Spencer Alexander

Multipole tensors - Analytic expression for multipole tensors of arbitrary order based on pattern recognition derivation

Notations used:

1. Three-dimensional Cartesian indices are represented as a_i. For instance, r_{a_1} can represent r_x, r_y, or r_z.
2. Summations are implied over repeated lower indices. For instance, $R_{a_1} r_{a_1} = R_x r_x + R_y r_y + R_z r_z$.
3. $\int_V (\cdots)\, dv := \int_{\text{all space}} (\cdots)\, dxdydz$.

Vocabulary used: In what follows, I will define tensors and contants $T^{(l)}$, $q^{(l)}$, $J^{i(l)}$, $m^{(l)}$, and $d^{(l)}$. People refer to both $T^{(l)}$ and $q^{(l)}$ (or $J^{i(l)}$) as multipole tensors, or multipole moments. For example, if someone mentions the electrostatic dipole moment, then depending on context they may be referring to $\rho^{(1)} = \boldsymbol{e}_{a_1} \int_V r_{a_1} \rho(\boldsymbol{r})\, dv$ or $T^{(1)} = \boldsymbol{e}_{a_1} R_{a_1}$. Furthermore, in most contexts, $J^{i(l)}$ is not used directly - instead, $m^{(l)}$ is used, where e.g.,

$$\begin{aligned} \boldsymbol{m} &= m^{(1)} \\ &= \frac{1}{2} \int_V \boldsymbol{r} \times \boldsymbol{J} dv. \end{aligned} \tag{1}$$

As with $T^{(l)}$ and $J^{i(l)}$, people refer to $m^{(l)}$ is as multipole tensors or multipole moments. Note that the $m^{(l)}$'s are defined such that there is always a single tensor at order l, as opposed to three $J^{i(l)}$ tensors at the same order.
On this page, multipole tensor refers to either $q^{(l)}$ or $J^{i(l)}$, prefactor tensor refers to $T^{(l)}$, and denominator constant refers to $d^{(l)}$.

Multipole expansion (Cartesian tensor formalism), and the main result of this webpage: The electric potential $\Phi(\boldsymbol{R})$ can be written as

$$\begin{aligned} \Phi(\boldsymbol{R}) &= \frac{1}{4\pi\varepsilon_0} \left[\frac{\int_V \rho(\boldsymbol{r})\, dv}{R} + \frac{R_{a_1} \int_V r_{a_1} \rho(\boldsymbol{r})\, dv}{R^3} + \frac{\left(3R_{a_1} R_{a_2} - R^2 \delta_{a_1 a_2}\right) \int_V r_{a_1} r_{a_2} \rho(\boldsymbol{r})\, dv}{2R^5} + \ldots \right] \\ &= \frac{1}{4\pi\varepsilon_0} \sum_{l=0} \frac{T^{(l)} q^{(l)}}{d^{(l)} R^{2l+1}}, \end{aligned} \tag{2}$$

and similarly the components of the magnetic vector potential $\boldsymbol{A}(\boldsymbol{R})$ can be written as

$$A^i(\boldsymbol{R}) = \frac{\mu_0}{4\pi} \sum_{l=0} \frac{T^{(l)} J^{i(l)}}{d^{(l)} R^{2l+1}}, \tag{3}$$

where

$$q^{(l)} := \int_V r_{a_1} \times \cdots \times r_{a_l} \rho(\boldsymbol{r})\, dv, \tag{4}$$

$$J^{i(l)} := \int_V r_{a_1} \times \cdots \times r_{a_l} J^i(\boldsymbol{r})\, dv, \tag{5}$$

$$d^{(l)} := \text{constants in denominator of } l'\text{th term}. \tag{6}$$

Some analytical expressions for the denominator constant are

$$\begin{aligned} d^{(l)} &= 2^{\sum_{n=1}^{\text{floor}(\log_2 l)} \text{floor}(l/2^n)} \\ &= 2^{\sum_{n=1}^{\infty} \text{floor}(l/2^n)} \\ &= \gcd\left(l!, 2^l\right). \end{aligned} \tag{7}$$

