1. While statement (a) holds true, statement (b) does not; as a counterexample, let $M = (E, \mathcal{I})$ be the martini glass-shaped graph matroid we frequently use in class. Choose $A, B \subseteq E$ such that $A = ace$ and $B = db$. Becuase $A \cap B = \emptyset$, the closure of $A \cap B$, $\overline{A \cap B}$, is d, so $r(\overline{A \cup B}) = 0$. However, $\overline{A} = abcdef$, and $\overline{B} = db$ (db in itself is already a flat in M), so $r(\overline{A} \cap \overline{B}) = r(db) = 1$, so $r(\overline{A \cap B}) \neq r(\overline{A} \cap \overline{B})$. Rather, the statement (b$'$) $r(\overline{A \cap B}) \leq r(\overline{A} \cap \overline{B})$ holds.

Proof. (for (a)) Since $A \subseteq A \cup B$, by Axiom (cl2) it follows that $\overline{A} \subseteq \overline{A \cup B}$, and by similar reasoning $\overline{B} \subseteq \overline{A \cup B}$. So, it must be that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Thus, by Axiom (r2), we can conclude that $r(\overline{A \cup B}) \geq r(\overline{A} \cup \overline{B})$.

Now, all we need to do is show that $r(\overline{A \cup B}) \leq r(\overline{A} \cup \overline{B})$. Note that $\overline{A} \cup \overline{B}$ contains $A \cup B$ ($A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, so $A \cup B \subseteq \overline{A} \cup \overline{B}$). But because $\overline{A \cup B}$ is the closure of $A \cup B$ – meaning that it is the set of elements whose addition to $A \cup B$ does not increase the rank – it must be that $r(\overline{A \cup B}) = r(A \cup B)$. And because $A \cup B \subseteq \overline{A} \cup \overline{B}$, by rank Axiom (r2) it follows that $r(\overline{A \cup B}) = r(A \cup B) \leq r(\overline{A} \cup \overline{B})$.

Since both $r(\overline{A \cup B}) \leq r(\overline{A} \cup \overline{B})$ and $r(\overline{A \cup B}) \geq r(\overline{A} \cup \overline{B})$, it follows that $r(\overline{A \cup B}) = r(\overline{A} \cup \overline{B})$. ■

Proof. (for (b$'$)) We already know, via counterexample, that $r(\overline{A \cap B}) \ngeq r(\overline{A} \cap \overline{B})$. However, because $A \cap B \subseteq A$, $\overline{A \cap B} \subseteq \overline{A}$ (follows from Axiom (cl2)), and by similar reasoning $\overline{A \cap B} \subseteq \overline{B}$. Thus, we can conclude that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, and by Axiom (r2) it follows that $r(\overline{A \cap B}) \leq r(\overline{A} \cap \overline{B})$. ■

2. (a) *Proof.* First note that by local rank Axiom (r2$'$) we have $r(A) \leq r(A \cup x) \leq r(A) + 1$. This means that $r(A \cup x) = r(A) + 1$ *or* $r(A \cup x) = r(A)$ (this similarly applies to $r(A \cup y)$ as well).

Case 1. If $r(A \cup x) = r(A) + 1$ and $r(A \cup y) = r(A) + 1$, then we can apply local rank Axiom (r2$'$) to see that the inequality holds.

$$\begin{aligned} r(A \cup x) + r(A \cup y) - r(A) - r(A \cup xy) &= r(A) + 1 + r(A) + 1 - r(A) - r(A \cup xy) \\ &= r(A) + 2 - r(A \cup xy) \\ &= r(A) + 2 - 2 \\ &\geq 0. \end{aligned}$$

Case 2. If $r(A \cup x) = r(A \cup y) = r(A)$, we can apply local rank Axiom (r3$'$):

$$\begin{aligned} r(A \cup x) + r(A \cup y) - r(A) - r(A \cup xy) &= \underline{r(A)} + \underline{r(A)} - r(A) - \underline{r(A)} \\ &= 0 \geq 0. \end{aligned}$$

