Newtonian gravitation from scratch, for C++ programmers

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Abstract

This paper contains a short introduction to Newtonian gravitation. The main focus is on some C++ code.

1 Introduction

In this paper, we focus on Newtonian gravitation [1]. We cover the following six subjects:

1. Typedefs that can be used to specify the precision of the floating point variables.

2. A custom 3D vector class that is used to encapsulate the data and member functions using object-oriented paradigms.

3. Some constants that are used throughout the paper.

4. A brute force integer field line intersection count function.

5. A heuristic real field line intersection count function.

6. An application is demonstrated, where we model Mercury's orbit by using numerical integration. A full code is given.

The main goal is to acquaint the coder with the basic mathematics behind Mercury's orbit due to Newtonian gravitation.

The following two assumptions are made:

1. Like with the irradiance of light (e.g. WiFi strength) and the intensity of sound, we assume in this paper that the acceleration in Newton's gravity follows an inverse-square law:

$$g_N \propto \frac{1}{R^2}. \tag{1}$$

   See Fig. 1.

2. The other assumption that we make in this paper is that the gravitational field line count is given by the holographic principle [2, 3], for lack of a better method.

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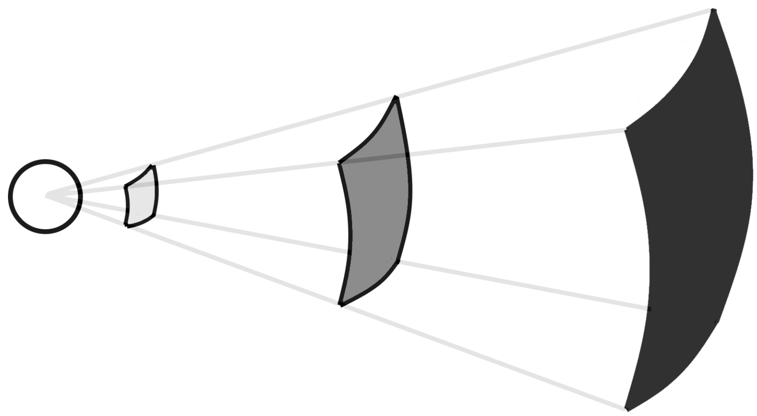

Figure 1: Inverse-square law visualized. The gravitational acceleration is $g_N \propto 1/R^2$.

2 Typedefs

In this tutorial, we leave the real number type up to the coder. For instance, we can use quad-precision long doubles (e.g. 16-byte floating point variables on Ubuntu):

```
typedef long double real_type;
```

On the other hand, we can use the Boost multiprecision library just as easily. Here we can use oct-precision variables (e.g. 32-byte floating point variables on all platforms):

```
#include <boost/multiprecision/cpp_bin_float.hpp>
using namespace boost::multiprecision;

typedef number<
        backends::cpp_bin_float<
                237,
                backends::digit_base_2,
                void,
                std::int32_t,
                -262142, // 2^18 == 262144
                262143>,
        et_off> cpp_bin_float_oct;

        // 237 significand bits
        // + 18 exponent bits
        // + 1 sign bit =
        // 256 bits (32 bytes)

typedef cpp_bin_float_oct real_type;
```

3 Custom 3D vector class

The tutorial also makes use of a 3D vector class:

```
class vector_3
{
public:
	real_type x, y, z;

	// Overloaded operators go here

	real_type dot(const vector_3& rhs) const
	{
		return x*rhs.x + y*rhs.y + z*rhs.z;
	}

	real_type self_dot(void) const
	{
		return x*x + y*y + z*z;
	}

	real_type length(void) const
	{
		return sqrt(self_dot());
	}

	vector_3& normalize(void)
	{
		const real_type len = length();

		if(len != 0)
		{
			x /= len;
			y /= len;
			z /= len;
		}

		return *this;
	}
};
```

4 Constants

The following constants will be used in this tutorial:

```
const real_type pi = 4.0 * atan(1.0);
const real_type G = 6.67430e-11; // Newton's constant
const real_type c = 299792458; // Speed of light in vacuum
const real_type c2 = c * c;
const real_type c3 = c * c * c;
const real_type c4 = c * c * c * c;
const real_type h = 6.62607015e-34; // Planck's constant
const real_type hbar = h / (2.0 * pi);
const real_type k = 1.380649e-23; // Boltzmann's constant
```