An analytical expression for the prefactor tensor is

$$T^{(l)} = (-1)^l \sum_{m=0}^{\mathrm{rd}(l/2)} d^{(l)} 2^{l-2m} \binom{-1/2}{l-m} \binom{l-m}{l-2m} R^{2m} \left(\prod_{\alpha=1}^{l-2m} R_{a_\alpha} \right) \left(\prod_{\beta=0}^{m-1} \delta_{a_{2\beta+l-2m+1} a_{2\beta+l-2m+2}} \right), \tag{8}$$

where $\mathrm{rd}\,(x)$ rounds x to the nearest integer, and rounds down if x is a half-integer, and where Euler products are 1 if the starting value is greater than the upper limit:

$$\mathrm{rd}\,(x) := \begin{cases} \mathrm{round}\,(x)\,, & x \neq n + \frac{1}{2}, n \in \mathbb{Z} \\ \mathrm{floor}\,(x)\,, & x = n + \frac{1}{2}, n \in \mathbb{Z} \end{cases} \quad , \quad \prod_{\gamma=\gamma_0}^{\gamma_1} x_\beta := \begin{cases} \prod_{\gamma=\gamma_0}^{\gamma_1} x_\beta, & \gamma_1 \geq \gamma_0 \\ 1, & \gamma_1 < \gamma_0 \end{cases}. \tag{9}$$

Note that the above analytical expression for $T^{(l)}$ is not unique - it is only the tensor products $T^{(l)}q^{(l)}$ and $T^{(l)}J^{i(l)}$ that are important, and there is freedom in the definition of $T^{(l)}$ that leaves these products invariant.

Derivation: What follows is half-derivation, half-pattern recognition. I'm only going to write it for the electric case, it is identical for the magnetic case.
Start with

$$\begin{aligned} \Phi\left(\boldsymbol{R}\right) &= \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{|\boldsymbol{R}-\boldsymbol{r}|} \rho\left(\boldsymbol{r}\right) dv \\ &= \frac{1}{4\pi\varepsilon_0} \int_V \left[\left(\boldsymbol{R}-\boldsymbol{r}\right)^2 \right]^{-1/2} \rho\left(\boldsymbol{r}\right) dv \\ &= \frac{1}{4\pi\varepsilon_0} \int_V \left[r^2 - 2\boldsymbol{R}\cdot\boldsymbol{r} + R^2 \right]^{-1/2} \rho\left(\boldsymbol{r}\right) dv. \end{aligned} \tag{10}$$

Now apply the generalized binomial theorem,

$$(a+b)^n = \sum_{k=0} \binom{n}{k} a^{n-k} b^k, \tag{11}$$

with

$$a = r^2 - 2\boldsymbol{R}\cdot\boldsymbol{r} \quad , \quad b = R^2 \quad , \quad n = -1/2. \tag{12}$$

This gives

$$\begin{aligned} \frac{1}{|\boldsymbol{R}-\boldsymbol{r}|} &= \left[r^2 - 2\boldsymbol{R}\cdot\boldsymbol{r} + R^2 \right]^{-1/2} \\ &= \frac{1}{R} + \frac{2\boldsymbol{R}\cdot\boldsymbol{r} - r^2}{2R^3} + \frac{3\left(-2\boldsymbol{R}\cdot\boldsymbol{r} + r^2\right)^2}{8R^5} + \ldots. \end{aligned} \tag{13}$$

Multiply terms by extra factors of R^2/R^2 such that $\boldsymbol{R}$ and $\boldsymbol{r}$ always show up multiplicatively in like powers. E.g., take $-r^2/R^3 \mapsto -r^2R^2/R^5$, etc. Then group terms so that they have like powers of R in the denominator, and multiply terms in a given numerator until no fractions show up in the numerators. This gives

$$\frac{1}{|\boldsymbol{R}-\boldsymbol{r}|} = \frac{1}{R} + \frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R^3} + \frac{3\left(\boldsymbol{R}\cdot\boldsymbol{r}\right)^2 - R^2r^2}{2R^5} + \frac{5\left(\boldsymbol{R}\cdot\boldsymbol{r}\right)^3 - 3R^2\left(\boldsymbol{R}\cdot\boldsymbol{r}\right)r^2}{2R^7} + \ldots. \tag{14}$$

Substitute this back into the expression for $\Phi\left(\boldsymbol{R}\right)$ and rearrange slightly,

$$\Phi\left(\boldsymbol{R}\right) = \frac{1}{4\pi\varepsilon_0} \left\{ \int_V \frac{1}{R} \rho\left(\boldsymbol{r}\right) dv + \int_V \frac{1}{R^3} \boldsymbol{R}\cdot\boldsymbol{r} \rho\left(\boldsymbol{r}\right) dv + \int_V \frac{1}{2R^5} \left[3\left(\boldsymbol{R}\cdot\boldsymbol{r}\right)^2 - R^2r^2 \right] \rho\left(\boldsymbol{r}\right) dv + \ldots \right\}. \tag{15}$$

