

# C++ Functions in Maxliklib Library

Shelby J. Haberman  
Haberman Statistics

## Abstract


The functions in the maxliklib repository are described. Arguments and their definitions are specified, and dependencies of functions are stated.

*Keywords:* Maximization procedures, quadrature procedures, maximum likelihood

The maxliklib repository consists of C++ functions helpful in estimation related to maximum likelihood. The functions should be appropriate for C++14. They rely on the Armadillo library (Sander-son & Curtin, 2016, 2018) at <http://arma.sourceforge.net>, the StatsLib library at <https://www.kthohr.com/statslib.html>, and the Boost library at <https://www.boost.org>. Unless otherwise noted, for the library members considered, it is assumed that users have verified that function arguments are valid. Namespaces assumed where relevant are *std*, *arma*, and *boost::math*. The following functions are found in the library, with files in the source code with the suffix .cpp.

- addsel
- berresp
- conjgrad
- contresp
- cumresp
- cx
- eaps

---

Shelby J. Haberman  <https://orcid.org/0000-0002-5490-0405>

Shelby Haberman is an independent statistical consultant whose website is <https://www.habermanstatistics.com>. He can also be reached at [haberman.statistics@gmail.com](mailto:haberman.statistics@gmail.com).

- fitquad
- genfact
- genprods
- genresp
- genresplik
- genrespmle
- gpcm
- gradascent
- gradresp
- gumbell
- gumbelu
- hermcoeff
- hermpoly
- hermpw
- intsel
- invcdf
- irtm
- irtmle
- irtms
- irtmsave
- irtmsaves
- ivecsel
- linsel
- loggamma
- logistic

- logit
- logitbeta
- logitdirichlet
- loglogl
- loglogu
- logmean
- lw
- lwm
- maxberresp
- maxf2vvar
- maxlinq2
- maxquad
- maxselect
- multlogit
- modit
- normal
- normalv
- nrv
- oneparamirt
- pack
- pcm
- posterior
- posteriors
- probit
- qnormpwe

- ranklogit
- rebound
- regprod
- regprodf
- savmaxf2v
- sintsel
- sivecsel
- startoneparamirt
- startgpcm
- startpcm
- starttwoparamirt
- svecsel
- truncresp
- twoparamirt
- unpack
- vecsel
- wmc

### **Distributions of Sums of Independent Multinomial Variables**

The functions in this section implement a modified and generalized version of the Lord-Wingersky algorithm (Lord & Wingersky, 1984; Thissen et al., 1995). The numerical procedures and their rationale are discussed in lw.pdf.

**lw**

The function *lw* finds the probability mass function of the sum  $S$  of mutually independent Bernoulli random variables  $X_j$ ,  $0 \leq j < n$ . The function declaration is

*vec lw(const double & c, const vec & p).*

In the function definition,  $c$  is a constant  $c$ , and  $p$  is an  $n$ -dimensional vector  $\mathbf{p}$ . The vector  $\mathbf{p}$  with elements  $p_j$ ,  $0 < p_j < 1$ , has dimension  $n$ . For  $0 \leq j < n$ ,  $p_j$  is the probability that  $X_j = 1$ . The variable  $c$  is normally a small positive number used as in *lw.pdf* to remove very small probabilities from consideration in order to speed computation. If  $c$  is not positive, then the modified Lord-Wingersky algorithm used by *lw* reduces to the conventional algorithm. The probability mass function is provided by *lw*, a vector with  $n + 1$  elements. For  $0 \leq k \leq n$ , element  $k$  of *lw* is the probability  $P(S = k)$  that  $S = k$ .

**lwm**

The function *lwm* finds the probability mass function of the sum  $S$  of  $n$  mutually independent random variables  $X_j$ ,  $0 \leq j < n$  with integer values from 0 to  $I_j - 1$  for an integer  $I_j > 1$ . The function declaration is

*vec lwm(const double & c, const field<vec> & p).*

Here  $p$  has  $n$  vector members  $\mathbf{p}_j$ ,  $0 \leq j < n$ . For each  $j$ ,  $\mathbf{p}_j$  has positive elements  $p_{ij}$ ,  $0 \leq i < I_j$ , that sum to 1. The probability  $P(X_j = i)$  is  $p_{ij}$ . The vector *lwm* has  $K + 1$  elements, where  $K = \sum_{j=1}^n (I_j - 1)$  and  $P(S = k)$  is element  $k$  of *lwm* for  $0 \leq k < K$ . The definition of  $c$  is the same as in *lw*.

## Tools for Line Searches

The functions in this section facilitate line searches during function maximization. Throughout discussions in this section and in Functions related to the Newton-Raphson algorithm and Functions Related to Gradient Methods, the theoretical background and the definitions of  $\eta$ ,  $\gamma_1$ ,  $\gamma_2$ , and  $\kappa$  are found in *convergence.pdf*. For some positive integer  $p$  and nonempty open convex set  $O$  of  $p$ -dimensional vectors, a continuously differentiable real function  $f$  on  $O$  is to be maximized by an iterative algorithm with a starting value in  $O$ . It is assumed that, for some real  $a$ , the set  $A$  of members of  $O$  at which  $f$  is at least  $a$  is closed and bounded, and the set  $A_0$  of members of  $O$  at which  $f$  exceeds  $a$  is nonempty. The function  $f$  is assumed to be strictly pseudoconcave on  $A_0$ . The starting values for algorithms are assumed to be in  $A_0$ . The convention is adopted that  $f(\mathbf{x}) = -\infty$  if  $\mathbf{x}$  is not in  $O$ .

**maxlinq2**

The function *maxlinq2* provides a line search for maximization algorithms. Only function values and gradients are used when *order* is 1, but Hessian matrices or their approximations are computed if *order* is greater than 1. The function declaration is

```
maxf2v maxlinq2(const int & order, const params & mparams, const vec & v,
const maxf2v & vary0, const function<f2v(const int &, const vec &)>f).
```

Here the definition of *maxf2v* is

```
struct maxf2v{vec locmax; double max; vec grad, mat hess};,
```

For a vector  $\mathbf{y}$ , the corresponding definitions of *maxf2v* are *maxf2v.locmax* equals  $\mathbf{y}$ , *maxf2v.max* equals  $f(\mathbf{y})$ , *maxf2v.grad* equals the gradient  $\nabla f(\mathbf{y})$  of  $f$  at  $\mathbf{y}$ , and, for *order* greater than 1, *maxf2v.hess* equals the Hessian matrix  $\nabla^2 f(\mathbf{y})$  of  $f$  at  $\mathbf{y}$ . The corresponding definition of *f2v* is

```
struct f2v{double value; vec grad; vec hess};,
```

where, for the vector  $\mathbf{y}$ , *f2v.value* is  $f(\mathbf{y})$ , *f2v.grad* is  $\nabla f(\mathbf{y})$ , and, for *order* greater than 1, *f.hess* is  $\nabla^2 f(\mathbf{y})$ . In the arguments of *maxlinq2*, *vary0.locmax* is the starting vector  $\mathbf{x}$  for the line search, the search direction is  $\mathbf{v}$ , and  $f$  is the function  $f$  together with its gradient and, for *order* greater than 1, its Hessian matrix. Maximization details are defined by *mparams*, where the definition of *params* is

```
struct params{bool print; int maxit; int maxits; double eta;
double gamma1; double gamma2; double kappa; double tol};.
```

Here *mparams.print* is used for output of the iteration number and function value at the end of the iteration, *mparams.maxit* is the number of primary iterations, *mparams.maxits* is the maximum number of uses of *maxquad* permitted for each primary iteration, *mparams.eta* is  $\eta$ , *mparams.gamma1* is  $\gamma_1$ , *mparams.gamma2* is  $\gamma_2$ , and *mparams.kappa* is  $\kappa$ . Iterations cease if the function value changes less than *mparams.tol* after a primary iteration.

In *maxlinq2*, *maxlinq2.locmax* is an approximate location  $y = \mathbf{x} + \alpha \mathbf{v}$  of the maximum over  $\alpha$  of  $f(\mathbf{x} + \alpha \mathbf{v})$ . The functions *maxf2vvar*, *maxquad*, *modit*, and *rebound* are all used.

**maxquad**

The function *maxquad* finds the location  $x$  of the maximum of a quadratic function  $q$  on the real line with value  $q(x_0)$  and derivative  $q_1(x_0)$  at  $x_0$  and value  $q(x_1)$  at  $x_1 \neq x_0$ . It is assumed that  $q(x_0) \geq q(x_1)$ . For a given  $s > 0$ , the function returns  $x$  unless  $|x - x_0| > s$ . If  $|x - x_0| > s$  and  $q_1(x_0) > 0$ , then  $x_0 + s$  is returned. If  $|x - x_0| > s$  and  $q_1(x_0) < 0$ , then  $x_0 - s$  is returned. The function declaration is

*double maxquad(const double & x0, const double & x1, const double & f0,  
const double & f1, const double & g0, const double & stepmax).*

Here  $x0$  is  $x_0$ ,  $x1$  is  $x_1$  used,  $f0$  is  $q(x_0)$ ,  $f1$  is  $q(x_1)$ ,  $g0$  is  $q_1(x_0)$ , and *stepmax* is  $s$ .

**modit**

The function *modit* truncates an iteration to conform to limits on step size and bounds in the case of a real function of one variable with a unique critical point and a limit of  $-\infty$  as the absolute value of the function argument approaches  $\infty$ . The function declaration is

*double modit(const double & eta, const double & alpha0, const double & alpha1,  
const double & stepmax, const bounds & b),*

and the struct *bounds* is defined as

*struct bounds {double lower; double upper;}.*

Here *eta* corresponds to  $\eta$ , *alpha0* is the previous location  $\alpha_0$ , *alpha1* is the proposed new location  $\alpha_1$ , *stepmax* is the positive limit  $s$  on step size, *b.lower* is the lower bound  $b_L \geq -\infty$ , and *b.upper* is the upper bound  $b_U \leq \infty$ . It is assumed that  $\alpha_0 \neq \alpha_1$  and  $b_L \leq \alpha_0 \leq b_U$ . If  $\alpha_1 > \alpha_0$ , then  $b_U > \alpha_0$  and the function returns the minimum of  $\alpha_1$ ,  $\alpha_0 + s$ , and  $\alpha_0 + \eta(b_U - \alpha_0)$ . If  $\alpha_1 < \alpha_0$ , then  $b_L < \alpha_0$  and the function returns the maximum of  $\alpha_1$ ,  $\alpha_0 - s$ , and  $\alpha_0 + \eta(b_L - \alpha_0)$ .

**rebound**

The function *rebound* updates the lower and upper bounds for maximization of a differentiable real function  $f$  on the real line with a unique critical point and a limit of  $-\infty$  as the absolute value of the function argument approaches  $\infty$ . The function declaration is

*bounds rebound(const double & y, const double & der, const bounds & b).*

The struct *bounds* is defined as in *modit*. Here  $y$  is the current location  $y$ , *der* is the function derivative  $f_1(y)$  at  $y$ , *b.lower* is the current lower bound  $L$ , and *b.upper* is the current upper bound  $U$ . It is assumed that  $f_1(y) \neq 0$ . If  $f_1(y) > 0$ , the returned lower bound is  $y$  and the returned upper bound is  $U$ . If  $f_1(y) < 0$ , the returned lower bound is  $L$  and the returned upper bound is  $y$ .

### Functions related to the Newton-Raphson algorithm

In this section, functions are discussed that are related to the Newton-Raphson algorithm. It should be noted that references to function values, gradients, and Hessian matrices do not address computational methods. In fact, the function values, gradients, and Hessian matrices employed may be approximations derived by numerical differentiation or large-sample approximations. In this section,  $f$  is assumed to be twice continuously differentiable.

#### **maxf2vvar**

The function *maxf2vvar* is used to combine information on a location and on a function's value, gradient, and Hessian matrix at the location. The function *maxf2vvar* has declaration

*maxf2v maxf2vvar(const int & order, const vec & y, const f2v & fy);*

The structs *f2v* and *maxf2v* are defined as in *maxlinq2*. At the vector  $\mathbf{y}$ , *fy.value* is  $f(\mathbf{y})$ , *fy.grad* is  $\nabla f(\mathbf{y})$  if *order* is positive, and *fy.hess* is  $\nabla^2 f(\mathbf{y})$  if *order* is greater than 1. The returned value *maxf2vvar.locmax* is  $\mathbf{y}$ , while *maxf2vvar.max* is  $f(\mathbf{y})$ , *maxf2vvar.grad* is  $\nabla f(\mathbf{y})$  if *order* is positive, and *maxf2vvar.hess* is  $\nabla^2 f(\mathbf{y})$  if *order* is greater than 1.

