5.

System Response

One of the essential steps of system analysis is understanding of dynamic behavior of the system. Frequently engineers rely on mathematical analysis and numerical simulation of the system model rather than experimentation with the actual physical system. Once the behavior of the uncontrolled system is understood, the engineer proceeds to design a controller so as to regulate its response in a desired manner. Then response of the controlled system is investigated under various operating conditions before a physical implementation of the controller is carried out. This chapter introduces the fundamental concepts of response analysis of dynamic systems described by state space model.

5.1 System Modeling

A model of a system is a mathematical representation of its dynamic behavior, and typically has the form of a set of linear or nonlinear differential equations. Models of dynamic systems are usually based on physical laws of nature, such as Newton's laws, Kirchhoff's laws, etc., and certain idealizing assumptions. Inaccuracies in the model may arise from various sources including poor understanding of the underlying process, and unpredictable randomness within the system or in the environment in which the process evolves. The orbital motion of a satellite can be fairly predicted; however, unpredictable geomagnetic variations and solar pressure can easily disrupt the stable orientation of a communication satellite. In hot strip mills, steel plates are cooled using cold water spray; but exact process of heat transfer between water spray and the hot steel surface is poorly understood.

Mathematically, time evolution of dynamic systems is described in terms of a set of differential equations. Given the current state (i.e., initial condition), and external inputs, the future time evolution of the system can then be predicted by solving this set of differential equations. Dynamic systems can be classified in several different ways depending on complexity of the underlying process.

1) Linarity
   a) Linear system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$
   b) Nonlinear system: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$

2) Time dependence
   a) Linear Time Invariant system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$
   b) Linear Time Varying system: $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}$
   c) Nonlinear Time Varying system: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$
3) Effects of noise
   a) Deterministic system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$
   b) Stochastic system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \eta(t)$, $\quad \eta(t)$ is a noise process
4)
   a) Lumped parameter system
   b) Distributed parameter system

A linear model is the simplest that can be used for mathematical analysis of dynamic systems,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, &\quad& \text{State Equation} \\ \mathbf{x}(t_0) &= \mathbf{x}_0, && \text{Initial state} \end{aligned} \tag{5.1}$$

where $\mathbf{x} \in R^n$ is the state vector, and $\mathbf{u} \in R^m$ is the control input. The matrices $\mathbf{A}$ and $\mathbf{B}$ are of compatible dimension. Most dynamic systems in engineering applications are however nonlinear, and can be described by the nonlinear model

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

where f is a nonlinear function satisfying appropriate properties.

In case the right hand side of the equation is an explicit function of time, it is called a time varying system. For example, the matrices $\mathbf{A}$ and $\mathbf{B}$ could be varying with time. Most physical processes of engineering interest are affected by noise, either from the environment or from within itself. Such systems can be represented as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{C}\eta$$

where η is a Gaussian noise process.

The state variable of certain processes must be expressed as a function of time and a spatial coordinate. For example, lateral displacement of a power line varies with time as well as with distance from anchored end-point. These types of systems often require partial differential equations as their mathematical model, and are call distributed parameter systems.

An Electro-Mechanical System

Frequently a system is an interconnection of several subsystems. Operation of each subsystem must be clearly understood before making any attempt of modeling the system. A position servo system is a simple yet useful practical example that shows interaction between a mechanical system and an electrical system. Position servo systems are used to regulate the orientation of satellite disk antenna, the inclination of anti-aircraft guns, etc. An armature controlled dc motor is frequently used as the force actuator which drives the motor shaft to a new position.

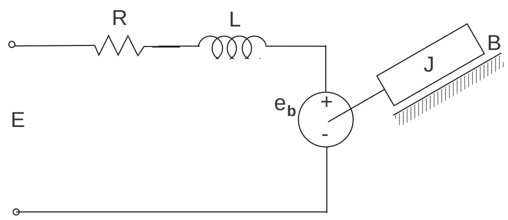

Figure 5.1 Schematic of a Position Servo System

The system model is easily derived using the Kirchoff's voltage law and the Newton's law

$$\begin{aligned} Ri + L\frac{di}{dt} + e_b &= E \\ J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} &= T_m, \end{aligned} \tag{5.2}$$

where $e_b k_b = \frac{d\theta}{dt}$ is the back emf induced in the armature, and $T_m = k_t i$ is the developed mechanical torque, where k_b and k_t are constant. To express the system in state space form, denote $x_1 = \theta, x_2 = \omega, x_3 = i$. Then we have

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & \frac{k_t}{J} \\ 0 & -\frac{k_b}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} E \\ y &= [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned} \tag{5.3}$$

This completes the state space model of the position servo system. ▷

Modeling of a Gantry Crane

Cranes running on a horizontal track are often used for cargo handling at shipyards and factories. Unlike cranes used at construction sites that have rotary motions, gantry cranes transport payload along a straight line. The load is suspended from a trolley using a cable, and the trolley moves on a horizontal track by an electric motor. During transport, the payload may oscillate like a pendulum, which can be hazardous to personnel and equipment. The control problem is to transport the payload with minimum oscillations. Reference [51] considers optimal control of gantry cranes; some other references on the subject are [53, 13]. In this section we develop a simple mathematical model of the gantry crane.

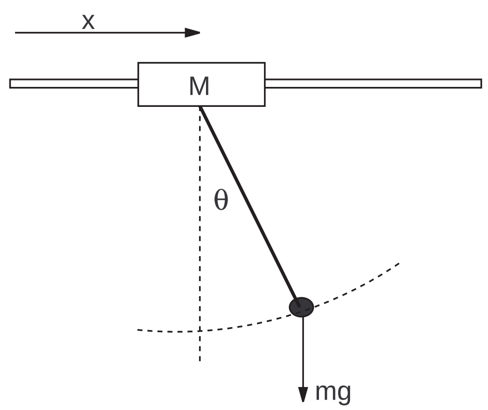

Figure 5.2 Schematic of a Gantry Crane

Let M and m denote the mass of the trolley and the payload, respectively, and let ℓ is the length of the cable suspending the payload. Acceleration due to gravity is denoted by g. The cable is assumed to be massless, and inelastic, and that the length of the cable is fixed. We also assume that there is a friction between the trolley rollers and the track; the corresponding damping coefficient is denoted by D. We assume that a control force is applied only to the trolley, but not to the payload. Denote the position of the trolley by x, and the corresponding velocity and acceleration by $\dot{x}$ and $\ddot{x}$, respectively. Then acceleration of the payload with respect to the ground is given by $x + \ell\theta$, where θ is the angular position of the payload. Then by Newton's law,

$$m\frac{d^2(x + \ell\theta)}{dt^2} + mg\sin\theta = 0 \tag{5.4}$$

The cable is assumed to be inextensible. We also assume that there is no transverse oscillation of the cable itself, which is true if the payload is heavy and the tension on the cable is high. Thus the trolley and the payload will be simultaneously acted upon by the control force u. Thus for the trolley motion we write

$$M\frac{d^2x}{dt^2} + m\frac{d^2(x+\ell\theta)}{dt^2} + D\frac{dx}{dt} = u \tag{5.5}$$

where D is the friction coefficient between the roller of the trolley and the rail.

Equations (5.4) and (5.5) constitute a nonlinear model of the gantry crane. Assume small angle oscillation of the payload, i.e., $\sin\theta = \theta$. To express the system in the state space form, we define $x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta}$ Then we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{D}{M} & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{D}{M\ell} & -\frac{g}{\ell}\frac{M+m}{M} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix} u \tag{5.6}$$

This completes the development of a simple dynamic model of the gantry crane.

5.2 Linearization of Nonlinear Systems

The general form of a nonlinear state space model is given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{5.7}$$

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}), \tag{5.8}$$

where $\mathbf{x}$ is the state vector, and $\mathbf{f}$ is the vector containing the right hand side of the differential equations which is a function of the state $\mathbf{x}$ and the control $\mathbf{u}$. The second equation (5.8) is the output equation. For the sake of simplicity and ease of analysis, nonlinear models are often simlified to a linear model. It must be noted that the linearized model is only an approximation of the actual nonlinear dynamics, and a control system based on the linearized model may not provide the desired performance.

The linearized model is obtained by a simple application of the Taylor series expansion of the nonlinear functions in the right hand side of the system model. Let

$$\begin{aligned} \mathbf{x} &= \mathbf{x}^* + \Delta\mathbf{x} \\ \mathbf{u} &= \mathbf{u}^* + \Delta\mathbf{u}, \end{aligned} \tag{5.9}$$

where $\mathbf{x}^*$ and $\mathbf{u}^*$ denote equilibrium state and the control corresponding to the equilibrium state, respectively, and $\Delta\mathbf{x}$ and $\Delta\mathbf{u}$ denote small variation with respect to these equilibrium.

Then from equation (5.7), we have

$$\frac{d}{dt}(\mathbf{x}^* + \Delta\mathbf{x}) = \mathbf{f}(\mathbf{x}^* + \Delta\mathbf{x}, \mathbf{u}^* + \Delta\mathbf{u}).$$

By expanding the right hand side of the above equation in terms of Taylor's series, we obtain

$$\frac{d}{dt}(\mathbf{x}^*+\Delta\mathbf{x}) = \mathbf{f}(\mathbf{x}^*,\mathbf{u}^*)+\left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\mathbf{x}^*,\mathbf{u}^*}\Delta\mathbf{x}+\left.\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\right|_{\mathbf{x}^*,\mathbf{u}^*}\Delta\mathbf{u}+\text{ higher order terms} \quad (5.10)$$

Since $\frac{d\mathbf{x}^*}{dt} = \mathbf{f}(\mathbf{x}^*,\mathbf{u}^*)$, the above equation simplifies to

$$\frac{d\Delta\mathbf{x}}{dt} = \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\mathbf{x}^*,\mathbf{u}^*}\Delta\mathbf{x}+\left.\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\right|_{\mathbf{x}^*,\mathbf{u}^*}\Delta\mathbf{u}, \quad (5.11)$$

where we have approximated the infinite series using first order terms only. Equation (5.9) is the linearized state equation for the nonlinear system (5.7). Note that this equation is only an approximation since we have dropped all higher order terms from the infinite series. In (5.11), $\frac{\partial\mathbf{f}}{\partial\mathbf{x}}$ and $\frac{\partial\mathbf{f}}{\partial\mathbf{u}}$ are matrices defined below:

$$\begin{aligned}
\frac{\partial\mathbf{f}}{\partial\mathbf{x}} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \\
\frac{\partial\mathbf{f}}{\partial\mathbf{u}} &= \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix},
\end{aligned} \quad (5.12)$$

which are evaluated at the equilibrium $\mathbf{x}^*$ and $\mathbf{u}^*$. The matrix $\frac{\partial\mathbf{f}}{\partial\mathbf{x}}$ is known as the Jacobian matrix. The linearized version of the output equation is also obtained in a similar manner. This is given by

$$\Delta\mathbf{y} = \left.\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\right|_{\mathbf{x}^*,\mathbf{u}^*}\Delta\mathbf{x}+\left.\frac{\partial\mathbf{g}}{\partial\mathbf{u}}\right|_{\mathbf{x}^*,\mathbf{u}^*}\Delta\mathbf{u}. \quad (5.13)$$

Finally, the linearized model is expressed in the standard form as

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}(t)\mathbf{x}+\mathbf{B}(t)\mathbf{u} \\
\mathbf{y} &= \mathbf{C}(t)\mathbf{x}+\mathbf{D}(t)\mathbf{u}
\end{aligned}$$

where the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are appropriately defined in light of (5.11) and (5.13). Note that the linearized model is not necessarily a time invariant model since $\mathbf{x}^*$ is not necessarily a constant vector. It is possible that a system be linearized with respect to a time varying trajectory rather than a constant rest state. Likewise, $\mathbf{u}^*$ may also be a time varying function.

