# 7.
# Partial Differential Equations

Linear operators theory plays a significant role in the analysis of dynamic systems governed by partial differential equations. Systems that require partial differential equations as dynamic model include bending of a beam, propagation of electromagnetic waves, heat conduction through a medium, etc. These types of systems can be effectively described using linear operators over certain function spaces much like the way finite dimensional systems are described using matrices. In this section we investigate fundamentals of linear operator theory, and their applications in the analysis of dynamic systems.

## 7.1 Linear Operator

An operator is a mapping, defined as, $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ that assigns each element form the space $\mathcal{U}$ to a corresponding element in the function space $\mathcal{V}$.

$$v = \mathcal{A}(u), \qquad u \in \mathcal{U}$$

The space on which the operator $\mathcal{A}$ operates is called the domain space, which may not be all of $\mathcal{U}$, i.e., $D_{\mathcal{A}} \subset \mathcal{U}$. The range space is defined as

$$R_{\mathcal{A}} = \{v : v = \mathcal{A}u \text{ for some } u \in D_{\mathcal{A}}\}$$

Consdier the Laplacian operator

$$\mathcal{A}\varphi = -\Delta\varphi \tag{7.1}$$

For simplicity we shall consider the one dimensional problem so that

$$\mathcal{A}\varphi = -\frac{\partial^2 \varphi}{\partial x^2} \tag{7.2}$$

with the boundary conditions $\varphi(0) = 0$ and $\varphi(\ell) = 0$. The underlying space is $L_2(0, \ell)$. The domain of the operator $\mathcal{A}$ is defined as

$$D_{\mathcal{A}} = \{\varphi \in C^2(0, \ell) : \varphi(0) = 0, \varphi(\ell) = 0\} \tag{7.3}$$

The domain $D_{\mathcal{A}} \subset L_2(0, \ell)$, and its range is $R_{\mathcal{A}} = C(0, \ell)$.

An operator $\mathcal{A}$ is linear if

$$\mathcal{A}(\alpha u + \beta v) = \alpha \mathcal{A}(u) + \beta \mathcal{A}(v)$$

The Laplacian operator given above is linear. To prove this statement, note that

$$\begin{aligned} \mathcal{A}(\alpha\varphi + \beta\psi) &= -\frac{\partial^2}{\partial x^2}(\alpha\varphi + \beta\psi) \\ &= -\alpha\frac{\partial^2\varphi}{\partial x^2} - \beta\frac{\partial^2\psi}{\partial x^2} \\ &= \alpha\mathcal{A}\varphi + \beta\mathcal{A}\psi \end{aligned}$$

We consider another example. Define an integral operator

$$\mathcal{A}u = \int_0^t u(\tau)\, d\tau$$

for $0 \le t \le T$. The domain $D_{\mathcal{A}}$ of this opeator is taken as $L_2(0, T)$, and its range is given by

$$R_{\mathcal{A}} = \{v \in C(0, T) : \mathcal{A}(u) = v, u \in L_2(0, T)\}$$

One can easily verify that the integral operator defined above is also linear. We take another example. Consider the integral operator

$$\mathcal{A}u = \int_0^x u^2(\xi)\, d\xi$$

with domain $L_2(0, \ell)$. It is easily verified that this is a nonlinear operator.

## Bounded Operator

An operator $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ is said to be bounded if it maps bounded sets in $\mathcal{U}$ to bounded sets in $\mathcal{V}$, i.e.,

$$\|u\|_{\mathcal{U}} \le c_1 \qquad \text{implies} \qquad \|\mathcal{A}u\|_{\mathcal{V}} \le c_2$$

This is equivalent to this next statement. A linear operator $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ is said to be bounded if there exists a nonegative real number $c > 0$ such that

$$\|\mathcal{A}u\|_{\mathcal{V}} \le c\|u\|_{\mathcal{U}}$$

where $u \in D_{\mathcal{A}} \subset \mathcal{U}$. The smallest value of $c$ for which the above condition is satisfied is the norm of the operator.

$$\|\mathcal{A}\| = \sup_{u \neq 0} \frac{\|\mathcal{A}u\|}{\|u\|}$$

This also gives

$$\|\mathcal{A}u\| \leq \|\mathcal{A}\| \, \|u\|$$

A linear operator $\mathcal{A}$ is said to be continuous if

$$\lim_{n \to \infty} \|\mathcal{A}u^n - \mathcal{A}u\| = 0 \text{ whenever } \lim_{n \to \infty} \|u^n - u\| = 0$$

A linear operator is continuous in and only if it is bounded. Furthermore, if an operator is continuous at 0, then it is continuous on the entire domain.

The integral operator deined above is bounded, and is proved as follows:

$$\begin{aligned}
\|\mathcal{A}u\|^2 &= \int_0^T \left( \int_0^t u(\tau) \, d\tau \right)^2 dt \\
&\leq \int_0^T \big( \int_0^t 1^2 \, d\tau \big) \big( \int_0^t u^2(\tau) \, d\tau \big) \, dt \\
&\leq \int_0^T \big( \int_0^T 1^2 \, d\tau \big) \big( \int_0^T u^2(\tau) \, d\tau \big) \, dt \\
&\leq T^2 \|u\|^2
\end{aligned}$$

which shows that

$$\|\mathcal{A}u\| \leq T \, \|u\|$$

Thus $\mathcal{A}$ is a bounded linear operator, and $\|\mathcal{A}\| = T$.

Next we consider the Laplacian operator given earlier. Consider a sequence

$$\varphi_n = \frac{\sqrt{2}}{n\pi^2} \sin n\pi x$$

with norm

$$\begin{aligned}
\|\varphi_n\| &= \left[ \int_0^1 \left( \frac{\sqrt{2}}{n\pi} \sin n\pi x \right)^2 dx \right]^{\frac{1}{2}} \\
&= \frac{1}{n\pi}
\end{aligned}$$

This gives

$$\mathcal{A}\varphi_n = -\sqrt{2} n \sin n\pi x$$

and

$$\begin{aligned}
\|\mathcal{A}\varphi_n\| &= \left[ \int_0^1 \big( -\sqrt{2} n \sin n\pi x \big)^2 \, dx \right]^{\frac{1}{2}} \\
&= n\pi
\end{aligned}$$

This shows that as $n \to \infty$, the sequence $\|\varphi_n\| \to 0$, but $\|\varphi_n\| \to \infty$, so that $\mathcal{A}$ is an unbounded operator. Furthermore, $\mathcal{A}$ is not a continuous operator.

## Positive Operator

An operator $\mathcal{A}$ in a Hilbert space $L_2$ is said to be positive if

$$\langle \mathcal{A}u, u\rangle \geq 0$$

for all $u \in D_{\mathcal{A}} \subset L_2$, and $\mathcal{A}$ is strictly positive if

$$\langle \mathcal{A}u, u\rangle > 0$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2$. For the Laplacian operator, using the basic definition of scalar product and integrating by parts, we have

$$\begin{aligned}\langle \mathcal{A}\varphi, \varphi\rangle &= -\int_0^\ell \frac{\partial^2 \varphi}{\partial x^2}\varphi \, dx \\ &= -\frac{\partial \varphi}{\partial x}\varphi\Big|_0^\ell + \int_0^\ell |\frac{\partial \varphi}{\partial x}|^2 \, dx \\ &\geq 0\end{aligned}$$

where we have used the given boundary conditions. Thus the Laplacian operator is a positive operator.

