Jinseob Sohn

1. (a) The instant $(E, \mathcal{I})$ pair does not define a matroid, as it violates Axiom (I2). As a counterexample, note that $ac \in \mathcal{I}$, however $c \notin \mathcal{I}$ as required by (I2).

(b) This $(E, \mathcal{I})$ pair also does not define a matroid, as it violates Axiom (I3). This can be verified easily; while $c, ab \in \mathcal{I}$, it is not the case that ac nor $bc \in \mathcal{I}$.

(c) Here as well, this $(E, \mathcal{I})$ pair does not define a matroid. Axiom (I2) is violated in the instant case; $abc \in \mathcal{I}$, $bc \subseteq abc$ but $bc \notin \mathcal{I}$!

(d) This $(E, \mathcal{I})$ pair also does not define a matroid, as it violates Axiom (I3). Observe, for instance, that while $ac, def \in \mathcal{I}$, there is no element $x \in def - ac$ s.t. $ac \cup \{x\} \in \mathcal{I}$ (neither acd, ace nor acf are present in $\mathcal{I}$).

2. *All maximal independent sets in a matroid have the same size.*

Proof. Let M be a matroid, with $M = (E, \mathcal{I})$. Assume (by way of contradiction) that not all of the maximal independent sets in M are of the same size, and pick two maximal independent sets, I_1 and I_2, that have different sizes ($|I_1| < |I_2|$). As M is a matroid, the three Axioms must hold for all subsets in $\mathcal{I}$. The implication of Axiom (I3), however, is that given two independent subsets of different sizes (in our case, I_1 and I_2), one can always find an element $x \in I_2 - I_1$ s.t. $I_1 \cup \{x\} \in \mathcal{I}$. This is runs counter to our assumption that I_1 is a maximal subset, and is thus a *contradiction!* $\rightarrow\leftarrow$

Expanding this notion to all maximal independent sets in $\mathcal{I}$, it is clear that if any two maximal independent sets in $\mathcal{I}$ have different sizes, the presence of Axiom (I3) means that maximality will always be broken for the smaller independent set. To preserve maximality, then, it must be that all maximal independent sets in M *must* have the same size. ∎

3. *For all integers* $0 \leq r \leq n$, $U_{r,n}$ *is a matroid.*

Proof. Let $U_{r,n} = (E, \mathcal{I})$, with $E = \{e_1, e_2, \dots, e_n\}$ and $\mathcal{I}$ be all subsets of $\mathcal{I}$ with r or fewer elements.

It is important to first assume that $n > 0$ so that the ground set is non-empty. With this out of the way, determining whether $U_{r,n}$ is a matroid is a matter of verifying whether the independence set $\mathcal{I}$ satisfies all three Axioms (I1), (I2) and (I3).

Case 1: If $r = 0$, then $\mathcal{I} = \{\emptyset\}$. This is not the same as saying that $\mathcal{I} = \emptyset$; $\mathcal{I}$ has one element, the empty set, so the size of $\mathcal{I}$ is 1. Therefore, $\mathcal{I}$ satisfies Axiom (I1). Trivially, Axioms (I2) and (I3) also hold.

Case 2: If $0 < r \leq n$, then the independent set $\mathcal{I}$ contains all subsets of the ground set up to size r. This means that Axiom (I1) holds trivially, and because all subsets of size r are included in $\mathcal{I}$ (I2) is also satisfied. To see whether $U_{r,n}$ also satisfies Axiom (I3), take any two $I, J \in \mathcal{I}$ (with $|I| < |J|$). The fact that $\mathcal{I}$ contains *all* subsets of the ground set E with r or fewer elements means that we can always find an $x \in J - I$ such that $I \cup \{x\} \in \mathcal{I}$. Hence Axiom (I3) holds as well.

For all cases, it has been shown that all Axioms (I1), (I2) and (I3) are all satisfied by $U_{r,n}$. Therefore, $U_{r,n}$ is, by definition, a matroid! ∎

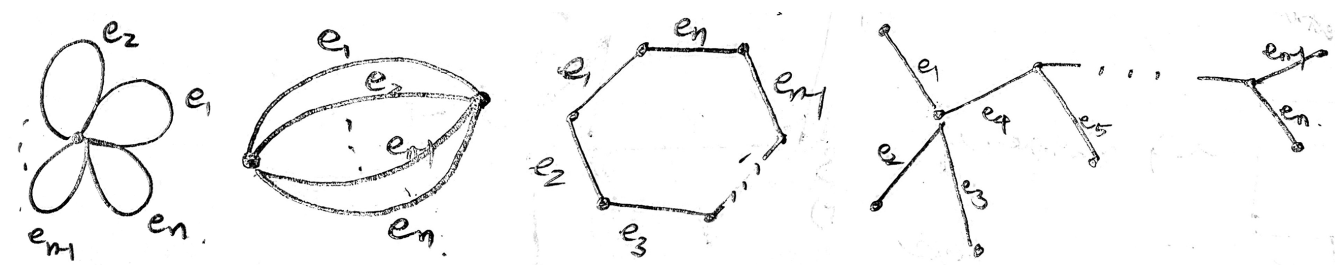
4. Let $U_{r,n} = (E, \mathcal{I})$, with $E = \{e_1, e_2, \dots, e_n\}$. For this problem assume, again, that $n > 0$, so that the ground set is non-empty. Let the graph from which the matroid is defined be G.