5.3 Linear Time Invariant Systems

Consider the linear time invariant system described by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, && \text{State Equation} \\ \mathbf{x}(t_0) &= \mathbf{x}_0, && \text{Initial state} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, && \text{Output equation} \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} \mathbf{x} &= n \times 1 \text{ state vector} \\ \mathbf{u} &= m \times 1 \text{ control vector} \\ \mathbf{y} &= r \times 1 \text{ output vector} \\ \mathbf{A} &= n \times n \text{ system matrix} \\ \mathbf{B} &= n \times m \text{ control matrix} \\ \mathbf{C} &= r \times n \text{ output matrix} \\ \mathbf{D} &= r \times m \text{ control feedforward matrix.} \end{aligned} \tag{5.15}$$

It is our objective to find the state $\mathbf{x}(t)$ and the output $\mathbf{y}(t)$ at an arbitrary time $t > t_0$. Premultiplying the first equation of (5.14) by $e^{-\mathbf{A}t}$, we have

$$e^{-\mathbf{A}t}\dot{\mathbf{x}} - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x} = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}, \tag{5.16}$$

which is equivalent to

$$\frac{d(e^{-\mathbf{A}t}\mathbf{x})}{dt} = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}. \tag{5.17}$$

Equivalency of (5.16) and (5.17) is not obvious, and requires some justification. This will be shown at the end of this section. Integrating both sides of the above equation from t_0 to t,

$$e^{-\mathbf{A}t}\mathbf{x}(t) - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\, d\tau. \tag{5.18}$$

Premultiplication of both sides of the above equation by $e^{\mathbf{A}t}$ gives

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau, \tag{5.19}$$

which is the state response of the system at an arbitrary time t. The output response $\mathbf{y}(t)$ follows easily from (5.14) and (5.19) as:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \mathbf{C}\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau + \mathbf{D}\mathbf{u}, \tag{5.20}$$

which again has three components, the first part is due to the initial state $\mathbf{x}_0$, the second part is due to the control input $\mathbf{u}(t)$, and the third part $\mathbf{D}\mathbf{u}$ is the control feed forward component. Note however that this last component may be zero for many systems.

Equivalency of equation (5.17) and equation (5.16)

Equivalency of equation (5.17) from equation (5.16) is based on the fact that

$$\frac{d}{dt}e^{-\mathbf{A}t} = -e^{-\mathbf{A}t}\mathbf{A} = -\mathbf{A}e^{-\mathbf{A}t}$$

This can be shown using the infinite series expansion of the matrix exponential:

$$\begin{aligned}
\frac{d}{dt}e^{-\mathbf{A}t} &= \frac{d}{dt}\big[\mathbf{I} - \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} - \frac{\mathbf{A}^3t^3}{3!} + \frac{\mathbf{A}^4t^4}{4!} + \cdots\big] \\
&= -\mathbf{A} + \mathbf{A}^2t - \frac{\mathbf{A}^3t^2}{2!} + \frac{\mathbf{A}^4t^3}{3!} + \cdots \\
&= -\mathbf{A}\big[\mathbf{I} - \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} - \frac{\mathbf{A}^3t^3}{3!} + \cdots\big] = -\big[\mathbf{I} - \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} - \frac{\mathbf{A}^3t^3}{3!} + \cdots\big]\mathbf{A} \\
&= -\mathbf{A}e^{-\mathbf{A}t} = -e^{-\mathbf{A}t}\mathbf{A}
\end{aligned}$$

This analysis also shows that the matrices $\mathbf{A}$ and $e^{\mathbf{A}t}$ commute, i.e., $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$ holds for any matrix $\mathbf{A}$.

Numerical or analytical evaluation of the response of a dynamic system is far more complex than the compact mathematical expressions given above. First of all, for a given system, the state transition matrix, $e^{\mathbf{A}(t-t_0)}$ must be computed. Furthermore, the control input $\mathbf{u}(t)$, being a time function, introduces additional complexity in the evaluation of the integration of (5.19). Nevertheless, for simple control inputs analytic expressions for the system response can be derived; however, for more complex types of control functions it is more common to use numerical methods.

Computation of State Transition Matrix: Diagonalization Method

The diagonalization method of computation of the matrix exponential is based on a similarity transformation that diagonalizes the matrix $\mathbf{A}$. Consider the n-th order system

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\
\mathbf{x}(0) &= \mathbf{x}_0.
\end{aligned} \tag{5.21}$$

Suppose there exists a nonsingular matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix. Now define the transformation

$$\mathbf{x} = \mathbf{T}\mathbf{q}, \tag{5.22}$$

where $\mathbf{T}$ is a $n \times n$ matrix and $\mathbf{q}$ is a $n \times 1$ vector. Substituting (5.22) into (5.21), we obtain

$$\mathbf{T}\dot{\mathbf{q}} = \mathbf{A}\mathbf{T}\mathbf{q},$$

so that

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{q}\\ &= \mathbf{D}\mathbf{q}.\end{aligned} \tag{5.23}$$

and

$$\mathbf{q}(t) = e^{\mathbf{D}t}\mathbf{q}(0). \tag{5.24}$$

Substituting the above result into (5.22), we then have

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{T}e^{\mathbf{D}t}\mathbf{q}(0)\\ &= \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}\mathbf{x}(0),\end{aligned} \tag{5.25}$$

which we may compare with the standard result

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0).$$

This shows that

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}. \tag{5.26}$$

Although the right hand side of the above equation is apparently more complex, in reality, its numerical evaluation is significantly easy. This is because of the fact that *the exponential of a diagonal matrix is easily obtained by taking the exponential of its diagonal entries.*

In order to carry out the calculations as discussed above, it is necessary first to find a transformation matrix **T** that diagonalizes the matrix **A**. This is easily done if **A** has all distinct eigenvalues. In that case one can simply take **T** as the matrix of eigenvectors of the matrix **A**. Note also that since the eigenvectors are linearly independent, the matrix **T** is always nonsingular. We illustrate the method by the following example.

Example 5.1

Given the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -2\\ 1 & -2 & -1\\ 0 & 0 & -3 \end{bmatrix},$$

it is easily verified that the eigenvalues of **A** are $\{-1, -2, -3\}$. Using the corresponding eigenvectors, we obtain the transformation matrix **T** as

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & -1\\ -1 & 1 & 1\\ 0 & 0 & 1 \end{bmatrix}.$$

This gives us

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 & -e^{-t}+e^{-3t} \\ -e^{-t}-e^{-2t} & e^{-2t} & e^{-t}+e^{-2t} \\ 0 & 0 & e^{-3t} \end{bmatrix}.$$

▷

In a numerical setup the calculations shown above can be carried out even more easily. Let $\mathbf{A}$ be an $n \times n$ matrix so that there are n-eigenvalues and n-eigenvectors, and let the eigenvalues are distinct. Let us define

$$\mathbf{T} = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,], \qquad \text{and} \qquad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{w}_1' \\ \mathbf{w}_2' \\ \vdots \\ \mathbf{w}_n' \end{bmatrix}, \tag{5.27}$$

where $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n$ are the eigenvectors of $\mathbf{A}$, and $\mathbf{w}_1', \mathbf{w}_2', \cdots \mathbf{w}_n'$ are the rows of the matrix $\mathbf{T}^{-1}$. Then we have

$$\begin{aligned} e^{\mathbf{A}t} &= [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,] \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1' \\ \mathbf{w}_2' \\ \vdots \\ \mathbf{w}_n' \end{bmatrix} \\ &= \sum_{i=1}^{n} e^{\lambda_i t}\, \mathbf{v}_i \mathbf{w}_i', \end{aligned} \tag{5.28}$$

in which the multiplications $\mathbf{v}_i \mathbf{w}_i'$ are easily numerically evaluated. Note however that the above equation is valid only if the eigenvalues are distinct. ▷

Example 5.2

For the matrix given in Example 7.4, we have

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{aligned} \mathbf{w}_1 &= [\;\;1 \quad 0 \quad -1]' \\ \mathbf{w}_2 &= [-1 \quad 1 \quad\;\; 1]' \\ \mathbf{w}_3 &= [\;\;0 \quad 0 \quad\;\; 1]'. \end{aligned}$$

Using (5.28), we obtain

$$e^{\mathbf{A}t} = e^{-t}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + e^{-2t}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + e^{-3t}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which simplifies to the result given in Example 7.4. ▷

If the matrix $\mathbf{A}$ has repeated eigenvalues, it may not always be possible to find a transformation matrix that diagonalizes the matrix $\mathbf{A}$. In that case one can use the matrix of eigenvectors and generalized eigenvectors as the transformation matrix $\mathbf{T}$ so that $\mathbf{T}^{-1}\mathbf{AT}$ is a Jordan canonical matrix. As in the case of distinct eigenvalues, the exponential $e^{\mathbf{A}t}$ is given by

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1}, \tag{5.29}$$

where $\mathbf{J}$ is a Jordan canonical matrix.

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

For simplicity of presentation, let there are m repeated eigenvalues. Then the $e^{\mathbf{J}t}$ is given by

$$e^{\mathbf{J}t} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

(Prove the above equation using Cayley-Hamilton theorem.) Note that in the above matrix, the main diagonal is 1, first upper diagonal is t, second upper diagonal is $\frac{t^2}{2!}$, and so on.

Example 5.3

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

which has the eigenvalues $\{-1, -1, -1\}$. Using the eigenvectors and generalized eigenvectors, we take

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0.5 & -0.25 \end{bmatrix},$$

giving rise to

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

The corresponding exponential is obtained as

$$e^{\mathbf{J}t} = e^{-t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the above results, we then have

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1} = e^{-t}\begin{bmatrix} 1 & t+t^2 & 2t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix}.$$

▷

Find an expression similar to (5.28) to express exponential of matrices with repeated eigenvalues, and use it for the above example.

Computation of State Transition Matrix: Using Cayley-Hamilton Theorem

The Cayley-Hamilton theorem is extremely powerful in evaluating the matrix exponential $e^{\mathbf{A}t}$. Since

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots + \frac{\mathbf{A}^{n-1}t^{n-1}}{(n-1)!} + \frac{\mathbf{A}^nt^n}{n!} + \frac{\mathbf{A}^{n+1}t^{n+1}}{(n+1)!} + \cdots, \quad (5.30)$$

and the matrices $\mathbf{A}^n, \mathbf{A}^{n+1}, \mathbf{A}^{n+2}, \cdots$ can be expressed in terms of matrices $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \cdots \mathbf{A}^{n-1}$, we can write (5.30) as a finite sum

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \cdots \alpha_{n-1}(t)\mathbf{A}^{n-1}, \quad (5.31)$$

where the coefficients $\alpha_0, \alpha_1, \cdots \alpha_{n-1}$ are suitable time functions. It can be shown that (5.31) is also satisfied by each eigenvalue of the matrix $\mathbf{A}$; in other words, if λ_i is a eigenvalue of $\mathbf{A}$, then

$$e^{\lambda_i t} = \alpha_0(t) + \alpha_1(t)\lambda_i + \alpha_2(t)\lambda_i^2 + \cdots + \alpha_{n-1}(t)\lambda^{n-1}. \quad (5.32)$$

Suppose the matrix $\mathbf{A}$ has distinct eigenvalues. Then since (5.32) holds for each eigenvalue of the matrix $\mathbf{A}$, we obtain a set of n-algebraic equations which can be simultaneously solved to obtain the coefficients $\alpha_0, \alpha_1, \cdots \alpha_{n-1}$. We illustrate the method by the following example:

Example 5.4

Consider the matrix given in Example 7.1,

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -2 \\ 1 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix},$$

which has the eigenvalues $-1, -2, -3$. Since $\mathbf{A}$ is a 3×3 matrix, corresponding to (5.31) and (5.32), we have