Get the multipole moments $q^{(l)}$ to show up explicitly by writing dot products between $\boldsymbol{R}$ and $\boldsymbol{r}$ and between $\boldsymbol{r}$ and itself using index convention, e.g.,

$$R^2\left(\boldsymbol{R}\cdot\boldsymbol{r}\right)r^2 = R^2 R_{a_1} r_{a_1} r_{a_2} r_{a_2}. \tag{16}$$

Then get all r indices in a given term to not repeat themselves using Kronecker deltas, e.g.,

$$R^2 R_{a_1} r_{a_1} r_{a_2} r_{a_2} = R^2 R_{a_1} \delta_{a_2 a_3} r_{a_1} r_{a_2} r_{a_3}. \tag{17}$$

Substituting,

$$\Phi(\boldsymbol{R}) = \frac{1}{4\pi\varepsilon_0}\left[\int_V \frac{1}{R}\rho(\boldsymbol{r})\,dv + \int_V \frac{R_{a_1}}{R^3} r_{a_1}\rho(\boldsymbol{r})\,dv + \int_V \frac{3R_{a_1}R_{a_2} - R^2\delta_{a_1 a_2}}{2R^5} r_{a_1} r_{a_2}\rho(\boldsymbol{r})\,dv + \ldots\right]. \tag{18}$$

Move all terms not dependent on $\boldsymbol{r}$ outside of the integrals,

$$\Phi(\boldsymbol{R}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{R}\int_V \rho(\boldsymbol{r})\,dv + \frac{R_{a_1}}{R^3}\int_V r_{a_1}\rho(\boldsymbol{r})\,dv + \frac{3R_{a_1}R_{a_2} - R^2\delta_{a_1 a_2}}{2R^5}\int_V r_{a_1} r_{a_2}\rho(\boldsymbol{r})\,dv + \ldots\right]. \tag{19}$$

Now identify the integrals with $q^{(l)}$, the numerators outside of the integrals with $T^{(l)}$, the constants in the denominators outside of the integrals as $d^{(l)}$, and the powers of R in the denominators as R^{2l+1},

$$\begin{aligned}\Phi(\boldsymbol{R}) &= \frac{1}{4\pi\varepsilon_0}\left[\frac{T^{(0)}q^{(0)}}{d^{(0)}R^{2\times 0+1}} + \frac{T^{(1)}q^{(1)}}{d^{(1)}R^{2\times 1+1}} + \frac{T^{(2)}q^{(2)}}{d^{(2)}R^{2\times 2+1}} + \ldots\right] \\ &= \frac{1}{4\pi\varepsilon_0}\sum_{l=0} \frac{T^{(l)}q^{(l)}}{d^{(l)}R^{2l+1}}.\end{aligned} \tag{20}$$

The above procedure was coded into the Mathematica document Multipole expansion.nb. Using the procedure, any $T^{(l)}$ or $d^{(l)}$ can be found algorithmically, but the procedure is not yet an analytical expression for these (although we already have an analytical expression for $q^{(l)}$ by $q^{(l)}$'s definition). However, the procedure is handy for quickly exploring arbitrarily high-order prefactor tensors and denominator constants and looking for patterns to determine an analytical expression, and that's what I did.

Let's start with the denominator constants. The first several are:

l	0	1	2	3	4	5	6	7	8	9	10	11	12
$d^{(l)}$	1	1	2	2	8	8	16	16	128	128	256	256	1024

A handy tool for exploring nonobvious integer patterns like this is The On-Line Encyclopedia of Integer Sequences, OEIS. Throwing it in there, it is sequence number A060818, and a few expressions for the sequence are given as

$$\begin{aligned}d^{(l)} &= 2^{\sum_{n=1}^{\infty}\mathrm{floor}(l/2^n)} \\ &= \gcd\left(l!, 2^l\right) \\ &= \mathrm{denominator}\left(\binom{2l}{l}/2^l\right).\end{aligned} \tag{21}$$

The first and last of these are also equivalent to

$$\begin{aligned}d^{(l)} &= 2^{\sum_{n=1}^{\mathrm{floor}(\log_2 l)}\mathrm{floor}(l/2^n)} \\ &= \mathrm{denominator}\left(2^l\binom{-1/2}{l}\right),\end{aligned} \tag{22}$$

since $\binom{2l}{l}/2^l = -2^l\binom{-1/2}{l}$.

