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Is the anisotropic interaction of luminous matter responsible for the extrinsic gravitation usually attributed to exotic dark matter?

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A B S T R A C T

It is considered that the extrinsic gravitational attraction related to the spiral galaxy flat rotation curve problem is an emergent property of any non-spherically symmetric gravitationally bound distribution of luminous matter. Also discussed briefly within the context of this model are the hierarchy problem, the fractal nature of interaction, as well as rigid body rotation and asymptotic freedom.

1. Introduction

As modeled and discussed in [1–3], the luminous matter within the disc of a spiral galaxy undergoes an increasingly extraneous amount of gravitational attraction as distance from the galactic core is increased. This results in a flat rotation curve that is not predicted by either standard Keplerian dynamics or general relativity [4]. On the contrary, the galactic core is much closer to a spherically symmetric configuration, and the luminous matter within it is observed to exhibit standard orbit behaviour.

In the spirit of five-dimensional models of gravitation, a modification of the Einstein field equations is considered, in that the strength of standard gravitational attraction is proportional to E/c^5, instead of E/c^4.

When considering the interaction of energy along two spatial dimensions to be equivalent to luminous matter ($m = E/c^2$), the quantity E/c^5 is then equivalent to the interaction of luminous matter along three spatial dimensions (m/c^3).

This modification is then applied to cases where the interaction of luminous matter is anisotropic, constrained to occur along less than three spatial dimensions. Considered are gravitationally bound discs ($m/c^3 \rightarrow m/c^2$) and filaments ($m/c^3 \rightarrow m/c$).

2. Method: establishing a regular scale of spacetime

In standard general relativity, the strength of gravitational attraction generated by a spherically symmetric pressureless perfect fluid (dust), as approximated as a single point particle, is related to the speed of light in vacuum $c = 299792458$ ($\mathrm{m}^1\ \mathrm{s}^{-1}$), the universal gravitational constant $\mathrm{G} \approx 6.67 \cdot 10^{-11}$ ($\mathrm{kg}^{-1}\ \mathrm{m}^3\ \mathrm{s}^{-2}$), and the density of energy as defined by a unit 3-volume of space $T_{00} = E/1^3$ ($\mathrm{J}^1\ \mathrm{m}^{-3}$):

$$G_{00} = \frac{8\pi \mathrm{G}}{c^4} T_{00} \quad (\mathrm{m}^{-2}). \tag{1}$$

From the perspective of a five-dimensional model of gravitation, the strength of standard gravitational attraction is related to the isotropic interaction of luminous matter (E/c^5, $\mathrm{kg}^1\ \mathrm{s}^3\ \mathrm{m}^{-3}$), and so the following numerically equivalent modification is presented:

$$G_5 = \frac{8\pi}{c^5 \alpha_5} T_{00} \quad (\mathrm{m}^{-2}), \tag{2}$$

where the coupling constant is

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$$\alpha_5 = \frac{8\pi}{c^5 G_5} T_{00} \approx 50 \quad (\mathrm{N}^1\ \mathrm{s}^5\ \mathrm{m}^{-5}), \tag{3}$$

which relates to the Planck force $F_P \approx 1.21 \cdot 10^{44}$ (N^1) by

$$c^5 \alpha_5 = F_P \quad (\mathrm{N}^1). \tag{4}$$

By constraining the number of spatial dimensions along which the interaction of luminous matter occurs to less than three (e.g., $E/c^4,\ \mathrm{kg}^1\ \mathrm{s}^2\ \mathrm{m}^{-2}$ or $E/c^3,\ \mathrm{kg}^1\ \mathrm{s}^1\ \mathrm{m}^{-1}$), the strength of gravitational attraction increases:

$$G_4 = \frac{8\pi}{c^4 \alpha_4} T_{00} \quad (\mathrm{m}^{-2}), \tag{5}$$

$$\alpha_4 = \frac{8\pi}{c^4 G_4} T_{00} \approx 50 \quad (\mathrm{N}^1\ \mathrm{s}^4\ \mathrm{m}^{-4}), \tag{6}$$

$$G_3 = \frac{8\pi}{c^3 \alpha_3} T_{00} \quad (\mathrm{m}^{-2}), \tag{7}$$

$$\alpha_3 = \frac{8\pi}{c^3 G_3} T_{00} \approx 50 \quad (\mathrm{N}^1\ \mathrm{s}^3\ \mathrm{m}^{-3}). \tag{8}$$