#### **nrv**

The function *nrv* applies a modified version of the Newton-Raphson algorithm to maximization of the function  $f$ . The function *nrv* has declaration

*maxf2v nrv(const int & order, const params & mparams, const vec & start, const function<f2v(const int &, vec &)> f).*

The structs *f2v*, *maxf2v*, and *params* are defined as in *maxlinq2*. The algorithm is found in *convergence.pdf*. The starting vector  $\mathbf{y}_0$  must be in  $O$ . The function returns the approximation  $\mathbf{y}$  for the location of the maximum, together with  $f(\mathbf{y})$ ,  $\nabla f(\mathbf{y})$  and  $\nabla^2 f(\mathbf{y})$ .

The function *nrv* uses *maxf2vvar*, *maxlinq2*, *maxquad*, *modit*, and *rebound*. The value of *order* should be at least 2. The starting vector is *start*, and the function, gradient, and Hessian matrix are provided in *f*.



**savmaxf2v**

The function *savmaxf2v* is used to save basic results of maximization. The declaration is

```
void savmaxf2v(const int & order , const maxf2v & vlm, const string & out,
const bool & fflag, const bool & pflag).
```

The value of *order* is 1, 2, or 3. The struct *maxf2v* is defined as in *maxlinq2*. If *fflag* is *true*, then the output file is defined by *out*. The output file must be a binary file used by Armadillo for storage. If *pflag* is *true*, then standard output is employed. Both *fflag* and *pflag* may have the same value. If *order* is 1, output is *vlm.max*, *vlm.locmax*, and *vlm.grad*. If *order* exceeds 2, then output also includes *vlm.hess*, the inverse of  $-1$  times *vlm.hess*, and the square roots of the diagonal elements of this inverse.

**Functions Related to Gradient Methods**

In this section, functions are considered based on gradient-based methods.

**conjgrad**

The function *conjgrad* implements a conjugate gradient algorithm for maximization of the function *f*. The function declaration is

```
maxf2v conjgrad(const int & order, const params & mparams,
const vec & start, const function<f2v(const int & , const vec & )> f).
```

Arguments are defined as *nrv*. The value of *order* must be at least 1. If *order* is at least 2, Hessian matrices are computed even though not used in the algorithm.

The function *conjgrad* uses *maxf2vvar*, *maxlinq2*, *maxquad*, *modit*, and *rebound*.

**gradascent**

The function *gradascent* uses a gradient-ascent algorithm for maximization of *f*. The function declaration for *gradascent* is

```
maxf2v gradascent(const order & , const params & mparams,
const vec & start, const function<f2v(const int & , const vec & )> f).
```

The functions *maxf2vvar*, *maxlinq2*, *maxquad*, *modit*, and *rebound* are used. Definitions are as in *conjgrad*.

**maxselect**

The function *maxselect* uses either the conjugate-gradient, gradient-ascent, or modified Newton-Raphson algorithm for maximization of  $f$ . The function declaration for *maxselect* is

*maxf2v maxselect(const order & , const params & mparams, const char & algorithm  
const vec & start, const function<f2v(const int & , const vec & )> f).*

The functions *conjgrad*, *gradascent*, *maxf2vvar*, *maxlinq2*, *maxquad*, *modit*, *nrv*, and *rebound* are used. Definitions are as in *conjgrad*, except that *algorithm* is *C* for conjugate gradient, *G* for gradient ascent, and *N* for Newton-Raphson.

**Log-likelihood Components**

In this section, components of log-likelihood functions are provided. A component has the form  $\ell_c(\beta; \mathbf{Y}, A, F, q, r)$ . Here the character  $A$  defines the type of model component involved,  $F$  is a distribution function with a positive and twice-continuously differential derivative  $F_1$  such that  $\log F_1$  has a negative second derivative. The integer  $q > 0$  is the parameter dimension, and the integer  $r > 0$  is the data dimension. The character  $A$  is in the set  $\mathcal{A}$  with elements  $B$ (logit beta),  $C$ (cumulative),  $D$ (continuous),  $E$ (logit Dirichlet),  $G$ (graded),  $H$ (log gamma),  $L$ (multinomial logit),  $M$ (maximum of two independent Bernoulli variables),  $N$ (multivariate normal),  $P$ (log-mean Poisson case),  $R$ (rank logit),  $S$ (Bernoulli), and  $T$ (censored continuous). Distribution functions used in this section are in the set  $\mathcal{F}$  with four members,  $G_L$ , the standard minimum Gumbel distribution function with value  $G_L(y) = 1 - \exp(-\exp(y))$  for  $y$  real (Gumbel, 1935),  $G_U$ , the standard maximum Gumbel distribution with value  $G_U(y) = \exp(-\exp(-y))$  for  $y$  real,  $\Psi$ , the standard logistic distribution function with value  $\Psi(y) = 1/[1 + \exp(-y)]$  for  $y$  real, and  $\Phi$ , the standard normal distribution function with derivative  $\Phi_1(y) = \exp(-y^2/2)/(2\pi)^{1/2}$  for real  $y$ . The value of  $F$  is only relevant in the cumulative, continuous, graded, Bernoulli, and censored continuous cases. The variables  $M$ ,  $F$ ,  $q$ , and  $r$  then define an open convex subset  $O(A, F, q, r)$  of  $q$ -dimensional vectors and a set  $\mathcal{Y}(A, F, q, r)$  of  $r$ -dimensional vectors. The vector  $\beta$  is in  $O(A, F, q, r)$ , and  $\mathbf{Y}$  is in  $\mathcal{Y}(A, F, q, r)$ .

To treat both continuous and discrete log-likelihood components, the integral symbol  $\int$  is used in the following sense. Consider a real function  $g$  on a nonempty finite-dimensional set  $C$ . If  $C$  is convex and has a nonempty interior and  $g$  is integrable, then  $\int(g)$  denotes the integral of  $g$  over  $C$ . If  $C$  is finite or countably infinite and  $g$  is summable, then  $\int(g)$  is the sum of  $g(\mathbf{c})$  over  $\mathbf{c}$  in  $C$ . More generally, let  $\mathcal{D}$  be a finite or countably infinite collection of nonempty disjoint sets  $D$  that are either convex sets with nonempty interior or finite or countably-infinite sets. Let  $C$  be the union of the sets in  $\mathcal{D}$ , and let  $g_D$  denotes the restriction of  $g$  to  $D$  in  $\mathcal{D}$ . Let  $\int(g_D)$  be defined for  $D$  in  $\mathcal{D}$ , and let the  $\int(g_D)$ ,  $D$  in  $\mathcal{D}$ , be summable. Then  $\int(g)$  is the sum of  $\int(g_D)$  over

$D$  in  $\mathcal{D}$ . Similar conventions apply if  $g$  is vector-valued or matrix-valued. The requirement is imposed here that, for  $\beta$  in  $O(A, F, q, r)$ ,  $\int(\exp(\ell_c(\beta; \cdot, A, F, q, r))) = 1$ . Here  $\exp(\ell_c(\beta; \cdot, A, F, q, r))$  is the function on  $\mathcal{Y}(A, F, q, r)$  equal to  $\exp(\ell_c(\beta; \mathbf{y}, A, F, q, r))$  if  $\mathbf{y}$  is in  $\mathcal{Y}(A, F, q, r)$ . The gradient function of  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  is  $\nabla \ell_c(\cdot; \mathbf{y}, A, F, q, r)$  and the corresponding Hessian matrix is  $\nabla^2 \ell_c(\cdot; \mathbf{y}, A, F, q, r)$ .

For a positive integer  $n$  and an observation  $i$ ,  $0 \leq i < n$ , positive integers  $q_i$  and  $r_i$  and character variables  $A_i$  in  $\mathcal{A}$  and  $F_i$  in  $\mathcal{F}$  are given. The component of the log likelihood for observation  $i$  involves the predicted random vector  $\mathbf{Y}_i$  in  $\mathcal{Y}(A_i, F_i, q_i, r_i)$ , the  $q_i$  by  $p$  predicting matrix  $\mathbf{X}_i$  in a nonempty set  $\mathcal{X}_i$ , the  $q_i$ -dimensional vector  $\mathbf{o}_i$ , and the positive real weight  $w_i$ . If  $\beta$  is in  $O$ , then let  $\lambda_i(\beta) = \mathbf{o}_i + \mathbf{X}_i \beta$  be in  $O(A_i, F_i, q_i, r_i)$  for  $0 \leq i < n$ , and let the log-likelihood function under study have the form

$$\ell(\beta) = \sum_{i=0}^{n-1} w_i \ell_c(\lambda_i(\beta); \mathbf{Y}_i, A_i, F_i, q_i, r_i). \quad (1)$$

It follows that the gradient of  $\ell$  at  $\beta$  in  $O$  is

$$\nabla \ell(\beta) = \sum_{i=0}^{n-1} w_i \mathbf{X}_i^T \nabla \ell_c(\lambda_i(\beta); \mathbf{Y}_i, A_i, F_i, q_i, r_i), \quad (2)$$

and the Hessian matrix of  $\ell$  at  $\beta$  is

$$\nabla^2 \ell(\beta) = \sum_{i=0}^{n-1} w_i \mathbf{X}_i^T \nabla^2 \ell_c(\lambda_i(\beta); \mathbf{Y}_i, A_i, F_i, q_i, r_i) \mathbf{X}_i. \quad (3)$$

The Hessian matrix  $\nabla^2 \ell(\beta)$  has the approximation

$$\tilde{\nabla}^2 \ell(\beta) = - \sum_{i=0}^{n-1} w_i \mathbf{X}_i^T \nabla \ell_c(\lambda_i(\beta); \mathbf{Y}_i, A_i, F_i, q_i, r_i) [\nabla \ell_i(\lambda_i(\beta); \mathbf{Y}_i, A_i, F_i, q_i)]^T \mathbf{X}_i \quad (4)$$

(Haberman, 2013; Louis, 1982) .

The functions  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  used are considered in this section. Some are examined in the literature on survival analysis (Cox, 1972; Kalbfleisch & Prentice, 2002), generalized linear models (McCullagh & Nelder, 1989), multivariate analysis (Anderson, 2003), and discrete choice (McFadden, 1973). It should be noted that names for models are somewhat variable in different references, especially for graded and cumulative cases. In addition, graded and cumulative cases are defined to be consistent with the Bernoulli cases. The following C++ functions are employed for common examples. The structs `f2v` are defined as in `maxliq2`. If the argument `beta` is not in  $O$ , then all values returned equal  $-\infty$ . It is assumed that the user of the function has verified that the input vector  $\mathbf{y}$  is in  $\mathcal{Y}_i$ . In the cases under study in this section, unless otherwise stated, the components are strictly concave, so that  $\ell$  is strictly concave whenever  $\mathbf{X}_i$ ,  $0 \leq i < n$ , spans a space of dimension  $p$ . Conditions for a unique  $\hat{\beta}$  in  $O$  such that  $\ell(\hat{\beta})$  equals the supremum of  $\ell$  over  $O$  are relatively

complex (Haberman, 1974, 1977, 1980). It is worth noting that in cases in which  $\hat{\beta}$  in  $O$  satisfies the conditions that  $\nabla\ell(\hat{\beta})$  is the  $p$ -dimensional vector  $\mathbf{0}_p$  with all elements 0 and  $\nabla^2\ell(\hat{\beta})$  is negative definite, then  $O$  can be restricted to ensure that  $\ell$  is strictly concave on  $O$  and  $\hat{\beta}$  is the only member of  $O$  such that  $\ell(\hat{\beta})$  equals the supremum of  $\ell$  on  $O$  and, for  $\beta$  in  $O$ ,  $\nabla\ell(\beta)$  is the unique vector with all elements 0 if  $\beta$  equals  $\hat{\beta}$ . In all component functions, *order* is less than 1 if only the component value is computed, 1 if the component value and gradient are found, and greater than 1 if the component value, gradient, and Hessian are found. If *order* exceeds 2, the approximation of the Hessian by Equation 4 is employed. Repeated use is made of the struct *resp* with *ivec* component *iresp* and *vec* component *dresp*. In typical cases, *resp.dresp* or *resp.iresp* has no elements; however, exceptions do exist.

