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Topics: LCCDE

1. Consider the linear constant-coefficient difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1]$$

Determine $y[n]$ for $n \geq 0$ when $x[n] = \delta[n]$ and $y[n] = 0, n < 0$.

The difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1]$$

To solve, we take the Fourier transform of both sides.

$$Y(e^{j\omega}) - \frac{3}{4}Y(e^{j\omega})e^{-j\omega} + \frac{1}{8}Y(e^{j\omega})e^{-j2\omega} = 2 \cdot X(e^{j\omega})e^{-j\omega}$$

The system function is given by:

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{2e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} \end{aligned}$$

The impulse response (for $x[n] = \delta[n]$) is the inverse Fourier transform of $H(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{-8}{1 + \frac{1}{4}e^{-j\omega}} + \frac{8}{1 - \frac{1}{2}e^{-j\omega}}$$

Thus,

$$h[n] = -8(\frac{1}{4})^n u[n] + 8(\frac{1}{2})^n u[n].$$

2. (a) Determine the frequency response $H(e^{j\omega})$ of the LTI system whose input and output satisfy the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

(b) (b) Write a difference equation that characterizes a system whose frequency response is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}$$

(a) **The difference equation:**

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \end{aligned}$$

(b) **A system with frequency response:**

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} \\ &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \end{aligned}$$

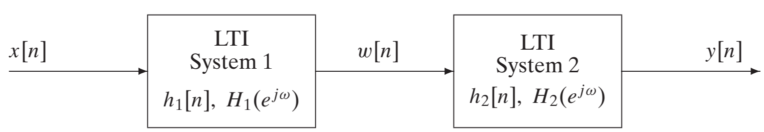
cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3].$$

3. Consider the cascade of LTI discrete-time systems shown in Figure. The first system is described by the frequency



response

$$H_1(e^{j\omega}) = e^{-j\omega} \begin{cases} 0, & |\omega| \leq 0.25\pi \\ 1, & 0.25\pi < |\omega| \leq \pi \end{cases}$$

and the second system is described by

$$h_2[n] = 2\frac{\sin(0.5\pi n)}{\pi n}$$

(a) Determine an equation that defines the frequency response, $H(e^{j\omega})$, of the overall system over the range $-\pi \leq \omega \leq \pi$.

(b) Sketch the magnitude, $|H(e^{j\omega})|$, and the phase, $\angle H(e^{j\omega})$, of the overall frequency response over the range $-\pi \leq \omega \leq \pi$.

(c) Use any convenient means to determine the impulse response $h[n]$ of the overall cascade system.

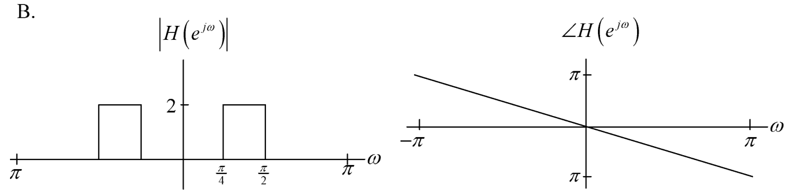
A. The impulse response $h_2[n] = 2\frac{\sin(0.5\pi n)}{\pi n}$ corresponds to a frequency response of

$$H_2(e^{j\omega}) = \begin{cases} 2, & \omega < \pi/2 \\ 0, & \pi/2 < \omega \le \pi \end{cases}$$

Then

$$\begin{aligned} H(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\ &= e^{-j\omega} \times \begin{cases} 0, & |\omega| < \pi/4 \\ 2, & \pi/4 < |\omega| < \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi. \end{cases} \end{aligned}$$

B.



C. Method 1 (Easiest):
The overall cascade system can be viewed as the difference of two lowpass filters with a one-sample delay.

$$h[n] = 2\frac{\sin\left(\frac{\pi}{2}(n-1)\right)}{\pi(n-1)} - 2\frac{\sin\left(\frac{\pi}{4}(n-1)\right)}{\pi(n-1)}$$

Method 2 (Harder):
The overall cascade system can be viewed as having a lowpass response modulated up to frequency $3\pi/8$.

$$h[n] = 4\frac{\sin\left(\frac{\pi}{8}(n-1)\right)}{\pi(n-1)}\cos\left(\frac{3\pi}{8}(n-1)\right)$$

Method 3 (Direct):
Just evaluate

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

4. A linear time-invariant system is described by the input-output relation

$$y[n] = x[n] + 2x[n-1] + x[n-2].$$

(a) Determine $h[n]$, the impulse response of the system.

(b) Is this a stable system?

(c) Determine $H(e^{jw})$, the frequency response of the system. Use trigonometric identities to obtain a simple expression for $H(e^{jw})$

(d) Plot the magnitude and phase of the frequency response.

(e) Now consider a new system whose frequency response is $H_1(e^{jw}) = H(e^{j(w+\pi)})$. Determine $h_1[n]$, the impulse response of the new system.

Sol)

(a)

$$\begin{aligned} y[n] &= x[n] + 2x[n-1] + x[n-2] \\ &= x[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2]) \\ h[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

(b) For given $h[n]$, the system is stable.

(c)

$$H(e^{jw}) = 1 + 2e^{-jw} + e^{-2jw}$$

(d) Plot the magnitude and phase of the frequency response.

i. Magnitude

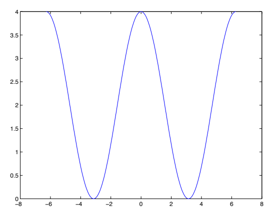


Figure 1: Magnitude

ii. Phase

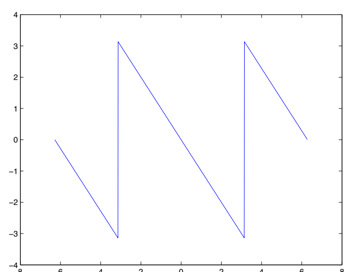


Figure 2: Phase

(e)

$$\begin{aligned} h_1[n] &= \frac{1}{2\pi} \int H_1(e^{jw}) e^{jw} dw \\ &= \frac{1}{2\pi} \int H_1(e^{j(w+\pi)}) e^{jwn} dw \\ &= e^{-j\pi n} \frac{1}{2\pi} \int H_1(e^{jw}) e^{jwn} dw \\ &= -1^n h[n] \\ &= \delta[n] - 2\delta[n-1] + \delta[n-2] \end{aligned}$$