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 \tag{5.33}$$

$$e^{\lambda_i t} = \alpha_0(t) + \alpha_1(t)\lambda_i + \alpha_2(t)\lambda_i^2, \qquad i = 1, 2, 3. \tag{5.34}$$

Substituting the three eigenvalues $-1, -2, -3$ into (5.34) we obtain

$$\begin{aligned} e^{-t} &= \alpha_0 - \alpha_1 + \alpha_2 \\ e^{-2t} &= \alpha_0 - 2\alpha_1 + 4\alpha_2 \\ e^{-3t} &= \alpha_0 - 3\alpha_1 + 9\alpha_2, \end{aligned}$$

which are solved to obtain

$$\begin{aligned} \alpha_0 &= 3e^{-t} - 3e^{-2t} + e^{-3t} \\ \alpha_1 &= \frac{5}{2}e^{-t} - 4e^{-2t} + \frac{3}{2}e^{-3t} \\ \alpha_2 &= \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}. \end{aligned}$$

Substituting these coefficients into (5.33) then gives

$$\begin{aligned} e^{\mathbf{A}t} &= \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 \\ &= \begin{bmatrix} \alpha_0 - \alpha_1 + \alpha_2 & 0 & -2\alpha_1 + 8\alpha_2 \\ \alpha_1 - 3\alpha_2 & \alpha_0 - 2\alpha_1 + 4\alpha_2 & -\alpha_1 + 3\alpha_2 \\ 0 & 0 & \alpha_0 - 3\alpha_1 + 9\alpha_2 \end{bmatrix} \end{aligned}$$

which simplifies to the same result as given in Example 7.4. ▷

This method can also be used when the matrix $\mathbf{A}$ has repeated eigenvalues. Suppose the eigenvalues λ_i is repeated m times. Then corresponding to the first λ_i we have

$$e^{\lambda_i t} = \alpha_0(t) + \alpha_1(t)\lambda_i + \alpha_2(t)\lambda_i^2 + \cdots + \alpha_{n-1}(t)\lambda^{n-1}. \tag{5.35}$$

An additional $m - 1$ equations are obtained by repeated differentiation of (5.35) with respect to λ_i. In other words, we have the following chain of equations for the repeated eigenvalue λ_i:

$$\begin{aligned}
te^{\lambda_i t} &= \alpha_1 + 2\alpha_2\lambda_i + 3\alpha_3\lambda_i^2 + \cdots + (n-1)\alpha_{n-1}\lambda_i^{n-2} \\
t^2e^{\lambda_i t} &= 2\alpha_2 + 6\alpha_3\lambda_i + \cdots + (n-1)(n-2)\alpha_{n-1}\lambda_i^{n-3} \\
\vdots \qquad & \qquad\qquad \vdots \\
t^{m-1}e^{\lambda_i t} &= \frac{d^{m-1}}{d\lambda_i^{m-1}}\{\alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \cdots + \alpha_{n-1}\lambda_i^{n-1}\}.
\end{aligned}$$

The above set of equations together with (5.35) are then solved to obtain the unknown coefficients in (5.33). This can be seen from the following example:

Example 5.5

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

The eigenvalues of the matrix $\mathbf{A}$ are $\{0, 0, -1\}$. In view of the above discussions, we have

$$\begin{aligned}
e^{\mathbf{A}t} &= \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 \\
e^{\lambda_i t} &= \alpha_0(t) + \alpha_1(t)\lambda_i + \alpha_2(t)\lambda_i^2.
\end{aligned}$$

Using the three eigenvalues we then obtain

$$\begin{aligned}
e^{0t} &= \alpha_0 + \alpha_1 0 + \alpha_2 0^2 \\
te^{0t} &= \alpha_1 + 2\alpha_2 0 \\
e^{-t} &= \alpha_0 + \alpha_1(-1) + \alpha_2(-1)^2,
\end{aligned}$$

which yield

$$\begin{aligned}
\alpha_0 &= 1 \\
\alpha_1 &= t \\
\alpha_2 &= e^{-t} - 1 + t.
\end{aligned}$$

Using these coefficients we obtain

$$\begin{aligned}
e^{\mathbf{A}t} &= \begin{bmatrix} \alpha_0 & -2\alpha_1 & \alpha_1 - 3\alpha_2 \\ 0 & \alpha_0 & \alpha_1 - \alpha_2 \\ 0 & 0 & \alpha_0 - \alpha_1 + \alpha_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -2t & 3e^{-t} - 2t + 3 \\ 0 & 1 & -e^{-t} + 1 \\ 0 & 0 & e^{-t} \end{bmatrix}.
\end{aligned}$$

▷

Example 5.6

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y = [\,1 \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Suppose a unit step input is applied to the system at time $t = 0$. We would like to investigate the system performance at an arbitrary time.

We first find the state transition matrix

$$e^{\mathbf{A}(t-t_0)} = \begin{bmatrix} 3e^{-2(t-t_0)} - 2e^{-3(t-t_0)} & e^{-2(t-t_0)} - e^{-3(t-t_0)} \\ -6e^{-2(t-t_0)} + 6e^{-3(t-t_0)} & -2e^{-2(t-t_0)} + 3e^{-3(t-t_0)} \end{bmatrix}$$

so that the response of the system is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+ \int_0^t \begin{bmatrix} 3e^{-2(t-\tau)} - 2e^{-3(t-\tau)} & e^{-2(t-\tau)} - e^{-3(t-\tau)} \\ -6e^{-2(t-\tau)} + 6e^{-3(t-\tau)} & -2e^{-2(t-\tau)} + 3e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 \; d\tau$$

$$= \begin{bmatrix} 3e^{-2t} - 2e^{-3t} \\ -6e^{-2t} + 6e^{-3t} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} - \frac{5}{2}e^{-2t} - \frac{5}{3}e^{-3t} \\ -5e^{-2t} + 5e^{-3t} \end{bmatrix}.$$

The output $\mathbf{y}(t)$ is then obtained as

$$\mathbf{y}(t) = [\,1 \quad 0\,] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$= \frac{1}{6} - \frac{5}{2}e^{-2t} - \frac{5}{3}e^{-3t}.$$

▷

Example 5.7

Consider the system given in Example 7.6, and assume that the initial state of the system is zero. Let the control input is a unit impulse applied at $t = 1$ second. Then the response of the system is obtained as

$$
\begin{aligned}
y(t) &= \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau-1)\,d\tau \\
&= [1 \quad 0]\int_0^t \begin{bmatrix} 3e^{-2(t-\tau)} - 2e^{-3(t-\tau)} & e^{-2(t-\tau)} - e^{-3(t-\tau)} \\ -6e^{-2(t-\tau)} + 6e^{-3(t-\tau)} & -2e^{-2(t-\tau)} + 3e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta(\tau-1)\,d\tau \\
&= \int_0^t \left[e^{-2(t-\tau)} - e^{-3(t-\tau)}\right]\delta(\tau-1)\,d\tau \\
&= \begin{cases} 0 & \text{for } t \le 1 \\ e^{-2(t-1)} - e^{-3(t-1)} & \text{for } t > 1 \end{cases}
\end{aligned}
$$

where the last equality is obtained using

$$\int_a^b f(t)\delta(t-t_0)\,dt = f(t_0), \text{ where } a \le t_0 \le b$$

Note that prior to $t = 1$, the control input is zero so that the state of the system remained at the zero state, and the state evolved with time after the application of the impulse at $t = 1$.

▷

Example 5.8

The state $\mathbf{x}(t)$ of the following system at time $t = 2$ has been given. It is our objective to determine the system state at time $t = 0$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1(2) \\ x_2(2) \\ x_3(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

From (5.19), we have

$$\mathbf{x}(t_0) = e^{-\mathbf{A}(t-t_0)}\mathbf{x}(t),$$

so that

$$\mathbf{x}(0) = e^{-\mathbf{A}t}\mathbf{x}(t).$$

This gives

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 & -t & \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}_{t=2} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}_{t=2}$$

$$= \begin{bmatrix} 1 & -2 & \frac{4}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

▷

5.4 System Mode

As shown in the previous section, the response of the time invariant system is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0). \tag{5.36}$$

Also as shown in equation (5.28), the matrix $e^{\mathbf{A}t}$ can be expressed in terms of the eigenvalues and the eigenvectors of the matrix $\mathbf{A}$. If the matrix $\mathbf{A}$ has distinct eigenvalues, then from (5.28), we have

$$e^{\mathbf{A}t} = \sum_{i=1}^{n} e^{\lambda_i t}\, \mathbf{v}_i \mathbf{w}_i', \tag{5.37}$$

where λ_i's are the (distinct) eigenvalues, $\mathbf{v}_i$'s are the corresponding eigenvectors, and $\mathbf{w}_i$'s are the left eigenvectors of the matrix $\mathbf{A}$. This gives

$$\begin{aligned} \mathbf{x}(t) &= \sum_{i=1}^{n} e^{\lambda_i t}\, \mathbf{v}_i \mathbf{w}_i' \mathbf{x}(0) \\ &= \sum_{i=1}^{n} \alpha_i \, e^{\lambda_i t}\, \mathbf{v}_i, \qquad \text{where} \quad \alpha_i = \mathbf{w}_i'\mathbf{x}(0), \end{aligned} \tag{5.38}$$

where the terms $e^{\lambda_i t}\mathbf{v}_i$ are called the system mode. This last equation shows that the response of a system can be expressed as a linear combination of the various modes of the system.

A similar analysis can be carried out also for nonhomogeneous time invariant systems. In that case, we shall see that the coefficients $q_i(t)$ depend also on the external input in addition to the initial state $x(0)$. Consider the time invariant system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ \mathbf{x}(t_0) &= \mathbf{x}(t_0),\end{aligned} \tag{5.39}$$

The solution of this equation is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}x(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Using the equation (5.37) in the above equation, we have

$$\begin{aligned}\mathbf{x}(t) &= \sum_{i=0}^{n} e^{\lambda_i t}\mathbf{v}_i\mathbf{w}_i' x(0) + \int_0^t \sum_{i=0}^{n} e^{\lambda(t-\tau)}\mathbf{v}_i\mathbf{w}_i'\mathbf{B}\mathbf{u}(\tau)\,d\tau\\ &= \sum_{i=1}^{n} e^{\lambda_i t}\mathbf{v}_i\big[\mathbf{w}_i' x(0) + \sum_{i=1}^{n}\int_0^t e^{-\lambda_i\tau}\mathbf{w}_i'\mathbf{B}\mathbf{u}(\tau)\,d\tau\big]\end{aligned} \tag{5.40}$$

This last equation can be further expanded into

$$\begin{aligned}\mathbf{x}(t) &= \sum_{i=1}^{n}\alpha_i e^{\lambda_i t}\mathbf{v}_i + \sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^t e^{\lambda_i(t-\tau)}\mathbf{v}_i\mathbf{w}_i'\mathbf{b}_j u_j(\tau)\,d\tau\\ &= \sum_{i=1}^{n}\left[\alpha_i e^{\lambda_i t} + \sum_{j=1}^{m}\int_0^t e^{\lambda_i(t-\tau)}\mathbf{w}_i'\mathbf{b}_j u_j(\tau)\,d\tau\right]\mathbf{v}_i\\ &= \sum_{i=1}^{n}\left[\alpha_i + \sum_{j=1}^{m}\int_0^t e^{-\lambda_i\tau}\mathbf{w}_i'\mathbf{b}_j u_j(\tau)\,d\tau\right]e^{\lambda_i t}\mathbf{v}_i\end{aligned} \tag{5.41}$$

where $\mathbf{b}_j$ is the j-th column of $\mathbf{B}$, and $\mathbf{w}_j'\mathbf{b}_j$ is a scalar.