### berresp

The function *berresp* is used to handle standard models for Bernoulli random variables. Here  $q = r = 1$ ,  $A$  is  $S$ ,  $\mathcal{Y}(A, F, q, r)$  is the set of one-dimensional vectors  $\mathbf{y}$  with element  $y_0$  equal 0 or 1, and  $O(A, F, q, r)$  is the set of all one-dimensional vectors, and  $F$  is in  $\mathcal{F}$ . For  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$  and  $\beta$  in  $O(A, F, q, r)$  with element  $\beta_0$ ,

$$\ell_c(\beta; \mathbf{y}, A, F, q, r) = \begin{cases} \log(F(\beta_0)), & y_0 = 1, \\ \log(1 - F(\beta_0)), & y_0 = 0. \end{cases} \quad (5)$$

The function declaration is

```
f2v berresp(const int & order, const char & transform, const resp & y,
const vec & beta).
```

If *transform* is  $G$ , then  $F = G_L$ . If *transform* is  $H$ , then  $F = G_U$ . If *transform* is  $L$ , then  $F = \Psi$ . If *transform* is  $N$ , then  $F = \Phi$ . The function *berresp.value* is  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\beta$ .

The function *berresp* requires loglogl, loglogu, logit, and probit.

### contresp

The function *contresp* computes the function value, gradient, and Hessian matrix associated with the distribution of a location and scale model for a continuous random vector. Here  $r = 1$ ,  $q = 2$ ,  $A$  is  $D$ ,  $\mathcal{Y}(A, F, q, r)$  is the set of all one-dimensional vectors,  $O(A, F, q, r)$  is the set of all two-dimensional vectors  $\beta$  with element  $\beta_1 > 0$ , and  $F$  is in  $\mathcal{F}$ . For  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$  and  $\beta$  in  $O(A, F, q, r)$ ,

$$\ell_c(\beta; \mathbf{y}, A, F, q, r) = \log(\beta_1) + \log(F_1(\beta_0 + \beta_1 y_0)). \quad (6)$$

These cases correspond to a model that a random variable  $Y$  has a distribution function  $F(\beta_0 + \beta_1 y)$ , where  $F$  is the distribution function of a random variable  $Z$ . Here  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  is concave, and the function is strictly concave if  $y_0$  is not 0.

For all cases, the function declaration is

*f2v contresp(const int & order, const char & transform, const resp & y,  
const vec & beta).*

The variable *transform* is defined as in *berresp*. The function *contresp.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.dresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ . The function *contresp* requires gumbell, gumbelu, logistic, and normal.

### **cumresp**

The function *cumresp* computes the function value, gradient, and Hessian matrix associated with a cumulative response transformation. Here  $r = 1$ ,  $q \geq 1$ ,  $A$  is  $C$ ,  $F$  is in  $\mathcal{F}$ ,  $\mathcal{Y}(A, F, q, r)$  is the set of one-dimensional vectors  $\mathbf{y}$  such that  $y_0$  is a nonnegative integer no greater than  $q$ ,  $O(A, F, q, r)$  is the set of all vectors of dimension  $q$ , and  $F$  is defined as in *berresp*. For  $\boldsymbol{\beta}$  in  $O(A, F, q, r)$  and  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$ , if  $y_0 = k$ , then

$$\ell_i(\boldsymbol{\beta}; \mathbf{y}) = \begin{cases} \log(1 - F(\beta_0)), & k = 0, \\ \log(1 - F(\beta_k)) + \sum_{j=0}^{k-1} \log(F(\beta_j)), & 0 < k < q, \\ \sum_{j=0}^{q-1} \log(F(\beta_j)), & k = q. \end{cases} \quad (7)$$

The function declaration is

*f2v cumresp(const int & order, const char & transform, const resp & y,  
const vec & beta).*

Here *transform* is defined as in *berresp*. The function *cumresp.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ . The function *cumresp* requires *berresp*, *loglogl*, *loglogu*, *logit*, and *probit*. If  $r = 1$ , then use of *cumresp* is equivalent to use of *berresp*. In general,  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  is concave. Strict concavity holds if  $q - y_0$  does not exceed 1.

### **gradresp**

The function *gradresp* computes the function value, gradient, and Hessian matrix associated with a graded response transformation. Then  $r = 1$ ,  $q \geq 1$ ,  $A$  is  $G$ ,  $F$  is in  $\mathcal{F}$ ,  $O(A, F, q, r)$  is the set of all vectors of dimension  $q$  with strictly decreasing elements,  $\mathcal{Y}(A, F, q, r)$  is the set of one-dimensional vectors  $\mathbf{y}$  with  $y_0$  a nonnegative integer no greater than  $q$ , and, for  $\boldsymbol{\beta}$  in  $O(A, F, q, r)$  and  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$ , if

$$\ell_i(\boldsymbol{\beta}; \mathbf{y}) = \begin{cases} \log(1 - F(\beta_0)), & k = 0, \\ \log(F(\beta_{k-1}) - F(\beta_k)), & 0 < k < q, \\ \log(F(\beta_{k-1})), & k = q. \end{cases} \quad (8)$$

The function declaration is

*f2v gradresp(const int & order, const char & transform, const resp & y,  
const vec & beta).*

Here *transform* is defined as in *berresp*. The function *gradresp.value* is  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\beta$ . If  $q = 1$ , then *berresp*, *cumresp* and *gradresp* yield the same result. The function  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  is concave. Strict concavity only holds if  $q$  is 1 or  $q$  is 2 and  $y_0 = 1$ .

### **gumbell**

The function *gumbell* provides the computations required in *contresp* for  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  for the minimum Gumbel case of  $F = G_L$ ,  $A$  with value  $D$ ,  $q = 2$ ,  $r = 1$ ,  $\mathcal{Y}(A, F, q, r)$  the set of real numbers, and  $O(A, F, q, r)$  the set of two-dimensional vectors  $\beta$  with  $\beta_1 > 0$ . The function declaration is

*f2v gumbell(const int & order, const resp & y, const vec & beta).*

The function *gumbell.value* is then  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.dresp* is  $\mathbf{y}$  and *beta* is  $\beta$ .

### **gumbelu**

The function *gumbelu* provides the computations required in *contresp* for  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  for the maximum Gumbel case of  $F = G_U$ ,  $A$  with value  $D$ ,  $q = 2$ ,  $r = 1$ ,  $\mathcal{Y}(A, F, q, r)$  the set of real numbers, and  $O(A, F, q, r)$  the set of two-dimensional vectors  $\beta$  with  $\beta_1 > 0$ . The function declaration is

*f2v gumbelu(const int & order, const resp & y, const vec & beta).*

The function *gumbelu.value* is then  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.dresp* is  $\mathbf{y}$  and *beta* is  $\beta$ .

### **loggamma**

The function *loggamma* provides the computations required for  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  for the log-gamma distribution with  $A$  with value  $H$ ,  $q = 2$ ,  $r = 1$ ,  $\mathcal{Y}(A, F, q, r)$  the set of real numbers, and  $O(A, F, q, r)$  the set of two-dimensional vectors  $\beta$  with positive elements. The function declaration is

*f2v loggamma(const int & order, const resp & y, const vec & beta).*

The function *loggamma.value* is then  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.dresp* is  $\mathbf{y}$  and *beta* is  $\beta$ .

**logistic**

The function *logistic* provides the computations required in *contresp* for  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  for the logistic case  $F = \Psi$  in *contresp* with  $A$  with value  $D$ ,  $q = 2$ ,  $r = 1$ ,  $\mathcal{Y}(A, F, q, r)$  the set of real numbers, and  $O(A, F, q, r)$  the set of two-dimensional vectors  $\beta$  with  $\beta_1 > 0$ . The function declaration is

*f2v logistic(const int & order, const resp & y, const vec & beta).*

The function *logistic.value* is  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.dresp* is  $\mathbf{y}$  and *beta* is  $\beta$ .

**logit**

The function *logit* computes the function value, gradient, and Hessian matrix associated with the logit case in *berresp* with  $A$  equal to  $S$ ,  $F = \Psi$ , and  $q = r = 1$ . The function declaration is

*f2v logit(const int & order, const resp & y, const vec & beta).*

The function *logit.value* is  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\beta$ .

**logitbeta**

The function *logitbeta* computes the function value, gradient, and Hessian matrix associated with the logit of a beta distribution with a two-dimensional parameter vector  $\beta$  with positive elements. Here  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  has  $A$  with value  $H$ ,  $q = 2$ ,  $r = 1$ ,  $\mathcal{Y}(A, F, q, r)$  the set of real numbers, and  $O(A, F, q, r)$  the set of two-dimensional vectors  $\beta$  with positive elements. The function declaration is

*f2v logitbeta(const int & order, const resp & y, const vec & beta).*

The function *logitbeta.value* is  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if  $q = 2$ ,  $r = 1$ , *y.iresp* is  $\mathbf{y}$ , *A* is  $B$ , and *beta* is  $\beta$ .

**logitdirichlet**

The function *logitdirichlet* computes the function value, gradient, and Hessian matrix associated with the logits of a Dirichlet distribution with a  $q = r + 1$ -dimensional parameter vector  $\beta$  with positive elements. Here  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  has  $A$  with value  $E$ ,  $\mathcal{Y}(A, F, q, r)$  the set of  $r$ -dimensional vectors, and  $O(A, F, q, r)$  the set of  $q$ -dimensional vectors  $\beta$  with positive elements. The function declaration is

*f2v logitdirichlet(const int & order, const resp & y, const vec & beta).*

The function *logitdirichlet.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$ , *beta* is  $\boldsymbol{\beta}$ , and *A* is *E*. The  $q$ -dimensional random variable  $\mathbf{u}$  has a Dirichlet distribution with parameter vector  $\boldsymbol{\beta}$  if 1 is the sum of the elements of  $\mathbf{u}$  and  $y_i = \log(u_i/u_q)$  for integers  $i$  from 1 to  $r$ . If  $r = 1$ , then the logit Dirichlet case reduces to the case of a logit beta.

### loglogl

The function *loglogl* computes the function value, gradient, and Hessian matrix associated with the complementary log-log case of berresp with *A* equal to *S*,  $F = G_L$ , and  $q = r = 1$ . The function declaration is

*f2v loglogl(const int & order, const resp & y, const vec & beta).*

The function *loglogl.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ .

### loglogu

The function *loglogu* computes the function value, gradient, and Hessian matrix associated with the log-log case of berresp with *A* equal to *S*,  $F = G_U$ , and  $q = r = 1$ . The function declaration is

*f2v loglogu(const int & order, const resp & y, const vec & beta).*

The function *loglogu.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ .

### logmean

The function *logmean* computes the function value, gradient, and Hessian matrix associated with a log-mean transformation for a Poisson random variable. In this case,  $q = r = 1$ , *A* is *P*, the value of *F* in  $\mathcal{F}$  is irrelevant,  $\mathcal{Y}(A, F, q, r)$  is the set of one-dimensional vectors  $\mathbf{y}$  such that  $y_0$  is a nonnegative integer, and  $O(A, F, Q, R)$  is the set of all one-dimensional vectors. For  $\boldsymbol{\beta}$  in  $O(A, F, q, r)$  and  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$ ,

$$\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r) = y_0 \beta_0 - \exp(\beta_0) - \log([y_0]!). \quad (9)$$

The function declaration is

*f2v logmean(const int & order, const resp & y, const vec & beta).*

The function *logmean.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ .

### maxberresp

The function *maxberresp* finds the log likelihood component, gradient, and Hessian matrix for the maximum of two unobserved Bernoulli random variables. Here



$q = 2$ ,  $r = 1$ ,  $A$  is  $M$ ,  $F$  is in  $\mathcal{F}$ ,  $O(A, F, q, r)$  is the set of two-dimensional vectors, and  $\mathcal{Y}(A, F, q, r)$  is the set of one-dimensional vectors  $\mathbf{y}$  with  $y_0$  equal 0 or 1. For  $y$  in  $\mathcal{Y}(A, F, q, r)$  and  $\boldsymbol{\beta}$  in  $O(A, F, q, r)$ ,

$$\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r) = \begin{cases} \log(F(\beta_0) + F(\beta_1) - F(\beta_0)F(\beta_1)), & y_0 = 1, \\ \log(1 - F(\beta_0)) + \log(1 - F(\beta_1)), & y_0 = 0. \end{cases} \quad (10)$$

It should be noted that

$$F(\beta_0) + F(\beta_1) - F(\beta_0)F(\beta_1) = 1 - [1 - F(\beta_0)][1 - F(\beta_1)] \quad (11)$$

and

$$\log(1 - F(\beta_0)) + \log(1 - F(\beta_1)) = \log([1 - F(\beta_0)][1 - F(\beta_1)]). \quad (12)$$

The function  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  is not necessarily concave if  $y_0 = 1$ .