To check which r, n pairs make the uniform matroid graphic, first let $r = 0$. This implies that the independent set only has one element, $\emptyset$, which means that the circuits of $U_{r,n}$ have size 1. In graphical terms, this means that all the edges of G are *loops*. In this case then, $U_{0,n}$ is definitely graphical.

What about $r = 1$? This means that the base of the matroid $U_{1,n}$ has size 1, the circuits all have size 2. The graph G would therefore be composed of multiple edges on 2 vertices.

Now, let $r = n$. There are no circuits, thus no cycles in G, and the bases of $U_{n,n}$ have size n; this describes a graph that is, in essence, an acyclic graph with $n + 1$ vertices with n edges. (A *spanning tree!*)

What about $r = n - 1$? There is only one circuit in $U_{n-1,n}$; the entire ground set. This means that in graph terms, there is a single cycle, that is formed by all n edges. Therefore, $U_{n-1,n}$ is also graphic, and describes the graph C_n. (An exception to this is when $n = 2$: in that case, the graph that is formed is a multiple edge on 2 vertices. Also when $n = 1$, the graph would be a single loop. And of course, in both instances the matroid would still be graphical regardless.)



From left to right: graphs associated with $U_{0,n}, U_{1,n}, U_{n-1,n}, U_{n,n}$.

Assume now, that $1 < r < n - 1$ ($n - 1 \geq 3$, or $n \geq 4$). The definition of the uniform matroids means that $U_{r,n}$ has base size r (also its rank), and since r has to be greater than or equal to 2, the graph G cannot have any loops/multiple edges. Meanwhile, because the rank of the matroid is r, G must have $r + 1$ vertices, while also having n edges in total. However, if G is to not have any loops/multiple edges, it must be that G has at most $\binom{r+1}{2}$ edges (the number of edges of K_{r+1}).

But, since $n - 1 \geq 3$ (or, $n \geq 4$) and $r + 1 < n$,

$$\binom{r+1}{2} < \binom{n}{2} < n,$$

meaning that there are more edges in G than what would be allowed if G is to have no loops/multiple edges. This is a *contradiction*! $\rightarrow\leftarrow$

Hence, the values of (r, n) for which $U_{r,n}$ is graphic is $(0, n), (1, n), (n - 1, n), (n, n)$, with $n > 0$.

5. *The "Escher Matroid" is not a matroid.*

Proof. The provided point-line incidence structure violates (I3) and cannot be a matroid as a result. Consider the two following elements of $\mathcal{I}$, efg and $acef$. Per Axiom (I3), there must be an $x \in acef - efg$ such that $efg \cup \{x\} \in \mathcal{I}$; however this is not the case.

Take any element from $acef - efg$ and we observe that no such element that satisfies (I3) exists. For example, $a \cup efg = aefg \notin \mathcal{I}$ (coplanar points), and $c \cup efg = cefg \notin \mathcal{I}$ (also coplanar points).

Therefore, by definition, the "Escher Matroid" cannot be a matroid. ■

6. (a) *The Fano plane F_7 is not graphic.*

Proof. Assume, by way of contradiction, that the Fano plane F_7 is actually graphic, and call this potential graph G. Because the ground set of F_7 is $\{a, b, c, d, e, f, g\}$, its corresponding graph G should also have a total of 7 edges. In addition, the independent set of F_7 would have $\emptyset$, all single elements, all pairs, all triples *except* $abe, acf, adg, bdf, bcg, cde, efg$. (All subsets of the ground set with size 4 or bigger are automatically not in $\mathcal{I}$).

The bases of the Fano plane have size 3 (the matroid has rank 3), meaning that G would have to have 4 vertices (the spanning tree of G has 3 edges, so by definition G would have to have $3 + 1 = 4$ vertices). Also, because all subsets of the ground set of sizes less than 3 are all in the independent set, this means that there can be absolutely no loops/multiple edges in G. However, having 4 vertices with no loops/multiple edges means that G can only have at most 6 edges (K_4 has $\binom{4}{2} = 6$ edges). As G *must* have 7 edges, it cannot exist, so this is a *contradiction!* $\rightarrow\leftarrow$

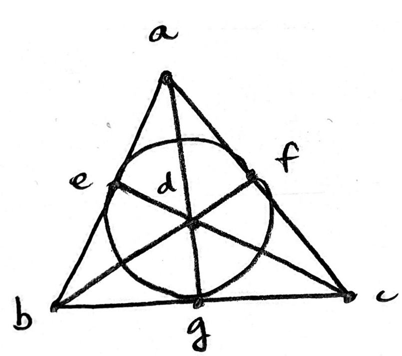
Therefore, F_7 cannot be graphic. ■

(b) *The non-Fano plane F_7^- is also not graphic.*

Proof. Again assume, by way of contradiction, that the non-Fano plane F_7 is actually graphic, and call this potential graph G'. The ground set of F_7^- would be the same, with $\{a, b, c, d, e, f, g\}$, and the independent set would have all the same elements that F_7 had (with the addition of efg). Therefore, its bases have size 3 and G' would also have 4 vertices and 7 edges in total without any loops/multiple edges present.

By the same reasoning as in (a), this is a *contradiction*, as we cannot make a graph G' that has 4 vertices with 7 total edges without any loops/multiple edges (again, see K_4). $\rightarrow\leftarrow$

Therefore, F_7^- is also non-graphic. ■



Used this as the Fano plane. F_7^- is the same without the "inner ring".