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Topics: Discrete-time LTI systems and DTFT

1. An LTI system has impulse response $h[n] = 5(-1/2)^n u[n]$. Use the Fourier transform to find the output of this system when the input is $x[n] = (1/3)^n u[n]$.

We take the Fourier transform of both $h[n]$ and $x[n]$, and then use the fact that convolution in the time domain is the same as multiplication in the frequency domain.

$$\begin{aligned}
H(e^{j\omega}) &= \frac{5}{1+\frac{1}{2}e^{-j\omega}} \\
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
&= \frac{5}{1+\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{1}{3}e^{-j\omega}} \\
&= \frac{3}{1+\frac{1}{2}e^{-j\omega}} + \frac{2}{1-\frac{1}{3}e^{-j\omega}} \\
y[n] &= 2(\frac{1}{3})^n u[n] + 3(-\frac{1}{2})^n u[n]
\end{aligned}$$

2. Consider an LTI system with frequency response

$$H(e^{j\omega}) = \frac{1-e^{-j2\omega}}{1+\frac{1}{2}e^{-j4\omega}}$$

Determine the output $y[n]$ for all n if the input $x[n]$ for all n is

$$x[n] = \sin\left(\frac{\pi n}{4}\right)$$

.

SOL: First we re-write $x[n]$ as a sum of complex exponentials:

$$x[n] = \sin(\frac{\pi n}{4}) = \frac{e^{jwn/4} - e^{-jwn/4}}{2j}$$

Since complex exponentials are eigenfuntions of LTI systems

$$y[n] = \frac{H(e^{j\pi/4})e^{j\pi n/4} - H(e^{-j\pi/4})e^{-j\pi n/4}}{2j}$$

Evaluation the frequency response at $\omega = \pm\pi/4$

$$H(e^{j\frac{\pi}{4}}) = \frac{1-e^{-j\pi/2}}{1+1/2e^{-j\pi}} = 2(1+j) = 2\sqrt{2}e^{j\pi/4}$$

$$H(e^{-j\frac{\pi}{4}}) = \frac{1-e^{j\pi/2}}{1+1/2e^{j\pi}} = 2(1-j) = 2\sqrt{2}e^{-j\pi/4}$$

we get

$$y[n] = \frac{2\sqrt{2}e^{j\pi/4}e^{j\pi n/4} - 2\sqrt{2}e^{-j\pi/4}e^{-j\pi n/4}}{2j} = 2\sqrt{2}\sin(\frac{\pi n}{4} + \frac{\pi}{4})$$

3. (a) Determine the Fourier transform of the sequence

$$r[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

(b) Consider the sequence

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right], & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Sketch $w[n]$ and express its DTFT $W(e^{j\omega})$ in terms of $R(e^{j\omega})$ which is DTFT of $r[n]$.

(c) Sketch the magnitude of $R(e^{j\omega})$ and $W(e^{j\omega})$ for the case $M = 4$.

(a) We have

$$r[n] = \begin{cases} 1, & \text{for } 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Taking the Fourier transform

$$\begin{aligned} R(e^{j\omega}) &= \sum_{n=0}^{M} e^{-j\omega n} \\ &= \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} \\ &= e^{-j\frac{M}{2}\omega}\left(\frac{e^{j\frac{M+1}{2}\omega}-e^{-j\frac{M+1}{2}\omega}}{e^{j\omega}-e^{-j\omega}}\right) \\ &= e^{-j\frac{M}{2}\omega}\left(\frac{\sin(\frac{M+1}{2}\omega)}{\sin(\omega/2)}\right) \end{aligned}$$

(b) We have

$$w[n] = \begin{cases} \frac{1}{2}(1+\cos(\frac{2\pi n}{M}), & \text{for } 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