The function declaration is

*f2v maxberresp(const int & order, const char & transform, const resp & y, const vec & beta).*

The variable *transform* is defined as in *berresp*. The function *maxberresp.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ . The functions *berresp*, *logit*, *loglogl*, *loglogu*, and *probit* are required.

### multlogit

The function *multlogit* computes the function value, gradient, and Hessian matrix associated with a multinomial logit transformation. In this case,  $r = 1$ ,  $q \geq 1$ ,  $F$  is irrelevant,  $A$  is  $L$ ,  $\mathcal{Y}(A, F, q, r)$  is the set of one-dimensional vectors  $\mathbf{y}$  such that  $y_0$  is a nonnegative integer no greater than  $q$ , and  $O(A, F, q, r)$  is the set of all  $q$ -dimensional vectors. For  $\boldsymbol{\beta}$  in  $O(A, F, q, r)$  and  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$ , if  $y_0 = k$ , then

$$\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r) = \begin{cases} -\log\left(1 + \sum_{k=0}^{q-1} \exp(\beta_k)\right), & k = 0, \\ \beta_{k-1} + \ell_c(\boldsymbol{\beta}; \mathbf{0}_1, A, F, q, r), & k > 0. \end{cases} \quad (13)$$

The function declaration is

*f2v multlogit(const int & order, const resp & y, const vec & beta).*

The function *multlogit.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ . If  $q = 1$ , use of *multlogit* gives the same result as use of *logit* and as use of *berresp*, *cumresp*, or *gradresp* with *transform* equal  $L$ .

**normal**

The function *normal* computes the function value, gradient, and Hessian matrix associated with the normal case in *contresp*. Thus  $A$  is  $D$ ,  $F = \Phi$ ,  $q = 2$ ,  $r = 1$ ,  $\mathcal{Y}(A, F, q, r)$  is the space of one-dimensional vectors, and  $O(A, F, q, r)$  is the set of two-dimensional vectors  $\beta$  with  $\beta_1 > 0$ . The function declaration is

*f2v normal(const int & order, const vec & y, const vec & beta).*

The function *normal.value* is  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.dresp* is  $\mathbf{y}$  and *beta* is  $\beta$ .

**normalv**

The function *normalv* computes the function value, gradient, and Hessian matrix associated with the log-likelihood component associated with a multivariate normal model with  $r$  positive,  $q = r(r+3)/2$ ,  $A$  equal to  $N$ ,  $F$  irrelevant,  $\mathcal{Y}(A, F, q, r)$  the set of all  $r$ -dimensional real vectors, and  $O(A, F, q, r)$  the set of  $q$ -dimensional vectors  $\beta$  with elements  $\beta_h$ ,  $0 \leq h < q$  such that  $\beta_h > 0$  if  $h = r + j(j+3)/2$  and  $0 \leq j < r$ . For such  $\beta$ , let  $\mathbf{a}(\beta)$  be the  $r$ -dimensional vector with elements  $a_j(\beta) = \beta_j$  for  $0 \leq j < r$ , and let  $\mathbf{B}(\beta)$  be the symmetric positive-definite  $r$  by  $r$  matrix with row  $j$  and column  $k$  equal to  $\beta_h$  if  $0 \leq k \leq j < r$  and  $h = r + k + (j(j+1))/2$ . For an  $r$ -dimensional vector  $\mathbf{z}$  with elements  $z_j$ ,  $0 \leq j < q$ , let  $\phi(\mathbf{z}; r)$  be the product of the  $\Phi_1(z_j)$ ,  $0 \leq j < r$ .

For  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$ ,

$$\ell_c(\beta; \mathbf{y}) = \sum_{j=0}^{r-1} \log(\beta_{r+j(j+3)/2}) + \log(\phi(\mathbf{a}(\beta) + \mathbf{B}(\beta)\mathbf{y}; r)). \quad (14)$$

This case corresponds to a model that a random vector has a distribution  $\mathbf{a}(\beta) + \mathbf{B}(\beta)\mathbf{Z}$ , where  $\mathbf{Z}$  is an  $r$ -dimensional multivariate normal random vector with zero mean and with covariance matrix equal to the identity matrix. The function  $\ell_c(\cdot; \mathbf{y}, A, F, q, r)$  is always concave but is not strictly concave. The function declaration is

*f2v normalv(const int & order, const resp & y, const vec & beta).*

The function *normalv.value* is  $\ell_c(\beta; \mathbf{y}, A, F, q, r)$  if *y.dresp* is  $\mathbf{y}$  and *beta* is  $\beta$ . If  $r$  is 1, then *normalv* reduces to *normal*. The function *normalv* requires *pack* and *unpack*.

**pack**

For the struct *vecmat* defined by

*struct vecmax{vec v; mat m;},,*

the function *pack* converts a  $d$ -dimensional vector *pack.v* and a  $d$  by  $d$  symmetric matrix *pack.m* to a vector with dimension  $d(d+3)/2$  with  $d$  initial elements the vector *pack.v* and element  $h = d + k + (j(j+1))/2$  equal to row  $j$  and column  $k$  of *pack.m* for nonnegative  $k$  no greater than  $j < d$ . The function declaration is

*vec pack(const vecmat & u).*

Diagonal elements of the matrix equal the corresponding elements of  $u$ , and off-diagonal elements are twice the corresponding elements of the vector.

### probit

The function *probit* computes the function value, gradient, and Hessian matrix associated with a probit transformation in berresp with  $A$  equal to  $S$ ,  $F = \Psi$ , and  $q = r = 1$ . The function declaration is

*f2v probit(const int & order, const resp & y, const vec & beta).*

The function *probit.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ .

### ranklogit

The function *ranklogit* computes the function value, gradient, and Hessian matrix associated with a model for discrete choice in which  $q+1$  objects are ranked for some positive integer  $q$  and the  $r$  most-preferred objects are recorded for some positive integer  $r \leq q$ . Here  $A$  has value  $R$ ,  $F$  is irrelevant, the set  $\mathcal{Y}(A, F, q, r)$  consists of the vectors  $\mathbf{y}$  of dimension  $r$  with distinct nonnegative integer elements that are no greater than  $q$ , and  $O(A, F, q, r)$  is the set of all  $q$ -dimensional vectors. To describe the model, consider  $\boldsymbol{\beta}$  in  $O(A, F, q, r)$ . Let  $U_j$ ,  $0 \leq j \leq q$ , be independent random variables such that  $U_0$  and  $U_j - \beta_j$ ,  $1 \leq j \leq q$ , have the common distribution function  $G_U$ . Let  $v$  denote a random permutation of the integers from 1 to  $q$  such that  $U_{v(j)}$ ,  $1 \leq j \leq q$ , is nonincreasing in  $j$ . For  $\boldsymbol{\beta}$  in  $O(A, F, q, r)$  and  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$ , the log-likelihood  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  is the logarithm of the probability that  $v^{-1}(k) = y_k$  for  $0 \leq k \leq r$ . To find  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$ , let  $\boldsymbol{\alpha}(\boldsymbol{\beta})$  be the vector of dimension  $q+1$  such that element  $j$ ,  $0 \leq j \leq q$ , is  $\alpha_j(\boldsymbol{\beta}) = 0$  if  $j = 0$  and  $\alpha_j(\boldsymbol{\beta}) = \beta_{j-1}$  if  $j > 0$ . Let  $K_j(\mathbf{y})$  be the set of nonnegative integers no greater than  $q$  that are not equal to  $y_k$  for any nonnegative integer  $k < j$ . Thus  $K_0(\mathbf{y})$  is the set of nonnegative integers no greater than  $q$ . Then the log-likelihood component is

$$\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r) = \sum_{j=0}^{r-1} \left\{ \alpha_{y_j}(\boldsymbol{\beta}) - \log \left[ \sum_{k \in K_j(\mathbf{y})} \exp(\alpha_k(\boldsymbol{\beta})) \right] \right\}. \quad (15)$$

The function declaration is

*f2v ranklogit(const int & order, const resp & y, const vec & beta).*

The function *ranklogit.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$  if *y.iresp* is  $\mathbf{y}$  and *beta* is  $\boldsymbol{\beta}$ . If  $r = 1$ , use of *ranklogit* gives the same result as use of *multlogit*.

### **truncresp**

The function *truncresp* computes the function value, gradient, and Hessian matrix associated with a right-censored continuous random variable with the distribution of  $\beta_0 + \beta_1 Z$  for some real  $\beta_0$  and positive real  $\beta_1$ , where, as in *contresp*,  $Z$  has distribution function  $F$  in  $\mathcal{F}$ . In this case,  $q = r = 2$ ,  $A$  is  $T$ ,  $\mathcal{Y}(A, F, q, r)$  consists of two-dimensional vectors  $\mathbf{y}$  such that  $y_0$  is a real number and  $y_1$  is 0 or 1, and  $O(A, F, q, r)$  is the set of all two-dimensional vectors  $\boldsymbol{\beta}$  with element  $\beta_1 > 0$ . For  $\boldsymbol{\beta}$  in  $O(A, F, q, r)$  and  $\mathbf{y}$  in  $\mathcal{Y}(A, F, q, r)$ , if  $y_1 = 0$ , then the observation is not censored and the corresponding log-likelihood component is

$$\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r) = \log(\beta_1) + \log(F_1(\beta_0 + \beta_1 y_0)), \quad (16)$$

while in the case of  $y_1 = 1$ , the the observation is censored at  $y_0$  and the log-likelihood component is

$$\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r) = \log(1 - F(\beta_0 + \beta_1 y_0)). \quad (17)$$

The function declaration is

*f2v truncresp(const int & order, const char & transform, const resp & y, const vec & beta).*

Here *y.iresp* has the single element  $y_1$ , *y.dresp* has the single element  $y_0$ , *beta* is  $\boldsymbol{\beta}$ , *transform* is defined as in *berresp*, and *truncresp.value* is  $\ell_c(\boldsymbol{\beta}; \mathbf{y}, A, F, q, r)$ . Functions required are *berresp*, *contresp*, and their respective required functions.

### **unpack**

The function *unpack* converts a vector of dimension  $d(d+3)/2$  to the *vecmat* format described in *pack*. The function declaration is

*vecmat unpack(const int & d, const vec & beta).*

The vector *unpack.v* contains the first  $d$  elements of *beta* and row  $j$  and column  $k$  of *unpack.m* is element  $d + k + j(j+3)/2$  of *beta* for nonnegative integers  $k \leq j < d$ .

## Computation of Log Likelihood Functions

### genresp

The function *genresp* provides a general tool for computation of a component of a log-likelihood function, its gradient, and its Hessian matrix. The function declaration is

```
f2v genresp(const int & order, const model & choice, const resp & y,
const vec & beta).
```

Here *model* has the definition

```
struct model{char type; char transform}.
```

In *choice*, *choice.type* has value *B* for logit beta, *C* for a cumulative case, *D* for a continuous case, *E* for logit Dirichlet, *G* for a graded response, *H* for log gamma, *L* for the multinomial logit case, *M* for the maximum of two independent Bernoulli variables, *N* for the multivariate normal case, *P* for the log-mean Poisson case, *R* for the rank-logit case, *S* for the Bernouli case, and *T* for the censored continuous case. For discrete cases, *choice.transform* has possible values *G* for log-log cases, *L* for logit cases, and *N* for probit cases. For continuous cases, *G* is for the Gumbel distribution, *L* is for the logistic case, and *N* is for the normal case. For example, *choice.type* is *C* and *choice.transform* is *G* for the cumulative log-log case, while *choice.type* is *S* and *choice.type* is *N* in the probit case. The variable *choice.transform* is only relevant if *choice.type* is *C*, *D*, *G*, *M*, *S*, or *T*.

The function *genresp* uses *berresp*, *contresp*, *cumresp*, *gradresp*, *loggamma*, *logitbeta*, *logitdirichlet*, *logmean*, *maxberresp*, *multlogit*, *ranklogit*, and *truncresp*, together with the functions they in turn require.