## Symmetric Operator

An operator $\mathcal{A} \in L_2$ is said to be symmetric if

$$\langle \mathcal{A}\varphi, \psi\rangle = \langle \varphi, \mathcal{A}\psi\rangle$$

where $\varphi, \psi \in \mathcal{D}(\mathcal{A})$.

The Laplacian operator is a symmetric operator. Indeed, starting with the basic definition, and integration by parts,

$$\begin{aligned}\langle \mathcal{A}\varphi, \psi\rangle &= \int_0^\ell -\frac{\partial^2 \varphi}{\partial x^2}\psi \, dx \\ &= -\frac{\partial \varphi}{\partial \psi}\Big|_0^\ell + \int_0^\ell \frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x} \, dx \\ &= \varphi\frac{\partial \psi}{\partial x}\Big|_0^\ell - \int_0^\ell \varphi\frac{\partial^2 \psi}{\partial x^2} \, dx \\ &= \langle \varphi, \mathcal{A}\psi\rangle\end{aligned}$$

## 7.2 Eigenvalue Problem

The operator eigenvalue problem is defined in the same way as in the case of matrices. Given an operator in a Hilbert space $H$, if there exists a scalar $\lambda$ and a nontrivial $\varphi \in H$ satisfying

$$\mathcal{A}\varphi = \lambda\varphi$$

then $\lambda$ is an eigenvalue of $\mathcal{A}$, and $\varphi$ is the corresponding eigenfunction.

Because of its importance in engineering applications, we shall study the eigenvalue problem of the Laplacian operator. Consider the Laplacian operator

$$\mathcal{A}\varphi = -\frac{\partial^2\varphi}{\partial x^2}$$

with domain

$$D_{\mathcal{A}} = \{\varphi \in C^2(0,\ell) : \varphi(0) = 0,\, \varphi(\ell) = 0\}$$

Then general solution of the eigenvalue problem is given by

$$\varphi(x) = c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x)$$

Using the boundary conditions, we clearly obtain

$$\varphi(0) = 0 = c_2$$

and

$$\varphi(\ell) = c_1 \sin(\sqrt{\lambda}\ell) = 0$$

so that

$$\sin(\sqrt{\lambda}\ell) = 0 = \sin(k\pi), \qquad k = 0, 1, 2, \cdots$$

This gives us a sequence of infinitely many eigenvalues for the operator as

$$\lambda_k = \frac{k^2\pi^2}{\ell^2}, \qquad k = 0, 1, 2, \cdots$$

The corresponding eigenfunctions are obtained from the above as

$$\varphi_k(x) = c_1 \sin(\frac{k\pi}{\ell}x), \qquad k = 1, 2, \cdots$$

Note that we have discarded the first solution of the eigenvalue problem since the corresponding eigenfunction since $\lambda = 0$ leads to the trivial solution $\varphi = 0$.

The constant $c_1$ can be chosen to normalize the eigenfunctions. For example, taking the norm of $\varphi_k(x)$, we obtain

$$\begin{aligned}\|\varphi_k\|^2 &= \int_0^\ell c_1^2 \sin^2(\frac{k\pi}{\ell}x)\, dx \\ &= c_1^2 \frac{\ell}{2}\end{aligned}$$

Thus taking $c_1 = \sqrt{\frac{2}{\ell}}$, we obtain the normalized eigenfunctions as

$$\varphi_k(x) = \sqrt{\frac{2}{\ell}} \sin(\frac{k\pi}{\ell})x, \qquad k = 1, 2, \cdots \tag{7.4}$$

It is also known that these eigenfunctions are orthonormal. This can also be proved directly using the basic definition of the eigenvalue problem. Let $\lambda_m$ and $\lambda_n$ be two different eigenvalues for the opearator $\mathcal{A}$. Then we have

$$\begin{aligned} \mathcal{A}\varphi_m &= \lambda_m \varphi_m \\ \mathcal{A}\varphi_n &= \lambda_n \varphi_n \end{aligned}$$

Scalar multiplying (in $L_2$) the first equation by $\varphi_n$, and the second equation by $\varphi_m$, and taking the difference, we obtain

$$\begin{aligned} \langle \lambda_m \varphi_m, \varphi_n \rangle - \langle \lambda_n \varphi_n, \varphi_n \rangle &= \langle \mathcal{A}\varphi_m, \varphi_n \rangle - \langle \mathcal{A}\varphi_n, \varphi_m \rangle \\ (\lambda_m - \lambda_n)\langle \varphi_m, \varphi_n \rangle &= \langle \mathcal{A}\varphi_m, \varphi_n \rangle - \langle \mathcal{A}\varphi_n, \varphi_m \rangle \end{aligned}$$

Integrating the first term by parts in the right hand side of the above equation, we obtain

$$\begin{aligned} (\lambda_m - \lambda_n)\langle \varphi_m, \varphi_n \rangle &= -\frac{\partial \varphi_m}{\partial x}\varphi_n \Big|_0^\ell + \varphi_m \frac{\partial \varphi_n}{\partial x}\Big|_0^\ell + \langle \varphi_m, \mathcal{A}\varphi_n \rangle - \langle \mathcal{A}\varphi_n, \varphi_m \rangle \\ &= 0 \end{aligned}$$

This shows that the eigenfunctions are orthonormal.

Thus we observe that the Laplacian operator has an infinite sequence of eigenfunctions that span the space $L_2(0, \ell)$. Consequently, any arbitrary function in $L_2$ can be expressed in a Fourier series

$$f(x) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \tag{7.5}$$

The concept developed above can be extended to similar eigenvalue problems in multiple spatial domains. In these cases, the eigenvalues and the eigenfunctions are certainly going to be different from those computed above.

## 7.3 Applications

### Example 7.1: Electric Field Problem

Consider a capacitor-type device with a non-unoform charge distribution between the two plates. We know that electric field in the region between the plates is given by the Poisson's equation

$$-\frac{d^2V}{dx^2} = \frac{\rho}{\varepsilon} \tag{7.6}$$

For brevity, we denote $\frac{\rho}{\varepsilon} = f$, which is a function of $x$. Let's also assume that $V(0) = 0$ and $V(\ell) = 0$. Then the field distribution between the two plates is obtained by solving the equation

$$\begin{aligned} -\frac{d^2V}{dx^2} &= f \\ V(0) = 0, \qquad & V(\ell) = 0 \end{aligned}$$

Assume that the field distribution can be expanded by an infinite series

$$V(x) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \tag{7.7}$$

where $a_k$ are some unknown constants, and the expansion functions, $\varphi_k$ satisty the eigenvalue problem

$$\begin{aligned} -\frac{d^2\varphi_k}{dx^2} &= \lambda_k \varphi_k \\ \varphi_k(0) = 0, \qquad & \varphi_k(\ell) = 0 \end{aligned} \tag{7.8}$$

We define the operator $\mathbf{A}$ as

$$\begin{aligned} \mathbf{A}\varphi &= -\frac{\partial^2\varphi}{\partial x^2} \\ \varphi(0) = 0, \qquad & \varphi(\ell) = 0 \end{aligned}$$

in $L_2(0, \ell)$ and the domain as defined earlier.