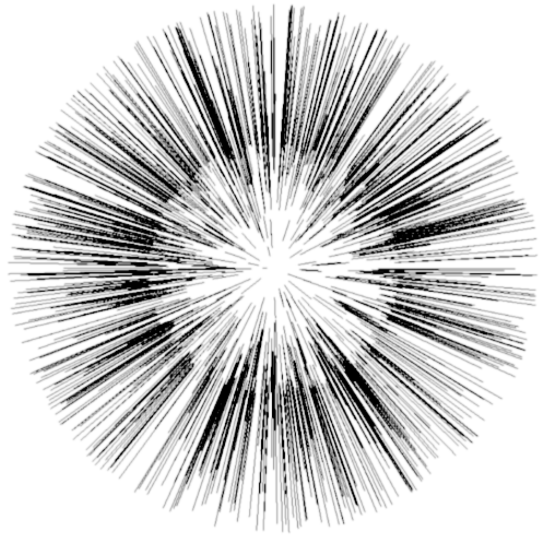

Figure 2: The emitter is spherical. The field line starting positions are placed pseudorandomly on a 2-sphere, and the normals (e.g. field line directions) are calculated using the same positions.

5 Brute force: integer field line count

The main idea behind this tutorial is that there is a finite number of field lines extending out from a gravitating body. See Fig. 2.

We use the field line intersection count to signify field strength, from which we can obtain the gradient.

Where r is the receiver radius, R is the distance from the centre of the emitter, β is the get intersecting line count function, and n is the field line count, the gradient (e.g. directional derivative) is:

$$\alpha = \frac{\beta(R+\epsilon) - \beta(R)}{\epsilon}. \tag{2}$$

The gradient strength is:

$$g = \frac{-\alpha}{r^2}. \tag{3}$$

```
long long unsigned int get_intersecting_line_count(
        const vector<vector_3>& unit_vectors,
        const vector_3& sphere_location,
        const real_type sphere_radius)
{
        long long unsigned int count = 0;

        // Get cross-section edge direction
        vector_3 cross_section_edge_dir(sphere_location.x, sphere_radius, 0);
        cross_section_edge_dir.normalize();

        // Get receiver direction
        vector_3 receiver_dir(sphere_location.x, 0, 0);
        receiver_dir.normalize();

        // The minimum threshold for intersection
```

```
        const real_type min_dot = cross_section_edge_dir.dot(receiver_dir);

        for (size_t i = 0; i < unit_vectors.size(); i++)
                if (unit_vectors[i].dot(receiver_dir) >= min_dot)
                        count++; // Intersection occurred

        return count;
}
```

```
int main(int argc, char** argv)
{
        // Field line count
        const size_t n = 1000000000;

        cout << "Allocating memory for field lines" << endl;
        vector<vector_3> unit_vectors(n);

        for (size_t i = 0; i < n; i++)
        {
                unit_vectors[i] = pseudorandom_unit_vector();

                static const size_t output_mod = 10000;

                if (i % output_mod == 0)
                        cout << "Getting pseudorandom locations: "
                        << static_cast<float>(i) / n << endl;
        }

        string filename = "newton.txt";
        ofstream out_file(filename.c_str());
        out_file << setprecision(30);

        const real_type start_distance = 10.0;
        const real_type end_distance = 100.0;
        const size_t distance_res = 1000;

        const real_type distance_step_size =
                (end_distance - start_distance)
                / (distance_res - 1);

        for (size_t step_index = 0; step_index < distance_res; step_index++)
        {
                const real_type r =
                        start_distance +
                        step_index * distance_step_size;

                const vector_3 receiver_pos(r, 0, 0);
                const real_type receiver_radius = 1.0;

                const real_type epsilon = 1.0;

                vector_3 receiver_pos_plus = receiver_pos;
                receiver_pos_plus.x += epsilon;

                const long long signed int collision_count_plus =
                        get_intersecting_line_count(
```

```
                                unit_vectors,
                                receiver_pos_plus,
                                receiver_radius);

                const long long signed int collision_count =
                        get_intersecting_line_count(
                                unit_vectors,
                                receiver_pos,
                                receiver_radius);

                const real_type gradient =
                        static_cast<real_type>
                        (collision_count_plus - collision_count)
                        / epsilon;

                const real_type gradient_strength =
                        -gradient
                        / (receiver_radius * receiver_radius);

                cout << "r: " << r << " gradient strength: "
                << gradient_strength << endl;

                out_file << r << " " << gradient_strength << endl;
        }

        out_file.close();

        return 0;
}
```