This is meant to subscribe to the principle that space and time are inextricably connected by their mutual exclusivity. For a spiral galaxy, the spatial extent of gravitational interaction within the disc is increasingly constrained to occur along less than three spatial dimensions as distance from the galactic core is increased. The result is a corresponding increase in the gravitational interaction's temporal extent, leading to a flat rotation curve.

3. Results: the modified Einstein field equations applied to the spiral galaxy flat rotation curve problem

For a spherically symmetric dust of energy, the Schwarzschild radius R_S is

$$R_S = \frac{1^3 G_5}{4\pi} = \frac{2E}{F_P} \quad (\mathrm{m}^1). \tag{9}$$

The luminous matter within the spiral galaxy's disc is considered to be sufficiently far enough from the Schwarzschild radius of the galactic core so that the rates of proper time $d\tau$ and coordinate time dt are nearly identical. In this weak-field limit, Newtonian approximation may be used to determine the strength of gravitational attraction required to maintain a circular orbit of constant speed at arbitrary distance:

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{R_S}{r}} \approx 1, \tag{10}$$

$$\frac{d\tau}{dtdr} \approx \frac{R_S}{2r^2} \quad (\mathrm{m}^{-1}), \tag{11}$$

$$a = c^2 \frac{d\tau}{dtdr} = c^2 \frac{E}{F_P r^2} \quad (\mathrm{m}^1\ \mathrm{s}^{-2}), \tag{12}$$

$$v = c\sqrt{\frac{E}{F_P r}} \quad (\mathrm{m}^1\ \mathrm{s}^{-1}), \tag{13}$$

$$\frac{1}{c^n \alpha_n} = \frac{v^2 r}{c^2 E} \quad (\mathrm{N}^{-1}). \tag{14}$$

Considering a massive test particle of negligible energy in circular orbit at a distance of $r_a = 5$ (kpc^1) around a spherically symmetric dust of energy $E_{Core} = 10^{58}$ (J^1) (roughly $5.6 \cdot 10^{10}$ Solar masses), standard Newtonian approximation ($n \equiv 5$) gives an orbit speed of $v \approx 220{,}000$ $(\mathrm{m}^1\ \mathrm{s}^{-1})$.

Assuming a flat rotation curve within the disc that accompanies this model galactic core, then as the orbit distance is increased to $r_b = 10$ (kpc^1), speed remains constant ($dv/dr = 0$).

Rearranging the components of Eq. (14), the total energy E_{Total} required to generate a circular orbit of constant speed at arbitrary distance is

$$E_{Total} = v^2 c^3 \alpha_5 r_b \quad (\mathrm{J}^1), \tag{15}$$

where $r_b = 10$ (kpc^1), $E_{Total} = 2 \cdot 10^{58}$ (J^1).

Taking the total amount of luminous matter contained within the spherical shell defined by the pair of inner r_a and outer r_b radii to be negligible, the remainder of the required energy can be accounted for by an approximately homogeneous distribution of dark matter E_{DM} contained within the shell:

$$E_{DM} = E_{Total} - E_{Core} = 10^{58} \quad (\mathrm{J}^1). \tag{16}$$

Instead of explicitly relying on an energy contribution from dark matter, it is considered that the increase in the strength of gravitational attraction is generated when the gravitational interaction of luminous matter within the galaxy is constrained to occur along $(n-2) < 3$ spatial dimensions:

$$c^n\alpha_n = F_P\frac{E_{Core}}{E_{Total}} = \frac{c^2E_{Core}}{v^2r_b} \approx 6.05\cdot 10^{43}\quad (\mathrm{N}^1), \tag{17}$$

$$n = \log_c c^n = \log_c\frac{c^2E_{Core}}{v^2\alpha_5 r_b} \approx 4.96. \tag{18}$$