Great, now let's move onto the prefactor tensors. Start by making a table of the first several:

l	$T^{(l)}$
0	1
1	$-R_{a_1}$
2	$3R_{a_1}R_{a_2} - 3R^2 R_{a_1}\delta_{a_1 a_2}$
3	$5R_{a_1}R_{a_2}R_{a_3} - 3R^2 R_{a_1}\delta_{a_2 a_3}$
4	$35R_{a_1}R_{a_2}R_{a_3}R_{a_4} - 30R^2 R_{a_1}R_{a_2}\delta_{a_3 a_4} + 3R^4\delta_{a_1 a_2}\delta_{a_3 a_4}$
5	$63R_{a_1}R_{a_2}R_{a_3}R_{a_4}R_{a_5} - 70R^2 R_{a_1}R_{a_2}R_{a_3}\delta_{a_4 a_5} + 15R^4 R_{a_1}\delta_{a_2 a_3}\delta_{a_4 a_5}$

In general, we can write each $T^{(l)}$ as

$$T^{(l)} = \sum_{m=1}^{\text{upper limit}} \kappa_m \lambda_m, \tag{23}$$

with κ_m representing the constants, and λ_m representing everything with R's and δ's. First, what is upper limit? For $l = 0, 1$ it is 1, for $l = 2, 3$, it is 2, and for $l = 4, 5$, it is 3. This pattern continues, a fact made obvious since the first term must always have R_{a_i}'s only, and each subsequent term must trade a pair of R_{a_i}'s for an $R^2\delta_{a_i a_{i+1}}$. So we conclude

$$\text{upper limit} = \text{rd}\left[(l+1)/2\right]. \tag{24}$$

The pattern for the κ_m's is very nonobvious, so skip for now. The pattern for the λ_m's is what I mentioned above. It is clear that we can write a given λ_m using Euler products, and the rest is just being careful about how we write all of the indices. It comes to

$$\lambda_m = R^{2(m-1)} \left(\prod_{\alpha=1}^{l-2(\alpha-1)} R_{a_\alpha} \right) \left(\prod_{\beta=0}^{m-2\geq 0} \delta_{l-\beta, l-\beta+1} \right). \tag{25}$$

So,

$$\begin{aligned} T^{(l)} &= \sum_{m=1}^{\text{rd}[(l+1)/2]} \kappa_m R^{2(m-1)} \left(\prod_{\alpha=1}^{l-2(m-1)} R_{a_\alpha} \right) \left(\prod_{\beta=0}^{m-2\geq 0} \delta_{a_{2\beta+l-2m+3} a_{2\beta+l-2m+4}} \right) \\ &= \sum_{m=0}^{\text{rd}(l/2)} \kappa_m R^{2m} \left(\prod_{\alpha=1}^{l-2m} R_{a_\alpha} \right) \left(\prod_{\beta=0}^{m-1\geq 0} \delta_{a_{2\beta+l-2m+1} a_{2\beta+l-2m+2}} \right). \end{aligned} \tag{26}$$

I'll use a different route to figure out the κ_m's:

1. Write out the terms in the binomial expansion of $1/|\boldsymbol{R} - \boldsymbol{r}|$ given in equations 11 and 12, where the analytical form of the constants in front of each term will be obvious.
2. Figure out a general pattern for how these terms are regrouped for the multipole expansion.
3. Screw around algebraically as necessary to get the constants to appear exactly as they are in the multipole expansion.
4. Identify the resulting constants with the κ_m's.

We have

$$\frac{1}{|\boldsymbol{R} - \boldsymbol{r}|} = \sum_{k=0} \binom{-1/2}{k} \left(R^2\right)^{-1/2-k} \left(r^2 - 2\boldsymbol{R}\cdot\boldsymbol{r}\right)^k. \tag{27}$$

Again using the generalized binomial theorem, this time on the $\left(r^2 - 2\boldsymbol{R}\cdot\boldsymbol{r}\right)$ term,

$$\begin{aligned} \frac{1}{|\boldsymbol{R} - \boldsymbol{r}|} &= \sum_{k=0} \binom{-1/2}{k} \left(R^2\right)^{-1/2-k} \sum_{j=0} \binom{k}{j} \left(r^2\right)^{k-j} \left(2\boldsymbol{R}\cdot\boldsymbol{r}\right)^j \\ &= \sum_{k=0} \sum_{j=0}^{k} 2^j \binom{-1/2}{k} \binom{k}{j} \frac{1}{R^{2k+1}} \left(r^2\right)^{k-j} \left(\boldsymbol{R}\cdot\boldsymbol{r}\right)^j. \end{aligned} \tag{28}$$