Substituting the solution (7.7) in the equation (7.6), we obtain

$$\sum_{k=0}^{\infty} a_k \mathbf{A}\varphi_k = f$$

using (7.8) and scalar multiplying the above equation by $\varphi_n$, we obtain

$$\begin{aligned} \langle \sum_{k=0}^{\infty} a_k \lambda_k \varphi_k, \varphi_n \rangle &= \langle f, \varphi_n \rangle \\ \sum_{k=0}^{\infty} a_k \lambda_k \langle \varphi_k, \varphi_n \rangle &= \langle f, \varphi_n \rangle \end{aligned}$$

Since the eigenfunctions are orthonormal, the above equation simplifies to

$$\begin{aligned} \lambda_n a_n &= \langle f, \varphi_n \rangle \\ &= \int_0^\ell f(x) \sqrt{\frac{2}{\ell}} \sin \frac{n\pi}{\ell} x \, dx \end{aligned} \tag{7.9}$$

Once the expansion coefficients, $a_n$, are computed, the complete solution of the voltage distribution is obtained from (7.7).

As an example consider

$$f(x) = \sin \frac{\pi}{\ell} x$$

Then using (7.9), we compute

$$a_1 = \sqrt{\frac{\ell}{2}} \frac{\ell^2}{\pi^2}$$

and $a_2 = a_3 = \cdots = 0$ This gives the voltage distribution as

$$\begin{aligned} V(x) &= a_1 \varphi_1(x) \\ &= \frac{\ell^2}{\pi^2} \sin \frac{\pi}{\ell} x \end{aligned}$$

The same answer could also be calculated using (7.6).

## Example 7.2: Heat conduction problem

Consider a steel rod with two ends immersed in an ice bath. The temperature distribution of the rod is described by the equation

$$\begin{aligned} &\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < \ell \\ &T(t, 0) = 0, \qquad T(t, \ell) = 0 \end{aligned} \tag{7.10}$$

In the above equation $T$ denotes the temperature as a function of position $x$, and time $t$, and the various constants are $\rho$ is the mass density, $c_p$ is the specific heat, and $k$ is the thermal conductivity. Let's us assume that at time $t = 0$, local heat is applied to the rod, and then the heat source is removed. Let the initial temperature distribution of the rod is as follows:

$$T(0, x) = g(x) = \begin{cases} T_0 & 0.4\ell \le x \le 0.6\ell \\ 0 & \text{elsewhere} \end{cases}$$

Since the heat source is removed, we expect that the initial heat energy will slowly dissipate to the entire length of the rod, and eventually the entire rod will come to zero temperature; note that the two ends of the rod are in an ice bath.

To simplify the analysis, we shall consider only the normalized equation

$$\begin{aligned} &\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \\ &T(t,0) = 0, \qquad T(t,\ell) = 0, \qquad \text{boundary condition} \\ &T(0,x) = 0, \qquad \text{initial condition} \end{aligned} \tag{7.11}$$

We shall seek the solution of the form

$$T(t,x) = Y(t)X(x) \tag{7.12}$$

where $Y$ is only a function of $t$ and $X$ is only a function of $x$. Substituting this into the equation (7.11), we obtain

$$\dot{Y}X = YX''$$

where, for simplicity, we have used dot for the time derivative and prime for derivative with respect to $x$. Dividing both side of the above equation by $YX$, we obtain

$$\frac{\dot{Y}}{Y} = \frac{X''}{X}$$

Since the left hand side of the above equation is only a function of $t$ and the right hand side is only a function of $X$, the only way the above equality could hold if the two sides are equal to a constant value independent of both $t$ and $x$. Thus we denote

$$\frac{\dot{Y}}{Y} = \frac{X''}{X} = \beta \tag{7.13}$$

This gives

$$\dot{Y} = \beta Y$$

and

$$X'' = \beta X$$

At this point we don't know the constant $\beta$. Nevertheless, we can make some initial investigation. Suppose, $\beta$ is positive. This would mean that

$$Y(t) = c_0 e^{\beta t}$$

where $c_0$ is a constant. Since $T(t,x) = Y(t)X(x)$, this would mean that the temperature of the rod will increase to infinity as time increases, which is impossible. Suppose, $\beta = 0$. Then the temperature of the rod will stay constant for all time, which is also impossible. Suppose $\beta$ is negative. Then clearly, the temperature will decrease with time. Thus we take

$$\frac{\dot{Y}}{Y} = \frac{X''}{X} = -\gamma^2 \tag{7.14}$$

This gives the temperature distribution

$$T(t,x) = Y(t)X(x) \tag{7.15}$$

where $Y$ is given by

$$Y(t) = c_0 e^{-\gamma^2 t}$$

and the function $X(x)$ satisfies the equation

$$-\frac{d^2X}{dx^2} = \gamma^2 X$$

The boundary conditions for the above equation are also easily obtained from (7.11)

$$T(t,0) = 0 = Y(t)X(0)$$

so that $X(0) = 0$. Similarly, one obtains $X(\ell) = 0$. This gives us the eigenvalue problem

$$\begin{aligned} -\frac{d^2X}{dx^2} &= \gamma^2 X \\ X(0) = 0, \qquad & X(\ell) = 0 \end{aligned}$$

The solution of the eigenvalue problem has been computed above. It has been shown that there are infinitely many solutions

$$X_n(x) = \sqrt{\frac{2}{\ell}} \sin \frac{n\pi}{\ell} x, \qquad n = 1, 2, \cdots \infty \tag{7.16}$$

and

$$\gamma_n = \frac{n\pi}{\ell}$$

Since there are many solutions, the complete general solution is obtained as

$$\begin{aligned} T(t,x) &= \sum_{n=1}^{\infty} Y_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{\ell}}\, e^{-\frac{n^2\pi^2}{\ell^2}t}\, \sin \frac{n\pi}{\ell} x \end{aligned} \tag{7.17}$$

where the constants $c_n$ are yet to be determined.

To find the constants $c_n$, we use the initial condition

$$\begin{aligned} T(0,x) = g(x) &= \sum_{n=1}^{\infty} Y_n(0) X_n(x) \\ &= \sum_{n=1}^{\infty} c_n X_n(x) \end{aligned}$$

Scvalar multiplying the above equation in $L_2$ by $X_m$, and using the fact that the eigenfunctions are orthonormal, we obtain

$$\begin{aligned}\langle g, X_m\rangle &= \langle \sum_{n=1}^{\infty} c_n X_n, X_m\rangle \\ &= c_m\end{aligned}$$

Thus the coefficients, $c_n$ are computed using the above equation, i.e.,

$$c_n = \int_0^{\ell} g(x)\sqrt{\frac{2}{\ell}}\sin(\frac{n\pi}{\ell}x)\,dx$$

This completes the analysis of temperature distribution in the rod.

## Example 7.3: Heat Conduction

Consider the heat conduction problem discussed above, and suppose there is a heating coil wrapped around the steel rod. Let's also assume that the current in the coil is the control variable. We would like to investigate the temperature distribution along the length of the rod.

The problem can be formally represented as

$$\begin{aligned}&\rho c_p\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + f_1(x)u(t), \qquad 0 < x < \ell \\ &T(t,0) = 0, \qquad T(t,\ell) = 0\end{aligned} \tag{7.18}$$

where $u(t)$ is the current, and $f_1(x)$ represents the heat generation due to the current.

Again, for simplicity we shall consider the normalized problem given by

$$\begin{aligned}&\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + f(x)u(t), \qquad 0 < x < \ell \\ &T(t,0) = 0, \qquad T(t,\ell) = 0\end{aligned} \tag{7.19}$$

Suppose the rod was at zero degrees at time $t = 0$.