### genrespplik

The function *genrespplik* computes the log-likelihood function and its gradient and Hessian matrix. The function declaration is

```
f2v genrespplik(const int & order, const field<pattern> & patterns,
const xsel & patternnumber, const field<resp> & data, const resp & theta,
const field<xsel> & selectbeta, const xsel & selectbetano,
const field<xsel> & selectbetac, const xsel & selectbetacno,
const field<xsel> & selectthetai, const xsel & selectthetaino,
const field<xsel> & selectthetad, const xsel & selectthetadno,
const field<xsel> & selectthetac, const xsel & selectthetacno,
const vec & w, const xsel & obssel, const vec & beta).
```

The definitions of arguments and structs rely the the definition of *model* in *genresp* and the definition of *resp* in *truncresp*. The variable *theta* is a supplemental variable often used with item-response models.

In the case of *xsel*, the struct is defined as

```
struct xsel{bool all; uvec list}.
```

The struct is applied to a finite and nonempty collection of  $k$  objects numbered from 0 to  $k - 1$ . If *all* is *true*, all members of the collection are considered. Otherwise, only members in *list* are used. Thus in the case of *obsisel*, the observation numbers  $i$  to be used are specified. The function *intsel* is used for selections based on *xsel*. If *xsel.bool* is *true*, then an integer  $i$  is mapped by *intsel* to  $i$ , whereis if *bool* is *false*, then  $i$  is mapped to element  $i$  of *xsel.list*.

If *patternnumber* maps the integer  $i$  to  $j(i, 1)$ , then element  $j(i, 1)$  of *patterns* defines the function  $\lambda_i$  and the model used for the response  $\mathbf{Y}_i$ . In addition, the weight  $w_i$  assigned to  $\mathbf{Y}_i$  is element  $i$  of  $w$ , while element  $j(i, 2)$  of *selectbeta* specifies the elements of *beta* used to define  $\lambda_i$  if *selectbetano* maps  $i$  to  $j(i, 2)$ .

The struct *pattern* is defined as

```
struct pattern{model choice; vec o; mat x; cube c;}.
```

Thus for element  $j(i, 1)$  of *patterns*, the component *choice* defines the model for  $\mathbf{Y}_i$ , *o* defines  $\mathbf{o}_i$ , while *x* and *c* are used to specify  $\mathbf{X}_i$ . Definition of  $\mathbf{X}_i$  is somewhat complex. Let *selectbetacno* map  $i$  to  $j(i, 3)$ , and let *selectthetacno* map  $i$  to  $j(i, 4)$ . If  $k$  is not selected by element  $j(i, 2)$  of *selectbeta*, then column  $k$  of  $\mathbf{X}_i$  is the zero vector. Otherwise, let  $k(i, 2)$  be element  $m(i, 2)$  selected by element  $j(i, 2)$  of *selectbeta*. If  $m(i, 2)$  is not selected by element  $j(i, 3)$  of *selectbetac*, then column  $k(i, 2)$  of  $\mathbf{X}_i$  is column  $m(i, 2)$  of component *x* of element *patterns*. Otherwise, let  $m(i, 2)$  be element  $m(i, 3)$  selected by element  $j(i, 3)$  of *selectbetac*. Then column  $k(i, 2)$  of  $\mathbf{X}_i$  is the sum of column  $m(i, 2)$  of component *x* of element  $j(i, 1)$  of *patterns* and column  $m(i, 3)$  of the sums of the products of element  $k(i, 4)$  of component *dresp* of *theta* times slice  $m(i, 4)$  of component *c* of element  $j(i, 1)$  of *patterns* for pairs  $k(i, 4)$  and  $m(i, 4)$  such that  $k(i, 4)$  is integer  $m(i, 4)$  selected by element  $j(i, 4)$  of *selectthetac*.

In the case of  $\mathbf{Y}_i$ , let element  $i$  of *thetaino* select  $j(i, 5)$  and element  $i$  of *thetadno* select  $j(i, 6)$ . If element  $j(i, 5)$  of *thetaino* does not select anything, component *iresp* of element  $j(i, 5)$  of *data* is component *iresp* of  $\mathbf{Y}_i$ . Otherwise, component *iresp* of  $\mathbf{Y}_i$  consists of the elements of the component *iresp* of element  $j(i, 5)$  of *theta* selected by element  $j(i, 5)$  of *thetaino*. If element  $j(i, 6)$  of *thetadno* does not select anything, component *dresp* of element  $i$  of *data* is the double component of  $\mathbf{Y}_i$ . Otherwise, the double component of  $\mathbf{Y}_i$  consists of the elements of *theta.dresp* selected by element  $j(i, 6)$  of *thetadno*. The function *genrespplik* uses *addsel*, *cx*, *intsel*, *linsel*, *sintsel*, *sivecsel*, *svecsel*, *vecsel*, *genresp*, and all C++ functions *genresp* requires.

**genrespmle**

The function *genrespmle* applies maximizes the log-likelihood function, gradient, and Hessian matrix of *genresplik*. The function declaration is

```
maxf2v genrespmle(const int & order, const params & mparams,
const char & algorithm, const field<pattern> & patterns,
const xsel & patternnumber, const field<resp> & data, const resp & theta,
const field<xsel> & selectbeta, const xsel & selectbetano,
const field<xsel> & selectbetac, const xsel & selectbetacno,
const field<xsel> & selectthetai, const xsel & selectthetaino,
const field<xsel> & selectthetad, const xsel & selectthetadno,
const field<xsel> & selectthetac, const xsel & selectthetacno,
const vec & w, const xsel & obssel, const vec & start).
```

Definitions are as in *maxlinq2*, *maxf2vvar*, *maxselect*, and *genresplik*. The functions *maxselect* and *genresplik* are required, together with all C++ functions that these two functions need.

**Tools for Computation of Log Likelihood Functions****addsel**

The function *addsel* is used to add *f2v* structures. The function declaration is

```
void addsel(const int & order, const xsel & xselect,
const f2v & x, f2v & y, const double & a).
```

Here *order* and *xselect* are defined as in Log-likelihood Components. The struct *y* is modified by use of the struct *x* and the multiplier *a*. In all cases, *ax.value* is added to *y.value*. If *xselect.all* is *true*, then *x* and *y* have compatible dimensions, *ax.grad* is added to *y.grad* if *order* is at least 1, and *ax.hess* is added to *y.hess* if *order* is at least 2. If *xselect.all* is *false*, then *ax.grad* is added to *y.grad.elem(xselect.list)* if *order* is at least 1 and *ax.hess* is added to *y.hess.submat(xselect.list,xselect.list)* if *order* is at least 2.

**cx**

The function *cx* is used to multiply a cube by a vector to yield a matrix. The function declaration is

```
mat cx(const cube & c, const vec & x).
```

Let the  $c_{ijk}$  be the elements of the cube defined by  $c$ , and let  $x_k$  be the elements of the vector defined by  $x$ . The result is that row  $i$  and column  $j$  of  $cx$  is the sum over  $k$  of  $c_{ijk}x_k$ .

### **intsel**

The function *ivecsel* selects a nonnegative integer from a struct *xsel* defined as in *genresplik*. The function declaration is

*int intsel(const xsel & xselect, const int & i).*

The value of *intsel* is the  $i$ th integer defined by *xselect*.

### **ivecsel**

The function *ivecsel* is employed to create a new integer vector from an old vector of integers by extracting of elements of the old vector. The function declaration is

*ivec ivec sel(const xsel & xselect, const ivec & y).*

Here the struct *xsel* is defined as in *genresplik*. If *xselect.all* is *true*, then *ivecsel* is *y*. Otherwise, *ivecsel* is a vector with the number of elements in component *list* of *xselect*, if  $j(i)$  is element  $i$  of component *list* of *xselect*, then element  $i$  of *ivecsel* is element  $j(i)$  of *y*.

### **linsel**

The function *linsel* is used to apply a linear transformation to an *f2v* struct. The function declaration is

*f2v linsel(const int & order, const f2v & x, const mat & a).*

Here *order* is defined as in Log-likelihood Components. Let  $\mathbf{a}$  be the matrix defined by *a*. It is always the case that *linsel.value* is *x.value*. If *order* is positive and  $\mathbf{g}$  is the vector specified by *x.grad*, then *linsel.grad* is  $\mathbf{a}'\mathbf{g}$ . If *order* exceeds 1 and  $\mathbf{H}$  is the matrix specified by *x.hess*, then *linsel.hess* is  $\mathbf{a}'\mathbf{H}\mathbf{a}$ .

### **sintsel**

The function *sintsel* counts the number of integers selected from the first  $n$  nonnegative integers. The function declaration is

*int sintsel(const xsel & xselect, const int & n).*



The integers from 0 to  $n - 1$  are selected according to *xselect*, where *xsel* is defined as in *genresplik*.

### **sivecsel**

The function *sivecsel* counts the number of integer vector elements selected. The function declaration is

*int sivecsel(const xsel & xselect, const ivec & y).*

The integers in *y* are selected according to *xselect*, where *xsel* is defined as in *genresplik*.

### **svecsel**

The function *svecsel* counts the number of vector elements selected. The function declaration is

*int svecsel(const xsel & xselect, const vec & y).*

The integers in *y* are selected according to *xselect*, where *xsel* is defined as in *genresplik*.

### **vecsel**

The function *vecsel* is employed to create a new vector from an old vector by extracting of elements of the old vector. The function declaration is

*vec vecsel(const xsel & xselect, const vec & y).*

Here the struct *xsel* is defined as in *genresplik*. If *xselect.all* is *true*, then *vecsel* is *y*. Otherwise, *vecsel* is a vector with the number of elements in *xselect.list*, and element *i* of *vecsel* is element *j(i)* of *y* if *j(i)* is element *i* of component *list* of *xselect*.

## **Latent Structures**

In this section, functions useful for analysis of latent structures are considered. The log-likelihood function in this section is defined based on the definitions in Log-likelihood Components; however, use of latent variables is involved. In typical cases, data involve multiple responses for each individual observation. For a positive integer  $m$ ,  $m$  observations are present. For observation  $h$ ,  $0 \leq h < m$ , the observation has weight  $w_{h*} > 0$ , and  $n_h$  responses are observed. In addition, a latent vector appears in the model such the observed responses are conditionally independent given the latent vector and the predicting variables. Associated with the latent vector are positive integers  $q_*$  and  $r_*$ ,  $A_*$  in  $\mathcal{A}$ , and  $F_*$  in  $\mathcal{F}$ . The latent vector  $\theta_h$  is in  $\mathcal{Y}(A_*, F_*, q_*, r_*)$

and is predicted by the  $q_*$  by  $p$  predicting matrix  $\mathbf{X}_{h*}$  in the nonempty set  $\mathcal{X}_*$  and the fixed  $q_*$ -dimensional vector  $\mathbf{o}_*$ . It is assumed that  $\boldsymbol{\lambda}_*(\boldsymbol{\beta}) = \mathbf{o}_* + \mathbf{X}_*\boldsymbol{\beta}$  is in  $O(A_*, F_*, q_*, r_*)$  as long as  $\mathbf{X}_*$  is in  $\mathcal{X}_*$  and  $\boldsymbol{\beta}$  is in  $O$ . For response  $i$ ,  $0 \leq i < n_h$ , positive integers  $q_{hi}$  and  $r_{hi}$  are given. The variable  $A_{hi}$  is in  $\mathcal{A}$  and  $F_{hi}$  is in  $\mathcal{F}$ . The component of the log likelihood for response  $i$  involves the predicted random vector  $\mathbf{Y}_{hi}$  in  $\mathcal{Y}(A_{hi}, F_{hi}, q_{hi}, r_{hi})$ , the latent vector  $\boldsymbol{\theta}_h$ , the  $q_{hi}$  by  $p$  predicting matrix  $\mathbf{X}_{hi}$  in a nonempty set  $\mathcal{X}_{hi}$ , the  $q_{hi}$ -dimensional vector  $\mathbf{o}_{hi}$ , the  $q_{hi}$  by  $q_*$  matrix  $\mathbf{D}_{hi}$ , the positive real weight  $w_{hi}$ , the  $q_{hi}$  by  $p$  matrix  $\mathbf{D}_{hik}$ ,  $0 \leq k < q_*$ , and the function  $\ell_c(\cdot; \mathbf{y}, A_{hi}, F_{hi}, q_{hi}, r_{hi})$  on  $O(A_{hi}, F_{hi}, q_{hi}, r_{hi})$  defined for  $\mathbf{y}$  in  $\mathcal{Y}(A_{hi}, F_{hi}, q_{hi}, r_{hi})$ . For any  $\boldsymbol{\beta}$  in  $O$ ,  $\mathbf{X}$  in  $\mathcal{X}_{hi}$ , and  $\boldsymbol{\theta}$  in  $\mathcal{Y}(A_*, F_*, q_*, r_*)$ ,

$$\boldsymbol{\lambda}_{hi}(\boldsymbol{\beta}|\boldsymbol{\theta}) = \mathbf{o}_{hi} + \mathbf{X}_{hi}\boldsymbol{\beta} + \mathbf{D}_{hi}\boldsymbol{\theta} + \sum_{k=0}^{p-1} \theta_k \mathbf{D}_{hik}\boldsymbol{\beta} \quad (18)$$

is in  $O(A_{hi}, F_{hi}, q_{hi}, r_{hi})$ . A helpful general reference to applications in psychometrics is provided by van der Linden (2018). A large collection of references to other applications provided in Muthén (2002)

