MQP Proposal: Techniques for Functional Solid Modeling and Numerical Solutions to Boundary Value Problems

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April 5, 2014

Abstract

The rise of democratized and digital manufacturing has lead to a change in the demands of solid modeling tools. Functional modeling representations appear to address these new requirements. Prior work has established techniques to simplify the preparation of numerical analysis using functional solids. I will lay out a motivation for a Major Qualifying Project studying the mathematics and application of techniques for functional modeling and numerical solutions to boundary value problems.

1 Introduction

1.1 Application Principles

With the advent of consumer level 3D printing, there has been an increased demand for Computer Aided Design (CAD) software. 3D printing allows for the user to customize designs before production. In order for this to be possible, geometric models need to be parametric. Typically the designer will expose a few parameters for the user to adjust and thus customize to their liking.

Today many engineers use parametric design to adjust for manufacturing tolerances and component interfaces. Parametric design allows digital designs to adjust to local material availability. An example is exposing hardware sizes as Metric or Imperial measure in order to suit international markets. This allows for generalized application of appropriate technology and ecological sustainability.

Mathematically the fundamentals of parametric design corresponds well with a functional solid representation. For example with a function representing a half space or boundary,$f(x_1, ..., x_n, p_1, ..., p_n) \geq 0$, the parameters x_n represent spatial dimensions or time varying quantities and p_n are the user parameters established as constants in the function. We will see later this can become a logical predicate.

1.2 Solid Modeling

Today the dominant paradigm for solid modeling has remained unchanged since the 1970's, relying primarily on Boundary Representation (B-Rep). It relies primarily on the manipulation and representation of edges, vertices, and faces to build a model. While B-Rep is intuitive for users of a graphical environment, it is unwieldy as a textual and functional representation. This methods is natural for engineers and designers, but sacrifices parametric design. In addition B-Rep requires the use of a geometry kernel to handle the interpretation of constraints and geometric construction. [1]

The use of a geometry kernel removes the solid from a mathematical representation. For numerical analysis this decouples the geometric description from the problem formulation. [2] This middle step of Computer Aided Engineering (CAE) is known as pre-processing. Research has shown that functional representations can simplify or eliminate this step.

In addition, designers targeting parametric design have turned to the methods of Constructive Solid Geometry (CSG), which works using manipulation of geometric primitives (half-spaces) as a level of abstraction. This enables parametric solids to be represented using operations and relations on primitive solids. CSG has been growing in popularity due to programs such as OpenSCAD[1], CoffeeSCAD[2], POVRay[3] and Thingiverse Customizer[4]. These programs are particularly popular for collaboration in conjunction with version control systems such as Git.

2 Mathematical Theory

2.1 Rvachev Functional Representation

In the 1960's Vladimir Rvachev produced a method for handling the "inverse problem of analytic geometry". His theory consists of functions which provide a link between logical and set operations in geometric modeling and analytic geometry.[3] I believe the following anecdote helps elucidate the theory. While attempting to solve boundary value problems, Rvachev formulated an equation of a square as

$$a^2 + b^2 - x^2 - y^2 + \sqrt{(a^2 - x^2)^2 + (b^2 - y^2)^2} = 0.$$

Implicitly, the sides of a square can be defined as $x = \pm a$ and $y = \pm b$. The union of these two is a square. By reducing the formulation of the square we can generalize an expression for the union between two functions.

$$\cup : f_1 + f_2 + \sqrt{f_1^2 + f_2^2} = 0$$

Likewise we can see that intersections and negations can be formed for logical completion.

$$\cap : f_1 + f_2 - \sqrt{f_1^2 + f_2^2} = 0$$

$$\neg : -f_1$$

These formulations can be modified for C^m continuity for any m.[4] In addition Pasko, et. al. have shown that Rvachev functions can serve to replace

