BESSEL-GAUSS BEAMS

F. GORI, G. GUATTARI
Dipartimento di Fisica, Università di Roma "La Sapienza", P. le A. Moro, 2, 00185 Roma, Italy

and

C. PADOVANI
Dipartimento di Ingegneria Elettrica, Università di L'Aquila, Monteluco di Roio, L'Aquila, Italy

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A new type of solution of the paraxial wave equation is presented. It encompasses as limiting cases both the diffraction-free beam and the gaussian beam. The propagation features of this solution are discussed.

1. Introduction

Propagation of coherent or partially coherent light in the form of beams is of obvious importance from both the theoretical and the experimental point of view. Although light beams of one form or another are simple to produce in practice, the corresponding analytical description is not that easy. This occurs because the evaluation of the integrals involved in the propagation formulas seldom leads to closed expressions. As a celebrated example, we recall that the deceptively simplest way of shaping a light pencil, namely limiting a plane wave by a circular aperture, prompted the introduction of a new family of special functions, known as Lommel functions [1], for its paraxial analysis. Nevertheless, a number of types of light beams both useful and analytically simple are known today. They range from the ubiquitous gaussian beam of zero order [2,3], to more sophisticated forms of both coherent [2–7] and partially coherent [8–26] beams.

Recently, new types of coherent light beams, called diffraction-free beams, have been studied [27,28]. They have the peculiar property of conserving the same disturbance distribution (apart from a phase factor) across any plane orthogonal to the direction of propagation, say the z-axis. An intuitive understanding of these beams can be gained by thinking of them as a superposition of plane waves whose wave vectors lie on a cone around the z-axis. All these plane waves have the same component of the wave vector along the z-axis. Accordingly, they all suffer the same phase change for any given pathlength along the z-axis. The mutual phase relations among the various plane waves do not change on propagation, so that the overall interference pattern has one and the same shape at any plane z=constant. For the simplest beam of this type, which is circularly symmetric, the transverse disturbance distribution has the form of a Bessel function of the first kind and zero order, $J_0(\beta r)$, where r denotes the distance from the z-axis and β is the length of the component, orthogonal to the z-axis, of any wave vector belonging to one of the plane waves producing the beam. As well known, the maxima of the oscillating function $J_0(x)$ tend to decrease like $1/x^{1/2}$ when x goes to infinity. Because of this slow decrease law, it is impossible to realize experimentally a beam giving everywhere a good approximation to the ideal model. Note also that the ideal beam should carry an infinite power because of the divergence of the norm of $J_0(\beta r)$. In a laboratory test [27], an experimental beam was realized whose disturbance in the plane $z=0$ was of the form $J_0(\beta r)$ within a circular aperture and vanished elsewhere. It

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was shown through geometrical considerations [27] that the experimental beam could exhibit the main propagation features of the ideal beam, provided that only a suitably limited path along the beam axis was considered. These provisions were fully confirmed by the experiment. However, an analytical description of that experimental beam would not be trivial (see the above remark on the diffraction from a circular aperture).

In this paper, we consider the case in which the disturbance in the plane $z=0$ is of the form $J_0(\beta r)$ multiplied by a gaussian profile. The corresponding beam will be termed a Bessel–Gauss beam (of zero order). Unlike the diffraction-free beam, the Bessel–Gauss beam carries a finite power and can be realized experimentally to a good approximation because of the fast decrease law of the gaussian profile. We will examine the propagation of this beam in the paraxial approximation and we will show that the disturbance distribution throughout the whole space can be given a, rather simple, closed form (sec. 2). A discussion of the main features of the Bessel–Gauss beam will be given in section 3.

2. The Bessel–Gauss beam

Let us suppose that a certain monochromatic wave produces the following circularly symmetric disturbance across the plane $z=0$

$$V(r,0)=A\,J_0(\beta r)\exp[-(r/w_0)^2]\,, \tag{2.1}$$

where A is a (possibly complex) amplitude factor and where the positive constant w_0 gives a measure of the width of the gaussian term. We dropped the usual term $\exp(-\mathrm{i}\omega t)$ expressing the temporal dependence of the disturbance. It is easily seen that the function $V(r,0)$ has a finite norm. Thus, a finite power is carried by the wave through the plane $z=0$.

