbf{A} [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_n]$$

$$\begin{aligned}
&= [\mathbf{A}\mathbf{v}_1 \mathbf{A}\mathbf{v}_2 \cdots \mathbf{A}\mathbf{v}_n] \\
&= [\lambda_1 \mathbf{v}_1 \lambda_2 \mathbf{v}_2 \cdots \lambda_n \mathbf{v}_n] \\
&= [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\
&= \mathbf{T}\mathbf{D}.
\end{aligned}$$

Thus we have $\mathbf{AT} = \mathbf{TD}$, from which it follows that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$. Note also that since \mathbf{T} is nonsingular it follows that \mathbf{T}^{-1} exists \square

The ordering of the eigenvectors is not important in diagonalizing a matrix. In case the eigenvalues of a matrix are complex, the transformation matrix \mathbf{T} will also be complex, and the corresponding diagonal matrix \mathbf{D} will have complex entries on its diagonal.

In case the eigenvalues of a matrix are repeated, there may not exist a full set of linearly independent eigenvectors. These matrices cannot be diagonalized. However, if there exist a full set of linearly independent eigenvectors for repeated eigenvalues, then the matrix is diagonalizable.

If a matrix is in the controllable canonical form and has distinct eigenvalues, a simple alternative to the transformation matrix for diagonalization is the following

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}. \quad (17)$$

The matrix in (17) is known as the **Vandermond matrix**. This transformation matrix is simple to construct since only the eigenvalues of the matrix are needed.

7.4. Generalized Eigenvectors

7.5. Jordan Canonical Form

7.6. Left Eigenvector

7.7. Eigenvalue Sensitivity

7.8. Spectral Decomposition

7.9. Functions of Matrices

8. System Response

9. Partial Differential Equations

Exercise 9.1. Show that the eigenvectors of a symmetric matrix are orthogonal.

Proof:

□

Exercise 9.2. Consider the matrix in the controllable canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \dots & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

Show that the characteristic equation of this matrix is

$$\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda_1 = 0.$$

Proof:

□

Exercise 9.3. Consider the matrix in the observable canonical form

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & 0 & \dots & 0 & -c_1 \\ \vdots & \vdots & \vdots & \ddots & 0 & -c_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}.$$

Show that the characteristic equation of this matrix is

$$\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda_1 = 0.$$

Proof:

□

Exercise 9.4. Consider the matrix

$$M = \begin{bmatrix} A & \mathbf{0} \\ -Q & -A' \end{bmatrix}$$

where A and Q are square matrices. This matrix is known as the Hamiltonian matrix in control system theory. Show that the eigenvalues of M are symmetrically located with respect to the imaginary axis

Proof:

□

Exercise 9.5.

Proof:

□

Exercise 9.6.

Proof:

□

Exercise 9.7. Given the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}.$$

Using Cayley-Hamilton theorem, find the matrix $\mathbf{B} = \sqrt{\mathbf{A}}$. Also compute the eigenvalues of the matrices \mathbf{A} and \mathbf{B} and verify that $\lambda_{\mathbf{B}} = \sqrt{\lambda_{\mathbf{A}}}$.

Proof:

□

Exercise 9.8. For the matrix given above, find the function $\mathbf{B} = \sin(\mathbf{A})$. Also verify that $\lambda_{\mathbf{B}} = \sin(\lambda_{\mathbf{A}})$.

Proof:

□

Exercise 9.9. Compute the matrix exponential $e^{\mathbf{A}}$ for the following matrices:

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Proof:

□

Exercise 9.10. Using the Cayley Hamilton theorem compute the inverse of the following matrices:

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Proof:

□

Exercise 9.11. Show that the eigenvalues of the matrices \mathbf{A} and \mathbf{A}' are identical but the eigenvectors are different. Find a relation between the two sets of eigenvectors.

Proof:

□

Exercise 9.12. Supposed \mathbf{T} is a transformation matrix that diagonalizes the matrix \mathbf{A} , i.e. $\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. Show that $\mathbf{A}^k = \mathbf{T}\mathbf{D}^k\mathbf{T}^{-1}$. Note that in this method the computation of matrix power becomes relatively simple since \mathbf{D}^k is easily computed. Note: The same concept holds for any other matrix function as well.

Proof:

□

Exercise 9.13. Find the eigenvalues and eigenvectors of the matrix \mathbf{A}^n where n is arbitrary

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Proof:

□

Exercise 9.14. Show that if λ is an eigenvalue of \mathbf{A} then $f(\lambda)$ is an eigenvalue of the matrix $f(\mathbf{A})$, where f is analytic function of its argument. Hint: Use Cayley-Hamilton theorem and the basic definition of eigenvalue problem.

Proof:

□

Exercise 9.15. Show that for a matrix \mathbf{A} with eigenvalues λ_i , $i = 1, 2, \dots, n$, show that

$$|\mathbf{A}| = \prod_{i=1}^n \lambda_i$$
$$\text{Tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$$

Proof:

□

Exercise 9.16. Suppose \mathbf{A} is an $n \times n$ matrix whose eigenvalues are not necessarily distinct. Show that the set of eigenvectors and generalized eigenvectors form a basis for the space \mathbb{R}^n .

Proof:

□

Exercise 9.17. Find the eigenvalues and eigenvectors for the following matrices:

$$\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Proof:

□

Exercise 9.18. Suppose \mathbf{A} be a block diagonal matrix given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{M}_1 & 0 & 0 \\ 0 & \mathbf{M}_2 & 0 \\ 0 & 0 & \mathbf{M}_3 \end{bmatrix}$$

where the diagonal blocks are square matrices. Show that the eigenvalues of \mathbf{A} are those matrices \mathbf{M}_1 , \mathbf{M}_2 , and \mathbf{M}_3

Proof:

□

Exercise 9.19.

Proof:

□

Exercise 9.20. Show that the transfer function of a system is invariant under similarity transformation.

Proof:

□

Exercise 9.21. *Let λ be an eigenvalue of the matrix \mathbf{A} . Then show that λ^ϕ is an eigenvalue of the matrix \mathbf{A}^ϕ , where ϕ is not necessarily an integer.*

Proof:

□