The dark matter and anisotropic interaction models are compared by idealizing the entire gravitational source as a single point particle with a Schwarzschild radius of

$$R'_S = \frac{2(E_{Core}+E_{DM})}{F_P} \approx \frac{2E_{Core}}{c^{4.96}\alpha_{4.96}}\quad (\mathrm{m}^1), \tag{19}$$

which both result in the required constant orbit speed at $r_b = 10$ (kpc^1).

The instantaneous rate of change in dimension with respect to distance is independent of system scale:

$$f(r) = \frac{dn}{dr} = -\frac{1}{r}\quad (\mathrm{m}^{-1}), \tag{20}$$

$$n = 5 + \int_{r_a}^{r_b} f(r)dr = 5 - \log_c\frac{r_b}{r_a}, \tag{21}$$

so regardless of E_{Core}, where $n = 5$ at r_a, then $n = 4$ at $r_a c$:

$$\int_{r_a}^{r_b=r_ac^x} f(r)dr = -x. \tag{22}$$

As the ratio r_b/r_a increases beyond 1, the number of spatial dimensions $(n-2)$ along which gravitational interaction occurs within the galaxy (as defined by the network of stars inside of r_b) reduces, resulting in a flat rotation curve. In regard to the model galaxy described by E_{Core}, the radius $r_a c \approx 1.5$ (tpc^1) (where $n = 4$) is far beyond the edge of any observed galactic disc.

On the other hand, a strictly Newtonian approximation is not entirely necessary. Acknowledging that in the weak-field limit the rates of proper and coordinate time are not entirely equal $(d\tau/dt \neq 1)$, a practically circular orbit of constant speed (where eccentricity exists, but is negligible) may be approximated by accounting for gravitational time dilation:

$$v = c\sqrt{1-\frac{d\tau}{dt}} = c\sqrt{1-\sqrt{1-\frac{R_S}{r_a}}} \approx 220{,}000\ (\mathrm{m}^1\ \mathrm{s}^{-1}), \tag{23}$$

which is analogous to kinematic time dilation in special relativity

$$v = c\sqrt{1-\frac{d\tau_{sr}^2}{dt^2}} \approx c\sqrt{1-\left(1-\frac{R_S}{2r_a}\right)} \approx 220{,}000\ (\mathrm{m}^1\ \mathrm{s}^{-1}), \tag{24}$$

$$\frac{d\tau}{dt} \approx \frac{d\tau_{sr}^2}{dt^2} \approx 0.99999946. \tag{25}$$

As hinted at, but intentionally neglected in Eqs. (11)–(13), the cause of orbit is gravitational time dilation $d\tau/dt \neq 1$ (and its gradient), which in turn causes kinematic time dilation $d\tau_{sr}/dt \neq 1$ in the orbiting massive test particle. The two forms of time dilation effectively generate an orbital period which is slightly shorter than that predicted by Newtonian approximation, manifesting as a deviation from Keplerian circular orbit in the form of relativistic precession [4,5]:

$$\delta \approx 2\pi\left(1-\frac{d\tau^2}{dt^2}\right) + 2\pi\left(1-\frac{d\tau_{sr}^2}{dt^2}\right) \approx 2\pi\left(\frac{R_S}{r_a}+\frac{R_S}{2r_a}\right) \tag{26}$$

$$\approx \frac{6\pi E_{Core}}{F_P r_a} \approx 10^{-5}\ (\mathrm{rad}^1)\ \text{per orbit}, \tag{27}$$

or $\sim 1.5\cdot 10^{-6}$ (arcsec^1) per Earth century.

