Management of the Angular Momentum of Light: Preparation of Photons in Multidimensional Vector States of Angular Momentum

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We put forward schemes to prepare photons in multidimensional vector states of orbital angular momentum. We show realizable light distributions that yield prescribed states with finite or infinite normal modes. In particular, we show that suitable light vortex pancakes allow the add-drop of specific vector projections. We suggest that such photons might allow the generation of *engineered quNits* in multidimensional quantum information systems.

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Light carries energy and both linear and angular momenta. The total angular momentum can contain a spin contribution associated with polarization [1,2], and an orbital contribution associated with the spatial profile of the light intensity and phase [3]. Such angular momentum can be transferred to trapped suitable material particles causing them to rotate, a property with important applications in optical tweezers and spanners in fields as diverse as biosciences [4] and micromechanics [5]. The angular momentum of light can also be used to encode quantum information that is carried by the corresponding photon states [6]. In this regard, exploitation of the orbital contribution to the angular momentum opens the door to the generation and manipulation of multidimensional quantum entangled states [7], with an arbitrarily large number of entanglement dimensions. Such multidimensional entanglement should allow the exploration of deeper quantum features and might guide the elucidation of capacity-increased quantum information processing schemes.

Allen and co-workers showed a decade ago that paraxial Laguerre-Gaussian laser beams carry a well-defined orbital angular momentum associated to their spiral wave fronts [8]. The formal analogy between paraxial optics and quantum mechanics implies that such modes are the eigenmodes of the quantum mechanical angular momentum operator [9,10]. The Laguerre-Gaussian modes form a complete Hilbert set and can thus be used to represent the quantum photon states within the paraxial regime of light propagation. The quantum angular momentum number carried by the photon is then represented by the topological charge, or winding number m, of the corresponding mode, and each mode carries an orbital angular momentum of $m\hbar$ per photon. In this Letter, we show how to construct multidimensional vector photon states with controllable projections into modes with well-defined winding numbers. In particular, we put forward schemes based on suitable light vortex pancakes made of Gaussian laser beams with distributions of nested topological screw wave front dislocations, which allow the manipulation, including the addition and removal, of specific projections of the vector states. We also anticipate that the schemes developed by Nienhuis and Courtial and co-workers [11] to measure the rotational frequency shift imparted to a vortex light beam might be employed to verify our predictions.

Only the total angular momentum, containing spin and orbital contributions, is a quantum mechanical physical observable. Here we restrict ourselves to the orbital contribution. We thus consider the slowly varying electric field envelope $u(x, y; z)$ of a cw paraxial beam propagating in free space; z is the propagation direction, and x and y are the spatial transverse coordinates. The time averaged energy per unit length carried by the beam is $U = 2\epsilon_0 \int \int |u|^2 \, dx \, dy$, where ϵ_0 is the permittivity of vacuum. The time averaged z component of the orbital angular momentum per unit length carried by the light beam is given by $L_z = \int \int [\vec{r}_\perp \times \vec{p}] \, dx \, dy$, where $\vec{r}_\perp$ is the vector position in the X-Y plane, $\vec{p} = (i\epsilon_0/\omega)[u\nabla_\perp u^* - u^*\nabla_\perp u]$, and ω is the angular frequency. The Laguerre-Gauss (LG) modes u_{mp} form a complete, infinite-dimensional basis for the solutions of the paraxial wave equation; thus any field distribution can be represented as a vector state in that basis. They are characterized by the two integer indices p and m. The index p can take any non-negative value and determines the radial shape, or node number, of the light distribution. The index m can take any integer number and determines the azimuthal phase dependence of the mode. When $m \neq 0$ the LG modes contain screw wave front dislocations, or optical vortices, with topological charge m nested on them. To elucidate the angular momentum content of a field distribution $u(x, y; z)$ one has to compute its projection into the spiral harmonics $\exp(in\varphi)$ [9], where n is the winding number. We thus let

$$u(\rho, \varphi; z) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} a_n(\rho, z) \exp(in\varphi), \quad (1)$$