For example, in a simple two-parameter item-response model for dichotomous responses, a pool of  $T$  items numbered from 0 to  $T - 1$  is used, and  $p = 2T$ . The element  $\beta_{2t}$  is the item intercept for item  $t$ , and  $\beta_{2t+1}$  is the corresponding item slope. The response  $\mathbf{Y}_{hi}$ , which corresponds to pool member  $t(h, i)$ , is a one-dimensional vector with the single element  $Y_{0hi}$  equal to 0 or 1, the  $t(h, i)$  are all different for the same observation  $h$ , each  $q_{hi}$  and each  $r_{hi}$  is 1,  $r_*$  is 1,  $q_*$  is 2,  $\mathbf{o}_*$  is the two-dimensional vector with elements 0 and 1, all elements of the  $\mathbf{X}_{h*}$  are 0, the only nonzero element of  $\mathbf{X}_{hi}$  is row 0 and column  $2t(h, i)$ , which is 1, the  $\mathbf{D}_{hk}$  have all 0 elements, the  $\mathbf{D}_{hik}$  are 1 by 1 matrices with the single element 0 except for  $\mathbf{D}_{hi(2t(h,i)+1)}$ , which is the 1 by 1 matrix with the single element 1, and  $\mathbf{o}_{hi}$  is the vector with the single element 0. Thus  $\boldsymbol{\lambda}_{hi}(\boldsymbol{\beta}|\boldsymbol{\theta})$  is the vector with the single element  $\beta_{2t(h,i)} + \beta_{2t(h,i)+1}\theta_0$ . The value of  $A_{hi}$  is  $S$ , and each  $F_{hi}$  is the same. If each  $F_{hi}$  is  $\Psi$ , then the model is a two-parameter logistic (2PL) model. If each  $F_{hi}$  is  $\Phi$ , then the model is a normal ogive (2PN) model.

For  $\boldsymbol{\beta}$  in  $O$ , the log-likelihood has the form

$$\ell(\boldsymbol{\beta}) = \sum_{h=0}^{m-1} w_{h*} \ell_h(\boldsymbol{\beta}), \quad (19)$$

where  $\ell_h(\boldsymbol{\beta})$  is the component of the log-likelihood for observation  $h$ . Thus the gradient function of  $\ell$  at  $\boldsymbol{\beta}$  satisfies

$$\nabla \ell(\boldsymbol{\beta}) = \sum_{h=0}^{m-1} w_{h*} \nabla \ell_h(\boldsymbol{\beta}), \quad (20)$$

where  $\nabla \ell_h(\boldsymbol{\beta})$  is the gradient function of  $\ell_h$  at  $\boldsymbol{\beta}$ . The Hessian function of  $\ell$  at  $\boldsymbol{\beta}$  satisfies

$$\nabla^2 \ell(\boldsymbol{\beta}) = \sum_{h=0}^{m-1} w_{h*} \nabla^2 \ell_h(\boldsymbol{\beta}), \quad (21)$$

where  $\nabla^2 \ell_h(\boldsymbol{\beta})$  is the Hessian function of  $\ell_h$  at  $\boldsymbol{\beta}$ . The approximation

$$\tilde{\nabla}^2 \ell(\boldsymbol{\beta}) = - \sum_{h=0}^{m-1} w_{h*} \nabla \ell_h(\boldsymbol{\beta}) [\nabla \ell_h(\boldsymbol{\beta})]^T, \quad (22)$$

may also be considered.

In turn,  $\ell_h(\boldsymbol{\beta})$  involves the product

$$\ell_h(\boldsymbol{\beta}|\boldsymbol{\theta}) = \ell_c(\boldsymbol{\lambda}_*(\boldsymbol{\beta}); \boldsymbol{\theta}, A_*, F_*, q_*, r_*) \sum_{i=0}^{n_h-1} w_{hi} \ell_c(\boldsymbol{\lambda}_{hi}(\boldsymbol{\beta}|\boldsymbol{\theta}); \mathbf{Y}_{hi}, A_{hi}, F_{hi}, q_{hi}, r_{hi}) \quad (23)$$

for  $\boldsymbol{\theta}$  in  $\mathcal{Y}(A_*, F_*, q_*, r_*)$ . The component

$$\ell_h(\boldsymbol{\beta}) = \log \int (\exp(\ell_h(\boldsymbol{\beta}|\cdot))), \quad (24)$$

where  $\exp(\ell_h(\boldsymbol{\beta}|\cdot))$  is the function with value  $\exp(\ell_h(\boldsymbol{\beta}|\boldsymbol{\theta}))$  for  $\boldsymbol{\theta}$  in  $\mathcal{Y}(A_*, F_*, q_*, r_*)$ . In practice,  $\ell_h(\boldsymbol{\beta})$  is evaluated by

$$\tilde{\ell}_h(\boldsymbol{\beta}) = \log \left[ \sum_{k=1}^Q u_{hk} \exp(\ell_h(\boldsymbol{\beta}|\boldsymbol{\theta}_{hk})) \right], \quad (25)$$

for some positive weights  $u_{hk}$  and elements  $\boldsymbol{\theta}_{hk}$  in  $\mathcal{Y}(A_*, F_*, q_*, r_*)$ .

For computations for latent-structure models, the following functions are employed.

### gpcm

The main function *gpcm* finds maximum-likelihood estimates for a generalized partial credit (GPCM) model (Muraki, 1992) with a standard normal latent variable. As in the two-parameter item-response model for dichotomous responses, a pool of  $T$  items numbered from 0 to  $T - 1$  is used. For each nonnegative integer  $t < T$ ,  $I_t$  is a positive integer greater than 1 that corresponds to pool member  $t$ , and  $p$  is the sum of the integers  $I_t$  for  $0 \leq t < T$ . In the GPCM case, response  $\mathbf{Y}_{hi}$  is a one-dimensional vector with the single element  $Y_{0hi}$ . The response corresponds to item  $t(h, i)$  from the pool, where  $t(h, i)$  is a nonnegative integers less than  $T$ . Item  $t(h, i)$  has  $I_{t(h, i)}$  possible nonnegative integer values from 0 to  $I_{t(h, i)} - 1$ , the latent vector  $\boldsymbol{\theta}_h$  has dimension 1, and  $\theta_{h0}$  has a standard normal distribution. The  $t(h, i)$  are all different for the same observation  $h$ . Let  $J_{hi0}$  be the sum of the  $I_u$  for  $u < t(h, i)$ , let  $J_{hi1}$  be the sum of the  $I_u$  for  $u \leq t(h, i)$ , and let the nonnegative integer  $k$  be less than  $I_{t(h, i)} - 1$ . Conditional on  $\theta_{h0}$ ,  $Y_{0hi}$  has a multinomial logit distribution with element  $k$  of the parameter vector  $\boldsymbol{\lambda}_{hi}(\boldsymbol{\beta}|\boldsymbol{\theta})$  equal to  $\beta_a + \beta_b k \theta_{h0}$ , where  $a = J_{hi0} + k$  and  $b = J_{hi1}$ .

The function *gpcm* considers the simplified case of  $n_h$  constant and  $t(h, i) = i$  for nonnegative integers  $i < n_0$  and  $I_i - 1$  equal to the maximum value of  $Y_{0hi}$  for

$0 \leq h < m$ . The function uses a control file *controlfile* read from standard input. The file is a text file with any line consisting of a pair of entries separated by a space. The entries contain no blank characters. The following cases exist.

The variable *data* has string value *infile* that specifies the input file for the data. The default is *infile.csv*. Data are read as an integer matrix with row *h* and column *i* equal to  $Y_{0hi}$ .

The variable *sf* is associated with the string *startvalue* that specifies a file containing the vector of starting values. The default is to apply startgpcm.

The variable *outfile* has string value that specifies the name of the output file for results. The default is *outfile*.

The bool variable *fflag* indicates whether an output file is used. The default is *true*.

The bool variable *pflag* indicates whether anything is printed in ascii form in standard output. The default is *true*.

The character variable *method* specifies the algorithm applied. The possibilities are *G* for gradient ascent, *C* for conjugate gradient ascent, *N* for modified Newton-Raphson, and *L* for Louis approximation. The default is *N*.

The positive double variable *tol* specifies the convergence criterion *mparams.tol*. The default is 0.001.

The bool variable *adapt* indicates whether adaptive quadrature is used. The default is *true*.

The character variable *quadrature* indicates the quadrature procedure used, with *G* for Gauss-Hermite and *Q* for normal quantiles. The default is *G*.

The positive integer variable *points* that exceeds 1 gives the number of quadrature points, with the default 9.

The program uses irtmle and all its required functions. In addition, startgpcm and its required functions, hermpw, qnormpwe, and savmaxf2v are needed.

## invcdf

The function *invcdf* finds the inverse of the cumulative distribution function and its derivative at *prob* for *F* in  $\mathcal{F}$ . The function declaration is

*f1 invcdf(const char & cdf, const double & prob).*

The struct *f1* is

```
struct f1{double value; double der}.
```

The variable *cdf* is defined as in berresp. At *prob*, the value of the inverse is *invcdf.value*, and the derivative is *invcdf.der*.

**irtm**

The function *irtm* finds the log likelihood component  $\ell_h(\beta)$  and associated gradient and Hessian matrix for a latent structure model. The function uses numerical integration if  $\mathcal{Y}(A_*, F_*, q_*, r_*)$  is not finite or countably infinite. The function declaration is

```
f2v irtm (const int & order, const field<pattern> & patterns,
const xsel & patternnumber, const field<resp> & data, const field <pwr> & thetas,
const adq & scale, dovecmat & obsscale,
const field<xsel> & selectbeta, const xsel & selectbetano,
const field<xsel> & selectbetac, const xsel & selectbetacno,
const field<xsel> & selectthetai, const xsel & selectthetaino,
const field<xsel> & selectthetad, const xsel & selectthetadno,
const field<xsel> & selectthetac, const xsel & selectthetacno,
const vec & w, const xsel & obssel, const vec & beta).
```

In this declaration, almost all arguments are defined as in *genresplik*. The exceptions are *thetas*, *scale*, and *obsscale*. These arguments rely on the structs *pwr*, *adq*, and *dovecmat* with the following definitions:

```
struct pwr{double weight; double kernel; resp theta;},
```

```
struct adq{bool adapt; xsel linselect;xsely quadselect;},
```

and

```
struct dovecmat{double s; vec v; mat m;}.
```

The structs *thetas* and *obsscale* define the  $u_{hk}$  and  $\theta_{hk}$  of Equation 25, while *scale* is used in *irtmle* for computation of maximum-likelihood estimates of  $\beta$ . For element  $k$  of *thetas*,  $u_{hk}$  is obtained by multiplying *thetas(h).weight* by *obsscale.s* and dividing by *thetas(h).kernel*, the integer elements of  $\theta_{hk}$  are the same as in *thetas(h).theta.iresp*, and the remaining elements of  $\theta_{hk}$  are found by adding *obsscale.v* to the product of the matrix *obsscale.m* and the vector *thetas(h).theta.dresp*. Use of *scale* is discussed in the description of *irtmle*. In addition to functions needed by *genresplik*, *fitquad* and its required functions are needed.