The modal decomposition (5.39) can be used to obtain a better insight of the dynamic behavior of the system.

a) An unstable system is easily identified from the presence of an unstable mode. Indeed if the system has a positive eigenvalue, then as seen from (5.40), the corresponding mode amplitude grows to infinity with time so that the state becomes unstable.

b) Suppose for a certain system the term $\mathbf{w}_i'\mathbf{B}$ is zero so that according to (5.40), the control input $\mathbf{u}$ will have no effect on the i-th mode. This means that the i-th mode of the system cannot be controlled, which implies that the system cannot be completely controlled by any control input.

c) Modal analysis is also useful in reducing the model order of large dimensional systems. Suppose for a system, the eigenvalue λ_k is very large and negative. This implies that, again from (5.40), the mode response $q_k(t)$ will quickly decay to zero, and will not have any significant effect on the system response $x(t)$.

d) If $\mathbf{w}_i'\mathbf{b}_j = 0$, then the control u_j will not have any effect on the i-th mode.

e) Finally, mode analysis also tells us about the inter-relationship among the various components of the state vector through the eigenvectors.

Example 5.9

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 1 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

We shall investigate the response of this system for various initial conditions. As shown in Example 5.4, the system matrix has three distinct eigenvalues $\{-1, -2, -3\}$. Various eigenvectors and reciprocal eigenvectors have also been computed in Example 5.4. Consider the transformation

$$\mathbf{x} = \mathbf{T}\mathbf{q},$$

where $\mathbf{T}$ is matrix of eigenvectors of the matrix $\mathbf{A}$. More explicitly, we take

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

Using this transformation, the mode equaiton of the system is expressed as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u,$$

which clearly shows three decoupled equations that govern the time variation of mode amplitudes. For simplicity, we shall assume that $u = 0$. Then using (5.38), we can express the system response as

$$\mathbf{x}(t) = \alpha_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 e^{-3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

where $\alpha_i = \mathbf{w}_i'\mathbf{x}(0)$ can be easily computed for any given initial condition. Considering the initial state $\mathbf{x}(0) = [\,2 \quad 0 \quad 2\,]'$, we obtain $\alpha = [\,0 \quad 0 \quad 2\,]'$ so that the response of the system will be given by

$$\mathbf{x}(t) = 2e^{-3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which shows that the response contains only the third mode. On the other hand for the initial state $\mathbf{x}(0) = [\,2 \quad 1 \quad 1\,]'$, one has $q(0) = [\,1 \quad 0 \quad 1\,]'$ resulting in the response

$$\mathbf{x}(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which contains the first and the third modes.

As mentioned above, system modes are excited also by control inputs. In this example, it is clear from the modal equation that the third mode will not be excited by the control input. Thus, no matter what control is applied, if the initial condition is zero, only the first and second modes will be excited. Recall that as shown in equation (5.40), if $\mathbf{w}_i'\mathbf{B} = 0$, then the control will have no effect on the i-th mode, and the system cannot be arbitrarily controlled, in other words, the system is 'uncontrollable'.

It is also interesting to notice that the output response of a system for certain initial conditions. Consider, for example, the initial state $\mathbf{x}(0) = [\,0 \quad 2 \quad 0\,]'$. For this initial state we have $\alpha = \mathbf{T}^{-1}\mathbf{x}(0) = [\,0 \quad 2 \quad 0\,]'$ which shows that the state response will consists of only the second mode giving

$$\mathbf{x}(t) = 2e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and the resulting output response will be

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{C}\mathbf{x} \\ &= e^{-2t}[\,1 \quad 0 \quad 0\,]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= 0 \qquad \text{for all } t > 0. \end{aligned}$$

In general, we have

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{C}\mathbf{x} \\ &= \sum_{i=1}^{n} \mathbf{C}e^{\lambda_i t}\mathbf{v}_i\mathbf{w}_i'\mathbf{x}(0) \\ &= \sum_{i=1}^{n} \alpha_i e^{\lambda_i t}\mathbf{C}\mathbf{v}_i \end{aligned}$$

Clearly if $\mathbf{C}\mathbf{v}_i = 0$, the corresponding mode will not have any effect in the system output, and therefore will not be 'visible' to the measurement system. This shows that under certain conditions, the system output may remain completely hidden from all measurement devices. Under such situations the system is known as "unobservable".

Mode analysis is also useful for disturbance rejecting controllers. In multi-input systems, it is possible to design controllers so that the closed loop system has an appropriate set of eigenvalues along with the corresponding eigenvectors and left eigenvectors. In particular, it is possible to choose left eigenvectors appropriately so as to make the control system insensitive to certain inputs or disturbances.

5.5 Linear Time Varying Systems

In this section we investigate the dynamic response of linear varying systems. Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{5.42}$$

where $\mathbf{x} \in \mathcal{R}^n$ is the state vector, and $\mathbf{u} \in \mathcal{R}^m$ is the control vector. The matrices $\mathbf{A}$ and $\mathbf{B}$ are of size $n \times n$ and $n \times m$, respectively. It is assumed that the entries of the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are piece-wise continuous. The response of the above system for $t \geq t_0$ is given by the following result:

Theorem 5.1. *The solution of the linear time varying system (5.42) is given by*

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau, \tag{5.43}$$

where the state transition matrix $\Phi(t, \tau)$ is the unique solution of the equation

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(t, \tau) &= \mathbf{A}(t)\Phi(t, \tau) \\ \Phi(\tau, \tau) &= \mathbf{I}, \end{aligned} \tag{5.44}$$

where $\mathbf{I}$ is the identity matrix of dimension $n \times n$.

Proof: The result can be verified by direct substitution. Indeed, differentiating (5.43) and using (5.44), we obtain

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{d}{dt}\Phi(t, t_0)\mathbf{x}(t_0) + \frac{d}{dt}\int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \\ &= \mathbf{A}(t)\Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{A}(t)\Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \Phi(t, t)\mathbf{B}(t)\mathbf{u}(t) \\ &= \mathbf{A}(t)\big[\Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau\big] + \mathbf{B}(t)\mathbf{u}(t) \\ &= \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}. \end{aligned}$$

In the above analysis we have used the Leibnitz rule for differentiation

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)\,d\tau = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,\tau)\,d\tau + \frac{db}{dt} f(t,\tau)|_{\tau=b(t)} - \frac{da}{dt} f(t,\tau)|_{\tau=a(t)}. \quad (5.45)$$

The first term on the right hand side of (5.43) is the response of the system due to the initial state $\mathbf{x}_0$. This is also known as the **zero input response**, which means that this is the response of the system if the control input was zero. The second term of (5.43) is due to the specific control input $\mathbf{u}$. This part of the response is also known as the **zero state response**, which means that this is the response for zero initial condition.

The state transition matrix $\Phi(t, t_0)$ plays a significant role in the transition of the system state $\mathbf{x}(t_0)$ at time t_0 to the state $\mathbf{x}(t)$ at time t. The matrix $\Phi(t, t_0)$ "carries" the initial state $\mathbf{x}(t_0)$ to the final state $\mathbf{x}(t)$ through the operation $\Phi(t, t_0)\mathbf{x}(t_0)$; in other words, $\Phi(t, t_0)$ "causes" the transition of the system state from $\mathbf{x}(t_0)$ to $\mathbf{x}(t)$, and hence it is called the state transition matrix.

The state transition matrix $\Phi(t, \tau)$ satisfies the general properties:

1) $\Phi(t_2, t_1)\Phi(t_1, t_0) = \Phi(t_2, t_0)$
2) $\Phi(t, \tau)$ is nonsingular, and $\phi^{-1}(t, \tau) = \Phi(\tau, t)$
3) $\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau)$
4) $\frac{\partial}{\partial \tau}\Phi(t, \tau) = -\Phi(t, \tau)A(\tau)$

An important property of the state transition matrix is the semigroup property. Let the state of the system at time t_0, t_1, and t_2 be denoted as $\mathbf{x}(t_0), \mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$, respectively. Then in light of (5.43) with $u = 0$, we have

$$\begin{aligned} \mathbf{x}(t_2) &= \Phi(t_2, t_0)\mathbf{x}(t_0) \\ \mathbf{x}(t_2) &= \Phi(t_2, t_1)\mathbf{x}(t_1) \\ \mathbf{x}(t_1) &= \Phi(t_1, t_0)\mathbf{x}(t_0). \end{aligned}$$

Combining the last two equations and comparing with the first equation, we obtain

$$\mathbf{x}(t_2) = \Phi(t_2, t_1)\Phi(t_1, t_0)\mathbf{x}(t_0) = \Phi(t_2, t_0)\mathbf{x}(t_0),$$

which shows that for arbitrary time t_0, t_1, t_2,

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0). \quad (5.46)$$

This proves the semigroup property. For the second property of the state transition matrix, we observe that

$$\mathbf{I} = \Phi(t_0, t_0) = \Phi(t_0, t)\Phi(t, t_0.)$$

This clearly shows that we have

$$\Phi^{-1}(t, t_0) = \Phi(t_0, t). \tag{5.47}$$

To prove the last property, note that the following identity holds for any t and τ:

$$\mathbf{I} = \Phi(t, \tau)\Phi(\tau, t).$$

Therefore, differentiating the above equation with respect to τ, and using the properties (2) and (3) stated above, we obtain

$$\begin{aligned} 0 &= \frac{d}{d\tau}\big[\Phi(t, \tau)\Phi(\tau, t)\big] \\ &= \big[\frac{\partial}{\partial \tau}\Phi(t, \tau)\big]\Phi(\tau, t) + \Phi(t, \tau)\big[\frac{\partial}{\partial \tau}\Phi(\tau, t)\big] \\ &= \big[\frac{\partial}{\partial \tau}\Phi(t, \tau)\big]\Phi(\tau, t) + \Phi(t, \tau)A(\tau)\Phi(\tau, t). \end{aligned}$$

Thus we have

$$\big[\frac{\partial}{\partial \tau}\Phi(t, \tau)\big]\Phi(\tau, t) = -\Phi(t, \tau)\mathbf{A}(\tau)\Phi(\tau, t),$$

so that

$$\begin{aligned} \frac{\partial}{\partial \tau}\Phi(t, \tau) &= -\Phi(t, \tau)\mathbf{A}(\tau)\Phi(\tau, t)\Phi^{-1}(\tau, t) \\ &= -\Phi(t, \tau)\mathbf{A}(\tau)\Phi(\tau, t)\Phi(t, \tau) \\ &= -\Phi(t, \tau)\mathbf{A}(\tau)\Phi(\tau, \tau) \\ &= -\Phi(t, \tau)\mathbf{A}(\tau). \end{aligned}$$

In general, it is very difficult to compute $\Phi(t, \tau)$ for time varying systems except for some simple cases. Analytical expression of the state transition matrix can be derived for a class of systems for which the matrices $\mathbf{A}(t)$ and $\int_\tau^t \mathbf{A}(\theta)\, d\theta$ commute, i.e., if

$$\mathbf{A}(t)\bigg(\int_\tau^t \mathbf{A}(\theta)\, d\theta\bigg) = \bigg(\int_\tau^t \mathbf{A}(\theta)\, d\theta\bigg)\mathbf{A}(t), \tag{5.48}$$

the state transition matrix is given by

$$\Phi(t, \tau) = \exp\bigg[\int_\tau^t \mathbf{A}(\theta)\, d\theta\bigg]. \tag{5.49}$$

The result may be proved by direct differentiation. Note that the solution of the homogeneous system is given by $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$. By differentiation and using infinite series expansion for exponential, we obtain