where $a_n(\rho, z) = 1/(2\pi)^{1/2} \int_0^{2\pi} u(\rho, \varphi, z) \exp(-in \times \varphi) \, d\varphi$. The energy carried by the corresponding light beam can be written as $U = 2\epsilon_0 \sum_{-\infty}^{\infty} C_n$, where $C_n = \int_0^\infty |a_n(\rho, z)|^2 \rho \, d\rho$, can be shown to be constants independent of z. The angular momentum carried by

the light beam is thus given by $L_z = (2\epsilon_0/\omega) \sum_{-\infty}^{\infty} n C_n$. When the energy U is measured in units of $\hbar\omega$ (i.e., $\tilde{U} = U/\hbar\omega$), the ratio $\tilde{L}_z = L_z/\tilde{U}$, which is usually referred to as the angular momentum per photon [3], is given by $\hbar \sum_{-\infty}^{\infty} n C_n / \sum_{-\infty}^{\infty} C_n$.

Within the paraxial regime of light propagation the LG modes are the eigenmodes of the quantum mechanical energy $\hat{E}$ and angular momentum $\hat{L}_z$ operators [3,9,10], so photons represented by a single LG mode are in a quantum state $|m\rangle$ with well-defined values of energy ($\hat{E}|m\rangle = \hbar\omega|m\rangle$) and orbital angular momentum ($\hat{L}_z|m\rangle = m\hbar|m\rangle$). State vectors which are not represented by a pure LG mode correspond to photons in a superposition state, and the weights of the quantum superposition $\{P_n\}$ are determined by the array $\{C_n\}$, by the expression

$$P_n = \frac{C_n}{\sum_{-\infty}^{\infty} C_l}. \tag{2}$$

The mean value of the orbital angular momentum per photon, obtained by a full quantum average over many realizations, is $\tilde{L}_z = \hbar \sum_{-\infty}^{\infty} n P_n$. The central idea put forward in this Letter is to control the series $\{P_n\}$, hence to prepare photons in a superposition state of modes $|m\rangle$, by elucidating suitable light field distributions. Such superposition can be restricted to a finite number of modes, which are to be chosen, or it can consist of an infinite, but discrete, number of modes.

Photons that carry angular momentum in a superposition state of an infinite, but controllable, number of normal modes can be prepared in a variety of ways. An experimentally important scheme is obtained by passing a pure state $|m\rangle$ through astigmatic optical components [10,12]. A light beam prepared in a pure m-order LG mode with the beam center located at $(x, y) = (0, 0)$, but whose orbital angular momentum is measured relative to an origin located at $(x_0, 0)$, constitutes another important example of an infinite-dimensional superposition state. The angular momentum operator $\hat{L}_z$ relative to the displaced origin is given by $\hat{L}_z = \hat{L}_{oz} + \exp(ix_0\hat{P}_x)[\hat{L}_{oz}, \exp(-ix_0\hat{P}_x)]$, where $\hat{L}_{oz}$ is the operator associated to the point $(0, 0)$, $\hat{P}_x$ is the linear momentum operator, and $[,]$ is a Poisson bracket. Thus, even though the photons are prepared in a pure $|m\rangle$ state for the operator $\hat{L}_{oz}$, such is not the case for the operator $\hat{L}_z$. A direct extension of the above are multipearl necklace light fields [13], but with nested vortices in each pearl [14]. Consider the simplest case of a two-pearl vortex necklace with $u(x, y, z) = A_{m0}u_{m0}(x + x_0/2, y, z) + B_{m0}u_{m0}(x - x_0/2, y, z)$, which corresponds to the coherent superposition of two LG beams separated a distance $d = x_0$. The relative amplitudes and phases between both modes, and the length d, are the control parameters to act on $\{P_n\}$. Let $m = 1$, and $A_{10} = B_{10}$. Then, when the pearls are superimposed ($d = 0$), photons are prepared in a pure $|m = 1\rangle$ state. However, when $d \neq 0$ photons are prepared in an infinite-dimensional superposition state of *odd modes only*. Figure 1 shows typical
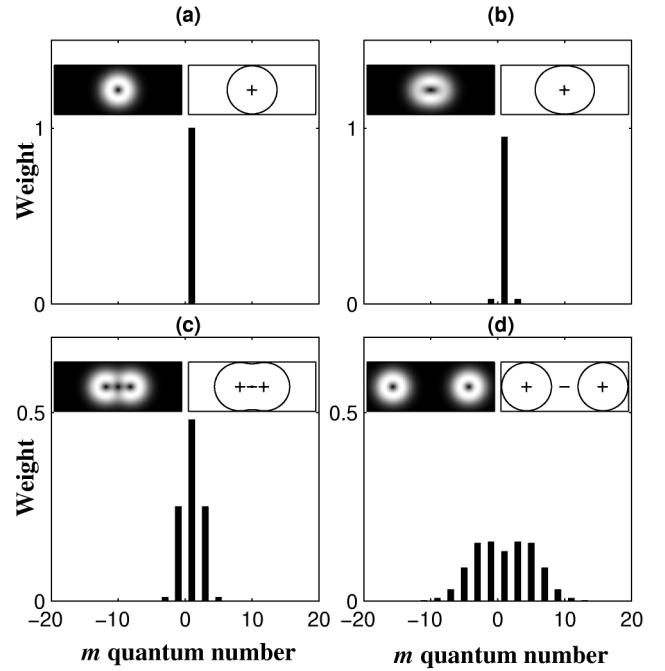

