CS524: Introduction to Optimization
Lecture 16

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Modeling Trick (#1)

Last Time

- Suppose we wish to have a constraint hold if an associated indicator variable δ is flipped to 1. That is...

$$\boxed{\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b}$$

- This can be represented by the constraint
  - $\sum_{j \in N} a_j x_j + \mathcal{M}\delta \leq \mathcal{M} + b$
  - $\mathcal{M}$ is an upper bound for the expression $\sum_{j \in N} a_j x_j - b$.

The Logic

$$\boxed{\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b \Leftrightarrow \sum_{j \in N} a_j x_j + \mathcal{M}\delta \leq \mathcal{M} + b}$$

- Equivalent to $\sum_{j \in N} a_j x_j - b \leq \mathcal{M}(1 - \delta)$
- $\delta = 0 \Rightarrow \sum_{j \in N} a_j x_j - b \leq \mathcal{M}$
  - (true by definition of $\mathcal{M}$)
- $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j - b \leq 0$

Modeling Trick #2: Converse of First

$$\boxed{\sum_{j\in N} a_j x_j \leq b \Rightarrow \delta = 1}$$

- $\delta = 0 \Rightarrow \sum_{j\in N} a_j x_j \not\leq b$
- $\delta = 0 \Rightarrow \sum_{j\in N} a_j x_j > b$
- $\delta = 0 \Rightarrow \sum_{j\in N} a_j x_j \geq b + \epsilon$, so this is equivalent to above statement
  - If a_j, x_j are integer, we can choose $\epsilon = 1$
- Model as $\sum_{j\in N} a_j x_j - (m - \epsilon)\delta \geq b + \epsilon$
  - m is a lower bound for the expression $\sum_{j\in N} a_j x_j - b$
- Note: if $\delta = 0$ then $\sum_{j\in N} a_j x_j \geq b + \epsilon$ (as required above)
- but $\delta = 1$ results in $m \leq \sum_{j\in N} a_j x_j - b$ (i.e. nothing)

Some Last Modeling Tricks

$$\boxed{\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \geq b}$$

- Model as $\sum_{j \in N} a_j x_j + m\delta \geq m + b$

$$\boxed{\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1}$$

- Model as $\sum_{j \in N} a_j x_j - (\mathcal{M} + \epsilon)\delta \leq b - \epsilon$

Slide of Tricks: $m \leq \sum_{j \in N} a_j x_j - b \leq \mathcal{M}$

1. $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b$
   - $\sum_{j \in N} a_j x_j + \mathcal{M}\delta \leq \mathcal{M} + b$
2. $\sum_{j \in N} a_j x_j \leq b \Rightarrow \delta = 1$
   - $\sum_{j \in N} a_j x_j - (m - \epsilon)\delta \geq b + \epsilon$
3. $\sum_{j \in N} a_j x_j < b \Rightarrow \delta = 1$
   - $\sum_{j \in N} a_j x_j - m\delta \geq b$
4. $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \geq b$
   - $\sum_{j \in N} a_j x_j + m\delta \geq m + b$
5. $\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1$
   - $\sum_{j \in N} a_j x_j - (\mathcal{M} + \epsilon)\delta \leq b - \epsilon$
6. $\sum_{j \in N} a_j x_j > b \Rightarrow \delta = 1$
   - $\sum_{j \in N} a_j x_j - \mathcal{M}\delta \leq b$

More Tricks

- The first tricks we learned can be derived from these general tricks.
- $x > 0 \Rightarrow \delta = 1$
  - $x \leq \mathcal{M}\delta$ (Trick 6)
- $\delta = 1 \Rightarrow x \geq b$
  - $x \geq b\delta$ (Trick 4)

  (Note that we require a lower bound m on $x - b$ and this is $m = -b$. Substituting this into Trick 4 expression above gives the result.)

Fall Into the...

- GAP: Generalized Assignment Problem

- We have a set $M = \{1, 2, \ldots, m\}$ of machines
- and a set $N = \{1, 2, \ldots, n\}$ of jobs that must be performed on the machines.
- Each machine i has a capacity of b_i units of work
- Each job j requires a_{ij} units of work to be completed if it is scheduled on machine i.
- All jobs must be assigned to exactly one machine.
- Suppose there is a fixed cost h_i of assigning any jobs to machine i

GAP Models

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{i \in M} h_i z_i$$

$$\sum_{j \in N} a_{ij} x_{ij} \leq b_i \qquad \forall i \in M$$

$$\sum_{i \in M} x_{ij} = 1 \qquad \forall j \in N$$

z_i is 1 if machine i is on

$$x_{ij}, z_i \in \{0, 1\} \qquad \forall i \in M, j \in N$$

Fixed Cost Logic

- Logic 1:

$$\sum_{j \in N} x_{ij} > 0 \Rightarrow z_i = 1 \quad \forall i \in M$$

- Slide of Trix (6):

$$\sum_{j \in N} x_{ij} \leq \mathcal{M} z_i \quad \forall i \in M$$

  - `vub_eq_1`: $\mathcal{M} = |N| \quad (\forall i \in M)$
  - `vub_eq_2`: $\mathcal{M}_i = |\{j : j \text{ may be assigned to } i\}|$

- Logic 2:

$$x_{ij} > 0 \Rightarrow z_i = 1 \quad \forall i \in M, \forall j \in N$$

- Slide of Trix (6): `vub_eq_3`

$$x_{ij} \leq z_i \quad \forall i \in M, \ \forall j \in N$$

More Fixed Cost Logic

- Logic 3:

$$\sum_{j \in N} a_{ij} x_{ij} > 0 \Rightarrow z_i = 1 \quad \forall i \in M$$

- Slide of Trix (6): `vub_eq_4`

$$\sum_{j \in N} a_{ij} x_{ij} \leq \mathcal{M} z_i \quad \forall i \in M$$

  - $\mathcal{M}_i = b_i$

Which is Best?!?!?