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \frac{d}{dt}\Phi(t,t_0)\mathbf{x}(t_0)\\
&= \frac{d}{dt}\left[\exp\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau\right]\mathbf{x}(t_0)\\
&= \frac{d}{dt}\left[\mathbf{I}+\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau+\frac{1}{2!}\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau\int_{t_0}^{t}\mathbf{A}(\theta)\,d\theta+\cdots\right]\mathbf{x}_0\\
&= \left[\mathbf{A}(t)+\frac{1}{2!}\mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}(\theta)\,d\theta+\frac{1}{2!}\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau\mathbf{A}(t)+\cdots\right]\mathbf{x}(t_0)\\
&= \left[\mathbf{A}(t)+\mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau+\frac{1}{2!}\mathbf{A}(t)\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau\int_{t_0}^{t}\mathbf{A}(\theta)\,d\theta+\cdots\right]\mathbf{x}(t_0)\\
&= \mathbf{A}(t)\left[\mathbf{I}+\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau+\frac{1}{2!}\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau\int_{t_0}^{t}\mathbf{A}(\theta)\,d\theta+\cdots\right]\mathbf{x}(t_0)\\
&= \mathbf{A}(t)\left[\exp\int_{t_0}^{t}\mathbf{A}(\tau)\,d\tau\right]\mathbf{x}(t_0)\\
&= \mathbf{A}(t)\mathbf{x}(t).
\end{aligned}$$

Some time varying systems that satisfy the commutativity property are as follows:

1) $\mathbf{A}$ is a constant matrix. In this case the result simplifies to that of time invariant system, i.e., $\Phi(t,\tau)=e^{\mathbf{A}(t-\tau)}$.
2) $\mathbf{A}$ is a diagonal time varying matrix.
3) $\mathbf{A}=\bar{\mathbf{A}}f(t)$ where $\bar{\mathbf{A}}$ is a constant matrix and f is a scalar time varying function. In this case $\Phi(t,\tau)=e^{\bar{\mathbf{A}}\int_{\tau}^{t}f(s)\,ds}$.
4) The matrix $\mathbf{A}$ can be expressed as $\mathbf{A}(t)=\sum_i^k\mathbf{M}_i f_i(t)$, where for each i, $\mathbf{M}_i$ is a constant matrix satisfying $\mathbf{M}_i\mathbf{M}_j=\mathbf{M}_j\mathbf{M}_i$, and $f_i(t)$ is a scalar time varying function. Then

$$\Phi(t,\tau)=\Pi_i^k\exp\big(\mathbf{M}_i\int_{\tau}^{t}f_i(s)\,ds\big) \tag{5.50}$$

Example 5.10

Suppose

$$\mathbf{A}(t) = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}.$$

Then one can easily verify that the matrix $\mathbf{A}(t)$ and $\int_{t_0}^{t} \mathbf{A}(\theta)\, d\theta$ commute. Therefore the state transition matrix is given by

$$\begin{aligned} \Phi(t,\tau) &= \exp\left[\int_{\tau}^{t} \mathbf{A}(\theta)\, d\theta\right] \\ &= \exp\begin{bmatrix} t-\tau & 0 \\ 0 & \frac{1}{2}(t^2-\tau^2) \end{bmatrix} \\ &= \begin{bmatrix} e^{t-\tau} & 0 \\ 0 & e^{\frac{1}{2}(t^2-\tau^2)} \end{bmatrix}. \end{aligned}$$

Example 5.11

Consider the matrix

$$\mathbf{A}(t) = \begin{bmatrix} 2t & 1 \\ -1 & 2t \end{bmatrix}.$$

Then we verify that

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} t,$$

which is of the form $\mathbf{A}(t) = \mathbf{M}_1 f_1(t) + \mathbf{M}_2 f_2(t)$ with $f_1 = 1$ and $f_2 = t$. Furthermore, $\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2\mathbf{M}_1$. Thus the state transition matrix is obtained as

$$\begin{aligned} \Phi(t,\tau) &= \exp\left(\mathbf{M}_1 \int_{\tau}^{t} f_1(s)\, ds\right) \exp\left(\mathbf{M}_2 \int_{\tau}^{t} f_2(s)\, ds\right) \\ &= \exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \int_{\tau}^{t} 1\, ds\right) \exp\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \int_{\tau}^{t} s\, ds\right) \\ &= \exp\left(\begin{bmatrix} 0 & t-\tau \\ -(t-\tau) & 0 \end{bmatrix}\right) \exp\left(\begin{bmatrix} t^2-\tau^2 & 0 \\ 0 & t^2-\tau^2 \end{bmatrix}\right) \\ &= \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix} \begin{bmatrix} e^{t^2-\tau^2} & 0 \\ 0 & e^{t^2-\tau^2} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix} e^{t^2-\tau^2}. \end{aligned}$$

5.6 Solution of Nonlinear Systems

Most practical systems of engineering interest are nonlinear, and linearity is actually an exception than a rule. Nevertheless, we tend to use the linear model if the nonlinearity is 'mild', and more importantly one can easily derive simple solutions for linear systems. In this section we investigate the response of nonlinear dynamic systems.

Consider the system

$$\begin{aligned} \dot{x} &= f(t,x) \\ x(0) &= x_0 \end{aligned} \tag{5.51}$$

where $x \in R^n$ is the state vector, and f is a nonlinear function which is continuous in both t and x. Then the response of the system can be expressed as

$$x(t) = x_0 + \int_0^t f(t, x(\tau))\, d\tau \tag{5.52}$$

Note however that continuity of f in both t and x is too strong for many engineering applications. One can show that the system (5.51) has a solution under milder conditions, such as

$$\begin{aligned} &\|f(t,x) - f(t,y)\| \leq k_1 \|x - y\| \qquad \text{Lipschitz condition} \\ &\|f(t,x)\| \leq k_2 \qquad \text{for all } t, x \end{aligned}$$

Furthermore, the solution is unique, and may be contructed using the sequence, known as Picard approximation:

$$\begin{aligned} x_1(t) &= x_0 \\ x_{k+1} &= x_0 + \int_0^t f(\tau, x_k(\tau))\, d\tau \end{aligned}$$

where k is the iteration counter. The solution $x_k(t)$ converges to the true colution as $k \to \infty$.

In the following example we derive the solution of linear system using Picard method. For nonlinear systems, it is almost impossible to find a closed form analytical solution, which necessiates the use of numerical methods.

Example 5.12 Picard Approximation

Consider the linear system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

The above equation is equivalent to

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{A}\mathbf{x}(\tau)\, d\tau$$

Using this equation, one can form an approximating sequence

$$\mathbf{x}_n(t) = x_0 + \int_0^t \mathbf{A}\mathbf{x}_{n-1}(\tau)\, d\tau$$

to generate the correct solution, where $\mathbf{x}_n(t)$ is the n-th approximation of the solution $\mathbf{x}(t)$.

Using the above sequence, we find successive approximations as

$$\begin{aligned}
\mathbf{x}_1(t) &= \mathbf{x}_0 + \int_0^t \mathbf{A}\mathbf{x}_0\, d\tau \\
&= \mathbf{x}_0 + \mathbf{A}t\mathbf{x}_0 \\
&= (\mathbf{I} + \mathbf{A}t)\mathbf{x}_0
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_2(t) &= \mathbf{x}_0 + \int_0^t \mathbf{A}\mathbf{x}_1\, d\tau \\
&= \mathbf{x}_0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}\tau)\mathbf{x}_0\, d\tau \\
&= (\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2})\mathbf{x}_0
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_3(t) &= \mathbf{x}_0 + \int_0^t \mathbf{A}\mathbf{x}_2\, d\tau \\
&= \mathbf{x}_0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}\tau + \frac{\mathbf{A}^2 t^2}{2})\mathbf{x}_0\, d\tau \\
&= (\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2} + \frac{\mathbf{A}^3 t^3}{3!})\mathbf{x}_0
\end{aligned}$$

Then by induction,

$$\begin{aligned}
\mathbf{x}_n(t) &= \mathbf{x}_0 + \int_0^t \mathbf{A}\mathbf{x}_{n-1}\, d\tau \\
&= (\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots + \frac{\mathbf{A}^n t^n}{n!})\mathbf{x}_0
\end{aligned}$$

In the limit as $n \to \infty$, one then obtains

$$\begin{aligned}
\mathbf{x}(t) &= \lim_{n\to\infty} (\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots + \frac{\mathbf{A}^n t^n}{n!})\mathbf{x}_0 \\
&= e^{\mathbf{A}t}\mathbf{x}(0)
\end{aligned}$$

5.7 Stochastic Systems

There are many dynamic systems of engineering interest that are driven by random noise. For example, a satellite in geosynchronous orbit is perturbed by random solar pressure. The motion of a satellite in low earth orbit is affected by variations in earth's magnetic field. The load applied to a motor is often random. The jerking of a car on an uneven road surface is random. Analysis of these types of systems requires the use of stochastic differential equations. This section introduces the fundamentals of stochastic systems. First we review some fundamentals concepts from probability and random processes.

Random Variable

Let S denote the sample space. A probability distribution function $F_x : R \to R$ for the random variable x is defined as

$$F_x(c) = P\{\xi \in S : x(\xi) \leq c\}$$

The probability density function of a random variable is defined by the property

$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f_x(x)\, dx \tag{5.53}$$

where $P(x_1 \leq x \leq x_2)$ denotes the probability of event $x_1 \leq x \leq x_2$. For Gaussian or normal random variables, the probability density function is given by

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\hat{x})^2}{2\sigma^2}\right] \tag{5.54}$$

where $\hat{x}$ and σ^2 are the mean and variance of x defined by

$$\begin{aligned} \hat{x} = E\{x\} &= \int_{-\infty}^{\infty} x\, f_x(x)\, dx \\ \operatorname{var} x = \sigma^2 = E\{(x-\hat{x})^2\} &= \int_{-\infty}^{\infty} (x-\hat{x})^2\, f_x(x)\, dx \end{aligned} \tag{5.55}$$

For vector valued random variables, mean and variance are defined in a similar way. Let $\mathbf{x} = [\,x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_n\,]'$ be a random vector. Then the mean is defined as

$$\hat{\mathbf{x}} = \begin{bmatrix} E\{x_1\} \\ E\{x_2\} \\ \vdots \\ E\{x_n\} \end{bmatrix} \tag{5.56}$$

and the covariance matrix is

$$\mathbf{P} = E\{(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})'\} \tag{5.57}$$

The standard deviation of the random vector is given by

$$\text{Trace } \mathbf{P} = E\{\sum_{i}^{n}(x_i - \hat{x}_i)^2\} \tag{5.58}$$

The density function of a Gaussian random vector is given by

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n|\mathbf{P}|}} \exp\left[-\frac{1}{2}(\mathbf{x}-\hat{\mathbf{x}})\mathbf{P}^{-1}(\mathbf{x}-\hat{\mathbf{x}})'\right] \tag{5.59}$$

$|\mathbf{P}|$ is the determinant of the covariance matrix $\mathbf{P}$.

Random process

A stochastic process $\{x(t), t \in T\}$ is a collection of random variables defined on the probability space with t usuallly denoting time, and is defined using the basic concepts of random variables. Let $\mathbf{x}(t)$ be a n-vector values stochastic process. Then we have

$$\begin{aligned} \hat{\mathbf{x}}(t) &= E\{\mathbf{x}(t)\} \\ \text{cov } [\mathbf{x}(t)\mathbf{x}(s)] &= R(t,s) = E\{(\mathbf{x}(t) - \hat{\mathbf{x}}(t)(\mathbf{x}(s) - \hat{\mathbf{x}}(s))'\} \end{aligned} \tag{5.60}$$

The process $\mathbf{x}(t)$ is completely characterized by the mean and the covariance matrix. Note also that $\sigma^2(t) = R(t,t)$. For stationary processes, the mean and the covariance matrix R are independent of time, and R depends only on $t - s$ so that $\sigma^2 = R(0)$.