FIG. 1. Weight of the $|m\rangle$ states as a function of the m-quantum number, for a two-pearl vortex-necklace light distribution, for different separations d between the pearls. (a) $d = 0$; (b) $d = w_0$; (c) $d = 2w_0$; (d) $d = 6w_0$. Each pearl features a LG_{10} shape. The insets show the wave function amplitude and a sketch of the location of the existing screw wave front dislocations in each case. Notice that in (a) and (b), only one dislocation is present in the whole wave front, whereas in (c) and (d) the wave front contains three dislocations. The net topological charge is always $+1$.

examples. Vortex necklaces that prepare photons in states with *even modes only* can also be constructed.

The ultimate goal in the effort to prepare the state of photons carrying orbital angular momentum is to elucidate a light signal that yields a superposition into a *finite, in principle, arbitrary large, number of* $|m\rangle$ *states,* which in addition allows *adding-dropping specific projections.* In what follows, we show that such a goal can be achieved by using light vortex pancakes made of properly distributed single-charge screw dislocations nested into a Gaussian host. A pancake with N single-charged dislocations is given by [15]

$$u_N(\rho, \varphi; z = 0) = A_0 \prod_{l=1}^{N} [\rho \exp(i\varphi) - \rho_l \exp(i\varphi_l)] \times \exp(-\rho^2/w_0^2), \tag{3}$$

where A_0 is a constant, ρ_l, φ_l are the radial and azimuthal positions of the lth vortex in cylindrical coordinates, respectively, and w_0 is the beam waist. Projection of (3) onto LG modes u_{mp} yields

$$u_N(\rho, \varphi; z) = A_0\sqrt{\pi} \sum_{l=0}^{N} (-1)^{N-l} \left(\frac{w_0}{\sqrt{2}}\right)^{l+1} \sqrt{l!} \times B_{N-l}u_{l0}(\rho, \varphi; z), \tag{4}$$

where $B_n = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_n} \prod_{l=1}^{n} \rho_{j_l} \exp(i\varphi_{j_l})$, with $j_l \in [1, N]$, and $j_l < j_{l+1}$. Computation of the array $\{C_n\}$ for the distribution (3) yields