While this method works, it is both memory and processor intensive. This method is meant to be a stepping stone for the next section.

6 Heuristic: real field line count

Rather than allocating gigabytes of RAM to store some unit vectors, we can instead use a heuristic approach to solve the problem from the previous section. This heuristic solution instead uses basic geometry to obtain the intersection count.

Where r is the receiver radius, R is the distance from the centre of the emitter, β is the get intersecting line count function, and n is the field line count, the gradient is:

$$\alpha = \frac{\beta(R+\epsilon) - \beta(R)}{\epsilon}. \tag{4}$$

Here we assume that the number of field lines is given by the holographic principle:

$$n = \frac{Akc^3}{4G\hbar \log 2}. \tag{5}$$

The gradient strength is:

$$g = \frac{-\alpha}{r^2} \approx \frac{n}{2R^3}. \tag{6}$$

From this we can get the Newtonian gradient, in terms of either n, g, A, or GM:

$$g_N = \frac{nc\hbar \log 2}{k4\pi M R^2} = \frac{gRc\hbar \log 2}{k2\pi M} = \frac{Ac^4}{16\pi GMR^2} = \frac{GM}{R^2}. \tag{7}$$

We will use $g_N = GM/R^2$, the simplest version of g_N, in the next section.

```
real_type get_intersecting_line_count(
        const real_type n,
        const vector_3& sphere_location,
        const real_type sphere_radius)
{
        const real_type sphere_area =
                4 * pi * sphere_location.x * sphere_location.x;

        const real_type circle_area =
                pi * sphere_radius * sphere_radius;

        const real_type ratio =
                circle_area / sphere_area;

        return n * ratio;
}
```

```
int main(int argc, char** argv)
{
        const real_type emitter_radius = 1.0;

        const real_type emitter_area =
                4.0 * pi * emitter_radius * emitter_radius;

        // Field line count
        // re: holographic principle:
        const real_type n =
                (k * c3 * emitter_area)
                / (log(2.0) * 4.0 * G * hbar);

        const real_type emitter_mass = c2 * emitter_radius / (2.0 * G);

        // 2.39545e47 is the 't Hooft-Susskind constant:
        // the number of field lines for a black hole of
        // unit Schwarzschild radius
        //
        //const real_type G_ =
        //      (k * c3 * pi)
        //      / (log(2.0) * hbar * 2.39545e47);

        const string filename = "newton.txt";
        ofstream out_file(filename.c_str());
        out_file << setprecision(30);

        const real_type start_distance = 10.0;
        const real_type end_distance = 100.0;
        const size_t distance_res = 1000;

        const real_type distance_step_size =
                (end_distance - start_distance)
                / (distance_res - 1);

        for (size_t step_index = 0; step_index < distance_res; step_index++)
        {
                const real_type r =
```

```
                    start_distance + step_index * distance_step_size;

            const vector_3 receiver_pos(r, 0, 0);
            const real_type receiver_radius = 1.0;

            const real_type epsilon = 1.0;

            vector_3 receiver_pos_plus = receiver_pos;
            receiver_pos_plus.x += epsilon;

            const real_type collision_count_plus =
                    get_intersecting_line_count(
                            n,
                            receiver_pos_plus,
                            receiver_radius);

            const real_type collision_count =
                    get_intersecting_line_count(
                            n,
                            receiver_pos,
                            receiver_radius);

            const real_type gradient =
                    (collision_count_plus - collision_count)
                    / epsilon;

            real_type gradient_strength =
                    -gradient
                    / (receiver_radius * receiver_radius);

            const real_type gradient_strength_ =
                    n / (2.0 * pow(receiver_pos.x, 3.0));

            const real_type newton_strength =
                    n * c * hbar * log(2.0)
                    /
                    (k * pow(receiver_pos.x, 2.0)
                            * emitter_mass * 4.0 * pi);

            const real_type newton_strength_ =
                    G * emitter_mass / pow(receiver_pos.x, 2.0);

            cout << "r: " << r << " gradient strength: "
                    << gradient_strength << endl;

            out_file << r << " " << gradient_strength << endl;
        }

        out_file.close();

        return 0;
}
```