At any point (r,z) in the half-space $z>0$, the disturbance can be evaluated in the paraxial approximation by using the following Fresnel diffraction integral

$$V(r,z)=(-\mathrm{i}k/z)\exp[\mathrm{i}(kz+kr^2/(2z))]$$
$$\times\int_0^\infty V(\rho,0)\exp[\mathrm{i}k\rho^2/(2z)]\,J_0(k\rho r/z)\,\rho\,\mathrm{d}\rho\,, \tag{2.2}$$

where $k=2\pi/\lambda$, λ being the wavelength of the radiation. Let us now recall that the following integral formula holds [29]

$$\int_0^\infty \exp(-\alpha x)\,J_0(2\gamma\sqrt{x})\,J_0(2\delta\sqrt{x})\,\mathrm{d}x$$
$$=(1/\alpha)\,I_0(2\gamma\delta/\alpha)\exp[-(\gamma^2+\delta^2)/\alpha]\,, \tag{2.3}$$

for any choice of the (possibly complex) parameters α, γ and δ. Here, I_0 denotes the modified Bessel function of the first kind and zero order, connected to the ordinary Bessel function J_0 through the relation [30]

$$I_0(x)=J_0(\mathrm{i}x)\,. \tag{2.4}$$

On inserting from eq. (2.1) into eq. (2.2) and on using eqs. (2.3) and (2.4) we obtain

$$V(r,z)=-(\mathrm{i}kA/2zQ)\exp[\mathrm{i}k(z+r^2/2z)]$$
$$\times J_0(\mathrm{i}\beta kr/2zQ)\exp[-(1/4Q)\,(\beta^2+k^2r^2/z^2)]\,, \tag{2.5}$$

where

$$Q=1/w_0^2-\mathrm{i}k/2z\,. \tag{2.6}$$

Through simple algebraic manipulations and taking into account that J_0 is an even function, eq. (2.5) can be transformed into the following form

$$V(r,z)=(Aw_0/w(z))$$
$$\times\exp\{\mathrm{i}[(k-\beta^2/2k)z-\Phi(z)]\}$$
$$\times J_0[\beta r/(1+\mathrm{i}z/L)]\exp\{[-1/w^2(z)$$
$$+\mathrm{i}k/2R(z)]\,(r^2+\beta^2z^2/k^2)\}\,, \tag{2.7}$$

where the parameter L is given by

$$L=kw_0^2/2\,, \tag{2.8}$$

and where the functions $w(z)$, $\Phi(z)$ and $R(z)$ are as follows

$$w(z)=w_0\,[1+(z/L)^2]^{1/2}\,, \tag{2.9}$$

$$\Phi(z)=\arctan(z/L)\,, \tag{2.10}$$

$$R(z)=z+L^2/z\,. \tag{2.11}$$

Formulas (2.8)–(2.11) are identical to those occurring in the description of the ordinary gaussian beams. For those beams, L is the so-called Rayleigh distance, whereas w, Φ and R give the spot-size, the phase-shift and the radius of curvature of the gaussian wavefront, respectively.

Finally, it is to be noted that eq. (2.7), although formally derived in the half space $z>0$, is meaningful also for $z<0$. In other words, eq. (2.7) gives a solution of the paraxial wave equation throughout the whole space.