Let's go to the GAMS

Some Additional Problems on GAP

1. If you use k or more machines, then you must pay a penalty cost of λ
2. If you operate machine one or two, then you may not operate both machines 3 and 4
3. If you operate both machine 1 and machine 2, then you may use no more than 50% of the capacity of machine 3
4. Each job $j \in N$ has a duration (or length) d_j. Minimize makespan.

GAP 1

- If you use k or more machines, then you must pay a penalty cost of λ

- Need (earlier) "fixed cost" logic for z_i
- Model $\sum_{i \in M} z_i \geq k \Rightarrow \delta_1 = 1$
- Add $\lambda\delta_1$ to objective function

- Appropriate trick (5) is

$$\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1 \Leftrightarrow \sum_{j \in N} a_j x_j - (\mathcal{M} + \epsilon)\delta \leq b - \epsilon$$

- $\mathcal{M} = |M| - k$[1]. $\epsilon = 1$:

$$\sum_{i \in M} z_i - (|M| - k + 1)\delta_1 \leq k - 1.$$

[1] bad notation clash, sorry!

GAP 2

- If you operate machine one or two, then you may not operate both machines 3 and 4

- May want to model (iff): $z_i = 1 \Rightarrow \sum_{j \in N} x_{ij} \geq 1$
- Then need to model $z_1 + z_2 \geq 1 \Rightarrow z_3 + z_4 \leq 1$
  - $z_1 + z_2 \geq 1 \Rightarrow \delta_2 = 1$
  - $\delta_2 = 1 \Rightarrow z_3 + z_4 \leq 1$

- Trick for $z_i = 1 \Rightarrow \sum_{j \in N} x_{ij} \geq 1$ is

$$\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \geq b \Leftrightarrow \sum_{j \in N} a_j x_j + m\delta \geq m + b$$

$m = -1$, $\epsilon = 1$:

$$\sum_{j \in N} x_{ij} - z_i \geq 0.$$

Gap 2, cont.

- First trick is:

$$\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1 \Leftrightarrow \sum_{j \in N} a_j x_j - (\mathcal{M} + \epsilon)\delta \leq b - \epsilon$$

$\mathcal{M} = 1$, $\epsilon = 1$:

$$z_1 + z_2 - 2\delta_2 \leq 0.$$

(Note: could also model (better) as $\delta_2 \geq z_1, \delta_2 \geq z_2$)

- Second trick is:

$$\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b \Leftrightarrow \sum_{j \in N} a_j x_j + \mathcal{M}\delta \leq \mathcal{M} + b$$

$\mathcal{M} = 1, \epsilon = 1$:

$$z_3 + z_4 + \delta_2 \leq 2.$$

GAP 3

- If you operate both machine 1 and machine 2, then you may use no more than 50% of the capacity of machine 3

- May want to model: $z_i = 1 \Rightarrow \sum_{j \in N} x_{ij} \geq 1 \quad \forall i \in M$
- Then model $z_1 + z_2 \geq 2 \Rightarrow \sum_{j \in N} a_{3j} x_{3j} \leq 0.5 b_3$
  - Then model $z_1 + z_2 \geq 2 \Rightarrow \delta_3 = 1$
  - $\delta_3 = 1 \Rightarrow \sum_{j \in N} a_{3j} x_{3j} \leq 0.5 b_3$

GAP 3, Cont.

- Trick for $z_1 + z_2 \geq 2 \Rightarrow \delta_3 = 1$ is

$$\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1 \Leftrightarrow \sum_{j \in N} a_j x_j - (\mathcal{M} + \epsilon)\delta \leq b - \epsilon$$

$\mathcal{M} = 0$, $\epsilon = 1$:

$$z_1 + z_2 - \delta_3 \leq 1$$

- Trick for $\delta_3 = 1 \Rightarrow \sum_{j \in N} a_{3j} x_{3j} \leq 0.5 b_3$ is

$$\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b \Leftrightarrow \sum_{j \in N} a_j x_j + \mathcal{M}\delta \leq \mathcal{M} + b$$

$\mathcal{M} = b_3 - 0.5 b_3 = 0.5 b_3$:

$$\sum_{j \in N} a_j x_j + 0.5 b_3 \delta_3 \leq b_3$$

GAP 4

- Each job $j \in N$ has a duration (or length) d_j. Minimize makespan.

- MINIMAX again. (No integer variables needed)
- Let $t \geq \max_{i \in M}\{\sum_{j \in N} d_j x_{ij}\}$.

$$\min t$$

$$t \geq \sum_{j \in N} d_j x_{ij} \quad \forall i \in M$$