Appendix A: Proof of Proposition 1.

Recall that an adaptive algorithm for ASRN or ASR can be viewed as a decision tree. We will show that any feasible decision tree for the ASR instance $\mathcal{J}$ is also feasible for the ASRN instance $\mathcal{I}$ with the same objective, and vice versa.

In one direction, consider a feasible decision tree $\mathbb{T}$ for the ASR instance $\mathcal{J}$. For any expanded scenario $(i,\omega) \in H$, let $P_{i,\omega}$ be the unique path traced in $\mathbb{T}$, and $S_{i,\omega}$ the elements selected along $P_{i,\omega}$. Note that by definition of a feasible decision tree, at the last node ("leaf") of path $P_{i,\omega}$, it holds $f_{i,\omega}(S_{i,\omega}) = 1$ which, in the notation of the original ASRN instance, translates to $f_i(\{(e,\omega_e) : e \in S_{i,\omega}\}) = 1$.

In the other direction, let $\mathbb{T}'$ be any decision tree for ASRN instance $\mathcal{I}$. Suppose the target scenario is $i \in [m]$ and element-outcomes are given by $\omega \in \Omega^n$ on the $\star$-elements for i, which is unknown to the algorithm. Then a *unique* path $P'_{i,\omega}$ is traced in $\mathbb{T}'$. Let $S'_{i,\omega}$ denote the elements on this path. Since i is covered at the end of $P'_{i,\omega}$ we have $f_i(\{(e,\omega_e) : e \in S'_{i,\omega}\}) = 1$. Now consider $\mathbb{T}'$ as a decision tree for ASR instance $\mathcal{J}$. Under scenario i,ω, it is clear that path $P'_{i,\omega}$ is traced and so elements $S'_{i,\omega}$ are selected. It follows that $f_{i,\omega}(S'_{i,\omega}) = f_i(\{(e,\omega_e) : e \in S'_{i,\omega}\}) = 1$ which means that scenario (i,ω) is covered at the end of $P'_{i,\omega}$. Therefore $\mathbb{T}'$ is a also feasible decision tree for $\mathcal{J}$. Taking expectations, the cost for $\mathcal{J}$ is at most that for instance $\mathcal{I}$. $\square$

Appendix B: Details in Section 3

The non-adaptive SFRN algorithm (Algorithm 1) involves two phases. In the first phase, we run the SFR algorithm using sampling to get estimates $\overline{G_E}(e)$ of the scores. If at some step, the maximum sampled score is "too low" then we go to the second phase where we perform all remaining elements in an arbitrary order. The number of samples used to obtain each estimate is polynomial in m, n, ε^{-1}, so the overall runtime is polynomial.

Pre-processing. We first show that by losing an $O(1)$-factor in approximation ratio, we may assume that $\pi_i \geq n^{-2}$ for all $i \in [m]$. Let $A = \{i \in [m] : \pi_i \leq n^{-2}\}$, then $\sum_i \pi_i \leq n^{-2} \cdot n \leq n^{-1}$. Replace all scenarios in A with a single dummy scenario "0" with $\pi_0 = \sum_{i \in A} \pi_i$, and define f_0 to be any f_i where $i \in A$. By our assumption that each f_i must be covered irrespective of the noisy outcomes, it holds that $f_{i,\omega}([n]) = 1$ for each $\omega \in \Omega(i)$, and hence the cover time is at most n. Thus, for any permutation σ, the expected cover time of the old and new instance differ by at most $O(n^{-1} \cdot n) = O(1)$. Therefore, the cover time of any sequence of elements differs by only $O(1)$ in this new instance (where we removed the scenarios with tiny prior densities) and the original instances.

We now present the formal proof of Theorem 3, with proofs of the lemmas deferred to Appendix B. To analyze the our randomized algorithm, we need the following sampling lemma, which follows from the standard Chernoff bound.

LEMMA 7. *Let X be a $[0,1]$-bounded random variable with $\mathbb{E}X \geq m^{-2}n^{-4}\varepsilon$. Let $\bar{X}$ denote the average of $m^3n^4\varepsilon^{-1}$ many independent samples of X. Then* $\Pr\left[\bar{X} \notin [\frac{1}{2}\mathbb{E}X, 2\mathbb{E}X]\right] \leq e^{-\Omega(m)}$.

The next lemma shows that sampling does find an approximate maximizer unless the score is very small, and also bounds the *failure* probability.

LEMMA 8. *Consider any step in the algorithm with $S = \max_{e\in[n]} G_E(e)$ and $\bar{S} = \max_{e\in[n]} \overline{G_E}(e)$ with $\overline{G_E}(e^*) = \bar{S}$. Call this step a failure if (i) $\bar{S} < \frac{1}{4}m^{-2}n^{-4}\varepsilon$ and $S \geq \frac{1}{2}m^{-2}n^{-4}\varepsilon$, or (ii) $\bar{S} \geq \frac{1}{4}m^{-2}n^{-4}\varepsilon$ and $G_E(e^*) < \frac{S}{4}$. Then the probability of failure is at most $e^{-\Omega(m)}$.*

Based on Lemma 8, in the remaining analysis we condition on the event that our algorithm never encounters failures, which occurs with probability $1 - e^{-\Omega(m)}$. To conclude the proof, we need the following key lemma which essentially states that if the score of the greediest element is low, then the elements selected so far suffices to cover *all* scenarios with high probability, and hence the ordering of the remaining elements does not matter much.