[1] `http://www.openscad.org`
[2] `http://coffeescad.net/`
[3] `http://www.povray.org/`
[4] `http://www.thingiverse.com`

a geometry kernel by creating logical predicates. [5] Their research also establishes the grounds for user interfaces and environment description. For this work a practical implementation will most likely leverage their insights. Rvachev and Shapiro have also shown that using the POLE-PLAST and SAGE systems a user can generate complex semi-analytic geometry as well.[6]

2.2 Numerical Analysis

While a functional representation for geometry is mathematically enticing on its own, the power it gives for numerical analysis might be its greatest virtue. Numerical analysis justified the initial investigation by Rvachev early on. A boundary value problem on a R-Function-predicate domain allows for analysis without construction of a discrete mesh.[6]

One of the most general expositions in the English language of R-Functions applied to BVPs is Vadim Shapiro's "Semi-Analytic Geometry with R-Functions". [4] Unfortunately, no monographs about R-Functions exist in the English literature. Most literature is in Russian, however many articles presenting applied problems using the R-Function Method. [7] This is the topic of this project I stand to gain the most insight.

2.3 Closed Loop Optimization

Optimization could be studied using mathematical solid modeling. I have not completely researched existing literature on this topic, so I include this idea as a "stretch goal" for the project. Given a parametrized solid and numerical analysis, optimization might prove useful. Possible ideas include the minimization of mechanical stress on a solid or reduction of material volume.

3 Technologies

3.1 Julia

Julia is a programming language first released in early 2012 by a group of developers from MIT. The language targets technical computing by providing a dynamic type system with near-native code performance. This is accomplished by using three concepts: a Just-In-Time (JIT) compiler to target the LLVM framework, a multiple dispatch system, and code specialization.[8] The syntactical style is similar to MATLAB and Python. The language implementation and many libraries are available under the permissive MIT license.[5]

Software development could elucidate the mathematics of this project and contribute unique libraries to the Julia ecosystem. Benchmarks have shown the language can consistently perform within a factor of two of native C and FORTRAN code.[6] This is enticing for a solid modeling application and for numerical analysis, as the code abstraction can grow organically without performance penalty. In fact, the authors of Julia call this balance a solution to the "two language problem". The problem is encountered when abstraction in

[5] `http://opensource.org/licenses/MIT`

[6] `http://julialang.org/benchmarks`

a high-level language will disproportionately affect performance unless implemented in a low-level language.

I would like to use Julia as a media of exploration for the mathematical concepts and techniques during this project. While the language is relatively new and in development, sufficient documentation and an active community can help support software development. Many advanced and mature packages for fields such as operations research exist.[9]

3.2 Open Source

Along with Julia, I believe it is important to publish software frequently and openly. In the past the open source community has been very receptive to new ideas. I expect that over the course of the project we can attract external contributions to the software developed. This will help strengthen and quicken developments.

As such, I believe it will be important to improve the accessibility to the mathematical theory. Along with a report, I would like to improve and write Wikipedia pages to enable a greater audience. The existing Wikipedia page for Rvachev-Functions[7] contains simple concepts compared to the richness of the academic literature existing on the topic.

4 Conclusion

The motivation of this project is social. 3D printing has promise to be disruptive to economics and technological application. However it is only the physical media in which collaboration between people can change. The underlying mathematics of the *digital* media for collective action will be the focus of this project.

The intellectual challenge for this project will be mathematical. In particular, the question is *how can techniques for functional solid modeling and numerical analysis be applied to improve the state of 3D modeling?* My previous insight into the importance of parametric design and engineering analysis will drive the development of a solid mathematical understanding.

The application of this project will be through distillation of mathematical theory into software and documentation. In particular, libraries and scripts will provide means for others to experiment and improve our work. Documentation and explanations of theory will enrich the open source 3D modeling community.

I first encountered Rvachev functions in October 2013 while researching mathematical techniques to implement a CAD program. The existing literature is vast, however few practical implementations exist for the maker community to use. I expect this project to take 4/3 units of work during the 2014-2015 academic year. My hope is this will permit me to review the literature and develop applications of utility that might inspire more research. I appreciate your time in reviewing this proposal.

References

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