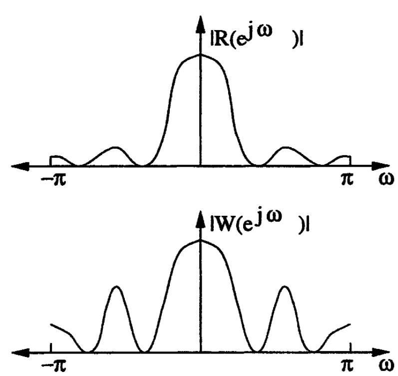
We note that,

$$w[n] = r[n] \cdot \frac{1}{2}[1 + cos(\frac{2\pi n}{M})].$$

Thus,

$$\begin{aligned} W(e^{j\omega}) &= R(e^{j\omega}) * \sum_{n=-\infty}^{\infty} \frac{1}{2}(1+cos(\frac{2\pi n}{M}))e^{-j\omega n} \\ &= R(e^{j\omega}) * \sum_{n=-\infty}^{\infty} \frac{1}{2}(1+\frac{1}{2}e^{j\frac{2\pi n}{M}}+\frac{1}{2}e^{-j\frac{2\pi n}{M}})e^{-j\omega} \\ &= R(e^{j\omega}) * (\frac{1}{2}\delta(\omega)+\frac{1}{4}\delta(\omega+\frac{2\pi}{M})+\frac{1}{4}\delta(\omega-\frac{2\pi}{M})) \end{aligned}$$

(c)



4. An LTI discrete-time system has frequency response given by

$$H(e^{j\omega}) = \frac{1}{1-0.8e^{-j\omega}} + \frac{e^{-j2\omega}}{1-0.8e^{-j\omega}}$$

(a) Use one of the above forms of the frequency response to obtain an equation for the impulse response $h[n]$ of the system.

(b) From the frequency response, determine the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$ of the system.

(c) If the input to this system is

$$x[n] = 4 + 2\cos(\omega_0 n)$$

for what value of ω_0 will the output be of the form $y[n] = A =$ constant for $-\infty < n < \infty$? What is the constant A?

A.

$$H(e^{j\omega}) = \frac{1}{1-0.8e^{-j\omega}} + \frac{e^{-j2\omega}}{1-0.8e^{-j\omega}}$$
$$h[n] = (0.8)^n u[n] + (0.8)^{n-2} u[n-2]$$

B.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+e^{-j2\omega}}{1-0.8e^{-j\omega}}$$
$$Y(e^{j\omega}) - 0.8e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) + e^{-j2\omega}X(e^{j\omega})$$
$$y[n] - 0.8y[n-1] = x[n] + x[n-2]$$
$$y[n] = 0.8y[n-1] + x[n] + x[n-2]$$

C. Using the frequency response we can write the output as

$$y[n] = H(e^{j0})4 + 2\left|H(e^{j\omega_0})\right|\cos\left(\omega_0 n + \angle H(e^{j\omega_0})\right).$$

To get $y[n] =$ constant we need $\left|H(e^{j\omega_0})\right| = 0$, which means $1 + e^{-j2\omega_0} = 0$, or $\omega_0 = \pi/2$.

Then $y[n] = 4\frac{1+1}{1-0.8} = 40$.

5. Which of the following disrete-time signals could be eigenfunctions of any stable LTI system?

(a) $5^n u[n]$

(b) e^{j2wn}

(c) $e^{jwn} + e^{j2wn}$

(d) 5^n

(e) $5^n e^{j2wn}$

(a)

$$\begin{aligned}
x[n] * h[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]5^{n-k}u[n-k] \\
&= 5^n \sum_{k=-\infty}^{n} h[k]5^{-k} \\
&\Rightarrow \text{Because the summation depends on } n\text{, this is not an eigenfunction.}
\end{aligned}$$

(b) Sinusoid function is an eigenfunction of LTI system.

(c)

$$\begin{aligned}
x[n] * h[n] &= \sum_{k=-\infty}^{\infty} h[k]e^{jw(n-k)} + \sum_{k=-\infty}^{\infty} h[k]e^{j2w(n-k)} \\
&= e^{jw}H(e^{jw}) + e^{j2w}H(e^{j2w}) \\
&\Rightarrow \text{Not an eigenfunction.}
\end{aligned}$$

(d)

$$\begin{aligned}
x[n] * h[n] &= \sum_{k=-\infty}^{\infty} h[k]5^{n-k} \\
&= 5^n \sum_{k=-\infty}^{\infty} 5^{-k} \\
&\Rightarrow \text{Eigenfunction.}
\end{aligned}$$

(e)

$$\begin{aligned}
x[n] * h[n] &= \sum_{k=-\infty}^{\infty} h[k]5^{n-k}e^{j2w(n-k)} \\
&= 5^n e^{j2wn} \sum_{k=-\infty}^{\infty} 5^{-k}e^{-j2wk} \\
&\Rightarrow \text{Eigenfunction.}
\end{aligned}$$