$$C_n = |A_0|^2 \pi n! \left(\frac{w_0^2}{2}\right)^{n+1} |B_{N-n}|^2. \tag{5}$$

According to Eq. (4), the vortex pancakes (3) prepare photons in superpositions of a maximum of $N + 1$, $|m\rangle$ states. The actual number of states and their weights are given by the positions of the vortices nested in the beam at $z = 0$. Notice that even though such positions vary with z, together with the beam waist and wave front curvature, the weights of the $|m\rangle$ states do not. Calculation of the weights $\{P_n\}$ associated to a given light distribution is straightforward (even though in general it has to be done numerically), but the interesting problem is just the opposite: Finding a light vortex pancake that yields a desired distribution $\{P_n\}$. In the case of small N, the problem can be solved in an analytical form. $N = 1$ corresponds to a Gaussian beam with a nested off-axis vortex [6,16] and prepares photons in a superposition of the states $|m = 0\rangle$ and $|m = 1\rangle$, only. One finds that the mean value of the angular momentum per photon can take any value between 0 and $\hbar$. The two limiting cases correspond to the vortex located at the center of the host beam (then $\tilde{L}_z = \hbar$) and to the vortex located very far from the center, which obviously yields $\tilde{L}_z \to 0$. We will consider in detail the case $N = 2$ which corresponds to a Gaussian beam with waist w_0 with two single-charge vortices nested off-axis. The corresponding photons are set in three-dimensional vector states, and the elements of the array $\{C_n\}$ are found to be given by

$$C_0 = \tfrac{1}{2} w_0^2 \pi \rho_1^2 \rho_2^2 |A_0|^2, \tag{6}$$

$$C_1 = \tfrac{1}{4} w_0^4 \pi |A_0|^2 [\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos(\varphi_1 - \varphi_2)], \tag{7}$$

$$C_2 = \tfrac{1}{4} w_0^6 \pi |A_0|^2. \tag{8}$$

Most vortex locations lead to photon states with the three possible dimensions occupied, so that $P_0 \neq 0$, $P_1 \neq 0$, $P_2 \neq 0$. For example, equidistributed populations, with $P_0 = P_1 = P_2 = 1/3$, are obtained with a pancake where the vortices are located at

$$\rho_1^2 \rho_2^2 = \frac{w_0^4}{2}, \tag{9}$$

$$\varphi_1 - \varphi_2 = \pi - \cos^{-1}\left(\frac{w_0^2 - \rho_1^2 - \rho_2^2}{2\rho_1\rho_2}\right). \tag{10}$$

Photon states with one of the projections suppressed can also be realized. One gets $P_0 = 0$ states by letting $\rho_1 = 0$. Then, any (P_1, P_2) arbitrary combination, with $P_1 + P_2 = 1$, is obtained when $\rho_2 = w_0[(1 - P_2)/P_2]^{1/2}$. $P_1 = 0$ states can be realized by letting $\varphi_1 - \varphi_2 = \pi$, and $\rho_1 = \rho_2$. Then, any (P_0, P_2) combination, with $P_0 + P_2 = 1$, is obtained when $\rho_1 = \rho_2 =$

$w_0[(1 - P_2)/2P_2]^{1/4}$. $P_2 \to 0$ states can be achieved only asymptotically, by locating one of the vortices very far from the beam center (i.e., $\rho_2 \to \infty$). Then, any (P_0, P_1) combination, with $P_0 + P_1 \to 1$, is obtained when the remaining off-axis vortex is located at $\rho_1 = w_0[(1 - P_1)/P_1]^{1/2}$. Figure 2 illustrates the actual light vortex pancakes and the corresponding $\{P_n\}$ weights for the above cases. In principle, multidimensional vector states with large values of N can be managed by solving numerically the inverse problem posed by (2) and (5). At present we cannot give a definite solving scheme for arbitrary N. However, the plots displayed in Fig. 3, which correspond to $N = 10$, show that preparation of fairly large multidimensional L-managed photon states is possible indeed.