LEMMA 9. *Assume that there are no failures. Consider the end of phase 1 in our algorithm, i.e. the first step with $\overline{G_E}(e^*) < \frac{1}{4}m^{-2}n^{-4}\varepsilon$. Then, the probability that the realized scenario is not covered is at most m^{-2}.*

The above is essentially a consequence of the submodularity of the target functions. Suppose for contradiction that there is a scenario i that, with at least m^{-2} probability over the random outcomes, remains *uncovered* by the currently selected elements. Recall that by our feasibility assumption, if all elements were selected, then f_i is covered with probability 1. Thus, by submodularity, there exists an individual element $\tilde{e}$ whose inclusion brings more coverage than the average coverage over all elements in $[n]$, and hence $\tilde{e}$ has a "high" score.

Proof of Theorem 3. Assume that there are no failures. We proceed by bounding the expected costs (number of elements) from phase 1 and 2 separately. By Lemma 8, the element chosen in each step of phase 1 is a 4-approximate maximizer (see case (ii) failure) of the score used in the SFR algorithm. Thus, by Theorem 4, the expected cost in phase 1 is $O(\log m)$ times the optimum. On the other side, by Lemma 9 the probability of performing phase 2 is at most $e^{-\Omega(m)}$. As there are at most n elements in phase 2, the expected cost is only $O(1)$. Therefore, Algorithm 1 is an $O(\log m)$-approximation algorithm for SFRN. □

B.1. Proof of Lemma 7.

Let $X_1, ..., X_N$ be i.i.d. samples of random variable where $N = m^3 n^4 \varepsilon^{-1}$ is the number of samples. Letting $Y = \sum_{i\in[N]} X_i$, the usual Chernoff bound implies for any $\delta \in (0,1)$,

$$\Pr\left(Y \notin [(1-\delta)\mathbb{E}Y, (1+\delta)\mathbb{E}Y]\right) \le \exp(-\frac{\delta^2}{2} \cdot \mathbb{E}Y).$$

The lemma follows by setting $\delta = \frac{1}{2}$ and using the assumption $\mathbb{E}Y = N \cdot \mathbb{E}X_1 = \Omega(m)$. □

B.2. Proof of Lemma 8

We will consider the two types of failures separately. For the first type, suppose $S \ge \frac{1}{2} m^{-2} n^{-4} \varepsilon$. Using Lemma 7 on the element $e \in [n]$ with $G_E(e) = S$, we obtain

$$\Pr[\bar{S} < \frac{1}{4} m^{-2} n^{-4} \varepsilon] \le \Pr[\overline{G_E}(e) < \frac{1}{4} m^{-2} n^{-4} \varepsilon] \le e^{-\Omega(m)}.$$

So the probability of the first type of failure is at most $e^{-\Omega(m)}$.

For the second type of failure, we consider two further cases:

- $S < \frac{1}{8} m^{-2} n^{-4} \varepsilon$. For any $e \in [n]$ we have $G_E(e) \le S < \frac{1}{8} m^{-2} n^{-4} \varepsilon$. Note that $\overline{G_E}(e)$ is the average of N independent samples each with mean $G_E(e)$. We now upper bound the probability of the event $\mathcal{B}_e$ that $\overline{G_E}(e) \ge \frac{1}{4} m^{-2} n^{-4} \varepsilon$. We first artificially increase each sample mean to $\frac{1}{8} m^{-2} n^{-4} \varepsilon$: note that this only increases the probability of event $\mathcal{B}_e$. Now, using Lemma 7 we obtain $\Pr[\mathcal{B}_e] \le e^{-\Omega(m)}$. By a union bound, it follows that $\Pr[\bar{S} \ge \frac{1}{4} m^{-2} n^{-4} \varepsilon] \le \sum_{e\in[n]} \Pr[\mathcal{B}_e] \le e^{-\Omega(m)}$.
- $S \ge \frac{1}{8} m^{-2} n^{-4} \varepsilon$. Consider now any $e \in U$ with $G_E(e) < S/4$. By Lemma 7 (artificially increasing $G_E(e)$ to $S/4$ if needed), it follows that $\Pr[\overline{G_E}(e) > S/2] \le e^{-\Omega(m)}$. Now consider the element e' with $G_E(e') = S$. Again, by Lemma 7, it follows that $\Pr[\overline{G_E}(e') \le S/2] \le e^{-\Omega(m)}$. This means that element e^* has $\overline{G_E}(e^*) \ge \overline{G_E}(e') > S/2$ and $G_E(e^*) \ge S/4$ with probability $1 - e^{-\Omega(m)}$. In other words, assuming $S \ge \frac{1}{8} m^{-2} n^{-4} \varepsilon$, the probability that $G_E(e^*) < S/4$ is at most $e^{-\Omega(m)}$.

Adding the probabilities over all possibilities for failures, the lemma follows. □

B.3. Proof of Lemma 9

Let E denote the elements chosen so far and p the probability that E does *not* cover the realized scenario-copy of H. That is,

$$p = \Pr_{(i,\omega)\in H}(f_{i,\omega}(E) < 1) = \sum_{i=1}^{m} \pi_i \cdot \Pr_{\omega\in\Omega(i)}(f_{i,\omega}(E) < 1).$$

It follows that there is some i with $\Pr_{\omega\in\Omega(i)}(f_{i,\omega}(E) < 1) \ge p$. By definition of separability, if $f_{i,\omega}(E) < 1$ then $f_{i,\omega}(E) \le 1 - \varepsilon$. Thus,

$$\sum_{\omega\in\Omega(i)} \pi_{i,\omega} f_{i,\omega}(E) \le \sum_{\omega: f_{i,\omega}(E)=1} \pi_{i,\omega} \cdot 1 + \sum_{\omega: f_{i,\omega}(E)<1} \pi_{i,\omega} \cdot f_{i,\omega}(E) \le (1 - \varepsilon p)\pi_i.$$