While this result is practically identical to Newtonian approximation, it does highlight the direct use of the time–time component of the Schwarzschild metric:

$$g_{tt} = 1-\frac{R_S}{r_a} = \frac{d\tau^2}{dt^2} \approx \frac{d\tau_{sr}^4}{dt^4}. \tag{28}$$

In regard to the flat rotation curve at r_b, the idealized Schwarzschild radius, total energy, force, and dimension are obtained by rearranging the components of Eq. (23):

$$R'_S = r_b\left(1-\left(1-\frac{v^2}{c^2}\right)^2\right) = r_b(1-g_{tt}) = R_S\frac{r_b}{r_a} \quad (\mathrm{m}^1), \tag{29}$$

$$E_{Total} = \frac{R'_S F_P}{2} = E_{Core}\frac{r_b}{r_a} \approx 2\cdot 10^{58} \quad (\mathrm{J}^1), \tag{30}$$

$$c^n\alpha_n = \frac{2E_{Core}}{R'_S} = F_P\frac{r_a}{r_b} \approx 6.05\cdot 10^{43} \quad (\mathrm{N}^1), \tag{31}$$

$$n = \log_c\frac{2E_{Core}}{R'_S\alpha_5} = \log_c\left(c^5\frac{r_a}{r_b}\right) \approx 4.96. \tag{32}$$

Although Eq. (29) may also be written in terms of c^n, like in Eq. (19), doing so would create a superficial incompatibility between this model and geometrized units (where $c = G = 1$), since a logarithm of unit base ($\log_c = \log_1$) is not mathematically valid.

For the gravitationally bound model filament consisting of a core and two opposing arms of negligible energy contribution, the ratio r_b^2/r_a^2 represents a reduction in gravitational interaction along two spatial dimensions instead of one, resulting in a flat acceleration curve ($da/dr = 0$):

$$g(r) = \frac{dn}{dr} = -\frac{2}{r} \quad (\mathrm{m}^{-1}), \tag{33}$$

$$n = 5 + \int_{r_a}^{r_b} g(r)dr = 5 - \log_c\frac{r_b^2}{r_a^2}, \tag{34}$$

$$\int_{r_a}^{r_b=r_a c^x} g(r)dr = -2x, \tag{35}$$

$$R'_S = R_S\frac{r_b^2}{r_a^2} \quad (\mathrm{m}^1). \tag{36}$$

It should be noted at this point that utmost care needs to be taken when relying on the approximation of orbit velocity given in Eqs. (23) and (24), since it will fail outside of the weak-field limit. This is most readily illustrated by the fact that the approximated velocities diverge in value as distance from the Schwarzschild radius is reduced. In order to avoid this limitation entirely, see [4] (specifically, Chapter 25) for an in-depth discussion on how to accurately model orbit behaviour in regions of both weak and strong gravitation alike.

4. Discussion

Up to this point, the model presented here deals strictly with luminous matter bound mainly by gravitation. As discussed in the work of Cooperstock and Tieu [3], the interface between the galaxy's luminous matter and "outer space" is an approximately continuous surface from centre to edge, and so the core itself actually extends out to the edge of the galaxy. As distance increases beyond the radius where the flat rotation curve commences (where Keplerian falloff fails to commence), the spatial extent of gravitational interaction within the galaxy decreases, resulting in an increase in its temporal extent.

For a protoplanetary disc, interaction in the form of isotropic pressure between its constituent bodies of luminous matter is non-negligible at all distances, resulting in standard Keplerian dynamics ($n \approx 5$, $dn/dr \approx 0$).

Also discussed in [3], the Sun's surface does not extend out to the orbit radii of the planets in the form of a gravitationally bound disc, and so anisotropic interaction within the Solar System is negligible ($n \approx 5$, $dn/dr \approx 0$).

Is then the standard gravitational field of an individual star or planet ($n \approx 5$, $dn/dr \approx 0$) a direct reflection of the isotropic nuclear and electromagnetic interaction within it? If a spherically symmetric non-rotating body were to oscillate in a sufficiently anisotropic manner, so that the interaction within it occurs along significantly less than three spatial dimensions, would the gravitational field increase in temporal extent?