Gaussian Process

Gaussian (also known as white noise) process is the most widely used stochastic process in engineering literature because of its analytical simplicity. A stochastic process is Gaussian if every linear combination $\sum_{i=1}^{n} a_i x(t_i), t_i \in T$ is a Gaussian random variable. The correlation function of a Gaussian process is given by a delta function,

$$\sigma^2 = R(t,\tau) = E\{x(t)x(s)\} = q\,\delta(\tau)$$

where $\tau = |t - s|$, i.e., the processes $x(t)$ and $x(s)$ are uncorrelated for $t \neq s$. Secondly the spectral density of a Gaussian process is calculated as

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} q\,\delta(t)\,dt = \frac{q}{2\pi}$$

which means the spectral density is constant for all frequencies. A white noise process thus spreads uniformly over all frequencies, and because of this property the phrase 'white' is used to denote Gaussian processes much like the way 'white' light is the contribution of lights of all color. Also note that $\sigma^2(0) = \infty$. There is no such process that has these properties, nevertheless, one can argue that the noise process is uniform over the frequency band that is of interest for the particular application. In any case, because of its simplicity, Gaussian white noise it has been widely used in engineering literature for analysis of many applications.

Dynamic Systems in the Presence of Gaussian Noise

Dynamic systems are frequently acted upon by external disturbance. Following the standard notation of deterministic systems, in engineering literature it is customary to write models of dynamic systems in the presence of noise as

$$\dot{x}(t) = A(t)x(t) + H(t)u(t) + B(t)v(t) \tag{5.61}$$

where $x \in R^n$ is the state vector, $u \in R^r$ is the control vector, and $v \in R^m$ is the Gaussian noise process with zero mean, and covariance $Q(t)$. We also assume that the noise process is independent of the initial state x_0.

$$\begin{aligned} E\{v(t)v'(s)\} &= Q(t)\delta(t-s) \\ E\{x_0 v'(t)\} &= 0 \qquad \text{for all } t \end{aligned} \tag{5.62}$$

In what follows we shall study the response of the uncontrolled system, and suppress the term involving the control input u. The matrices A and B are assumed to be deterministic, i.e., there is no randomness in the elements of these matrices, but are possibly time varying. Then the response of the system can be given in terms of the state transition matrix of the unperturbed system:

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)\,v(\tau)\,dt \tag{5.63}$$

where $\Phi(t,\tau)$ is the state transition matrix corresponding to the system matrix A. As shown in previous section, in case the system matrix A is time invariant, the state transition matrix can be given in terms of the matrix exponential, $\Phi(t,\tau) = e^{A(t-\tau)}$. Taking expectation of equation (5.63), we have

$$E\{x(t)\} = \Phi(t,0)E\{x_0\} + E\{\int_0^t \Phi(t,\tau)B(\tau)\,v(\tau)\,dt\}$$

Since Φ and B are deterministic, the above equation simplifies to

$$E\{x(t)\} = m(t) = \Phi(t,0)m_0 \tag{5.64}$$

where $m_0 = E\{x_0\}$ is the mean of the initial state. Since $\Phi(t,\tau)$ is the state transition matrix corresponding to the system matrix A, differentiating the above equation, it follows that the mean of the process $x(t)$ satisfies

$$\begin{aligned} \frac{dm}{dt} &= A(t)m \\ m(0) &= m_0 \end{aligned} \tag{5.65}$$

Next we derive covariance of the process $x(t)$. Denote

$$P(t) = E\{x(t)x'(t)\} \tag{5.66}$$

Substituting (5.63) into the above equation, we obtain

$$
\begin{aligned}
P(t) &= E\{[\Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)\,v(\tau)\,d\tau][\Phi(t,0)x_0 + \int_0^t \Phi(t,s)B(s)\,v(s)\,ds]'\} \\
&= E\{\Phi(t,0)x_0x_0'\Phi'(t,0)\} \\
&\quad + E\{\int_0^t \Phi(t,\tau)B(\tau)\,v(\tau)\,x_0'\Phi'(t,0)\,d\tau\} + E\{\int_0^t \Phi(t,0)x_0\,v'(s)\,B'(s)\Phi'(t,s)\,ds\} \\
&\quad + E\{\int_0^t \int_0^t \Phi(t,\tau)B(\tau)\,v(\tau)v'(s)\,B'(s)\Phi'(t,s)\,d\tau\,ds\}
\end{aligned}
$$

First of all note that x_0 is independent of the noise process $v(t)$ so that the second and third terms of the above equation reduce to zero. Thus we have

$$
P(t) = E\{\Phi(t,0)x_0x_0'\Phi'(t,0)\} + E\{\int_0^t \int_0^t \Phi(t,\tau)B(\tau)\,v(\tau)v'(s)\,B'(s)\Phi'(t,s)\,d\tau\,ds\}
$$

Next we note that

$$
E\{v(\tau)\,v'(s)\} = Q(\tau)\,\delta(\tau - s), \quad \tau > s
$$

so that we have

$$
P(t) = E\{\Phi(t,0)x_0x_0'\Phi'(t,0)\} + \int_0^t \int_0^t \Phi(t,\tau)B(\tau)Q(\tau)\delta(\tau - s)\,B'(s)\Phi'(t,s)\,d\tau\,ds
$$

Carrying out the integration with respect to τ, the above equation further simplifies to

$$
\begin{aligned}
P(t) &= E\{\Phi(t,0)x_0x_0'\Phi'(t,0)\} + \int_0^t \Phi(t,s)B(s)Q(s)\,B'(s)\Phi'(t,s)\,ds \\
&= \Phi(t,0)P(0)\Phi'(t,0) + \int_0^t \Phi(t,s)B(s)Q(s)\,B'(s)\Phi'(t,s)\,ds
\end{aligned}
\tag{5.67}
$$

Although this equation gives the covariance of the process $x(t)$, it is not of much use because of the its computational complexities. In particular, one has to compute the state transition matrix $\Phi(t,\tau)$, which is a very difficult job for time varying systems. In what follows we derive a simpler expression for the covariance matrix. Indeed, by direct differentiation of the above equation, and using the properties of the state transition matrix, we obtain

$$\begin{aligned}
\dot{P}(t) = {} & A(t)\Phi(t,0)P_0\Phi'(t,0) + \Phi(t,0)P_0\Phi'(t,0)A'(t) \\
& + \int_0^t A(t)\Phi(t,\tau)B(\tau)Q(\tau)B'(\tau)\Phi'(t,\tau)\,d\tau \\
& + \int_0^t \Phi(t,\tau)B(\tau)Q(\tau)B'(\tau)\Phi'(t,\tau)A'(t)\,d\tau \\
& + \Phi(t,t)B(t)Q(t)B'(t)\Phi'(t,t)
\end{aligned}$$

$$\begin{aligned}
& = A(t)[\Phi(t,0)P_0\Phi'(t,0) + \int_0^t \Phi(t,\tau)B(\tau)Q(\tau)B'(\tau)\Phi'(t,\tau)\,d\tau] \\
& \quad + [\Phi(t,0)P_0\Phi'(t,0) + \int_0^t \Phi(t,\tau)B(\tau)Q(\tau)B'(\tau)\Phi'(t,\tau)\,d\tau]A'(t) + B(t)Q(t)B'(t) \\
& = A(t)P(t) + P(t)A'(t) + B(t)Q(t)B'(t)
\end{aligned}$$

This shows that the covariance matrix satisfies

$$\begin{aligned}
\dot{P}(t) &= A(t)P(t) + P(t)A'(t) + B(t)Q(t)B'(t) \\
P(0) &= P_0 = E\{x_0x_0'\}
\end{aligned} \tag{5.68}$$

In case A, B, and Q are time invariant, and A has all eigenvalues with negative real parts, the variance P becomes a constant as $t \to \infty$, and satisfies

$$AP + PA + BQB' = 0$$

Example 5.13

Consider the linear system

$$\dot{x} = -\alpha x + \beta\, v(t)$$

where α and β are constant. We assume that $E\{v(t)\,v(\tau)\} = q\,\delta(t-\tau)$, and $E\{x^2(0)\} = p_0$. Then using the result given above,

$$\dot{p}(t) = -2\alpha p + \beta^2 q$$

whose solution is

$$p(t) = p_0e^{-2\alpha t} + \frac{\beta^2}{2\alpha}q(1 - e^{-2\alpha t})$$

This shows that covariance of the process evolves with time, and becomes stationary at $p(t \to \infty) = \frac{\beta^2}{2\alpha}q$.

5.8 Stochastic Differential Equation

The dynamic model (5.61) is mathematically not very precise for various reasons. First of all in order for the system to be causal, it is necessary that the noise $v(t)$ and $v(s)$ be independent for $t \neq s$ so that $x(t)$ depends only on current state and current forcing term, and not on any previous terms. This requires that $v(t)$ be a white noise. Next we note that if the noise $v(t)$ is white Gaussian, which has infinite variance, the variance of $\dot{x}(t)$ is also expected to be infinite. This cannot be accepted since this will require that rate of change of $x(t)$ will be infinite at least for some samples. Furthermore calculation of $x(t)$ requires that the right hand side of (5.61) be integrated with respect to time, which is mathematically difficult to justify because of the presence of the Gaussian noise as the integrand. Since the covariance of a Gaussian process is a delta function, it is possible that some samples of $v(t)$ will be infinity, and therefore it is not possible to do integration in the Riemann sense. An alternate but mathematically clean approach for analyzing systems in the presence of noise can be given using the Wiener process. The following is a brief introduction of Wiener process and its application for analysis of dynamic systems.

Wiener Process

In 1828, botanist Robert Brown observed under microscope that pollen particles suspended in fluid exhibit extremely irregular, random jittary motions. Then in 1905 Albert Einstein suggested that the motion of pollen particles is due to bombardment of the particles (about 1μm in diameter) by molecules of water (1 nm in diamter), and obtained mathematical equations describing their motion. Mathematical foundation of the Brownian motion was given by Norman Wiener in 1931 so that Brownian motion is also frequently known as Wiener process.

A normally distributed process whose increments are white Gaussian is a Wiener process. Some of the properties of Wiener process are:

a) it is continuous with probability 1.

b) its sample path is nowhere differentiable with probability 1

c) its increments, $dW(t_i) = W(t_{i+1}) - W(t_i)$, is a Gaussian process with mean zero, and variance equal to the time increment $t_{i+1} - t_i$, that is, $dW(t_i) = N(0, (t_{i+1} - t_i))$.

d) over disjoint intervals, its increments are independent random variables, i.e., for $0 \leq s < t \leq u < v$, the increments $W(t) - w(s)$ and $W(v) - W(u)$ are independent.

The following properties are satisfied by the Wiener process:

$$\begin{aligned}
W(0) &= 0 \\
E\{dW(t_i)\} = E\{W(t_{i+1}) - W(t_i)\} &= 0 \\
E\{(W(t) - W(\tau))(W(t) - W(\tau))'\} &= (t - \tau)\, Q, \quad t > \tau, \quad Q \\
E\{W(t)W'(s)\} &= Q \ \min\,(t, s) \\
E\{dW(t)dW'(t) &= Q(t)\, dt
\end{aligned}$$

where Q is the incremental covariance matrix, which is also known as 'diffusion', and for standard Wiener processes, Q is an identity matrix.

Formally the Wiener process $W(t)$ is the derivative of the white noise process $v(t)$, i.e.,

$$dW(t) = v(t)dt$$

Note however that this equation should not be interpreted as

$$v(t) = \frac{dW(t)}{dt}$$

since Wiener is nowhere differentiable.