On the other hand, taking all the elements we have $f_{i,\omega}([n])=1$ for all $\omega\in\Omega(i)$. Thus,

$$\sum_{\omega\in\Omega(i)}\pi_{i,\omega}f_{i,\omega}([n])=\sum_{\omega\in\Omega(i)}\pi_{i,\omega}=\pi_i.$$

Taking the difference of the above two inequalities, we have

$$\sum_{\omega\in\Omega(i)}\pi_{i,\omega}\cdot(f_{i,\omega}([n])-f_{i,\omega}(E))\ge\pi_i\cdot\varepsilon p.$$

Consider function $g(S):=\sum_{\omega\in\Omega(i)}\pi_{i,\omega}\cdot(f_{i,\omega}(S\cup E)-f_{i,\omega}(E))$ for $S\subseteq[n]$, which is also submodular. From the above, we have $g([n])\ge\pi_i\cdot\varepsilon p$. Using submodularity of g,

$$\max_{e\in[n]}g(\{e\})\ge\frac{\varepsilon p\pi_i}{n}\implies\exists\tilde e\in[n]:\sum_{\omega\in\Omega(i)}\pi_{i,\omega}\cdot(f_{i,\omega}(E\cup\{\tilde e\})-f_{i,\omega}(E))\ge\frac{\varepsilon p\pi_i}{n}.$$

It follows that $G_E(\tilde e)\ge\frac{\varepsilon p\pi_i}{n}\ge n^{-3}\varepsilon p$, where we used that $\min_i\pi_i\ge n^{-2}$. Now, suppose for a contradiction that $p\ge m^{-2}$. Since there is no failure and $G_E(\tilde e)\ge n^{-3}m^{-2}\varepsilon\ge\frac{1}{4}n^{-4}m^{-2}\varepsilon$, by case (ii) of Lemma 8 , we deduce that $\overline{G_E}(e^*)\ge\frac{1}{4}m^{-2}n^{-4}$, which is contradiction. $\square$

Appendix C: Details in Section 4

C.1. Proof of Lemma 1.

By decomposing the summation in the left hand side of (3) as $H'=\cup_iH'\cap H_i$, and noticing that $f_{i,\omega}(E)=f_i(\nu_E)$, the problem reduces to showing that for each $i\in[m]$,

$$\sum_{(i,\omega)\in H'\cap H_i}\pi_{i,\omega}\cdot\big(f_{i,\omega}(e\cup E)-f_{i,\omega}(E)\big)=p_i\cdot\mathbb{E}_{i,\nu_e}[f_i(\nu_E\cup\{\nu_e\})-f_i(\nu_E)].$$

Recall that $p_i=n_i\cdot\frac{\pi_i}{|\Omega|^{c_i}}$ and $\pi_{(i,\omega)}=\frac{\pi_i}{|\Omega|^{c_i}}$, the above simplifies to

$$\frac{1}{n_i}\sum_{(i,\omega)\in H'\cap H_i}\big(f_{i,\omega}(e\cup E)-f_{i,\omega}(E)\big)=\mathbb{E}_{i,\nu_e}[f_i(\nu_E\cup\{\nu_e\})-f_i(\nu_E)].$$

Note that $n_i=|H'\cap H_i|$, so the above is equivalent to

$$\frac{1}{n_i}\sum_{(i,\omega)\in H'\cap H_i}f_{i,\omega}(e\cup E)=\mathbb{E}_{i,\nu_e}[f_i\big(\nu_E\cup\{\nu_e\}\big)].\tag{7}$$

It is straightforward to verify that the above by considering the following are two cases.

- If $r_i(e)=\nu_e\in\Omega\setminus\{*\}$, then the outcome ν_e is deterministic conditional on scenario i, and so is $f_i\big(\nu_E\cup\{\nu_e\}\big)$, the value of f_i after selecting e. On the left hand side, for every $\omega\in H_i$, by definition of H_i it holds $\nu_e=\omega_e$, and hence $f_{i,\omega}(e\cup E)=f_i(\nu_E\cup\{\nu_e\}$ for *every* $(i,\omega)\in H_i$. Therefore all terms in the summation are equal to $f_i(\nu_E\cup\{\nu_e\}$ and hence (7) holds.

- If $r_i(e) = *$, then each outcome $o \in \Omega$ occurs with equal probabilities, thus we may rewrite the right hand side as

$$\begin{aligned}\mathbb{E}_{i,\nu_e}[f_i(\nu_E \cup \{\nu_e\})] &= \sum_{o\in\Omega} \mathbb{P}_i[\nu_e = o] \cdot f_i(\nu_E \cup \{\nu_e\}) \\ &= \frac{1}{|\Omega|} \sum_{o\in\Omega} f_i(\nu_E \cup \{(e,o)\}).\end{aligned}$$

To analyze the other side, note that by definition of H_i and H', there are equally many expanded scenarios (i,ω) in $H' \cap H_i$ with $\omega_e = o$ for each outcome $o \in \Omega$. Thus, we can rewrite the left hand side as

$$\begin{aligned}\frac{1}{n_i} \sum_{(i,\omega)\in H'\cap H_i} f_{i,\omega}(e \cup E) &= \frac{1}{n_i} \sum_{o\in\Omega} \sum_{\substack{(i,\omega)\in H'\cap H_i,\\ \omega_e = o}} f_{i,\omega}(e \cup E) \\ &= \frac{1}{n_i} \sum_{o\in\Omega} \frac{n_i}{|\Omega|} f_{i,\omega}(e \cup E) \\ &= \frac{1}{|\Omega|} \sum_{o\in\Omega} f_i(\nu_E \cup \{(e,o)\}),\end{aligned}$$

which matches the right hand side of (7) and completes the proof. □

C.2. Application of Algorithm 2 and Algorithm 3 to ODTN.