Following the method discussed in the previous example, let the temperature distribution be desctibed by

$$T(t,x) = \sum_{n=1}^{\infty} Y_n(t)X_n(x) \tag{7.20}$$

where $X_n(x)$ are the mode shape functions satisfying the eigenvalue problem

$$\begin{aligned}&-\frac{d^2X_n}{dx^2} = \gamma_n^2 X_n \\ &X_n(0) = 0, \qquad X_n(\ell) = 0\end{aligned} \tag{7.21}$$

Then substituting the above expression into the equation (7.19), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \dot{Y}_n(t) X_n(x) &= \sum_{n=1}^{\infty} Y_n(t) X_n''(x) + f(x)u(t) \\ &= -\sum_{n=1}^{\infty} Y_n(t)\gamma_n X_n(x) + f(x)u(t)\end{aligned}$$

Scalar multiplying the above equation by $X_m$, and using the fact that the eigenfunctions are orthonormal, the above equation reduces to

$$\dot{Y}_m(t) = -\gamma_m Y_m(t) + \langle f, X_m \rangle u(t) \tag{7.22}$$

The initial condition also follows from the initial condition of the original problem, and is given by

$$Y_n(0) = 0$$

The solution of (7.22) is obtained as

$$Y_m(t) = e^{-\gamma_m t} Y_m(0) + \int_0^t e^{-\gamma_m (t-\tau)} b_m u(\tau)\, d\tau \tag{7.23}$$

where $b_m = \langle f, X_m \rangle$.

The complete solution of the problem is then obtained by substituting the above solution into the general expansion (7.20).

## Example 7.4: Vibration Problem

The lateral vibration of a vibrating string is described by

$$\begin{aligned}&\rho \frac{\partial^2 y}{\partial t^2} = S \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < \ell \\ &y(t,0) = 0, \qquad y(t,\ell) = 0\end{aligned} \tag{7.24}$$

where the lateral displacement $y$ is a function of time $t$ and distance $x$ along the length of the string, and $S$ is the tension on the string, and $\rho$ is the mass density. The two ends of the string are assumed to be tightly fixed so that they cannot vibrate. For simplicity, we assume that $\rho = 1$ and $S = 1$. The string is assumed to be plucked at the middle at $t = 0$ so that we have

$$y(0,x) = f(x), \qquad \frac{\partial y}{\partial t}(0,x) = 0 \tag{7.25}$$

Following the method of separation of variables, we take

$$y(t,x) = T(t)X(x) \tag{7.26}$$

Substituting the above expression into the differential equation, we obatin

$$\frac{\ddot{T}}{T} = \frac{X''}{X} = -\gamma^2 \tag{7.27}$$

from which we obtain

$$X(x) = c_1 \sin \gamma x + c_2 \cos \gamma x \tag{7.28}$$

and

$$T(t) = a_1 \sin \gamma t + a_2 \cos \gamma t \tag{7.29}$$

Using the boundary conditions (7.24), we obtain

$$X(0) = 0 = c_2$$

and

$$X(\ell) = 0 = \sin \gamma \ell$$

so that we have infinitely many solutions for $\gamma$,

$$\gamma_n = \frac{n\pi}{\ell}$$

The complete solution of the vibration problem is thus obtained as

$$y(t,x) = \sum_{n=1}^{\infty} (p_n \sin \gamma_n t + q_n \cos \gamma_n t) \sin \gamma_n x \tag{7.30}$$

The constants $p_n$ and $q_n$ are computed using the initial conditions for the string, i,e, (7.25). This gives

$$y(0,x) = f(x) = \sum_{n=1}^{\infty} q_n \sin \gamma_n x$$

Scalar multiplying the above equation by $\sin \gamma_m$, and using the orthogonality condition, we have

$$q_m = \langle f, \sin \gamma_m x \rangle$$

Similarly, differentiating equation (7.30) with respect to $t$,

$$\frac{\partial y}{\partial t}(t,x) = \sum_{n=1}^{\infty} (\gamma_n p_n \cos \gamma_n t - q_n \gamma_n \sin \gamma_n t) \sin \gamma_n x \tag{7.31}$$

Again, using the boundary conditions,

$$0 = \sum_{n=1}^{\infty} (\gamma_n p_n) \sin \gamma_n x$$

Scalar multiplying by $\sin \gamma_m x$, and using the orthogonality condition, we have

$$p_n = 0$$

for all $n$.

The complete solution of the vibration problem is thus obtained as

$$y(t,x) = \sum_{n=1}^{\infty} q_n \cos \gamma_n t \sin \gamma_n x \tag{7.32}$$

This completes the analysis.

### Example 7.5: Vibrating Beam

The figure shows the schematic of an infinitesimal segment of a beam. Let $L$ be the length of the beam, $x$, the spatial coordinate, and $y(x,t)$, the transverse displacement of the beam as a function of time $t$ and axial position $x$. Let $\rho$ denote the mass density, $E$ the Young's modulus, $I$, the moment of area, and $f$ be a distributed force applied to the beam.

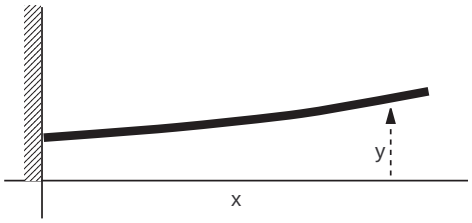


Figure 7.1 A Simple Beam

The dynamic model for transverse vibrations of a cantilever beam is given by

$$\rho\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = f, \tag{7.33}$$

with the boundary conditions

$$\begin{aligned} y(0,t) &= 0 \\ \frac{\partial y}{\partial x}(0,t) &= 0 \\ \frac{\partial^2 y}{\partial x^2}(L,t) &= 0 \\ \frac{\partial^3 y}{\partial x^3}(L,t) &= 0, \end{aligned} \tag{7.34}$$

and the initial conditions

$$\begin{aligned} y(x,0) &= g(x) \\ \frac{\partial y}{\partial t} &= h(x). \end{aligned} \tag{7.35}$$

The system model derived above is a typical example for a class of systems known as distributed parameter system in the literature. There is a big collection of books and articles in the literature on control theory for this class of systems. The subject matter requires strong understanding of mathematical analysis which is beyond the scope of this discussion. Instead, in what follows, we shall approximate the dynamic model (7.33) in terms of a finite dimensional model, which is popular among engineering professionals. This analysis is based on the concept of generalized Fourier series discussed in Section 5.2.

We shall assume that the response $y(\cdot,t) \in L_2(0,L)$ and express the response $y(x,t)$ as an infinite sum

$$y(x,t) = \sum_{i=0}^{\infty} y_i(t)\varphi_i(x), \tag{7.36}$$

where $\varphi_i, i = 1,2,\cdots$, is a basis for $L_2(0,L)$. In engineering literature these functions are known as mode shape functions, and $y_i(t)$ as the mode amplitudes. The eigenfunction or mode shape functions satisfy the equation

$$\begin{aligned} &\frac{\partial^4\varphi_i}{\partial x^4} = \beta^4\varphi_i \\ &\varphi_i(0) = 0, \qquad \frac{\partial^2\varphi_i}{\partial x^2}(L) = 0 \\ &\frac{\partial\varphi_i}{\partial x}(0) = 0, \qquad \frac{\partial^3\varphi_i}{\partial x^3}(L) = 0. \end{aligned} \tag{7.37}$$

In mathematical terms, this is an eigenvalue problem for the operator

$$\mathcal{A}\varphi = \frac{\partial^4\varphi}{\partial x^4}, \tag{7.38}$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to L_2(0,L)$ with the domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \big\{\varphi : \varphi, \varphi_x, \varphi_{xx}, \varphi_{xxx}, \varphi_{xxxx} \in L_2(0,L), \text{ and} \\ \varphi(0) = 0, \varphi_x(0) = 0, \varphi_{xx}(L) = 0, \varphi_{xxx}(L) = 0\big\}. \end{aligned} \tag{7.39}$$