**irtmle**

The function *irtmle* finds the maximum likelihood estimate for a latent structure model. As in *irtm*, the function uses numerical integration if  $\mathcal{Y}(A_*, F_*, q_*, r_*)$  is not finite or countably infinite. The function declaration is

```

maxf2v irtmle (const int & order, const params & mparams,
const char & algorithm, const field<patterns> & patterns,
const field<xsel> & patternnumber, const xsel & patno,
const field<field<resp> > & data, const field<field <pwr> > & thetas,
const xsel & thetano,
const field<adq> & scale, const xsel & scaleno, field<dovecmat> & obsscale,
const field<xsel> & selectbeta, const field<xsel> & selectbetano,
const xsel & selbetano,
const field<xsel> & selectbetac, const field<xsel> & selectbetacno,
const xsel & selbetacno,
const field<xsel> & selectthetai, const field<xsel> & selectthetaino,
const xsel & selthetaino,
const field<xsel> & selectthetad, const field<xsel> & selectthetadno,
const xsel & selthetadno,
const field<xsel> & selectthetac, const field<xsel> & selectthetacno,
const xsel & selthetacno,
const field<vec> & w, const xsel & wno,
const field<xsel> & obsel, const xsel & obselno,
const vec & obsweight, const xsel & dataset,
const field<xsel> & betasel, const xsel & betaselno, const vec & start).

```

In this declaration, *maxf2v* and *mparams* are defined as in *maxliq2* and *maxf2vvar*, while *order* is defined as in *genresplik*, whereas *mparams* is used for the basic iterations used for determination of the maximum-likelihood estimate, The variables *algorithm* and *start* are defined as in *genrespmle*. The vector *obsweight* provides the weights  $w_{h*}$  for  $0 \leq h \leq m - 1$ . The variables *patterns*, *selectbeta*, *selectbetac*, *selectthetai*, *selectthetad*, and *selectthetac* are defined as in *genresplik*. For each observation  $h$  and each nonnegative integer  $i < n_h$ , if  $m(h, 1)$  is *patno*( $h$ ), then *patternno*( $m(h, 1)$ ) assigns  $i$  to the member of *patterns* that corresponds to  $\lambda_{hi}$ . Similar arguments apply to triples such as *selectbeta*, *selectbetano*, and *selbeta*. In the case of *w* and *wno*, for each observation  $h$ , *wno* assigns a vector in *w* of length  $n_h$  that corresponds to the  $w_{hi}$  for  $0 \leq i < n_h$ . A similar relationship exists between *obsel* and *obselno*. Selection of observations  $h$  is determined by *dataset*, and *betasel* and *betaselno* determine which subvector of the parameter vector  $\beta$  applies to  $h$ .

The functions *irtms* and its required functions and the functions that are prerequisites for *genrespmle* are required by *irtmle*.

### **irtms**

The function *irtms* finds the log likelihood component  $\ell(\beta)$  and associated gradient and Hessian matrix for a latent-structure model. As in *irtm*, the function uses numerical integration if  $\mathcal{Y}(A_*, F_*, q_*, r_*)$  is not finite or countably infinite. The function declaration is

```

f2v irtms (const int & order, const field<pattern> & patterns,
const field<xsel> & patternnumber, const xsel & patno,
const field<field<resp> > & data, const field<field <pwr> > & thetas,
const xsel & thetano,
const field<adq> & scale, const xsel & scaleno, field<dovecmat> & obsscale,
const field<xsel> & selectbeta, const field<xsel> & selectbetano,
const xsel & selbetano,
const field<xsel> & selectbetac, const field<xsel> & selectbetacno,
const xsel & selbetacno,
const field<xsel> & selectthetai, const field<xsel> & selectthetaino,
const xsel & selthetaino,
const field<xsel> & selectthetad, const field<xsel> & selectthetadno,
const xsel & selthetadno,
const field<xsel> & selectthetac, const field<xsel> & selectthetacno,
const xsel & selthetacno,
const field<vec> & w, const xsel & wno, const field<xsel> & obssel,
const xsel & obsselno,
const vec & obsweight, const xsel & dataset,
const field<xsel> & betasel, const xsel & betaselno, const vec & beta).

```

Definitions are as in *irtmle*, except that *beta* is the general function argument of the log likelihood rather than a starting vector. The function *irtm* and its required functions are used by *irtms*.

### **irtmsave**

The function *irtmsave* finds the log likelihood components of  $\ell_h(\beta)$  and associated gradient and Hessian matrix for a latent structure model. The function declaration is

```

field<pwr>f2v irtmsave (const int & order, const field<pattern> & patterns,
const xsel & patternnumber, const field<resp> & data, const field <pwr> & thetas,
dovecmat & obsscale,
const field<xsel> & selectbeta, const xsel & selectbetano,
const field<xsel> & selectbetac, const xsel & selectbetacno,
const field<xsel> & selectthetai, const xsel & selectthetaino,
const field<xsel> & selectthetad, const xsel & selectthetadno,
const field<xsel> & selectthetac, const xsel & selectthetacno,
const vec & w, const xsel & obssel, const vec & beta).

```

In this declaration, almost all arguments are defined as in *irtm*. The struct *pwr* is defined as follows:

*struct pwr2v*{double weight; double kernel; resp theta; double value; vec grad; mat hess;}.

For element  $i$  of *thetas*, element  $i$  of *irtmsave* has *theta* equal to *thetas*( $i$ ).*theta*, *weight* equal to *thetas*( $i$ ).*weight*, *kernel* equal to *thetas*( $i$ ).*kernel*, *value* equal to  $\ell_h(\beta|\theta)$  for  $\theta$  equal to *point* and  $\beta$  equal to *beta*, *grad* equal to the corresponding gradient at  $\beta$  (if *order* is positive), and *hess* equal to the corresponding Hessian at  $\beta$  (if *order* exceeds 1). Functions needed by *genresplik* are also needed in *irtmsave*.

### **irtmsaves**

The function *irtmsaves* finds the log likelihood components of all the  $\ell_h(\beta)$  and associated gradient and Hessian matrix for a latent structure model. The function declaration is

```
field<pwr2v> irtmsave s(const int & order, const field<pattern> & patterns,
const xsel & patternnumber, const xsel & patno,
const field<field<resp> > & data, const field<field <pwr> > & thetas,
const xsel & thetano,
field<dovecmat> & obsscale,
const field<xsel> & selectbeta, const field<xsel> & selectbetano,
const xsel & selbetano,
const field<xsel> & selectbetac, const field<xsel> & selectbetacno,
const xsel & selbetacno,
const field<xsel> & selectthetai, const field<xsel> & selectthetaino,
const xsel & selthetaino,
const field<xsel> & selectthetad, const field<xsel> & selectthetadno,
const xsel & selthetadno,
const field<xsel> & selectthetac, const field<xsel> & selectthetacno,
const xsel & selthetacno,
const field<vec> & w, const xsel & wno, const field<xsel> & obssel,
const xsel & obsselno,
const vec & obsweight, const xsel & datasel,
const field<xsel> & betasel, const xsel & betaselno, const vec & beta).
```

Definitions are as in *irtms* and *irtmsave*. Element  $i$  of *irtmsaves*( $h$ ) corresponds to  $\ell_h(\beta|\theta_i)$ . Functions required are *irtmsave* and the functions used in *irtmsave*.

### **oneparamirt**

The main function *oneparamirt* finds maximum-likelihood estimates for a one-parameter item-response model with dichotomous responses and a univariate standard normal latent variable. In this model, a pool of  $T$  items numbered from 0 to  $T - 1$  is



used. The model dimension  $p = T + 1$ . Response  $\mathbf{Y}_{hi}$  is a one-dimensional vector with the single element  $Y_{0hi}$ . The response corresponds to item  $t(h, i)$  from the pool, where  $t(h, i)$  is a nonnegative integers less than  $T$ . Item  $t(h, i)$  has possible values 0 and 1, the latent vector  $\boldsymbol{\theta}_h$  has dimension 1, and  $\theta_{h0}$  has a standard normal distribution. The  $t(h, i)$  are all different for the same observation  $h$ . Conditional on  $\theta_{h0}$ ,  $Y_{0hi}$  has a Bernoulli distribution with probability of a value 1 equal to  $F(\beta_{t(h,i)} + \beta_T \theta_{h0})$  for a distribution function  $F$  in  $\mathcal{F}$ .

The main function *oneparamirt* considers the simplified case of  $n_h$  constant and  $t(h, i) = i$  for nonnegative integers  $i < n_0$ . The function uses a control file *controlfile* read from standard input. The file is a text file with any line consisting of a pair of entries separated by a space, The entries contain no blank characters. The following cases exist.

The variable *data* has string value *infile* that specifies the input file for the data. The default is *infile.csv*. Data are read as an integer matrix with row  $h$  and column  $i$  equal to  $Y_{0hi}$ .

The character variable *dist* specifies the cumulative distribution associated with the binary responses. The possibilities are *N* for normal, *L* for logistic, and *G* for Gumbel. The default is *L*.

The variable *sf* is associated with the string *startvalue* that specifies a file containing the vector of starting values. The default is to apply *startoneparamirt*.

The variables *outfile*, *fflag*, *pflag*, *method*, *tol*, *adapt*, *iquadrature*, and *points* are defined as in *gpcm*.

The program uses *irtmle* and all its required functions. In addition, *startoneparamirt* and its required functions, *hermpw*, *qnormpw*, and *savmaxf2v* are needed.

## **pcm**

The main function *pcm* finds maximum-likelihood estimates for a partial credit model (Masters, 1982) with a standard normal latent variable. The pool of  $T$  items and the item responses and latent vectors are defined as in *gpcm*; however, in the case of the partial credit model, the model dimension  $p$  is 1 plus the sum of the  $I_t - 1$  for  $0 \leq t < T$ . In addition,  $J_{hi0}$  is now the sum of the  $I_u - 1$  for  $u < t(h, i)$ , and  $J_{hi1}$  is the sum of the  $I_u - 1$  for  $u \leq t(h, i)$ . For the nonnegative integer  $k < I_{t(h,i)} - 1$ , conditional on  $\theta_{h0}$ ,  $Y_{0hi}$  has a multinomial logit distribution with element  $k$  of the parameter vector  $\boldsymbol{\lambda}_{hi}(\boldsymbol{\beta}|\boldsymbol{\theta})$  equal to  $\beta_{p-1} + \beta_b k \theta_{h0}$ , where  $b = J_{hi1}$ .

The function *pcm* considers the simplified case of  $n_h$  constant and  $t(h, i) = i$  for nonnegative integers  $i < n_0$  and  $I_i - 1$  equal to the maximum value of  $Y_{0hi}$  for  $0 \leq h < m$ . Input is defined as in *gpcm*.

**posterior**

For an observation, the function *posterior* finds the posterior distribution of the latent vector given the observed responses. The function declaration is

*field<pwr> posterior (const field<pwrf2v> & irtcomps*

The definition of *pwr* is in *irtm*, and the definition of *pwrf2v* is in *irtmsave*. The posterior is presented as  $q$  quadrature weights and points, where *irtcomps* has  $q$  members.

**posteriors**

For a latent-structure model, the function *posteriors* finds the posterior distributions of the latent vectors given the observed responses for each observation. The function declaration is

*field<field<pwr> > posteriors (const field<field<pwrf2v> > & irtcompsm*

The definition of *pwr* is in *irtm*, and the definition of *pwrf2v* is in *irtmsave*. The posterior for each observation  $h$  is presented as  $q(h)$  quadrature weights and points, where *irtcompsm*( $h$ ) has  $q(h)$  members.

**regprod**

The function *regprod* minimizes the sum of squares

$$d(\mathbf{y}, \boldsymbol{\beta}) = \sum_{j=0}^{k-1} [y(i(0, j), i(1, j)) - \beta(i(0, j))\beta(i(1, j))]^2 \quad (26)$$

over  $\beta(h)$ ,  $0 \leq h < p$ , where  $k$  and  $p$  are positive integers, and, for each nonnegative integer  $j < k$ ,  $i(0, j)$  and  $i(1, j)$  are distinct nonnegative integers less than  $p$  and  $y(i(0, j), i(1, j))$  is a given real number. The function declaration is

*maxf2v regprod(const int & order, const params & mparams, const char & algorithm, const field<resp> & y, const vec & start),*

where *maxf2v* and the initial three function arguments are defined as in *irtmle*,  $y$  has  $p$  members,  $y(j).iresp$  has elements  $i(0, j)$  and  $i(1, j)$ , and  $y(j).dresp$  has the single element  $y(i(0, j), i(1, j))$ . In the selected algorithm, the starting values for the  $\beta(h)$  are provided by the  $p$  elements of *start*.

The algorithm uses *regprodf* together with *maxselect* and its required functions.