For concreteness, we provide a closed-form formula for Score_c and Score_r in the ODTN problem using Lemma 1, which were used in our experiments for ODTN. In §2.3, we formulated ODTN as an ASRN instance. Recall that the outcomes $\Omega = \{+1, -1\}$, and the submodular function f (associated with each hypothesis i) measures the proportion of hypotheses eliminated after observing the outcomes of a subset of tests.

As in §4, at any point in Algorithm 2 or 3, after selecting set E of tests, let $\nu_E : E \to \pm 1$ denote their outcomes. For each hypothesis $i \in [m]$, let n_i denote the number of surviving expanded-scenarios of i. Also, for each hypothesis i, let p_i denote the total probability mass of the surviving expanded-scenarios of i. For any $S \subseteq [m]$, we use the shorthand $p(S) = \sum_{i\in S} p_i$. Finally, let $A \subseteq [m]$ denote the compatible hypotheses based on the observed outcomes ν_E (these are all the hypotheses i with $n_i > 0$). Then, $f(\nu_E) = \frac{m-|A|}{m-1}$. Moreover, for any new test/element T,

$$f(\nu_E \cup \{\nu_T\}) = \begin{cases} \frac{m-|A|+|A\cap T^-|}{m-1} & \text{if } \nu_T = +1 \\ \frac{m-|A|+|A\cap T^+|}{m-1} & \text{if } \nu_T = -1 \end{cases}.$$

Recall that T^+, T^- and T^* denote the hypotheses with $+1$, -1 and $*$ outcomes for test T. So,

$$\frac{f(\nu_E \cup \{\nu_T\}) - f(\nu_E)}{1 - f(\nu_E)} = \begin{cases} \frac{|A\cap T^-|}{|A|-1} & \text{if } \nu_T = +1 \\ \frac{|A\cap T^+|}{|A|-1} & \text{if } \nu_T = -1 \end{cases}.$$

It is then straightforward to verify the following.

Proposition 2. *Consider implementing Algorithm 2 on an ODTN instance. Suppose after selecting tests E, the expanded-scenarios H' (and original scenarios A) are compatible with the parameters described above. For any test T, if $b_T \in \{+1,-1\}$ is the outcome corresponding to $B_T(H')$ then the second term in* $\mathrm{Score}_c(T;E,H')$ *and* $\mathrm{Score}_r(T;E,H')$ *is:*

$$\left(\frac{|A\cap T^-|}{|A|-1}+\frac{|A\cap T^+|}{|A|-1}\right)\cdot\frac{p\left(A\cap T^*\right)}{2}+\frac{|A\cap T^-|}{|A|-1}\cdot p\left(A\cap T^+\right)+\frac{|A\cap T^+|}{|A|-1}\cdot p\left(A\cap T^-\right).$$

The above expression has a natural interpretation for ODTN: conditioned on the outcomes ν_E so far, it is the expected number of newly eliminated hypotheses due to test T (normalized by $|A|-1$).

The first term of the score $\pi\left(L_T\left(H'\right)\right)$ or $\pi\left(R_T\left(H'\right)\right)$ is calculated as for the general ASRN problem. Finally, observe that for the submodular functions used for ODTN, the separation parameter is $\varepsilon=\frac{1}{m-1}$. So, by Theorem 7 we immediately obtain a polynomial time $O(\min(r,c)+\log m)$-approximation for ODTN.

C.3. Proof of Theorem 6

The proof is similar to the analysis in Navidi et al. (2020).

With some foresight, set $\alpha := 15(r+\log m)$. Write Algorithm 3 as ALG and let OPT be the optimal adaptive policy. It will be convenient to view ALG and OPT as decision trees where each node represents the "state" of the policy. Nodes in the decision tree are labelled by elements (that are selected at the corresponding state) and branches out of each node are labelled by the outcome observed at that point. At any state, we use E to denote the previously selected elements and $H'\subseteq M$ to denote the *expanded-scenarios* that are (i) compatible with the outcomes observed so far and (ii) uncovered. Suppose at some iteration, elements E are selected and outcomes ν_E are observed, then a scenario i is said to be *covered* if $f_i(E\cup\nu_E)=1$, and *uncovered* otherwise.

For ease of presentation, we use the phrase "at time t" to mean "after selecting t elements". Note that the cost incurred until time t is exactly t. The key step is to show

$$a_k \le 0.2a_{k-1}+3y_k, \qquad \text{for all } k\ge 1, \tag{8}$$

where

- $A_k\subseteq M$ is the set of uncovered expanded scenarios in ALG at time $\alpha\cdot 2^k$ and $a_k=p(A_k)$ is their total probability,
- Y_k is the set of uncovered scenarios in OPT at time 2^{k-1}, and $y_k=p(Y_k)$ is the total probability of these scenarios.

As shown in Section 2 of Navidi et al. (2020), (8) implies that Algorithm 3 is an $O(\alpha)$-approximation and hence Theorem 6 follows. To prove (8), we consider the total score collected by ALG between iterations $\alpha 2^{k-1}$ and $\alpha 2^k$, formally given by

$$Z := \sum_{t>\alpha 2^{k-1}}^{\alpha 2^k} \sum_{(E,H')\in V(t)} \max_{e\in[n]\setminus E} \left(\sum_{(i,\omega)\in R_e(H')} \pi_{i,\omega} + \sum_{(i,\omega)\in H'} \pi_{i,\omega} \cdot \frac{f_{i,\omega}(e\cup E) - f_{i,\omega}(E)}{1 - f_{i,\omega}(E)} \right) \tag{9}$$

where $V(t)$ denotes the set of states (E, H') that occur at time t in the decision tree ALG. We note that all the expanded-scenarios seen in states of $V(t)$ are contained in A_{k-1}.