Case 3. If either $r(A \cup x)$ or $r(A \cup y) = r(A)$ and the other equals $r(A) + 1$, then

$$\begin{aligned} r(A \cup x) + r(A \cup y) - r(A) - r(A \cup xy) &= r(A) + r(A) + 1 - r(A) - (r(A) + 1) \\ &= 0 \geq 0, \end{aligned}$$

as desired. ■

(b) *Proof.* *(r1)*: From (r1′), we know that r(∅) - 0, and from (r2′) we know that for all $A \subseteq E$ and $x \in E$ $r(A) \leq r(A \cup x) \leq r(A) + 1$. So, starting from ∅, we can derive that $0 \leq r(\emptyset) \cup x \leq r(\emptyset) + 1$. This means that every time we "build up" from the empty set and add an element to it one by one, (r2′) is telling us that every time we add an element the rank can *only* increase by 1; meaning, if we define $A = \{\emptyset \cup x_1 \cup x_2 \cup \ldots\}$, the rank of A can never exceed its total number of elements, so $0 \leq r(A) \leq |A|$.

(r2): Choose $A \subseteq E$, and choose an $x \in E$ (it doesn't matter whether $x \in A$ or $x \notin A$). Call $A \cup x = B$; by how we defined B it must be that $A \subseteq B$. We must now see whether $r(A) \leq r(B) = r(A \cup x)$; and by directly applying (r2′) we find that for all $A \subseteq E$ and $x \in E$, $r(A) \leq r(A \cup x) \leq r(A) + 1$. And because $r(A \cup x) = r(B)$, we can now conclude that if $A \subseteq B$, then $r(A) \leq r(B)$.

(r3): To prove (r3), we will induct on the size of $|B - A| = k$.

First, note that from (a), Lemma 2.47 directly follows, as we were shown in class.

Base: $(k = 0)$ If $k = 0$, then it follows that $r(A \cup B) + r(A \cap B) = r(A) + r(A \cap B) = r(A) + r(B)$ (it holds that $A \cap B = B$, since $|B - A| = 0$). So, the inequality is satisfied.

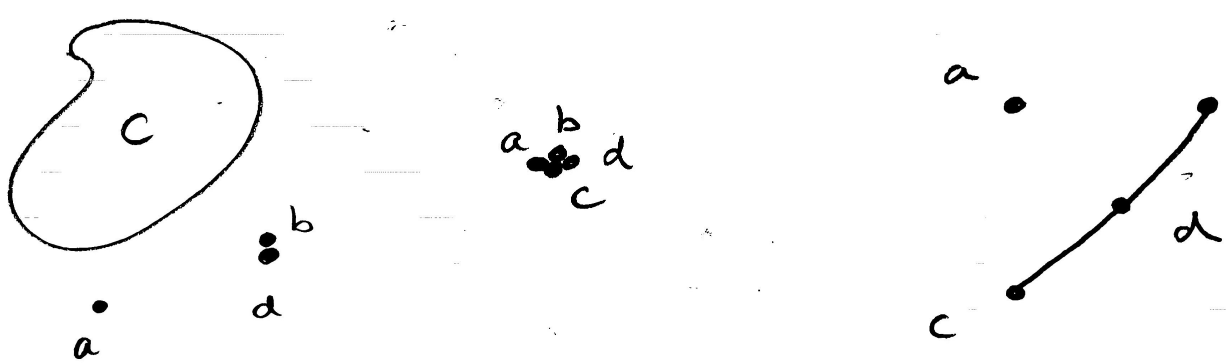
Inductive: (applies to all $|B - A| < k$). Assume that the inequality holds for all $|B - A| < k$.