The general solution of equation (7.37) can be given by

$$\varphi(x) = A\sinh\beta x + B\cosh\beta x + C\sin\beta x + D\cos\beta x. \tag{7.40}$$

Then using the boundary conditions as given in (7.39), we obtain

$$\begin{aligned} 0 &= B + D \\ 0 &= A + C \\ 0 &= A\sinh\beta L + B\cosh\beta L - C\sin\beta L - D\cos\beta L \\ 0 &= A\cosh\beta L + B\sinh\beta L - C\cos\beta L + D\sin\beta L, \end{aligned} \tag{7.41}$$

or equivalently

$$\begin{bmatrix} \sinh\beta L + \sin\beta L & \cosh\beta L + cos\beta L \\ \cosh\beta L + \cos\beta L & \sinh\beta L - \sin\beta L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0. \tag{7.42}$$

A nontrivial solution for the unknown coefficients $A$ and $B$ can be found only if the determinant of the coefficient matrix is zero. Taking the determinant of the coefficient matrix in the above equation, and simplyfing, one obtains

$$\cosh\beta L\cos\beta L + 1 = 0. \tag{7.43}$$

This trancendental equation has infinitely many solutions for $\beta$. This can be seen from the following graphics. Denote $\beta L = z$ so that equation (7.43) is equivalent to

$$\cos z = -\frac{1}{\cosh z}$$

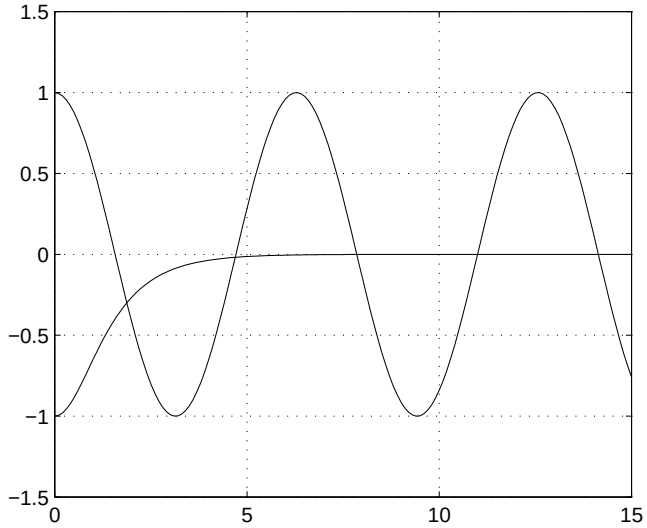


Fig 7.2 Solution of Trancendental Equation (7.43)

Figure 5.5 shows graphical representation of the functions $\cos z$ and $-\frac{1}{\cosh z}$, and the intersection points of these two functions are the solutions of the above equation. Clearly there are infinitely many values of $z = \beta L$ for which equation (7.43) has a solution of which the following is a partial list:

$$\begin{aligned} z_1 &= 1.8751 \\ z_2 &= 4.6940 \\ z_3 &= 7.854 \\ z_4 &= 10.996, \end{aligned}$$

and $z_n = \frac{2n-1}{2}\pi$ as $n \to \infty$. Thus the eigenvalue problem (7.37) are has infinitely many eigenvalues given by $\beta_n = \frac{z_n}{L}$. The corresponding eigenfunctions are

$$\varphi_n = A_n\big(\sinh \beta_n x - \sin \beta_n x\big) + B_n(\cosh \beta_n x - \cos \beta_n x). \tag{7.44}$$

Since from the first equation of (7.42)

$$\frac{A_n}{B_n} = -\frac{\cosh \beta_n x + \cos \beta_n x}{\sinh \beta_n x + \sin \beta_n x} = -\gamma_n,$$

the unnormalized eigenfunctions are obtained as

$$\varphi_n(x) = (\cosh\beta_n x - \cos\beta_n x) - \gamma_n(\sinh\beta_n x - \sin\beta_n x). \tag{7.45}$$

Orthogonality of the above eigenfunctions plays an important role in the analysis of the response of the system. In what follows, we prove that the eigenfunctions (7.45) are orthogonal. Indeed this can be proved right from the definition of the eigenvalue problem (7.37) rather than algebraic work using (7.45). It is known that the eigenvectors of a symmetric matrix are orthogonal. Similarly, the eigenfunctions of a self-adjoint operator are also orthogonal as shown below.

Let $\beta_n^4$ and $\beta_m^4$ be two eigenvalues of the eigenvalue problem (7.37) with $\varphi_n$ and $\varphi_m$ being the corresponding eigenfunctions. For notational simplicity, we shall use $'$ to denote partial derivative with respect to $x$, i.e., $\varphi' = \frac{\partial\varphi}{\partial x}$. We shall also use $\langle\cdot,\cdot\rangle$ to denote scalar product in $L_2(0,L)$, for example, $\langle\phi,\psi\rangle = \int_0^L \phi\,\psi\,dx$. Then we have

$$\begin{aligned} \varphi_n'''' &= \beta_n^4\varphi_n \\ \varphi_m'''' &= \beta_m^4\varphi_m. \end{aligned} \tag{7.46}$$

Scalar multiplying the first equation by $\varphi_m$ and the second equation by $\varphi_n$ and taking the difference

$$(\beta_n^4 - \beta_m^4)\langle\varphi_n,\varphi_m\rangle = \langle\varphi_m,\varphi_n''''\rangle - \langle\varphi_n,\varphi_m''''\rangle.$$

Integrating the first term in the right hand side,

$$(\beta_n^4 - \beta_m^4)\langle\varphi_n,\varphi_m\rangle = -\varphi_m''\varphi_n'\Big|_0^L - \varphi_m'''\varphi_n\Big|_0^L + \langle\varphi_m'''',\varphi_n\rangle - \langle\varphi_n,\varphi_m''''\rangle.$$

Using the boundary conditions of (7.37), this gives

$$(\beta_n^4 - \beta_m^4)\langle\varphi_n,\varphi_m\rangle = 0.$$

Since for $\beta_n \neq \beta_m$ for $n \neq m$, this gives

$$\langle\varphi_n,\varphi_m\rangle = 0,$$

which shows that the eigenfunctions $\{\varphi_i\}$ are orthogonal. We can always normalize the eigenfunctions so that $\|\varphi_n\| = 1$.

$$\varphi_n(x) = \frac{1}{\sqrt{L}}\big[(\cosh\beta_n x - \cos\beta_n x) - \gamma_n(\sinh\beta_n x - \sin\beta_n x)\big]. \tag{7.47}$$

At this point we return to the system model (7.33) and its approximation using the expansion (7.36). Substituting (7.36) into (7.33), one obtains

$$\rho\sum_{i=1}^{\infty}\ddot{y}_i(t)\varphi_i(x) + EI\sum_{i=1}^{\infty} y_i(t)\frac{\partial^4\varphi_i}{\partial x^4} = f. \tag{7.48}$$

We shall assume that the distributed control $f(x,t)$ be given by

$$f(x,t) = \Psi(x)u(t),$$

where $\Psi(x)$ is a spatial distribution function and $u(t)$ is a finite dimensional control. Scalar multiplying the above equation by $\varphi_n$ in $L_2(0,L)$, and using the eigenvalue problem (7.37)

$$\sum_{i=1}^{\infty} \rho\ y_i\ \langle \varphi_i, \varphi_n \rangle + \sum_{i=1}^{\infty} EI\ \beta_i^4\ y_i\ \langle \varphi_i, \varphi_n \rangle = \langle \Psi, \varphi_n \rangle u(t).$$