For example, if a body were to travel as close as possible to $v \approx c$ (where $d\tau_{sr}/dt \approx 0$, kinematically) back and forth between two fixed points separated by a distance as close as possible to the Planck length ($r \approx l_P$), would the body's gravitational field correspond to $n \approx 3$, $dn/dr \approx 0$, $R'_S \approx R_S c^2$?

Would the same effect occur if a body were to be fully gravitationally time dilated by two opposing and equidistant point sources of equal energy contribution ($d\tau/dt \approx 0$, $r \equiv l_P$, $a \approx 0$, $v \approx 0$, $n \approx 3$, $dn/dr \approx 0$, $R'_S \approx R_S c^2$)? Similarly, if a body were to be fully gravitationally time dilated at the centre of a ring of very many point sources of equal energy contribution, would the body's gravitational field correspond to $n \approx 4$, $dn/dr \approx 0$, $R'_S \approx R_S c$?

Hopefully the answers to these questions will be found through future experimentation, if not specifically due to interest in this model, then at least with regard to gravitational wave emission [4], the high-energy regime of supersymmetric string theory [6,7], and loop quantum gravity [8].

Although Cooperstock and Tieu's observations regarding the structure of several astrophysical systems have been reiterated here, there exists a possible incompatibility between this model and theirs. This is made evident in their model of the Coma cluster [9], which predicts non-Keplerian dynamics during the collapse of a spherically symmetric dust of energy in the weak-field limit. Contrarily, the model presented here predicts a lack of such behaviour in this specific idealized case, since asymmetry of system shape (the generator of anisotropic interaction) is not present. This leads to the question of whether or not the Coma cluster is actually close to being spherical in shape, as investigated in the work of Schipper and King [10]. If the cluster is not spherical, then to what extent does the asymmetry of system shape account for the observed non-Keplerian dynamics? As such, future work on this model will focus on the dynamics of galactic clusters.

One additional property of the model presented here is that it serves to eliminate the larger-than-expected gap in strength between the weak nuclear and standard gravitational interactions in the low-energy limit. When comparing approximations [11] of the low-energy strengths of the strong (~ 1), electromagnetic ($\sim 1/137$), and weak ($\sim 10^{-14}$) interactions against standard gravitation ($\sim 10^{-39}$), the resulting regular logarithmic hierarchy hints at an underlying fractal [12] nature. Using G_2, G_1, and G_0 to represent the low-energy strengths of the weak, electromagnetic, and strong interactions:

$$G_4 \approx G_5 c, \tag{37}$$

$$G_3 \approx G_5 c^2, \tag{38}$$

$$G_2 \approx G_5 c^3 \cdot 0.37, \tag{39}$$

$$G_1 \approx G_5 c^4 \cdot 904, \tag{40}$$

$$G_0 \approx G_5 c^5 \cdot 0.0004. \tag{41}$$

Because the interaction hierarchy exhibits this logarithmic self-similarity at various scales, n is taken to represent Hausdorff dimension [13]. This highlights the space-filling nature of the flux-diffusion of energy (E/c^n), insomuch that spatiotemporal extent is

$$c^n \quad (\mathrm{m}^n\ \mathrm{s}^{-n}), \tag{42}$$

and interaction strength decreases as spatial extent increases (as temporal extent decreases)

$$G_n \propto E/c^n \quad (\mathrm{J}^1\ \mathrm{s}^n\ \mathrm{m}^{-n}), \tag{43}$$

and so the more space-filling (more diffuse) an interaction is, the less strength it exhibits. This is reflected in the rotation curve of a spiral galaxy, where gravitation increases in strength (temporal extent) as n runs from 5 to 4. In effect, gravitational interaction becomes less space-filling (less diffuse) as n decreases. As stated in Section 2, this is meant to subscribe to the principle that space and time are inextricably connected by their mutual exclusivity.