Let $B' \subseteq B$ and choose x such that $B' \cup x = B$ and $A \cap B = A \cap B'$ (the intersection stays the same). Then, it follows that

$$\begin{aligned}
r(A \cup B) + r(A \cap B) &= r(A \cup B' \cup x) + r(A \cap B) \\
&= r(A \cup B' \cup x) + r(A \cap B') \\
&\leq \underline{r(A \cup B' \cup x) - r(A \cup B')} + r(A) + r(B') \text{ (Induction Hypothesis)} \\
&\leq r(B' \cup x) - r(B') + r(A) + r(B') \text{ (direct app. of Lemma 2.47)} \\
&= r(B) + r(A),
\end{aligned}$$

as desired. So, (r3) holds! ■

3. (a) *Proof.* Here, we must show that for any $I \in \mathcal{I}_1 \cup \mathcal{I}_2$, I is independent in $M_1 \vee M_2$. For I to be independent it must also be an element of $I_1 \cup I_2$, where $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. Consider $I = \emptyset \cup I_k$, where I_k is any independent set from either I_1 or I_2. And because it is a union of the independent sets, $\emptyset \cup I_k \in \mathcal{I}_1 \cup \mathcal{I}_2$. But also, since $\emptyset \in I_1, I_2$, and I_k is an independent subset in either M_1 or M_2, $\emptyset \cup I_k = I_k$ *must* also be in $I_1 \cup I_2$, which describes all the independent sets of $M_1 \vee M_2$. So this must mean that $\mathcal{I}_1 \cup \mathcal{I}_2 \subseteq$ *(all independent subsets of $M_1 \vee M_2$)*. ■



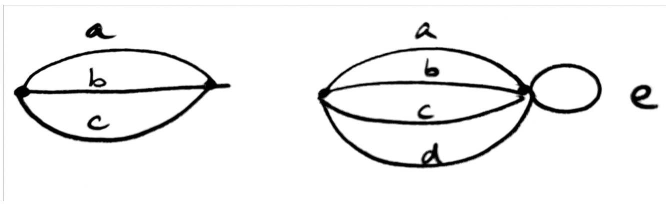
M_1, M_2, and $M_1 \vee M_2$ respectively.

(b) Consider these two matroids, $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_1)$. ($M_1$ is the matroid with the loop c, and M_2 is the matroid with co-linear points a, b, c and d). The union of these two matroids, $M_1 \vee M_2$, is the right-most PLIS.

First, note that $\mathcal{I}_1 = \{\emptyset, a, b, d, ab, ad\}$, and $\mathcal{I}_2 = \{\emptyset, a, b, c, d\}$. Their union, $\mathcal{I}_1 \cup \mathcal{I}_2$, is $\{\emptyset, a, b, c, d, ab, ad\}$.

However, there are clearly way more independent sets in the matroid *union*, as the set of all $I_1 \cup I_2$ such that $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ is $\{\emptyset, a, b, c, d, ab, ad, \underline{abc, abd, acd}\}$.

So in this scenario $\mathcal{I}_1 \cup \mathcal{I}_2 \subset$ *(independent sets of* $M_1 \vee M_2$*)*.



G_1, G_2 respectively. $M(G_1)$ and $M(G_2)$ follows from these graphs.

(c) Consider these two graphic matroids. Let the first matroid be $M(G_1)$, and the second be $M(G_2)$. Since $\mathcal{I}_1$ is $\{\emptyset, a, b, c\}$ and $\mathcal{I}_2$ is $\{\emptyset, a, b, c, d\}$ (Since e is a loop it is not in $\mathcal{I}_2$).

The independent sets of the matroid union is $\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd\}$. The union of the two matroids has bases size 2, so this implies that (if the union were to be graphic) the spanning tree must have 3 vertices. In addition, all sets size 2 are independent.

Hence, for $M(G_1) \vee M(G_2)$ to be graphic, it must be that the resulting graph G' has

1. 3 vertices, with 5 edges (ground set of the matroid union is $\{a, b, c, d, e\}$), and
2. Since all subsets size 1 and 2 are independent, there are no loops nor edges.

This construction is impossible; we cannot have a graph with 3 vertices and 5 edges without having any multiple edges nor loops. Hence, in this scenario $M(G_1) \vee M(G_2)$ is *not* graphic.

This graph, however, is impossible to draw; the way the independent set of the matroid union is defined does not allow for any selection of three edges of $M_1 \vee M_2$ to always create a cycle.