Using the orthonormality property of the eigenfunctions, the above equation reduces to

$$\rho \ddot{y}_n + EI\beta_n^4 y_n = b_n u(t), \tag{7.49}$$

where $b_n = \langle \Psi, \varphi_n \rangle$. Note that the above equation holds for all modes $n = 1, 2, \cdots$, and is a linear ordinary differential equation that describes the mode amplitude for each mode. Recall that there are infinitely many modes in the beam vibrations corresponding to $n = 1, 2, \cdots, \infty$. In engineering literature, it is a common practice to approximate the beam vibrations in terms of a finite number of modes rather than the infinite series as in (7.36), i.e.,

$$y(x,t) \simeq \sum_{i=1}^{N} y_i(t)\varphi_i(x). \tag{7.50}$$

Although there is some loss of accuracy because of truncation of the infinite series, it makes the entire finite dimensional control theories available for (approximate) analysis of distributed parameter systems. Considering the $N$ modes as in (7.49), the finite dimensional approximate model of the transverse vibration of an elastic beam is therefore given by

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_N \\ \cdots \\ \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_N \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ -\frac{EI\beta_1^4}{\rho} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{EI\beta_2^4}{\rho} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{EI\beta_N^4}{\rho} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \\ \cdots \\ v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} u, \tag{7.51}$$

where $v_1, v_2, \cdots$ are the mode velocities corresponding to the mode amplitudes $y_1, y_2, \cdots$ respectively. This is a linear time invariant finite dimensional model so that the entire finite dimensional control theories can be used for its analysis and control system design.

## 7.4 Finite Difference Method

An analytical solution of this partial differential equation can be found only if the domain of interest (i.e., the region in which the solution is to be computed) is simple, such as a rectangle or a circle. For finding the voltage distribution in a region of nonuniform geometry, engineers frequently use numerical techniques. Numerical solution of partial differential equations can be easily computed using the finite difference method and the finite element method. While finite difference method is easy to implement, finite element method is the preferred approach for practical applications since it usually leads to more accurate solution. This section gives an overview of the finite element method.

Finite difference approximation is a very powerful tool used in applied engineering analysis including for solution of two-point-boundary-value problems, and partial differential equations. It originates from Taylor series expansion of a variable with respect to a given point. The finite difference method consists of three steps:

a) Discretize the domain as a grid.

b) Find an algebraic equation for each node in the grid.

c) Solve the set of algebraic equations.

**Grid**

For finite difference solution, we direct our attention only at some finite number of points in the domain. These points are referred to as grid (or mesh or node) points. For domains in Cartesian geometry, a rectangular grid is usually used. For cylindrical geometry, the region is discretized utilizing cylindrical coordinates.

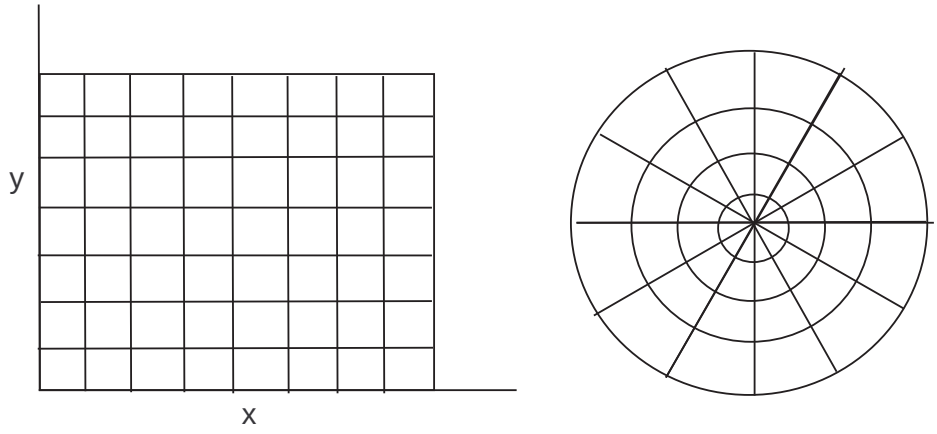


Figure 7.3 Discretization

It is not necessary that the grid size be uniform for the $x$ and $y$ coordinates, however a uniform grid results in some simplification in the numerical method. Note

however that for domains with irregular shaped geometry, a uniform grid cannot be used. Nevertheless, it is possible to extend the method with good accuracy.

**Difference Equation**

This step approximates each node in the domain in terms of an algebraic equation. For simplicity of presentation, we shall consider numerical solution of Laplace equation arising from the electric field distribution in a capacitor.

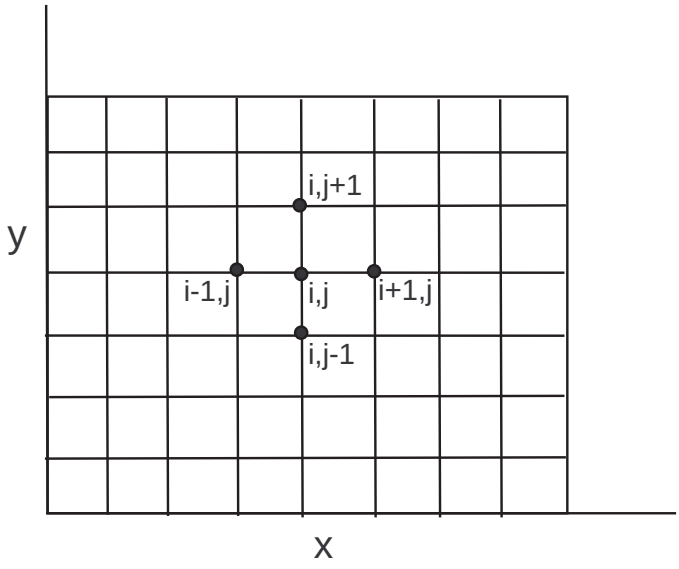


Fig 7.4 Discretization

The potential $V$ at the boundary nodes are given. The objective is to find the potential $V$ for the interior nodes. Consider the derivative $\frac{\partial V}{\partial x}$, and suppose we would like to approximate it at an arbitrary point $x$. Here the variable $V$ is a function of the independent variables $x$ and $y$.

We discretize the $x$-domain into a number of equal intervals as shown in the figure, and number the discrete points on the $x$-axis as $1, 2, 3, \cdots, N$. Then by Taylor series expansion, we have

$$V(x+\Delta x) = V(x) + \frac{\partial V}{\partial x}\bigg|_x \Delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\bigg|_x (\Delta x)^2 + \frac{1}{3!}\frac{\partial^3 V}{\partial x^3}\bigg|_x (\Delta x)^3 + \cdots \quad (7.52)$$

Similarly, we can write

$$V(x-\Delta x) = V(x) - \frac{\partial V}{\partial x}\bigg|_x \Delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\bigg|_x (\Delta x)^2 - \frac{1}{3!}\frac{\partial^3 V}{\partial x^3}\bigg|_x (\Delta x)^3 + \cdots \quad (7.53)$$

From these equations we can derive a number of basic relations for various approximations of derivatives of $V$ with respect to $x$. For example, considering equation (7.52), we have

$$\left.\frac{\partial V}{\partial x}\right|_x = \frac{V(x+\Delta x) - V(x)}{\Delta x} + o(\Delta x) \tag{7.54}$$

This is the well known first order approximation of derivative of a function. This equation is also known as the forward difference formula of $\frac{d}{dV}x$. In a similar way, we can derive the backward difference formula for the derivative using equation (7.53):

$$\left.\frac{\partial V}{\partial x}\right|_x = \frac{V(x) - V(x-\Delta x)}{\Delta x} + o(\Delta x) \tag{7.55}$$

The last term in the above equations means that the remainder of the infinite series is proportional to $\Delta x$, and as $\Delta x \to 0$, the remainder $o(\Delta x) \to 0$.