We want to get this to look like the multipole expansion, in which $\boldsymbol{R}$ and $\boldsymbol{r}$ always show up in equal powers in the numerators. To do this, rewrite the above expression as

$$\frac{1}{|\boldsymbol{R} - \boldsymbol{r}|} = \sum_{k=0} \sum_{j=0}^{k} 2^j \binom{-1/2}{k} \binom{k}{j} \frac{1}{R^{4k-2j+1}} \left(r^2\right)^{k-j} \left(\boldsymbol{R}\cdot\boldsymbol{r}\right)^j \left(R^2\right)^{k-j}. \tag{29}$$

Now we group terms in the above expression as they are grouped in the multipole expansion by noting that grouped terms will have like powers of R in the denominator. Let

$$\mu_{k,j} := 2^j \binom{-1/2}{k} \binom{k}{j} \frac{1}{R^{4k-2j+1}} \left(r^2\right)^{k-j} \left(\boldsymbol{R}\cdot\boldsymbol{r}\right)^j \left(R^2\right)^{k-j}. \tag{30}$$

The table below makes obvious which $\mu_{k,j}$ correspond to which $R^{4k-2j+1} = R^{2l+1}$ in the denominators of the multipole expansion:

$R^{4k-2j+1} = R^{2l+1}$ in denominator	R^1	R^3	R^5	R^7	R^9	R^{11}
$\mu_{k,j}$ terms	$\mu_{0,0}$	$\mu_{1,1}$	$\mu_{2,2},\ \mu_{1,0}$	$\mu_{3,3},\ \mu_{2,1}$	$\mu_{4,4},\ \mu_{3,2},\ \mu_{2,0}$	$\mu_{5,5},\ \mu_{4,3},\ \mu_{3,1}$

Overall, the $\mu_{k,j}$ corresponding to R^n are $\left\{\mu_{\frac{1}{2}(n-1)-n',\frac{1}{2}(n-1)-2n'}\middle| n' \in \left\{0,1,\ldots,\mathrm{rd}\left[\frac{1}{4}(n-1)\right]\right\}\right\}$, or equivalently, the $\mu_{k,j}$ corresponding to R^{2l+1} are $\left\{\mu_{l-m,l-2m}\middle| m \in \{0,1,\ldots,\mathrm{rd}(l/2)\}\right\}$. The fact that m goes from 0 to $\mathrm{rd}(l/2)$ is good: Once we substitute $1/(\boldsymbol{R}-\boldsymbol{r})$ as phrased in terms of $\mu_{l-m,l-2m}$'s back into the expression for $\Phi(\boldsymbol{R})$, we can identify the constants in each $\mu_{l-m,l-2m}$ as being some mixture of κ_m's and $d^{(l)}$'s, and the symbolic parts of the expression as being the λ_m's and $q^{(l)}$'s. Doing this, substituting back in for $\mu_{l-m,l-2m}$, and writing the symbolic parts of the expression briefly as λ_m's and $q^{(l)}$'s, the sum over m is

$$\sum_{m=0}^{\mathrm{rd}(l/2)} k^{(l)}T^{(l)}q^{(l)} = \sum_{m=0}^{\mathrm{rd}(l/2)} 2^{l-2m}\binom{-1/2}{l-m}\binom{l-m}{l-2m}R^{2m}\left(\prod_{\alpha=1}^{l-2m}R_{a_\alpha}\right)\left(\prod_{\beta=0}^{m-1\geq 0}\delta_{a_{2\beta+l-2m+1}a_{2\beta+l-2m+2}}\right)q^{(l)}. \tag{31}$$

Equivalently,

$$T^{(l)} = \sum_{m=0}^{\mathrm{rd}(l/2)} k_m^{(l)}2^{l-2m}\binom{-1/2}{l-m}\binom{l-m}{l-2m}R^{2m}\left(\prod_{\alpha=1}^{l-2m}R_{a_\alpha}\right)\left(\prod_{\beta=0}^{m-1\geq 0}\delta_{a_{2\beta+l-2m+1}a_{2\beta+l-2m+2}}\right), \tag{32}$$

i.e., the expresion on the RHS is equal to $T^{(l)}$, except that there is still a multiplicative constant in front of each term in $T^{(l)}$. To figure out how to adjust the expression further, define

$$U^{(l)} := \sum_{m=0}^{\mathrm{rd}(l/2)} 2^{l-m}\binom{-1/2}{l-m}\binom{l-m}{l-2m}R^{2m}\left(\prod_{\alpha=1}^{l-2m}R_{a_\alpha}\right)\left(\prod_{\beta=0}^{m-1\geq 0}\delta_{a_{2\beta+l-2m+1}a_{2\beta+l-2m+2}}\right), \tag{33}$$