4. (a) *Proof.* To prove this statement, we must verify two inequalities:

1. Is $(x \vee y) \geq x \vee (y \wedge z)$? This inequality holds, because at best $y \geq (y \wedge z)$ (the greatest lower bound of y and z can only be y or lower).
2. Is $(x \vee z) \geq x \vee (y \wedge z)$? Yes. Again, by similar reasoning as above, at best $z \geq (y \wedge z)$, so the inequality holds.

Since these two inequalities hold, we can now say that $x \vee (y \wedge z)$ serves as a lower bound (not necessarily greatest!) for both $x \vee z$ and $x \vee y$. And because $(x \vee y) \wedge (x \vee z)$ describes the *greatest* lower bound of these two factors, the lower bound $x \vee (y \wedge z)$ can be no greater than the *greatest* lower bound $(x \vee y) \vee (x \vee z)$. Hence, $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$. ∎

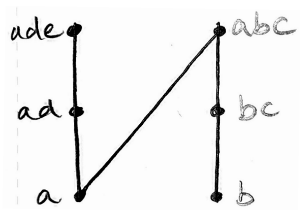
(b) *Proof.* Since the lattice L is semimodular, it follows that $\rho(x \vee y) + \rho(x \wedge y) \leq \rho(x) + \rho(y)$ for the rank function ρ of the lattice. Because both x and y cover $x \wedge y$ in L, it must be that $\rho(x \wedge y) =$

$\rho(x) - 1 = \rho(y) - 1$. So, if we set $\rho(x \wedge y) = \rho(x) - 1$,

$$\rho(x \wedge y) + \rho(x \vee y) = \rho(x) - 1 + \rho(x \vee y) \leq \rho(x) + \rho(y)$$

which means that $\underline{\rho(x \vee y) \leq \rho(y) + 1}$. By similar reasoning, setting $\rho(x \vee y) = \rho(y) - 1$ we get $\underline{\rho(x \vee y) \leq \rho(x) + 1}$. These two results imply that $x \vee y$ cover x and y. ■

5. Consider the following poset $P = (E, \preceq)$, with $E = \{a, b, ad, bc, ade, abc\}$, and $A \preceq B$ if $A \subseteq B$ for any $A, B \in E$.



$P = (E, \preceq)$.

In this poset P, the longest chain has length 3, and these chains are $\{a, ad, ade\}$ and $\{b, bc, abc\}$; so all $t \in P$ is contained in a chain of length $l = 3$. However, there also exists a *maximal* chain $\{a, abc\}$, which itself is not contained in any longer chains (in our case, chains of length 3) and has length $2 < l$. ■

6. Claim: For the divisor lattice D_n to be *atomic*, it must be that n is an integer consisting of only *distinct* prime factors without any squares (in other words, n must be of the format $n = p_1 p_2 \dots p_{k-1} p_k$, where $p_1 \dots p_k$ denote unique primes in ascending order).

Proof. Assume BWOC that at least one of the primes in the prime factorisation of n has an exponent that is 2 or larger (call that prime p_i,with $1 \leq i \leq k$ This means that the prime factorisation of n will be $n = p_1 p_2 \dots p_i^2 \dots p_k$).

In the lattice D_n, any elements that cover $\hat{0}$ (atoms of the lattice) are prime factors, as only prime numbers have 1 as the divisor other than themselves.

If this is the case, then because p_1 divides p_1^2 (and also divides p_1^3 ... and so on) in the divisor lattice of D_n, a chain p_i —— p_i^2 (with $p_i \leq p_i^2$ in the poset) can be established. But because p_i^2 (and p_i^3 ... and so forth) do *not* have any other prime factors, the exponentiated prime divisor cannot never be the result of a join of any two distinct atoms; this runs counter to the assumption that the divisor lattice D_n is atomic. $\Rightarrow\Leftarrow$

Therefore, only when n can be factorized into form $n = p_1 p_2 \dots p_{k-1} p_k$, with k *distinct* primes, is the divisor lattice D_n atomic!