Ito Stochastic Integral

Ito stochastic integrals (named after Kiyoshi Ito) form the foundation of analysis of systems in the presence of noise. The most basic form of Ito ingral is given as

$$I(t) = \int_{t_0}^{t} f(\tau)\, dW(\tau) \tag{5.69}$$

where $f(\tau)$ is a L_2 function, and W is a Wiener process. The above integral cannot be interpreted as a Lebesgue integral. It cannot be interpreted also as a Stieltje's integral since $W(t)$ is not a function of bounded variation. In fact, Wiener process is a function of unbounded variation. This integration may be formally interpreted (in the sense of Reimann-Stieltje's integral) as the limit

$$I(t) = \lim_{N\to\infty}\left[\sum_{i=0}^{N-1} f(t_i)(W(t_{i+1}) - W(t_i))\right]$$

where the time domain is partitioned in N sub-intervals as $t_0 < t_1 < t_2 \cdots < t_{N_1} < t_N = t$. Here the Since W is a Wiener process, for eact t, $I(t)$ is a stochastic process as well, and has the following properties:

a) $I(t)$ is a Markov process, $E\{I(t+dt)\} = I(t)$.

b) E{I(t)} $= 0$

c) The inverse of the Ito integral is expressed as
$dI(t) = I(t+dt) - I(t) = f(t)\, dW(t)$

d) $E\left(\int_0^t f(\tau)\, dW(\tau)\right)^2 = \int_0^t E\{|f(\tau)|^2\}, d\tau$

In case f is a function of the random process, the stochastic integral (5.69) is also well defined, but more complex.

Stochastic Differential Equation

Representing the noise as a Wiener process, the dynamic system (5.61) is expressed as

$$\begin{aligned} dx &= A(t)x\,dt + B(t)dW(t) \\ x(0) &= x_0 \end{aligned} \tag{5.70}$$

Then the response of the system can be given in terms of the state transition matrix of the unperturbed system:

$$x(t) = \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)\,dW(\tau) \tag{5.71}$$

where $\Phi(t,\tau)$ is the state transition matrix corresponding to A. In what follows we show that the results derived earlier remain valid for the system model with the Wiener process.

Taking expectation of equation (5.71), we have

$$E\{x(t)\} = \Phi(t,0)E\{x_0\} + E\{\int_0^t \Phi(t,\tau)B(\tau)\,dW(\tau)\}$$

which simplifies to

$$E\{x(t)\} = m(t) = \Phi(t,0)m_0 \tag{5.72}$$

where $m_0 = E\{x_0\}$ is the mean of the initial state as shown earlier. Next we derive covariance of the process $x(t)$. Denote

$$P(t) = E\{x(t)x'(t)\} \tag{5.73}$$

Substituting equation (5.71) into the above equation

$$\begin{aligned} P(t) &= E\{[\Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)\,dW(\tau)][\Phi(t,0)x_0 + \int_0^t \Phi(t,s)B(s)\,dW(s)]'\} \\ &= E\{\Phi(t,0)x_0x_0'\Phi'(t,0)\} \\ &\quad + E\{\int_0^t \Phi(t,\tau)B(\tau)\,dW(\tau)x_0'\Phi'(t,0)\} + E\{\int_0^t \Phi(t,0)x_0 dW(s)B'(s)\Phi'(t,s)\} \\ &\quad + E\{\int_0^t \int_0^t \Phi(t,\tau)B(\tau)dW(\tau)dW'(s)B'(s)\Phi'(t,s)\} \end{aligned}$$

Since the processes x_0 and W are assumed to be independent, we have

$$E\{\int_0^t \Phi(t,\tau)B(\tau)\,dW(\tau)x_0'\Phi'(t,0)\} = 0$$

$$E\{\int_0^t \Phi(t,0)x_0 dW(s)B'(s)\Phi'(t,s)\} = 0$$

Furthermore, by virtue of the properties of Wiener process, it follows from the above equation that

$$P(t) = \Phi(t,0)P_0\Phi'(t,0) + \int_0^t \Phi(t,\tau)B(\tau)Q(\tau)B'(\tau)\Phi'(t,\tau)\,d\tau \tag{5.74}$$

which was also derived earlier.

Example 5.14

The Black-Scholes model is the most commonly used model for stock prices. It assumes that the stock has a riskless 'return' μ, and a 'volatility' σ. Both μ and σ are assumed to be constant. Then the overall return $R(t)$ on the stock is given by

$$dR(t) = \mu\, dt + \sigma dW(t)$$

where W is a Brownian motion process. Defining 'return' as the per unit change in stock price,

$$dR(t) = \frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW$$

simplifies to

$$dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dW(t) \tag{5.75}$$

where S is the price of the stock.

To find the solution of the abive equation, we use Ito formula

$$\begin{aligned} d(\log S) &= \frac{dS}{S} - \frac{1}{2}\frac{\sigma^2 S^2 dW}{S^2} \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dW \end{aligned}$$

so that

$$S(t) = S(0)\exp\big\{(\mu - \frac{\sigma^2}{2})\, t + \sigma W(t)\big\}$$

Also integrating (5.75)

$$S(t) = S(0) + \int_0^t \mu\, S(\tau)\, d\tau + \int_0^t \sigma\, S(\tau)\, dW(\tau)$$

Taking expectation of the above equation, and noting that $E\{\int_0^t \sigma\, S\, dW\} = 0$, we have

$$E\{S(t)\} = S(0) + E\{\int_0^t \mu\, S(\tau)\, d\tau\}$$

so that

$$E\{S(t)\} = S(0)e^{\mu\, t}$$

which shows that the expected value of the stock price is same as the deterministic solution corresponding to $\sigma = 0$.

Example 5.15

Motion of pollen particles in a fluid can be given by

$$dx = -\alpha x \, dt + \beta \, dW$$

where α and β are constant. The coefficient α is the friction coefficient between the particle and the fluid molecules, and β is the diffution coefficient. This equation is also known as the Langevin's equation. We assume that $E\{dW(t) \, dW(t)\} = q \, dt$, and $E\{x^2(0)\} = p_0$. Then using the result given above,

$$p(t) = -2\alpha p + \beta^2 q$$

whose solution is

$$p(t) = p_0 e^{-2\alpha t} + \frac{\beta^2}{2\alpha} q(1 - e^{-2\alpha t})$$

This shows that covariance of the process evolves with time, and becomes stationary at $p(t \to \infty) = \frac{\beta^2}{2\alpha} q$.

Also a direct integration of the Langevin's equation shows that

$$x(t) = e^{-\alpha t} x_0 + \int_0^t e^{-\alpha(t-\tau)} \beta \, dW(\tau)$$

and taking expectation

$$E\{x(t)\} = e^{-\alpha t} E\{x_0\}$$

so that

$$E\{x(t)\} \to 0 \quad \text{as} \quad t \to \infty$$

This shows that motion of pollen particles settles to a Gaussian process $N(0, \frac{\beta^2}{2\alpha} q)$.

Ito's formula

Ito formula is one of the powerful tools of analysis of stochastic systems, and allows us find the expected value of a function of the stochastic process $x(t)$. Let f be a function of the noise process w. Then by Taylor's theorem,

$$f(w + dw) = f(w) + f'(w)dw + \frac{1}{2} f''(w)(dw)^2 + \cdots$$

If w is deterministic, then as a first order approximation, we can neglect the third term $(dw)^2$, and all higher order terms. On the other hand if w is stochastic, we have

$$dw(t) = w(t + dt) - w(t)$$

so that

$$(dw)^2 = [w(t + dt) - w(t)]^2 = dt$$

by the property of Wiener process. This shows that the third term of the Taylor expansion cannot be neglected.

Now consider a scalar stochastic process $x(t)$ defined by the equation

$$dx = f\,dt + \sigma\,dw$$

Then the stochastic differential

$$\begin{aligned} dg(t,x(t)) &= g(t+dt, x+dx) - g(t,x) \\ &= g_t\,dt + g_x\,dx + \frac{1}{2}g_{tt}(dt)^2 + g_{tx}\,dt\,dx + \frac{1}{2}g_{xx}(dx)^2 + \cdots \\ &= g_t\,dt + g_x\,(f\,dt + \sigma\,dw) + \frac{1}{2}g_{tt}(dt)^2 + f_{tx}\,dt\,(f\,dt + \sigma\,dw) + \frac{1}{2}g_{xx}(f\,dt + \sigma\,dw)^2 + \cdots \end{aligned}$$

Note that $(dt)^2$ and $dt\,dw$ are second order terms, but $(dw)^2$ is a first order term. Thus retaining only first order terms in the above equation we have

$$dg(t,x) = (g_t + g_x\,f + \frac{1}{2}\sigma^2 g_{xx})\,dt + \sigma\,g_x dw$$

This is what is known as the Ito formula.

This approach can be extended for n-vector valued processes governed by

$$d\mathbf{x}(t) = \mathbf{f}(t)dt + \mathbf{G}(t)dW(t) \tag{5.76}$$

where W is a m-vector Wiener process, and $\mathbf{f}$ is a n-vector valued function, and $\mathbf{G}$ is a $n \times m$ matrix. We also assume that $\mathbf{f}$ and $\mathbf{G}$ are sufficiently differentiable with respect to t and $\mathbf{x}$.

Suppose $F(t,\mathbf{x})$ is a smooth deterministic function of the process $\mathbf{x}(t)$. Then stochastic differential of F is given by

$$dF = \left[F_t + \mathbf{f}'F_{\mathbf{x}} + \frac{1}{2}\text{ trace }(\mathbf{G}\mathbf{G}'F_{\mathbf{xx}}\right]dt + F'_{\mathbf{x}}\mathbf{G}\,dW \tag{5.77}$$

where F_t and $F_{\mathbf{x}}$ are partial derivative of F with respect to t and $\mathbf{x}$ respectively, and $F_{\mathbf{xx}}$ is the second partial of F with respect to $\mathbf{x}$.

Example 5.16

Consider the linear system given by scalar differential equation

$$dx(t) = a(t)x(t)dt + b(t)x(t)dW(t)$$

where a and b are continuous function of t, and let $F(x) = x^2$. We shall supress the symbol t for notational simplicity. Then the stochastic differential of F is obtained as

$$\begin{aligned} dx^2 &= (2ax^2 + b^2x^2)\,dt + 2bx^2dW \\ &= (2a + b^2)\,x^2\,dt + 2b\,x^2\,dW \end{aligned}$$

Let's assume that $a(t) = -\alpha$, a constant, and $b(t) = \beta$, a constant. Also denote q denote the x^2, the norm square of the solution. Then we have

$$dq = (-2\alpha + \beta^2)q\,dt + 2\beta q\,dW$$

Integrating the above equation from $t = 0$ to t, and taking expectation

$$\begin{aligned} E\{q(t)\} &= E\{q(0)\} + E\{\int_0^t (-2\alpha + \beta^2)q(t)\,dt\} + E\{\int_0^t 2\beta q(t)dW(t)\} \\ &= E\{q(0)\} + E\{\int_0^t (-2\alpha + \beta^2)q(t)\,dt\} \end{aligned}$$

so that

$$E\{q(t)\} \leq E\{q(0)\}e^{(-2\alpha+\beta^2)t}$$

This is an interesting result. Note that the noise free system, i.e., with $\beta = 0$ is stable, but if there is a noise acting on the system, stability is not always guaranteed. The system is stable only if the noise is not too strong, i.e., if $\beta^2 < 2\alpha$.

Ito Product Rule

Suppose

$$\begin{aligned} dx_1 &= A_1x_1dt + B_1dW \\ dx_2 &= A_2x_2dt + B_2dW \end{aligned}$$

Assume that the product x_1x_2 is dimensionally compatible. Then

$$d(x_1x_2) = dx_1x_2 + x_1dx_2 + B_1QB_2dt$$

Note that the last term is the correction that is required due to Ito calculus. This can be proved from

$$d\{x_1x_2\} = x_1(t+dt)x_2(t+dt) - x_1(t)x_2(t)$$

and following Taylor's expansion, and using the properties of the Wiener process. We leave the details as an exercise for the reader.