**regprodf**

The function *regprodf* computes Equation 26. The function declaration is

$$f2v \text{ regprodf}(\text{const int } \& \text{ order}, \text{const field} \langle \text{resp} \rangle \& y, \text{const vec } \& \text{ beta}).$$

The function arguments are defined as in *regprod*.

**startgpcm**

The function *startgpcm* finds starting values for *gpcm*. The function declaration is

$$\text{vec startgpcm}(\text{const int } \& \text{ order}, \text{const params } \& \text{ mparams}, \text{const char } \& \text{ algorithm}, \text{const imat } \& \text{ responses})$$

The first three arguments are defined as in *irtmle*, and *responses* is the  $n$  by  $m$  matrix of responses. Required functions are *regprod* and its required functions.

**startoneparamirt**

The function *startoneparamirt* finds starting values for *oneparamirt*. The function declaration is

$$\text{vec startoneparamirt}(\text{const char } \& \text{ cdf}, \text{const imat } \& \text{ responses}).$$

The variable *cdf* is defined as in *berresp*, and *responses* is defined as in *startgpcm*. The required function is *invcdf*.

**startpcm**

The function *startpcm* finds starting values for *pcm*. The function arguments are

$$\text{vec startpcm}(\text{const imat } \& \text{ responses}).$$

The function argument *response* is defined as in *startgpcm*.

**starttwoparamirt**

The function *starttwoparamirt* finds a starting vector for use in *twoparamirt*. The function declaration is

$$\text{vec starttwoparamirt}(\text{const int } \& \text{ order}, \text{const params } \& \text{ mparams}, \text{const char } \& \text{ algorithm}, \text{const char } \& \text{ cdf}, \text{const imat } \& \text{ responses}).$$

Arguments are defined as in `startgpcm` and `startoneparamirt`. The function `regprod` and its required functions are needed.

### **twoparamirt**

The program *twoparamirt* provides a basic analysis of a two-parameter item-response model for binary responses with a univariate standard normal latent variable. For each observation, the same items apply. It uses a control file *controlfile* read from standard input. The file is a text file with any line consisting of a pair of entries separated by a space. The entries contain no blank characters. The following cases exist.

The variable *data* has string value *infile* that specifies the input file for the data. The default is *infile.csv*. Each line of the input file represents a different observation and contains the same number of entries, with each entry the response to an item. Items are presented in the same order for each observation.

The variable *sf* is associated with the string *startvalue* that specifies a file containing the vector of starting values. The default is to apply `starttwoparamirt`.

The variable *outfile* has string value that specifies the name of the output file for results. The default is *outfile*.

The bool variable *fflag* indicates whether an output file is used. The default is *true*.

The bool variable *pflag* indicates whether anything is printed in ascii form in standard output. The default is *true*.

The character variable *dist* specifies the cumulative distribution associated with the binary responses. The possibilities are *N* for normal, *L* for logistic, and *G* for Gumbel. The default is *L*.

The character variable *method* specifies the algorithm applied. The possibilities are *G* for gradient ascent, *C* for conjugate gradient ascent, *N* for modified Newton-Raphson, and *L* for Louis approximation. The default is *N*.

The positive double variable *tol* specifies the convergence criterion *mparams.tol*. The default is 0.001.

The bool variable *adapt* indicates whether adaptive quadrature is used. The default is *true*.

The character variable *quadrature* indicates the quadrature procedure used, with *G* for Gauss-Hermite and *Q* for normal quantiles. The default is *G*.

The positive integer variable *points* that exceeds 1 gives the number of quadrature points, with the default 9.

The program uses `irtmle` and all its required functions. In addition, `starttwoparamirt` and its required functions, `hermpw`, `qnormpw`, and `savmaxf2v` are needed.

## Integration Tools

### eaps

For each observation  $i$ , the function *eaps* generates a posterior weighted mean  $eaps(i).v$  and covariance matrix  $eaps(i).m$  that correspond to a discrete posterior distribution with points  $posts(i).m.col(j)$  with probabilities  $posts(i).v(j)$  for  $j$  from 1 to  $posts(i).m.n\_cols$ . The function declaration is

*field<vecmat> eaps(const field<vecmat> & posts).*

The definition of *vecmat* is as in *pack*. The function *wmc* is used.

### fitquad

The function *fitquad* fits a quadratic function to function values and quadrature points and finds the linear transformation that maps the origin onto the location of the maximum of the quadratic function and has a symmetric Jacobian matrix equal to the positive-definite and symmetric square root of the inverse of the Hessian matrix of the fitted quadratic function. The function declaration is

*dovecmat fitquad(const field<f2v> & cresults, const field<pwr> & newthetas, const adq & scale, dovecmat & obsscale).*

The structs are defined as in *irtm*. The linear components of the fitted quadratic not set to 0 are specified by *scale.linselect*, and the quadratic components not set to 0 are specified by *scale.quadselect*. If the fitted quadratic function is not strictly concave, then *fitquad* is *obsscale*.

### genfact

For a vector *sizes* of positive integers, the function *genfact* generates all vectors  $i$  of nonnegative integers with the same number of elements as *sizes* such that each element of  $i$  is less than the corresponding element of *sizes*. The function declaration is

*imat genfact(const ivec & sizes).*

The columns of *genfact* are the possible vectors  $i$ . For example, if the elements of *sizes* are 2 and 3, then Column 0 of *genfact* has elements 0 and 0, and Column 1 has elements 1 and 0. In all, *sizes* has 6 columns, and Column 5 has elements 1 and 2.

### genprods

The function *genprods* generates a collection of quadrature points and quadrature weights for a multivariate integral from quadrature weights and quadrature points

for a univariate integral. The function declaration is

*vecmat genprods(const imat & indices, const field<pw> & pws).*

The struct *vecmat* is defined as in *pack*, and the struct *pw* has *vec* elements *points* and *weights*. Consider the case of  $Q$  quadrature points for a multidimensional integral on the space of  $D$ -dimensional vectors, where  $Q$  and  $D$  are positive integers. Then *genprods.m* has  $Q$  columns and *genprods.v* has  $Q$  elements. The matrix *genprods.m* has  $D$  rows. The array *pws* has  $D$  members. For  $0 \leq d < D$ , *pws(d).points* and *pws(d).weights* have  $m(d) > 1$  members, and the members of *pws(d).weights* are positive. The matrix *indices* specifies the quadrature vectors and quadrature weights to construct from *pws*. If *indices* has  $p$  columns,  $0 \leq k < p$ , and  $0 \leq d < D$ , then row  $d$  and column  $k$  of *indices* is nonnegative and less than  $m(d)$  and the corresponding row and column of *genprods.m* is *pws(d).points(indices(d,k))*. Element  $k$  of *genprods.v* is the product of *pws(d).weights(indices(d,k))* for  $0 \leq d < D$ .

### hermcoeff

The function *hermcoeff* finds the coefficients of a Hermite polynomial of a given degree. The function declaration is

*vec hermcoeff(const int & n).*

The integer variable  $n$  is the nonnegative order. The vector *hermcoeff* has  $n+1$  elements. The polynomial is  $H_n(x) = \sum_{i=0}^n \alpha_i x^{n-i}$  for real  $x$ , and element  $i$  of *hermcoeff* is  $\alpha_i$ . For example, if  $n$  is 2, then the elements of *hermcoeff* are 1, 0, and  $-1$ .

### hermpoly

The function *hermpoly* evaluates the Hermite polynomials up to a given degree at a specified real value. The function declaration is

*vec hermpoly(const int & n, const double & x).*

The degree is the nonnegative integer variable  $n$ , and the real value is  $x$ . The vector *hermpoly* has  $n+1$  elements. For  $0 \leq k \leq n$ , element  $k$  of *hermpoly* is the value of  $H_k$  at  $x$ .

### hermpw

The function *hermpw* uses the algorithm of Golub and Welsch (1969) to find the quadrature points and quadrature weights for Gauss-Hermite quadrature. The function declaration is

*pw hermpw(const int & n).*

The struct *hermpw* has vector elements *hermpw.points* and *hermpw.weights*. The number of quadrature points is *n*. The ordered quadrature points are in *hermpw.points*. The corresponding weights are in *hermpw.weights*. The weights are relative to the standard normal density.

### **qnormpwe**

The function *qnormpwe* provides normal-scores quadrature of a given order. The function declaration is

*pw qnormpwe(const int & n).*

The struct *qnormpwe* has vector elements *qnormpwe.points* and *qnormpwe.weights*. The number of quadrature points is *n*. The ordered quadrature points are in *qnormpwe.points*. The corresponding weights are in *qnormpwe.weights*.

### **wmc**

The function *wmc* computes a weighted mean vector and covariance matrix. The function declaration is

*vecmat wmc(const vecmat & wx).*

The elements of *wx.v* are probabilities corresponding to the rows of *wx.m*.

## **References**

- Anderson, T. W. (2003). *An introduction to multivariate statistical analysis* (3rd ed.). Wiley-Interscience.
- Cox, D. R. (1972). Regression models and life-tables. *Journal of the Royal Statistical Society: Series B (Methodological)*, 34, 187–202. <https://doi.org/10.1111/j.2517-6161.1972.tb00899.x>.
- Golub, G. H., & Welsch, J. H. (1969). Calculation of gauss quadrature rules. *Mathematics of Computation*, 23, 221–s10. <https://doi.org/10.2307/2004418>.
- Gumbel, E. J. (1935). Les valeurs extrêmes des distributions statistiques. *Annales de l'Institut Henri Poincaré*, 5, 115–158.
- Haberman, S. J. (1974). *The analysis of frequency data*. University of Chicago Press.
- Haberman, S. J. (1977). Maximum likelihood estimates in exponential response models. *The Annals of Statistics*, 5, 815–841. <https://doi.org/10.1214/aos/1176343941>.
- Haberman, S. J. (1980). Discussion of *regression models for ordinal data*, by P. McCullagh. *Journal of the Royal Statistical Society, Series B*, 42, 136–137.

- Haberman, S. J. (2013). *A general program for item-response analysis that employs the stabilized Newton-Raphson algorithm* (ETS Research Report No. RR-13-32). Educational Testing Service. <https://doi.org/10.1002/j.2333-8504.2013.tb02339.x>.
- Kalbfleisch, J. D., & Prentice, R. L. (2002). *The statistical analysis of failure time data* (2nd ed.). John Wiley. <https://doi.org/10.1002/9781118032985>.
- Lord, F. M., & Wingersky, M. S. (1984). Comparison of IRT true-score and equipercentile observed-score “equatings”. *Applied Psychological Measurement*, 8, 453–461. <https://doi.org/10.1177/014662168400800409>.
- Louis, T. (1982). Finding the observed information matrix when using the *em* algorithm. *Journal of the Royal Statistical Society, Ser. B*, 44, 226–233. <https://doi.org/10.2307/2345828>.
- Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrika*, 47, 149–174. <https://doi.org/10.1007/bf02296272>.
- McCullagh, P., & Nelder, J. A. (1989). *Generalized linear models* (2nd ed.). Springer US. <https://doi.org/10.1007/978-1-4899-3242-6>.
- McFadden, D. L. (1973). Conditional logit analysis of qualitative choice behavior. In P. Zarembka (Ed.), *Frontiers in econometrics* (pp. 105–142). Academic Press.
- Muraki, E. (1992). A generalized partial credit model: Application of an EM algorithm. *Applied Psychological Measurement*, 16, 159–176. <https://doi.org/10.1177/014662169201600206>.
- Muthén, B. O. (2002). Beyond SEM: General latent variable modeling. *Behaviormetrika*, 29, 81–117. <https://doi.org/10.2333/bhmk.29.81>.
- Sanderson, C., & Curtin, R. (2016). Armadillo: A template-based C++ library for linear algebra. *The Journal of Open Source Software*, 1, 26. <https://doi.org/10.21105/joss.00026>.
- Sanderson, C., & Curtin, R. (2018). A user-friendly hybrid sparse matrix class in C++. In *Mathematical software – ICMS 2018* (pp. 422–430). [https://doi.org/10.1007/978-3-319-96418-8\\_50](https://doi.org/10.1007/978-3-319-96418-8_50).
- Thissen, D., Pommerich, M., Billeaud, K., & Williams, V. S. L. (1995). Item response theory for scores on tests including polytomous items with ordered responses. *Applied Psychological Measurement*, 19, 39–49. <https://doi.org/10.1177/014662169501900105>.
- van der Linden, W. J. (Ed.). (2018). *Handbook of item response theory, three volume set*. Chapman; Hall/CRC. <https://doi.org/10.1201/9781315119144>.