To experimentally verify our predictions, a scheme that resolves the LG spectrum of a light beam must be elucidated. We anticipate that a setup similar to that implemented by Courtial and co-workers [11] should serve that purpose. Courtial et al.'s setup uses a rotating Dove prism that imparts a Doppler frequency shift to the light signal given by $\Delta\omega_m = 2m\Omega$, where Ω is the angular velocity of the prism and m is the topological charge of the vortex. Thus, dislocations with different topological charges m induce different frequency shifts. Therefore, the superposition of LG modes discussed in this Letter is expected to generate a frequency spectrum consisting of sidebands, with an amplitude given by the mode weight, around the unshifted frequency. Resolving such sidebands,
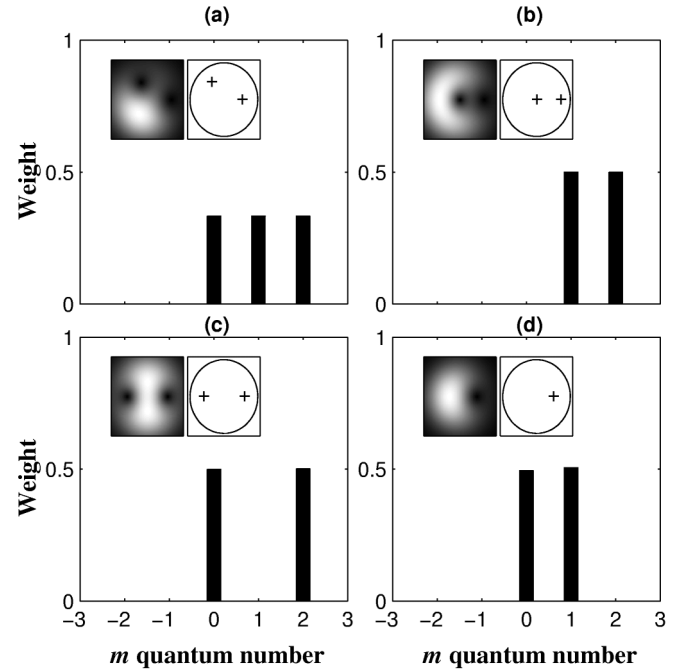

FIG. 2. Preparation of L-managed three-dimensional photon states with a $N = 2$ vortex pancake. Features as in Fig. 1. In (a) $P_0 = P_1 = P_2 = 1/3$; in (b) $P_0 = 0$; $P_1 = P_2 = 1/2$; in (c) $P_1 = 0$; in (d): $P_2 = 0$.
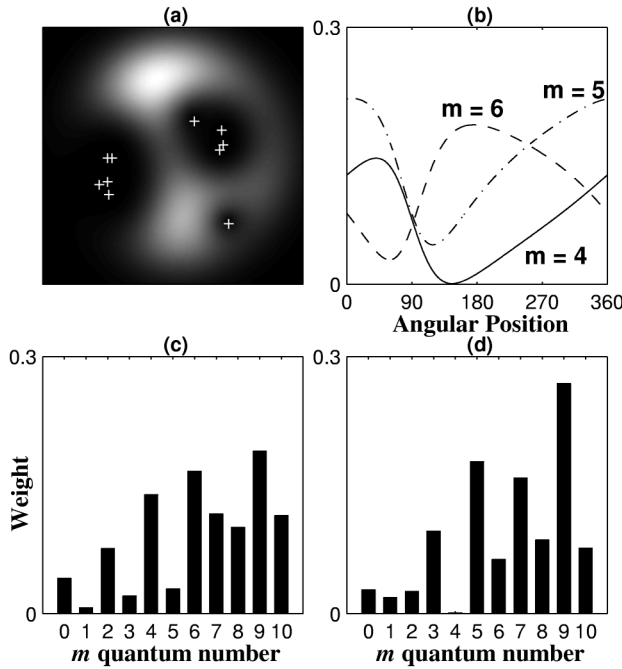
FIG. 3. Analogous to Fig. 2, but for 11-dimensional photon states prepared with a $N = 10$ vortex pancake. (a) Light amplitude; (b) mode weight versus azimuthal location of one of the vortices (in the particular case shown, φ_1); (c),(d) mode weight versus m-quantum number for two different pancakes. (c) corresponds to (a), and in (d) φ_1 was chosen so that the $|m = 4\rangle$ mode is almost suppressed, as dictated by (b).

e.g., by a Michelson-Morley interferometer, should reveal the LG spectrum of the light signal.

A generalized version of the scheme invented by Mair and co-workers [6], to spatially separate the different m orders with suitable holograms and filter them out with single-mode optical fibers, should serve the same purpose with single photons.