Consider any state (E, H') at time t in the algorithm. Recall that H' are the expanded-scenarios and let $S \subseteq [m]$ denote the original scenarios in H'. Let $T_{H'}(k)$ denote the subtree of OPT that corresponds to paths traced by expanded-scenarios in H' up to time 2^{k-1}. Note that each node (labeled by any element $e \in [n]$) in $T_H(k)$ has at most $|\Omega|$ outgoing branches and one of them corresponds to the outcome $o_e(S)$ defined in Algorithm 3. We define $\mathsf{Stem}_k(H')$ to be the path in $T_{H'}(k)$ that at each node (labeled e) follows the $o_e(S)$ branch. We also use $\mathsf{Stem}_k(H') \subseteq [n] \times \Omega$ to denote the observed element-outcome pairs on this path.

Definition 1. Each state (E, H') is exactly one of the following types:

- **bad** if the probability of uncovered scenarios in H' at the end of $\mathsf{Stem}_k(H')$ is at least $\frac{\Pr(H')}{3}$.
- **okay** if it is not bad and $\Pr(\cup_{e\in \mathsf{Stem}_k(H')} R_e(H'))$ is at least $\frac{\Pr(H')}{3}$.
- **good** if it is neither bad nor okay and the probability of scenarios in H' that get covered by $\mathsf{Stem}_k(H')$ is at least $\frac{\Pr(H')}{3}$.

Crucially, this categorization of states is well defined. Indeed, each expanded-scenario in H' is **(i)** uncovered at the end of $\mathsf{Stem}_k(H')$, or **(ii)** in $R_e(H')$ for some $e \in \mathsf{Stem}_k(H')$, or **(iii)** covered by some prefix of $\mathsf{Stem}_k(H')$, i.e. the function value reaches 1 on $\mathsf{Stem}_k(H')$. So the total probability of the scenarios in one of these 3 categories must be at least $\frac{\Pr(H)}{3}$.

In the next two lemmas, we will show a lower bound (Lemma 10) and an upper bound (Lemma 11) for Z in terms of a_k and y_k, which together imply (8) and complete the proof.

Lemma 10. *For any $k \geq 1$, it holds $Z \geq \alpha \cdot (a_k - 3y_k)/3$.*

Proof. The proof of this lower bound is identical to that of Lemma 3 in Navidi et al. (2020) for noiseless-ASR. The only difference is that we use the scenario-subset $R_e(H') \subseteq H'$ instead of subset "$L_e(H) \subseteq H$" in the analysis of Navidi et al. (2020). □

Lemma 11. *For any $k \geq 1$, $Z \leq a_{k-1} \cdot (1 + \ln \frac{1}{\epsilon} + r + \log m)$.*

Proof. This proof is analogous to that of Lemma 4 in Navidi et al. (2020) but requires new ideas, as detailed below. Our proof splits into two steps. We first rewrite Z by interchanging its double summation: the outer layer is now over the A_{k-1} (instead of times between $\alpha 2^{k-1}$ to $\alpha 2^k$ as in the original definition of Z). Then for each fixed $(i,\omega)\in A_{k-1}$, we will upper bound the inner summation using the assumption that there are at most r original scenarios with $r_i(e)=\star$ for each element e.

Step 1: Rewriting Z. For any uncovered $(i,\omega)\in A_{k-1}$ in the decision tree ALG at time $\alpha 2^{k-1}$, let $P_{i,\omega}$ be the path traced by (i,ω) in ALG, starting from time $\alpha 2^{k-1}$ and ending at time $\alpha 2^k$ or when (i,ω) is covered.

Recall that in the definition of Z, for each time t between $\alpha 2^{k-1}$ and $\alpha 2^k$, we sum over all states (E,H') at time t. Since $t\geq \alpha 2^{k-1}$, and the subset of uncovered scenarios only shrinks at t increases, for any $(E,H')\in V(t)$ we have $H'\subseteq A_{k-1}$. So, only the expanded scenarios in A_{k-1} contribute to Z. Thus we may rewrite (9) as

$$\begin{aligned} Z &= \sum_{(i,\omega)\in A_{k-1}} \pi_{i,\omega}\cdot \sum_{(e;E,H')\in P_{i,\omega}} \left(\frac{f_{i,\omega}(e\cup E)-f_{i,\omega}(E)}{1-f_{i,\omega}(E)} + \mathbf{1}[(i,\omega)\in R_e(H')]\right) \\ &\leq \sum_{(i,\omega)\in A_{k-1}} \pi_{i,\omega}\cdot \left(\sum_{(e;E,H')\in P_{i,\omega}} \frac{f_{i,\omega}(e\cup E)-f_{i,\omega}(E)}{1-f_{i,\omega}(E)} + \sum_{(e;E,H')\in P_{i,\omega}} \mathbf{1}[(i,\omega)\in R_e(H')]\right). \end{aligned} \tag{10}$$

Step 2: Bounding the Inner Summation. The rest of our proof involves upper bounding each of the two terms in the summation over $e\in P_{i,\omega}$ for any fixed $(i,\omega)\in A_{k-1}$. To bound the first term, we need the following standard result on submodular functions.