Subtracting equation (7.53) from equation (7.52), we have

$$\left.\frac{\partial V}{\partial x}\right|_x = \frac{V(x+\Delta x) - V(x-\Delta x)}{2\Delta x} + o((\Delta x)^2) \tag{7.56}$$

This equation is known as the central difference formula, and has the second order accuracy in approximation of the derivative. Note that the remainder is proportional to $(\Delta x)^2$. We can derive similar equations for higher order derivatives of $y$ with respect to $x$. For example, adding equations (7.52) and equation (7.53), we have

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_x = \frac{V(x+\Delta x) - 2V(x) + V(x-\Delta x)}{(\Delta x)^2} + o((\Delta x)^2) \tag{7.57}$$

These equations form the basics for approximation of derivatives in various ordinary and partial differential equations. For computer applications, it is more convenient to rewrite the above equations in terms of the index $i$ that denotes the various points of discretization along the $x$-coordinate. For example, the central difference formula for derivative at an arbitrary node $(i, j)$ is written as

$$\left.\frac{\partial V}{\partial x}\right|_{i,j} = \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta x} \tag{7.58}$$

and for second order derivative we have

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{i,j} = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta x)^2} \tag{7.59}$$

where $V_{i,j}$ denotes $V(x, y)$.

Central difference formula for partial derivatives in the $y$ coordinate can be derived in a similar way as:

$$\left.\frac{\partial V}{\partial y}\right|_{i,j} = \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta y} \tag{7.60}$$

and for second order derivative we have

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_{i,j} = \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{(\Delta y)^2} \tag{7.61}$$

where $V_{i,j+1}$ denotes $V(x, y + \Delta y)$.

Care must be taken if the point where the derivative is approximated happens to be on the boundary. For example, equation (7.58) cannot be used to approximate $\frac{\partial V}{\partial x}|_1$ which is at the boundary. This is because of the fact that the second term of (7.58) does not exist for $i = 1$. An alternate expression can be obtained if we go back to the fundamentals. For example,

$$V_2 = V_1 + \left.\frac{\partial V}{\partial x}\right|_1 \Delta x + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_1 (\Delta x)^2 + \cdots$$

and

$$V_3 = V_1 + \left.\frac{\partial V}{\partial x}\right|_1 2\Delta x + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_1 (2\Delta x)^2 + \cdots$$

from which we easily obtain

$$\left.\frac{\partial V}{\partial x}\right|_1 = \frac{4V_2 - V_3 - 3V_1}{2\Delta x} + o((\Delta x)^2) \tag{7.62}$$

Similar expressions can be derived for derivative at the right boundary, and for second order derivatives.

In what follows we shall use these concepts in solving several numerical problems.

## Example 7.5: Electric Field Distribution

One of the fundamental equations governing electric field distribution in a region is the Laplace equation. In fact, Laplace equation is also a fundamental equation used in many other branches of science and engineering. For example, the temperature distribution on a metal plate under steady state condition is governed by the Laplace equation.

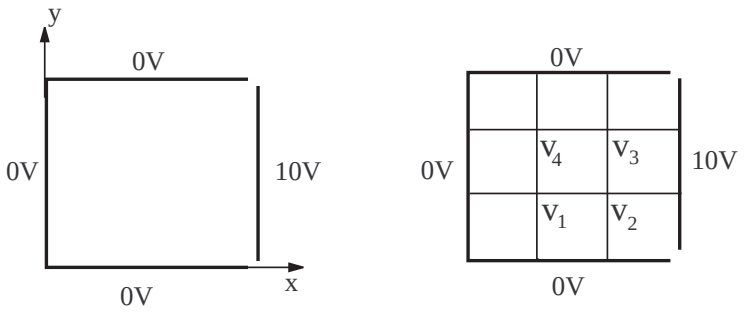


Figure 7.4 Discretization of Domain

The electric field distribution problem in the region shown in figure 2 is governed by the Laplace equation, which is a partial differential equation given by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{7.63}$$

where $V = V(x, y)$ is the unknown potential at various points in the domain as a function of the coordinates $x$ and $y$. We discretize the domain into a grid as shown in the figure- in this case we have to discretize both the $x$-domain and the $y$-domain. Let $(i, j)$ denote an arbitrary point on the grid in the $x$ and $y$ direction respectively. Then at an arbitrary point $(i, j)$ within the domain the partial differential equation (7.63) can be approximated as

$$\frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta x)^2} + \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{(\Delta y)^2} = 0, \tag{7.64}$$

For simplicity we also assume that the discretization interval $\Delta x$ and $\Delta y$ in the two directions are equal. In that case the above equation simplifies to

$$V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j} = 0 \tag{7.65}$$

that is, the electric field at an arbitrary point is the average of electric field potential of four equidistant points around it.

For illustration, consider only four points inside the domain. Using the concept of (7.65) for the four interior points and writing the corresponding equations as a matrix equation, we obtain

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 10 \\ 0 \end{bmatrix}. \tag{7.66}$$

The electric field distribution at the four interior points are then obtained by solving the above matrix equation. It must, however, be noted that for a reasonably accurate result one must discretize the domain into a finer grid considering many more points within the domain instead of just four point as done in this example. This would obviously result in a large matrix equation requiring a digital computer for its solution.

## Example 7.6. Solution of Heat Equation

Consider the heat equation

$$\begin{aligned} &\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < 1 \\ &y(t,0) = 0, \qquad y(t,1) = 0 \\ &y(0,x) = y_0(x) \end{aligned} \tag{7.67}$$

In this case we have to discretize both the spatial domain as well as the time domain. Let $i$ denotes the node points for the space domain, and $j$ the node points for the time domain. We are interested to find the solution $y_{i,j}$ for each node point. Note however that since the solution evolves in time, we march along the time coordinate one step at a time.

Discretize the space domain into nodes $i = 1, 2, 3, \cdots, N-1, N$, where $i = 1$ and $i = N$ denote the boundary points. For the time domain we use the index $j$. Discretizing the above equation for an arbitrary point we have

$$\frac{y_{i,j+1} - y_{i,j}}{\Delta t} = \frac{y_{i+1,j} - 2y_{i,j} + y_{i-1,j}}{(\Delta x)^2} \tag{7.68}$$

which holds for all interior node points, $i = 2, 3, \cdots, N-1$. Note also that we have used first order approximation for the derivative in the time instead of a more accurate second order central difference formula. This is necessary since at any time index $j$, the solution is known, and one intends to find the solution for the next time index $j+1$. Clearly the above equation can be rewritten as

$$y_{i,j+1} = y_{i,j} + \frac{\Delta t}{(\Delta x)^2}\left(y_{i+1,j} - 2y_{i,j} + y_{i-1,j}\right) \tag{7.69}$$

This equation is used for each interior node point in the $x$ domain, which are expressed in the form of a matrix equation. The various terms in the right hand side of the above equation are known at any time step $j$. Then one obtains the solution at the next time step $j+1$. To begin the process, one uses the initial condition given in the original problem. The solution is continued for the entire time period of simulation.