While looking for such experimental verification, we conclude noticing the implications of the results to the generation of quantum entangled photons with angular momentum [6], by parametric down-conversion of vortex signals in quadratic nonlinear crystals [17]. The idea is to generate entangled states $\psi = \sum_{m_1,m_2} B_{m_1 m_2}|m_1\rangle|m_2\rangle$, with modified $B_{m_1 m_2}$ probability amplitudes, by using L-managed pump photons. Such multidimensional entanglement should allow the experimental exploration of quantum features realizable only in N-dimensional Hilbert spaces. The degree of violation of Bell's inequalities as a function of N [18], generated by different L-managed vector states, constitutes a fascinating possibility. Finally, we anticipate that L-managed photon states should find

applications in spintronics [19] and in capacity-increased quantum information schemes [20,21].

[1] J. H. Poynting, Proc. R. Soc. London A **82**, 560 (1909).
[2] R. A. Beth, Phys. Rev. **50**, 115 (1936).
[3] L. Allen, M. J. Padget, and B. Babiker, in *Progress in Optics,* edited by E. Wolf (Elsevier, Amsterdam, 1999), Vol. XXXIX, pp. 291–372.
[4] A. Ashkin, Opt. Photonics News **10**, 41 (1999); M. E. J. Friese *et al.,* Nature (London) **394**, 348 (1998); M. J. Padgett and L. Allen, Contemp. Phys. **41**, 275 (2000); L. Paterson *et al.,* Science **292**, 912 (2001).
[5] P. Galajda and P. Ormos, Appl. Phys. Lett. **78**, 249 (2001).
[6] A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger, Nature (London) **412**, 313 (2001).
[7] M. Reck *et al.,* Phys. Rev. Lett. **73**, 58 (1994); M. Zukowski, A. Zeilinger, and M. A. Horne, Phys. Rev. A **55**, 2564 (1997); M. Bourennane, A. Karlsson, and G. Björk, Phys. Rev. A **64**, 012306 (2001).
[8] L. Allen *et al.,* Phys. Rev. A **45**, 8185 (1992).
[9] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley and Hermann, Paris, 1977).
[10] S. J. van Enk and G. Nienhuis, Opt. Commun. **94**, 147 (1992); S. J. van Enk, Opt. Commun. **102**, 59 (1993); G. Nienhuis and L. Allen, Phys. Rev. A **48**, 656 (1993).
[11] G. Nienhuis, Opt. Commun. **132**, 8 (1996); J. Courtial *et al.,* Phys. Rev. Lett. **80**, 3217 (1998); **81**, 4282 (1998).
[12] E. Abramochkin and V. Volostnikov, Opt. Commun. **83**, 123 (1991); M. W. Beijersbergen *et al.,* Opt. Commun. **96**, 123 (1993); G. Molina-Terriza *et al.,* Phys. Rev. Lett. **87**, 023902 (2001).
[13] M. Soljacic and M. Segev, Phys. Rev. E **62**, 2810 (2000); Phys. Rev. Lett. **86**, 420 (2001); A. S. Desyatnikov and Yu. S. Kivshar, Phys. Rev. Lett. **87**, 033901 (2001).
[14] G. Molina-Terriza, J. Recolons, and L. Torner, Opt. Lett. **25**, 1135 (2000).
[15] G. Indebetouw, J. Mod. Opt. **40**, 73 (1993).
[16] M. V. Vasnetsov, V. V. Slyusar, and M. S. Soskin, Kvant. Elektron. **31**, 464 (2001) [Quantum Electron. **31**, 464 (2001)].
[17] P. Di Trapani *et al.,* Phys. Rev. Lett. **81**, 5133 (1998); J. Arlt *et al.,* Phys. Rev. A **59**, 3950 (1999).
[18] D. Kaszlikowski *et al.,* Phys. Rev. Lett. **85**, 4418 (2000), and references therein.
[19] S. D. Ganichev *et al.,* Phys. Rev. Lett. **86**, 4358 (2001), and references therein.
[20] D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer, Berlin, 2000).
[21] E. Knill, R. Laflamme, and G. J. Milburn, Nature (London) **409**, 46 (2001).