(so each term of $U^{(l)}$ differs from the corresponding term in $T^{(l)}$ by factors of $k_m^{(l)}$ and 2^m. Write a table of l, $T^{(l)}$, and $U^{(l)}$:

l	$T^{(l)}$	$U^{(l)}$
0	1	1
1	R_{a_1}	$-R_{a_1}$
2	$3R_{a_1}R_{a_2} - R^2\delta_{a_1a_2}$	$\frac{3}{2}R_{a_1}R_{a_2} - R^2\delta_{a_1a_2}$
3	$5R_{a_1}R_{a_2}R_{a_3} - 3R^2R_{a_1}\delta_{a_2a_3}$	$-\frac{5}{2}R_{a_1}R_{a_2}R_{a_3} + 3R^2R_{a_1}\delta_{a_2a_3}$
4	$35R_{a_1}\cdots R_{a_4} - 30R^2R_{a_1}R_{a_2}\delta_{a_3a_4} + 3R^4\delta_{a_1a_2}\delta_{a_3a_4}$	$\frac{35}{8}R_{a_1}\cdots R_{a_4} - \frac{30}{4}R^2R_{a_1}R_{a_2}\delta_{a_3a_4} + \frac{3}{2}R^4\delta_{a_1a_2}\delta_{a_3a_4}$
5	$63R_{a_1}\cdots R_{a_5} - 70R^2R_{a_1}R_{a_2}R_{a_3}\delta_{a_4a_5} + 15R^4R_{a_1}\delta_{a_2a_3}\delta_{a_4a_5}$	$-\frac{63}{8}R_{a_1}\cdots R_{a_5} + \frac{70}{4}R^2R_{a_1}R_{a_2}R_{a_3}\delta_{a_4a_5} - \frac{15}{2}R^4R_{a_1}\delta_{a_2a_3}\delta_{a_4a_5}$

The tensors are exactly the same, except for the following differences:

- The are off by a factor of $(-1)^l$.
- Letting $c^{(l)}$ be the leading denominator in $U^{(l)}$, The (l,m)'th terms are off by factors of $c^{(l)}/2^m$.

So, if we can figure out the pattern for the terms $c^{(l)}$, we can write the appropriate summation to convert from $U^{(l)}$ to $T^{(l)}$, and we are done. $c^{(l)}$ is just the denominator in the $m=0$ constant in $U^{(l)}$, i.e.,

$$\begin{aligned}
c^{(l)} &= \text{denominator}\left(2^{l-m}\binom{-1/2}{l-m}\binom{l-m}{l-2m}\bigg|_{m=0}\right) \\
&= \text{denominator}\left(2^{l}\binom{-1/2}{l}\binom{l}{l}\right) \\
&= \text{denominator}\left(2^{l}\binom{-1/2}{l}\right) \\
&= d^{(l)},
\end{aligned} \tag{34}$$

by equation 22. So, we have effectively determined that

$$\begin{aligned} k_m^{(l)} &= (-1)^l \, 2^m \frac{d^{(l)}}{2^m} \\ &= (-1)^l \, d^{(l)}. \end{aligned} \tag{35}$$

From this and equations 26 and 32, this is equivalent to saying that

$$\kappa_m = (-1)^l \, d^{(l)} 2^{l-2m} \binom{-1/2}{l-m} \binom{l-m}{l-2m}. \tag{36}$$

Substituting back into equation 26 or 32, we have

$$T^{(l)} = (-1)^l \sum_{m=0}^{\mathrm{rd}(l/2)} d^{(l)} 2^{l-2m} \binom{-1/2}{l-m} \binom{l-m}{l-2m} R^{2m} \left(\prod_{\alpha=1}^{l-2m} R_{a_\alpha} \right) \left(\prod_{\beta=0}^{m-1 \geq 0} \delta_{a_{2\beta+l-2m+1} a_{2\beta+l-2m+2}} \right). \tag{37}$$

The results of the above derivation are summarized by equations 2 through 9.