3. Propagation features

In this section, we analyze how the cross-section of the Bessel–Gauss beam changes during propagation. Let us start with a few geometrical considerations. We saw in sec. 1 that a diffraction-free beam can be thought of as a superposition of plane waves whose wave vectors lie on a cone. Similarly, it could be proved that a Bessel–Gauss beam is produced by the superposition of gaussian beams whose axes are uniformly distributed on a cone. The angular half-aperture of the cone, say θ, is connected to the parameter β by the relation

$$\beta=k\sin\theta\ . \tag{3.1}$$

The behaviour of the Bessel–Gauss beam is determined by two competing effects. On the one hand, because of the angular separation of their axes, any two component gaussian beams tend to overlap lesser and lesser on increasing z. On the other hand, the spot-size of each component gaussian beam grows with respect to z (see eq. (1.9)) thus contributing to the overlapping of the various beams. In the far-zone, the resulting overlapping is determined by the ratio between the angular aperture of the cone and the angular spread of a single gaussian beam. The latter, say θ_G, is given by [2]

$$\theta_G=\lambda/\pi w_0\ . \tag{3.2}$$

According to eqs. (3.1) and (3.2), the ratio between the two angles is

$$\theta/\theta_G=(\pi w_0/\lambda)\ \arcsin(\beta/k)\simeq\beta w_0/2\ , \tag{3.3}$$

the last approximation holding if θ is small. If the ratio (3.3) is less than one, all the component gaussian beams keep overlapping and interfering even in the far-zone ($z\to\infty$). In this case, we could expect that any cross-section of the Bessel–Gauss beam resembles the central part of a diffraction-free beam. In order to have a more complete picture, let us refer to the plane $z=0$ (eq. (2.1)) and recall that the central core of the function $J_0(\beta r)$ has a radius equal to $2.4/\beta$. Now, it is seen from eq. (3.3) that if θ/θ_G is smaller than one then the spot-size w_0 is smaller than such a radius. Accordingly, the outer oscillations of the function J_0 are strongly damped and the actual cross-section at $z=0$ is quite similar to that of a gaussian beam. As a consequence, when $\theta/\theta_G<1$ the Bessel–Gauss beam should not be too different from an ordinary gaussian beam.

Let us turn to the opposite case, namely $\theta/\theta_G>1$. In the far-zone, the component gaussian beams will give rise to an annular illumination leaving the central region near to the z-axis practically obscure. A significant superposition of all the component gaussian beams will survive up to a certain distance, say D, from $z=0$. An estimate of D can be obtained by requiring that at $z=D$ the center of a typical component gaussian beam has receded from the z-axis for a distance equal to w_0, or

$$D=w_0/\theta\ . \tag{3.4}$$

This is a conservative estimate because the spot-size actually increases along the z-axis (see eq. (2.9)). In conclusion, we can expect that for $\theta/\theta_G>1$ the Bessel–Gauss beam behaves like a portion of a diffraction-free beam for a distance D only (on both sides of $z=0$).

The above considerations could be substantiated by an analysis of eq. (2.7). We shall limit ourselves to some graphical results and a few remarks. In figs. 1a–1e, plots are given of the radial intensity distribution, at a few planes $z=$const, for a Bessel–Gauss beam with the following parameters: $\theta=10^{-3}$ rad; $w_0=1$ mm; $\lambda=632.8$ nm. This implies $\theta/\theta_G=4.96$ and $D=1$ m (see eqs. (3.3) and (3.4)). The curves have been drawn by evaluating numerically the squared modulus of $V(r,z)$ (see eq. (2.7)). For all the planes, the intensity has been normalized to one. The behaviour illustrated by these plots agrees with the provisions made above. We do not report plots of the intensity for cases in which $\theta/\theta_0<1$ because they actually resemble those of an ordinary gaussian beam.