6. Consider an LTI system with frequency response

$$H(e^{jw}) = e^{-j(w-\frac{\pi}{4})}\Big(\frac{1+e^{-j2w}+4e^{-j4w}}{1+\frac{1}{2}e^{-j2w}}\Big)$$

Determine the output $y[n]$ for all n if the input for all n is

$$x[n] = \cos(\frac{\pi n}{2}).$$

Note we can write $x[n]$ as a linear combination of complex exponentials:

$$x[n] = \frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2}$$

This means that for an LTI system with frequency response $H(e^{j\omega})$, the output to the input $e^{j\pi n/2}$ is $H(e^{j\pi n/2})e^{j\pi n/2}$ and $e^{-j\pi n/2}$ is $H(e^{-j\pi n/2})e^{-j\pi n/2}$, therefore the output to $x[n]$ is

$$\frac{H(e^{j\pi/2})e^{j\pi n/2} + H(e^{-j\pi/2})e^{-j\pi n/2}}{2}$$

Evaluating $H(e^{j\omega})$ at $\omega = \pm\pi/2$ and plugging it into the above, we have the output

$$y[n] = 8e^{j\pi/4}\cos(\frac{\pi}{2}n - \frac{\pi}{2})$$

7. For $X(e^{jw}) = 1/(1 - ae^{-jw})$, with $-1 < a < 0$, determine and sketch the following as a function of w :
   (a) $Re\{X(e^{jw})\}$
   (b) $Im\{X(e^{jw})\}$
   (c) $|X(e^{jw})|$
   (d) $Arg\{X(e^{jw})\}$

   Sol)

   (a) $Re\{X(e^{jw})\} = \frac{1}{2}\big\{X(e^{jw}) + X^*(e^{jw})\big\} = \frac{1-a\cos(w)}{1-2a\cos(w)+a^2}$
   (b) $Im\{X(e^{jw})\} = \frac{1}{2j}\big\{X(e^{jw}) - X^*(e^{jw})\big\} = \frac{-a\sin(w)}{1-2a\cos(w)+a^2}$
   (c) $|X(e^{jw})| = \big\{X(e^{jw})X^*(e^{jw})\big\}^{\frac{1}{2}} = \frac{1}{1-2a\cos(w)+a^2}^{\frac{1}{2}}$
   (d) $Arg\{X(e^{jw})\} = \arctan(\frac{Im\{X(e^{jw})\}}{Re\{X(e^{jw})\}}) = \arctan(\frac{-a\sin(w)}{1-a\sin(w)})$

8. A sequence has the discrete-time Fourier transform

$$X(e^{jw}) = \frac{1-a^2}{(1-ae^{-jw})(1-ae^{jw})}, \quad |a| < 1.$$

   (a) Find the sequence $x[n]$.
   (b) Calculate $\int_{-\pi}^{\pi} X(e^{jw})\cos(w)dw/2\pi$.

   (a) There are several ways to solve this problem.
   (1) Let us define $x_1[n]$ as

$$x_1[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

   and let

$$x_1[n] = a^n u[n],\ X_1(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

Then

$$X(e^{j\omega}) = (1-a^2)X_1(e^{j\omega})X_1(e^{-j\omega})$$

This implies

$$x[n] = (1-a^2)x_1[n] * x_1[-n]$$

$$\begin{aligned} x_1[n] * x_1[-n] &= \sum_{k=-\infty}^{\infty} a^k u[k] a^{k-n} u[k-n] \\ &= a^{-n} \sum_{k=\max[0,n]}^{\infty} a^{2k} \end{aligned}$$

So we have

$$x[n] = \begin{cases} a^n & n \geq 0 \\ a^{-n} & n < 0 \end{cases}$$

Or simply put, $x[n] = a^n u[n] + a^{-n} u[-n-1]$.
(2) Alternatively we can use partial fraction expansion,

$$\begin{aligned} X(e^{jw}) &= \frac{1-a^2}{(1-ae^{-jw})(1-ae^{jw})} \\ &= \frac{1}{1-ae^{-jw}} + \frac{ae^{jw}}{1-ae^{jw}} \end{aligned} \tag{1}$$