This method is faster and less memory intensive when compared to the integer field count method. This method is meant to be a stepping stone for the next section.

For reference, if you know n, and you wish to know the emitter radius from that, then the equation is:

$$r_{emitter} = \sqrt{\frac{nG\hbar \log 2}{kc^3\pi}}. \tag{8}$$

Using this radius, one can ensure that the results from this section match the results of the previous section, where n is relatively small anyway (e.g. $n = 10^7$). As for predictability, for instance where $n = 10^7$ (e.g. $r_{emitter} = 6.46109 \times 10^{-21}$ metres), it is found that at a distance of 170.521 metres that the gradient $-\alpha = 0.999615$ dips below 1.0. This distance value depends entirely on our use of the holographic principle.

7 Application: modeling Mercury's orbit using numerical integration

In essence, the numerical calculation of the Newtonian orbit of Mercury is as follows:

1. Place Mercury at the aphelion to start.
2. Calculate the orbit path by repeatedly taking steps in time.

The constant time step [4] is:

```
const real type dt = 10000; // 2.77777 hours
```

The initial conditions are:

```
vector_3 Mercury_pos(0, 69817079000.0, 0); // Aphelion location
vector_3 Mercury_vel(-38860, 0, 0); // Aphelion velocity
```

The orbit code is as follows. Here we use Eq. 7 (e.g. $g_N = GM/R^2$) to calculate the acceleration from Newtonian gravitation:

```
vector_3 Newtonian_acceleration(
        const real_type emitter_mass,
        const vector_3& pos, // Receiver pos
        const real_type G)
{
        // Sun's position is fixed at the origin
        vector_3 grav_dir = vector_3(0, 0, 0) - pos;
        const real_type distance = grav_dir.length();
        grav_dir.normalize();

        vector_3 accel = grav_dir * G * emitter_mass / pow(distance, 2.0);

        return accel;
}
```

Here we show the Euler integration [5], which is extremely simple. The acceleration is calculated, then it is added (e.g. integrated) to the velocity. Once that's done, the velocity is added to the position. See Figs. 3 and 4.

```
void proceed_Euler(
        vector_3& pos,
        vector_3& vel,
        const real_type G,
        const real_type dt)
```

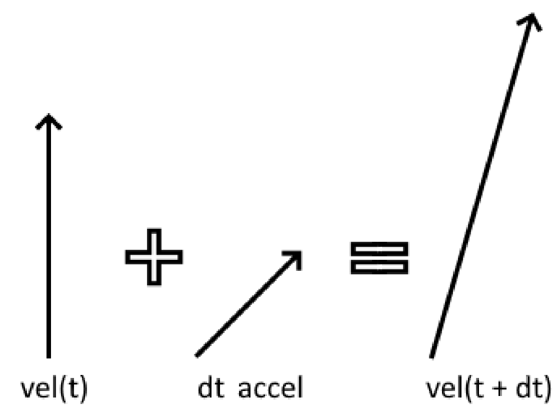

Figure 3: A diagram of the Euler integration of velocity.