Let us now add a few remarks on eq. (2.7). First,

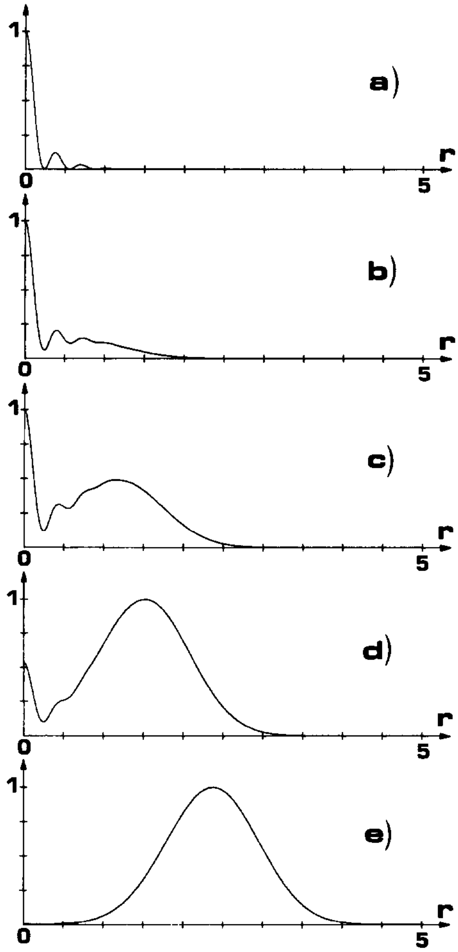

Fig. 1. Intensity distribution along a radius of a Bessel–Gauss beam at various planes orthogonal to the axis of the beam (z-axis); (a) $z=0$; (b) $z=1.0$ m; (c) $z=1.4$ m; (d) $z=1.7$ m; (e) $z=2.5$ m. The radial coordinate is measured in millimeters whereas the intensity is normalized to one at each plane. The beam parameters are: $w_0=1$ mm; $\theta=1$ mrad; $\lambda=632.8$ nm.

it is immediately seen that for $\beta=0$ the Bessel- Gauss beam reduces to an ordinary gaussian beam. Second, when $w_0\to\infty$ we have L, w, $R\to\infty$, $w_0/w\to 1$ and $\Phi\to 0$. Accordingly, eq. (2.7) becomes

$$V(r,z)=A\exp[\mathrm{i}(k-\beta^2/2k)z]\,J_0(\beta r)\,. \qquad (3.5)$$

Eq. (3.5) is in the form of a diffraction-free beam. The only difference with respect to the form discussed in refs. [27] and [28] is that the component of the wave vectors along the z-axis appears here in the paraxial approximation $k-\beta^2/2k$.

An asymptotic form of eq. (2.7) can be obtained by noting that for $z\gg L$ the following relations hold: $w\simeq w_0 z/L$; $\Phi\simeq\pi/2$; $R\simeq z$. In addition, the asymptotic expression of the Bessel function can be used, namely

$$I_0(x)=\exp(x)/\sqrt{2\pi x}\,,\quad (x\to\infty)\,. \qquad (3.6)$$

On using the previous expressions, eq. (2.7) can be given the asymptotic form (valid for large z and r)

$$V(r,z)=(-\mathrm{i}Aw_0/\sqrt{4\pi\beta rz/k})$$
$$\times\exp[\mathrm{i}(kz+kr^2/2z)]\exp[-((r-\beta z/k)/w)^2]\,, \qquad (3.7)$$

accounting for the shifted gaussian appearance of the intensity curve in the far-zone (see fig. 1e).

4. Conclusions

We introduced a new solution of the paraxial wave equation. Such a solution is obtained by modulating with a gaussian profile the transverse disturbance distribution of a diffraction-free beam of zero order. The corresponding beam has been termed a Bessel–Gauss beam. In a more complete way, it could be termed a Bessel–Gauss beam of zero order. In fact, the gaussian transverse modulation could be applied to diffraction-free beams of order higher than zero.

The Bessel–Gauss beam is characterized by two real parameters β and w_0 whose values determine the propagation features of the beam. By a suitable choice of β and w_0 both the gaussian beam and the diffraction-free beam are obtained as particular cases. Except for the limiting case $w_0\to\infty$, the Bessel–Gauss beam represents a disturbance distribution that can be well approximated in practice. This makes possible an experimental study of the propagation of a nearly diffraction-free beam, that is analytically expressible in a closed form.

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