When we try to invert the second term of (1), $\frac{ae^{j\omega}}{1-ae^{j\omega}}$ we have to be careful because $|a| < 1$. By defining

$$X_1(e^{j\omega}) = \frac{ae^{-j\omega}}{1-ae^{-j\omega}}$$

whose inverse DTFT is $x_1[n] = a^n u[n-1]$. But we see that the second term of (1) is $X_1(e^{-j\omega})$, which means its inverse DTFT is $x_1[-n] = a^{-n} u[-n-1]$. So we have

$$x[n] = a^n u[n] + a^{-n} u[-n-1]$$

(b)

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\cos(w)dw &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\frac{e^{jw}+e^{-jw}}{2}dw \\ &= \frac{1}{2}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})e^{jw}dw + \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})e^{-jw}dw\Big) \\ &= \frac{1}{2}(x[-1]+x[1]) \\ &= a \end{aligned}$$

1. **(MATLAB):**

   Try to avodig using `for` or `while` loops in the code – use vectors and matrices if possible

   (a) **Plotting discrete signals:** Construct a sequence $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Also consider impulse response of a moving average system

   $$h[n] = \begin{cases} \frac{1}{M+1} & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

   When you plot discrete sequences, always use `stem` function rather than the usual `plot` function. Let $N = 20$ and $M = 5$. Plot the sequences $x[n]$ and $h[n]$ for index vector $0 \leq n \leq N-1$ where $N = 20$. Also use `subplot` to plot two sequences in one page: $x[n]$ above and $h[n]$ below. (Hint: Some part of your code will look like
   ‘`subplot(2,1,1); stem(...); subplot(2,1,2); stem(...)`’)

   (b) **Discrete convolution:** Let $y[n]$ be the output of the system with impulse response $h[n]$ and input sequence $x[n]$. We would like to find out $y[n]$, and you may want to use built-in `conv` function. Also you will find that the length of output $y[n]$ will be different from $x[n]$ or $h[n]$. Plot $y[n]$ against the index vector nn that starts from 0 and has the same length as $y[n]$. You might find `length` or `size` function useful.

   (c) **Computing DTFT:** Plot the magnitude and phase of the DTFT of $x[n]$, $h[n]$ and $y[n]$. Use subplot to plot magnitude and phase of each signal in one page respectively against ω ranging from -3π to 3π with the step size of 0.01π. You should use normal `plot` function for plotting in the frequency domain. (Warning: remember vectorized coding – you should use ONLY vectors and matrices to compute DTFT!) (Hint: When computing DTFT, try to express the complex exponentials as a *matrix*.)

```
%Description: M-file for assignment 2
clear all
clc

% (a) Plotting discrete signals
N=20;
M=5;
n=0:N-1;
u=(n>=0);
ut=(n>M);
x=((1/2).^n).*u;
h=1/(M+1).*(u-ut);
figure(1);
suptitle('(a)');
subplot(2,1,1);
stem(n,x);
xlabel('n')
ylabel('x[n]')
subplot(2,1,2);
stem(n,h);
xlabel('n')
```

```
ylabel('h[n]')


% (b) Discrete convolution
y=conv(x,h);
nn=0:length(y)-1;
figure(2);
stem(nn,y);
xlabel('nn')
ylabel('y[n]')
title('(b)');

% (c) Computing DTFT
W=-3*pi:0.01*pi:3*pi;
X=x*(exp(-1i).^(n'*W));
H=h*(exp(-1i).^(n'*W));
Y=y*(exp(-1i).^(nn'*W));

figure(3);
suptitle('X(jw)');
subplot(2,1,1);
plot(W,abs(X));
xlabel('W');
ylabel('Magnitude');
subplot(2,1,2);
plot(W,angle(X) );
xlabel('W');
ylabel('Angle');


figure(4);
suptitle('H(jw)');
subplot(2,1,1);
plot(W,abs(H));
xlabel('W');
ylabel('Magnitude');
subplot(2,1,2);
plot(W,angle(H));
xlabel('W');
ylabel('Angle');


figure(5);
suptitle('Y(jw)');
subplot(2,1,1);
plot(W,abs(Y));
xlabel('W');
ylabel('Magnitude');
```

```
subplot(2,1,2);
plot(W,angle(Y));
xlabel('W');
ylabel('Angle');
```