We can use this theory to derive the covariance matrix of the process xxx. Given

$$dx = Axdt + BdW$$

we also have

$$dx' = x'A'dt + dW'B'$$

so that using the Ito product rule,

$$\begin{aligned} d(xx') &= dxx' + xdx' + BQBdt \\ &= (Axdt + BdW)x' + x(x'A' + dW'B') + BQB'dt \\ &= (Axx' + xx'A' + BQB')dt + BdWx' + xdW'B' \end{aligned}$$

Integrating the above equation, and taking expectation

$$\begin{aligned} E\{xx'\}(t) &= E\{x_0x_0'\} + E\int_0^t (Axx' + xx'A' + BQB')d\tau + E\int_0^t BdWx' + E\int_0^t xdW'B' \\ &= E\int_0^t (Axx' + xx'A' + BQB')d\tau \end{aligned}$$

which is equivalent to

$$\begin{aligned} \dot{P}(t) &= AP + PA' + BQB' \\ P(0) &= E\{x_0x_0'\} \end{aligned}$$

Numerical Solution of Stochastic Differential Equations

Numerical solution of stochastic differential equations is based on the Ito stochastic equation (5.70) rather than the model (5.61). Consider the model

$$\begin{aligned} dx &= f(t,x)\,dt + g(t,x)\,dW \\ x(0) &= x_0 \end{aligned} \tag{5.78}$$

Assume that f and g satisfy appropriate conditions to guarantee the existence of a solution. Then given the solution at time t_n, we compute the solution at time t_{n+1} using

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(t,x(t)\,dt + \int_{t_n}^{t_{n+1}} g(t,x(t))\,dW$$

Assume that the step size $\Delta t = t_{n+1} - t_n$ is sufficiently small, and is constant for the entire time interval. Then the simplest method for evaluating the above equation is the Euler's formula. For notational simplicity, we shall use x_n to denote $x(t_n)$.

$$x_{n+1} = x_n + f(t_n,x_n)\,\Delta t + g(t_n,x_n)\,\Delta W_n$$

where ΔW_n are the increment of the Wiener process.

$$\Delta W_n = W(t_{n+1}) - W(t_n)$$

Also recall that Wiener increments are Gaussian. It can be shown that the numerical solution converges to the true solution in the mean square sense, and has the global error $o(\Delta t)$.

Numerically the Euler's method works well, and gives a solution that is fairly close to the true solution only if Δt is very small. A numerically more stable method with higher order accuracy is derived from the Runge-Kutta method. Following the fourth order Runge-Kutta method, we have

$$x_{t+1} = x_n + \frac{1}{6}[F_0 + 2F_1 + 2F_2 + F_3]\,\Delta t + \frac{1}{6}[G_0 + 2G_1 + 2G_2 + G_3]\,\Delta W_n$$

where

$$
\begin{aligned}
F_0 &= f(t_n,\, x_n)\\
F_1 &= f(t_n + \frac{1}{2}\Delta t,\, x_n + \frac{1}{2}F_0\Delta t + \frac{1}{2}G_0\Delta W_n)\\
F_2 &= f(t_n + \frac{1}{2}\Delta t,\, x_n + \frac{1}{2}F_1\Delta t + \frac{1}{2}G_1\Delta W_n)\\
F_3 &= f(t_n + \Delta t,\, x_n + F_2\Delta t + G_2\Delta W_n)
\end{aligned}
$$

$$
\begin{aligned}
G_0 &= g(t_n,\, x_n)\\
G_1 &= g(t_n + \frac{1}{2}\Delta t,\, x_n + \frac{1}{2}F_0\Delta t + \frac{1}{2}G_0\Delta W_n)\\
G_2 &= g(t_n + \frac{1}{2}\Delta t,\, x_n + \frac{1}{2}F_1\Delta t + \frac{1}{2}G_1\Delta W_n)\\
G_3 &= g(t_n + \Delta t,\, x_n + F_2\Delta t + G_2\Delta W_n)
\end{aligned}
$$

The stochastic analog of Runga-Kutta method also has the accuracy of the order of $(\Delta t)^4$ in the mean square sense.

5.9 Exercise

5.1 A linear system is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The state transition matrix of the system is given by

$$e^{\mathbf{A}(t-\tau)} = \begin{bmatrix} 1 & 0 \\ 1 - e^{-(t-\tau)} & e^{-(t-\tau)} \end{bmatrix}$$

a) Find the state $\mathbf{x}(t)$ for a unit step input applied at $t = 1$.

b) Find the state at $t = 0$.

5.2 A linear system is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = [\,1 \quad 0 \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

a) Find the state transition matrix.

b) Find the response $y(t)$ for an impulse input applied at $t = 2$ if the initial state is $x(1) = [\,1 \quad 0 \quad 0\,]'$.

c) Find $x(0)$.

5.3 A linear system is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find the state $x(t)$ as a function of t, if the input is $u(t) = e^{-t}$ applied at $t = 0$.

5.4 A linear system is described by

$$\begin{aligned} \dot{x}_1 &= x_2, \qquad & x_1(0) &= 1 \\ \dot{x}_2 &= -x_1 + u \qquad & x_2(0) &= 0 \end{aligned}$$

Suppose a square pulse of width 1 second at $t = 0$. Find the response of the system for all $t > 0$.

5.5 Find an expression for the response of the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} + \mathbf{Ev} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

where $\mathbf{u}$ is a control input and $\mathbf{v}$ is an external disturbance.

5.6 Find the impulse response of the linear time invariant system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Show that Laplace transform of this impulse response is given by

$$\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

which is same as the transfer function matrix of the system. Hint: First solve the problem assuming that u is a scalar input. Then generalize the concept for multiple input case. The main concern for multi-input case is that what is the definition of a unit impulse? Is it a vector of unit impulses corresponding to each control or is u is the superposition of a number of vectors with each vector defined appropriately by taking one impulsive u at a time?

5.7 A linear system is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find the impulse response of the system. Compute the transfer function of the system. Show that the Laplace transform of the impulse response is same as the transfer function. Note: Assume that the initial state of the system is zero, and the impulse is applied at $t = 0$.

5.8 Find the state transition matrix for the time varying system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

5.9 A control system is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + f(t, \mathbf{x})$$

Assume that $\mathbf{A}$ is a $n \times n$ matrix, and $f(t)$ is a $n \times 1$ vector. Express the solution $\mathbf{x}(t)$ in terms of its state transition matrix.

5.10 A system is described by the bilinear model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + u\mathbf{B}\mathbf{x}$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{x} \in \mathcal{R}^n$, and $\mathbf{A}$ and $\mathbf{B}$ are matrices of dimension $n \times n$, and u is a scalar control input. Find an expression for the solution $\mathbf{x}(t)$. Hint: Use state transition matrix corresponding to the matrix $\mathbf{A}$.

5.11 An adjoint dynamic system is described by

$$\begin{aligned} \dot{\mathbf{z}} &= -\mathbf{A}'\mathbf{z} + \mathbf{C}'w \\ \mathbf{z}(T) &= \mathbf{z}_T \end{aligned}$$

Find the response of the system for $0 \leq t \leq T$.

5.12 A damped second order system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Find the state transition matrix. Using the state transition matrix, find the response of the system for $u(t) = 1$ for all $t > 0$.

5.13 Suppose $\mathbf{A}$ is nonsingular. Show that

$$\int_0^t e^{\mathbf{A}\tau}\, d\tau = \left[e^{\mathbf{A}t} - \mathbf{I}\right]\mathbf{A}^{-1} = \mathbf{A}^{-1}\left[e^{\mathbf{A}t} - \mathbf{I}\right]$$

Hint: Use infinite series expansion of the matrix exponential, and do term-by-term integration.

5.14 Find condition under which the following equalities hold:

$$\begin{aligned} (\mathbf{A} + \mathbf{B})^2 &= \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 \\ e^{\mathbf{A}t}e^{\mathbf{B}t} &= e^{(\mathbf{A}+\mathbf{B})t} \end{aligned}$$

5.15 Using the Picard approximation, show that solution of the time varying system

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{A}(t)\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

can be obtained as

$$\mathbf{x}(t) = \Phi(t, 0)\mathbf{x}_0$$

where

$$\begin{aligned} \Phi(t, 0) = \mathbf{I} + \int_0^t \mathbf{A}(s_1)\, ds_1 + \int_0^t \mathbf{A}(s_1) \int_0^{s_1} \mathbf{A}(s_2)\, ds_2\, ds_1 + \\ \int_0^t \mathbf{A}(s_1) \int_0^{s_1} \mathbf{A}(s_2) \int_0^{s2} \mathbf{A}(s_3)\, ds_3\, ds_2\, ds_1 + \cdots \end{aligned}$$

This method is known as the Peano-Baker integral series. Note that the above infinite series provides an iterative method of generating the solution of time varying linear systems. However it is clear that difficulties may exist in evaluating successive integrations.

5.16 Using the Peano-Baker series (or any other method), find the state transition matrix of the following matrices:

$$\mathbf{A}(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$
$$\mathbf{A}(t) = \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix}$$
$$\mathbf{A}(t) = \begin{bmatrix} 2t & 1 \\ 1 & 2t \end{bmatrix}$$
$$\mathbf{A}(t) = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}$$
$$\mathbf{A}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$$
$$\mathbf{A}(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

5.17 Consider the linear system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

where

$$\mathbf{A}(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a) Find the state transition matrix, and the response of the system for the initial condition $\mathbf{x}(0) = [1 \quad 1]'$, and $u = 0$. Is the system stable?

b) Using the concept of eigenvalues, determine stability of the uncontrolled system. Is this conclusion consistent with that of part(a) above? Why or why not?

5.18 Show that

$$R(t) = \Phi(t, s)R(s)$$

where R is the covariance matrix for the linear system

$$dx = Ax\, dt + B\, dW$$

and Φ is the state transition matrix corresponding to the matrrix A.

5.19 The simplest model of a car on uneven road surface can be given by a mass, M (representing the car body) supported by a spring, K (representing the tire). The eneven road surface represents a random displacement N to the spring.

$$M\ddot{x} + Kx = N$$

Assume that $M = 250$ Kg, $K = 150 \times 10^3$ N m^{-1}. Find the equation describing the covariance matrix. Simulate the model, and compute the covariance numerically. Compare your numerical solution with analytical solution.

5.20 A spherical satellite in space can be described by

$$\ddot{\theta} = u + N$$

where θ is the attitude angle, u is the control torque, and N represents the random disturbance arising from solar pressure. Find the solution θ and its covariance.

5.21 The dynamic model of a simple pendulum supported from a shaky surface can be given by

$$\ddot{\theta} + \alpha\dot{\theta} + \theta = N$$

What are the variance of the processes θ and $\dot{\theta}$?

5.22 A simplified model of roll angle motion of a ship can be given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.01 & -0.02 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} N$$

where x_1 is the roll angle, and x_2 is the roll velocity, and N is the disturbance torque due to sea waves. Simulate the ship motion assuming that N is a Gaussian process with variance 0.01 rad^2 s^{-4}.

5.23 Show that the solution of the equation

$$dx = xdW$$

is given by

$$x(t) = x_0 e^{-\frac{1}{2}t + W(t)}$$

5.24 Prove the following equalities:

$$dW^2(t) = 2W(t)dW(t) + dt$$

$$\int_0^t W\, dW = \frac{1}{2}W^2 - \frac{1}{2}t$$

5.25 Consider the linear system

$$dx = Ax\, dt + Bx\, dW$$

with $E\{dW\, dW'\} = dt$. Show that

$$\dot{P} = AP + PA + BPB'$$

where $P(t) = E\{x(t)x(t)'\}$