LEMMA 12 **(Azar and Gamzu (2011))**. *Let $f: 2^U\to[0,1]$ be any monotone function with $f(\emptyset)=0$ and $\varepsilon=\min\{f(S\cup\{e\})-f(S): e\in U, S\subseteq U, f(S\cup\{e\})-f(S)>0\}$ be the separability parameter. Then for any nested sequence of subsets $\emptyset=S_0\subseteq S_1\subseteq\cdots S_k\subseteq U$, it holds*

$$\sum_{t=1}^{k}\frac{f(S_t)-f(S_{t-1})}{1-f(S_{t-1})} \leq 1+\ln\frac{1}{\varepsilon}.$$

It follows immediately that

$$\sum_{(e;E,H')\in P_{i,\omega}} \frac{f_{i,\omega}(e\cup E)-f_{i,\omega}(E)}{1-f_{i,\omega}(E)} \leq 1+\ln\frac{1}{\varepsilon}. \tag{11}$$

Next we consider the second term $\sum_{(e;E,H')\in P_{i,\omega}} \mathbf{1}[(i,\omega)\in R_e(H')]$. Recall that $S\subseteq[m]$ is the subset of original scenarios with at least one expanded scenario in H'. Consider the partition of scenarios

S into $|\Omega|+1$ parts based on the response entries (from $\Omega \cup \{*\}$) for element e. From Algorithm 3, recall that $U_e(S)$ denotes the part with response $*$ and $C_e(S)$ denotes the largest cardinality part among the non-$*$ responses. Also, $o_e(S) \in \Omega$ is the outcome corresponding to part $C_e(S)$. Moreover, $R_e(H') \subseteq H'$ consists of all expanded-scenarios that *do not* have outcome $o_e(S)$ on element e. Suppose that $(i,\omega) \in R_e(H')$. Then, it must be that the observed outcome on e is *not* $o_e(S)$. Let $S' \subseteq S$ denote the subset of original scenarios that are also compatible with the observed outcome on e. We now claim that $|S'| \le \frac{|S|+r}{2}$. To see this, let $D_e(S) \subseteq S$ denote the part having the *second largest cardinality* among the non-$*$ responses for e. As the observed outcome is not $o_e(S)$ (which corresponds to the largest part), we have

$$|S'| \le |U_e(S)| + |D_e(S)| \le |U_e(S)| + \left(\frac{|S| - |U_e(S)|}{2}\right) = \frac{|S| + |U_e(S)|}{2} \le \frac{|S|+r}{2}.$$

The first inequality above uses the fact that S' consists of $U_e(S)$ (scenarios with $*$ response) and some part (other than $C_e(S)$) with a non-$*$ response. The second inequality uses $|D_e(S)| \le \frac{|D_e(S)|+|C_e(S)|}{2} \le \frac{|S|-|U_e(S)|}{2}$. The last inequality uses the upper-bound r on the number of $*$ responses per element. It follows that each time $(i,\omega) \in R_e(H')$, the number of compatible (original) scenarios on path $P_{i,\omega}$ changes as $|S'| \le \frac{|S|+r}{2}$. Hence, after $\log_2 m$ such events, the number of compatible scenarios on path $P_{i,\omega}$ is at most r. Finally, we use the fact that the number of compatible scenarios reduces by at least one whenever $(i,\omega) \in R_e(H')$, to obtain

$$\sum_{(e;E,H') \in P_{i,\omega}} \mathbf{1}[(i,\omega) \in R_e(H')] \le r + \log_2 m. \tag{12}$$

Combining (10), (11) and (12), we obtain the lemma. □

Appendix D: Details in Section 5

D.1. A Low-Cost Membership Oracle

Note that Steps 3, 9 and 18 are well-defined because the ODTN instance is assumed to be identifiable. If there is no new test in Step 3 with $T^+ \cap Z' \neq \emptyset$ and $T^- \cap Z' \neq \emptyset$, then we must have $|Z'| = 1$. If there is no new test in Step 9 with $z \notin T^*$ then we must have identified z uniquely, i.e. $Y = \emptyset$. Finally, in step 18, we use the fact that there are tests that deterministically separate every pair of hypotheses.

Proof. If $\bar{i} \in Z$ then it is clear that $i = \bar{i}$ in step 6 and Member(Z) declares $\bar{i} = i$. Now consider the case $\bar{i} \notin Z$. Recall that $i \in Z$ denotes the unique hypothesis that is still compatible in step 6, and that Y denotes the set of compatible hypotheses among $[m] \setminus \{i\}$, so it always contains $\bar{i}$. Hence, $Y \neq \emptyset$ in step 14, which implies that $k = 4\log m$. Also recall the definition of set S and J from (13).

Algorithm 5 Member(Z) oracle that checks if $\bar{i} \in Z$.

1: Initialize: $Z' \leftarrow Z$.

2: **while** $|Z'| \geq 2$ **do** % While-loop 1: Finding a suspect – reducing $|Z'|$ to 1

3: Choose any new test $T \in \mathcal{T}$ with $T^+ \cap Z' \neq \emptyset$ and $T^- \cap Z' \neq \emptyset$, observe outcome $\omega_T \in \{\pm 1\}$.

4: Let R be the set of hypotheses ruled out, i.e. $R = \{j \in [m] : M_{T,j} = -\omega_T\}$.

5: Let $Z' \leftarrow Z' \backslash R$.

6: Let z be the unique hypothesis when the while-loop ends. ▷ Identified a "suspect".

7: Initialize $k \leftarrow 0$ and $Y = H$.

8: **while** $Y \neq \emptyset$ and $k \leq 4 \log m$ **do** ▷ While-loop 2: choose deterministic tests for z.