It is encouraging to note that a velocity curve akin to rigid body rotation ($v \propto r$, $v \ll c$) is generated by

$$\int_{r_a}^{r_b = r_a c^x} h(r)dr = -3x, \tag{44}$$

and that a naive form of asymptotic freedom [14] is generated by both

$$\int_{r_a}^{r_b = r_a c^x} i(r)dr = -4x, \tag{45}$$

and

$$\int_{r_a}^{r_b = r_a c^x} j(r)dr = -5x. \tag{46}$$

A preliminary review of the work of others on the topic of quantum fractal spacetime uncovers a strong ideological similarity between this model and both the logarithmic scale relativity presented by Nottale [15], and the variable gravitational strength presented by Iovane, Giordano, and Laserra [16]. As such, future work on this model will focus on establishing a connecting framework that continues to rely foremost on the power law inherent to the flux-diffusion of energy. The work of Svozil [17,18] should also prove useful during this model's transition from the classical to the quantum regime, since it provides a fractal-centric approach to such fundamentally important topics as dimensional reduction, regularization, and compactification. Of course, this small selection of works is by no means a complete picture of recent progress made in regard to the study of quantum fractal spacetime. This is evidenced by the vast collection of works in the journal Chaos, Solitons & Fractals by authors such as El Naschie and Iovane on the relationship between $E^{(\infty)}$ Cantorian spacetime theory and many advanced topics, including the holographic principle [19], exceptional Lie groups [20], large scale structure [21], and cosmology [22].

5. Conclusions

The following list summarizes the main aspects of the model presented here:

1. The interaction hierarchy is fractal in nature, and the low-energy interaction strengths are partially governed by the flux-diffusion of energy ($G_n \propto E/c^n$). As Hausdorff dimension n is increased, interaction increases in spatial extent (diffusion), and decreases in temporal extent (strength).

2. The strength of standard gravitational interaction G_5 is proportional to E/c^5, which is equivalent to the interaction of luminous matter along three spatial dimensions (isotropic interaction).
3. Regardless of whether a system of luminous matter is gravitationally or hydrodynamically bound, an increase in the strength of gravitational interaction occurs when the interaction of luminous matter occurs along less than three spatial dimensions (anisotropic interaction). Where the interaction of luminous matter is constrained to occur along two spatial dimensions, the strength of gravitational interaction G_4 is proportional to E/c^4. Where the interaction of luminous matter is constrained to occur along one spatial dimension, the strength of gravitational attraction G_3 is proportional to E/c^3.
4. The strength of the weak interaction G_2 is proportional to E/c^2, the strength of the electromagnetic interaction G_1 is proportional to E/c^1, and the strength of the strong interaction G_0 is proportional to E/c^0.
5. For a system where the interaction of luminous matter remains isotropic at all distances, a Keplerian velocity curve results ($dn/dr = -0/r$).
6. For a spiral galaxy, the spatial extent of gravitational interaction is reduced along one dimension as distance from the galactic core is increased ($dn/dr = -1/r$). For a gravitationally bound filament, the reduction is along two dimensions ($dn/dr = -2/r$).
7. An increase in interaction strength via a decrease in Hausdorff dimension is also applicable to non-gravitational dynamics, such as rigid body rotation ($dn/dr = -3/r$) and asymptotic freedom ($dn/dr = -4/r$, $dn/dr = -5/r$).
8. An asymmetry exists within the low-energy interaction hierarchy. Only four of six interactions are generated concurrently:
   (a) The three forms of gravitational interaction (G_3, G_4, G_5) are mutually exclusive. In terms of a set: $G_{\{g\}} = \{G_n\}$, where $3 \le n \le 5$. Only one strength is generated.
   (b) The strong, electromagnetic, and weak interactions (G_0, G_1, G_2) are not mutually exclusive: $G_{\{sew\}} = \{G_0, G_1, G_2\}$, where $n = \{0, 1, 2\}$. All three strengths are generated.

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