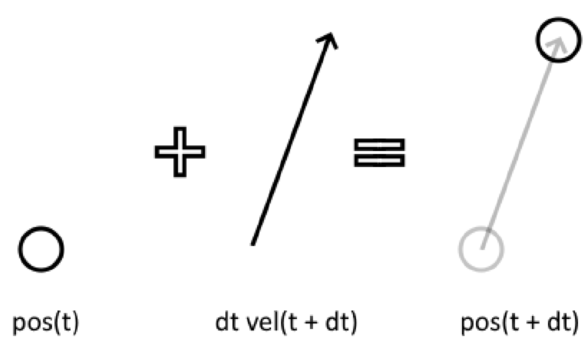

Figure 4: A diagram of the Euler integration of position.

```
{
        vector_3 accel =
                Newtonian_acceleration(
                        emitter_mass,
                        pos,
                        G);

        vel += accel * dt;
        pos += vel * dt;
}
```

The passage of time is computed whenever the window manager (e.g. OpenGL/GLUT) is not busy drawing or processing input:

```
void idle_func(void)
{
        proceed_Euler(Mercury_pos, Mercury_vel, G, dt);
}
```

On the other hand, rather than using Euler integration, the 4th-order symplectic integration does a better job at conserving energy, but at a speed cost:

```
void proceed_symplectic_order_4(
        vector_3& pos,
        vector_3& vel,
        real_type G,
        real_type dt)
{
        static const real_type cr2 =
                pow(2.0, 1.0 / 3.0);

        static const real_type c[4] =
        {
                1.0 / (2.0 * (2.0 - cr2)),
                (1.0 - cr2) / (2.0 * (2.0 - cr2)),
                (1.0 - cr2) / (2.0 * (2.0 - cr2)),
                1.0 / (2.0 * (2.0 - cr2))
        };

        static const real_type d[4] =
        {
                1.0 / (2.0 - cr2),
                -cr2 / (2.0 - cr2),
                1.0 / (2.0 - cr2),
                0.0
        };

        pos += vel * c[0] * dt;
        vel += Newtonian_acceleration(
                        emitter_mass,
                        pos,
                        G) * d[0] * dt;

        pos += vel * c[1] * dt;
        vel += Newtonian_acceleration(
                        emitter_mass,
                        pos,
                        G) * d[1] * dt;
```

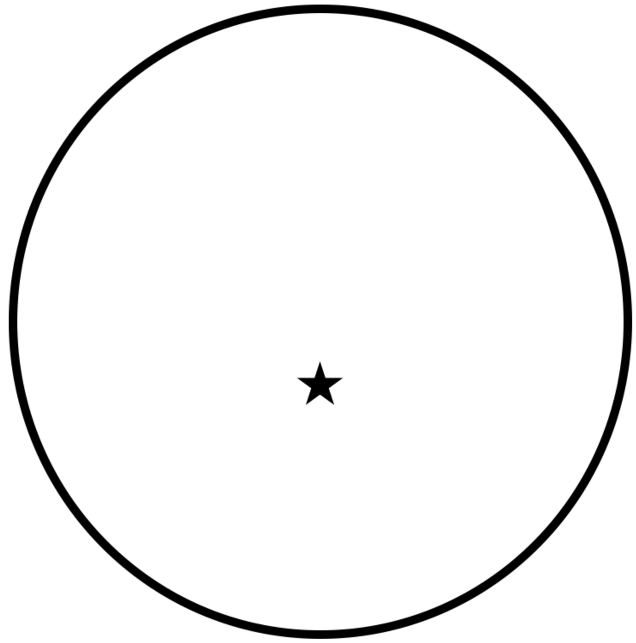

Figure 5: Mercury in orbit around the Sun. Note that the orbit path is slightly elliptical.

```
        pos += vel * c[2] * dt;
        vel += Newtonian_acceleration(
                        emitter_mass,
                        pos,
                        G) * d[2] * dt;

        pos += vel * c[3] * dt;
        // last element d[3] is always 0
}
```

See Fig. 5.

A full code, which models the orbit of Mercury, is at:
`https://github.com/sjhalayka/mercury_orbit_glut`

References

[1] Misner et al. Gravitation. (1970)

[2] ‘t Hooft. Dimensional reduction in quantum gravity. (1993)

[3] Susskind. The World as a Hologram. (1994)

[4] Fiedler. Fix Your Timestep! (2004)

[5] Fiedler. Integration Basics. (2004)