9: Choose any new test T with $M_{T,i} \neq *$ and observe outcome $\omega_T \in \{\pm 1\}$.

10: **if** $\omega_T = -M_{T,i}$ **then** ▷ i ruled out.

11: Declare "$\bar{i} \notin Z$" and stop.

12: **else**

13: Let R be the set of hypotheses ruled out, $Y \leftarrow Y \setminus R$ and $k \leftarrow k+1$.

14: **if** $Y = \emptyset$ **then**

15: Declare "$\bar{i} = i$" and terminate.

16: **else**

17: Let $W \subseteq \mathcal{T}$ denote the tests performed in step 9 and ▷ Now consider the "bad" case.

$$\begin{aligned} J &= \{j \in Y : M_{T,j} = M_{T,i} \text{ for at least } 2 \log m \text{ tests } T \in W\} \\ &= \{j \in Y : M_{T,j} = * \text{ for at most } 2 \log m \text{ tests } T \in W\}. \end{aligned} \tag{13}$$

18: For each $j \in J$, choose a test $T = T(j) \in \mathcal{T}$ with $M_{T,j}, M_{T,i} \neq *$ and $M_{T,j} = -M_{T,i}$

19: let $W' \subseteq \mathcal{T}$ denote the set of these tests.

20: **if** no tests in $W \cup W'$ rule out i **then** ▷ Let i duel with hypotheses in J.

21: Declare "$\bar{i} = i$".

22: **else**

23: Declare "$\bar{i} \notin Z$".

- Case 1. If $\bar{i} \in J$ then we will identify correctly that $\bar{i} \neq i$ in step 20 as one of the tests in W' (step 18) separates $\bar{i}$ and i deterministically. So in this case we will always declare $\bar{i} \notin Z$.
- Case 2. If $\bar{i} \notin J$, then by definition of J, we have $\bar{i} \in T^*$ for at least $2\log m$ tests $T \in W$. As i has a deterministic outcome for each test in W, the probability that all outcomes in W are consistent with i is at most m^{-2}. So with probability at least $1 - m^{-2}$, some test in W must have an outcome (under $\bar{i}$) inconsistent with i, and based on step 20, we would declare $\bar{i} \notin Z$.

In order to bound the cost, note that the number of tests performed are at most: $|Z|$ in step 3, $4\log m$ in step 9 and $|J| \le |Z|$ in step 18, and the proof follows. $\square$

Proof. For the first statement, fix any $x = (i, \omega) \in \Omega$. Recall that P_x only contains tests from step 7. We only need to consider the case that $t_x < |P_x|/2$. Let $t'_x = 2 \cdot t_x$ which is a power-of-2. By (6) we know that there is some k with $t_x < k \le t'_x$ and $\theta_x(k) < 1/\rho$. Hence $\theta_x(t'_x) < \frac{2}{\rho} < \frac{1}{2}$.

Consider the point in the algorithm after performing the first t'_x tests (call them S) on P_x. Because t'_x is a power-of-two, the algorithm calls the member oracle in this iteration. Let $X \subseteq [m]$ be the compatible hypotheses after the t'_x-th test on P_x. Because $\theta_x(t'_x) < 1/2$, at most $|S|/2$ tests in S are $\star$-tests for hypothesis i: in other words the weight $w_x \ge |S|/2$ at this point in the algorithm. Let

$$X' = \left\{ y \in X : S \text{ has at most } \frac{|S|}{2} \ast\text{-tests for } y \right\} = \left\{ y \in X : w_y \ge \frac{|S|}{2} \right\}.$$

Using Lemma 6 with S and X, it follows that $|X'| \le 2Cm^\alpha$. Hence the number of hypotheses $y \in X$ with $w_y \le |S|/2 \le w_x$ is at most $2Cm^\alpha$, and so $i \in Z$ (recall that Z consists of $2Cm^\alpha$ hypotheses with the lowest weight). This means that after step 4, we would have correctly identified $\bar{i} = x$ and so P_x ends. Hence $|P_x| \le t'_x$. The first part of the lemma now follows from the fact that $t'_x \le 2 \cdot t_x$.

The second statement in the lemma follows by taking expectation over all $x \in H$. $\square$

D.2. Proof of Lemma 2.

Consider any feasible decision tree $\mathbb{T}$ for the ODTN instance and any hypothesis $i \in [m]$. If we *condition* on $\bar{i} = i$ then $\mathbb{T}$ corresponds to a feasible adaptive policy for $SSC(i)$. This is because:

- for any expanded hypothesis $(\omega, i) \in \Omega(i)$, the tests performed in $\mathbb{T}$ must rule out all the hypotheses $[m] \backslash i$, and
- the hypotheses ruled-out by any test T (conditioned on $\bar{i} = i$) is a random subset that has the same distribution as $S_T(i)$.

Formally, let $P_{i,\omega}$ denote the path traced in $\mathbb{T}$ under test outcomes ω, and $|P_{i,\omega}|$ the number of tests performed along this path. Recall that u_i is the number of unknown tests for i, and that the probability of observing outcomes ω when $\bar{i} = i$ is 2^{-u_i}, so this policy for $SSC(i)$ has cost $\sum_{(i,\omega)\in\Omega(i)} 2^{-u_i} \cdot |P_{i,\omega}|$. Thus, $OPT_{SSC(i)} \le \sum_{(i,\omega)\in\Omega(i)} 2^{-u_i} \cdot |P_{i,\omega}|$. Taking expectations over $i \in [m]$ the lemma follows. $\square$

D.3. Proof of Lemma 3.