Note that equation (7.69) has the structure of a difference equation. As is well known, for stability the eigenvalues of the matrix generated from the right hand side of (7.69) must be within unit circle. This can be assured if $\Delta t$ and $\Delta x$ are chosen so

that $\frac{\Delta t}{(\Delta x)^2} < 0.5$. Note that the original system as given is not unstable, rather the numerical method that we have developed can be unstable because of accumulation of round-off error and truncation error during computation. A detailed discussion of stability condition is beyond the scope of this presentation.

Round-off error is the error that originates due to finite precision of data in computer memory. Note that computers hold data in the memory for up to a certain number of digits only whereas a true representation of a data may require infinite number of digits. Truncation error is the error that originates from truncation of infinite series of Taylor series expansion to a few terms only. In deriving the difference equations, we have neglected all higher order terms in the Taylor series.

## Example 7.6. Solution of Wave Equation

The solution of the wave equation is fundamentally similar to the method discussed above. Consider the equation

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < 1 \\ &y(t,0) = 0, \qquad y(t,1) = 0 \\ &y(0,x) = y_0(x), \qquad \frac{\partial y}{\partial t}(0,x) = v_0(x) \end{aligned} \tag{7.70}$$

Since approximation of a second order differential requires data of two consecutive steps, it is more convenient to rewrite the system equation as two first order equations. Define $y_1 = y$, and $y_2 = \frac{\partial y}{\partial t}$. Then the above system can be written as

$$\begin{aligned} &\frac{\partial y_1}{\partial t} = y_2 \\ &\frac{\partial y_2}{\partial t} = \frac{\partial^2 y_1}{\partial x^2} \\ &y_1(t,0) = 0, \qquad y_1(t,1) = 0 \\ &y_2(t,0) = 0, \qquad y_2(t,1) = 0 \\ &y_1(0,x) = y_0(x), \qquad y_2(0,x) = v(x) \end{aligned} \tag{7.71}$$

Using the forward difference formula for time, and central difference formula for $x$, we obtain

$$\begin{aligned} &\frac{y_1(i,j+1) - y_1(i,j)}{\Delta t} = y_2(i,j) \\ &\frac{y_2(i,j+1) - y_2(i,j)}{\Delta t} = \frac{y_1(i+1,j) - 2y_1(i,j) + y_1(i-1,j)}{(\Delta x)^2} \end{aligned} \tag{7.72}$$

which hold for all nodes $i = 2, 3, \cdots, N-1$. We rewrite these equations as

$$\begin{aligned} &y_1(i,j+1) = y_1(i,j) + \Delta t y_2(i,j) \\ &y_2(i,j+1) = y_2(i,j) + \frac{\Delta t}{(\Delta x)^2}(y_1(i+1,j) - 2y_1(i,j) + y_1(i-1,j)) \end{aligned} \tag{7.73}$$

The loop starts with $j = 1$ at the initial time and uses the initial values for $y_1(0,x)$ and $y_2(0,x)$. Once again, for stability reasons, it is necessary that $\frac{\Delta t}{\Delta x} < 1$.

## 7.7. Semidiscretization Method

The semidiscretization method entails converting the partial differential equations to a set of ordinary differential equations. Consider the heat equation given by equation (7.67). Then for an arbitrary node point $(i, j)$, we discretize the right hand side only to obtain

$$\frac{dy_{i,j}}{dt} = \frac{y_{i+1,j} - 2y_{i,j} + y_{i-1,j}}{(\Delta x)^2} \tag{7.74}$$

This leads to a set of ordinary differential equations that can be solved using any ode solver. This method is not susceptible to discretization error as in the case of the finite difference method since numerical errors are usually minimized by standard ode solvers.

## 7.5 Exercise

**7.1** Find the eigenvalues and eigenfunctions of the following operator:

$$\begin{aligned}\mathcal{A}\varphi &= -\frac{\partial^2\varphi}{\partial x^2}\\ \varphi(0) = 0, &\qquad \frac{\partial\varphi}{\partial x}(\ell) = 0\end{aligned}$$

**7.2** Find the eigenvalues and eigenfunctions of the following operator:

$$\begin{aligned}\mathcal{A}\varphi &= -\frac{\partial^2\varphi}{\partial x^2}\\ \frac{\partial\varphi}{\partial x}(0) = 0, &\qquad \frac{\partial\varphi}{\partial x}(\ell) = 0\end{aligned}$$

**7.3** Find the eigenvalues and eigenfunctions of the following operator:

$$\begin{aligned}\mathcal{A}\varphi &= \frac{\partial^4\varphi}{\partial x^4}\\ \varphi(0) = 0, &\qquad \varphi(\ell) = 0\\ \frac{\partial\varphi}{\partial x}(0) = 0, &\qquad \frac{\partial\varphi}{\partial x}(\ell) = 0\end{aligned}$$

**7.4** Find the electric field distibution, $V(x, y)$, in a rectangular region of dimension $0 \le x \le 1, \quad 0 \le y \le 1$ with zero voltage applied on all four sides, and charge distribution in the region described by $\rho(x, y) = \varepsilon_0 xy(1 - x)(1 - y)$, where $\varepsilon_o$ is the permittivity of free space.

**7.5** Electric field distribution in a region is described by the Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho}{\varepsilon}$$

where $\rho$ is the charge density, and $\varepsilon$ is the dielectric constant. Determine the potential within the region if $0 \le x \le 1, \quad 0 \le y \le 1$. Assume that the voltage at the boundary is zero.

**7.6** Find the temperature distribution on a rod described by the following:

$$\begin{aligned}\frac{\partial y}{\partial t} &= \frac{\partial^2 y}{\partial x^2}, &\qquad 0 < x < 1\\ y(0, t) = 0, &\qquad \frac{\partial y}{\partial x}(1, t) = 0\\ y(x, 0) &= 0.1x(1 - 0.5x)\end{aligned}$$

**7.7** Find the displacement of the wave equation described by

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < 1$$
$$y(0,t) = 0, \quad \frac{\partial y}{\partial x}(1,t) = 0$$
$$y(x,0) = 0.1x(1-0.5x), \frac{\partial y}{\partial t}(x,0) = 0$$

**7.8** Find the displacement of the wave equation described by

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + x(1-0.5x), \qquad 0 < x < 1$$
$$y(0,t) = 0, \quad \frac{\partial y}{\partial x}(1,t) = 0$$
$$y(x,0) = 0, \quad \frac{\partial y}{\partial t}(x,0) = 0$$

**7.9** Find the displacement of the simply supported beam described by

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial^4 y}{\partial x^4}, \qquad 0 < x < 1$$
$$y(0,t) = 0, \quad y(1,t) = 0$$
$$\frac{\partial^2 y}{\partial x^2}(0,t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(1,t) = 0$$
$$y(x,0) = 0.1(x - 2x^3 + x^4), \quad \frac{\partial y}{\partial t}(x,0) = 0$$

For numerical solution, assume $y_1 = \frac{\partial^2 y}{\partial x^2}, y_2 = \frac{\partial y}{\partial t}$.

**7.10** The charge distribution inside a capacitor with a leaky dielectric is given by $\rho(x) = \varepsilon \sin \pi x$, where $\varepsilon$ is the dielectric constant. Suppose the capacitor is connected to a source so that the voltage of one plate is zero, i.e., $V(0) = 0$, and that for the other plate is $V(1) = V_0$. Find the electric field distribution within the capacitor.

Hint: Convert the nonhomogeneous boundary value problem to a homogeneous boundary value problem. Assume that $\varphi(x) = x(1-x)V(x)$.