For simplicity write $(T')^+$ as T'_+ (similarly define T'_-, T'_*). Note that $\mathbb{E}[|S_T(i) \cap (A\backslash i)|] = \frac{1}{2}\left(|T^+ \cap A| + |T^- \cap A|\right)$ because $i \in T^*$. We consider two cases for test $T' \in \mathcal{T}$.

- If $M_{T',i} = *$, then

$$\mathbb{E}[|S_{T'}(i) \cap (A\backslash i)|] = \frac{1}{2}\left(|T'_+ \cap A| + |T'_- \cap A|\right) \leq \frac{1}{2}\left(|T^+ \cap A| + |T^- \cap A|\right),$$

  by the "greedy choice" of T in step 7.
- If $i \in T'_+ \cup T'_-$ then

$$\mathbb{E}[|S_{T'}(i) \cap (A\backslash i)|] \leq \max\{|T'_+ \cap A|, |T'_- \cap A|\} \leq |T'_+ \cap A| + |T'_- \cap A|,$$

  which is at most $|T^+ \cap A| + |T^- \cap A|$ by the choice of T.

In either case the claim holds, and the lemma follows. □

D.4. Proof of Lemma 4

By definition of α-sparse instances, the maximum number of candidate hypotheses that can be eliminated after performing a single test is m^α. As we need to eliminate $m-1$ hypotheses irrespective of the realized hypothesis $\bar{i}$, we need to perform at least $\frac{m-1}{m^\alpha} = \Omega(m^{1-\alpha})$ tests under every $\bar{i}$, and the proof follows. □

Appendix E: Extension to Non-identifiable ODT Instances

Previous work on ODT problem usually imposes the following *identifiability* assumption (e.g. Kosaraju et al. (1999)): for every pair hypotheses, there is a test that distinguishes them deterministically. However in many real world applications, such assumption does not hold. Thus far, we have also made this identifiability assumption for ODTN (see §2.1). In this section, we show how our results can be extended also to non-identifiable ODTN instances.

To this end, we introduce a slightly different stopping criterion for non-identifiable instances. (Note that is is no longer possible to stop with a unique compatible hypothesis.) Define a *similarity graph* G on m nodes, each corresponding to a hypothesis, with an edge (i,j) if there is *no* test separating i and j deterministically. Our algorithms' performance guarantees will now also depend on the maximum degree d of G; note that $d = 0$ in the perfectly identifiable case. For each hypothesis $i \in [m]$, let $D_i \subseteq [m]$ denote the set containing i and all its neighbors in G. We now define two stopping criteria as follows:

- The *neighborhood* stopping criterion involves stopping when the set K of compatible hypotheses is contained in *some* D_i, where i might or might not be the true hypothesis $\bar{x}$.

- The *clique* stopping criterion involves stopping when K is contained in some clique of G.

Note that clique stopping is clearly a stronger notion of identification than neighborhood stopping. That is, if the clique-stopping criterion is satisfied then so is the neighborhood-stopping criterion. We now obtain an adaptive algorithm with approximation ratio $O(d+\min(h,r)+\log m)$ for clique-stopping as well as neighborhood-stopping.

Consider the following two-phase algorithm. In the first phase, we will identify some subset $N \subseteq [m]$ containing the realized hypothesis $\bar{i}$ with $|N| \le d+1$. Given an ODTN instance with m hypotheses and tests $\mathcal{T}$ (as in §2.1), we construct the following ASRN instance with hypotheses as scenarios and tests as elements (this is similar to the construction in §2.3). The responses are the same as in ODTN: so the outcomes $\Omega = \{+1,-1\}$. Let $U = \mathcal{T} \times \{+1,-1\}$ be the element-outcome pairs. For each hypothesis $i \in [m]$, we define a submodular function:

$$\widetilde{f}_i(S) = \min\left\{\frac{1}{m-d-1}\cdot \Big| \bigcup_{T:(T,+1)\in S} T^- \bigcup \bigcup_{T:(T,-1)\in S} T^+ \Big|, 1\right\}, \quad \forall S \subseteq U.$$

It is easy to see that each function $\widetilde{f}_i : 2^U \to [0,1]$ is monotone and submodular, and the separability parameter $\varepsilon = \frac{1}{m-d-1}$. Moreover, $\widetilde{f}_i(S) = 1$ if and only if at least $m-d-1$ hypotheses are incompatible with at least one outcome in S. Equivalently, $\widetilde{f}_i(S) = 1$ iff there are at most $d+1$ hypotheses compatible with S. By definition of graph G and max-degree d, it follows that function $\widetilde{f}_i$ can be covered (i.e. reaches value one) irrespective of the noisy outcomes. Therefore, by Theorem 7 we obtain an $O(\min(r,c)+\log m)$-approximation algorithm for this ASRN instance. Finally, note that any feasible policy for ODTN with clique/neighborhood stopping is also feasible for this ASRN instance. So, the expected cost in the first phase is $O(\min(r,c)+\log m)\cdot OPT$.

Then, in the second phase, we run a simple splitting algorithm that iteratively selects any test T that splits the current set K of consistent hypotheses (i.e., $T^+ \cap K \ne \emptyset$ and $T^- \cap K \ne \emptyset$). The second phase continues until K is contained in (i) some clique (for clique-stopping) or (ii) some subset D_i (for neighborhood-stopping). Since the number of consistent hypotheses $|K| \le d+1$ at the start of the second phase, there are at most d tests in this phase. So, the expected cost is at most $d \le d \cdot OPT$. Combining both phases, we obtain the following.

THEOREM 10. *There is an adaptive $O(d+\min(c,r)+\log m)$-approximation algorithm for ODTN with the clique-stopping or neighborhood-stopping criterion.*